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## Analysis, Geometry and Topology of Positive Scalar Curvature Metrics

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ABSTRACT. One of the fundamental problems in Riemannian geometry is to understand the relation of locally defined curvature invariants and global properties of smooth manifolds. This workshop was centered around the investigation of scalar curvature, addressing questions in global analysis, geometric topology, relativity and minimal surface theory.

*Mathematics Subject Classification (2010)*: Primary: 53C20 (Global Riemannian Geometry).  
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### Introduction by the Organisers

The workshop *Analysis, Geometry and Topology of Positive Scalar Curvature Metrics* was attended by some fifty participants from Europe, the US, South America and Japan, including a number of young scientists on a doctoral or postdoctoral level. Rather than representing a single mathematical discipline the workshop aimed at bringing together researchers from different areas, but working on similar questions. Hence special emphasis was put on the exchange of ideas and methods from mathematical physics, global analysis and topology, providing an attractive and diverse scientific program.

The foundations of the subject were laid in the sixties, seventies and eighties in the work of Kazdan-Warner, Lichnerowicz, Hitchin, Schoen-Yau and Witten, among others: On the one hand it is well known that each smooth function on a closed smooth manifold  $M$  of dimension at least 3 can be realized as the scalar curvature of a Riemannian metric on  $M$ , if and only if  $M$  admits a metric of positive scalar curvature. On the other hand, by combining the Weitzenböck formula

from spin geometry with the Atiyah-Singer index theorem, a closed spin manifold with non-vanishing  $\hat{A}$ -genus cannot carry a metric of positive scalar curvature. In combination with the Kazdan-Warner result, which relies on the analysis of geometric PDEs, this reveals a deep interplay of the theory of geometric PDEs and subtle differential-topological invariants. Additional obstructions based on minimal hypersurfaces and the positive mass theorem establish close connections to variational methods and general relativity.

All of these, by now classical, aspects play important roles in today's research and were represented in the scientific activities during the workshop. After two ninety minutes survey lectures on index theory and general relativity the state of the art was unfolded in sixteen one hour lectures and eight short contributions.

Among the broad range of subjects a major theme was the discussion of the Einstein constraint equations from general relativity including the optimal localisation of asymptotically flat metrics of non-negative scalar curvature on Euclidean space.

A number of talks was devoted to invariants detecting the topology of the space of positive scalar curvature metrics on a fixed manifold based on higher  $\rho$ -invariants and up to date methods from differential topology.

The discussion of the Schoen-Yau minimal hypersurface obstruction for non-smooth metrics of positive scalar curvature, of the indices of minimal surfaces and of large scale obstructions to positive scalar curvature metrics like macroscopic dimension were the content of three more talks.

Further topics of interest were the construction of Riemannian metrics satisfying certain criticality conditions on connected sums of Einstein manifolds and the discussion of the Yamabe invariant on products of manifolds.

The long time properties of Ricci solitons, metrics with invertible Dirac operators on spin manifolds and their behaviour under surgery, spectral properties of the conformal Laplacian and metrics of almost non-negative sectional curvature also attracted special attention.

Shorter contributions by young participants dealt with equivariant aspects of the positive scalar curvature problem, its relation to stable homotopy theory and higher index theory, and the index theory of pseudodifferential operators on open manifolds, among others.

An informal problem session at the end of the conference collected important topics for future investigations.

The interdisciplinary character of the workshop was reflected by the fact that the lectures were not on a highly specialized, technical level, but accessible to an audience with different mathematical backgrounds.

The atmosphere during the workshop was both relaxed and inspiring, with many questions and discussions during and outside the lectures. A perfect organization and management by the staff of the Oberwolfach institute created an optimal working environment.

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## Abstracts

### Localizing solutions of the Einstein constraint equations

ALESSANDRO CARLOTTO

(joint work with Richard M. Schoen)

The object of Mathematical General Relativity is the study of spacetimes, namely of Lorentzian manifolds  $(\mathbb{L}, \gamma)$  with signature  $(-+++)$  solving the Einstein equation  $G = T$  where  $G = Ric(\gamma) - \frac{1}{2}R(\gamma)\gamma$  (or possibly with an extra cosmological term) and  $T$  is the stress-energy tensor of the matter fields. In such investigation, and more specifically in the investigation of the *large time behaviour* of  $(\mathbb{L}, \gamma)$ , for instance of its geodesic completeness, a great deal of information can be extracted from the analysis of Riemannian slices  $(M, g, k)$  consisting of all those events that happen at a given instant in time for some physical observer. Results of this flavour, that go back to Penrose and Hawking, motivate the study of solutions to the Einstein constraint equations

$$\begin{cases} \frac{1}{2} \left( R(g) + (tr_g k)^2 - \|k\|_g^2 \right) = \mu \\ div_g (k - (tr_g k) g) = J, \end{cases}$$

that any such slice  $(M, g, k)$  is required to satisfy because of the Gauss and Codazzi equations for submanifolds. Here  $g$  (resp.  $k$ ) are the first (resp. second) fundamental form of  $M$  in  $(\mathbb{L}, \gamma)$ , while  $\mu$  (the *mass density*) and  $J$  (the *current density*) are suitable components of the tensor  $T$ . Furthermore, this differential system comes together with the functional inequality  $\mu \geq |J|_g$  which follows from a basic physical axiom, known as *dominant energy condition*, that prescribes the matter density measured by any physical observer to be non-negative at each point. In the most basic of all cases, that is when  $k = 0$  (in which case we will say that the data  $(M, g, k)$  are *time-symmetric*) the Einstein constraints reduce to the single equation

$$R(g) = 2\mu$$

namely to a scalar curvature prescription problem, where the function  $\mu$  is non-negative. More generally, a similar conclusion applies to the Hamiltonian constraint in the *maximal case* namely when  $tr_g k = 0$ , for in that case one studies the problem  $R(g) = 2\mu + \|k\|_g^2$  (coupled with  $div_g(k) = J$ ).

In this report, we are concerned with a special (yet fundamental) class of solutions of the Einstein constraints, the *asymptotically flat* ones and for the sake of simplicity we will mostly refer to the aforementioned time-symmetric setting. An easy way to describe the presence of gravitational fields, namely the displacement of a given asymptotically flat manifold  $(M, g)$  from the trivial ground state  $(\mathbb{R}^3, \delta)$  is by introducing a scalar invariant called ADM mass, which can be computed as a certain flux integral at infinity in  $(M, g)$ . The celebrated *Positive Mass Theorem* by Schoen-Yau [13] precisely states that the *local* constraints above do have dramatic global consequences: if  $(M, g)$  is asymptotically flat and  $R(g) \geq 0$ ,

then  $\mathcal{M}_{ADM} \geq 0$  with equality if and only if  $(M, g)$  is the Euclidean space. For instance, this implies the following remarkable

**Corollary.** *Let  $g$  be a Riemannian metric on  $\mathbb{R}^3$  having non-negative scalar curvature. Suppose that there exists a compact set  $K$  such that  $g$  is exactly Euclidean outside of  $K$ . Then  $g$  is the Euclidean metric on the whole  $\mathbb{R}^3$ .*

With a little bit more effort, one can get a sharper form of the previous statement. Given an asymptotically metric  $g$  on  $\mathbb{R}^n$  we let  $U(g) = \{x \in \mathbb{R}^n \mid Ric(g)(x) \neq 0\}$  and define the *content at infinity* of  $g$  to be the asymptotic size of the set  $U$

$$\Theta(g) = \liminf_{\sigma \rightarrow \infty} \sigma^{1-n} \mathcal{H}^{n-1}(U \cap B(\sigma))$$

for  $B(\sigma)$  the ball centered at the origin and having radius  $\sigma$ . With this notation introduced, one can then easily show that the following implication holds:

$$\mathcal{M}_{ADM} > 0 \implies \Theta(g) > 0$$

for  $\mathcal{M}_{ADM}$  the ADM mass of  $(\mathbb{R}^n, g)$  (here we need  $3 \leq n \leq 7$  or  $M$  spin). Roughly speaking, this says that an asymptotically flat metric cannot be localized inside sets that are asymptotically too small, for instance a cylinder or a slab between two parallel planes. Thus one is naturally led to the following basic question.

**Problem.** *What is the optimal localization of an asymptotically flat metric of non-negative scalar curvature?*

For instance, can we construct an asymptotically flat metric of non-negative scalar curvature, positive ADM mass and which is exactly trivial in a half-space?

In joint work with Richard Schoen [5], we give an essentially complete answer to such question, by developing a systematic method to localize a given scalar flat, asymptotically flat metric inside a cone of arbitrarily small aperture. To state our main theorem precisely, we need some notation.

Let  $(M, \check{g}, \check{k})$  be an initial data set for the *vacuum* Einstein equation, with decay assumption of the form  $|\check{g}_{ij}(x) - \delta_{ij}| \lesssim |x|^{-\check{p}}$ , and  $|\check{k}_{ij}(x)| \lesssim |x|^{-\check{p}-1}$  for  $(n-2)/2 < \check{p} \leq n-2$  (hence we are assuming that the Einstein constraint equations are satisfied for  $\mu = J = 0$ ). Given an angle  $0 < \theta < \pi$  and a point  $a \in \mathbb{R}^n$  with  $|a| \gg 1$  we denote by  $C_\theta(a)$  the region of  $M$  consisting of the compact part together with the set of points  $x$  in the exterior region which make an angle less than  $\theta$  with the vector  $-a$ . If we are given two angles  $0 < \theta_0 < \theta_1 < \pi$  we consider the region between the cones. By regularizing this region near the vertex  $a$  we get two smooth hypersurfaces  $\Sigma_0$  and  $\Sigma_1$  such that  $M \setminus (\Sigma_0 \cup \Sigma_1)$  is a disjoint union of three domains  $\Omega_I$ ,  $\Omega$ , and  $\Omega_O$  where we refer to  $\Omega_I$  as the inner region,  $\Omega$  the transition region, and  $\Omega_O$  the outer region.

**Theorem.** *Assume that we are given a set of initial data  $(M, \check{g}, \check{k})$  as above together with angles  $\theta_0, \theta_1$  satisfying  $0 < \theta_0 < \theta_1 < \pi$ . Furthermore, suppose  $(n-2)/2 < p < \check{p}$ . Then there exists  $a_\infty$  so that for any  $a \in \mathbb{R}^n$  such that  $|a| \geq a_\infty$  we can find a metric  $\hat{g}$  and a symmetric  $(0, 2)$ -tensor  $\hat{k}$  so that  $(M, \hat{g}, \hat{k})$  satisfies*

the vacuum Einstein constraint equations,  $\hat{g}_{ij} = \delta_{ij} + O(|x|^{-p})$   $\hat{k}_{ij} = O(|x|^{-p-1})$  and

$$(\hat{g}, \hat{k}) = \begin{cases} (\check{g}, \check{k}) & \text{in } \Omega_I(a) \\ (\delta, 0) & \text{in } \Omega_O(a). \end{cases}$$

A few remarks concerning our gluing scheme are in order. First of all, let us point out that, as a function of  $|a|$ , our construction satisfies an interesting continuity property in suitable weighted Sobolev spaces, so that the ADM 4-vector of  $(M, \hat{g}, \hat{k})$  converges to the ADM 4-vector of  $(M, \check{g}, \check{k})$  as  $|a| \rightarrow \infty$ . Second: while the conceptual scheme of the proof of this theorem goes along the lines of [10] and [11] (see also [8]), it turns out that such proof relies on rather subtle improved functional inequalities, of independent interest. In addition, the whole construction happens to be truly delicate as one needs to work in *doubly weighted* Sobolev/Hölder spaces because of the need of controlling the regularity at the boundary of the gluing region as well as the decay at infinity at the same time. Third: for any  $\theta_1 < \pi/2$  the data that we produce are flat on a half-space and therefore they contain plenty of stable (in fact: locally area-minimizing) minimal hypersurfaces, in striking contrast with various recent scalar curvature rigidity results both in the closed and in the free-boundary case (see the works [2, 12, 1]). Together with our rigidity counterparts, contained in [3] and [4], the theorem above also sheds some light on the problem of existence of stable minimal hypersurfaces in asymptotically flat manifolds (more generally: marginally outer trapped hypersurfaces in initial data sets).

One can then essentially iterate the construction and get a new class of  $N$ -body solutions for the Einstein constraint equations, which the reader should compare to the results in [6, 7, 9]. More precisely, we can assembly together conical regions  $\Omega^{(1)}, \dots, \Omega^{(N)}$ , belonging to given data  $(M_1, g_1, k_1), \dots, (M_N, g_N, k_N)$  respectively, and we can perform this construction in a way that the vertices of such cones (with respect to the Euclidean coordinates in the background) occupy, possibly modulo rescaling, a pre-assigned configuration (the axes of the cones also being given).

These  $N$ -body solutions exhibit, following a definition by P. Chruściel, the phenomenon of *gravitational shielding* in the sense that one can prepare subclasses of data that do not have any interaction for finite but arbitrarily long times when evolved by means of the Einstein field equations, in striking contrast with the Newtonian gravity scenario. The fine geometric properties of such exotic data, for instance the characterization of their large isoperimetric domains and more specifically the question whether the large stable constant mean curvature spheres in the Euclidean regions are in fact isoperimetric are under our current investigation.

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## Indices of Pseudo-Differential Operators on Open Manifolds

ALEXANDER ENGEL

For a metric space  $X$  we define *uniform  $K$ -theory* by

$$K_u^*(X) := K_{-*}(C_u(X)),$$

where  $C_u(X)$  denotes the  $C^*$ -algebra of all bounded, uniformly continuous functions on  $X$ .

If  $X = M$  is a Riemannian manifold of bounded geometry (i.e., its injectivity radius is uniformly positive and the curvature tensor and all its derivatives are bounded in sup-norm), then we may prove the following:

$$K_u^0(M) \cong \{[E] - [F]: E, F \rightarrow M \text{ are vector bundles of bounded geometry}\}.$$

Uniform  $K$ -theory is the dual theory to uniform  $K$ -homology  $K_*^u(X)$  which was introduced by Spakula in [1]. If  $M^m$  is a  $\text{Spin}^c$ -manifold of bounded geometry, then we have a Poincaré duality theorem:

$$K_u^*(M) \cong K_{m-*}^u(M)$$

Using this we may now generalize John Roe's Index Theorem from [2] from operators of Dirac type to pseudo-differential operators on open manifolds.

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## Surgery and the Positive Mass Conjecture

ANDREAS HERMANN

(joint work with E. Humbert)

Let  $(M, g)$  be a closed Riemannian manifold of dimension  $n \geq 3$  and let  $L_g$  be the conformal Laplace operator for the metric  $g$ . We assume that

- all eigenvalues of  $L_g$  are strictly positive and
- the metric  $g$  is flat on an open neighborhood of a point  $p \in M$ .

Then we define the mass of  $(M, g)$  at  $p$  as the constant term in the expansion of the Green function of  $L_g$  at  $p$ . Using surgery methods from an article by Gromov and Lawson [1] we prove the following result (see [2]). Assume that there exists a closed simply-connected non-spin manifold  $M$  of dimension  $n \geq 5$  such that for all metrics  $g$  on  $M$  with the two properties stated above the mass of  $(M, g)$  is non-negative. Then the mass is non-negative for all such metrics on every closed manifold of the same dimension as  $M$ . It follows from known results ([5], [3]) that in this case one would also have a proof of the Positive Mass Conjecture for all asymptotically flat Riemannian manifolds of the same dimension as  $M$ . The Positive Mass Conjecture has been proved in all dimensions less or equal to 7 by Schoen and Yau [4]. It is an open problem whether for every  $n \geq 8$  one can find a closed simply-connected non-spin manifold  $M$  of dimension  $n$  such that for every metric as above on  $M$  the mass is non-negative. Possible candidates are  $\mathbb{C}\mathbb{P}^{2k}$  or  $\mathbb{C}\mathbb{P}^{2k} \times S^\ell$ ,  $k \geq 1$ ,  $\ell \geq 2$ .

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## A compactness theorem for embedded minimal hypersurfaces with bounded area and index

BEN SHARP

We present a compactness theorem for embedded minimal hypersurfaces in a compact Riemannian manifold. The predecessor of this is a paper of Schoen-Simon [2] where they prove a regularity and compactness theorem for stable hypersurfaces, allowing for a general existence theory for minimal hypersurfaces and extending the fundamental work of Pitts [1]. Here we weaken the hypothesis of Schoen-Simon to include hypersurfaces of bounded index.

A simplified statement of the Schoen-Simon compactness theory is that if  $(N^{n+1}, g)$  is an orientable and closed Riemannian manifold with  $\{M_k^n\} \subset N$  a sequence of stable, smooth, embedded minimal hypersurfaces and  $\mathcal{H}^n(M_k) \leq \Lambda < \infty$ , then (up to subsequence) there exists a stable (possibly singular) embedded hypersurface  $M \subset N$  such that  $M_k \rightarrow M$  in the varifold sense and

- if  $n < 7$  then  $M$  is smooth
- if  $n = 7$  then  $M$  is smooth away from a finite set of points
- if  $n > 7$  then  $\mathcal{H}^\alpha(\text{sing}(M)) = 0$  for all  $\alpha > n - 7$  and the singular set  $\text{sing}(M) \subset M$  is closed.

Our result is that the same conclusions hold if we weaken the stability hypothesis to  $\text{Index}(M_k) \leq I \in \mathbb{N}$ , and the limit has  $\text{Index}(M) \leq I$ . We note that the index of the limit is well defined due to the relative smallness of the singular set. We prove furthermore that the convergence is smooth and graphical away from a set  $S$  formed of the singular set of  $M$  and at most  $I$  points on the regular part of  $M$ . Moreover if the number of approximating sheets is  $\geq 2$  then  $M$  must be stable.

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## Topology of the space of metrics with positive scalar curvature

BORIS BOTVINNIK

(joint work with Johannes Ebert, Oscar Randal-Williams)

For a closed Spin  $d$ -manifold  $W$ , we let  $\mathcal{Riem}^+(W)$  be the space of psc-metrics. Let  $h \in \mathcal{Riem}^+(W)$  be a base point, then for another psc-metric  $g$  we can form the path of metrics  $g_t = (1-t) \cdot h + t \cdot g$ ,  $t \in [0, 1]$ . This gives a path of associated Dirac operators in the space  $\text{Fred}^d$  of  $\mathbf{CI}^d$ -linear Fredholm operators on Hilbert space, and it starts and ends in the subspace of invertible operators, which is contractible. This gives a well-defined homotopy class of maps

$$\text{inndiff}_h : \mathcal{Riem}^+(W) \longrightarrow \Omega^{\infty+d+1} \mathbf{KO}$$

**Theorem A.** *Let  $W$  be a Spin manifold of dimension  $d \geq 6$ , and fix  $h \in \mathcal{Riem}^+(W)$ . If  $k = 4s - d - 1 \geq 0$  then the map*

$$(\text{inddiff}_h)_k \otimes \mathbf{Q} : \pi_k(\mathcal{Riem}^+(W), h) \otimes \mathbf{Q} \longrightarrow KO_{4s}(\ast) \otimes \mathbf{Q} = \mathbf{Q}$$

*is surjective. If  $e = 1, 2$  and  $k = 8s + e - d - 1$ , then the map*

$$(\text{inddiff}_h)_k : \pi_k(\mathcal{Riem}^+(W), h) \longrightarrow KO_{8s+e}(\ast) = \mathbf{Z}/2$$

*is surjective. In other words, the map  $(\text{inddiff}_h)_k$  is nontrivial if  $k \geq 0$ ,  $d \geq 6$  and the target is nontrivial.*

In order to prove Theorem A, we show that for even dimensional manifolds (of dimension at least six), the map  $\alpha : \Omega^\infty \mathbf{MTSpin}(d) \rightarrow \Omega^{\infty+d} \mathbf{KO}$  representing the  $KO$ -theory orientation of the Madsen–Tillmann–Weiss spectrum  $\mathbf{MTSpin}(d)$  factors through the space of psc metrics on such manifolds, i.e. there exists a map  $\rho : \Omega^{\infty+1} \mathbf{MTSpin}(2n) \rightarrow \mathcal{Riem}^+(W)$  such that the composition

$$\Omega^{\infty+1} \mathbf{MTSpin}(2n) \xrightarrow{\rho} \mathcal{Riem}^+(W) \xrightarrow{\text{inddiff}_h} \Omega^{\infty+2n+1} \mathbf{KO}$$

is weakly homotopic to  $\Omega\alpha$ .

We prove a similar result for odd-dimensional manifolds (of dimension at least seven), which is derived from the even-dimensional case, with the help of the (Clifford-linear) family version of the spectral-flow theorem.

Besides standard surgery Gromov-Lawson technique for psc-metrics, we use new key ingredient provided by the work by the third named author and Galatius on moduli spaces of high-dimensional manifolds.

### Vanishing of the $\hat{A}$ -Genus

BURKHARD WILKING

*Does the  $\hat{A}$ -genus of an abstract non-negatively curved spin manifold vanish?*

We report on some progress towards an affirmative answer of the above question, which was first posed by John Lott. More precisely, we give a generalization of a theorem of Gromov: Given  $n$  and  $D$ , there exists  $\varepsilon(n, D)$  such that any manifold with curvature  $K > -1$ ,  $\text{diam}(M) \leq D$  and

$$\frac{1}{\text{vol}(M)} \int_M \|\text{Rm}\| < \varepsilon(n, D)$$

is finitely covered by a nilmanifold. We then explain why one might hope that this should give an affirmative answer to the above question.

## Small Eigenvalues of the Conformal Laplacian

CHRISTIAN BÄR

(joint work with Mattias Dahl)

This talk is based on the paper [1]. Throughout the talk let  $M$  be a compact oriented differentiable manifold of dimension  $n \geq 3$ . Given a Riemannian metric  $g$  on  $M$  the *conformal Laplacian*  $L_g$  is defined as

$$L_g = \Delta_g + \frac{n-2}{4(n-1)} \cdot \text{Scal}_g: C^\infty(M) \rightarrow C^\infty(M) \subset L^2(M),$$

where  $\Delta_g = d^*d$  is the Laplacian and  $\text{Scal}_g$  is the scalar curvature of  $g$ . The operator  $L_g$  is an elliptic differential operator of second order, essentially self-adjoint in  $L^2(M)$ . Let  $\mu_0(g) \leq \mu_1(g) \leq \mu_2(g) \leq \dots$  be the spectrum of  $L_g$ , the eigenvalues being repeated according to their multiplicities. Let  $f$  be a positive smooth function on  $M$ . The conformal Laplacian of the conformally related metric  $\bar{g} = f^{\frac{4}{n-2}}g$  is given by

$$(1) \quad L_{\bar{g}}u = f^{-\frac{n+2}{n-2}}L_g(fu).$$

Applying (1) to the function  $u \equiv 1$  gives the formula

$$(2) \quad \text{Scal}_{\bar{g}} = \frac{4(n-1)}{n-2}f^{-\frac{n+2}{n-2}}L_g f$$

for the scalar curvature of  $\bar{g}$ .

We now introduce a differential topological invariant of a compact manifold by counting the number of small eigenvalues of the conformal Laplacian.

**Definition.** Let  $M$  be a compact differentiable manifold. The  $\kappa$ -invariant  $\kappa(M)$  is defined to be the smallest integer  $k$  such that for every  $\varepsilon > 0$  there is a Riemannian metric  $g_\varepsilon$  on  $M$  for which

$$\begin{cases} \mu_k(g_\varepsilon) = 1, \\ |\mu_i(g_\varepsilon)| < \varepsilon, & 0 \leq i < k. \end{cases}$$

If no such integer exists set  $\kappa(M) := \infty$ .

Heuristically,  $\kappa(M)$  is the dimension of the “almost-kernel” of the conformal Laplace operator.

By rescaling the metrics  $g_\varepsilon$  accordingly one sees that  $\kappa(M)$  is also the smallest integer  $k$  such that for each constant  $C > 0$  there exists a Riemannian metric  $g_C$  for which

$$\begin{cases} \mu_k(g_C) > C, \\ |\mu_i(g_C)| \leq 1, & 0 \leq i < k. \end{cases}$$

Hence  $\kappa(M)$  tells us which is the first eigenvalue that can be made arbitrarily large for appropriate metrics while keeping the preceding ones bounded.

If we made this definition using the Laplace operator acting on  $p$ -forms instead of the Conformal Laplacian, then by Hodge theory the resulting invariant would be nothing but the  $p^{\text{th}}$  Betti number  $b_p(M)$ .

**Open Question.** Does the case  $\kappa(M) = \infty$  occur?

We will see that for simply connected manifolds of dimension  $\geq 5$  the  $\kappa$ -invariant is always finite.

From the fact that the spectrum of the disjoint union  $M_1 \amalg M_2$  is the disjoint union of the spectra of  $M_1$  and of  $M_2$  it follows that

$$(3) \quad \kappa(M_1 \amalg M_2) = \kappa(M_1) + \kappa(M_2).$$

The next proposition concerns the relation between  $\kappa(M)$  and scalar curvature.

**Proposition 1.** *Let  $M$  be a compact differentiable manifold of dimension  $n \geq 3$ . Then*

- (1)  $\kappa(M) = 0$  if and only if  $M$  carries a metric of positive scalar curvature.
- (2) If  $M$  has a metric with  $\text{Scal} \geq 0$ , then  $\kappa(M) \leq b_0(M)$ .

*Proof.* If  $\kappa(M) = 0$ , then there is a metric with  $\mu_0 = 1$ . The corresponding eigenfunction  $f_0$  can be chosen to be positive. From Equation (2) it follows that  $\tilde{g} = f_0^{\frac{4}{n-2}} g$  has positive scalar curvature. Conversely, if  $g$  is a metric of positive scalar curvature on  $M$ , then  $L_g > 0$  and we can rescale so that  $\mu_0 = 1$ . Hence  $\kappa(M) = 0$ .

For a metric  $g$  with  $\text{Scal} \geq 0$  on  $M$  we have  $L_g = \Delta_g \geq 0$  and the zero eigenspace consists of the locally constant functions. Hence  $\mu_0(g) = \dots = \mu_{b_0(M)-1}(g) = 0$  and  $\mu_{b_0(M)}(g) > 0$ . □

The following theorem controls the spectrum of  $L_g$  under surgeries of codimension at least three. This enables us to examine the behavior of  $\kappa(M)$  under such surgeries.

**Theorem 2.** *Let  $(M, g)$  be a closed Riemannian manifold. Let  $\tilde{M}$  be obtained from  $M$  by surgery in codimension at least three. Then for each  $k \in \mathbb{N}$  and for each  $\varepsilon > 0$  there exists a Riemannian metric  $\tilde{g}$  on  $\tilde{M}$  such that the first  $k + 1$  eigenvalues of the operators  $L_g$  and  $L_{\tilde{g}}$  are  $\varepsilon$ -close, that is*

$$|\mu_j(g) - \mu_j(\tilde{g})| < \varepsilon$$

for  $j = 0, \dots, k$ .

As an immediate consequence we obtain

**Corollary 3.** *Let  $M$  be a compact differentiable manifold of dimension  $n \geq 3$ . Suppose  $\tilde{M}$  is obtained from  $M$  by surgery of codimension  $\geq 3$ . Then*

$$\kappa(\tilde{M}) \leq \kappa(M).$$

Hence for any  $\kappa_0 \in \mathbb{N}_0$  the property of having  $\kappa \leq \kappa_0$  is preserved under surgery of codimension at least three. For  $\kappa_0 = 0$  this means that the property of admitting a metric of positive scalar curvature is preserved under such surgeries. This is a famous by now classical result of Gromov and Lawson [3]. We do not give a new proof of this fact since we use the work of Gromov and Lawson when we prove Theorem 2.

As to the case  $\kappa_0 = 1$  it is interesting to note that the property of allowing a scalar flat metric is not preserved under such surgeries. It follows that the converse of statement (2) in Proposition 1 does not hold. For example, the  $n$ -dimensional torus  $T^n$  has a flat metric but no metric of positive scalar curvature [4]. Thus  $\kappa(T^n) = 1$ . Performing surgery in codimension at least three on  $T^n$  yields a manifold  $M^n$  not admitting metrics with positive or zero scalar curvature. Yet we have  $\kappa(M^n) = 1$ .

Also note that the condition  $\kappa = 0$  is not preserved under surgery of codimension 2. Like any compact connected 3-manifold the 3-torus  $T^3$  can be obtained from  $S^3$  by a sequence of surgeries in codimension 2. But we have  $\kappa(T^3) = 1 > \kappa(S^3) = 0$ . This also shows that Theorem 2 cannot hold for surgeries in codimension less than three.

The  $\kappa$ -invariant measures how close  $L$  can come to being a positive operator for some Riemannian metric on  $M$ . Since  $L$  is positive if and only if  $M$  allows a metric of positive scalar curvature one can also view  $\kappa$  as a measure of how close one can get to having positive scalar curvature. Therefore it is not unreasonable to suspect that  $\kappa$  is related to the  $\hat{A}$  or  $\alpha$ -genus of  $M$ , the primary obstruction to allowing metrics of positive scalar curvature. We show that this indeed is the case. On the one hand we have

**Theorem 4.** *Let  $M$  be a compact spin manifold of dimension  $n = 4m$ . Then*

$$|\hat{A}(M)| \leq 2^{2m-1} \kappa(M).$$

**Open Question.** Can one replace the factor  $2^{2m-1}$  in Theorem 4 by  $2^m$ ?

Theorem 4 together with a classical eigenvalue estimate by Cheeger [2] implies the following isoperimetric result.

**Corollary 5.** *Let  $M$  be a compact spin manifold of dimension  $n = 4m$  with  $|\hat{A}(M)| > 2^{2m-1}$ . Then there exists a constant  $C = C(M)$  such that for each Riemannian metric with  $|\text{Scal}| \leq 1$  there exists a hypersurface  $S \subset M$  dividing  $M$  into two connected components  $M_1$  and  $M_2$  such that*

$$\text{vol}_{n-1}(S) \leq C \cdot \min\{\text{vol}_n(M_1), \text{vol}_n(M_2)\}.$$

On the other hand, we can bound  $\kappa(M)$  from above in terms of the dimension and the  $\alpha$ -genus, at least for simply connected manifolds of dimension  $n \geq 5$ . First we make the following

**Observation.** *Let  $M$  be a simply connected compact differentiable manifold of dimension  $n \geq 5$ . If  $M$  is non-spin or if  $n \equiv 3, 5, 6, 7 \pmod{8}$  then*

$$\kappa(M) = 0.$$

This comes from the fact that in these cases  $M$  is well known to carry a metric of positive scalar curvature, see [3], [5].

In dimensions  $n \equiv 1, 2 \pmod{8}$  we have  $\alpha(M) \in \text{KO}^{-n}(\text{pt}) \cong \mathbb{Z}/2\mathbb{Z}$ . By  $|\alpha(M)| \in \mathbb{Z}$  we mean 0 if  $\alpha(M)$  is trivial and 1 otherwise.

**Theorem 6.** *Let  $M$  be a simply connected spin manifold of dimension  $n = 8l + 1$  or  $8l + 2$ ,  $l \geq 1$ . Then*

$$\kappa(M) = |\alpha(M)|.$$

This shows that  $\kappa(M)$  can distinguish certain exotic spheres. In particular,  $\kappa(M)$  is *not* invariant under homeomorphisms, only under diffeomorphisms.

In dimensions  $n \equiv 0 \pmod{4}$  the  $\alpha$ -genus of a spin manifold is integer-valued and it essentially coincides with the  $\hat{A}$ -genus. In these dimensions one can bound the  $\kappa$ -invariant in terms of the  $\hat{A}$ -genus. In these remaining dimensions  $\text{KO}^{-n}(\text{pt}) = 0$  and  $\kappa(M) = 0$  for simply connected  $M$ .

Even though Theorem 4 shows that  $\kappa(M)$  can become arbitrarily large it turns out that in a stable sense it takes only the values 0 and 1. More precisely, let  $B$  be a compact simply connected 8-dimensional spin manifold with  $\hat{A}(B) = 1$ . Then  $\alpha(M \times B) = \alpha(M)$  for all spin manifolds  $M$ .

**Theorem 7.** *Let  $M$  be a simply connected spin manifold. Then*

$$\kappa(M \times B^p) \leq 1$$

*for all sufficiently large  $p$ .*

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### A Positive Mass Theorem for Metrics with Weakened Regularity

DAN A. LEE

(joint work with Philippe G. LeFloch)

In this work we prove a version of the positive mass theorem for metrics with regularity  $C^0 \cap W_{loc}^{1,n}$ , where  $n$  is the dimension of the manifold and  $W_{loc}^{1,n}$  is a Sobolev space. With this level of regularity, the scalar curvature can be defined as a distribution. Recall that the positive mass theorem was established by Schoen and Yau in dimensions  $n$  less than eight [18, 17] and Witten for spin manifolds [20], under the assumption that the underlying metric is regular. Bartnik [1] showed that Witten’s spinor argument works whenever the metric is  $W_{loc}^{2,p}$  with  $p > n$ . For the slightly weaker integrability class  $C^0 \cap W_{loc}^{2,n/2}$ , see [5]. As far as solely “piecewise regular” metrics are concerned, Miao [15] used a smoothing plus

conformal deformation (following Bray [3]) and proved a version of the positive mass theorem for metrics that are singular only along a hypersurface. Similar results were also proved by Shi and Tam [19] (using Witten's spinor method) and McFeron and Székelyhidi [14] (using the Ricci flow). The conformal deformation method was also used by Lee [6] to treat metrics with low-dimensional singular sets.

Our result generalizes *all* of those previous results in the spin case. It also fits together with and was motivated by earlier work by LeFloch and collaborators [9, 10, 11, 13], who defined and investigated the Einstein equations within the broad class of metrics with  $L^\infty \cap W_{loc}^{1,2}$  regularity and established existence results for the Cauchy problem at this level of regularity.

**Theorem 1 (The positive mass theorem for distributional curvature).**

*Let  $M$  be a smooth  $n$ -manifold ( $n \geq 3$ ) endowed with a spin structure and a  $C^0 \cap W_{-q}^{1,n}$  regular and asymptotically flat, Riemannian metric  $g$ , with  $q \geq (n-2)/2$ . If the distributional scalar curvature  $R_g$  of  $g$  is non-negative, then its generalized ADM mass is non-negative. Moreover, the mass is zero only when  $(M, g)$  is isometric to Euclidean space.*

We find it convenient to choose a smooth background metric  $h$  on  $M$  that is identically equal to the Euclidean metric outside a compact set. Using this metric, it is a standard matter to define the Lebesgue spaces and Sobolev spaces, which do not depend upon the choice of  $h$ . The  $q$  subscript in  $W_{-q}^{1,n}$  refers to the decay rate of the metric  $g - h$ .

Let  $g$  be a  $C^2$  metric and define

$$\Gamma_{ij}^k := \frac{1}{2} g^{kl} (\bar{\nabla}_i g_{jl} + \bar{\nabla}_j g_{il} - \bar{\nabla}_l g_{ij}),$$

where  $\bar{\nabla}$  denotes the Levi-Civita connection of the background metric  $h$ . In general, we use the background metric  $h$  for computing barred quantities. A straightforward computation shows that

$$R_g = \bar{\nabla}_k V^k + F,$$

where

$$V^k := g^{ij} g^{kl} (\bar{\nabla}_j g_{il} - \bar{\nabla}_l g_{ij}),$$

$$F := \bar{R} - \bar{\nabla}_k g^{ij} \Gamma_{ij}^k + \bar{\nabla}_k g^{ik} \Gamma_{ji}^j + g^{ij} (\Gamma_{kl}^k \Gamma_{ij}^l - \Gamma_{jl}^k \Gamma_{ik}^l).$$

This calculation motivates the following definition.

**Definition 2.** *Let  $M$  be a smooth manifold endowed with a smooth background metric  $h$ . Given any Riemannian metric  $g$  with  $L_{loc}^\infty \cap W_{loc}^{1,2}$  regularity and locally bounded inverse  $g^{-1} \in L_{loc}^\infty$ , the **scalar curvature distribution**  $R_g$  is defined, for every compactly supported smooth (test-) function  $u : M \rightarrow \mathbb{R}$  by*

$$\langle R_g, u \rangle := \int_M \left( -V \cdot \bar{\nabla} \left( u \frac{d\mu_g}{d\mu_h} \right) + F u \frac{d\mu_g}{d\mu_h} \right) d\mu_h,$$

*in which the dot product is taken using the metric  $h$  and  $d\mu_h$  and  $d\mu_g$  denote the volume measures associated with  $h$  and  $g$ , respectively.*

This definition makes sense because the assumption that  $g \in L_{loc}^\infty \cap W_{loc}^{1,2}$  implies that  $\Gamma \in L_{loc}^2$ ,  $V \in L_{loc}^2$ ,  $F \in L_{loc}^1$ , and  $\frac{d\mu_g}{d\mu_h} \in L_{loc}^\infty \cap W_{loc}^{1,2}$ . Although this level of regularity is enough to show that the scalar curvature distribution is well-defined, we need more regularity to carry our Witten’s spinor proof of the positive mass theorem. In particular, we have the following:

**Proposition 3 (A Lichnerowicz-Weitzenböck identity for metrics with distributional curvature).** *Assume that  $g$  is a  $C^0 \cap W_{loc}^{1,n}$  metric on a smooth  $n$ -manifold  $M$ . If  $\psi$  and  $\phi$  are  $W_{loc}^{1,2}$  spinors and  $\phi$  has compact support, then*

$$0 = -\langle \mathcal{D}\psi, \mathcal{D}\phi \rangle_{L^2} + \langle \nabla\psi, \nabla\phi \rangle_{L^2} + \frac{1}{4} \langle R_g, \langle \psi, \phi \rangle \rangle,$$

where all quantities are computed using  $g$ , and  $\mathcal{D}$  is the Dirac operator.

One last point worth mentioning is that our regularity of  $g$  is not strong enough for the boundary integrals appearing in the usual definition of ADM mass to be well-defined. So instead, we make the following more general definition that only involves integration over an annulus.

**Definition 4.** *Let  $g$  be a  $C^0 \cap W_{-q}^{1,n}$  regular and asymptotically flat, Riemannian metric on  $M^n$  with  $q \geq (n - 2)/2$ . The **generalized ADM mass** of such a manifold is then defined as*

$$m := \frac{1}{2(n - 1)\omega_{n-1}} \inf_{\epsilon > 0} \liminf_{\rho \rightarrow +\infty} \left( \frac{1}{\epsilon} \int_{\rho < r < \rho + \epsilon} V \cdot \bar{\nabla} r \, d\mu_h \right),$$

where  $V$  is the vector field defined above,  $r$  is the radial coordinate of the asymptotically flat coordinate chart, and  $\omega_{n-1}$  is the volume of the standard unit  $(n - 1)$ -sphere.

Under the hypotheses of Theorem 1, the  $\liminf_{\rho \rightarrow +\infty}$  in the formula always exists and is independent of  $\epsilon$ , though it could be  $+\infty$ .

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## On the topology and index of minimal surfaces

DAVI MAXIMO

(joint work with Otis Chodosh)

The work of Schoen and Yau on the Positive Mass Theorem [11] and also on metrics of positive scalar curvature [12] made well known that the existence of minimal submanifolds of certain topological and Morse index type can impose restrictions on the curvature of the ambient manifold, and vice-versa. On several instances since, the index of minimal submanifolds have played a key role in the study of curvature: manifolds with positive isotropic curvature [8], the finite time extinction of three-dimensional Ricci flow [3], the Willmore conjecture [7], to cite a few.

Granted all this, the connection between the index of minimal submanifolds, their topology, and the curvature of the ambient remains elusive – even when the ambient manifold has constant curvature. In recent work with Otis Chodosh [2], we investigated the relationship between the index and the topology of minimal surfaces in  $\mathbb{R}^3$ . We showed a lower bound for the index of a minimal surface in terms of its genus and number of ends:

**Theorem 1.** *Suppose that  $\Sigma \rightarrow \mathbb{R}^3$  is an immersed complete two-sided minimal surface of genus  $g$  and with  $r$  ends. Then*

$$\text{Index}(\Sigma) \geq \frac{2}{3}(g + r) - 1.$$

The main consequence of Theorem 1 is that it allows one to extend the classification of minimal surfaces with “small” index. Previously, minimal surfaces of  $\mathbb{R}^3$  with index zero ([5],[4],[9],[10]) and index one [6] were classified. One can use Theorem 1 to show that there are no embedded minimal surfaces of index two, as conjectured by Choe [1]:

**Theorem 2.** *There are no embedded minimal surfaces of index two in  $\mathbb{R}^3$ .*

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### A Positive Mass Theorem for Asymptotically Flat Manifolds with a Non-Compact Boundary

EZEQUIEL BARBOSA

(joint work with Sérgio Almaraz, Levi Lopes de Lima)

We consider an oriented asymptotically flat Riemannian manifold  $(M^n, g)$  with boundary  $\Sigma$  and dimension  $n \geq 3$ , modeled on the half-space  $\mathbb{R}_+^n$  (see [1] for precise definitions). We denote by  $R_g$  the scalar curvature of  $(M, g)$ . We also

assume that  $\Sigma$  is oriented by an outward pointing unit normal vector  $\eta$ , so that its mean curvature is  $H_g = \operatorname{div}_g \eta$ . In terms of asymptotically flat coordinates, we define a mass-type quantity, which is the analogue to the ADM mass, by

$$\mathfrak{m}_{(M,g)} := \lim_{r \rightarrow +\infty} \left\{ \int_{S_{r,+}^{n-1}} (g_{ij,j} - g_{jj,i}) \mu^i dS_{r,+}^{n-1} + \int_{S_r^{n-2}} g_{\alpha n} \vartheta^\alpha dS_r^{n-2} \right\},$$

where  $S_{r,+}^{n-1} \subset M$  is a large coordinate hemisphere of radius  $r$  with outward unit normal  $\mu$ , and  $\vartheta$  is the outward pointing unit co-normal to  $S_r^{n-2} = \partial S_{r,+}^{n-1}$ , viewed as the boundary of the bounded region  $\Sigma_r \subset \Sigma$ . We prove that  $\mathfrak{m}_{(M,g)}$  is a geometric invariant, in the sense that it does not depend on the asymptotically flat coordinates chosen. Moreover, assuming that  $R_g \geq 0$  and  $H_g \geq 0$ , and either  $3 \leq n \leq 7$  or  $n \geq 3$  and  $M$  is spin, we obtain that  $\mathfrak{m}_{(M,g)} \geq 0$ , with the equality  $\mathfrak{m}_{(M,g)} = 0$  occurring if and only if  $(M, g)$  is isometric to  $(\mathbb{R}_+^n, \delta)$ .

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### Low Dimensional Polar Actions

FRANCISCO J. GOZZI

A proper isometric action of a Lie group  $G$  on a Riemannian manifold  $M$  is called *polar* if it admits a *section*, i.e., an immersed complete submanifold  $\Sigma$  of  $M$  intersecting every orbit orthogonally. These actions are special in that they can be reconstructed from its orbit space together with a marking of isotropy groups along strata, as it was shown by Grove and Ziller [1].

In the case of cohomogeneity one actions Hoelscher [2] gives an equivariant classification of the actions on compact simply-connected manifolds of dimension 7 or less. We finish the classification of polar actions on compact simply-connected manifold up to dimension 5, by addressing the case of cohomogeneity at least two. This classification is both equivariant and topological.

Our main contribution is the description of effective polar  $T^2$ -actions as the result of equivariant surgery operations starting from linear polar actions on spheres and products of spheres, or biquotient  $T^2$ -actions on either of the two  $S^3$  bundles over  $S^2$ . The equivariant surgeries correspond to connected sums at fixed points and surgeries along regular orbits.

As an application of our classification and the explicit description of what the sections are in each case, we are able to determine which actions admit invariant non-negative sectional curvature.

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**Twisted Spin Cobordism**

FABIAN HEBESTREIT

(joint work with Michael Joachim)

The surgery theorem of Gromov and Lawson in [1] showed that the existence of a metric of positive scalar curvature (psc) on a given connected, closed, smooth manifold  $M$  of dimension  $n$  greater than 4 depends only on its bordism class in a certain bordism group; precisely which bordism group is determined by the normal 1-type of  $M$  (compare e.g. [4, 8, 13]). For spin manifolds, for example, one has to consider the spin bordism group

$$\Omega_n^{Spin}(B\pi_1(M))$$

Their approach has lead to strong existence theorems for psc-metrics (see [9]): For example Gromov and Lawson used the well-known computation of the oriented bordism ring to conclude that any closed, simply connected, non-spin manifold of dimension greater than 4 does admit a psc metric. For spin manifolds on the other hand the indices of various Dirac-operators associated with the spin structure give well-known obstructions to the existence of such metrics. These indices were coalesced into a single class in the operator  $K$ -theory group  $KO_n(C^*\pi_1(M))$  by Rosenberg in [2], which he conjectured to be a complete obstruction (but see [7, 10]); here  $C^*G$  denotes the (reduced)  $C^*$ -algebra associated to a given group  $G$ . As it is invariant under spin bordism one obtains a map

$$\Omega_n^{Spin}(B\pi_1(M)) \rightarrow KO_n(C^*\pi_1(M))$$

which factors over the connective  $K$ -homology group  $ko_n(B\pi_1(M))$  via the Atiyah-Bott-Shapiro-orientation  $\alpha$  and an assembly map. The strongest general theorem available in this case is due to Stolz and Fühling (see [5, 12]): The vanishing of  $\alpha(M) \in ko_n(B\pi_1(M))$  is sufficient for the existence of a psc-metric on  $M$ . Since the assembly map alluded to above is an isomorphism in the case of  $\pi_1(M) = 0$ , this in particular covers the case of simply connected manifolds originally solved in [3].

While the dividing line for the existence of index-theoretic obstructions seems to be a spin structure *on the universal cover of  $M$*  (compare e.g. [11]), an analogue of the above result remains a conjecture (due to Stolz) for this greater class of manifolds. Various ingredients, however, are already in place: The correct bordism groups are twisted spin bordism groups  $\Omega_n^{Spin}(B\pi_1(M), w(M))$  (see [13]), Stolz constructed a generalisation of Rosenberg’s invariant (taking values in the  $K$ -theory of a  $C^*$  algebra associated to the fundamental super-group of  $M$ ) in

[15], Joachim provided a twisted version of the Atiyah-Bott-Shapiro orientation in [6] and we prove a factorisation of Stolz' invariant through twisted, connective  $K$ -homology in [13].

In my talk I reported on recent work (from my PhD-thesis [14]) giving both a homotopical and a homological description of the underlying parametrised spectrum  $M_2O$  representing twisted spin cobordism. Specifically, I presented a generalisation of the Anderson-Brown-Peterson splitting (which is the basis for the computation of the spin cobordism ring) and computed  $H^*(M_2O, \mathbb{Z}/2)$  as a module over an appropriately extended Steenrod algebra.

These computations should be regarded as a first step towards a proof of Stolz' conjecture.

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## Recent Progress on the Equivariant Yamabe Problem

FARID MADANI

**LICHNEROWICZ CONJECTURE.** *For every compact Riemannian manifold  $(M, g)$  of dimensions  $n \geq 3$ , which is not conformal to the round sphere  $S^n$ , there exists a metric  $\tilde{g}$  conformal to  $g$ , for which  $\text{Isom}(M, \tilde{g}) = \text{Conf}(M, g)$  and the scalar curvature  $\text{Scal}_{\tilde{g}}$  is constant.*

In order to solve this conjecture, Hebey and Vaugon [2] introduced the following problem:

*Let  $G$  be a subgroup of the isometry group  $\text{Isom}(M, g)$ . Is there a  $G$ -invariant metric  $g_0$  which minimizes the functional*

$$J(g') := \frac{\int_M \text{Scal}_{g'} dv_{g'}}{\left(\int_M dv_{g'}\right)^{\frac{n-2}{n}}},$$

where  $g'$  belongs to the  $G$ -invariant conformal class of metrics  $g$  defined by:

$$[g]^G := \{\tilde{g} = e^f g \mid f \in C^\infty(M), \sigma^* \tilde{g} = \tilde{g} \quad \forall \sigma \in G\}$$

Using the result of Lelong-Ferrand [3], which asserts that  $\text{Conf}(M, g)$  is compact if and only if  $(M, g)$  is not conformal to the round sphere  $S^n$ , Hebey–Vaugon proved that a solution to the equivariant Yamabe problem implies that the Lichnerowicz conjecture holds.

We define the integer  $\omega$  at the point  $p$  as

$$\omega = \inf\{i \in \mathbb{N} \mid \|\nabla^i \text{Weyl}_g(p)\| \neq 0\}$$

**HEBEY–VAUGON CONJECTURE.** *If  $(M, g)$  is not conformal to the round sphere  $S^n$ , or if the action of  $G$  has no fixed point, then the following inequality holds*

$$(1) \quad \inf_{g' \in [g]^G} J(g') < n(n-1) \text{vol}(S^n)^{2/n} \left(\inf_{q \in M} \text{card } G(q)\right)^{2/n}$$

This conjecture is a generalization of Aubin’s conjecture [1] for the Yamabe problem (it corresponds to  $G = \{\text{id}\}$ ).

Assuming the Positive Mass Theorem and using the test function of Aubin [1] and Schoen [6], Hebey and Vaugon [2] proved that the Hebey–Vaugon conjecture holds when the action of  $G$  is free, when the dimension of  $M$  is between 3 and 11, or when there exists  $p \in M$  with finite minimal orbit, such that  $\omega > (n-6)/2$  or  $\omega \leq 2$ .

In [4] and [5], the author proved recently that the Hebey–Vaugon conjecture holds if there exists  $p \in M$  with finite minimal orbit, such that  $\omega \leq (n-6)/2$ . Therefore, the Lichnerowicz conjecture holds and the equivariant Yamabe problem has solutions, if the positive mass theorem holds, for any dimension  $n \geq 3$ .

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## The Yamabe equation on Riemannian products and the Yamabe invariant

JIMMY PETEAN

(joint work with Guillermo Henry)

Consider the Einstein-Hilbert functional (or normalized total scalar curvature)  $\mathbf{S}$  on the space of Riemannian metrics on a closed smooth manifold  $M$  of dimension  $n \geq 3$  and restrict it to a conformal class  $[g]$ . If we let  $p = p_n = \frac{2n}{n-2}$  and write  $h \in [g]$  as  $h = f^{p-2}$  for a positive function  $f$  then

$$\mathbf{S}(h) = \frac{\int_M a_n \|\nabla f\|^2 + s_g f^2 \, dv_g}{\|f\|_p^2}.$$

Here  $a_n = 4\frac{n-1}{n-2}$ ,  $s_g$  is the scalar curvature of  $g$  and  $dv_g$  its volume element. Written in this way the Euler-Lagrange equation of  $\mathbf{S}|_{[g]}$  is

$$-a_n \Delta f + s_g f = \lambda f^{p-1}$$

where  $\lambda$  is a constant. And this means that  $h$  has constant scalar curvature if and only if  $f$  solves the previous equation, which is called the *Yamabe equation*.

By a fundamental result it is known that there is always a solution to the Yamabe equation since the minimum of  $\mathbf{S}|_{[g]}$  is always achieved. The infimum is called the *Yamabe constant* of  $[g]$ ,  $Y(M, [g])$ . This constant is positive if and only if there is a metric of positive scalar curvature in  $[g]$ . When  $Y(M, [g]) \leq 0$  the minimizing solution is the only solution, i.e. for a fixed volume there is exactly one metric of constant scalar curvature in the conformal class. But in the positive case the space of solutions can be very complicated. A particular case to consider is the conformal class of a Riemannian product of round spheres  $(S^n \times S^m, [g_0^n + Tg_0^m])$ , where  $g_0^k$  is the curvature one metric on the  $k$ -sphere and  $T$  is a positive number. Besides the case of conformal classes of Einstein metrics (where there is also only one solution by a classical theorem of M. Obata) the case where  $n = 1$  or  $m = 1$  is one of the few cases where one can completely describe the space of solutions). When  $n, m \geq 2$  the problems is already very difficult and we can give multiplicity results by looking for solutions which bifurcate from the product metric (the constant solution). These products are of course very natural to consider by themselves but they are also very important in the theory of the *Yamabe invariant*,  $Y(M)$ , which is defined as the supremum of the Yamabe

constants of all conformal classes of metrics on  $M$  ([1]). To study the invariant is important to understand its behavior under surgery and the conformal classes of these products play a fundamental role.

There is a value  $T = T_0$  such that the product is Einstein. For  $T$  small ( $T < T_0$ ) we can see that all solutions bifurcating from the constant solution depend only on  $S^n$  (the "big" variable). Moreover, by looking at solutions *adapted* to an isoparametric function (in particular radial functions) one can see that each value of  $T$  at which the linearized equation has a non-trivial kernel is a bifurcation point ([2, 3]). In this way one can prove existence of solutions whose level sets are any given isoparametric hypersurface. By using global bifurcation techniques one can give multiplicity results for solutions of the equation for all small values of  $T$  ([2, 3]). Moreover, by a careful study of the behavior of the family of solutions appearing at the bifurcation points one can also show the existence of degenerate solutions of the equation ([4]).

Several interesting questions can be asked related to these results. Of course it would be very important if one could understand all solutions of the Yamabe equation in these products. In particular one would like to know if there might be new bifurcating points appearing at the branches of solutions constructed. It would also be interesting to know how does the space of solutions look like around the bifurcating points (are there solutions whose level sets are not isoparametric hypersurfaces?). Is the space of solutions (parametrized by  $T$ ) connected? (do all solutions come from the constant solution by a sequence of bifurcations?). Are there solutions which depend non-trivially on both variables? This last question is particularly important for applications to the Yamabe invariant.

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### Stability and Instability of Ricci Solitons

KLAUS KRÖNCKE

A Riemannian metric  $(M, g)$  is called a Ricci soliton if it satisfies the equation

$$\text{Ric}_g + \frac{1}{2} \mathcal{L}_X g = c \cdot g$$

for some smooth vector field  $X$  and some constant  $c \in \mathbb{R}$ . If  $(M, g)$  is not Einstein, we call it nontrivial. We call a soliton gradient if  $X = \text{grad} f$  for some smooth function  $f$ . If  $c > 0$ , the soliton is called shrinking, if  $c = 0$ , it is steady and if

$c < 0$ , we call it expanding. Ricci solitons appear in Ricci flow theory as self-similar solutions of the Ricci flow

$$\dot{g}(t) = -2\text{Ric}_{g(t)}.$$

In this talk, we discuss stability properties of compact Ricci solitons under the Ricci flow. The Ricci flow is not a gradient flow in the strict sense, but Perelman made the remarkable discovery that it can be regarded as the gradient flow of the functional

$$\nu(g) = \inf_{\substack{\tau > 0, f \in C^\infty(M) \\ \int_M e^{-f} dV = (4\pi\tau)^{n/2}}} \frac{1}{(4\pi\tau)^{n/2}} \int_M [\tau(|\nabla f|_g^2 + \text{scal}_g) + f - n] e^{-f} dV_g$$

on the space of metrics modulo diffeomorphism and rescaling [5]. This functional, which is often called shrinker entropy, admits precisely shrinking Ricci solitons as its critical points.

In this talk, I prove the following two assertions (which are the main results of [4]):

**Theorem.** *Let  $(M, g)$  be a compact shrinking Ricci soliton. If  $g$  is a local maximizer of  $\nu$  in the space of metrics, it is dynamically stable, i.e. for any  $C^k$ -neighbourhood  $\mathcal{U}$ ,  $k \geq 3$ , there exists a  $C^{k+5}$ -neighbourhood  $\mathcal{V} \subset \mathcal{U}$  such that any volume-normalized Ricci flow starting in  $\mathcal{V}$  exists for all time and converges modulo diffeomorphism to a Ricci soliton in  $\mathcal{U}$  as  $t \rightarrow \infty$ .*

**Theorem.** *Let  $(M, g)$  be a compact shrinking Ricci soliton. If  $g$  is not a local maximizer of  $\nu$ , it is dynamically unstable, i.e. there exists a nontrivial ancient normalized Ricci flow converging modulo diffeomorphism to  $g$  as  $t \rightarrow -\infty$ .*

The converse implications also hold due to monotonicity of  $\nu$  along the flow. Observe that either of the above cases occur and we have a complete description of the Ricci flow as a dynamical system close to  $g$ . These theorems generalize results previously obtained in the Ricci-flat and the Einstein case [2, 3]. Compact expanding and steady Ricci solitons are known to be Einstein and so they are already covered by those results.

An important tool in the proof of both theorems is a Lojasiewicz-Simon inequality: For any compact shrinking Ricci soliton  $(M, g_0)$ , there exists a  $C^{2,\alpha}$ -neighbourhood  $\mathcal{U}$  in the space of metrics and constants  $C > 0$ ,  $\sigma \in [1/2, 1)$  such that

$$|\nu(g) - \nu(g_0)|^\sigma \leq C \left\| \text{Ric}_g + \nabla^2 f_g - \frac{1}{2\tau_g} g \right\|_{L^2}$$

for all  $g \in \mathcal{U}$ . Here,  $f_g$  and  $\tau_g$  denote the minimizers in the definition of  $\nu(g)$ .

The following theorem relates stability properties of a given Ricci soliton to the eigenvalues of an elliptic operator coming from the second variation of  $\nu$ , provided that an additional technical condition holds:

**Theorem.** *Let  $(M, g)$  be a compact shrinking Ricci soliton. Suppose that all infinitesimal solitonic deformations are integrable (i.e. for all  $h \in \ker(\nu'')$  there is a curve  $g(t)$  of Ricci soliton metrics such that  $g(0) = g$  and  $g'(0) = h$ ). Then,  $g$  is a local maximizer of  $\nu$  if and only if  $\nu'' \leq 0$ .*

Note that the implication “local maximizer of  $\nu \Rightarrow \nu'' \leq 0$ ” is immediate and does not need the integrability condition. For symmetric spaces of compact type, the largest eigenvalue of  $\nu''$  is known [1]. Thus, stability properties can be read off, provided that the integrability condition holds (e.g.  $S^n$  and  $Spin(n)$ ,  $n \geq 7$  are stable but  $\mathbb{H}P^n$  and  $Spin(5)$  are unstable). The  $\mathbb{C}P^n$  with the Fubini-Study metric satisfies  $\nu'' \leq 0$  but it violates the integrability condition and it is not a local maximizer of  $\nu$ , hence it is unstable.

All known nontrivial Ricci solitons are known to be unstable (because  $\nu''$  admits positive eigenvalues). It is an open question whether this property holds for all nontrivial Ricci solitons.

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### Critical metrics on connected sums of Einstein four-manifolds

MATTHEW GURSKY

(joint work with Jeff Viaclovsky)

In this talk I describe a gluing procedure designed to obtain canonical metrics on connected sums of Einstein four-manifolds. These metrics are critical points of the quadratic Riemannian functional which assigns to each metric  $g$  the quantity

$$(1) \quad \mathcal{B}_t[g] = \int |W_g|^2 dv + t \int R_g^2 dv,$$

where  $W = W_g$  is the Weyl tensor and  $R = R_g$  is the scalar curvature. Here,  $t$  is a free parameter. In the special case when  $t = 0$ , critical metrics are called *Bach-flat*, and examples include self-dual ( $W^- = 0$ ) and anti-self-dual ( $W^+ = 0$ ) metrics.

Note that (up to topological terms) any quadratic Riemannian functional is a linear combination of the three terms

$$\int |W_g|^2 dv, \int |Ric_g|^2 dv, \int R_g^2 dv,$$

where  $Ric$  is the Ricci tensor. However, due to the Chern-Gauss-Bonnet formula

$$32\pi^2\chi(M) = \int_M |W_g|^2 dv - 2 \int_M |Ric_g|^2 dv + \frac{2}{3} \int_M R_g^2 dv$$

one can write the Ricci term as linear combination of the Weyl and scalar terms, so that any quadratic functional can be written in the form of (1).

The Euler-Lagrange equations of  $\mathcal{B}_t$  are given by

$$(2) \quad B^t \equiv B + tC = 0,$$

where  $B$  is the *Bach tensor* defined by

$$(3) \quad B_{ij} \equiv -4 \left( \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R^{kl} W_{ikjl} \right),$$

and  $C$  is the tensor defined by

$$(4) \quad C_{ij} = 2\nabla_i \nabla_j R - 2(\Delta R)g_{ij} - 2RR_{ij} + \frac{1}{2}R^2 g_{ij}.$$

It follows that any Einstein metric is critical for  $\mathcal{B}_t$ . We will refer to such a critical metric as a  $B^t$ -flat metric. Note that by taking a trace of (2), it follows that the scalar curvature of a  $B^t$ -flat metric on a compact manifold is necessarily constant. Therefore a  $B^t$ -flat metric satisfies the equation

$$(5) \quad B = 2tR \cdot E,$$

where  $E$  denotes the traceless Ricci tensor. That is, the Bach tensor is a constant multiple of the traceless Ricci tensor.

The gluing problem for the anti-self-dual equations  $W^+ = 0$  in dimension four has been very successful; see for example [1, 2, 3, 4]. However, gluing for the  $B^t$ -flat equations is much more difficult because, as in the Einstein case, this is a self-adjoint problem. The parameter  $t$  is the key to overcoming this difficulty.

The main building blocks of our construction are the Fubini-Study metric  $(\mathbb{C}\mathbb{P}^2, g_{FS})$ , and  $(S^2 \times S^2, g_{S^2 \times S^2})$ , the product of 2-dimensional spheres with unit Gauss curvature. Both are Einstein, so are  $B^t$ -flat for all  $t$ . A key result we need is the rigidity of these metrics for certain ranges of  $t$ , which was proved in our previous work [5]. That is, these metrics admit no non-trivial infinitesimal  $B^t$ -flat deformations for certain ranges of  $t$  (other than scalings).

For the general gluing problem, even if the pieces are rigid, there can be nonzero infinitesimal kernel elements due to the presence of gluing parameters. In general, there are infinitesimal kernel elements corresponding geometrically to freedom of scaling the factors, rotating, and moving the base points of the gluing. Since these manifolds are toric, we can use the torus action plus a certain discrete symmetry, called a diagonal symmetry, to eliminate all gluing parameters except for the scaling parameter.

Our main result is:

**Main Theorem.** The following 4-manifolds admit a (toric-invariant)  $B^t$ -flat metric:

$$\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}, \mathbb{C}\mathbb{P}^2 \# 2\overline{\mathbb{C}\mathbb{P}^2}, 2\# S^2 \times S^2,$$

for some value of  $t < 0$ .

### Remarks and Questions

- (1) The value of  $t$  can be made arbitrarily close to specified values; see the complete statement in [6].
- (2) Other gluing configurations are possible, cf. [6].
- (3) In fact, the construction allows for a family of critical metrics which vary according to the choice of a gluing parameter. This implies the following interesting dichotomy: either (i) there is a critical metric at exactly one value of  $t = t_0$ , in which case there would necessarily be a 1-dimensional moduli space of solutions for this fixed  $t_0$  (this indeed happens for  $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ , in which case there is a 1-parameter family of self-dual metrics). Or, the other possibility (ii) is that for each value of the gluing parameter  $a$  sufficiently small, there will be a critical metric for a corresponding value of  $t_0 = t_0(a)$ . Trying to determine which holds is ongoing work.

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# Gromov Positive Scalar Curvature and Rationally Inessential Macroscopically Large Manifolds

MICHAL MARCINKOWSKI

## 1. OVERVIEW

Macroscopic dimension was defined by Gromov ([2]) in search of topological obstructions for manifolds to admit a Riemannian metric with positive scalar curvature (briefly PSC). He conjectured that such manifolds tend to have deficiency of macroscopic dimension.

There are different notions of small and large manifolds. E.g. manifolds with no deficiency of macroscopic dimension are called macroscopically large. On the other hand, vaguely speaking, rationally inessential manifolds are small in the homological sense. The purpose of this talk was to present examples of rationally inessential but macroscopically large manifolds ([1]). Such manifolds are counterexamples to Dranishnikov's rationality conjecture. Moreover, they do not admit a metric of positive scalar curvature, thus satisfy Gromov's positive scalar curvature conjecture. In the talk we outlined the construction which uses small covers of convex polyhedrons (or alternatively Davis complexes) and surgery. We discussed also the notion of right angled Coxeter groups, which are fundamental groups of the manifolds in question.

## 2. DETAILS

Let  $X$  be a metric space and let  $Y$  be a topological space. We say that a map  $f: X \rightarrow Y$  is uniformly cobounded if there exists real number  $C$  such that  $\text{diam}(f^{-1}(y)) < C$  for every  $y \in Y$ .

**Definition.** The macroscopic dimension of  $X$ , denoted  $\text{dim}_{mc}(X)$ , is the smallest number  $k$ , such that there exist a  $k$ -dimensional simplicial complex  $K$  and a continuous, uniformly cobounded map  $f: X \rightarrow K$ .

Let  $\widetilde{M}$  be the universal cover of  $M$ . Note that  $\text{dim}_{mc}(\widetilde{M})$  is never greater than topological dimension.

**Gromov Conjecture.** Let  $M$  be a closed  $n$ -dimensional manifold. If  $M$  admits a Riemannian metric of positive scalar curvature, then  $\text{dim}_{mc}(\widetilde{M}) \leq n - 2$ .

We always assume that a metric on  $\widetilde{M}$  is pulled back from some Riemannian metric on  $M$ . Macroscopic dimension of  $\widetilde{M}$  does not depend on metric chosen on  $M$ .

The  $n - 2$  in the conjecture comes from the following prototypical example: for any  $M^{n-2}$ , the manifold  $M' = M \times S^2$  admits a PSC metric. We have  $\text{dim}_{mc}(\widetilde{M}') = \text{dim}_{mc}(\widetilde{M} \times S^2) = \text{dim}_{mc}(\widetilde{M}) \leq n - 2$ . Thus inequality in the conjecture is sharp.

There is also a version of Gromov Conjecture, called **weak Gromov conjecture**, which asserts that if  $M$  admits PSC, then  $\dim_{mc}(\widetilde{M}) \leq n - 1$ .

The Gromov conjecture was proven for 3-dimensional manifolds ([3]) and for manifolds with some assumptions on their fundamental groups ([4, 5]). In the present state of the art, the Gromov conjecture (and even its weak version) is considered to be out of reach. It implies other longstanding conjectures, e.g., the Gromov-Lawson conjecture, which asserts that aspherical manifolds do not admit PSC metric.

Let us consider the following:

**Example.** Let  $M$  be a closed oriented manifold,  $\pi = \pi_1(M)$ , and let  $B\pi$  be a classifying space endowed with a structure of a CW-complex. Denote by  $f: M \rightarrow B\pi$  the map classifying the universal bundle. If  $f_*([M]) = 0 \in H_n(B\pi; \mathbf{Z})$ , then there is a homotopy of  $f$  to some map  $g: M \rightarrow B\pi^{[n-1]}$ . It follows that there exist an **equivariant homotopy** of a lift  $\tilde{f}: \widetilde{M} \rightarrow E\pi$  to  $\tilde{g}: \widetilde{M} \rightarrow E\pi^{[n-1]}$ . Then  $\tilde{g}$  is a cobounded map, thus  $M$  cannot be macroscopically large.

An  $n$ -dimensional manifold  $M$  is called **macroscopically large** if  $\dim_{mc}(\widetilde{M}) = n$ . One can ask if the property that a manifold  $M$  is large or not can be expressed in homological terms. To do that, let us introduce the following notions (using the notation from the example above). We call  $M$  **inessential** if  $f_*([M]) = 0 \in H_n(B\pi; \mathbf{Z})$  and **rationally inessential** if  $f_*([M]) = 0 \in H_n(B\pi; \mathbf{Q})$  (note that  $M$  is rationally inessential if and only if  $f_*([M]) \in H_n(B\pi; \mathbf{Z})$  is torsion). The example of rationally inessential (but essential) orientable manifold is  $M = \mathbb{R}P^3$ . Obviously  $\dim_{mc}(\widetilde{M}) = 0$ , thus being essential is not enough to be macroscopically large. Gromov expected, that if  $f_*([M])$  is rationally essential, then  $M$  is macroscopically large. A. Dranishnikov disproved this conjecture and found the right homology theory where one should place a fundamental class  $[M]$  to test if  $M$  is large just by checking if the class is non-trivial ([6]). Moreover, he showed that  $[M]$  is large if and only if there exist a **bounded homotopy** from  $\tilde{f}: \widetilde{M} \rightarrow E\pi$  to some map which ranges in  $E\pi^{[n-1]}$ . In [5] he conjectured the following:

**Rationality Conjecture.** If  $M$  is rationally inessential, then it is macroscopically small.

It would imply the weak Gromov conjecture for rationally inessential manifolds. In this paper we give counterexamples to this conjecture. In terms of homotopy theory, they are rationally inessential manifolds, such that  $\tilde{f}: \widetilde{M} \rightarrow E\pi$  can not be deformed by means of bounded homotopy to a map which ranges in  $E\pi^{[n-1]}$ . Our manifolds are spin and we also prove that they do not admit a PSC metric. Thus they satisfy the Gromov Conjecture.

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## Circle actions and positive scalar curvature

MICHAEL WIEMELER

In my talk I discussed the following question:

*Let  $G$  be a compact connected Lie group which acts effectively on a closed manifold  $M$ . Does there exist an invariant metric of positive scalar curvature on  $M$ ?*

It has been shown by Lawson and Yau [5] that  $M$  admits such a metric if  $G$  is non-abelian. Therefore in the following we restrict to the case where  $G$  is a torus or  $S^1$ . This case is more complicated. There exist manifolds which admit a non-trivial  $S^1$ -action but no metric of positive scalar curvature, e.g., there exist  $S^1$ -actions on certain exotic spheres not bounding spin manifolds [2], [4], [7]. Moreover, there are manifolds which admit a non-trivial  $S^1$ -action and a non-invariant metric of positive scalar curvature, but no invariant such metric [1].

Existence results for invariant metrics of positive scalar curvature on certain  $S^1$ -manifolds without fixed points have been obtained by Berard Bergery [1] and Hanke [3]. Therefore we restrict to  $S^1$ -actions with fixed points.

For the case that the  $S^1$ -action on a connected manifold  $M$  has a fixed point component of codimension two, we show in [8] that there is an invariant metric of positive scalar curvature on  $M$ .

Fixed point components of codimension two exist if, for example, one of the following two condition holds:

- $M$  has dimension four and the Euler-characteristic of  $M$  is negative.
- $M$  has dimension  $2n$  and there is an effective action of an  $n$ -dimensional torus  $T$  with fixed point on  $M$ . Then there is a circle subgroup of  $T$  which has a codimension two fixed point component. Moreover, in this case  $M$  admits an  $T$ -invariant metric of positive scalar curvature.

In [8] we also have existence results for the case that the  $S^1$ -action on a simply connected manifold  $M$ ,  $\dim M \geq 6$ , is semi-free and there is no fixed point component of codimension two. Here one has to distinguish between three cases:  $M$  is not spin;  $M$  is spin and the  $S^1$ -action is of odd type;  $M$  is spin and the  $S^1$ -action is of even type.

In the first two cases we show that the equivariant connected sum of two copies of  $M$  admits an invariant metric of positive scalar curvature. In the third case there is an obstruction against an invariant metric of positive scalar curvature on

$M$ . It is a generalised  $\hat{A}$ -genus of the orbit space  $M/S^1$  defined by Lott in [6]. We show that if it vanishes, then there is an invariant metric of positive scalar curvature on the equivariant connected sum of sufficiently many copies of  $M$ .

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**Invertible Dirac operators and handle attachments**

NADINE GROSSE

(joint work with Mattias Dahl)

Mathematicians got interested in Dirac operators and spin manifolds since they provide an obstruction to the existence of positive scalar curvature. More precisely, as soon as the scalar curvature of a closed Riemannian spin manifold  $(M^n, g)$  is everywhere positive, the Lichnerowicz formula implies that the Dirac operator  $D^g$  of  $M$  has no kernel. While in general the dimension of the kernel  $D^g$  depends on the manifold and the metric, one can obtain by the index theorem an – in general nontrivial – lower bound for  $\dim \ker D^g$  that no longer depends on the metric, cf. [4], [2, Sect. 3],

$$(1) \quad \dim \ker D^g \geq \begin{cases} |\hat{A}(M)| & \text{if } n \equiv 0 \pmod{4} \\ 1 & \text{if } n \equiv 1 \pmod{8} \text{ and } \alpha(M) \neq 0 \\ 2 & \text{if } n \equiv 2 \pmod{8} \text{ and } \alpha(M) \neq 0 \\ 0 & \text{else.} \end{cases}$$

Here, the  $\hat{A}$ -genus  $\hat{A}(M) \in \mathbb{Z}$  and the  $\alpha$ -genus  $\alpha(M) \in \mathbb{Z}_2$  are invariants of the spin bordism class of the differentiable spin manifold  $M$ .

Thus, one obtains a topological obstruction to the existence of elements in  $\ker D^g$ , so-called harmonic spinors, and, hence, to the existence of metrics of positive scalar curvature. In particular, the space of Riemannian metrics on  $M$  with positive scalar curvature,  $\text{Metr}^{\text{pos}}(M)$ , is contained in the space of Riemannian metrics on  $M$  with invertible Dirac operator,  $\text{Metr}^{\text{inv}}(M)$ .

While the topological obstruction obtained by the index theorem for  $D^g$  is not the only obstruction to positive scalar curvature, it actually is the only obstruction to invertibility of the Dirac operator:

**Theorem.** [1, Thm. 1.1] *Let  $M$  be a closed Riemannian spin manifold. Then, the space of Riemannian metrics for which Inequality (1) is saturated is dense in the  $C^\infty$ -topology and open in the  $C^1$ -topology in the space of all Riemannian metrics.*

The main technique in proving this result is surgery. In particular, the authors show that if a closed spin manifold  $N$  is obtained from a Riemannian spin manifold  $(M, g)$  by surgery of codimension  $\geq 2$ , then there is a metric  $h$  on  $N$  such that  $\dim \ker D^h \leq \dim \ker D^g$ .

We obtained a similar surgery result for manifolds with boundary and handle attachments.

**Theorem.** [3, Thm. 1.2] *Let  $(M, g)$  be a compact Riemannian spin manifold with boundary and invertible Dirac operator, see below for the meaning of invertible here. Let a compact spin manifold  $N$  with boundary be obtained from  $M$  by handle attachment of codimension  $\geq 2$ . Then,  $N$  admits a metric with invertible Dirac operator.*

In order to define the notion of invertible Dirac operators for manifolds with boundary, we restrict to metrics that have product structure near the boundary. This has several advantages. First, we have a good notion of invertible Dirac operator: For a Riemannian spin manifold  $(M, g)$  with boundary  $\partial M$  such that in a neighborhood  $U := \partial M \times (-\epsilon, 0]$  of  $\partial M$  the metric has product form  $g|_U = g|_{\partial M} + dt^2$  we can glue along the boundary the half-cylinder  $\partial M \times [0, \infty)$  with the metric  $g|_{\partial M} + dt^2$ . Thus, we obtain a manifold  $(M_\infty, g_\infty)$  with cylindrical end. We say that  $D^g$  is invertible if the Dirac operator  $D^{g_\infty}$  on  $M_\infty$  is invertible as an operator from  $L^2(M_\infty, g_\infty)$  to itself.

Second, if  $(M_i, g_i)$ ,  $i = 1, 2$ , are compact Riemannian spin manifold with boundary  $\partial M_1 = (\partial M_2)^-$  and  $g_1|_{\partial M_1} = g_2|_{\partial M_2}$  (Here  $(.)^-$  denotes the manifold with opposite orientation.) and if, moreover, both  $D^{g_i}$  are invertible, then the manifold  $(M_L, g_L) := (M_1, g_1) \cup_{\partial M_1} (\partial M_1 \times [0, L], g|_{\partial M} + dt^2) \cup_{\partial M_2} (M_2, g_2)$  has invertible Dirac operator for large enough  $L$ .

The gluing property allows to use the above surgery result in order to make statements on the space of invertible Dirac operators on closed manifolds by using handle decompositions. One application is for example that the space of metrics with invertible Dirac operators on the 3-sphere has infinitely many path components. This contrasts with the connectedness of the space of metrics on the 3-sphere with positive scalar curvature that was recently proven by Marques [5].

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**Dirac Eigenvalues of Higher Multiplicity**

NIKOLAI NOWACZYK

Let  $(M, \Theta)$  be a closed spin manifold of dimension  $m \geq 3$  with fixed topological spin structure  $\Theta$ . For any Riemannian metric  $g$ , one can construct the associated Dirac operator  $D^g$ . The spectrum of this Dirac operator depends on  $g$ , of course. In 2005, Dahl conjectured that  $M$  can be given a metric for which a finite part of the spectrum consists of arbitrarily prescribed eigenvalues of arbitrary (finite) multiplicity, see [1]. The only constraints one has to respect are the exception of the zero eigenvalue (due to the Atiyah-Singer index theorem) and in certain dimensions the quaternionic structure of the eigenspaces and also the symmetry of the spectrum. Dahl also proved his conjecture in case all eigenvalues have simple multiplicities. The question if one can prescribe eigenvalues of arbitrary multiplicity, or if the existence of eigenvalues of higher multiplicity might somehow be topologically obstructed, has been open ever since.

We will show the following result: *Let  $(M, \Theta)$  be a closed spin manifold of dimension  $m \equiv 0, 6, 7 \pmod{8}$ . There exists a Riemannian metric  $g$  on  $M$  such that the Dirac operator  $D^g$  has at least one eigenvalue of multiplicity at least two. In addition,  $g$  can be chosen such that it agrees with an arbitrary metric  $\tilde{g}$  outside an arbitrary small neighborhood on  $M$ .*

The proof introduces a technique which “catches” the desired metric with a loop in the space of all Riemannian metrics on  $M$ . We will first construct such a loop on  $S^m$  explicitly and then show that it is stable under certain surgeries, in particular the connected sum, by extending the results from [2]. This technique also requires a global enumeration of the Dirac spectrum by continuous functions on the Riemannian metrics as in [3].

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## Selected Mathematical Problems in General Relativity

PIOTR T. CHRUSCIEL

General relativity aims to explain how bodies move, and how they affect the motion of other bodies. According to Einstein, this is best described using a Lorentzian metric  $\mathbf{g}$ . Einstein postulates that the motion of test bodies takes place along timelike geodesics of  $\mathbf{g}$ , and that matter deforms the metric according to the equation

$$(1) \quad R_{\mu\nu} - \frac{1}{2}R \mathbf{g}_{\mu\nu} = 8\pi T_{\mu\nu} ,$$

where  $R_{\mu\nu}$  is the Ricci tensor of  $\mathbf{g}$ ,  $R$  is its curvature scalar,  $\Lambda$  is a constant called the *cosmological constant*, and  $T_{\mu\nu}$  is the energy-momentum tensor of matter fields. The vacuum equations are obtained when  $T_{\mu\nu}$  vanishes.

The theory has been very successful in verifying or predicting many astrophysical and cosmological phenomena. One of the most spectacular examples are the recent studies of Sagittarius  $A^*$ , an object of about four million solar masses located at the center of our galaxy, whose properties are best explained by assuming that the associated space-time metric is that of a black hole [20, 21].

The aim of *mathematical general relativity* (MGR) is to provide a firm mathematical basis for the associated physical theory, see [9] for a primer. Its foundations have been laid by Yvonne Choquet-Bruhat in her seminal 1952 paper [19], where she showed that the equations split into a set of hyperbolic evolution equations and a set of underdetermined elliptic constraint equations. A key later result is the proof, in 1962, of existence of *maximal globally hyperbolic developments* of vacuum Cauchy data by Choquet-Bruhat and Geroch [3]. A version of this result with metrics of low regularity can be found in [5].

The ultimate goal of MGR is a complete understanding of the dynamics of solutions of Einstein equations. A challenging problem is the Belinski-Lifschitz-Khalatnikov conjecture, which asserts that the dynamics of the gravitational field near singularities possess universal *chaotic* features, see [9, 16, 23] and references therein. Another is to unravel the fate of five-dimensional “black strings” subjected to the Gregory-Laflamme instabilities, as analyzed numerically by Lehner and Pretorius [25]. Yet another are the instabilities of anti-de Sitter space-time whose *generic* perturbations, no matter how small, seem to lead to black-hole formation [2], while *fine-tuned* perturbations lead, numerically, to time-periodic solutions [27]. This is in sharp contrast with Minkowski space-time, shown to be nonlinearly stable in the celebrated work of Christodolou and Klainerman [4]; a simplified proof has been subsequently given by Lindblad and Rodnianski [26].

One of the tools of MGR is the *causality theory*, whose early achievements include the *incompleteness theorems* of Hawking and Penrose [22]. (A Riemannian version of the *trapped surface incompleteness theorem* has been recently provided by Eichmair, Galloway and Pollack [17].) A new approach to causality theory has been initiated by Fathi and Siconolfi [18], in their study of manifolds equipped with a distributions of convex cones. The recent results of Minguzzi [28] lead to dramatic simplifications of the proof of the *Hawking area theorem* (compare [8]),

as well as the proof of smoothness of compact Cauchy horizon in space-times satisfying the dominant energy condition. This last result has been independently established by Larsson [24], using completely different methods.

The initial data for, say, the vacuum Einstein equations consist of a Riemannian metric  $g$  on the initial data manifold  $M$  and a symmetric two-covariant tensor field  $K$  on  $M$  representing the second-fundamental form of  $M$  when embedded in the associated space-time. The fields  $(g, K)$  have to satisfy the *general relativistic constraint equations*, discussed in detail by other speakers in this workshop. These constraints are a consequence of the Gauss-Codazzi-Mainardi embedding equations. Interesting classes of initial data sets include *constant mean curvature* (CMC) data, where  $\text{tr}K$  is a constant, or *maximal data*, where  $\text{tr}K$  vanishes. Impressive progress in the understanding of the constraint equations will be presented in Carlotto's and Gicquaud's lectures in this workshop.

The CMC condition is a non-linear elliptic equation on  $M$ , structurally reminiscent of the minimal surface equation [1]. The case of zero mean-curvature is especially relevant for this meeting, as then the *vacuum scalar constraint equation*,

$$R = |K|^2 + 2\Lambda - (\text{tr}K)^2 ,$$

implies that the initial-data metric  $g$  has non-negative scalar curvature. The existence of maximal hypersurfaces  $\text{tr}K = 0$  plays a key role in the theory of uniqueness of stationary black holes [6], and has only been settled for black-holes with bifurcate Killing horizons [15]. The case of degenerate Killing horizons remains open, and it would be of interest to fill this gap, see [7, 14].

The experimental verification of positivity of the cosmological constant  $\Lambda$  [29] led to new studies of the corresponding initial data: large families of initial data with  $\Lambda > 0$  and with asymptotic ends of cylindrical type have been constructed in [10, 11]. Gluing constructions with constant scalar curvature, where the final metric has constant scalar curvature and coincides with the original ones away from a small neighborhood of the gluing region have been carried-out in [12, 13].

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## Higher rho invariants and their geometric applications

PAOLO PIAZZA

(joint work with Thomas Schick)

In this talk, a sort of continuation of the survey-lecture given by Thomas Schick, I have reported on the content of the recently published paper [4] as well as on some more recent literature related to it. As the title suggests, the talk was centered around the definition of higher rho-invariants in K-theory, a notion due originally to Nigel Higson and John Roe, their main properties and some specific applications to the world of positive scalar curvature.

Recall first of all the main ideas leading to Dirac index classes on spin manifolds. Let  $(M, g)$  be a spin manifold with fundamental group  $\Gamma$ . We assume initially that  $M$  is even dimensional. For simplicity we assume that  $B\Gamma$  is a finite CW-complex. We denote by  $\mathcal{S}$  the associated spinor bundle and by  $\mathcal{D}_g$  the Dirac operator. In order to produce an index class one starts by producing a short exact sequence of  $C^*$ -algebras  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$  (needless to say, this sequence must be related to the given geometry) and then consider the associated six-term long exact sequence in K-theory:

$$\cdots \rightarrow K_1(A) \rightarrow K_1(A/I) \xrightarrow{\delta} K_0(I) \rightarrow \cdots .$$

In several examples one then produces the following data:

- a fundamental class  $[\mathcal{D}_g] \in K_1(A/I)$ ;
- an index class  $\text{Ind}(\mathcal{D}_g) := \delta[\mathcal{D}_g] \in K_0(I)$ ;
- a (specific) lift of  $[\mathcal{D}_g] \in K_1(A/I)$  to  $\rho(\mathcal{D}_g) \in K_1(A)$  if  $g$  has positive scalar curvature ( $\equiv$  PSC)

Notice that the last point implies, in particular, that the index class  $\text{Ind}(\mathcal{D}_g) \in K_0(I)$  vanishes if  $g$  has positive scalar curvature. Since, on the other hand, in all known examples,  $\text{Ind}(\mathcal{D}_g) \in K_0(I)$  can be proved to have a topological meaning, independent of  $g$ , it is clear that this method produces interesting obstructions to the existence of PSC metrics on  $M$ .

There are three classical examples of index classes that fit into this scheme when  $M$  is compact: (i) the classical numeric index of  $\mathcal{D}_g$ ; (ii) the  $\alpha$ -invariant of Hitchin; (iii) the Mishchenko-Fomenko index of the operator  $\mathcal{D}_g$  twisted by the Mishchenko bundle. All these three examples, however, don't produce interesting rho-invariants; indeed,  $K_1(A)$  vanishes in these three examples.

In order to get interesting rho-invariants we must enter into the world of coarse geometry and coarse index theory, a discipline initiated by Roe, see for example [5], and greatly developed by Higson and Roe.

Consider, quite generally, a complete riemannian manifold  $(X, g)$  and a generalized Dirac operator  $D$ . Let  $H$  be a Hilbert space, with a representation of  $C_0(X)$  in  $\mathcal{B}(H)$ . We consider  $D^*(X, H)$ , the closure in  $\mathcal{B}(H)$  of  $D_c^*(X, H)$ , the bounded operators that are of finite propagation and pseudolocal; we also consider  $C^*(X, H)$ , the closure in  $\mathcal{B}(H)$  of  $C_c^*(X, H)$ , the operators in  $D_c^*(X, H)$  that are, in addition, locally compact.  $C^*(X, H)$  is an ideal in  $D^*(X, H)$  and so we have a short exact

sequence of  $C^*$ -algebras  $0 \rightarrow C^*(X, H) \rightarrow D^*(X, H) \rightarrow D^*(X, H)/C^*(X, H) \rightarrow 0$ . The associated long exact sequence in  $K$ -theory is called the Higson-Roe surgery sequence. The  $K$ -theory groups of these  $C^*$ -algebras turn out to be independent of  $H$ . Moreover, they enjoy natural functoriality properties; in particular, given a coarse continuous map  $f : X \rightarrow Y$  we have homomorphisms  $f_* : K_*(C^*(X)) \rightarrow K_*(C^*(Y))$  and  $f_* : K_*(D^*(X)) \rightarrow K_*(D^*(Y))$ . If a discrete group  $\Gamma$  acts freely and isometrically on  $X$  then we also have  $D^*(X)^\Gamma$  and  $C^*(X)^\Gamma$  (obtained by closing-up the  $\Gamma$ -invariant elements in  $D_c^*(X)$  and  $C_c^*(X)$ ). It is also important to recall that if  $X/\Gamma$  is compact, then  $K_{*+1}(D^*(X)^\Gamma/C^*(X)^\Gamma) \simeq K_*(X/\Gamma)$  (Paschke duality); moreover  $K_*(C^*(X)^\Gamma) \simeq K_*(C_r^*\Gamma)$ , with  $C_r^*\Gamma$  the reduced group  $C^*$ -algebra.

Now, assuming for simplicity  $X/\Gamma$  to be compact and using the finite propagation property of the wave operator  $\exp(itD)$  one can produce the fundamental class  $[D] \in K_{\dim X+1}(D^*(X)^\Gamma/C^*(X)^\Gamma) = K_{\dim X}(X/\Gamma)$ ; the index class  $\delta[D] \in K_{\dim X}(C^*(X)^\Gamma) = K_{\dim X}(C_r^*\Gamma)$  and the rho class  $\rho(D) \in K_{\dim X+1}(D^*(X)^\Gamma)$  if  $D$  is  $L^2$ -invertible (for example if  $X$  is spin and has PSC). The definition of the fundamental class  $[D]$  is as follows: if  $X$  is even dimensional and the Dirac bundle  $E$  is equal to  $E^+ \oplus E^-$  then the fundamental class  $[D]$  is equal  $[U^*\chi(D)_+] \in K_1(D^*(X)^\Gamma/C^*(X)^\Gamma) = K_0(X/\Gamma)$ , with  $U$  a unitary operator  $L^2(X, E^+) \rightarrow L^2(X, E^-)$  and  $\chi$  an odd chopping function (a smooth odd function going to  $\pm 1$  as  $x \rightarrow \pm\infty$ ). If  $X$  is odd-dimensional then the fundamental class is equal to  $[\frac{1}{2}(1 + \chi(D))] \in K_0(D^*(X)^\Gamma/C^*(X)^\Gamma) = K_1(X/\Gamma)$ . If  $D$  is  $L^2$ -invertible and  $\chi$  equals 1 on the positive part of the spectrum, then we define the rho-invariants in even and odd dimension as

$$\rho(D) = [U^*\chi(D)_+] \in K_1(D^*(X)^\Gamma) \quad \text{and} \quad \rho(D) = [\frac{1}{2}(1 + \chi(D))] \in K_0(D^*(X)^\Gamma).$$

We can apply this definition to the universal cover  $\widetilde{M}$  of a compact spin manifold  $(M, g)$  with PSC and fundamental group  $\Gamma$ ; if  $u : M \rightarrow B\Gamma$  is the classifying map, covered by  $\tilde{u} : \widetilde{M} \rightarrow E\Gamma$ , we obtain

$$\rho(\not{D}_g) \in K_{\dim M+1}(D^*(\widetilde{M})^\Gamma) \quad \text{and} \quad \rho_\Gamma(\not{D}_g) := \tilde{u}_*(\rho(\not{D}_g)) \in K_{\dim M+1}(D_\Gamma^*)$$

with  $D_\Gamma^* := D^*(E\Gamma)^\Gamma$ . These are the invariants we wanted to define. Their connection with the classical Atiyah-Patodi-Singer rho-invariant is not obvious but it can be proved, see [2], that if  $M$  is odd dimensional and  $\alpha, \beta : \pi_1(M) \rightarrow U(\ell)$  are two unitary representations, then there exists a homomorphism  $\Theta_{\alpha, \beta} : K_0(D_\Gamma^*) \rightarrow \mathbb{R}$  such that  $\Theta_{\alpha, \beta}(\rho_\Gamma(\not{D}_g)) = \eta(\not{D}_{g, \alpha}) - \eta(\not{D}_{g, \beta})$  and the right hand side is, by definition, the classical rho invariant of Atiyah-Patodi-Singer. Recent results of Benamou and Roy extend this result to the Cheeger-Gromov rho-invariant.

The main properties of higher rho invariants with respect to problems related to PSC metrics is that they define a group homomorphism

$$(1) \quad \rho : \text{Pos}_n^{\text{spin}}(Z) \longrightarrow K_{n+1}(D^*(\widetilde{Z})^\Gamma)$$

with  $Z$  a compact topological space with fundamental group  $\Gamma$  and universal cover  $\widetilde{Z}$  (for example  $Z = M$  or  $Z = B\Gamma$ ). If  $[M, u : M \rightarrow Z, g_M] \in \text{Pos}_n^{\text{spin}}(Z)$  then

$\rho[M, u : M \rightarrow Z, g_M] := u_*(\rho(\mathcal{D}_{g_M}))$ . Similarly, one can consider the group  $P(M)$  of concordance classes of PSC metrics and define a group homomorphism

$$(2) \quad \rho : P(M) \rightarrow K_{\dim M+1}(D^*(\widetilde{M})^\Gamma).$$

See [6], [8].

Crucial to the proof of these results is the following theorem (the main technical contribution of [4]). Let  $(W, g_W)$  be an  $n$ -dimensional Riemannian spin-manifold with boundary  $\partial W$  such that  $g_{\partial W}$  has positive scalar curvature. Denote by  $\mathcal{D}_W$  and  $D_{\partial W}$  the Dirac operators on  $W$  and  $\partial W$ . Assume that  $\Gamma$  acts freely isometrically and  $W/\Gamma$  is compact. Then one can prove that there exists a well-defined index class  $\text{Ind}(\mathcal{D}_W) \in K_n(C^*(W)^\Gamma)$ .

**Theorem** (Delocalized APS-index theorem) *The following formula holds:*

$$(3) \quad \iota_*(\text{Ind}(\mathcal{D}_W)) = j_*(\rho(\mathcal{D}_{\partial W})) \quad \text{in } K_n(D^*(W)^\Gamma).$$

Here, we use  $j : D^*(\partial W)^\Gamma \rightarrow D^*(W)^\Gamma$  induced by the natural inclusion  $\partial W \rightarrow W$  and  $\iota : C^*(W)^\Gamma \rightarrow D^*(W)^\Gamma$  the inclusion.

This theorem was proved in even dimension in [4]. Later a new proof was provided by Xie and Yu in [7]; this latter proof establishes the result in all dimensions.

Using the delocalized APS index theorem and employing the fundamental class of a closed spin manifold  $N$ ,  $[\mathcal{D}_g]$ , the index class of a manifold with boundary  $W$ ,  $\text{Ind}(\mathcal{D}_W)$ , and the rho-class one can not only prove (1) but in fact map the whole surgery sequence of Stolz to the surgery sequence of Higson and Roe. See [4].

Recent results of Xie and Yu [8] show that these higher rho invariants do play a fundamental role in the problem of distinguishing metrics of PSC on an odd dimensional spin manifold that does carry one such metric. This is a classic problem, first tackled with these tools by Botvinnik and Gilkey for finite fundamental groups, see [1], and later by Piazza and Schick in general, see [3]. The results of Xie and Yu encompass all these previous results: indeed it is proved in [8] that the rank of the group of coinvariants of  $P(M)$  with respect to the action of the diffeomorphism group is at least 1. Under additional hypothesis on the group much sharper estimates on this rank, depending on the torsion elements of the fundamental group, are provided.

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## Recent results on the constraint equations in general relativity

ROMAIN GICQUAUD

General relativity describe the universe and its time evolution (the spacetime) as a manifold  $\mathcal{M}$  of dimension  $n + 1$  endowed with a Lorentzian metric  $\mathcal{G}$  i.e. a non-degenerate quadratic form with signature  $(n, 1)$  describing the gravitational field. The Einstein equation tell how non-gravitational fields (matter fields such as Dirac fields, electromagnetic fields...) imprint the curvature of  $\mathcal{G}$ :

$$\text{Ric}^{\mathcal{G}} - \frac{\text{Scal}^{\mathcal{G}}}{2}\mathcal{G} = T,$$

where  $T$  is the so-called stress-energy tensor which depends on all non-gravitational fields one wants to consider.

In the sequel, to simplify the exposition, we will consider no non-gravitational field, i.e. the vacuum case, and hence set  $T \equiv 0$ .

One of the major achievements in general relativity was the understanding of the well-posedness of the Cauchy problem by Yvonne Choquet-Bruhat in [5] and subsequently with Robert Geroch in [3].

Initial data for the Cauchy problem are usually given as a triple consisting of

- a manifold  $M$ ,
- a metric  $\widehat{g}$  on  $M$
- and a symmetric 2-tensor  $\widehat{K}$  on  $M$ .

The Cauchy problem consists then in finding a spacetime  $(\mathcal{M}^{n+1}, \mathcal{G})$  solving the (vacuum) Einstein equations

$$\text{Ric}^{\mathcal{G}} - \frac{\text{Scal}^{\mathcal{G}}}{2}\mathcal{G} = 0,$$

such that there exists an embedding  $M \hookrightarrow \mathcal{M}$  such that  $M$  is a 2-sided hypersurface in  $\mathcal{M}$  whose induced metric  $\mathcal{G}|_{TM} = \widehat{g}$  and whose second fundamental form is  $\widehat{K}$ .

It follows from the Gauss and Codazzi equations that part of the Einstein equations does not describe the evolution of the initial data but are instead constraints on the choice of  $\widehat{g}$  and  $\widehat{K}$ :

$$(1) \quad \begin{aligned} 0 &= \text{Scal}^{\widehat{g}} + \left(\text{tr}_{\widehat{g}}\widehat{K}\right)^2 - \left|\widehat{K}\right|_{\widehat{g}}^2, \\ 0 &= \text{div}^{\widehat{g}}\widehat{K} - d\text{tr}_{\widehat{g}}\widehat{K}. \end{aligned}$$

We refer the reader to the very nice survey article of Robert Bartnik and James Isenberg [1] and the beautiful book of Yvonne Choquet-Bruhat [2] for comprehensive introductions to the Cauchy problem in general relativity.

Constructing and classifying solutions to the system (1) then appears as a fundamental problem in general relativity. As such, this system is underdetermined and a natural idea to attack it is to split the variables  $(M, \widehat{g}, \widehat{K})$  into given (seed) data and unknowns that have to be adjusted so to satisfy (1).

One of the main such splitting is the conformal method described in great details in [9]. The idea is to make the following splitting:

Given data	Unknowns
<ul style="list-style-type: none"> <li>• A manifold <math>M</math></li> <li>• A metric <math>g</math></li> <li>• A function <math>\tau : M \rightarrow \mathbb{R}</math></li> <li>• A symmetric 2-tensor <math>\sigma</math> such that <math>\text{tr}_g \sigma \equiv 0</math> and <math>\text{div}^g \sigma = 0</math>,</li> </ul>	<ul style="list-style-type: none"> <li>• a conformal factor <math>\phi : M \rightarrow \mathbb{R}_+^*</math></li> <li>• a 1-form <math>W</math></li> </ul>

and cook out of it  $\widehat{g}$  and  $\widehat{K}$  as follows:

$$(2) \quad \widehat{g} = \phi^{N-2} g, \widehat{K} = \frac{\tau}{n} \widehat{g} + \phi^{-2} (LW + \sigma),$$

where  $N = \frac{2n}{n-2}$  and  $L$  is the conformal Killing operator (traceless part of  $L_{W\sharp}g$ ):

$$LW_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \nabla^k W_k g_{ij}.$$

Note that  $\tau = \text{tr}_{\widehat{g}} \widehat{K}$  corresponds to the mean curvature of the embedding  $M \hookrightarrow \mathcal{M}$ . The constraint equations (1) translate into the following system for  $\phi$  and  $W$ :

$$(3) \quad \begin{aligned} -\frac{4(n-1)}{n-2} \Delta \phi + \text{Scal}^g \phi &= -\frac{n-1}{n} \tau^2 \phi^{N-1} + \frac{|\sigma + LW|_g^2}{\phi^{N+1}} \\ -\frac{1}{2} L^* LW &= \frac{n-1}{n} \phi^N d\tau. \end{aligned}$$

For simplicity, we will assume from now on that the manifold  $M$  is compact.

The first equation, the Lichnerowicz equation, is an extension of the prescribed (non-positive) scalar curvature and is now well understood from the work of James Isenberg [8]: unless in some degenerate situations, given  $(M, g)$ ,  $\tau$ ,  $\sigma$  and  $W$ , the Lichnerowicz equation admits a unique solution.

The second equation, usually called the vector equation, is linear in  $W$  and, unless the manifold  $(M, g)$  has non-trivial conformal Killing vector fields, always admits a unique solution.

These results suffice to understand the case of constant mean curvature  $\tau$  and the reader can convince himself that perturbation arguments can be used to understand the case when  $\tau$  is close to being a constant. The situation when  $\tau$  is arbitrary appears much harder. Two important progresses were obtained by Michael Holst, Gabriel Nagy, Gantumur Tsogtgerel in [7] (subsequently improved

by David Maxwell [10]) and by Mattias Dahl, Emmanuel Humbert and the speaker in [4] (see also [11]).

The first one, [7], gives existence of a solution (very close to zero) to (3) if  $\sigma$  is non-zero but very small (for the  $L^2$ -norm in [11]). While the second method, [4], studies conditions under which one has a priori estimates for (3): If the limit equation

$$-\frac{1}{2}L^*LV = \lambda\sqrt{\frac{n-1}{n}}|LV|\frac{d\tau}{\tau}$$

has no non-zero solutions  $V$  for all  $\lambda \in [0, 1]$ , the set of solutions  $(\phi, W)$  to (3) is non-empty and compact.

While seemingly very different, the two methods are in fact two facets of a single idea described in [6].

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### On a Variational Characterization of the Exact Solutions for the Einstein-Maxwell Equation

SUMIO YAMADA

(joint work with Marcus Khuri, Gilbert Weinstein)

An Einstein metric  $g$  is an exact solution for the Einstein-Maxwell equation, which is

$$(1) \quad R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}$$

where  $R_{ab}$  is the Ricci curvature of the Lorentzian metric  $\mathbf{g}$ ,  $R$  is the scalar curvature of  $\mathbf{g}$  and  $T_{ab}$  is the stress-energy tensor. When  $T$  is identically zero,  $R_{ab} - \frac{1}{2}R\mathbf{g}_{ab} = 0$  is called the vacuum Einstein equation, and this is equivalent to  $R_{ab} = 0$ . Namely the vacuum universe is nothing but a Ricci flat Lorentz manifold. When the curvature tensor itself vanishes, then the rigidity theorem tells us that the Lorentz manifold is locally isometric to the Monkowski space-time  $\mathbb{R}^{3,1}$ . Hence the vacuum Einstein equation (VEE) can be regarded as the second simplest curvature condition, the simplest being  $R_{ijkl} = 0$ .

The simplest nontrivial solution to the VEE is the Schwarzschild solution, whose exterior region is given by

$$\mathbf{g} = -v^2 dt^2 + u^4 \delta, \quad N^4 = \mathbb{R} \times (\mathbb{R}^3 \setminus B)$$

where  $B = B_{m/2}(0)$  and

$$v = \frac{1 - m/2r}{1 + m/2r}, \quad u = 1 + \frac{m}{2r}$$

The Riemannian metric  $u^4 \delta$  is scalar flat.

Next we look at the Einstein-Maxwell equation

$$T_{ab} = - \left( F_{ac} F_b^c + \frac{1}{4} F_{cd} F^{cd} \mathbf{g}_{ab} \right)$$

where the electro-magnetic field tensor  $F$  behaves as a two form satisfying the Maxwell equation  $\text{div } F = \mathcal{J}, dF = 0$ . The Einstein-Maxwell equation is the Euler-Lagrange equation of the Hilbert functional [13]  $\mathcal{H}(\mathbf{g}) = \int_N R_{\mathbf{g}} + L d\mu_{\mathbf{g}}$ , where  $L$  is the Lagrangian for the electric and magnetic fields given by  $F^{ab} F_{ab}$ .

The simplest solution to the Einstein-Maxwell equation is the Reissner-Nordstrom (RN) metric, given by

$$\mathbf{g} = -v^2 dt^2 + u^4 \delta, \quad N^4 = \mathbb{R} \times (\mathbb{R}^3 \setminus B)$$

where  $B = B_{\sqrt{m^2 - q^2}/2}(0)$  and

$$v = \frac{1 - (m^2 - q^2)/4r^2}{1 + m/r + (m^2 - q^2)/4r^2}, \quad u = \sqrt{1 + \frac{m}{r} + \frac{m^2 - q^2}{4r^2}}$$

with electric and magnetic fields

$$E = u^{-6} \nabla \left( \frac{q}{r} \right), \quad B \equiv 0, \quad R(u^4 \delta) = 2|E|^2$$

The second static solution to the Einstein-Maxwell equation is the Mujumdar-Papapetrou(MP) (for a reference, see [1]) defined on the space-time  $[\mathbb{R}^3 \setminus \cup P_i] \times \mathbb{R}$ , where the punctures  $P_i$  represent at the location of the blackhole. It is a static solution and

$$\mathbf{g} = -u^2 dt^2 + u^{-2} \delta \quad N^4 = \mathbb{R} \times (\mathbb{R}^3 \setminus \cup_{i=1}^N \{p_i\})$$

where

$$u = \left( 1 + \sum_{i=1}^N m_i/r_i \right)^{-1}, \quad E = \nabla \log u, \quad B \equiv 0.$$

$m_i > 0$  is both the mass and charge of each black hole, and  $r_i$  is the Euclidean distance to the point  $p_i$ . We also have  $R(u^{-2}\delta) = 2|E|^2$ .

Recall that the Positive Mass Theorem [11, 12, 15] says that  $m \geq 0$  with the equality case realized only by the flat space  $\mathbb{R}^3$ . The Riemannian Penrose inequality [10, 3, 6] in turn gives a larger bound for the ASDM mass, which is of geometric origin:  $m \geq \frac{1}{2}r$  where  $r$  is the area radius of the blackhole defined as  $4\pi r^2 = A$  where  $A$  is the area of the outermost horizon. The equality for the Riemannian Penrose inequality holds if and only if the asymptotically flat three manifold  $(M, g)$  is isometric to the Schwarzschild space-like slice of the same mass and the area radius. Hence it is natural to hope [7, 4] that another exact solution, namely the Reissner-Nordstrom(RN) metric provides a lower bound for the initial data set for the Einstein-Maxwell equation:  $m \geq \frac{1}{2}\left(r + \frac{q^2}{r}\right)$  where  $q$  is the total charge hidden inside the black-hole horizon.

It turns out that the inequality can be violated [14] by the two-neck ( $N = 2$ ) MP metric, or rather a small perturbation of the two-neck RN metric, which is asymptotically flat. In a collaborative work [8] with Marcus Khuri and Gilbert Weinstein, we showed that

**Theorem** *Let  $(M, g, E)$  be an asymptotically flat, time-symmetric initial data set satisfying the dominant energy condition  $R \geq 2|E|^2$ , and having mass  $m$ , area radius  $r$ , and charge  $q$ . Then  $r \leq m + \sqrt{m^2 - q^2}$  with equality if and only if the data is RN.*

Note that the inequality is equivalent with the pair of the following inequalities

$$\begin{aligned} m &\geq \frac{1}{2}\left(r + \frac{q^2}{r}\right) && \text{if } r \geq |q| \\ m &\geq |q| && \text{if } r < |q| \end{aligned}$$

The second inequality is covered by the theorem [5] of Gibbons, Hawking, Horowitz, and Perry, which states  $m \geq \sqrt{q_e^2 + q_b^2}$ , where the equality holds iff there exists an isometric embedding  $(M, g, k, E, B) \hookrightarrow MP^4$ . The first inequality was established using the conformal deformation generalising the method first introduced by H. Bray [3].

An interpretation of our result is that the positive scalar curvature of the RN metric accounts for the bigger lower bound of the ADM mass when  $r \geq |q|$ , compared to the case for the scalar-flat Schwarzschild metric. Also, when  $r < |q|$  which causes the dominance of the repulsive forces between the charges over the attractive gravitational forces between the multiple components of the horizons, the MP metric provides an interesting situation where the Penrose-type inequality requires a modification, where the equality case  $m = |q|$  for the inequality is realised by the MP metric.

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## The topology of positive scalar curvature

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(joint work with Paolo Piazza)

Given a smooth closed manifold  $M$  we study the space of Riemannian metrics of positive scalar curvature on  $M$ . A long-standing question is: when is this space non-empty (i.e. when does  $M$  admit a metric of positive scalar curvature)? More generally: what is the topology of this space? For example, what are its homotopy groups?

Higher index theory of the Dirac operator is the basic tool to address these questions. This has seen tremendous development in recent years, and in the talk we present the underlying philosophy for this method.

The underlying observation to for this goes back to Erwin Schrödinger [4], rediscovered by André Lichnerowicz [1]: If  $M$  has positive scalar curvature and a spin structure then the Dirac operator on  $M$  is invertible. This forces its index (which is the super-dimension of the null space) to vanish.

On the other hand non-vanishing of the index follows from index theorems, giving rise to powerful *obstructions* to positive scalar curvature. For example, the Atiyah-Singer index theorem says that  $\text{ind}(D) = \hat{A}(M)$ , where the  $\hat{A}$ -genus is a fundamental differential topological invariant (not depending on the metric!).

In particular, we will show how advancements of *large scale index theory* (also called *coarse index theory*) give rise to new types of obstructions, and provide the

tools for a systematic study of the existence and classification problem via the K-theory of  $C^*$ -algebras. This is part of a program “mapping the topology of positive scalar curvature to analysis”, compare [2, 5].

The general pattern of the index method is the following:

- (1) The geometry of the manifold  $M$  produces the interesting Dirac operator  $D$ .
- (2) This operators defines an element in an operator algebra  $A$ , which depends on the precise context.
- (3) The operator satisfies a Fredholm condition, which means it is invertible module an ideal  $I$  of the algebra  $A$ , again depending on the context.
- (4) Finally, the above element which is invertible in  $A$  modulo an ideal  $I$  defines an element in  $K_{n+1}(A/I)$ , where  $n = \dim(M)$  (the appearance of  $n$  comes from additional Clifford symmetries).
- (5) We interpret the class defined by the Dirac operator as a fundamental class  $[M] \in K_{n+1}(A/I)$ . Homotopy invariance of  $K$ -theory implies that  $[M]$  does not depend on the full geometric data which goes in the construction of the operator  $D$ , but only on the topology of  $M$ .
- (6) The K-theory exact sequence of the extension  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$  contains the boundary map  $\delta$ . We call the image of  $[M]$  under  $\delta$  the index

$$\delta: K_{n+1}(A/I) \rightarrow K_n(I); [M] \mapsto \text{ind}(D).$$

- (7) Under positive scalar curvature, the operator  $D$  is positive and therefore invertible already in  $A$ , gives rise to a canonical lift of  $[M]$  to an element  $\rho(M, g) \in K_{n+1}(A)$ . Because of this, we think of  $K_{n+1}(A)$  as a *structure group* and  $\rho(M, g)$  is a *structure class*. It contains information about the underlying geometry.

Indeed, we want to advocate in the talk that the setup just described has quite a number of different manifestations, depending on the situation at hand. In Paolo Piazza’s talk some of these are discussed, more should be the goal of future research. More details of this are also given in the survey talk [3].

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## Exotic Fiber Bundles and Families of p.s.c. Metrics

WOLFGANG STEIMLE

(joint work with Bernhard Hanke, Thomas Schick)

It has been observed a long time ago [6] that non-trivial families of p.s.c. metrics on a closed manifold may be constructed using the action of the diffeomorphism group: For a closed smooth manifold  $M$ , if  $(\varphi_t)_{t \in S^n}$  represents an element in  $\pi_n(\text{Diff}(M), \text{id})$ , and  $g$  is a p.s.c. metric on  $M$ , then  $(\varphi_t^*g)_{t \in S^n}$  represents an element in the homotopy  $\pi_n(\mathcal{R}^+(M), g)$  of the space of p.s.c. metrics on  $M$  which may be non-trivial. Recently Crowley–Schick [3] have used specific constructions of exotic spheres to deduce the existence of non-trivial elements in certain higher homotopy groups of  $\text{Diff}(S^n)$ , which lead to non-trivial 2-torsion elements in the corresponding higher homotopy groups of  $\mathcal{R}^+(S^n)$ , detected by the  $\alpha$ -invariant.

In our work [5] we use the non-torsion part of the homotopy of  $KO$  to detect elements of infinite order in certain higher homotopy groups of  $\mathcal{R}^+(S^n)$ . These elements do however not come from the  $\text{Diff}(S^n)$ -action. Roughly speaking, our construction is as follows: Firstly, using techniques from surgery and pseudoisotopy theory, we construct non-trivial elements in  $\pi_*(\text{Diff}(Z), \text{id})$  for a high-dimensional spin manifold  $Z$  which is spin nullbordant. Any such element gives rise to a smooth  $Z$ -bundle over a sphere, and more precisely, we construct elements such that the total space of the corresponding bundle has non-trivial  $\widehat{A}$ -genus. Secondly, we use the  $\text{Diff}(Z)$ -action on a fixed  $g \in \mathcal{R}^+(Z)$  (such an element exists by the Gromov-Lawson construction [4]) to obtain an element in  $\pi_*(\mathcal{R}^+(Z), g)$ . By means of index theory, this element can be shown to be automatically non-trivial and of infinite order. Finally we apply the fiber-wise version of the Gromov-Lawson construction recently developed by Mark Walsh [7], which yields that  $\mathcal{R}^+(Z)$  is weakly homotopy equivalent to  $\mathcal{R}^+(S^n)$  in this situation.

Using a connected-sum construction, each of these elements gives rise to an infinite-order element in  $\pi_*(\mathcal{R}^+(M), g)$  for each closed spin manifold  $M$  of the same dimension and each  $g \in \mathcal{R}^+(M)$ .

*Remarks.* (1) The homotopy classes constructed this way remain non-zero under the Hurewicz map to  $H_*(\mathcal{R}^+(M); \mathbb{Q})$ .

(2) It turns out that for many spin manifolds  $M$ , the elements in  $\pi_*\mathcal{R}^+(M)$  constructed in this way cannot be obtained by letting  $\text{Diff}(M)$  act directly on  $\mathcal{R}^+(M)$  – in other words they survive in the homotopy of the Borel construction  $\mathcal{R}^+(M)//\text{Diff}(M)$ . For  $x \in M$  we let  $\text{Diff}_x(M) \subset \text{Diff}(M)$  be the subgroup of all diffeomorphisms which fix  $x$  and the tangent space at  $x$ . If  $M$  is connected, the action of  $\text{Diff}_x(M)$  on  $\mathcal{R}^+(M)$  is easily seen to be free and hence our elements also survive in the actual quotient space  $\mathcal{R}^+(M)/\text{Diff}_x(M)$ .

(3) For  $M = S^n$  and  $x \in S^n$  it was shown in [2] that the action of  $\text{Diff}_x(S^n)$  on  $\mathcal{R}^+(S^n)$  can only give rise to torsion classes in  $\pi_k\mathcal{R}^+(S^n)$ , provided  $0 < k \ll n$ .

(4) Recently Botvinnik–Ebert–Randal-Williams [1] have announced a construction using cobordism categories which also leads to elements of infinite order in

the homotopy of  $\mathcal{R}^+(S^n)$ , but with an improved dimension bound. It is unclear how these elements relate to the ones constructed by our method.

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