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Geometrie

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Abstract. The topics discussed at the meeting ranged from geometric evolution equations to minimal surfaces, Riemannian foliations and hyperbolic geometry. Because of a flexible schedule, the 53 participants had ample time for discussion.

Mathematics Subject Classification (2010): 53xx.

Introduction by the Organisers

The format of the meeting consisted of 18 one hour talks and four half hour after-dinner talks. The after-dinner talks were given by PhD students and recent PhDs.

Six of the talks were related to geometric flows. Gerhard Huisken investigated the mean curvature flow with surgery in 3 dimensional manifolds. If one starts with a mean convex initial surface then there are only finitely many singularities and, in the case of long time existence, the solution converges to a stable minimal hypersurface of the ambient manifold. Carlo Sinestrari established various results on ancient solutions of the mean curvature flow, e.g. he gave several characterizations of the shrinking sphere solutions. Anton Petrunin suggested that each compact polyhedral space with nonnegative curvature might be the initial singular metric of a smooth orbifold Ricci flow with nonnegative curvature operator. He provided a proof in the 3-dimensional case. Tobias Marxen talked on the asymptotics ($t \to \infty$) of the Ricci flow on a noncompact $(n + 1)$-dimensional manifold endowed with an isometric T^n-action. Peter Topping investigated the gradient flow of the Dirichlet energy of mappings from a surface S of genus ≥ 2 into a fixed Riemannian manifold M, letting both the map and the (hyperbolic) metric of the domain vary. For a nonpositively-curved target, he could establish long time existence of the flow. Valentino Tosatti talked about the Kähler Ricci-flow and proved a decade-old conjecture on the set of singularities, namely it forms an analytic variety.

Kähler geometry was also the subject of two other talks. Hans-Joachim Hein gave a characterization of Stenzel's metric, a Ricci flat Kähler metric on the tangent bundle of the sphere. Ben Weinkove generalized Yau's solution of the Calabi conjecture to certain Hermitian metrics on closed manifolds.

Karl-Theodor Sturm gave a survey on synthetic definitions of lower Ricci curvature bounds on metric measure spaces, in terms of properties of the optimal mass transport and solutions of the heat equation. André Neves showed how min-max techniques can be used to settle Yau's question about infinitely many minimal hypersurfaces in 3-manifolds, in the case of positive Ricci curvature, along with higher-dimensional generalizations. Claude LeBrun established optimal estimates for the L^2-norm of the positive part of the Weyl curvature, for various classes of 4-manifolds. Esther Cabezas-Rivas showed that a Riemannian manifold with a lower sectional curvature bound and an upper diameter bound is finitely covered by a nilmanifold, provided that the L^1-norm of the curvature operator is sufficiently small.

There were three talks related to (singular) Riemannian foliations and isometric group actions. Marco Radeschi explained how Clifford representations can be used to find many nonhomogenous singular Riemannian foliations of a round sphere. Alexander Lytchak proved rigidity statements for Riemannian foliations, ensuring that no exceptional fibers can occur, e.g. if the ambient space is a topological sphere and the leaf dimension is 7. Wolfgang Spindeler showed that fixed point homogeneous nonnegatively-curved manifolds admit a double disc bundle decomposition.

Ricardo Mendes explained why most known examples of homogeneous manifolds with positive sectional curvature also satisfy a certain stronger curvature condition, namely one can find metrics on these manifolds whose curvature operator can be modified by a four form in such a way that the modified curvature operator is nonnegative.

The geometry of hyperbolic space and higher rank symmetric spaces entered into three talks. Bernhard Leeb introduced the concept of a Morse action on a noncompact symmetric space, and used it to give a higher-rank substitute for convex cocompactness. Ursula Hamenstädt gave a simplified proof of the statement that any fundamental group of a hyperbolic 3-manifold contains a nontrivial surface subgroup, along with higher dimensional generalizations. Michelle Bucher gave integrality results for characteristic numbers of certain representations of lattices in hyperbolic isometry groups.

The remaining talks were given by Lange, Hensel and Gaifullin. Christian Lange answered the question of when the underlying topological space of an orbifold is in fact a manifold. Sebastian Hensel considered piecewise isometric self-maps of an interval to itself and showed that the uniquely ergodic maps form a path connected subset. Alexander Gaifullin presented various methods to construct flexible polyhedra and showed that the enclosed volume stays constant under deformation.

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Workshop: Geometrie

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Abstracts

Mean curvature flow of embedded meanconvex surfaces in 3-manifolds

Gerhard Huisken
(joint work with Simon Brendle)

We study solutions $F : M^2 \times [0,T) \to (N^3, \overline{g})$ of mean curvatrue flow

$$\frac{d}{dt}F = \overrightarrow{H} = -H \cdot \nu \qquad \text{(MCF)}$$

for closed, embedded initial surfaces M_0^2 of positive mean curvature $H > 0$. We prove that there is a solution of (MCF) with finitely many surgeries emanating from M_0^2 that either gets extinct in finite time and decomposes M_0^2 into finitely many copies of S^2 and $S^1 \times S^1$ or becomes smooth for $t > T_0$ and converges for $t \to \infty$ to a stable minimal surface of genus no larger than that of M_0^2. Here the Riemannian 3-manifold $(N^3, \overline{g})$ is assumed smooth and closed.

The proof follows the strategy for MCF with surgery developed by Huisken-Sinestrari for 2-convex surfaces $M^n \subset \mathbb{R}^{n+1}, n \geq 3$. It uses convexity estimates of Huisken-Sinestrari, a sharpening of Andrew's non-collapsing estimate by Brendle an interior gradient estimate for the curvature due to Haslhofer-Kleiner, the regularity theory for level-set solutions due to B. White and a new monotonicity formula due to Brendle to set up a surgery algorithm that uses curve-shortening flow in the cross-section of necks.

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Clifford Algebras and new singular Riemannian foliations in spheres

Marco Radeschi

A singular Riemannian foliation on a Riemannian manifold M is, roughly speaking, a partition of M into connected complete submanifold, not necessarily of the same dimension, that locally stay at a constant distance from each other. Singular Riemannian foliations on round spheres provide local models of general singular Riemannian foliations around a point.

An example of singular Riemannian foliation on round spheres is given by the decomposition into the orbits of an isometric group action, and such a foliation is called *homogeneous.*

Classifying non-homogeneous singular Riemannian foliations in spheres seems a very complex problem. A trivial way to obtain new foliations from old ones is called *spherical join.* Given singular Riemannian foliations $(\mathbb{S}^{n_i}, \mathcal{F}_i)$, $i = 1, 2$, the spherical join gives a new foliation $(\mathbb{S}^{n_1+n_2+1}, \mathcal{F}_1 \star \mathcal{F}_2)$. Any foliation that cannot be written as a spherical join is called *indecomposable*, and every foliation can be written in an essentially unique way as a spherical join of indecomposable ones. Because of this, our main interest lies in finding non-homogeneous, indecomposable singular Riemannian foliations.

The only known indecomposable non-homogeneous singular Riemannian foliations are either a family of codimension 1 (the so called *FKM examples*) or the foliation in $\mathbb{S}^{15}$ given by the fibers of the Hopf fibration $\mathbb{S}^{15} \to \mathbb{S}^8$. Recently A. Lytchak and B. Wilking proved, using a previous result of Wilking [5] and Grove-Gromoll [1], that this is the only non-homogeneous *regular* foliation, i.e., with leaves of the same dimension [3].

The aim of this talk is to discuss the main results of [4] where we show how to use Clifford systems to produce a new, large class of indecomposable, non-homogeneous singular Riemannian foliations of arbitrary codimension, which in particular includes all the previously known examples. Recall that a Clifford system can be thought of as a family $C = (P_0, \dots P_m)$ of symmetric matrices in $(\mathbb{R}^{2l}, \langle\, ,\, \rangle)$ such that $P_i^2 = Id$ for all $i = 0, \dots m$ and $P_iP_j = -P_jP_i$ for $i \neq j$. We define the map

$$\begin{aligned} \pi_C : \mathbb{S}^{2l-1} &\longrightarrow \mathbb{R}^{m+1} \\ x &\longmapsto \Big(\langle P_0 x, x\rangle, \dots \langle P_m x, x\rangle\Big). \end{aligned}$$

Theorem 1. *Let $C = (P_0, \dots P_m)$ be a Clifford system on $\mathbb{R}^{2l}$. Then the image of π_C is contained in the unit disk $\mathbb{D}_C$ around the origin in $\mathbb{R}^{m+1}$, and the following hold:*

(1) *The preimages of π_C define a singular Riemannian foliation $(\mathbb{S}^{2l-1}, \mathcal{F}_C)$ whose leaf space is either the m-sphere $\mathbb{S}_C = \partial\mathbb{D}_C$ (if $l = m$) or the disk $\mathbb{D}_C$ (if $l > m + 1$). In either case the induced metric on the quotient is a round metric of constant sectional curvature* 4.

(2) *The foliation $(\mathbb{S}^{2l-1}, \mathcal{F}_C)$ is homogeneous if and only if $m = 1, 2$ or $m = 4$ and $P_0 \cdot P_1 \cdot P_2 \cdot P_3 \cdot P_4 = \pm Id$, in which cases it is spanned by the orbits of the diagonal action of $\mathsf{SO}(k)$ on $\mathbb{R}^k \times \mathbb{R}^k$ ($m = 1$), $\mathsf{SU}(k)$ on $\mathbb{C}^k \times \mathbb{C}^k$ ($m = 2$) or $\mathsf{Sp}(k)$ on $\mathbb{H}^k \times \mathbb{H}^k$ (m=4).*

When the leaf space is a sphere one recovers the Hopf fibrations $\pi_C : \mathbb{S}^{2m-1} \to \mathbb{S}^m$, $m = 2, 4, 8$. When the leaf space is $\mathbb{D}_C$ with the round metric (also *hemisphere metric*) the π_C-preimages in $\mathbb{S}^{2l-1}$ of the concentric spheres in $\mathbb{D}_C$ give rise to the FKM family associated to the Clifford system C.

A singular Riemannian foliation $\mathcal{F}_0$ on the m-sphere $\mathbb{S}_C = \partial\mathbb{D}_C \subseteq \mathbb{R}^{m+1}$ extends by homotheties to a singular Riemannian foliation $\mathcal{F}_0^h$ on $\mathbb{D}_C$ (with the hemisphere metric) and the π_C-preimages of the leaves of $\mathcal{F}_0^h$ define a new foliation $\mathcal{F}_0 \circ \mathcal{F}_C$. This is a special case of a more general construction of Lytchak [2, Sect. 2.5].

Theorem 2. *Let C be a Clifford system on $\mathbb{R}^{2l}$ and let $(\mathbb{S}^{2l-1}, \mathcal{F}_C)$ be the associated Clifford foliation.*

(1) *If $\mathcal{F}_0$ is any singular Riemannian foliation on $\mathbb{S}_C$, then the foliation $(\mathbb{S}^{2l-1}, \mathcal{F}_0 \circ \mathcal{F}_C)$ is a singular Riemannian foliation as well.*
(2) *Let $C_{8,1}$ and $C_{9,1}$ denote, respectively, the unique Clifford systems $(P_0, \dots P_8)$ on $\mathbb{R}^{16}$ and $(P_0, \dots P_9)$ on $\mathbb{R}^{32}$. If $C \neq C_{8,1}$, $C_{9,1}$ then $(\mathbb{S}^{2l-1}, \mathcal{F}_0 \circ \mathcal{F}_C)$ is homogeneous if and only if both $\mathcal{F}_0$ and $\mathcal{F}_C$ are homogeneous. If $C = C_{9,1}$ and $(\mathbb{S}^{31}, \mathcal{F}_0 \circ \mathcal{F}_C)$ is homogeneous, then $\mathcal{F}_0$ is homogeneous.*

We call the foliations $\mathcal{F}_C$ described above *Clifford foliations*, and the foliations $\mathcal{F}_0 \circ \mathcal{F}_C$ *composed foliations.*

Unlike the FKM examples, inequivalent Clifford system give rise to different Clifford foliations. Moreover, Clifford foliations can be geometrically characterized as the only singular Riemannian foliations on spheres whose quotient is a sphere or a hemisphere of curvature 4. More precisely, let $\mathcal{G}$ the class of singular Riemannian foliations on a round sphere, whose quotient is a sphere or a hemisphere of curvature 4 and let $\mathcal{A}$ be the class of Clifford systems. Then the following holds.

Theorem 3. *The assignment $C \mapsto \mathcal{F}_C$ determines a bijection*

$$\mathcal{A}/\{\textit{geometric equivalence}\} \xrightarrow{\simeq} \mathcal{G}/\{\textit{congruence}\}$$

This is somewhat surprising, since it establishes an equivalence between purely algebraic and purely geometric objects.

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Morse actions of discrete groups on symmetric spaces

Bernhard Leeb
(joint work with Misha Kapovich and Joan Porti)

Let $X = G/K$ be a symmetric space of noncompact type, i.e. G is a noncompact semisimple Lie group and K a maximal compact subgroup. We consider discrete subgroups $\Gamma \subset G$; the isometric actions $\Gamma \curvearrowright X$ are then properly discontinuous. We are particularly interested in the case of infinite covolume and actions with "rank one behavior" on higher rank symmetric spaces.

Let $\overline{X} = X \cup \partial_\infty X$ denote the visual (ball) compactification of X; points in the visual boundary $\partial_\infty X$, the so-called ideal points at infinity, correspond to equivalence classes of asymptotic rays in X. The isometric action $G \curvearrowright X$ extends to a continuous action $G \curvearrowright \overline{X}$. The G-orbits in $\partial_\infty X$ are parametrized by the spherical model Weyl chamber $\Delta \cong \partial_\infty X/G$. The *regular* part $\partial_\infty^{reg} X \subset \partial_\infty X$ of the visual boundary consists of the ideal points which project to the interior of Δ. It is the union of the open (spherical Weyl) chambers at infinity; these are the top-dimensional simplices with respect to the spherical (Tits) building structure on $\partial_\infty X$. There is a natural projection $\partial_\infty^{reg} X \to \partial_F X \cong G/B$ to the space of chambers $\partial_F X$, the so-called Fürstenberg boundary, which is identified with the generalized full flag manifold G/B.

Given a discrete subgroup $\Gamma \subset G$, the set $\Lambda = \overline{\Gamma x} \cap \partial_\infty X$ is called the *limit set* of the action $\Gamma \curvearrowright X$. It does not depend on the orbit Γx. We restrict ourselves to subgroups satisfying certain regularity conditions. For simplicity of exposition, we assume here only the strongest such condition; in [KLP1, KLP2] we work with weaker conditions. We call the discrete subgroup $\Gamma \subset G$ *uniformly regular* if its limit set consists of regular ideal points, $\Lambda \subset \partial_\infty^{reg} X$. We call the projection $\Lambda_{ch} \subset \partial_F X$ of Λ the *chamber limit set* of Γ, cf. [B]. Furthermore, we call Γ *antipodal* if any two limit chambers in Λ_{ch} are opposite, and *non-elementary* if $|\Lambda_{ch}| \geq 3$. For non-elementary uniformly regular discrete subgroups $\Gamma \subset G$ we will compare several geometric and dynamical conditions which we now formulate.

The first condition is coarse geometric and a strengthening of quasiisometric embeddedness (undistorsion). We define the *Weyl hull* of a regular segment xy in X as the intersection of the euclidean Weyl chamber with tip x through y and the euclidean Weyl chamber with tip y through x; it is a subset of the unique maximal flat containing the segment. We call Weyl hulls of regular segments *Weyl diamonds.* We say that a quasigeodesic (segment) in X is *Morse* if all subsegments are uniformly close to Weyl diamonds. (In rank one, this is automatically satisfied as a consequence of the Morse Lemma for Gromov hyperbolic spaces which asserts that quasigeodesic segments are uniformly close to geodesic segments.) We call the subgroup Γ *Morse* if it is word hyperbolic and, for any $x \in X$, the orbit map $\Gamma \to \Gamma x \subset X$ sends uniform quasigeodesics in Γ to uniform Morse quasigeodesics in X.

The next three conditions generalize well established conditions for Kleinian groups to arbitrary rank.

A limit chamber $\sigma \in \Lambda_{ch}$ is called *conical* if an(y) orbit Γx has unbounded intersection with a sufficiently large tubular neighborhood of a(ny) euclidean Weyl chamber asymptotic to σ. The chamber limit set is called *conical* if all limit chambers are conical. This condition has been considered in [A]. We call the subgroup Γ *RCA* if it is antipodal and has conical chamber limit set.

Following Sullivan [S], we call the subgroup Γ *expanding at the chamber limit set* if for every limit chamber $\sigma \in \Lambda_{ch}$ there exists a neighborhood U of σ in $\partial_F X$ and an element $\gamma \in \Gamma$ which is uniformly strictly expanding on U (with respect to some Riemannian background metric on $\partial_F X$).

We call an antipodal subgroup *asymptotically embedded* if it is intrinsically word hyperbolic and the action $\Gamma \curvearrowright \Lambda_{ch}$ is a copy of the natural action $\Gamma \curvearrowright \partial_\infty \Gamma$ on the Gromov boundary.

Finally, we consider the Anosov condition introduced in [La] and generalized in [GW]. We formulate here our alternative definition of the Anosov condition which avoids using the geodesic flow of the group Γ; as proven in [KLP2] it is equivalent to the definition by Labourie, Guichard and Wienhard. We call the subgroup Γ *boundary embedded* if, intrinsically, it is non-virtually-cyclic word hyperbolic, and if there exists a Γ-equivariant embedding $\beta : \partial_\infty \Gamma \to \partial_F X$ which maps different ideal points to opposite chambers. We call Γ (non-uniformly) *Anosov* if, furthermore, for any normalized coarse geodesic ray $q : \mathbb{N} \to \Gamma$ asymptotic to $\zeta \in \partial_\infty \Gamma$ the elements $q(n)^{-1} \in \Gamma$ act on the tangent space $T_{\beta(\zeta)}(\partial_F X)$ with unbounded expansion rate as $n \to +\infty$.

Theorem ([KLP2]). These five conditions are equivalent.

Note that for Kleinian groups and, more generally, in rank one these conditions are equivalent to *convex cocompactness*. However, in rank ≥ 2, convex cocompactness is (much) stronger and too restrictive, as had been shown in [KlL]. The conditions discussed here can thus serve as a replacement for convex cocompactness in higher rank.

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Strongly positive curvature and Thorpe's trick

Ricardo Mendes
(joint work with Renato Bettiol)

We begin a systematic study of a curvature condition (strongly positive curvature) which lies strictly between positive curvature operator and positive sectional curvature. Originating from the work of Thorpe [3, 4], this condition was used in [2] to compute pinching of some homogeneous positively curved manifolds, and in [1] to produce a new manifold of positive curvature. Our main goals are to investigate which operations preserving positive sectional curvature also preserve strongly positive curvature; and which known examples of manifolds admitting a metric with positive sectional curvature also admit one with strongly positive curvature.

To define strongly positive curvature, let (M, g) be a Riemannian manifold with curvature operator R. Thus at each $p \in M$, $R(p) : \Lambda^2 T_pM \to \Lambda^2 T_pM$ is a self-adjoint linear map. A 4-form $\omega \in \Lambda^4 T_pM$ defines another self-adjoint linear map $S_\omega : \Lambda^2 T_pM \to \Lambda^2 T_pM$ by the formula

$$\langle S_\omega(X \wedge Y), Z \wedge W\rangle = \langle X \wedge Y \wedge Z \wedge W, \omega\rangle$$

The operator $R(p) + \omega$ is sometimes called the "modified curvature operator", and has the same sectional curvatures as $R(p)$. The manifold (M, g) is said to have strongly positive curvature if, at every $p \in M$, there is a 4-form ω such that $R(p)+\omega$ is positive-definite. In dimension 4 this condition is equivalent to positive sectional curvature by [4], while in dimensions greater than 4 it is strictly stronger, see [6].

Now we turn to operations preserving positive curvature. Let $\pi : \overline{M} \to M$ be a Riemannian submersion. It is well-known that if $\overline{M}$ has positive sectional curvature, then so does M. We prove that strongly positive curvature is also preserved by Riemannian submersions, using a convenient rewriting of the O'Neill formula for the curvature tensor of M in terms of the curvature tensor of $\overline{M}$ and the O'Neill tensor. Using this formula we also prove that strongly positive curvature is preserved under Cheeger deformations. In both cases we obtain explicit formulas for a modifying 4-form ω.

Next we consider examples, starting with compact rank one symmetric spaces. Spheres with the round metric have positive definite curvature operator, and therefore all complex and quaternionic projective spaces have strongly positive curvature, by the submersion result above. On the other hand, we prove that the (unique up to scaling) homogeneous metric on the Cayley plane does *not* have strongly positive curvature.

Finally we consider the remaining homogeneous manifolds with positive curvature. We show that they all admit a metric with strongly positive curvature, with the possible exception of the 24-dimensional octonionic flag manifold. For most of these our proof relies on a strenghtening of Wallach's theorem [5]: given compact Lie groups $H < K < G$, we list sufficient conditions for the total space G/H of the homogeneous fibration $K/H \to G/H \to G/K$ to admit a homogeneous metric with strongly positive curvature.

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Finite-time singularities of the Kähler-Ricci flow

Valentino Tosatti

(joint work with Tristan C. Collins)

Let (X, ω_0) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$. Let $\omega(t), t \in [0, T)$, be a smooth family of Kähler metric on X with $\omega(0) = \omega_0$, solving the Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)),$$

where $\mathrm{Ric}(\omega(t))$ denotes the Ricci form of $\omega(t)$. Since this is just the Ricci flow starting at a Kähler metric, classical work of Hamilton shows that the flow always has a unique smooth solution, defined on a (forward) maximal time interval $[0, T)$ with $0 < T \leq \infty$. Let us assume from now on that $T < \infty$, in which case the flow develops a finite-time singularity at time T. Thanks to [9] we know that

$$T = \sup\{t > 0 \mid [\omega_0] - 2\pi t c_1(X) \in \mathcal{C}_X\},$$

where $\mathcal{C}_X$ denotes the Kähler cone of X. Following [4] we define the singularity formation locus as

$$\Sigma = X \backslash \{x \in X \mid \exists U \ni x \text{ open}, \exists C > 0, \text{ s.t. } |\mathrm{Rm}(t)|_{g(t)} \leq C \text{ on } U \times [0, T)\}.$$

In [10] it is shown that we may replace the condition $|\mathrm{Rm}(t)|_{g(t)} \leq C$ by $R(t) \leq C$, where $R(t)$ is the scalar curvature of $\omega(t)$. In fact, we even have that Σ equals the complement of

$$\{x \in X \mid \exists U \ni x \text{ open}, \exists \omega_T \text{ Kähler metric on } U \text{ s.t. } \omega(t) \overset{C^\infty(U)}{\to} \omega_T \text{ as } t \to T^-\}.$$

The following conjecture was posed by Feldman-Ilmanen-Knopf in 2003:

Conjecture 1. *The singularity formation set Σ is an analytic subvariety of X.*

Let $[\alpha] = [\omega_0] - 2\pi T c_1(X) \in \partial\mathcal{C}_X$ be the limiting class along the flow. If $V \subset X$ is positive-dimensional irreducible analytic subvariety, $k > 0$, then we have

$$\int_V \alpha^{\dim V} \geq 0.$$

Define the null locus of the class $[\alpha]$ to be

$$\mathrm{Null}(\alpha) = \bigcup_{\int_V \alpha^{\dim V} = 0} V.$$

Note that $\int_V \alpha^{\dim V} = 0$ happens if and only if the volume of V with respect to $\omega(t)$ approaches zero as t approaches T. The following theorem gives a positive solution to Conjecture 1:

Theorem 2. *For any finite-time singularity of the Kähler-Ricci flow on a compact Kähler manifold we have*

$$\Sigma = \mathrm{Null}(\alpha),$$

which is a nonempty analytic subvariety of X.

In particular, if $\int_X \alpha^n = 0$ (which happens precisely when the volume of the whole manifold goes to zero), then $\Sigma = X$, and otherwise Σ is a proper analytic subvariety of X. The key tool in the proof of this result is the following analytic characterization of the null locus. Following [1] define the non-Kähler locus of the class $[\alpha]$ to be

$$E_{nK}(\alpha) = \bigcap_{Z \in [\alpha] \text{ Kähler current}} \mathrm{Sing}(Z),$$

where a Kähler current Z in the class $[\alpha]$ is a closed $(1,1)$ current $Z = \alpha + \sqrt{-1}\partial\overline{\partial}\varphi$, where φ is a quasi-plurisubharmonic function, which satisfies $Z \geq \varepsilon\omega_0$ weakly as currents, for some $\varepsilon > 0$. Here $\mathrm{Sing}(Z)$ denotes the set of points $x \in X$ such that φ is not smooth near x. $E_{nK}(\alpha)$ is an analytic subvariety of X. The following is the main result of [2], and generalizes and reproves algebro-geometric results in [6, 3].

Theorem 3. *Let X be a compact Kähler manifold and $[\alpha] \in \partial\mathcal{C}_X$, then*

$$\mathrm{Null}(\alpha) = E_{nK}(\alpha).$$

This is used to construct a good barrier function φ as above which is then employed to get uniform estimates for $\omega(t)$ outside $\mathrm{Null}(\alpha)$. In [2] we also prove a result similar to the one in Theorem 2 but for sequences of Ricci-flat Kähler metrics on a compact Calabi-Yau manifold.

We end this report with a list of open problems on finite-time singularities of the Kähler-Ricci flow, which are mostly well-known. The setup is the same as in Theorem 2.

(1) Show that the diameter of X with respect to $\omega(t)$ remains bounded as t approaches T.
(2) Prove or disprove that $|\mathrm{Rm}(t)|_{g(t)} \leq \frac{C}{T-t}$, for some constant C.

(3) Show that $(X, \omega(t))$ converges in Gromov-Hausdorff as t approaches T to the metric completion of the smooth limit of the metrics $\omega(t)$ on $X \backslash \mathrm{Null}(\alpha)$.
(4) Show that every irreducible component of $\mathrm{Null}(\alpha)$ is uniruled, i.e. covered by rational curves.
(5) Show that there exists a holomorphic map $\pi : X \to Y$ to a compact Kähler space Y, possibly singular, which is an isomorphism away from $\mathrm{Null}(\alpha)$, and such that $[\alpha]$ is the pullback of a Kähler class on Y.
(6) If (5) holds, show that the flow can be restarted on a variety birational Y, and the whole process is continuous in the Gromov-Hausdorff topology (cf. [8])
(7) Prove or disprove that the Kähler potential $\varphi(t)$ along the flow has a uniform L^∞ bound independent of t.
(8) Prove that the diameter of X with respect to $\omega(t)$ goes to zero as t approaches T if and only if $[\omega_0] = \lambda c_1(X)$ for some $\lambda > 0$ (cf. [7]).
(9) Prove or disprove that

$$\Sigma = X \backslash \{x \in X \mid \exists C > 0 \text{ s.t. } R(x,t) \leq C \text{ for all } 0 \leq t < T\}.$$

(10) Prove that blowup limits of rescalings of the flow as time approaches T are complete Kähler-Ricci solitons (shrinking and gradient if (2) holds).

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Incompressible surfaces in locally symmetric spaces

Ursula Hamenstädt

A conjecture of Gromov states that every one-ended hyperbolic group contains a *surface group*, i.e. the fundamental group of a closed oriented surface of genus $g \geq 2$.

In recent seminal work, J. Kahn and V. Markovic proved that the fundamental group of every closed hyperbolic 3-manifold contains a surface subgroup, so

Gromov's conjecture holds true for fundamental groups of hyperbolic 3-manifolds. This result is a crucial ingredient in the recent solution of the virtual fibred conjecture by Ian Agol.

In this lecture we report on the following extension of the result of Kahn and Markovic [2]

Theorem 1. *Let M be a closed rank one locally symmetric manifold. Then $\pi_1(M)$ contains surface subgroups.*

The case that M is an even dimensional hyperbolic manifold (i.e. the dimension of M is even and its universal covering is the hyperbolic space) is joint work with Jeremy Kahn [3]

The proof is based on the strategy developed by Kahn and Markovic, but it only uses differential geometric tools and standard tools from dynamical systems. The case of even dimensional hyperbolic manifolds relies on a more refined and substantially modified construction.

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Flexible polyhedra and their volumes

Alexander A. Gaifullin

A flexible polyhedron in the three-dimensional space can be thought about as a closed polyhedral surface with faces made of some rigid material and hinges at edges. Such polyhedral surface is allowed to flex so that all its faces remain constant during the flexion and its dihedral angles vary continuously. If a polyhedral surface admits a flex that is not induced by the ambient rotation of $\mathbb{R}^3$, then it is called *flexible*; otherwise, it is called *rigid*. Similarly, one can define flexible polyhedra in spaces of arbitrary dimensions.

In 1813 Cauchy proved his famous theorem claiming that any convex polytope is rigid. For non-convex polyhedra, the situation is much more interesting. In 1897 Bricard [1] constructed and classified flexible self-intersected octahedra in $\mathbb{R}^3$. Since then, for 80 years it had been unknown if there exist non-self-intersected flexible polyhedra. The first example of such polyhedron was constructed by Connelly [2]. The problem on existence of flexible polyhedra in spaces of higher dimensions turned out to be more complicated. Only few examples of flexible four-dimensional cross-polytopes have been constructed by Walz and Stachel, and no examples in dimensions 5 and higher have been known. First examples of flexible self-intersected polyhedra, namely, cross-polytopes in spaces of all dimensions have been obtained by the author [6]. These examples exist in all spaces of constant curvature, i. e., in Euclidean spaces, in spheres, and in Lobachevsky spaces. The tool for constructing these examples is the interpreting of the biquadratic relations between tangents of the halves of the dihedral angles of cross-polytopes as addition laws for Jacobi's elliptic functions. Using this approach, we have classified all flexible cross-polytopes in all spaces of constant curvature. For each flexible cross-polytope, we have obtained an explicit parametrization of the flexion via either elliptic or rational functions. Nevertheless, still there are no examples of non-self-intersected flexible polyhedra in dimensions 4 and higher.

Possibly, the most interesting problem concerning flexible polyhedra is the so-called Bellows conjecture posed by Connelly in 1978. This conjecture claims that the volume of any flexible polyhedron in the three-dimensional Euclidean space is preserved during the flexion. This conjecture was proved by Sabitov [9], [10]. He proved that the volume of any simplicial polyhedron in $\mathbb{R}^3$ satisfies a polynomial relation of the form

$$V^{2N} + a_1(\ell)V^{2N-2} + a_2(\ell)V^{2N-4} + \cdots + a_N(\ell) = 0, \tag{1}$$

where $a_j(\ell)$ are polynomials in the squares of edge lengths of the polyhedron depending on its combinatorial type. This result implies immediately the Bellows conjecture, since a root of a monic polynomial with fixed coefficients cannot vary continuously. In higher dimensions, the following generalization of Sabitov's theorem has been obtained by the author.

Theorem 1. ([4], [5]) *The volume of any polyhedron in $\mathbb{R}^n$, $n \geq 4$, with triangular two-dimensional faces satisfies a relation of the form (1), where the number N and the polynomials $a_j(\ell)$ depend on the combinatorial type of the polyhedron. Hence the volume of any flexible polyhedron in $\mathbb{R}^n$, $n \geq 4$, is constant during the flexion.*

Sabitov's proof of the Bellows conjecture in three dimensions uses a special technique of excluding the lengths of diagonals by means of resultants. An alternative proof using theory of places of fields was obtained by Connelly, Sabitov, and Walz [3]. Our proof of Theorem 1 is obtained by combining this approach using theory of places with some technique of combinatorial topology, namely, the theory of simplicial collapses.

There are two main directions for the generalization of results on existence of relations of the form (1). First, we can replace the edge lengths with, say, the areas of two-dimensional faces, and ask if there exists a relation of the form

$$V^{2M} + b_1(A)V^{2M-2} + b_2(A)V^{2M-4} + \cdots + b_M(A) = 0, \tag{2}$$

where $b_j(A)$ are polynomials in the squares of the areas of two-dimensional faces of the polyhedron. This question is highly non-trivial even in the simplest case of a simplex. In [7], we have proved that, for an n-dimensional simplex, a relation of the form (2) exists if and only if n is even and $n \geq 6$.

Second, we may ask if it is possible to replace the volume V in (1) with some other characteristic of the polyhedron. In this particular statement, nothing is known. Nevertheless, a related result has been obtained by S. A. Gaifullin and the author [8]. Instead of a polyhedron, we consider a doubly-periodic polyhedral surface $S \subset \mathbb{R}^3$ homeomorphic to $\mathbb{R}^2$. We mean that there exist two non-colinear vectors $\mathbf{a}$ and $\mathbf{b}$ such that S is invariant under the parallel translations along $\mathbf{a}$

and **b**. The surface S is allowed to flex so that it remains doubly-periodic and the period lattice varies continuously. Then the three Gram coefficients of the period vectors **a** and **b** are important characteristics of the doubly-periodic polyhedral surface. An analogue of Theorem 1 in this setting is as follows.

Theorem 2. ([8]) *For any doubly-periodic simplicial polyhedral surface $S \subset \mathbb{R}^3$ homeomorphic to the plane, the three Gram coefficients of its period vectors* **a** *and* **b** *satisfy two independent polynomial relations with coefficients being polynomials in the squares of edge lengths of the polyhedral surface depending on its combinatorial type. Hence only one-parametric deformations of the Gram matrix of the period vectors may occur during the flexion.*

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Ancient convex solutions of the mean curvature flow

Carlo Sinestrari
(joint work with Gerhard Huisken)

Let $\mathcal{M}_t$ be a family of smooth hypersurfaces of $\mathbb{R}^{n+1}$ evolving by mean curvature flow. The family is called an *ancient* solution if it is defined on a time interval of the form $(-\infty, 0)$. In our analysis we further assume that $\mathcal{M}_t$ is closed and convex for all t.

Ancient solutions are a special class among the solutions of mean curvature flow, which is in general ill-posed backward in time because of its parabolic nature. They are of interest for several reasons. They arise as tangent flows near singularities of the flow and therefore model the asymptotic singular profile of a general solution. They have also been of interest in theoretical physics where they appear as steady state solutions of boundary renormalisation-group-flow in the boundary sigma model [3, 10].

Examples of ancient solutions include all homothetically shrinking solutions, in particular the shrinking round sphere and the shrinking cylinders. An important example of a non-homothetic ancient solution is the *Angenent oval* [1], an ancient convex solution of curve shortening in the plane which arises by gluing together near $t \to -\infty$ two opposite translating (non-compact) solutions of curve shortening flow in the plane given by

$$y_1(t) = -\log\cos x + t, \quad y_2(t) = \log\cos x - t.$$

The translating solution above is known as the *grim reaper* curve in the mathematical community while it is known as the *hairpin* solution in the physics community, [3]. The *Angenent oval* is known as the *paperclip* solution in the physics literature [10]. Compact convex ancient solutions in the plane have been completely classified by Daskalopoulos-Hamilton-Sesum [5] to be either shrinking round circles or an Angenent oval. A solution analogous to the Angenent oval was constructed in higher dimensions by White [11]. Haslhofer and Hershkovits give a more detailed construction [6] and formal asymptotics of this solution was studied by Angenent in [2].

In our study, we aim at finding properties which characterize the shrinking sphere among all closed convex ancient solutions of the flow. We denote by $\lambda_1 \leq \cdots \leq \lambda_n$ the principal curvatures of the hyperusrface, by $H = \lambda_1 + \cdots + \lambda_n$ the mean curvature and by $|A|^2 = \lambda_1^2 + \cdots + \lambda_n^2$ the square norm of the second fundamental form. All the result we give in the following are contained in [9].

Theorem 1. *Let $\mathcal{M}_t$ be a closed convex ancient solution of mean curvature flow. Then the following properties are equivalent:*

(i) *$\mathcal{M}_t$ is a family of shrinking spheres.*
(ii) *The curvatures of $\mathcal{M}_t$ satisfies the uniform pinching condition $\lambda_i \geq \varepsilon H$ for some $\varepsilon > 0$.*
(iii) *The diameter of $\mathcal{M}_t$ satisfies* $\mathrm{diam}(\mathcal{M}_t) \leq C_1(1 + \sqrt{-t})$ *for some $C_1 > 0$.*
(iv) *The outer and inner radii of $\mathcal{M}_t$ satisfy $\rho_+(t) \leq C_2 \rho_-(t)$ for some $C_2 > 0$.*

(v) $\mathcal{M}_t$ *satisfies* $\max H(\cdot,t) \leq C_3 \min H(\cdot,t)$ *for some* $C_3 > 0$.
(vi) $\mathcal{M}_t$ *satisfies the reverse isoperimetric inequality* $|\mathcal{M}_t|^{n+1} \leq C_4 |\Omega_t|^n$ *for some* $C_4 > 0$*, where* Ω_t *is the region enclosed by* $\mathcal{M}_t$.
(vii) $\mathcal{M}_t$ *is of type I, that is,* $\limsup_{t\to-\infty} \sqrt{-t} \max H(\cdot,t) < \infty$.

We remark that Haslhofer and Hershkovitz [6] have proved, by a different approach, a related result (equivalence of (i), (ii), (iii) and (vii)) under the additional assumption that the solutions are α-noncollapsed in the sense of Andrews.

Concerning the optimality of the diameter growth in assumption (iii), we recall that the formal analysis of [2] supports the existence of ancient solutions whose diameter grows with rate $\sqrt{|t| \ln |t|}$. Thus suggests that the growth rate in our assumption is not far from being optimal.

The proof of Theorem 1 employs various tools which have been introduced during the last decades in the analysis of finite time singularities of the flow. For instance, to prove the equivalence between (i) and (ii) we consider, for a small $\sigma > 0$, the function, introduced in [7],

$$f_\sigma = \frac{|A|^2 - H^2/n}{H^{2-\sigma}}.$$

Such a function is nonnegative and vanishes exactly at the umbilical points. In [7] it was proved that the L^p norm of f_σ is decreasing in time for suitable values of p large and σ small. By refining the analysis in [7], we obtain the estimate

$$\left(\int_{\mathcal{M}_t} f_\sigma^p \, dt \right)^{\frac{2}{\sigma p}} \leq \frac{c_3}{|T_0|^{1-\frac{n}{\sigma p}} - |t|^{1-\frac{n}{\sigma p}}},$$

for suitable p, σ such that $\sigma p > n$, valid for any solution (not necessarily ancient) of the flow defined on a time interval $(T_0, 0)$. By letting $T_0 \to -\infty$, we obtain that $f_\sigma \equiv 0$, which implies that the $\mathcal{M}_t$'s are spheres. Such a result has some analogies with the analysis in [4], where it is shown that ancient solutions to the Ricci flow satisfying suitable pinching conditions are necessarily a shrinking sphere (up to quotients).

We now consider the case of ancient convex solutions which are in addition uniformly k-convex, that is, they satisfy $\lambda_1 + \cdots + \lambda_k \geq \alpha H > 0$ for some $\alpha > 0$. Using some techniques introduced in [8], we can prove

Theorem 2. *Let* $\mathcal{M}_t$, *be a convex closed ancient solution of the mean curvature flow, with* $n \geq 3$*, which is uniformly* k*-convex for some* $k = 2, \ldots, n-1$*. Then we have* $H^2 > (n-k+1)|A|^2$ *on* $\mathcal{M}_t$ *for all* t.

As a corollary, using the algebraic properties of the quotient $|A|^2/H^2$, we can prove a peculiar property enjoyed by this function.

Theorem 3. *Let* $\mathcal{M}_t$ *be a closed convex ancient solution of the mean curvature flow. Then there is an integer* $h = 1, \ldots, n$ *such that*

$$\sup_{t\in(-\infty,0)} \max_{\mathcal{M}_t} \frac{|A|^2}{H^2} = \frac{1}{h}.$$

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Nonnegatively curved fixed point homogeneous manifolds

Wolfgang Spindeler

A Riemannian manifold M is called *fixed point homogeneous* if it admits an isometric action by a Lie group G with nonempty fixed point set, such that the induced action of G on a normal sphere to some fixed point component is transitive. Equivalently, a component of the fixed point set of maximal dimension has codimension 1 in the orbit space. This definition was given in [1], where a classification of closed fixed point homogeneous manifolds of positive curvature was obtained.

A crucial structure result towards this classification is that a closed fixed point homogeneous manifold M of positive curvature decomposes as the union of the normal disc bundles over a maximal fixed point component F and the unique orbit $\mathsf{G}(p)$ of maximal distance to F. The ideas used in the proof of this structure result apply also to the case of nonnegative curvature leading to weaker conclusions than in the case of positive curvature. This approach has led to classifications of nonnegatively curved fixed point homogeneous manifolds in dimensions ≤ 4 ([2]) and in dimension 5 in the simply connected case ([3]).

In this talk I present a general structure result for closed nonnegatively curved fixed point homogeneous manifolds: given a maximal fixed point component F, there exists a smooth invariant submanifold N of M with empty boundary such that M is diffeomorphic to the unit normal bundles $D(F)$ and $D(N)$ of F and N, respectively, glued together along their boundaries;

$$M \cong D(F) \cup_\partial D(N).$$

As a corollary of this result it is shown that a simply connected torus manifold of nonnegative curvature is rationally elliptic (recall that a torus manifold is a closed orientable manifold of dimension $2n$ with an effective isometric action by the n-dimensional torus with nonempty fixed point set).

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Uniquely ergodic interval exchanges are path-connected

Sebastian Hensel

(joint work with Jon Chaika)

Interval exchange transformations (*IETs*) are piecewise isometric self-maps of an interval to itself, which rearrange subintervals by translations according to a permutation π. These maps form interesting and rich examples of dynamical systems, but also appear in geometric contexts. Maybe most importantly, first return maps of orientable foliations to transversals on Riemann surfaces are IETs. See [Z06], [Y10] or [V06] for good surveys.

In this work, we consider the set Δ_π of all IETs with a given (non-degenerate) permutation π defined on a unit interval. This set is naturally homeomorphic to the n-dimensional standard simplex in $\mathbb{R}^{n+1}$ (by considering the lengths of the subintervals).

We are interested in the subset of Δ_π of all those IETs which are uniquely ergodic (i.e. admit only one invariant measure). It is known (Masur [M82], Veech [V82]) that on the one hand this set has full measure in Δ_π. Yet, on the other hand, its complement has Hausdorff dimension at least $n-1$ (Masur-Smillie [MS91]).

We show

Theorem 1 (Chaika-Hensel [CH14])**.** *Let $k \geq 4$ and let π be any non-degenerate permutation on k letters. Then the set of uniquely ergodic unit length IETs with permutation π is path-connected.*

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Metric measure spaces with synthetic Ricci bounds – state of the art and recent results

Karl-Theodor Sturm

1. Curvature-dimension conditions for mms

A longstanding open problem in analysis and geometry on singular spaces was to find an appropriate notion of generalized lower Ricci curvature bounds for metric measure spaces (X, d, m). Lott&Villani [10] and Sturm [14] proposed such a concept based on the theory of optimal transportation. The so-called *curvature-dimension condition* $\mathrm{CD}(K, N)$ in the most easiest case $N = \infty$ states that the relative entropy $\mathrm{Ent}(.|m)$ regarded as a functional on the L^2-Wasserstein space $\mathcal{P}_2(X)$ is weakly K-convex (formally, $\mathrm{Hess\,Ent}(.|m) \geq K$.) That is, $\forall \mu_0, \mu_1 \in \mathcal{P}_2(X)$: $\exists$ geodesic $(\mu_t)_t$ s.t. $\forall t \in [0, 1]$:

$$\mathrm{Ent}(\mu_t|m) \quad \leq \quad (1-t)\mathrm{Ent}(\mu_0|m) + t\,\mathrm{Ent}(\mu_1|m) - \frac{K}{2}\,t(1-t)\,W_2^2(\mu_0, \mu_1).$$

Here $\mathrm{Ent}(\nu|m) = \int_X \rho \log \rho \, dm$ with $\rho = \frac{d\mu}{dm}$ if ν is absolutely continuous w.r.t. m and $\mathrm{Ent}(\nu|m) = +\infty$ if ν otherwise. The definition of $\mathrm{CD}(K, N)$ in the case of finite N – as originally introduced in [15] and adopted in [11] – is more involved. Its lack of a local-to-global property led to the definition of the *reduced curvature-dimension condition* $\mathrm{CD}^*(K, N)$ [4]. A powerful new characterization of the latter has been found recently.

Proposition [5]. *A non-branching metric measure space* (X, d, m) *satisfies the condition* $CD^*(K, N)$ *iff* $S = \mathrm{Ent}(.|m)$ *is weakly (K,N)-convex on* $\mathcal{P}_2(X, d)$*, formally*

$$\mathrm{Hess}\, S - \frac{1}{N}\,(\nabla S \otimes \nabla S) \geq K.$$

2. Heat flow on mms

The *heat equation* on (X, d, m) plays an crucial role in the study of refined properties of metric measures spaces (X, d, m). It can be defined

- either as gradient flow on $L^2(X, m)$ for the *energy*

$$\mathcal{E}(u) = \frac{1}{2}\int_X |\nabla u|^2 \, dm = \liminf_{v \to u \text{ in } L^2} \frac{1}{2}\int_X (\mathrm{lip}_x v)^2 \, dm(x)$$

  with $|\nabla u|$ = minimal weak upper gradient
- or as gradient flow on $\mathcal{P}_2(X)$ for the *relative entropy* $\mathrm{Ent}(.|m)$.

Theorem [1]. *For arbitrary metric measure spaces (X, d, m) satisfying $CD(K, \infty)$ both approaches coincide.*

Note that this also applies to Finsler spaces in which case the heat flow will be *non-linear* [12].

We say that a mms satisfies the *Riemannian curvature-dimension condtion* $\mathrm{RCD}^*(K, N)$ if it satisfies the condtion $\mathrm{CD}^*(K, N)$ and if its heat flow is linear. This condition again is *stable* under convergence [2, 5]. Moreover, it implies that the space is *essentially non-branching* [13].

3. Analysis on $\mathrm{RCD}^*(K, N)$-spaces

Theorem [5]. *For any metric measure space with linear heat flow the condition $CD^*(K, N)$ is equivalent to the Bakry-Emery condition $BE(K, N)$ (or "Bochner inequality")*

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \nabla \Delta u\rangle \geq K \cdot |\nabla u|^2 + \frac{1}{N} \cdot |\Delta u|^2.$$

This extends upon previous recent results by Ohta-Sturm for Finsler spaces, Gigli-Kuwada-Ohta, Zhang-Zhu for Alexandrov spaces and Ambrosio-Gigli-Savaré [3] for $CD(K, \infty)$-spaces. The Bochner inequality is the key ingredient for a refined heat kernel analysis on mms.

Corollary [6]. *On $RCD^*(K, N)$-spaces, the Li-Yau gradient estimate, the differential Harnack inequality and the Gaussian heat kernel estimates hold true in the same form as on Riemannian manifolds. In particular, for each $f \geq 0$*

$$\Delta(\log P_t f) \geq -\frac{N}{2t}.$$

4. Transformations of $\mathrm{RCD}^*(K, N)$-spaces

Given a ('smooth') mms (X, d, m) and ('smooth') functions V, W on X, let us consider the mms (X, d', m') with $m' = e^V m$ and

$$d'(x, y) = \inf \left\{ \int_0^1 |\dot{\gamma}_t| \cdot e^{W(\gamma_t)}\, dt : \ \gamma : [0, 1] \to X \text{ rectifiable, } \gamma_0 = x, \gamma_1 = y \right\}.$$

Theorem [16]. If (X, d, m) satisfies $RCD^*(K, N)$ then for each $N' > N$ there exists K' s.t. (X, d', m') satisfies $RCD^*(K', N')$.

The above transformation can also be expressed in terms of the associated Dirichlet forms: the form $\int |\nabla u|^2\, dm$ on $L^2(X, m)$ will be transformed into $\int |\nabla u|^2 e^{V-2W}\, dm$ on $L^2(X, e^V m)$. Three cases are of particular interest

- $W = 0$ ('drift transformation'): This is well studied, both in the context of Bakry-Emery conditions and in the context of Lott-Sturm-Villani conditions. The bound for the transformed space depends on $\mathrm{Hess}V$ and ∇V. Also $N = \infty$ is admissible.
- $V = 2W$ ('time change'): New transformation property. The bound for the transformed space depends on ΔW and ∇W. Finitenss of N is necessary.
- $V = NW$ ('conformal transformation'): This is the only case where N' can be chosen to coincide with N. Finitenss of N is necessary.

5. Geometry of $\mathrm{RCD}^*(K,N)$-spaces

Let us briefly mention some very recent breakthroughs which provide a deeper understanding of the geoemtry of mms satisfying a synthetic lower Ricci bound.

Theorem ('Splitting Theorem') [7]. *If (X,d,m) satisfies $RCD^*(0,N)$ and contains a line then*

$$X = \mathbb{R} \times X'$$

for some $RCD^*(0,N-1)$-space (X',d',m').

Theorem ('Maximal Diameter Theorem') [8]. *If (X,d,m) satisfies $RCD^*(N-1,N)$ and has diameter π then X is the spherical suspension of some $RCD^*(N-2,N-1)$-space (X',d',m').*

Proposition [8]. *For any $\kappa \geq 0$ and $N \geq 1$ the following are equivalent*

- *(X,d,m) satisfies $RCD^*(N-1,N)$ and has diameter $\leq \pi$*
- *The (κ,N)-cone over (X,d,m) satisfies $RCD^*(\kappa N, N+1)$.*

For $\kappa = 0$ this applies to Euclidean cones, for $\kappa = 1$ to spherical suspensions.

Theorem [9]. *If (X,d,m) satisfies $RCD^*(K,N)$ then $\exists$ integer $n \leq N$ s.t. for m-a.e. $x \in X$ the tangent cone at x is unique and isometric to $\mathbb{R}^n$.*

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Volume and characteristic numbers of representations of hyperbolic manifolds

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(joint work with Marc Burger, Alessandra Iozzi)

Let Γ be a lattice in $\mathrm{Isom}(\mathbb{H}^n)$ and $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be any representation. The definition of the volume $\mathrm{Vol}(\rho)$ of the representation ρ is classical in the cocompact case and has been extended to the non-cocompact case by various authors [Dun99, Fra04, BIW10, KK12a, BBI13]. The equivalence of these definitions has been recently established in [KK13].

With the definition introduced in [BBI13], which parallels the definition for surface groups given in [BIW10], it is easy to deduce that

$$|\mathrm{Vol}(\rho)| \leq \mathrm{vol}(\Gamma \setminus \mathbb{H}^n)\,. \tag{1}$$

One of the fundamental results concerning the volume of a representation is the volume rigidity theorem, according to which equality in (1) holds if and only if:

(1) $n = 2$ and ρ is the holonomy representation of a (possibly infinite volume) complete hyperbolization of the smooth surface underlying M, [Gol80, BI07, KM08], or
(2) $n \geq 3$ and ρ is conjugate to id_Γ, [Dun99, FK06, BCG07, BBI13].

If Γ is cocompact, one knows at least since [Rez96] that Vol is constant on the connected components of $\hom(\Gamma, \mathrm{Isom}(\mathbb{H}^n))$ and hence takes only finitely many values. In odd dimension the nature of these values is in general mysterious, while in even dimension $n = 2m$, the Chern–Gauss–Bonnet theorem implies that $\mathrm{Vol}(\rho)$ is, up to a universal constant, an integer.

If Γ is non-cocompact, the situation parallels the one above, at least in high dimension. In fact, using an approach via Schläfli's formula as in [BCG07], Kim and Kim proved that if $\Gamma \setminus \mathbb{H}^n$ is a finite volume hyperbolic manifold of dimension ≥ 4, the volume is constant on the connected components of $\hom(\Gamma, \mathrm{Isom}(\mathbb{H}^n))$ [KK13]. Like in the compact case, in odd dimension the nature of these values is mysterious.

Our main result is the integrality of $\mathrm{Vol}(\rho)$ in dimension $n = 2m \geq 4$; this generalizes the Harder–Gauss–Bonnet theorem according to which

$$\frac{2(-1)^m}{\mathrm{vol}(S^{2m})}\mathrm{vol}(\Gamma \setminus \mathbb{H}^n) = \chi(\Gamma \setminus \mathbb{H}^n)\,.$$

Theorem 1. *Let $n = 2m \geq 4$ be an even integer. Let $\Gamma < \mathrm{Isom}^+(\mathbb{H}^{2m})$ be a non-cocompact lattice and let $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^{2m})$ be any representation.*

(1) *If Γ is torsion free and the manifold $M = \Gamma \setminus \mathbb{H}^{2m}$ has only toric cusps, then*

$$\frac{2}{\mathrm{vol}(S^{2m})} \cdot \mathrm{Vol}(\rho) \in \mathbb{Z}\,.$$

(2) *If* Γ *is torsion free then*

$$\frac{2}{\mathrm{vol}(S^{2m})} \cdot \mathrm{Vol}(\rho) \in \frac{1}{B_{2m-1}} \cdot \mathbb{Z}\,,$$

where B_{2m-1} *is the Bieberbach number in dimension* $2m-1$.

(3) *There exists an integer* $B' = B'(\Gamma) \geq 1$ *such that*

$$\frac{2}{\mathrm{vol}(S^{2m})} \cdot \mathrm{Vol}(\rho) \in \frac{1}{B'} \cdot \mathbb{Z}\,.$$

It follows from (1) and Theorem 1 that, in fact, $\frac{2}{\mathrm{vol}(S^{2m})} \cdot \mathrm{Vol}(\rho)$ takes only a finite number of values.

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The Monge-Ampère equation for $(n-1)$-plurisubharmonic functions

Ben Weinkove

(joint work with Valentino Tosatti)

Let M be a compact complex manifold of complex dimension n. A Hermitian metric g on M is given in local complex coordinates $z^1, \ldots, z^n$ as a positive definite Hermitian metric $(g_{i\overline{j}})$. Associated to g is a real $(1,1)$ form $\omega = \sqrt{-1} g_{i\overline{j}} dz^i \wedge d\overline{z}^j$.

Note that ω and g contain the same information and we will refer to both as a metric. We say that g is Kähler if $d\omega = 0$.

The following fundamental result in Kähler geometry is due to Yau.

Theorem 1 (Yau [15])**.** *Let (M, ω) be a compact Kähler manifold. Let F be a smooth real-valued function M with $\int_M e^F \omega^n = \int_M \omega^n$. Then there exists a unique Kähler metric $\tilde{\omega}$ of the form $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}u > 0$, for u a smooth real-valued function, solving*

$$\tilde{\omega}^n = e^F \omega^n. \tag{1}$$

We call (1) the *complex Monge-Ampère equation.* An immediate consequence of Yau's theorem is the Calabi conjecture:

Corollary 1 (Yau [15])**.** *Let (M, ω) be a compact Kähler manifold. Given any representative $\psi \in c_1(M)$ there exists a unique Kähler metric $\tilde{\omega}$ of the form $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}u > 0$, for u a smooth real-valued function, solving*

$$\mathrm{Ric}(\tilde{\omega}) = \psi.$$

Here we recall that $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial} \log \det g$ is the Ricci curvature of ω and the first Chern class $c_1(M)$ is defined to be $c_1(M) = [\mathrm{Ric}(\omega)] \in H^{1,1}(M; \mathbb{R})$. To see that the corollary follows from the theorem, define F by $\mathrm{Ric}(\omega) = \psi + \sqrt{-1}\partial\bar{\partial}F$ with $\int_M e^F \omega^n = \int_M \omega^n$ and let $\tilde{\omega}$ solve (1).

Are there analogues of these results for non-Kähler metrics? In particular, we are interested in metrics satisfying one of the two natural conditions:

1. A Hermitian metric ω is *Gauduchon* if $\partial\bar{\partial}(\omega^{n-1}) = 0$. Such metrics always exist on compact complex manifolds [5].
2. A Hermitian metric ω is *balanced* if $d(\omega^{n-1}) = 0$ [10]. Such metrics are of recent interest in mathematical physics (see e.g. [9]).

There is an obvious difficulty in generalizing Yau's theorem to these metrics. Namely, if $n \geq 3$ then ω Gauduchon (balanced) does not imply in general that $\omega + \sqrt{-1}\partial\bar{\partial}u$ is Gauduchon (balanced).

First some definitions. We recall that a smooth function u on $\mathbb{C}^n$ is *plurisubharmonic* if $\sqrt{-1}\partial\bar{\partial}u \geq 0$ as a $(1,1)$ form. We say that u is $(n-1)$*-plurisubharmonic*, in the sense of Harvey-Lawson [7], if $\sqrt{-1}\partial\bar{\partial}u \wedge \omega_E^{n-2} \geq 0$ as an $(n-1, n-1)$-form, where $\omega_E = \sqrt{-1}\sum_i dz^i \wedge d\overline{z}^i$. Equivalently, u is subharmonic when restricted to every complex $(n-1)$-plane.

On a compact Hermitian manifold (M, ω) we give the following definition. Let ω_0 be another Hermitian metric. Define u to be $(n-1)$*-plurisubharmonic with respect to* ω *and* ω_0 if $\omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} \geq 0$. Note that an $(n-1, n-1)$ form Ψ defines a Hermitian matrix $(\Psi_{i\overline{j}})$ (look at the coefficients of $(\sqrt{-1})^{n-1}dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge \widehat{d\overline{z}^j} \wedge \cdots \wedge d\overline{z}^n)$ and we say that $\Psi \geq 0$ if this matrix is nonnegative definite.

There is an obvious Monge-Ampère equation associated to this notion, and our main result is that this equation always admits unique solutions.

Theorem 2 ([13, 14])**.** *Given any smooth function F there exists a unique pair (u,b) with u a smooth function and $b \in \mathbb{R}$ such that*

$$\begin{aligned} &\det(\omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2}) = e^{F+b}\det(\omega^{n-1}) \\ &\omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} > 0, \quad \sup_M u = 0. \end{aligned} \tag{2}$$

The equation (2) was first introduced by Fu-Wang-Wu [2], who solved it in the case when ω is Kähler with non-negative orthogonal bisectional curvature [3].

We now describe how equation (2) is related to Gauduchon and balanced metrics. Observe that the map $\omega \mapsto \omega^{n-1}$ is a bijection from positive definite $(1,1)$ forms to positive definite $(n-1,n-1)$ forms. We write its inverse as $\Psi \mapsto \Psi^{1/(n-1)}$.

Next, assume that ω is Kähler. Then

$$\omega_0 \text{ Gauduchon} \quad \Longrightarrow \quad \omega_u := (\omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2})^{1/(n-1)} \text{ is Gauduchon,}$$

if u is strictly $(n-1)$-plurisubharmonic. Moreover, the same is true if we replace "Gauduchon" with "balanced". Observe that equation (2) can be rewritten as

$$\omega_u^n = e^{\frac{F+b}{n-1}}\omega^n.$$

As a consequence of Theorem 2, we have:

Corollary 2 ([13])**.** *Let (M,ω) be a compact Kähler manifold and let ω_0 be a Gauduchon (balanced) metric. Then given a smooth function F there exists a Gauduchon (balanced) metric ω_u and $b \in \mathbb{R}$ such that*

$$\omega_u^n = e^{F+b}\omega^n.$$

Moreover, ω_u is the unique such metric of the form

$$\omega_u = (\omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2})^{1/(n-1)}.$$

Hence, on a compact Kähler manifold, we can solve the volume form equation for Gauduchon (balanced) metrics, up to a scaling factor. The result of Corollary 2 was conjectured by Fu-Xiao [4] and Popovici [11]. Note that for Gauduchon metrics, the assumption of ω being Kähler can be weakened to $\partial\bar{\partial}(\omega^{n-2}) = 0$ [14], a condition known as Astheno-Kähler [8].

Finally, it would be desirable to weaken the assumption that ω is Kähler. Indeed, the Gauduchon case was conjectured much earlier in the following form:

Conjecture 1 (Gauduchon [6])**.** *Let M be a compact complex manifold. Given $\psi \in c_1^{\mathrm{BC}}(M)$ there exists a Gauduchon metric $\tilde{\omega}$ with*

$$\mathrm{Ric}(\tilde{\omega}) = \psi.$$

Here, $\mathrm{Ric}(\omega) := -\sqrt{-1}\partial\bar{\partial}\log\det g$, which in general differs from the Riemannian Ricci curvature of g if g is non-Kähler. The first Bott Chern class $c_1^{\mathrm{BC}}(M)$ is the class of $\mathrm{Ric}(\omega)$ in the space $H^{1,1}_{\mathrm{BC}}(M;\mathbb{R}) := \{d\text{-closed real } (1,1) \text{ forms}\}/\mathrm{Im}\,\partial\bar{\partial}$.

Conjecture 1 is a natural generalization of the Calabi conjecture. It holds if $n = 2$ [1] or if M admits an Astheno-Kähler metric [14]. Conjecture 1 would follow from the solution of the Monge-Ampère equation obtained by taking the determinant of

$$\Phi_u := \omega^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} + \mathrm{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}\omega^{n-2})$$

(see [12, 14]). Indeed, the point is that Φ_u is $\partial\bar{\partial}$-closed if ω is Gauduchon, and this is exactly what is needed to obtain Gauduchon metrics with prescribed volume form. Conjecture 1 would then follow in the same way that Corollary 1 follows from Theorem 1.

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Uniqueness of Stenzel's metric

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(joint work with Ronan Conlon)

1. T^*S^n and Stenzel's metric

Stenzel's metric [9] is a complete Ricci-flat Kähler metric of cohomogeneity 1 on T^*S^n. To make sense of this statement, we must give T^*S^n a complex structure, and one very natural way to do so is to observe that $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $T^*S^n = \mathrm{SO}(n+1,\mathbb{C})/\mathrm{SO}(n,\mathbb{C})$. What makes this complex structure particularly interesting is that it appears "in the wild" as an affine hyperquadric,

$$T^*S^n \overset{\text{biholo}}{=} \{z_0^2 + ... + z_n^2 = 1\} \subset \mathbb{C}^{n+1}.$$

The $\mathrm{SO}(n+1,\mathbb{C})$-action is obvious in this picture, and the zero section $S^n \subset T^*S^n$ corresponds to the locus of real points of the quadric (all z_i real).

For an arbitrary but fixed $t \in \mathbb{C}$, consider the scaled quadric

$$M_t = \{z_0^2 + ... + z_n^2 = t\} \subset \mathbb{C}^{n+1}.$$

This is smooth for all $t \neq 0$, and $\lim_{t\to 0} M_t = M_0$. The latter space is invariant under multiplication by complex scalars, hence can be viewed as a complex cone, and has an isolated ("ordinary double point") singularity at the origin.

For $n \geq 2$, M_t $(t \neq 0)$ has only one end. Stenzel's metric is then the unique (up to a scale) $\mathrm{SO}(n+1)$-invariant Ricci-flat Kähler metric ω_t on M_t. We can give an explicit formula when $n = 2$ (this is the well-known Eguchi-Hanson metric):

$$\omega_t = i\partial\bar{\partial}\sqrt{|z|^2 + t}.$$

Observe that this converges to $\omega_0 = i\partial\bar{\partial}|z|$ if we either fix z and let $t \to 0$, or else if we fix t and let $|z| \to \infty$. Moreover, ω_0 defines a Ricci-flat Kähler cone metric on M_0; in fact, the cone (M_0, ω_0) is isometric to $\mathbb{C}^2/\{\pm 1\} = \mathcal{C}(\mathbb{R}P^3)$. When $n \geq 3$, there no longer exists a simple explicit formula, but we can still say that

$$\omega_t = i\partial\bar{\partial}u_t(|z|^2) = i\partial\bar{\partial}|z|^{2\frac{n-1}{n}} + O(|z|^{-2}),$$

and $\omega_0 = i\partial\bar{\partial}|z|^{2\frac{n-1}{n}}$ is a Ricci-flat (but no longer flat) Kähler cone metric on M_0. More precisely, $g_0 = dr^2 + r^2 g_{\mathcal{L}}$, where $g_{\mathcal{L}}$ is a homogeneous Einstein metric with $\mathrm{Ric}(g_{\mathcal{L}}) = (2n-2)g_{\mathcal{L}}$ on the link $\mathcal{L}$ of the cone (the unit cotangent bundle of S^n), and the radius function r of the cone metric g_0 satisfies $r = |z|^{(n-1)/n}$.

2. Main result

Theorem (Conlon-H [4]**).** *If $n \geq 4$, then, up to scaling, (M_1, ω_1) is the unique complete Ricci-flat Kähler manifold asymptotic to (M_0, ω_0) at infinity.*

Remarks. (1) (M, ω) is "asymptotic to (M_0, ω_0)" if there exists a diffeomorphism $\Phi : M_0 \setminus K_0 \to M \setminus K$ $(K_0, K$ compact) such that $|\nabla^j_{g_0}(\Phi^* g - g_0)|_{g_0} = O(r^{-\lambda-j})$ for some $\lambda > 0$ and all $j \in \mathbb{N}_0$.

(2) By Kronheimer's classification of ALE spaces [7], the theorem also holds for $n = 2$. In fact, our method yields an alternative proof of Kronheimer's theorem.

(3) If $n = 3$, then we show that there exists precisely one other example besides Stenzel: Candelas-de la Ossa's "small resolution of the conifold" [1].

(4) At least for $n = 2$, it is a folklore conjecture that the word "Kähler" in the statement of the theorem is unnecessary. This seems to be wide open.

3. Step 1 of the proof: M is biholomorphic to M_1

We compactify M as a complex manifold by using the asymptotic cone model, then show that the compactification X is projective algebraic, and finally reduce the problem to known classification results for algebraic varieties.

Theorem (Li [8]**).** *Let D be a compact complex manifold, $L \to D$ a holomorphic line bundle, and h a Hermitian fiber metric on L of positive curvature. Put $\omega_0 = i\partial\bar{\partial}h^{-\delta}$ on the total space of $L \setminus 0$ for some $\delta > 0$; this is a Kähler cone metric with radius function r given by $r^2 = h^{-\delta}$. Let J be a complex structure on $L \setminus 0$ such that $|\nabla^j_{g_0}(J - J_0)|_{g_0} = O(r^{-\lambda-j})$, where J_0 is the given complex structure of L. Then, up to diffeomorphism, J extends smoothly from $L \setminus 0$ to L.*

Remarks. (1) The level sets of h on $L \setminus 0$ are the geodesic spheres in (M_0, ω_0). The fact that these spheres are metrically convex corresponds to the positivity of the curvature of h, which is fundamental to everything that follows.

(2) Li's theorem is analogous to a compactification theorem for asymptotically *cylindrical* complex manifolds due to Haskins-H-Nordström [6]. Both results are proved by a Newlander-Nirenberg type construction at infinity.

We apply this result with $D = \{Z_0^2 + ... + Z_n^2 = 0\} \subset \mathbb{C}P^n$ (here $Z_0, ..., Z_n$ are homogeneous coordinates on $\mathbb{C}P^n$), $L = \mathcal{O}_{\mathbb{C}P^n}(1)|_D$, h the Fubini-Study metric, and $\delta = \frac{n-1}{n}$. The upshot is that M is biholomorphic to $X \setminus D$, where

(i) X is a compact complex manifold,

(ii) $D \subset X$ is a smooth divisor,

(iii) the holomorphic normal bundle $N_{D/X}$ is positive,

(iv) $-K_X = n[D]$.

Theorem (Conlon-H [3, 4]**).** *Assume* (i), (ii), (iii). *Then the following hold.*

(1) *There exists a degree 1 holomorphic map $p : X \to Y$ onto a normal projective variety Y such that p is an isomorphism onto its image near D, the singularities of Y are isolated, and the Cartier divisor $p_*[D]$ is ample on Y.*

(2) *If $-K_X = q[D]$ for some $q \in \mathbb{N}$, then $h^{0,i}(X) = 0$ for all $i > 0$, and all of the singularities of Y are canonical.*

(3) *If $q > 1$ and if $M = X \setminus D$ is Kähler, then X is projective and every Kähler form on M is cohomologous to the restriction to M of a Kähler form on X.*

Remark. (1) is an application of some powerful old results of Grauert [5].

In our example, we proceed by classifying Y. For this, we apply a construction from our earlier article [2] that turned out to be well-known in algebraic geometry ("deformation to the normal cone"). Choose a defining section s of the line bundle $p_*[D]$ on Y and define Y_t ($t \in \mathbb{C}^*$) to be the image of $\frac{s}{t}$ in the total space of $p_*[D]$. Then $\{Y_t \setminus p(D)\}_{t\in\mathbb{C}^*}$ extends to a flat algebraic family of affine algebraic varieties parametrized by $t \in \mathbb{C}$ whose central fiber $(t = 0)$ is precisely the cone M_0. Now the deformation theory of affine algebraic varieties, together with the fact that the versal deformation of our particular cone has a one-dimensional base (if not, there would exist more Ricci-flat examples!), allows us to conclude that either

(a) $Y_t \setminus p(D) = M_t$ for all t, or

(b) $Y_t \setminus p(D) = M_0$ for all t.

In (b), M is a quasiprojective crepant resolution of the cone M_0. Since $n \geq 3$, the singularity of M_0 is terminal, so the exceptional set $E = p^{-1}(0)$ of the blow-down map $p : M \to M_0$ satisfies $\mathrm{codim}_{\mathbb{C}} E \geq 2$. If $n \geq 4$, then $b^2(D) = 1$, and one can use an exact sequence written down in [2] to derive a contradiction.

4. Step 2 of the proof: ω is equal to ω_1

I didn't have time to talk about this step. It uses a Calabi-Yau type uniqueness theorem from [2] and the fact that $H^2(M_1) = 0$ because $n \geq 3$. The proof of this uniqueness theorem relies on a new Liouville theorem: *if M is an asymptotically conical Kähler manifold with* Ric ≥ 0*, and if u is a harmonic function on M with $u = o(r^2)$ as $r \to \infty$, then u is pluriharmonic.* In our example, the only harmonic functions with growth rate $o(r^2)$ are the restrictions of linear functions from $\mathbb{C}^{n+1}$ to M_1, which have growth rate $\frac{n}{n-1} \in (1, 2)$ and are clearly pluriharmonic.

5. Loose end: $n = 3$

For $n = 3$, we were left with the possibility that there exists a blow-down map $p : M \to M_0$ whose exceptional set $E = p^{-1}(0)$ is a complex curve; in fact, since $b^2(D) = 2$, our argument shows that E is irreducible, so that M must be the total space of a vector bundle $V \to E$. There is only one possibility for this: $E = \mathbb{C}P^1$ and $V = \mathcal{O}_{\mathbb{C}P^1}(-1)^{\oplus 2}$. Another classical and appealing way to write M is to make a linear change of coordinates such that $M_0 = \{xy - uv = 0\} \subset \mathbb{C}^4$ and then define $M = \{\left(\begin{smallmatrix} x & u \\ v & y \end{smallmatrix}\right)\left(\begin{smallmatrix} \lambda_1 \\ \lambda_2 \end{smallmatrix}\right) = 0\} \subset \mathbb{C}^4 \times \mathbb{C}P^1$. This is a fundamental example in algebraic geometry and carries a Ricci-flat Kähler metric [1] with rate $\lambda = 2$. By contrast, in Stenzel, the singularity of the cone gets replaced with a Lagrangian S^3 rather than a holomorphic S^2, and the rate of the metric is $\lambda = 2\frac{n}{n-1} = 3$.

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Riemannian orbifolds as quotients

Alexander Lytchak

Riemannian orbifolds are locally isometric to finite quotients of Riemannian manifolds by finite groups of isometries. Globally, not any Riemannian orbifold can be represented in such a way. However, any Riemannian orbifold can be globally described as a quotient of some Riemannian manifold by some compact group of isometries. In my talk I describe the following results relating such a global presentation of an orbifold and its the geometric and topological properties.

- Any Riemannian foliation on a homotopy sphere is either 1- or 3-dimensional or it is given by a Riemannian submersion with base and fiber being the 8-dimensional respectively the 7-dimensional homotopy spheres (joint with B. Wilking).
- Any orbifold with a contractible classifying space is a manifold.
- Any compact positively curved Riemannian orbifold which has dimension ≥ 3 and strata of codimension 1 is a good orbifold, which is orbi-covered by a sphere. (Joint with C. Gorodski).
- For an isometric group action of G on a Riemannian manifold M, the quotient M/G is a Riemannian orbifold if and only if an infinitesimal condition on all slice representations is fulfilled (joint with G. Thorbergsson).
- There is a classification of all quotients $\mathbb{S}^n/G$ which are Riemannian orbifolds. Beyond weighted projective spaces all such quotients in dimensions ≥ 3 either have constant curvature 1 or 4. The last case can only occur in dimensions ≤ 5. (Joint with C. Gorodski).

Existence of minimal hypersurfaces

André Neves

A question lying at the core of Differential Geometry, asked Poincaré in 1905, is whether every closed Riemann surface always admits a closed geodesic.

If the surface is not simply connected then we can minimize length in a nontrivial homotopy class and produce a closed geodesic. Therefore the question becomes considerably more interesting on a two-sphere, and the first breakthrough was in 1917, due to Birkhoff, who found a closed geodesic for any metric on a two-sphere.

Later, in a remarkable work, Lusternik and Schnirelmann showed that every metric on a 2-sphere admits three simple (embedded) closed geodesics. This result is optimal because there are ellipsoids which admit no more than three simple closed geodesics.

This suggests the question of whether we can find an infinite number of geometrically distinct closed geodesics in any closed surface. It is not hard to find infinitely many closed geodesics when the genus of the surface is positive.

The case of the sphere was finally settled in 1992 by Franks and Bangert. Their works combined imply that every metric on a two-sphere admits an infinite number of closed geodesics. Later, Hingston estimated the number of closed geodesics of length at most L when L is very large.

Likewise, one can ask whether every closed Riemannian manifold admits a closed minimal hypersurface. When the ambient manifold has topology one can find minimal hypersurfaces by minimization and so, like in the surface case, the question is more challenging when every hypersurface is homologically trivial. Using min-max methods, and building on earlier work of Almgren, Pitts in 1981 proved that every compact Riemannian $(n+1)$-manifold with $n \leq 5$ contains a smooth, closed, embedded minimal hypersurface. One year later, Schoen and Simon extended this result to any dimension, proving the existence of a closed, embedded minimal hypersurface with a singular set of Hausdorff codimension at least 7.

When M is diffeomorphic to a 3-sphere, Simon–Smith showed the existence of a minimal embedded sphere using min-max methods.

Motivated by these results, Yau made the following conjecture:

Every compact 3*-manifold* (M, g) *admits an infinite number of smooth, closed, immersed minimal surfaces.*

When M is a compact hyperbolic 3-manifold, Khan and Markovic found an infinite number of incompressible surfaces in M of arbitrarily high genus. One can then minimize energy in their homotopy class and obtain an infinite number of smooth, closed, immersed minimal surfaces.

Jointly with Fernando Marques we showed

Theorem 1. *Let* (M^{n+1}, g) *be a compact Riemannian manifold with* $2 \leq n \leq 6$ *and a metric of positive Ricci curvature. Then* M *contains an infinite number of distinct, smooth, embedded, minimal hypersurfaces.*

I explained the proof of this result.

Smoothing and Faceting

Anton Petrunin

(joint work with Nina Lebedeva)

We discuss bilateral approximations between Riemannian and polyhedral[1] spaces. We mean here approximations in Gromov–Hausdorff sense by the spaces with the same dimension.

The facetings, i.e., approximations of Riemannian manifolds by polyhedral spaces were considered in [1]. It was proved that if a compact Riemannian manifold M admits an faceting by polyhedral spaces with non negative curvature in the sense of Alexandrov then M satisfies peculiar curvature condition which we

[1]Polyhedral spaces are defined as metric spaces which admit finite triangulation such that each simplex is isometric to a simplex in a Euclidean space.

name *nonnegative cosectional curvature.* In the same paper, a partial converse was proved; it states that if cosectional curvature of a compact Riemannian manifold M is *strictly* positive then it admits faceting by polyhedral spaces with nonnegative curvature.

Here is a geometric way to see this curvature condition: a curvature tensor at a point has nonnegative cosectional curvature if it can apper as *the curvature tensor of a convex hypersurface in a convex hypersurface in ... in a Euclidean space.* The cosectional curvature at a point is *strictly* positive if the convex hypersurfaces above have positive definite second fundamental form.

The above results motivates the following conjecture.

Conjecture. *Any polyhedral space admits smoothing by Riemannian orbifolds*[2] *with nonnegative cosectional curvature.*

We prove that conjecture is true in 3-dimensional case. In fact we show that given 3-dimensional polyhedral space with nonnegative sectional curvature there is a continuous one parameter family of Riemannian orbifolds M^t for $t \in (0, T)$ such that $M^t \to P$ as $t \to 0$ in the sense of Gromov–Hausdorff and moreover M^t forms a solution of Ricci flow.

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When is the underlying space of an orbifold a manifold?

Christian Lange

The question posed by Davis "When is the underlying space of a smooth orbifold a topological manifold" [3] amounts to the classification of finite groups acting linearly on a Euclidean vector space such that the quotient space is homeomorphic to the original vector space. Two classes of examples for which this property holds are orientation preserving subgroups of real reflection groups and unitary reflection groups considered as real groups. These groups have the common property to be generated by transformations with codimension two fixed point subspace, so-called *pseudoreflections.* In general, a finite linear group generated by pseudoreflections is called a *pseudoreflection group.* These terminologies were introduced by Mikhaîlova who worked on the classification of pseudoreflection groups in the 1970s and 1980s. For irreducible pseudoreflection groups she obtained the following classification [2].

Theorem 1. *The irreducible pseudoreflection groups are given as follows.*

(1) *Orientation preserving subgroups of irreducible real reflection groups.*
(2) *Irreducible unitary reflection groups that are not the complexification of a real reflection group considered as real groups.*

[2] More formally *by the underlying metric spaces of Riemannian orbifolds....*

(3) *Two infinite families of pseudoreflection groups that are extensions of unitary reflection groups by another pseudoreflection.*
(4) *Many exceptional pseudoreflection groups in dimensions up to* 8.

The fact that reducible pseudoreflection groups in general do not split as products of irreducible components gives rise to many more nontrivial examples [2]. Based on her classification result Mikhaîlova proved that the quotient of a real vector space by a pseudoreflection group is homeomorphic to the original vector space [4]. Another example for which this property holds stems from Cannon's Double Suspension Theorem: The binary icosahedral group admits a faithful realization $P < SO(4)$ and it follows from Cannon's theorem that $\mathbb{R}^5/P = \mathbb{R}^4/P \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^5$ whereas $\mathbb{R}^4/P$ is not even a topological manifold. Since the quotient space S^3/P is Poincaré's homology sphere, we refer to P as a *Poincaré group*. In turned out that these are essentially the only examples in the following sense [1].

Theorem 2. *For a finite subgroup $G < O(n)$ the quotient space $\mathbb{R}^n/G$ is a topological manifold if and only if G has the form*

$$G = G_{ps} \times P_1 \times \ldots \times P_k$$

for a pseudoreflection group G_{ps} and Poincaré groups $P_i < SO(4)$, $i = 1, \ldots, k$, such that the factors act in pairwise orthogonal spaces and such that $n > 4$ if $k = 1$. In this case $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$.

It follows from the property of pseudoreflection groups, the double suspension theorem and the generalized Poincaré conjecture that $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$ for all groups described in the theorem.

The proof that there are no other examples is divided into three steps. In the first step we observe that if $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$ then, in particular, the quotient space $\mathbb{R}^n/\Gamma$ is a homology manifold. Using this observation we deduce that strata of $\mathbb{R}^n/G$ that are not contained in the closure of any higher dimensional singular stratum either have codimension two or codimension four and that the corresponding local groups are either cyclic groups or Poincaré groups. These local groups appear as subgroups of G fixing certain maximal subspaces of $\mathbb{R}^n$. Defining G' to be the normal subgroup of G generated by all of them yields a "sufficiently large" normal subgroup of G generated by pseudoreflections and Poincaré groups. A key ingredient in this step is a theorem by Zassenhaus characterizing the binary icosahedral group ([5], Theorem 6.2.1).

In the second step one can use an elementary fact about spherical triangles and the specific geometric structure of the 600-cell, i.e. the orbit of one point under the action of P on S^3, to show that the pseudoreflection group and all Poincaré groups generating G' act in pairwise orthogonal spaces.

Finally, in the last step one can show by induction that G/G' acts freely on S^{n-1}/G'. It follows from this that G/G' is a perfect group with periodic homology whose homology groups vanish in dimensions $i = 1, \ldots, n-1$. Such groups are classified and the classification allows us to identify G/G' as a trivial group.

Consequently, G coincides with G' and has thus a form as described in our theorem.

At the end of the talk we discussed the analogous question for which groups G the quotient space $\mathbb{R}^n/G$ is a manifold in other categories.

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Collapsing of a class of noncompact warped product manifolds under Ricci flow

Tobias Marxen

We study the evolution of a warped product manifold $\mathbb{R}\times N$, which is complete with bounded curvature, under Ricci flow, where (N, g_N) is a flat, complete, connected Riemannian manifold of dimension $n \geq 2$.

Let's start with the warped product manifold and some of its curvature properties: Let k be a Riemannian metric on $\mathbb{R}$, g_N as above and $g : \mathbb{R} \to \mathbb{R}$ a positive function. Then $h := k + g^2 g_N$ is a warped product metric on $\mathbb{R} \times N$, and since $k = f^2 dx^2$ for a unique positive function $f : \mathbb{R} \to \mathbb{R}$, we also write $h = f^2 dx^2 + g^2 g_N$. This means the following: Fix $x \in \mathbb{R}, q \in N$. We have a canonical isomorphism

$$T_{(x,q)}(\mathbb{R} \times N) \cong T_x\mathbb{R} \oplus T_q N.$$

Let $a, b \in T_x\mathbb{R}, v, w \in T_q N$. Then

$$\begin{aligned} h(x,q)(a+v, b+w) &= k(x)(a,b) + g^2(x) g_N(q)(v,w) \\ &= (f^2 dx^2)(x)(a,b) + g^2(x) g_N(q)(v,w). \end{aligned}$$

Moreover, assume that $(\mathbb{R} \times N, f^2dx^2 + g^2 g_N)$ is complete and has bounded curvature.

On $(\mathbb{R} \times N, f^2dx^2 + g^2 g_N)$ we have special sectional curvatures: The ones of planes tangent to a fiber $\{x\} \times N$ $(x \in \mathbb{R})$, and those of planes orthogonal to it (this is originally from [3, section 11], see [5] or [6]):

Proposition 1. *Fix $x \in \mathbb{R}, q \in N$ and let $a \in T_x\mathbb{R} \subset T_{(x,q)}(\mathbb{R} \times N)$ and $v, w \in T_q N \subset T_{(x,q)}(\mathbb{R} \times N)$, such that $\{a, v, w\}$ are linearly independent. Let $H :=$*

span$\{a, v\}, V := $ span$\{v, w\}$. *Then*

$$\sec H = -\frac{g_{ss}}{g}(x) =: K_H(x),$$
$$\sec V = -\frac{g_s^2}{g^2}(x) =: K_V(x),$$

where $g_s^2 := (g_s)^2$ *and, if* $u : \mathbb{R} \to \mathbb{R}$ *is a function,* $u_s := \frac{u'}{f}$, *and* $'$ *denotes standard differentiation.*

Now the full Riemann curvature tensor can be controlled by K_H and K_V (see [5] or [6]):

Proposition 2. *There exist positive integers* $a = a(n), b = b(n)$ *(*$n = \dim N$*), such that*

$$|Rm|^2(x, q) = a \cdot K_V^2(x) + b \cdot K_H^2(x), x \in \mathbb{R}, q \in N.$$

Now the question is: What happens with such a warped product manifold under Ricci flow, i.e. if $M = \mathbb{R} \times N$, if $h_0 = f_0^2 dx^2 + g_0^2 g_N$ with positive functions $f_0, g_0 : \mathbb{R} \to \mathbb{R}$ is a warped product metric on $\mathbb{R} \times N$ as above, and if $h(t), t \in [0, T_{\max})$ is the solution of Ricci flow, that is

$$\frac{d}{dt} h(t)(p) = -2Ric_{h(t)}(p)$$

for all $p \in M, t \in [0, T_{\max})$, on a maximal time interval with $h(0) = h_0$, what is the behaviour of $h(t)$?

But before that, let's mention a few related results (there are many more, see for example the introduction to my Ph.D. thesis [5] (German version) or [6] (English version)): In [3, section 11], R. Hamilton considered Ricci flow on closed three-manifolds with symmetries, leading to Ricci flow on warped product manifolds $S^1 \times T^2$ (T^2 is a flat torus). He proved that the warped product structure is preserved under the flow, longtime existence (i.e. $T_{\max} = \infty$), that the solution is of type III, i.e. the curvature estimate $|Rm|(x, q, t) \leq C/t$ holds for some $C \geq 0$ and all $x \in S^1, q \in T^2, t \in [1, \infty)$, and that the solution converges to a flat metric (as $t \to \infty$). In [8], M. Simon analysed Ricci flow on warped product manifolds $\mathbb{R} \times N$, where (N, g_N) is an Einstein manifold with positive Einstein constant, and also proved preservation of the warped product structure (using PDE methods), and showed for the first time, that neckpinch singularities (which are special finite time singularities) can occur under Ricci flow. Moreover, S. Angenent and D. Knopf considered in [1] and [2] Ricci flow on S^{n+1} (the $(n+1)$-dimensional sphere) with symmetries, corresponding to warped product metrics on $\mathbb{R} \times S^n$, and established, that neckpinch singularities can also occur on closed manifolds under Ricci flow, and they furthermore derived precise asymptotics for such Ricci flow neckpinches. Finally in [4] J. Lott and N. Sesum also analysed Ricci flow on closed three-manifolds with symmetries: They considered warped product manifolds $X \times S^1$, where X is a closed surface, and manifolds with a free local isometric T^2 action, and proved for example in several cases longtime existence, that the solution is of type III, and convergence of the solution to a flat metric.

So what happens now in our case? The following results are from my Ph.D. thesis ([5] or [6]) except for the last one, which is in [7]. First, the warped product structure is preserved under the flow, i.e. there exist positive functions $f, g : \mathbb{R} \times [0, T_{\max}) \to \mathbb{R}$ such that $h(t) = f^2(\cdot, t)dx^2 + g^2(\cdot, t)g_N, t \in [0, T_{\max})$. Second, we have longtime existence of the solution ($T_{\max} = \infty$); especially, no neckpinch or other finite time singularities occur. Third, we have the following curvature estimates:

Proposition 3.

$$\sup_{x\in\mathbb{R}} |K_V|(x,t) \leq \frac{1}{2nt + \frac{1}{\sup_{x\in\mathbb{R}} |K_V|(x,0)}}$$

for all $t \in [0, \infty)$.

Proposition 4. *There exist* $C = C(n, \sup_{x\in\mathbb{R}} |K_V|(x, 0), \sup_{x\in\mathbb{R}} |K_H|(x, 0)) > 0$ *and* $a = a(n, \sup_{x\in\mathbb{R}} |K_V|(x, 0)) > 0$*, such that*

$$|K_H|(x,t) \leq \frac{C}{t+a}$$

for all $x \in \mathbb{R}, t \in [0, \infty)$.

By Proposition 2 this implies

Proposition 5. *There exist* $C = C(n, \sup_{x\in\mathbb{R}} |K_V|(x, 0), \sup_{x\in\mathbb{R}} |K_H|(x, 0)) > 0$ *and* $a = a(n, \sup_{x\in\mathbb{R}} |K_V|(x, 0)) > 0$*, such that*

$$|Rm|(x,q,t) \leq \frac{C}{t+a}$$

for all $x \in \mathbb{R}, q \in N, t \in [0, \infty)$.

Especially, the solution is of type III, i.e. $|Rm|(x, q, t) \leq C/t$ for some $C \geq 0$ and all $x \in \mathbb{R}, q \in N, t \in [1, \infty)$. Longtime existence and the curvature estimates are obtained by applying an extended noncompact maximum principle to appropriate geometric quantities.

Finally, we get the following longtime behaviour of the solution, if h_0 additionally has finite volume:

Proposition 6. *If* $h_0 = f_0^2 dx^2 + g_0^2 g_N$ *is complete, has bounded curvature and finite volume, then* $g(\cdot, t) \to 0$ *(as* $t \to \infty$*) uniformly.*

In the proof we use the curvature estimate from Proposition 3 together with the fact that under these assumptions the volume of $(\mathbb{R} \times N, h(t))$ is nonincreasing in t (this was shown in [3, section 11] for the case $S^1 \times T^2$).

From this it follows that the solution is collapsing, i.e. the injectivity radius converges to 0 uniformly (as $t \to \infty$), while the curvatures (as follows from Proposition 5) stay bounded.

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Teichmüller harmonic map flow into nonpositively curved targets

Peter Topping

(joint work with Melanie Rupflin)

Given a smooth closed orientable surface $M := M_\gamma$ of genus $\gamma \geq 2$ and a smooth compact Riemannian manifold $N = (N, G)$ of any dimension, we consider the gradient flow of the harmonic map energy

$$E(u) = E(u,g) := \frac{1}{2}\int_M |du|_g^2 d\mu_g,$$

where $u : M \to N$ is a map and g is a metric on M. Allowing only u to vary, this gives the harmonic map flow of Eells-Sampson [1], which finds so-called harmonic maps. In contrast, we allow both u and g to flow, with g constrained to be hyperbolic (i.e. of constant Gauss curvature -1). Once the right geometric viewpoint is taken, this leads to the so-called Teichmüller harmonic map flow, introduced in [3], which for a given fixed parameter $\eta > 0$ is defined by

$$(1) \qquad \frac{\partial u}{\partial t} = \tau_g(u); \qquad \frac{\partial g}{\partial t} = \frac{\eta^2}{4} Re(P_g(\Phi(u,g))),$$

where $\tau_g(u)$ represents the tension field of u (i.e. $\operatorname{tr} \nabla du$), P_g represents the L^2-orthogonal projection from the space of quadratic differentials on (M, g) onto the space of *holomorphic* quadratic differentials, and $\Phi(u,g) = 4(u^*G)^{(2,0)}$ represents the Hopf differential. See [3] for further information. Note that the first equation here, governing the evolution of u, is simply the harmonic map flow. The direction that the metric g would move in order to reduce the energy as quickly as possible would be (a multiple of) $Re(\Phi(u,g))$. On the other hand, this direction will not in general preserve the hyperbolicity of the metric. Indeed, the tangent space to the space $\mathcal{M}_{-1}$ of hyperbolic metrics is given by

$$T\mathcal{M}_{-1} = \{\mathcal{L}_X g\} \oplus Re(\mathcal{H})$$

for vector fields X, where $\mathcal{H}$ is the space of *holomorphic* quadratic differentials. In particular, we see by inspection of (1) that g remains hyperbolic.

Critical points of E with respect to variations of both u and g turn out to be *weakly conformal* harmonic maps, which are then either constant maps or *branched minimal immersions*, as described in work of Gulliver, Osserman and Royden [2]. We might therefore reasonably hope that the flow (1) will find such objects. In the talk, we saw several senses in which this is the case. The simplest situation is that we start the flow with an incompressible map u_0, in which case the flow will exist for all time (in a well-understood weak sense) and converge modulo pull-back by diffeomorphisms to a branched minimal immersion (see [3]). For more general initial maps u_0, the domain can degenerate as the injectivity radius of (M, g) converges to zero. If this happens at infinite time, then a theorem joint with M. Zhu as well as Rupflin [4] says that the flow decomposes u_0 into a collection of branched minimal immersions as $t \to \infty$. However, the theory above leaves open the possibility that the domain degenerates in finite time, stopping the flow. The main result of the talk was that this cannot happen with an extra geometric hypothesis on the target. In fact, we prove in [5] the following result.

Theorem 1. *Suppose M, (N, G) and $\mathcal{M}_{-1}$ are as above, with (N, G) having nonpositive sectional curvature. Given any initial data $(u_0, g_0) \in C^\infty(M, N) \times \mathcal{M}_{-1}$, there exists a smooth solution $(u(t), g(t))$ to (1), for $t \in [0, \infty)$.*

Note that some sort of singularity must happen in the flow in general because a general map u_0 need not be homotopic to any harmonic map. Even with the nonpositive curvature hypothesis of the theorem, the domain can be seen to have to degenerate at some point because there may not be a branched minimal immersion (or constant map) homotopic to u_0. The theorem above says that the singularities must get pushed back to infinite time, by which time the whole flow has settled down, and we find that any map into a nonpositively curved target is decomposed into branched minimal immersions.

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A generalization of Gromov's almost flat manifold theorem

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(joint work with Burkhard Wilking)

A problem of great concern for differential geometers has classically been the understanding of relations between geometry and topology; in particular, we wonder if, given a certain curvature condition, it implies any topological information about the underlying manifold. In this framework, a prototypical result is:

Theorem 1 (Bieberbach 1912). *Let (M, g) be a compact flat Riemannian manifold. Then there exists a finite covering $\hat{M}$ of M such that $\hat{M}$ is isometric to a flat torus.*

The next natural step would be to study a small perturbation of the condition $K_g \equiv 0$ (here K_g denotes the sectional curvature of the metric g). However, if we multiply <u>any</u> Riemannian metric g by a constant $c \to \infty$, we get $K_{cg} = \frac{1}{c} K_g \to 0$; therefore, the mere existence of a metric with almost vanishing curvature yields no topological restriction. Consequently, we need a modified curvature assumption, whose concrete realization we find in the following celebrated theorem:

Theorem 2 (Gromov 1978 [2]). *For all $n \in \mathbb{N}$ there exists a universal constant $\varepsilon(n) > 0$ such that if a Riemannian n-dimensional manifold (M, g) satisfies*

$$diam_g(M)^2 |K_g| \leq \varepsilon(n), \tag{1}$$

then M admits a finite covering $\hat{M}$, so that $\hat{M}$ is diffeomorphic to a nilmanifold (that is, $\hat{M} \cong N/\Gamma$, where Γ is a a discrete subgroup acting cocompactly on a nilpotent Lie group N).

As our manifolds are compact, we can rescale the metric so that $\mathrm{diam}_g(M) = 1$; then the relevant curvature condition becomes $|K_g| \leq \varepsilon(n)$. Our generalization consists precisely in relaxing the latter assumption, which we regard as an L^∞-bound for the curvature. More precisely, we prove

Theorem 3. *For all $n \in \mathbb{N}$ and all $D < \infty$ there exists $\varepsilon = \varepsilon(n, D) > 0$ so that if (M, g) is a Riemannian n-dimensional manifold with*

$$\mathrm{diam}_g(M) \leq D, \qquad K_g \geq -1 \qquad \text{and} \qquad ⨍_M \|\mathrm{Rm}\|_g \, d\mu_g \leq \varepsilon, \tag{2}$$

then M is finitely covered by a nilmanifold.

Here Rm denotes the Riemannian curvature tensor, $d\mu_g$ is the Riemannian volume element and $⨍_B f d\mu$ represents the averaged integral $\frac{1}{\mathrm{vol}(B)} \int_B f d\mu_g$. Notice that the conclusion of this theorem fails if we replace the condition on the sectional curvature by $\mathrm{Rc}_g \geq -(n-1)$. Indeed, a counterexample is provided by $K3$ surfaces, since they admit sequences of Ricci-flat metrics converging to a flat orbifold (such examples were obtained by different methods in [6] and [4]).

About our curvature hypotheses, we highlight that Theorem 3 evidences that, in the presence of a lower bound for the sectional curvature, to impose an L^1-pinching condition is surprisingly rigid; indeed, typically such rigidity statements require L^p bounds for $p > n/2$. To illustrate this, let us mention that X. Dai, P. Petersen and G. Wei [1] used the Ricci flow to show that manifolds with $L^{p>n/2}$-bounds on the curvature after a short time become an L^∞-curvature control; and one can then apply Gromov's theorem. In our case, the assumptions on the curvature are so weak that the Ricci flow is helpless. Accordingly, we cannot reduce the proof to an eventual application of the classical result; this means that we also provide an alternative proof of Gromov's original theorem.

Hereafter we briefly sketch the main ideas, techniques and difficulties in the proof of Theorem 3. Arguing by contradiction, we take a sequence of n-manifolds (M_i, g_i) with

$$\mathrm{diam}_{g_i}(M_i) \leq D, \qquad K_{g_i} \geq -1 \qquad \text{and} \qquad \fint_{M_i} \|\mathrm{Rm}\|_{g_i} \to 0,$$

but which are not finitely covered by nilmanifolds. The proof involves a detailed study of the local structure of the Gromov-Hausdorff limit X of such a sequence. Indeed, Gromov's precompactness theorem guarantees the existence of this limit X and tells us that it has the structure of an Alexandrov space with curvature bounded below (in a comparison sense) by -1.

The **noncollapsing case** ($\dim(X) = n$) is the easiest scenario, since the stability theorem of Perelman ensures that X is furthermore a topological manifold. To address the proof of this case, we choose a point $p_\infty \in X$ and distinguish two possibilities: p_∞ is either a regular point (that is, its tangent cone is isometric to $\mathbb{R}^n$) or, otherwise, p_∞ is called singular. Using careful blow-up arguments, we manage to prove that the latter case cannot occur. On the other hand, to study the noncollapsed regular case, we proceed as follows:

Step 1. *Choose $p_i \in M_i$ converging to p_∞ and construct on a neighbourhood of p_i an orthonormal frame of vector fields $\{X^i_\alpha\}_{\alpha=1}^n$ which are almost parallel in an L^1-sense.*

Here we exploit the well-known fact (cf. [7]) that one can use distance coordinates to find (smooth) almost isometries f_i, which we call *almost linear coordinates*, between a neighbourhood of p_i (of uniform size) and a ball in Euclidean space. Next, we choose an orthonormal basis at a point x_i and consider its parallel transport along radial lines emanating from x_i. Notice that we cannot transport our vectors along Riemannian g_i-geodesics, due to the absence of injectivity radius estimates. Fortunately, we can use the coordinates f_i to move the basis along truly Euclidean lines (but defining the transport with respect to the Riemannian connection). Finally, exploiting the L^1-curvature condition and after a clever choice of x_i, we succeed in constructing a frame with the aforementioned properties.

Step 2. *The flows generated by the vector fields $\{X^i_\alpha\}_{\alpha=1}^n$ converge in a suitable weakly measured sense to local isometries defined on $B_\rho(p_\infty)$, for some $\rho > 0$.*

In this step we first prove convergence in a certain L^1-sense adapting to our setting arguments from [5, Lemma 3.7], but we have to face the additional technical difficulty that the flows corresponding to the vector fields from step 1 are in general not measure-preserving. After that, the weak convergence follows by applying a suitable generalization of Arzelà-Ascoli theorem.

Notice that the limiting isometries obtained here generate Killing fields $\{X^\infty_\alpha\}^n_{\alpha=1}$, which are are furthermore the limits (in an appropriate sense) of the vector fields constructed in step 1. As the latter are orthonormal, the limiting Killing fields are linearly independent and hence we conclude

Step 3. $B_\rho(p_\infty)$ *is a local homogeneous space.*
From this, by standard procedures, the distance function on $B_\rho(p_\infty)$ is induced by a smooth Riemannian metric g_∞, which is moreover the uniform limit of our original sequence g_i, when we write these metrics locally with respect to almost linear coordinates. Finally, one can easily prove that the fields $\{X^\infty_\alpha\}^n_{\alpha=1}$ are actually parallel and, consequently,

Step 4. $B_\rho(p_\infty)$ *is indeed flat.*

In short, we have shown that around any point in X we can find a neighbourhood which is smooth and moreover flat. This implies that for i large enough M_i is finitely covered by a torus (= 1-step nilmanifold), which contradicts our choice of the sequence.

The proof of Theorem 3 in the remaining **collapsed case** ($\dim(X) < n$) is much more involved. We argue by reverse induction on $\dim(X)$ and apply blow-up techniques to obtain a *less-collapsed* limit. Notice that this strategy does not allow to assume bounds on the diameter and hence we need to establish a localized version of the problem. More precisely, in this case our aim is to show that the manifolds of the sequence are Seifert fibered over a lower dimensional flat orbifold. We have to deal with the extra complication that singular points really appear and we have to analyze them; roughly speaking, we prove that one can *avoid singularities* by passing to a suitable finite cover.

Furthermore, this stage of the proof includes a result of independent interest giving some understanding of the local topology of collapsing when X has singular points of codimension bigger than 2; such a result improves (in the homology level) a theorem by Kapovitch in [3] (since the latter only works for isolated singularities).

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Weyl Curvature, Einstein Metrics, and 4-Dimensional Geometry

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The Weyl curvature tensor W of a smooth Riemanian n-manifold (M, g) is the totally-trace-free projection of the Riemann curvature tensor $\mathcal{R}$; thus, it is exactly the piece of $\mathcal{R}$ that is not algebraically determined by the Ricci tensor r. Its fundamental importance, however, stems from the fact that it is exactly the conformally invariant piece of $\mathcal{R}$. Indeed, when $n \geq 4$, a Riemannian metric is locally conformally flat iff its Weyl tensor W is identically zero.

If M is a smooth compact oriented n-manifold, $n \geq 4$, the *Weyl functional*

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class $[g] = \{u^2 g \mid u : M \stackrel{C^\infty}{\to} \mathbb{R}^+\}$ of the metric, and provides a natural measure of how far $[g]$ deviates from conformal flatness. It is thus natural and interesting to study the infimum of $\mathscr{W}$ among all metrics on a given manifold M, and to ask whether there is actually a minimizing metric which achieves this infimum.

It is a remarkable feature of the $n = 4$ case that the Weyl functional plays an important role in the theory of 4-dimensional Einstein manifolds. Indeed, Einstein metrics are critical points of the Weyl functional in dimension 4, whereas this is not true at all in higher dimensions. Moreover, in those few cases [3, 7, 8] where one can actually prove that the moduli space of Einstein metrics on a given 4-manifold is connected, one proceeds by showing that every Einstein metric on the manifold in question is necessarily a *minimizer* of the Weyl functional.

Nonetheless, even in dimension 4, most minimizers of the Weyl functional are not conformally Einstein. To see why, first note that the Hodge star operator on an oriented Riemannian 4-manifold induces a conformally invariant decomposition

$$W = W_+ + W_-$$

of the Weyl tensor into its self-dual and anti-self dual parts. In these terms, the 4-dimensional Thom-Hirzebruch signature formula $\tau = p_1/3$ can be written as

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|_g^2 - |W_-|_g^2\right) d\mu_g$$

for any metric g on M, and it therefore follows that

$$\mathscr{W}([g]) = -12\pi^2 \tau(M) + 2 \int_M |W_+|_g^2 d\mu_g.$$

In particular, the study of the Weyl functional on a fixed 4-manifold is completely equivalent to studying the L^2-norm of the self-dual Weyl curvature. Moreover, any metric with $W_+ = 0$ is an absolute minimizer of $\mathscr{W}$. Metrics of this latter type are called *anti-self-dual* [1], and turn out to exist in great profusion. For example, a result of Taubes [18] asserts that the connect sum $M = X \# k\overline{\mathbb{CP}}_2$ of an arbitrary smooth compact oriented 4-manifold X with k reverse-oriented complex projective planes admits anti-self-dual metrics for any sufficiently large integer $k \gg 0$. By contrast, the Hitchin-Thorpe inequality [7, 19] shows that $M = X\#k\overline{\mathbb{CP}}_2$ never admits Einstein metrics when k is very large. In short, anti-self-dual metrics and Einstein metrics seem to inhabit topologically different realms.

But while there are many known obstructions to the existence of Einstein metrics on 4-manifolds, the only known obstructions to the existence of anti-self-dual metrics are rather trivial: M can only admit anti-self-dual metrics if $\tau \leq 0$, and the inequality must moreover be strict if M has finite fundamental group and is not S^4. It would thus be of great interest to find new obstructions to the existence of anti-self-dual metrics. Certainly this would follow if one could show that certain non-anti-self-dual Einstein metrics actually minimize the Weyl functional.

Fortunately, a result of Gursky [6] provides an intriguing piece of evidence in this direction:

Theorem (Gursky). *Let M be a smooth compact oriented 4-manifold with $b_+ \neq 0$. Then any conformal class $[g]$ on M with Yamabe constant $Y_{[g]} > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality iff $[g]$ contains a Kähler-Einstein metric g with $\lambda > 0$.

Of course, this by no means answers the question, because the requirement that $[g]$ have positive Yamabe constant — i.e. that there exist a metric $g \in [g]$ of positive scalar curvature — excludes "most" conformal classes on M. My purpose here is to describe some results which provide further evidence for the conjecture that $\lambda > 0$ Kähler-Einstein metrics are actually minimizers of the Weyl functional in dimension 4. Moreover, these results extend to the somewhat broader class of *Hermitian* Einstein metrics with positive Einstein constant.

If a compact complex surface (M, J) admits an Einstein metric g which is Hermitian with respect to J and has Einstein constant $\lambda > 0$, then (M, J) has [9] ample anti-canonical line bundle K^{-1}, often abbreviated as $c_1 > 0$. Complex surfaces with $c_1 > 0$ are called *del Pezzo surfaces* [5, 14]; they are precisely the Fano manifolds of complex dimension 2. Every del Pezzo surface conversely admits [4, 12, 15, 20] a $\lambda > 0$ Einstein metric which is Hermitian with the specified complex structure, and this metric is unique [2, 11] up to complex automorphisms and rescalings. Any del Pezzo surface is biholomorphic either to $\mathbb{CP}_1 \times \mathbb{CP}_1$ or to a blow-up $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$ of the complex projective plane at k points in general position, $0 \leq k \leq 8$. In most cases, the relevant Einstein metric is actually Kähler-Einstein. In fact, there are just two cases in which this fails to be true: $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$ and $\mathbb{CP}_2\#2\overline{\mathbb{CP}}_2$. In these exceptional cases, the Einstein metric is not Kähler, but nonetheless belongs to the same conformal class as some Kähler metric.

Any del Pezzo surface (M, J) has $b_+(M) = 1$. Thus, for any conformal class $[g]$ on the oriented smooth 4-manifold M, there is, up to multiplication by a non-zero real constant, only one non-trivial self-dual harmonic 2-form ω. We will say that $[g]$ is of *symplectic type* if this harmonic self-dual 2-form satisfies $\omega \neq 0$ at every point of M. This condition is open in the $C^{2,\alpha}$ topology on the space of conformal classes.

In this context, we are able to prove the following variant of Gursky's Theorem:

Theorem 1. *Let M be the underlying smooth oriented 4-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality iff $[g]$ contains a Kähler-Einstein metric g.

The key point of interest is that most conformal classes of symplectic type have *negative* Yamabe constant. This indicates that Gursky's inequality is not some sort of aberration that is limited to the realm of positive scalar curvature.

In fact, the proof of Theorem 1 actually proves a stronger inequality:

Theorem 2. *Let M be the underlying 4-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3} \frac{(c_1 \cdot [\omega])^2}{[\omega]^2},$$

with equality iff $[g]$ contains a Kähler metric g of constant scalar curvature.

Here $c_1 \in H^2(M, \mathbb{R})$ is the first Chern class of the symplectic manifold (M, ω), and the dot product denotes the intersection pairing on $H^2(M, \mathbb{R})$. Note that Theorem 1 then implies Theorem 2 via the reverse Cauchy-Schwarz inequality for Minkowski space.

However, there are a couple of del Pezzo surfaces where equality is prohibited in Theorem 1, because neither $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ nor $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ carries a Kähler-Einstein metric. But, as previously noted, these two del Pezzo surfaces *do* still carry $\lambda > 0$ Einstein metrics that are conformally Kähler, and hence *Hermitian*, with respect to the complex structure. Now, both of these del Pezzo surfaces are *toric*: their complex automorphism groups both contain a 2-torus $T^2 = S^1 \times S^1$. Moreover, the associated Hermitian Einstein metrics are actually invariant under the relevant 2-torus action. This makes it natural to consider their conformal classes as belonging to the set of symplectic conformal classes which are T^2-invariant. In this narrower context, one can then show that these special Einstein metrics still minimize the Weyl functional:

Theorem 3. *Let M be the underlying 4-manifold of a toric del Pezzo surface, and let g be an Einstein metric on M which is* Hermitian *and invariant under the fixed torus action. Then its conformal class $[g]$ minimizes the Weyl functional among symplectic conformal classes which are invariant under the torus action. Moreover, up to diffeomorphism, $[g]$ is the* unique *such minimizer.*

The 4-manifolds $M = \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$ now seem to offer an ideal testing ground for exploring the Weyl functional. When $k \geq 10$, these manifolds have been shown to admit anti-self-dual metrics [13, 17]. On the other hand, when $k \leq 8$, we have seen that these manifolds admit $\lambda > 0$ Einstein metrics which are reasonable candidates for minimizers of the Weyl functional. In light of what we know so far, the following conjecture seems interesting and plausible:

Conjecture. *For any $k \leq 9$, the 4-manifold $M = \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$ does not admit any anti-self-dual metrics. Moreover, for $k = 0, \ldots, 9$,*

$$\inf_{[g]} \int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M) = \frac{4\pi^2}{3}(9-k)$$

with equality when $k \neq 1, 2$. For $k \leq 8$, minimizers do exist, and are exactly the conformal classes of conformally Kähler, Einstein metrics. By contrast, no minimizer exists when $k = 9$.

Of course, when $k = 0$, the relevant Einstein metric is the Fubini-Study metric, and this is obviously a minimizer because it has $W_- = 0$. The non-existence of anti-self-dual metrics is also known when $k = 1$ because the signature vanishes, and the relevant simply connected manifold is not diffeomorphic to S^4; but even here it has not been shown that the Page metric [16] minimizes $\mathscr{W}$. When $k = 9$, one can construct [10] collapsing sequences of metrics with $\int |W_+|^2 d\mu \to 0$, but the non-existence of anti-self-dual metrics still appears to be open in this case. Progress on this conjecture would thus be certain to add substantially to our understanding of the Weyl functional, especially in connection with the theory of 4-dimensional Einstein metrics.

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