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Mini-Workshop: Singularities in G_2-geometry

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Abstract. All currently known construction methods of smooth compact G_2-manifolds have been tied to certain singular G_2-spaces, which in Joyce's original construction are G_2-orbifolds and in Kovalev's twisted connected sum construction are complete G_2-manifolds with cylindrical ends. By a slight abuse of terminology we also refer to the latter as singular G_2-spaces, and in fact both construction methods may be viewed as desingularization procedures. In turn, singular G_2-spaces comprise a (conjecturally large) part of the boundary of the moduli space of smooth compact G_2-manifolds, and so their deformation theory is of considerable interest. Furthermore, singular G_2-spaces are also important in theoretical physics. Namely, in order to have realistic low-energy physics in M-theory, one needs compact singular G_2-spaces with both codimension 4 and 7 singularities according to Acharya and Witten. However, the existence of such singular G_2-spaces is unknown at present. The aim of this workshop was to bring reserachers from special holonomy geometry, geometric analysis and theoretical physics together to exchange ideas on these questions.

Mathematics Subject Classification (2010): 53C, 58C, 58J, 83E.

Introduction by the Organisers

This meeting was about singular G_2-spaces and the understanding of the boundary of the moduli space of G_2-metrics. Together with the organizers there were 16 participants, from Europe and the United States, forming a nice mixture of junior and more senior researchers. Their research expertises ranged from special holonomy geometry through geometric analysis and gauge theory to theoretical physics. The programme of the workshop alternated between overview talks, discussion sessions, informal communications and the more traditional research talks which are collected in this report. Furthermore, much time was dedicated to interaction between the participants.

Amongst the possible Riemannian holonomy groups appearing on Berger's list, the group G_2 is distinguished by the fact that it is carried by an odd-dimensional manifold. A G_2-metric on a 7-manifold is often specified through the choice of a G_2-structure which is torsion free, i.e., a special 3-form which satisfies a further nonlinear PDE. There are basically two known construction methods for compact G_2-manifolds: Joyce's generalized Kummer construction and Kovalev-Corti-Haskins-Nordström-Pacini's twisted connected sum construction.

In both constructions, the starting point is a singular or degenerate G_2-structure: a flat G_2-orbifold in the first and a pair of noncompact G_2-manifolds with cylindrical ends in the second. Conversely, both constructions provide a way of degenerating a family of non-singular G_2-manifolds into a singular one. In both cases these are noncollapsed limits, i.e. the limiting space is still 7-dimensional. In the first case, the diameter stays bounded and the family develops orbifold singularities in codimension 4 or higher. In the second, the manifold is stretched along a cylindrical neck so that the diameter of the family of metrics goes to infinity. Both constructions thus yield G_2-metrics close to the boundary of the moduli space of G_2-metrics.

The main focus of this workshop was on singular G_2-spaces, where by slight abuse of language we also want to include non-compact smooth ones. As pointed out above, they naturally appear at the boundary of the moduli space of compact G_2-manifolds, and so are intimately connected with existence and moduli of such.

Singular G_2-spaces and geometric analysis. Mark Haskins opened the meeting with an overview talk, explaining the two known construction methods of compact smooth G_2-manifolds alluded to earlier and setting the stage for the rest of the week. Jason Lotay introduced the class of compact G_2-spaces with isolated conic singularities and that of complete G_2-manifolds with asymptotically conic ends. He went on to explain his joint results with Spiro Karigiannis about the deformation theory of these spaces. Rafe Mazzeo gave an overview of techniques in geometric analysis which are likely to be useful in this context. This was exemplified by explaining his approach, together with Montcouquiol, to the deformation problem of hyperbolic conifolds in dimension three. In an informal evening talk, Jason Lotay explained a new approach to constructing compact smooth G_2-manifolds due to Joyce and Karigiannis, which eventually may also produce compact G_2-spaces with isolated conic singularities.

Nearly-Kähler geometry in dimension six. Compact G_2-spaces with isolated conic singularities are modelled on metric cones over nearly-Kähler 6-manifolds. Hence the geometry of nearly-Kähler 6-manifolds is of premier importance in this field. Uwe Semmelmann began by giving an overview over the basic features of nearly-Kähler 6-manifolds, highlighting the fact that until recently only four examples (all homogeneous) have been known. In a series of two talks Lorenzo Foscolo presented a new construction method for nearly-Kähler 6-manifolds which was obtained in joint work with Mark Haskins. Their method yields the first non-homogeneous examples, which are in fact of cohomogeneity one. In a second talk Uwe Semmelmann explained his joint work with Andrei Moroianu and Paul-Andi Nagy on the deformation space of a nearly-Kähler 6-manifold. It turns out that most of the homogeneous examples are rigid. Finally, Eleonora Di Nezza explained recent work of Martelli and Sparks describing explicit partial resolutions of 6-dimensional Calabi-Yau cones, the cross-section being a Sasaki-Einstein 5-manifold. This may have applications towards partial desingularisations of nearly-Kähler spaces with isolated conic singularities.

Singular G_2*-spaces and physics.* Apart from the interest in G_2-manifolds in mathematics, they are also of interest in theoretical physics; any G_2-metric is Ricci-flat and moreover such metrics constitute the simplest compactifying spaces in M-theory consistent with supersymmetry. As Bobby Acharya explained in his talk, one is interested in constructing G_2-manifolds that are solutions to M-theory which contains the Standard Model of Particle Physics. Since a smooth G_2-manifold cannot possibly provide such a solution – the Standard Model is a non-abelian gauge theory – one would need to construct singular G_2-spaces with singularities in codimension 4 and 7. He outlined how these singularities give rise to the required non-abelian gauge fields and chiral fermions that are necessary for this solution, and sketched a possible procedure for constructing them. In an informal evening talk, he explained the background of the Standard Model of Particle Physics to the mathematical audience.

Related topics. Sebastian Goette and Johannes Nordström gave a series of talks describing a new invariant of G_2-structures obtained in joint work with Diarmuid Crowley. While the initial definition of the ν-invariant due to Nordström and Crowley required a null bordism of the 7-manifold, an intrinsic definition can be given using η-invariants. An extension of this invariant is strong enough to distinguish components of the moduli space of G_2-holonomy metrics.

Gauge theory on G_2-manifolds is a subject initiated by Donaldson and aims at defining an invariant for G_2-manifolds by counting G_2-instantons. Goncalo Oliveira introduced the subject in an informal evening talk and went on to explain his results on monopoles on asymptotically conical G_2-manifolds the next day. Andriy Haydys talked about his joint work with Thomas Walpuski on compactness properties of the moduli space of G_2-instantons. This is related to the Seiberg-Witten equation with multiple spinors.

Frederik Witt explained joint work with Hartmut Weiß on the construction of a parabolic flow for G_2-structures, whose stationary points are the torsion-free ones which are dynamically stable under the flow. Jan Swoboda finished the meeting with a talk on joint work with Rafe Mazzeo, Hartmut Weiß and Frederik Witt on the asymptotic geometry of the hyperkähler metric on the moduli space of Higgs bundles.

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Mini-Workshop: Singularities in G_2-geometry

Table of Contents

Abstracts

Introduction to Nearly Kähler manifolds

Uwe Semmelmann

This talk gave a survey on nearly Kähler manifolds with a special emphasis on nearly Kähler manifolds in dimension 6. We gave several equivalent definitions, the most important properties and the standard examples.

1. Definition of Nearly Kähler manifolds. *Nearly Kähler* manifolds (short NK manifolds) are almost hermitian manifolds (M^{2n}, g, J) with the additional condition $(\nabla_X J)X = 0$ satisfied for all tangent vectors X. A nearly Kähler manifold is called *strict*, if $\nabla_X J \neq 0$ for all $p \in M$ and all $X \in T_pM$.

Pointwise the tensor ∇J of and almost hermitian manifold belongs to a sum of 4 irreducible $U(n)$-representations W_1, W_2, W_3, W_4. Accordingly one has the 16 classes of almost hermitian manifolds introduced by Gray and Hervella. The NK condition is equivalent to $\nabla J \in W_1$.

The *fundamental* 2*-form*, or *Kähler form*, is defined as $\omega(X, Y) = g(JX, Y)$. The NK condition is equivalent to $\nabla\omega$ being a 3-form, which can be written as $\nabla\omega = \frac{1}{3}d\omega$, ie. ω is a so-called *Killing 2 form.*

Almost hermitian manifolds admit a canonical connection $\bar\nabla$ satisfying $\bar\nabla J = 0 = \bar\nabla g$. Its intrinsic torsion is given by $\bar T(X, Y) = J(\nabla_X J)Y$. Hence NK manifolds are characterized by the fact that $\bar T$ is totally skew-symmetric. For NK manifolds it holds that $\bar\nabla \bar T = 0$, which is implicit in the work of A. Gray. Recall that by a theorem of Ambrose and Singer manifolds admitting a metric connection $\bar\nabla$ with $\bar\nabla\bar T = 0$ *and* $\bar\nabla R^{\bar\nabla} = 0$ have to be locally homogeneous.

2. Restriction to dimension 6. Nearly Kähler non-Kähler manifolds in dimension 6 have many special properties: they are strict NK, they are Einstein (the scalar curvature is usually normalized to scal = 30), they have $c_1(TM) = 0$ (thus in particular they are spin) and they are of constant type, ie.

$$\|(\nabla_X J)Y\|^2 \ = \ \alpha(\|X\|^2\|Y\|^2 - g(X, Y)^2 - g(JX, Y)^2)$$

for all tangent vectors X, Y and some real constant α, which is 1 if *scal* $= 30$.

Nearly Kähler manifolds in dimension 4 are automatically Kähler. Moreover P.-A. Nagy showed that locally any strict NK manifold is a product of 6-dimensional NK manifolds, twistor spaces of quaternionic Kähler manifolds or 3-symmetric spaces. The last two classes of manifolds admit a canonical NK structure. Examples in dimension 6 are

$$S^6, \quad S^3 \times S^3, \quad F_3 = SU(3)/T^2, \quad CP^3$$

Until the recent construction of NK metrics on S^6 and $S^3 \times S^3$ by M. Haskins and L. Foscolo, these were the only known examples. Moreover J.-B. Butruille showed that any homogenous 6-dimensional NK manifold is isometric to one of these examples.

From now on we restrict to complete 6-dimensional strict NK manifolds of scalar curvature $scal = 30$. The condition of constant type or equivalently the equation $\bar{\nabla}\bar{T} = 0$ translates in dimension 6 into the equation $\nabla_X d\omega = -3X^* \wedge \omega$ for all tangent vectors X, ie. the Kähler form ω is a so-called *special* Killing 2-form and in particular it holds that $\Delta\omega = 12\omega$.

One defines $\Psi^+ := \nabla\omega = \frac{1}{3}d\omega$ and $\Psi^- := *\Psi^+$. Then Ψ^+ is a 3-form of type $(0,3)+(3,0)$, ie. $\Psi^+(X, JY, JZ) = -\Psi^+(X,Y,Z)$ or equivalently $\Omega := \Psi^+ + i\Psi^-$ is a complex volume form. Thus NK manifolds in dimension 6 carry a $SU(3)$-structure. It was shown by Reyes-Carrion and N. Hitchin, that a $SU(3)$-structure (M^6, g, ω, Ψ^+) is strict NK iff

$$d\omega = 3\Psi^+ \qquad \text{and} \qquad d\Psi^- = -2\omega^2 \; .$$

Since the complex volume form Ω is $\bar{\nabla}$-parallel it follows that the holonomy of $\bar{\nabla}$ is contained in $SU(3)$. F. Belgun and A. Moroianu showed that the holonomy group is a strict subgroup, ie. the complex holonomy representation is reducible, only if the NK manifold is homothetic to $\mathbb{C}P^3$ or the flag manifold F_3. Moreover P.-A. Nagy showed that if the holonomy representation is complex irreducible but reducible as a real representation then the manifold is homothetic to $S^3 \times S^3$.

Another important property of 6-dimensional NK manifold is that the Riemannian curvature can be written as $R = R_{S^6} + R^{CY}$, where R_{S^6} is the curvature tensor of S^6 and R^{CY} is a curvature tensor of Calabi-Yau type, ie. $R^{CY} \in Sym^2(su(3))$.

3. Killing spinors. . Another equivalent definition of NK manifolds in dimension 6 can be given with Killing spinors. Since $c_1(TM) = 0$, or because of the existence of a $SU(3)$-structure, 6-dimensional NK manifolds are spin. Under the assumption $\pi_1(M) = 1$ the spin structure is unique. Then there exists a complex rank 8 vector bundle $S_M = S^+ \oplus S^-$ over M, the so-called spinor bundle.

A *Killing spinor*, is a section $\phi \in \Gamma(S_M)$ satisfying the equation $\nabla_X\phi = \lambda X \cdot \phi$ for all tangent vectors X and some real constant λ. Manifolds with Killing spinors are automatically Einstein, irreducible and non-symmetric.

R. Grunewald showed that the existence of a Killing spinor on a 6-dimensional manifold is equivalent to the NK condition. Let M^6 be a spin manifold not isometric to the standard sphere admitting a Killing spinor $\phi = \phi^+ + \phi^-$. Then the almost complex structure corresponding to ϕ is defined via the equation $J(X)\cdot\phi^+ = iX\cdot\phi^+$.

The description of 6-dimensional NK manifolds in terms of Killing spinors immediately implies the following result of Th. Friedrich. Let (M^6, g) be a Riemannian manifold not isometric to the standard sphere. Then there is at most one almost complex structure compatible with g and satisfying the NK condition.

A similar result of M. Verbitsky states that an almost complex manifold (M, J) admits up to scaling at most one NK metric g.

4. The cone construction. The metric cone of a Riemannian manifold (M, g) is defined as $\bar{M} = C(M) = M \times \mathbb{R}_+$ with the warped product metric $\bar{g} = r^2 g + dr^2$. A result of S. Gallot states that for a compact, simply connected manifold M, not isometric to the standard sphere, the cone $\bar{M}$ is irreducible.

For an even-dimensional spin manifold M the restriction of the spinor bundle of $\bar{M}$ to $M \subset \bar{M}$ can be identified with the spinor bundle of M. Under this identification Ch. Bär showed that Killing spinors on M are in 1-1 correspondence to parallel spinors on $\bar{M}$. This allows a characterization of Riemannian manifolds admitting Killing spinors. In particular 6-dimensional NK manifolds can be defined as manifolds for which the metric cone has holonomy contained in G_2.

Note that the first example of a (non-complete) G_2-metric was constructed by R. Bryant on the metric cone of the flag manifold F_3.

It is also possible to describe how parallel forms on the metric cone $\bar{M}$ restrict to forms on M. If $\rho \in \Omega^{p+1}(\bar{M})$. Then $\bar{\nabla}_{\partial_r}\rho = 0$ iff there exists forms $\omega \in \Omega^p(M)$ and $\psi \in \Omega^{p+1}(M)$ with $\rho = r^p dr \wedge \omega + \frac{1}{p+1} r^{p+1}\psi$ Using this decomposition one can characterize $\bar{\nabla}$-parallel forms as follows: $\bar{\nabla}\rho = 0$ if and only if

$$\nabla_X = \tfrac{1}{p+1} X \lrcorner \psi \qquad \text{and} \qquad \nabla_X \psi = -(p+1) X \wedge \omega$$

In particular, $\psi = d\omega$, ie. parallel forms on $\bar{M}$ correspond to special Killing forms on M.

As an application let (M, g, J) be a strict NK manifold, then $\rho = r^2 dr \wedge \omega + \frac{r^3}{3} d\omega$ is a parallel 3-form on $\bar{M}$ defining a G_2-structure on the metric cone. That ρ is a generic (or stable) 3-form can be seen using a special frame adapted to the NK structure. Conversely let $\rho \in \Omega^3(\bar{M})$ be a parallel generic form (ie. defining a G_2-structure on $\bar{M}$). Then $\omega := \partial_r \lrcorner \rho$ is a special Killing 2-form and J defined by $\omega(X, Y) = g(JX, Y)$ is an almost complex structure such that (M, g, J) is NK.

Let $M \subset \bar{M}$ be totally umbilic hypersurface, ie. the second fundamental form is multiple of the metric. E.g. this is the case for a manifold M considered as a hypersurface in its metric cone $\bar{M} = C(M)$. Then A. Gray remarked that a weak G_2-structure on $\bar{M}$, ie. a G_2-structure defined by a generic 3-form $\rho \in \Omega^3(\bar{M})$ with $d\rho = \lambda * \rho$ for some constant λ, induces a NK structure on M, defined by $\omega := N \lrcorner \rho$, where N is the normal vector of $M \subset \bar{M}$. Since $N = \partial_r$ is the normal vector in case of the metric cone, this corresponds to the construction above. Note that weak G_2-metrics are automatically Einstein.

However this way it is not possible to obtain new examples of NK manifolds due to a result of Koiso. In fact, let $M \subset \bar{M}$ be a totally umbilic hypersurface, $\bar{M}$ complete (which is not the case for the metric cone). Moreover assume the metric $\bar{g}$ on $\bar{M}$ and the induced metric g on M to be Einstein with $scal_g > 0$, then g and $\bar{g}$ have constant curvature, ie. $S^6 \subset \mathbb{R}^7$ is the only example obtained this way.

Deformations of $\mathbf{G_2}$ manifolds

Jason D. Lotay

(joint work with Spiro Karigiannis)

Given a G_2 manifold (M, φ) it is natural to ask about its moduli space: is it a smooth manifold and, if so, what is its dimension? In this talk I will review the situation when M is compact but focus on the case when M is either asymptotically conical (AC) (and so non-compact) or essentially the dual picture where M

is compact but has a conical singularity (CS), highlighting the similarities and differences which arise.

G_2 **conifolds.** In the AC and CS cases one has a conical model C with a link Σ and a rate of convergence ν to C: by convention, in the AC case $\nu < 0$ (so that if r is distance in the cone then r^ν decays as $r \to \infty$) and conversely in the CS case $\nu > 0$ (so that $r^\nu \to 0$ as $r \to 0$). Notice that we are always free to increase the rate in the AC case and decrease the rate in the CS case.

We can give a table for the known AC G_2 manifolds of Bryant-Salamon [1]:

	$\Lambda^2_+\mathcal{S}^4$	$\Lambda^2_+\mathbb{CP}^2$	$\mathbb{S}(\mathcal{S}^3)$
Σ	$\mathbb{CP}^3$	$\mathrm{SU}(3)/T^2$	$\mathcal{S}^3 \times \mathcal{S}^3$
ν	-4	-4	-3

(Here Σ is a nearly Kähler 6-manifold and $\mathbb{S}$ denotes the spinor bundle).

We know of no examples of CS G_2 manifolds, but they are expected to exist because they are natural models for how a family of compact G_2 manifolds degenerates. In particular, it is known [4] that if there is a CS G_2 manifold with a cone model at the singularities as in the table above, then it will arise as a limit of smooth compact G_2 manifolds. There is a proposal, following an idea of Joyce–Karigiannis [3], for constructing examples which have $\Sigma = \mathbb{CP}^3$: aspects of this construction are currently being investigated by myself and Karigiannis.

Moduli space. If we let $\mathcal{T}$ be the torsion-free G_2 structures on M and let $\mathcal{D}$ be the diffeomorphisms isotopic to the identity (with appropriate asymptotic behaviour for the G_2 structures and diffeomorphisms in the AC and CS cases), then the moduli space of deformations of (M, φ) we want to understand is $\mathcal{M} = \mathcal{T}/\mathcal{D}$.

The idea is to try to show that the $\psi \in \mathcal{T}$ near φ (modulo gauge) are in one-to-one correspondence with closed and coclosed 3-forms ζ on (M, φ). This correspondence is by identifying $\mathcal{M}$ with solutions to a nonlinear PDE, linearising and then applying the implicit function theorem. Thus the key points in the proof to look at surjectivity of the linearised operator in the deformation problem and to gauge-fix for the actions of diffeomorphisms.

Compact deformations. In the compact case, one has surjectivity by Hodge theory and gauge-fixing is possible, so one obtains the result first given by Joyce (c.f. [2]).

Theorem. If (M, φ) is compact then $\mathcal{M}$ is locally diffeomorphic to $H^3(M)$.

This result, though important, does not shed light on the global structure of the moduli space, and so it is natural to ask about the geometry and topology of $\mathcal{M}$. We now know that $\mathcal{M}$ can be disconnected, but we know little else. There are natural metrics on $\mathcal{M}$ so we can ask: what is the curvature of $\mathcal{M}$ and is the metric on $\mathcal{M}$ complete or not?

AC deformations. In the AC case we generalise and get a similar result to the compact case [5], which shows that (under certain assumptions on the rate ν) $\mathcal{M}$ is locally diffeomorphic to the closed and coclosed 3-forms of rate ν on M. Hence, we can determined $\dim \mathcal{M}$ from the topology of M and Σ and the spectrum of the Laplacian on 2-forms on Σ. The key is that surjectivity of the linearised operator still holds but there is some subtlety in the gauge-fixing: roughly speaking, one can have diffeomorphisms which actually grow (sub-linearly) on the conical end but still preserve the AC condition.

Using work in [6] on the spectrum on the Laplacian on the nearly Kähler 6-manifolds appearing in our table, we can deduce some interesting consequences.

Theorem. (a) The Bryant–Salamon AC G_2 manifolds are locally unique.
(b) If M is AC to a Bryant–Salamon cone then M is cohomogeneity one. Hence, the AC holonomy G_2 metrics on $\Lambda^2_+\mathcal{S}^4$ and $\Lambda^2_+\mathbb{CP}^2$ are unique.

We are therefore left with the open problem: is the AC holonomy G_2 metric on $\mathbb{S}(\mathcal{S}^3)$ unique? Certainly, the torsion-free G_2 structure is not unique as there three cohomologically distinct ones from the Bryant–Salamon construction, but they give the same metric. This question would be resolved, by our theorem, by a classification of cohomogeneity one G_2 manifolds, which is still open.

We can also ask whether we can extend this result to asymptotically local conical metrics: these naturally arise in the study of cohomogeneity one G_2 manifolds.

CS deformations. In the CS case, the deformation theory is rather different [5]. We do not find that $\mathcal{M}$ is a manifold in general but that it is given by the zero set of a smooth map π (which we can think of as a projection) from a finite-dimensional manifold $\mathcal{I}$ into a finite-dimensional vector space $\mathcal{O}$. The space $\mathcal{O}$ arises from the possible lack of surjectivity of the linearised operator, and we see immediately that if $\mathcal{O}$ is zero then $\mathcal{M}$ is smooth.

Unfortunately, it is not possible to interpret the space $\mathcal{I}$ in a satisfactory manner in general. This is due to issues arising in the gauge-fixing, caused by diffeomorphisms which decay sub-linearly at the singularity. It therefore leaves open the question: can we describe the full space $\mathcal{I}$ geometrically?

However, if we take ν near 0 then we can find a submanifold $\check{\mathcal{I}} \subseteq \mathcal{I}$ corresponding to the closed and coclosed 3-forms of rate ν plus the deformation arising from rescaling the identification of the conical singularity in M with the cone C. We thus obtain a subset $\check{\mathcal{M}}$ of $\mathcal{M}$ given by the zeros of π restricted to $\check{\mathcal{I}}$ and whose expected dimension we can compute explicitly as in the AC case.

Again using the spectral theory calculations from [6] we may deduce some useful results, including the following.

Theorem. Let M be CS with $\Sigma = \mathbb{CP}^3$ and let $\overline{M}$ be the compact G_2 manifold obtained from desingularizing M as in [4].
(a) $\mathcal{M} = \check{\mathcal{M}}$ is smooth.
(b) $\dim \mathcal{M} = b^3(\overline{M}) - 1$.

The result (b) shows that we can view the moduli space $\mathcal{M}$ as part of the top level stratum of the boundary of the moduli space of a compact G_2 manifold.

We have essentially the same result in the case where $\Sigma = \mathcal{S}^3 \times \mathcal{S}^3$, except that we do not know that $\mathcal{M}$ and $\check{\mathcal{M}}$ are equal: it would be interesting to see if there is a difference between them.

However, it is important to note that we do not have a corresponding result for the flag manifold $\Sigma = \mathrm{SU}(3)/T^2$. This is precisely because of the result in [6] which shows that this manifold has an 8-dimensional space of infinitesimal nearly Kähler deformations. These deformations can naturally be identified with the Lie algebra of SU(3), but there is currently no geometric interpretation for them. It is still an open question whether these nearly Kähler deformations are obstructed or not.

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The Physics of G_2-manifolds with Singularities

Bobby Samir Acharya

Compact manifolds with G_2-holonomy are used as models for the seven extra dimensions predicted by M-theory and this talk gives an overview of some of the properties that a G_2-manifold should have in order to provide a solution of M-theory which contains the Standard Model of Particle Physics.

The physical Universe is described, with a high degree of precision by Einstein's equations, Maxwell's equations, the Yang-Mill's equations and the Dirac equation. On the other hand all of these equations have played a distinguished role in geometry, analysis and topology over the last century. I described how superstring theories give rise to all of these equations and how the five consistent superstring theories (Type IIA, Type IIB, Type I, Heterotic $E_8 \times E_8$ and Heterotic $\frac{Spin(32)}{\mathbf{Z_2}}$) arise as limits of an eleven dimensional theory called M theory.

Next I went on to explain that *smooth* compact G_2-manifolds can not give rise to the Standard Model of Particle physics which is a non-Abelian gauge theory on $\mathbf{R^{3,1}}$ with fermions transforming in a complex representation of the $G_{SM} \equiv SU(3) \times SU(2) \times U(1)$ gauge group.

I first explained the origin of non-Abelian gauge fields: if a compact G_2-manifold (X, g) has a codimension four orbifold singularity along a three dimensional submanifold $Q^3 \subset X$ then one obtains non-Abelian gauge fields supported on Q^3. This picture can be derived by considering hyperkahler metrics on desingularisations of $\mathbf{R^4}/\Gamma_{ADE}$ where Γ_{ADE} is a finite subgroup of $SU(2) \subset SO(4)$ acting on $\mathbf{R^4}$. The gauge fields obtained in this way correspond to the gauge symmetry group of type A or D or E – corresponding to the ADE classification of the finite subgroups of $SU(2)$. One can see this by noting that physically a gauge field for a gauge group G corresponds to $\dim G$ *massless* particles transforming in the adjoint representation of G. These are represented by M theory membranes (M2-branes) which wrap 2-cycles in $H_2(\widetilde{\mathbf{R}^4}/\Gamma_{ADE}, \mathbf{Z})$ – which is isomorphic to the root lattice of the ADE Lie Algebra. These have zero mass in the singular orbifold $\mathbf{R^4}/\Gamma_{ADE}$.

I then described the origin of the chiral fermions, which, in the Standard Model are described by Dirac spinors transforming in a complex representation of G_{SM}. These arise from "more singular" ADE singularities in which the rank of the ADE type increases by one unit. A simple example illustrates this:

Example: Bryant-Salamon metric on $\mathbf{R}^+ \times \mathbf{CP^3}$.

Bryant and Salamon wrote an explicit, metric cone on the 7-manifold $Y = \mathbf{R}^+ \times \mathbf{CP^3}$ with holonomy group G_2. We will describe this space as a three dimensional family of smooth 4-manifolds parametrised by $\mathbf{R^3}$. Identify Y with $\mathbf{C^4}/U(1)$ in the standard way. Then, viewing $\mathbf{C^4}$ as $\mathbf{H^2}$ the hyperkahler moment map of the $U(1)$ action is a map $\mu : \mathbf{C^4} \to \mathbf{R^3}$ which commutes with $U(1)$. We therefore have: $\mu : Y \to \mathbf{R^3}$. The fibres of this map are, topologically T^*S^2 (which we imagine as a family of Eguchi-Hanson manifolds). At the origin of $\mathbf{R^3}$ the S^2 in the fibres collapses as $T^*S^2 \to \mathbf{R^4}/\mathbf{Z_2} = \mathbf{R^4}/\Gamma_{A1}$. In other words, the normal bundle of $\mathbf{R^3}$ in Y has generic fibre a smooth $\mathbf{R^4}$ (which we think of as $\mathbf{R^4}/\Gamma_{A0}$), but at one point on the three manifold, the fibre is $\mathbf{R^4}/\Gamma_{A1}$. So, in this example the rank of the ADE singularity increases from zero to one as we approach the origin. One interesting, open problem here is to re-write the Bryant-Salamon metric in coordinates which are compatible with this fibration.

By generalising this picture one conjectures the existence of G_2-holonomy conical metrics on other $U(1)$ quotients of $\mathbf{C^4}$. For instance, writing $\mathbf{C^4}$ as $\mathbf{C^2} \times \mathbf{C^2}$ we can consider a $U(1)$ action with degree (or charge) $(N, -1)$ on the two factors for positive integers N. The quotient is, topologically, $\mathbf{R}^+ \times \mathbf{WCP^3}_{N,N,1,1}$. Again, the hyperkahler moment map provides a map to $\mathbf{R^3}$, where, now the fibres are partial resolutions of $\mathbf{R^4}/A_N$ in which one S^2 has been resolved, leaving every generic fibre with an A_{N-1} singularity. At the origin of the $\mathbf{R^3}$ the S^2 collapses making the fibre at that point $\mathbf{R^4}/A_N$. If such a G_2-holonomy metric existed it would give rise to an $SU(N)$ gauge field with a chiral fermion transforming in the fundamental representation. Proceding in this fashion one conjectures that there exists a G_2-holonomy metric on $\mathbf{R}^+ \times \mathbf{WCP^3}_{p,p,q,q}$.

These ideas were first formulated in [1] and [2]

The recent new constructions of compact G_2-manifolds given in [3] produce of order 10^9 compact holonomy G_2 manifolds which are topologically $K3$-fibrations over the 3-sphere. These are produced from a twisted gluing construction starting with non-compact $K3$-fibered Calabi-Yau threefolds. The generic fibers of this fibration are smooth $K3$'s. We proposed a modification of this construction in which the generic $K3$-fibers have orbifold singularities. Though technically more difficult, this might potentially produce compact G_2 manifolds with codimension four ADE singularities.

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Aspects of linear elliptic theory on singular spaces

Rafe Mazzeo

This talk surveyed some techniques from geometric microlocal analysis which have been quite useful for understanding elliptic theory on singular and noncompact spaces. These ideas focus on understanding the pointwise polyhomogeneous behavior of the Schwartz kernels of parametrices for various types of degenerate linear elliptic operators, and on the analytic consequences following from such a description of these parametrices. The second part of the talk described one particular application, concerning the infinitesimal rigidity of three-dimensional hyperbolic conifolds; this illustrates how detailed knowledge about asymptotics of solutions can be used in practice.

There is a dictionary between various types of complete geometries, e.g. asymptotically locally Euclidean (or asymptotically conic), asymptotically hyperbolic, asymptotically cylindrical, etc., and various classes of degenerate elliptic operators which are modeled on the behaviors of the natural geometric operators on these spaces. There is a similar correspondence for certain incomplete geometries, including iterated edge metrics on stratified spaces. All of these types of spaces appear in the study of metrics with special holonomy, and it is expected that the ideas described here should be useful in the study of noncompact and/or singular G_2 spaces.

We give here only a few examples of these correspondences, but comment that many more complicated types of geometries have been handled by similar methods. The first setting is a set of three examples: metrics which are complete and either asymptotically cylindrical or asymptotically conical, or incomplete with isolated conic singularities. Suppose M^n is a compact manifold with boundary. Let x be a boundary defining function, so $0 \leq x \in \mathcal{C}^\infty$, $\partial M = \{x = 0\}$, and $dx \neq 0$ there. Let h be a metric on ∂M. Then consider three types of metrics:

$$g_b = \frac{dx^2}{x^2} + h, \qquad g_{ac} = \frac{dx^2}{x^4} + \frac{1}{x^2}h, \qquad g_c = dx^2 + x^2 h.$$

These are defined only near ∂M, but we assume they are extended to the rest of M. The first two are complete, i.e., ∂M is at infinite distance, while the third is incomplete. The first is a metric with a cylindrical end, as can be seen by setting $t = -\log x$, so $g_b = dt^2 + h$, $t \to \infty$; the second is a metric on the large end of a cone, which we see by setting $r = 1/x$, so that $g_{ac} = dr^2 + r^2 h$, $r \to \infty$; the third is a conic metric near the vertex of the cone ($x \to 0$). We also consider such metrics which include 'lower order' terms. Note that these metrics are conformal to one another: $g_{ac} = x^{-2}g_b$, $g_c = x^2 g_b$, so the underlying linear theories for each of these are closely related. Since this was not the main emphasis of the talk, we do not describe any of these results further.

Among the many other complete examples, we mention asymptotically hyperbolic (conformally compact) metrics, asymptotically complex hyperbolic metrics, QALE metrics, etc.

In the realm of incomplete spaces, conic metrics can be generalized to metrics with incomplete edges; these are stratified spaces with one singular stratum B which is itself a smooth closed manifold. A neighborhood of B is identified with a bundle of cones over B, and in this neighborhood,

$$g_e = dx^2 + h + x^2 k,$$

where h is a metric pulled up from B and $dx^2 + x^2 k$ restricts to a conic metric on each fiber. We may also consider spaces obtained by iterating the processes of taking cones or bundles of cones. From a differential topological perspective, these are the stratified spaces and one class of natural metrics on them are the iterated edge metrics. We illustrate with a 'depth 2' example: suppose that M is a space where the singular set decomposes into a simple edge, as above, and an isolated set of points near which M is identified with a cone over a space with conic points or simple edges. Thus if M is identified locally as a cone $C(Y)$ where Y itself has conic or edge singularities, and if x_2 is the radial function at the vertex, then

$$g_{ic} = dx_2^2 + x_2^2(dx_1^2 + x_1^2 h) = dx_2^2 + x_2^2 dx_1^2 + x_1^2 x_2^2 h,$$

where x_1 is a radial function near the cone point of Y and $dx_1^2 + x_1^2 h$ is conic metric on Y.

We now write out the model Laplacians for any one of these metrics:

$$\Delta_b = (x\partial_x)^2 + \Delta_h, \quad \Delta_{ac} = (x^2\partial_x)^2 + x^2\Delta_h + \text{l.o.t.}$$

$$\Delta_c = \partial_x^2 + \frac{n-1}{x}\partial_x + x^{-2}\Delta_h, \quad \Delta_{ic} = \partial_{x^2}^2 + \frac{n_2}{x_2}\partial_{x_2} + \frac{1}{x_2^2}\left(\partial_{x_1}^2 + \frac{n_1}{x_1}\partial_{x_1} + \frac{1}{x_1^2}\Delta_h\right).$$

This hierarchy of geometries and the correspondence with degenerate differential operators fits into the general framework of boundary fibration structures, see [5]. Many different boundary fibration structures, and numerous applications to geometric analysis, have been studied and used by many authors, starting with Melrose's pioneering work in the early 1980's.

The rest of the talk focused on the problem of rigidity and deformations of hyperbolic metrics with iterated conic singularities (the class of so-called hyperbolic conifolds). This setting was introduced by Thurston in the 1970's, and studied by many authors since that time; the results here appeared in joint work with Montcouquiol [4]. We suppose that M is a 3-dimensional space with a singular iterated conic hyperbolic metric. Thus each singular point of M either lies on an edge or else at a singular vertex; the metric in these two cases, takes the form

$$dr^2 + \beta^2 \sinh^2 r d\theta^2 + \cosh^2 r dy^2, \quad \text{or} \quad g = d\rho^2 + \rho^2 h,$$

where h is a metric of curvature $+1$ on a sphere with isolated conic singular points corresponding to the singular edges which meet at that singular vertex. The infinitesimal rigidity problem asks whether an infinitesimal variation $\dot{g}$ of these structures which leaves the dihedral angles along the singular edges fixed is necessarily trivial, i.e., arises by linearizing a one-parameter family of pullbacks $F_\epsilon^* g$. We regard $\dot{g}$ as a symmetric 2-tensor on M; the infinitesimal angle-fixing hypothesis means that $\dot{g}$ is bounded, and we may also assume that it is polyhomogeneous (i.e., admits a complete asymptotic expansion) at the singular set. The assertion that $\dot{g}$ is infinitesimally equivalent to a family of 'trivial' deformations $F_\epsilon^* g$ is equivalent to finding a bounded 1-form ω such that $\dot{g} = (\delta^g)^* \omega$ (or equivalently, showing that $\dot{g} = \mathcal{L}_X g$ for some bounded vector field X).

A hyperbolic metric $g' = g + h$ near to g (so h is small) is said to be in Bianchi gauge if

$$B^g(h) := \delta^g h + \tfrac{1}{2} \mathrm{tr}^g h = 0.$$

Given any g' near to g, there exists a unique diffeomorphism F so that $B^g(F^* g' - g) = 0$. Our main result asserts that if we first 'gauge' $\dot{g}$ by finding ω such that $\kappa = \dot{g} - (\delta^g)^* \omega$ satisfies $B^g \kappa = 0$, and then proving that necessarily $\kappa = 0$.

At a formal level, both of these steps are straightforward. To find ω, we recall the Weitzenböck formula

$$B^g (\delta^g)^* = (\nabla^* \nabla - \mathrm{Ric}),$$

and denote this operator by P^g. Thus to put $\dot{g}$ into gauge, we choose *omega* to satisfy $P^g \omega = B^g \dot{g}$, and consider the equivalent gauged infinitesimal deformation $\kappa = \dot{g} - (\delta^g)^* \omega$. This lies in the nullspace of the operator

$$L^g = \frac{1}{2} (\nabla^* \nabla - 2R),$$

which is the differential of the Bianchi gauged Einstein operator. Here R is the full curvature tensor regarded as a symmetric endomorphism on symmetric 2-tensors. That $\kappa = 0$ can then be proved by a Bochner argument.

The subtlety in all of this is showing that these steps can be carried out in this singular geometric setting. This involves choosing a proper (functional analytic) domain on which to define P^g, then proving a sharp regularity theorem for the solution ω, and then using this information to carry out the integration by parts needed to conclude that $\kappa = 0$. We use the Friedrichs domain, which has the property that the solution ω is bounded at the singular set (the key point is that it does not have logarithmic growth there). The microlocal structure of the generalized inverse shows that in fact ω is polyhomogeneous at the singular set since the same is true of $\dot{g}$. Along the way, we use a characterization of elements of the Friedrichs domain in terms of a certain partial regularity statement; this relies on an inductive procedure for defining the domains along the successive strata. This last step was later generalized considerably to a complete theory of iterated boundary conditions on stratified spaces [1].

This hyperbolic conifold example was chosen to illustrate how to handle iterated conic singularities (which also appear in the G_2 theory), and the role of parametrices and sharp regularity theory. There are many other closely related settings where these geometrical microlocal ideas have been useful in understanding metrics with special holonomy. We mention some recent examples: the existence of Kähler-Einstein edge metrics [3] and the regularity and deformation theory for asymptotically cylindrical Calabi-Yau metrics [2].

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New nearly Kähler 6-manifolds I and II

Lorenzo Foscolo

(joint work with Mark Haskins)

It is well known that the 6–sphere carries a non-integrable almost complex structure J defined by octonionic multiplication via the embedding $S^6 \subset \operatorname{Im}\mathbb{O}$. Since the almost complex structure J is compatible with the round metric g, it defines a $(1,1)$–form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. The almost Hermitian manifold (S^6, g, J, ω) has many remarkable properties. In particular,

$$\begin{cases} d\omega = 3\operatorname{Re}\Omega, \\ d\operatorname{Im}\Omega = -2\omega^2, \end{cases} \tag{1}$$

for a complex volume form Ω.

More in general, a 6–manifold M endowed with an SU(3)–structure (ω, Ω) satisfying (1) is called a (strict) *nearly Kähler* 6–manifold. The defining equations (1) are equivalent to the requirement that the Riemannian cone $C(M)$ over M has holonomy contained in the exceptional Lie group G_2. Here the Riemannian cone over (M, g) is $\mathbb{R}^+ \times M$ endowed with the incomplete Riemannian metric $dr^2 + r^2\, g$.

Since holonomy G_2–metrics are Ricci-flat, nearly Kähler 6–manifolds are Einstein with positive scalar curvature. In particular a complete nearly Kähler 6–manifold M is compact with finite fundamental group and by passing to the universal cover one can always assume that M is simply connected.

Besides the round 6–sphere, only three examples were known until now: CP^3, the flag manifold F_3 and $S^3 \times S^3$. These examples are all homogeneous (for a homogeneous metric distinct from the standard one) and were constructed in 1968 by Gray and Wolf as part of their classification of 3–symmetric spaces [1].

The scarcity of examples of nearly Kähler 6–manifolds is surprising when compared with geometries related to other special holonomy groups: there are infinitely many Calabi–Yau, hyperkäler and Spin(7)–cones [2].

In a recent joint work with Mark Haskins [3] we found the first inhomogeneous examples of complete nearly Kähler manifolds by constructing a cohomogeneity one nearly Kähler structure on S^6 and on $S^3 \times S^3$.

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G_2-Gauge Theory I: Compactness property for G_2-instantons

Andriy Haydys

(joint work with Thomas Walpuski)

In this talk I discuss the compactness property for the moduli space G_2-instantons. This turns out to be closely related to certain generalization of the Seiberg–Witten equations.

To understand why the moduli space of G_2-instantons may fail to be compact, it is instructive to consider the four-dimensional case first. Let a_n be a sequence of anti-self-dual instantons on a closed oriented Riemannian four-manifold X. Then either this sequence converges to an anti-self-dual instanton, or the energy of a_n concentrates near a finite set of points and the sequence converges to an "ideal instanton" [1]. In the latter case one says that the sequence a_n develops a bubble. A model for the bubble is an anti-self-dual instanton on $T_xX \cong \mathbb{R}^4$, where x is a point of energy concentration.

Let Y be a manifold of dimension bigger than 4. We concentrate on manifolds of dimension 7 in this talk, more specifically on those manifolds, which admit a Riemannian metric with holonomy group contained in G_2. Since G_2 is a subgroup of $SO(7)$, we can realize $\mathfrak{g}_2 = \mathrm{Lie}(G_2)$ as the subalgebra of $\mathfrak{so}(7)$. Hence, we obtain a splitting $\Lambda^2(\mathbb{R}^7)^* = \mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp$, where $\dim \mathfrak{g}_2 = 14$ and $\dim \mathfrak{g}_2^\perp = 7$. A global version of this decomposition is the splitting

$$\Lambda^2 T^*Y = \Lambda^2_{14} T^*Y \oplus \Lambda^2_7 T^*Y$$

of the bundle of 2-forms on a G_2-manifold, where $\mathrm{rk}\,\Lambda^2_{14}T^*Y = 14$ and $\mathrm{rk}\,\Lambda^2_7 T^*Y = 7$. Let $\pi_7(\omega)$ denote the $\Lambda^2_7 T^*Y$–component of ω.

Definition 1.1 ([3])**.** Given a principal G-bundle $P \to Y$, a connection A on P is called a G_2-instanton, if $\pi_7(F_A) = 0$, where F_A denotes the curvature of A.

Let A_n be a sequence of G_2-instantons over a closed G_2-manifold Y. Assume A_n does not converge. Then by the work of Uhlenbeck, Nakajima and Tian we know that A_n develops either a bubble along a subset M of Hausdorff-dimension at most 3 or an unremovable singularity. In this talk unremovable singularities are ignored. If the bubbling set M is smooth and three-dimensional, then M is associative. Moreover, for any $m \in M$ we obtain an anti-self-dual instanton a_m on the normal fiber $N_mM \cong \mathbb{R}^4$. Besides, the family $\{a_m \mid m \in M\}$ conjecturally yields [2, 4] a Fueter-section, which we describe in some details momentarily.

In the simplest case a Fueter-section is a map u from $\mathbb{R}^3$ to a hyperKähler manifold $(\mathcal{M}, I_1, I_2, I_3)$ satisfying

$$I_1 \frac{\partial u}{\partial x_1} + I_2 \frac{\partial u}{\partial x_2} + I_3 \frac{\partial u}{\partial x_3} = 0.$$

For the case of bubbles of G_2-instantons the relevant codomain is $\mathcal{M}_{asd}(\mathbb{R}^4)$ the framed moduli space of anti-self-dual instatons on $\mathbb{R}^4$. It is well known that $\mathcal{M}_{asd}(\mathbb{R}^4)$ has a natural hyperKähler structure. Notice that $\mathcal{M}_{asd}(\mathbb{R}^4)$ is equipped with an action of $\mathbb{R}_{>0}$ commuting with each complex structure I_j. This means that Fueter-sections always appear in 1-parameter families. Hence, existence of Fueter-sections is expected to be a codimension one phenomenon (say, in the space of all Riemannian metrics on M).

Another place where one meets Fueter-sections is a generalization of the Seiberg-Witten equations. To explain, let M be a closed oriented Riemannian three-manifold. Denote by S a spinor bundle of M. Fix also a $U(1)$–bundle L over M, a positive integer n and a $SU(n)$–bundle E together with a connection B.

We consider pairs $(A, \Psi) \in \mathcal{A}(L) \times \Gamma(\mathrm{Hom}(E, S \otimes L))$ consisting of a connection A on L and an n–tuple of twisted spinors Ψ satisfying the Seiberg–Witten equation with n spinors:

$$\begin{aligned} D_{A\otimes B}\Psi &= 0, \\ F_A &= \mu(\Psi). \end{aligned} \tag{1}$$

Here $\mu\colon \mathrm{Hom}(E, S \otimes L) \to i\mathfrak{su}(S)$ is defined by

$$\mu(\Psi) := \Psi\Psi^* - \frac{1}{2}|\Psi|^2 \,\mathrm{id}_S \tag{2}$$

and we identify $\Lambda^2 T^*M$ with $\mathfrak{su}(S)$.

Theorem 1.2 ([5])**.** *Let (A_i, Ψ_i) be a sequence of solutions of (1). Denote $\lambda_i = \|\Psi_i\|_{L^2(M)}$. If $\liminf \lambda_i < \infty$, then after passing to a subsequence and up to gauge transformations (A_i, Ψ_i) converges smoothly to a limit (A, Ψ). If $\lim \lambda_i = \infty$, then after passing to a subsequence the following holds:*

- *There is a closed nowhere-dense subset $Z \subset M$, a flat connection A on $L|_{M\setminus Z}$ with monodromy in $\mathbb{Z}_2$ and $\Psi \in \Gamma(M \setminus Z, \mathrm{Hom}(E, S\otimes L))$ such that*

(A, Ψ) *solves*

$$\|\Psi\|_{L^2} = 1, \quad D_{A\otimes B}\Psi = 0, \quad \mu(\Psi) = 0. \tag{3}$$

Moreover, $|\Psi|$ *extends to a Hölder continuous function on all of* M *and* $Z = |\Psi|^{-1}(0)$.

- *On* $M \setminus Z$, *up to gauge transformations,* A_i *converges to* A *in* C^∞_{loc} *and* $\lambda_i^{-1}\Psi_i$ *converges to* Ψ *in* C^∞_{loc}.

Equations (3) were studied in [4] in detail. In particular, it was established that there is a one-to-one correspondence between gauge equivalence classes of solutions of (3) and Fueter-sections with values in $\mathcal{M}_{asd}(\mathbb{R}^4)$.

Returning to the G_2-instantons, it is expected that for generic metric with holonomy in G_2 the moduli space of G_2-instantons is compact. However, in a 1-parameter family of such metrics there should be a finite number of points for which a G_2-instanton develops a bubble and dies leaving a trace, namely an associative submanifold $M^3 \subset Y^7$ and a Fueter section with values in $\mathcal{M}_{asd}(\mathbb{R}^4)$. These data can be conjecturally used to give a birth to a Seiberg–Witten monopole on M. Conversely, a Seiberg–Witten monopole on an associative submanifold of Y may degenerate to Fueter-section and by a result of Walpuski will be reborn as a G_2-instanton on Y. Consequences of these phenomena for invariants of G_2-manifolds will be discussed elsewhere.

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The extended Crowley-Nordström ν-invariant

Sebastian Goette

(joint work with D.Crowley, J. Nordström)

This talk and the following one by Johannes Nordström together show that there are manifolds whose G_2-moduli space is disconnected. Here, we present an invariant that can detect components of the G_2-moduli space.

1. The Crowley-Nordström ν-invariant

We represent a G_2-structure on a 7-manifold M by nonvanishing spinor $\sigma \in \Gamma(SM)$ up to homotopy and diffeomorphism. There exists a compact spin manifold W with $\partial W = M$. Extend σ to $\bar{\sigma} \in \Gamma(S^+W)$ transversal to 0. Then the *Crowley-Nordström ν-invariant* [2] is defined as

$$\nu(\sigma) = \chi(W) - 3\,\mathrm{sign}(W) - 2\#\bar{\sigma}^{-1}(0) \quad \in \mathbb{Z}/48\,.$$

It is independent of W and $\bar{\sigma}$, and it detects all 24 different G_2-structures on M if the topology of M is sufficiently simple.

2. The extended ν-invariant

We consider the Mathai-Quillen current $\psi(S^+W)$, which satisfies

$$d\big(\bar{\sigma}^*\psi(S^+W)\big) = e(S^+M) - \delta_{\bar{\sigma}^{-1}(0)}\,.$$

Let D_M denote the Dirac operator on spinors, and let B_M denote the odd signature operator on M. Combining the Atiyah-Patodi-Singer index theorem with the above, one gets an intrinsic description

$$\nu(\sigma) = 2\int_M \sigma^*\psi(SM) - 24(\eta + h)(D_M) + 3\eta(B_M) \in \mathbb{Z}/48\,.$$

Now assume that (M, g) has holonomy G_2, and let σ be the associated *parallel* spinor. Then the Mathai-Quillen term vanishes. Moreover, the η-invariants now depend continuously on g if g varies in the set of G_2-metrics. Hence we can define the refined invariant

$$\bar{\nu}(M, g) = -24\eta(D_M) + 3\eta(B_M) \in \mathbb{Z}\,.$$

3. Twisted connected sums

Let $M = M^+ \cup M^-$ be a twisted connected sum [4], [1], with $M^\pm \cong V^\pm \times S^1$ for Calabi-Yau manifolds $V^\pm$ with cyclindrical ends, and let $X \cong \partial M^\pm \cong \Sigma \times T^2$ denote the gluing hypersurface. By the Kirk-Lesch gluing theorem [3], one can decompose

$$\eta(B_M) = \eta_{\mathrm{APS}}(B_{M^+}, L_+) + \eta_{\mathrm{APS}}(B_{M^-}, L_-) + m(L_+, L_-)$$

for appropriate boundary conditions determined by

$$L_\pm = \mathrm{im}(H^\bullet(V^\pm; \mathbb{R}) \to H^\bullet(X; \mathbb{R}))\,.$$

Both $M^\pm$ admit reflexions that anticommute with $B_{M^\pm}$, hence the η-invariants vanish. The situation for D_M is similar. The Maslov index $m(L_+, L_-)$ depends on certain "angles" between L_+ and L_-. These vanish for all twisted connected sums, hence $\bar{\nu}(M, g) = 0$.

In certain examples, one can replace one or both halves by quotients $M^\pm/\mathbb{Z}_2$. Now, one can glue with an angle $\vartheta \in \{\pm\frac{\pi}{6}, \pm\frac{\pi}{4}, \pm\frac{\pi}{3}\}$ between the images of the external circle factors. The positions of $L_\pm$ allow one to determine an integer k such that

$$\bar{\nu}(M,g) = 72\,\frac{2\vartheta - \pi}{\pi} + 3k\ .$$

There are now examples of G_2-structures on diffeomorphic manifold with different values of $\bar{\nu}(M,g)$. In particular, their G_2-moduli spaces are disconnected.

Currently, the value of $\bar{\nu}$ for Joyce's examples is unknown, except for those that are representable as twisted connected sums.

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Disconnecting the G_2 moduli space

Johannes Nordström
(joint work with Diarmuid Crowley, Sebastian Goette)

The twisted connected sum construction studied by Kovalev [5] and Corti, Haskins, Pacini and the speaker [2] yields many examples of holonomy G_2 metrics on closed 7-manifolds. The simplest way to solve the "matching problem" that appears in the construction results in at least 10^8 examples on 7-manifolds that are 2-connected with torsion-free $H^4(M)$.

The only obvious remaining topological invariants of such 7-manifolds are the third Betti number $b_3(M)$ and the greatest integer $d(M)$ that divides the first Pontrjagin class $p_1(M) \in H^4(M)$. According to Wall and Wilkens [6], closed 2-connected 7-manifolds with torsion-free $H^4(M)$ are classified up to homeomorphism by $(b_3(M), d(M)) \in \mathbb{N} \times 4\mathbb{N}$. Further, one can obtain a diffeomorphism classification by adding an invariant $\mu(M)$ of the smooth structure (Crowley-N [3]), a generalisation of the Eells-Kuiper invariant, but $\mu(M)$ can be non-zero only when $d(M)$ is divisible by 7 or 16.

Twisted connected sums realise hundreds of pairs $(b_3(M), d(M))$ with

$$61 \le b_3(M) \le 239, \quad b_3(M) \text{ odd}, \quad d(M) \mid 48.$$

The analytic invariant $\bar{\nu}$ of the G_2-metric defined in Sebastian Goette's talk vanishes for all twisted connected sums. This talk describes the "extra-twisted connected sum construction", which produces a much smaller number of examples, but with non-zero $\bar{\nu}$. If we find some 2-connected extra-twisted connected sum M with torsion-free $H^4(M)$ and odd $b_3(M)$, then we have a good chance of finding a diffeomorphic ordinary twisted connected sum, and hence a closed 7-manifold with disconnected G_2 moduli space.

1. Basic picture of extra-twisted connected sums

Let V be a Calabi-Yau 3-fold with a single end, asymptotic to a product cylinder $\mathbb{R} \times S^1 \times \Sigma$, for Σ a K3 surface. Then $V \times S^1$ is a G_2-manifold, asymptotic to $\mathbb{R} \times T^2 \times \Sigma$.

Suppose V has an involution τ whose restriction the boundary at infinity $S^1 \times \Sigma$ is $a \times \mathrm{Id}_\Sigma$, for $a : S^1 \to S^1$ the antipodal map. Then $V \times S^1 / \tau \times a$ is an asymptotically cylindrical G_2-manifold, with end $\mathbb{R} \times T^2 \times \Sigma$. However, this $T^2 = S^1 \times S^1 / a \times a$ need *not* be isometric to a product $S^1 \times S^1$; that would only happen if the circumferences are chosen equal.

We want to construct closed G_2-manifolds M by gluing two asymptotically cylindrical G_2-manifolds of the above types using a product isometry $f \times r : T^2 \times \Sigma_+ \to T^2 \times \Sigma_-$, where

- $r : \Sigma_+ \to \Sigma_-$ is a "hyper-Kähler rotation",
- $f : T^2 \to T^2$ is an orientation-reversing isometry.

The key parameter of f is the angle θ between the "external S^1 directions", *i.e.* the directions in T^2 coming from the S^1 factor in $V \times S^1$. To get $\pi_1 M$ finite, which is necessary for M to admit metrics of full holonomy G_2, we need $\theta \neq 0, \pi$.

- Without involutions, *i.e.* in the ordinary twisted connected sum construction, the *only* possibility is $\theta = \pm\frac{\pi}{2}$.
- With an involution on one side, we can get $\theta = \pm\frac{\pi}{4}$ (set circumferences of internal and external S^1s to be equal to get a square T^2).
- With involutions on both sides we can get $\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{3}$ (set ratio of circumferences of internal and external S^1s to be $\sqrt{3}$ to get a hexagonal T^2).

The latter two cases give rise to what we call extra-twisted connected sums.

2. Hyper-Kähler rotations

The parallel $SU(3)$-structure on the asymptotically cylindrical Calabi-Yau 3-fold V can be described in terms of

- a holomorphic $(3,0)$-form Ω, asymptotic to $(-idt + du) \wedge (\omega^J + i\omega^k)$ on $\mathbb{R} \times S^1 \times \Sigma$; here $\omega^J + i\omega^k$ is a holomorphic $(2,0)$-form with respect to a unique complex structure I on the K3 surface Σ.
- a Kähler form ω, asymptotic to $dt \wedge du + \omega^I$, where ω^I is a Kähler form on Σ with respect to I.

The "hyper-Kähler triple" $(\omega^I, \omega^J, \omega^K)$ is equivalent to an $SU(2)$-structure on Σ.

The product G_2-structure on $V \times S^1$, where the external circle factor has coordinate v and circumference ℓ, is defined by the 3-form

$$\begin{aligned}\varphi &:= \ell dv \wedge \omega + \mathrm{Re}\,\Omega \\ &\sim \ell dv \wedge dt \wedge du + \ell dv \wedge \omega^I + du \wedge \omega^J + dt \wedge \omega^K \\ &= dt \wedge \left(\omega^K - \tfrac{i}{2} dz \wedge d\bar{z}\right) + \mathrm{Re}\,\left(dz \wedge (\omega^I - i\omega^J)\right),\end{aligned}$$

where $z := \ell v + iu$. For a pair V_+, V_- we can write an orientation-reversing isometry $T^2 \to T^2$ as $z \mapsto e^{i\theta}\bar{z}$, where θ corresponds to the angle between the external S^1 factors. We need $r : \Sigma_+ \to \Sigma_-$ such that

$$\begin{aligned}\mathbb{R} \times T^2 \times \Sigma_+ &\to \mathbb{R} \times T^2 \times \Sigma_-, \\ (t, z, x) &\mapsto (-t, e^{i\theta}\bar{z}, r(x))\end{aligned}$$

is an isomorphism of the cylindrical G_2-structures. This is equivalent to

$$\begin{aligned}r^*(\omega^I_- + i\omega^J_-) &= e^{i\theta}(\omega^I_+ - i\omega^J_-), \\ r^*\omega^K_- &= -\omega^K_-.\end{aligned} \tag{1}$$

If r satisfies this condition we call it a *hyper-Kähler rotation* with angle θ.

3. Matching problem

We can produce Calabi-Yau 3-folds V asymptotic to a cylinder $\mathbb{R} \times S^1 \times \Sigma$ by solving the complex Monge-Ampère equation on complex manifolds obtained by blow-up of a Fano or weak Fano 3-fold Y with an anticanonical divisor Σ (Haskins-Hein-N [4], Corti-Haskins-N-Pacini [1]). We can also produce asymptotically cylindrical Calabi-Yau 3-folds with the type of involution demanded for the extra-twisted connected sum construction from the (rather smaller) collection of Fano 3-folds with index 2, using branched double covers. But how can we find pairs (Y_+, Σ_+), (Y_-, Σ_-) with a hyper-Kähler rotation $r : \Sigma_+ \to \Sigma_-$?

Given r, we can identify both Σ_+ and Σ_- with a standard K3 Σ, and consider the sublattices $N_\pm := \mathrm{Im}\, H^2(V_\pm) \subset H^2(\Sigma)$ of the K3 lattice. A priori

$$[\omega^I_\pm] \in N_\pm \otimes \mathbb{R}$$

while the "period" of $\Sigma_\pm$, *i.e.* the 2-plane in $H^2(\Sigma; \mathbb{R})$ spanned by $[\omega^J_\pm]$ and $[\omega^K_\pm]$, is orthogonal to $N_\pm$. Let $\pi_\pm : N_\pm \otimes \mathbb{R} \to N_\mp \otimes \mathbb{R}$ denote the orthogonal projection (with respect to the intersection form). Then (1) implies

$$\pi[\omega^I_\pm] = (\cos\theta)[\omega^I_\mp].$$

We have the most degrees of freedom of finding appropriate triples of classes $[\omega^I_\pm], [\omega^J_\pm], [\omega^K_\pm] \in H^2(\Sigma; \mathbb{R})$ when N_+ and N_- are "at pure angle θ", *i.e.*

$$\pi_\pm \circ \pi_\mp = (\cos\theta)^2 \mathrm{Id}. \tag{2}$$

For $\theta \neq \frac{\pi}{2}$, the existence a pair of primitive isometric embeddings $N_\pm \hookrightarrow H^2(\Sigma)$ such that (2) holds imposes non-trivial arithmetic conditions on the pair of lattices N_+ and N_- (which are determined by the Fanos Y_+ and Y_-).

Generalising from the case $\theta = \frac{\pi}{2}$ [2, Proposition 6.18], we can use the Torelli theorem and deformation results for Fano manifolds to find hyper-Kähler rotations for asymptotically cylindrical Calabi-Yau manifolds with involution constructed from Fano 3-folds with index 2.

For extra-twisted connected sums whose hyper-Kähler rotation satisfies (2), the angles appearing in the computation of $\bar{\nu}$ in Goette's talk are all $\pm\theta$ or 0.

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G_2-Gauge Theory

Goncalo Oliveira

Yesterday Andriy Haydys explained that in order to define an invariant of G_2-manifolds by counting G_2-instantons one needs to understand the limiting behavior of instatons along associative submanifolds. Then, Andryi explained that one may then hope to define a joint weighted count of G_2-instatons and associatives instead. There are other interesting special submanifolds of G_2-manifolds, namely coassociative submanifolds and Dominic Joyce [3] conjectured that one may try to define an invariant of a G_2-manifold by counting rigid, compact coassociative submanifolds. On noncompact G_2-manifolds Donaldson and Segal, in [2] propose an invariant counting monopoles instead. They further suggest that this might be easier to define and possibly related to a direct coassociative count. In this talk we will report on the results of [4] for asymptotically conical (AC) G_2-manifolds.

Let (X, φ) be an AC G_2-manifold with asymptotic cone $((1, +\infty)_r \times \Sigma, g_C = dr^2 + r^2 g_\Sigma)$. For future reference recall that (Σ^6, g_Σ) comes equipped with a Nearly Kähler structure. Then, we shall consider a principal bundle P which at the conical end is modeled on P_∞ over Σ. Moreover, we equip $\mathfrak{g}_P$, $\mathfrak{g}_{P_\infty}$ with Ad-invariant inner products h, h_∞ which in combination with g_C are used to measure the growth rate of sections of $\Lambda^* \otimes \mathfrak{g}_P$. For example, if a denotes a section of $\Lambda^* \otimes \mathfrak{g}_P$ we shall by abuse of notation along the conical end pull it back to cone and still write a. Then we say it has rate $\delta \in \mathbb{R}$ with derivatives if along the conical end $|\nabla^j a| = O(r^{\delta - j})$ for all $j \in \mathbb{N}_0$; where $|\cdot|$ denotes a combination the norm g_C with h_∞, ∇ the connection obtained by twisting the Levi-Civita connection of g_C with ∇_∞.

Definition 1.1. Let P be a principal G-bundle as above. A monopole (A, Φ) on P is said to have finite mass if there is a connection A_∞ on P_∞ such that at the end $A - A_\infty$ has rate $-1-\epsilon$ with derivatives, for some $\epsilon > 0$ and

$$m(A,\Phi) = \lim_{\rho \to \infty} |\Phi|, \tag{1}$$

is well defined and constant. In this case $m(A,\Phi) \in \mathbb{R}^+$ is the mass of the monopole.

For a finite mass monopole (A, Φ) the Intermediate energy E^I on a precompact $U \subset X$ is defined by $E^I(U) = \frac{1}{2}\int_U |\nabla_A \Phi|^2 + |F_A \wedge \psi|^2$ and we proved that it may be rewritten as

$$E^I(U) \quad = \quad \int_{\partial U} \langle \Phi, F_A \rangle \wedge \psi + \frac{1}{2} \| F_A \wedge \psi - * \nabla_A \Phi \|^2_{L^2(U)}. \tag{2}$$

The Intermediate Energy is indeed the relevant Energy for finite mass monopoles and in fact if (A, Φ) is a finite mass, irreducible monopole on P, then:

- $\nabla_A \Phi \in L^2$ and there is $\Phi_\infty \in \Omega^0(X, \mathfrak{g}_{P_\infty})$, such that $\nabla_{A_\infty} \Phi_\infty = 0$ and $\Phi - \Phi_\infty$ has rate -5 with derivatives. In particular, the intermediate energy $E^I(X)$ is finite.
- If we further suppose $[A - A_\infty, \Phi_\infty]$ has rate $-6-\epsilon$ for some $\epsilon > 0$ with derivatives, then A_∞ is a pseudo-Hermitian-Yang-Mills connection for the nearly Kähler strucre on Σ, i.e. $F_A \wedge \omega^2 = F_A \wedge \Omega_2 = 0$.

If we are interested in studying finite mass monopoles we may as well just suppose that $(A, \Phi \neq 0)$ is a pair (not necessarily a monopole) satisfying the conclusion of the two previous bullets. We shall now suppose that G is either $SO(3)$ or $SU(2)$, then as $\nabla_{A_\infty} \Phi_\infty = 0$, the connection A_∞ is reducible to an HYM connection on a $\mathbb{S}^1$-sub1bundle $Q_\infty \subset P_\infty$, and

$$E^I \quad = \quad -2\pi m \langle c_1(L) \cup [i^* \psi], [\Sigma] \rangle + \frac{1}{2} \| F_A \wedge \psi - * \nabla_A \Phi \|^2_{L^2}, \tag{3}$$

where $[i^* \psi] \in H^4(\Sigma, \mathbb{R})$ denotes the restriction of $[\psi]$ to any cross section $\varphi^{-1}(\{r\} \times \Sigma)$ over the conical end $X \backslash K$ and L denotes the complex line bundle associated with Q_∞ with respect to the standard representation. Moreover, if one further supposes that $c_1(L) \cup [i^* \psi] = 0$ or (X, g) has rate $\nu < -4$, then there are no such finite mass, irreducible monopoles on P.
The classes $c_1(L) \in H^2(X, \mathbb{Z})$ are called monopole classes. Given a coassociative $N \subset X$, Poincaré duality gives a class $PD[N] \in H^3_{cs}(X, \mathbb{Z})$ and the long exact sequence

$$\ldots \to H^2(\Sigma, \mathbb{Z}) \xrightarrow{i} H^3_{cs}(X, \mathbb{Z}) \xrightarrow{j} H^3(X, \mathbb{Z}) \to \ldots,$$

shows that if $PD[N] \in \ker(j)$, then there is a monopole class α such that $i(\alpha) = PD[N]$. In the rest of the talk I shall try to convince you that there is a relation between monopoles on principal $SO(3)$ or $SU(2)$-bundles P asymptotic to a line bundle with class $\alpha \in H^2(\Sigma, \mathbb{Z})$ and coassociative submanifolds N, such that $PD[N] = i(\alpha)$. We will do this by looking to the 3 examples of AC G_2 manifolds we currently have at hand, namely the Bryant-Salamon manifolds [1].

The first example is the spin bundle over the 3-sphere $\mathcal{S}(\mathbb{S}^3)$, which has no compact coassociative submanifolds. As for monopoles, we can use the energy formula above to prove a vanishing theorem for monopoles.

Theorem 1. *Let P be an $SU(2)$ or $SO(3)$ bundle over $\mathcal{S}(\mathbb{S}^3)$. Then, there are no finite mass $m \neq 0$, irreducible monopoles (A, Φ) such that $|\nabla^j (\varphi^* A - \pi^* A_\infty)| = O(r^{-5-\epsilon-j})$, for some $\epsilon > 0$ and all $j \in \mathbb{N}_0$.*

This is an immediate consequence of the energy formula 3 and the fact that $\mathcal{S}(\mathbb{S}^3)$ is asymptotic to a cone over $\mathbb{S}^3 \times \mathbb{S}^3$ which has no second cohomoly. Hence, for any such monopole $E^I = 0$ and so $\nabla_A \Phi = 0$ and A is reducible (as $m \neq 0$). Alternatively, notice that $\mathcal{S}(\mathbb{S}^3)$ retracts onto $\mathbb{S}^3$, hence $H^4(\mathcal{S}(\mathbb{S}^3), \mathbb{R}) = 0$ and so $[\psi] = 0$.

Now turn to the other Bryant-Salamon manifolds, namely $\Lambda^2_- M$, i.e. the total spaces of the bundle of anti-self-dual 2-forms on $M = \mathbb{CP}$ or $\mathbb{S}^4$. Notice that in both cases These examples are very symmetric, in each case there is a compact Lie group K acting on $\Lambda^2_-(M)$ with cohomogeneity 1. Let P be a K-homogeneous principal G-bundle, i.e. the K-action on $\Lambda^2_-(M)$ lifts to the total space P. Then there is a notion of K-invariant connections and G=Higgs fields on P. Let $\mathcal{G}_{inv}$ denote the K-invariant gauge transformations, the moduli space of finite mass, invariant monopoles on $P \to \Lambda^2_-(M)$ is defined as

(4) $\mathcal{M}_{inv}(\Lambda^2_-(M), P) = \{\text{finite mass, } K\text{-invariant, irreducible monopoles}\}/\mathcal{G}_{inv}$.

The main result of [4] is

Theorem 2. *On $M = \mathbb{S}^4$ (respectively $M = \mathbb{CP}^2$) there are K-homogeneous principal $SU(2)$ (respectively $SO(3)$) bundles P, such that the moduli spaces $\mathcal{M}_{inv}(\Lambda^2_-(M), P)$ are non empty and the following hold:*

(1) *For all $(A, \Phi) \in \mathcal{M}_{inv}$, $\Phi^{-1}(0)$ is the zero section, and the mass gives a bijection*
$$m : \mathcal{M}_{inv}(\Lambda^2_-(M), P) \to \mathbb{R}^+.$$
(2) *Let $R > 0$, and $\{(A_\lambda, \Phi_\lambda)\}_{\lambda \in [\Lambda, +\infty)} \in \mathcal{M}_{inv}(\Lambda^2_-(M), P)$ be a sequence of monopoles with mass λ converging to $+\infty$. Then there is a sequence $\eta(\lambda, R)$ converging to 0 as $\lambda \to +\infty$ such that for all $x \in M$*
$$\exp^*_\eta(A_\lambda, \eta\Phi_\lambda)|_{\Lambda^2_-(M)_x}$$
converges uniformly to the BPS monopole (A^{BPS}, Φ^{BPS}) in the ball of radius R in $(\mathbb{R}^3, g_E)$. Here $\exp_\eta$ denotes the exponential map along the fibre $\Lambda^2_-(M)_x \cong \mathbb{R}^3$.

(3) *Let $\{(A_\lambda, \Phi_\lambda)\}_{\lambda \in [\Lambda, +\infty)} \subset \mathcal{M}_{inv}$ be the sequence above. Then the translated sequence*
$$\left(A_\lambda, \Phi_\lambda - \lambda \frac{\Phi_\lambda}{|\Phi_\lambda|} \right),$$

converges uniformly with all derivatives to a reducible, singular monopole on $\Lambda^2_-(M)$ with zero mass and which is smooth on $\Lambda^2_-(M)\backslash M$.

The main thing we should infer from this result is that for each fixed mass $m \in \mathbb{R}^+$, there is a unique invariant monopole (A, Φ) on P and that for this monopole $\Phi^{-1}(0)$ is the zero section M. This is a very promising result, indeed there is a unique compact coassociative submanifold on $\Lambda^2_-(M)$ and this is precisely the zero section M. Hence, in these examples a monopole count on P agrees with a count of rigid, compact, coassociative submanifolds. The remaining items investigate the large mass limit of finite mass monopoles. Combined these state that large mass monopoles concentrate on the coassociative submanifold M, with one BPS monopole bubbling off along the transverse directions to M and a reducible monopole left behind on $\Lambda^2_-(M)\backslash M$. More precisely, on a tubular neighborhood of M a large mass monopole (A, Φ) is close to a family of BPS monopoles on the transverse directions to M and outside such a neighborhood (A, Φ) is approximately reducible.

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Explicit "partial resolution" of Ricci-flat Kähler cones

Eleonora Di Nezza

This talk is based on the results in [2] and [3].
Let (L, g_L) a Sasaki-Einstein manifold of real dimension $\dim_{\mathbb{R}} L = 2n+1$. A Sasaki-Einstein manifold (L, g_L) is a complete Riemannian manifold whose metric cone

$$g_{C(L)} = dr^2 + r^2 g_L, \qquad C(L) = \mathbb{R}^+ \times L \tag{1}$$

is a Calabi-Yau manifold (i.e. Kähler and Ricci-flat). The metric in (1) is singular at $r = 0$ unless $L = S^{2n+1}$ and in that case L is the complex space.
The question we wonder about is whether there exists a resolution, namely a complete Ricci-flat Kähler metric on a non-compact manifold X that is asymptotically conical to the cone $C(L)$. In the following I will give an explicit construction of partial resolutions, where the word "partial" stands for the fact that X will still have singularities but at most orbifold singularities.
I will first present a countable infinite number of explicit quasi-regular and irregular Sasaki-Einstein manifolds, denoted by $Y^{p,q}$. Then, I will give an explicit expression of Kähler Ricci-flat metrics that asymptote to the metric cone $C(Y^{p,q})$.

The starting point is an explicit local metric depending on a parameter $a \in \mathbb{R}^+$ that expresses as

$$g = \frac{1-y}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}[d\gamma - cos\theta d\phi]^2$$

$$+w(y)\left[d\alpha + \frac{a-2y+y^2}{6(a-y^2)}(d\gamma - \cos\theta d\phi)\right]^2 \tag{2}$$

where

$$w(y) = \frac{2(a-y^2)}{1-y} \qquad q(y) = \frac{a-3y^2+2y^3}{a-y^2}.$$

One can check that this metric is Einstein and locally Sasaki. Let me observe that since we want g be a metric we work in the range $y \in [y_1, y_2]$, $y_1 < 0 < y_2 < 1$, where y_i are the zeros of the cubic $2y^3 + 3y^2 - a$. What we want to show is that the local expression of the metric extends to a global metric on a compact manifold, that will be topologically $S^2 \times S^3$ and furthermore we want to insure that the Sasaki structure extends globally.
The first step is to choose $\theta \in [0, \pi]$, $\phi, \gamma \in [0, 2\pi]$ and to show that the four-dimensional basis B_4 (we forget the coordinate α) is topologically $S^2 \times S^2$. Then, we look at the coordinate α and we rewrite the metric as

$$g = g_{B_4} + w(y)[d\alpha + B]^2$$

where $B = \frac{a-2y+y^2}{6(a-y^2)}(d\gamma - \cos\theta d\phi)$. In order to get a compact manifold we want α to describe an S^1-bundle over B_4. Thus, we set $\alpha \in [0, 2\pi l]$ and we ask $l^{-1}B$ to be a connection on a $U(1)$-bundle over $S^2 \times S^2$. Let us recall that an $U(1)$-bundle over $S^2 \times S^2$ is characterised by the generators of $H^2(S^2 \times S^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, say p and q. Therefore, the fact of having a $U(1)$-bundle over $S^2 \times S^2$ turns out to be equivalent to require

$$\frac{P_1}{P_2} = \frac{p}{q} \tag{3}$$

where $P_i := \frac{1}{2\pi}\int_{C_i} dB$ are the periods of the two curvature form dB over a basis of cycles C_i. Observe that P_i depends on the parameter a since B does. It can be proven that there is a countable infinite number of values of $a \in (0, 1)$ such that condition (3) is satisfied. Thus, for any such value of a, the Sasaki-Einstein manifold $Y^{p,q}$ is the total space of a $U(1)$-bundle over $S^2 \times S^2$.
Furthermore, the case when the two roots y_i are rational corresponds to a quasi-regular Sasaki-Einstein manifold, while when y_i are not rational the Sasaki-Einstein manifold is irregular. Let me point out that these are the first examples of irregular Sasaki-Einstein manifolds, which had been conjectured by Cheeger-Tian [1] not to exist. To summerize, we have the following result:

Theorem 1.1. *There exist a countably many Sasaki-Einstein manifolds on $S^2 \times S^3$ labelled by two positive integers $p, q \in \mathbb{Z}$, $q < p$, given explicitely in local coordinates by (2). The manifolds are cohomogeneity one. Furthermore, the Sasaki structeures are qquasi-regular if and only if $4p^2 - 3q^2 \in \mathbb{Z}$; otherwise they are irregular.*

Now, consider the cone over these Sasaki structures, $C(Y^{p,q})$. We want to look at Ricci-flat Kähler metrics that are asymptotic to the metric cone $g_{C(Y)} = dr^2 + r^2 g_{SE}$.
Once again the starting point is an explicit local expression of a Kähler metric

$$g = (1-x)(1-y)g_{\mathbb{P}^1} + \frac{y-x}{4X(x)}dx^2 + \frac{y-x}{4Y(y)}dy^2 + \frac{X(x)}{y-x}[d\tau + (1-y)(d\psi + A)]^2$$

$$+\frac{Y(y)}{y-x}[d\tau + (1-x)(d\psi + A)]^2 \tag{4}$$

and $dA = 2\omega_{FS}$, where ω_{FS} denotes the Fubini-Study form on $\mathbb{CP}^1$ and $g_{\mathbb{CP}^1}$ is the standard metric on $\mathbb{CP}^1$. Such a Kähler metric is Ricci-flat if and only if the metric functions are given by

$$X(x) = \frac{p_1(x)}{(x-1)}, \qquad p_1(x) = (x-1)^2 + \frac{2}{3}(x-1)^3 + 2\mu, \;\; \mu \in \mathbb{R}$$

$$Y(y) = \frac{p_2(y)}{(1-y)}, \qquad p_2(x) = (1-y)^2 - \frac{2}{3}(1-y)^3 - 2\nu, \;\; \nu \in \mathbb{R}.$$

Observe that the metric is symmetric in x and y but we break this symmetry by choosing x to be the "radial" coordinate and y the "polar" one. It can be proved, by changing variables ($x = \pm r^2$), that $g \to dr^2 + r^2 g_Y$ as $x \to \pm\infty$, where g_Y is the Sasaki-Einstein metric we considered above.
The goal is to extend the local metric in (4) to a metric on a non-compact manifold. In this case we fix the range $y \in [y_1, y_2]$ and $x < x_-$ or $x > x_+$, $y_1 \leq x_- < 0 < y_2 < 1 \leq x_+$ where y_i and $x_\pm$ are roots of $p_2(y)$ and $p_1(x)$ respectively. In a similar way of what we have described above it can be shown that, for any fixed $x < x_-$ or $x > x_+$, the local metric can be extended to the total space of a $U(1)$-bundle over $S^2 \times S^2$, that we will denote by $L^{p,k}$. Here p, k are positive integers such that $p < k < 2p$.
And, since the analysis was independent of x, the kähler Ricci flat metric g extends to a asymptocically conical metric on $\mathbb{R}^+ \times L^{p,k}$ and the link of the cone is the Sasaki-Einstein $Y^{p,k}$.
The last step is to examite the regularity of such a metric at $\{x = x_\pm, y = y_i\}$, locus of orbifold singularities. The generic situation that will happen is the conical singularity replaced by a divisor M with at most orbifold singularities. It can be also shown (thanks to the explicit description of the metric) that M is a the total space of a $\mathbb{WCP}^1_{[r,p-r]}$-fibration over $\mathbb{CP}^1$, $0 < r < k/2$, where $\mathbb{WCP}^1_{[r,p-r]}$ is the weighted projective space with singularities at $y = y_1$ and $y = y_2$.

Theorem 1.2. *For every $p, k, r \in \mathbb{N}$ with $p < k < 2p$, $0 < r < k/2$, there is an explicit Ricci-flat Kähler orbifold metric on the total space of the canonical line bundle K_M over the Fano orbifold*

$$M = \mathcal{O}(-m) \times_{U(1)} \mathbb{WCP}^1_{[r,p-r]}$$

where $m = k - 2r$. The metrics asymptotes to a cone over the Sasaki-Eisntein manifold $Y^{p,k}$.

This construction can be actually generalised to any dimension, namely one can start from any complete Fano manifold (V, g_V) of complex dimension n instead of $(\mathbb{P}^1, g_{\mathbb{P}^1})$. For some particular choice of the parameters p and k the partial resolution will be smooth and can be thought as a generalisation in higher dimension of the small resolution $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$ of the conifold $\{(z_0, z_1, z_2, z_3) : \sum_i z_i^2 = 0\}$ in complex dimension 3.

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Deformations of Nearly Kähler manifolds

Uwe Semmelmann

(joint work with Andrei Moroianu, Paul-Andi Nagy)

In this talk we presented the results of [1], [2] and [3] on the infinitesimal deformations of 6-dimensional nearly Kähler manifolds.

Let M^6 be a complete 6-dimensional nearly Kähler manifolds, ie. we have a $SU(3)$-structure (J, g, ω, Ψ^+), consisting of a Riemannian metric g, a compatible almost complex structure J, the fundamental 2-form ω, defined by $\omega(X,Y) = g(JX,Y)$ and a 3-form Ψ^+, which is the real part of a complex volume form $\Omega = \Psi^+ + i\Psi^-$, with $\Psi^- := *\Psi^+$. Moreover these tensors satisfy the equations

$$d\omega = 3\Psi^+ \qquad \text{and} \qquad d\Psi^- = -2\omega \wedge \omega \ .$$

The existence of such a structure implies that (M, g) is a compact Einstein manifold of scalar curvature $scal = 30$. A direct consequence of the structure equations are in particular the additional equations

$$d\Psi^+ = 0, \quad d^*\Psi^- = 0, \quad \Delta\omega = 12\omega \ .$$

In the following we want to derive equations characterizing the tangent vectors to curves $(g_t, J_t, \omega_t.\Psi_t^{\pm})$ of nearly Kähler structures, ie. infinitesimal deformations. From results of Friedrich and Verbitsky it is clear that one has to deform J and g simultaneously.

We first describe the infinitesimal deformations of a $SU(3)$-structure. For this we have to decompose the space of endomorphisms and forms as $U(3)$-representations. We write $End(TM) = End^+(TM) \oplus End^-(TM)$, where $End^+(TM)$ is the space of endomorphisms commuting with J and $End^-(TM)$ are the endomorphisms anit-commuting with J. Corresponding to the decomposition

$$End(TM) = Sym^2(TM) \oplus \Lambda^2(TM)$$

into symmetric and skew-symmetric endomorphisms. We have the splittings

$$End^+(TM) = Sym^+(TM) \oplus \Lambda^{11}(TM)$$

and

$$End^-(TM) = Sym^-(TM) \oplus \Lambda^{(2,0)+(0,2)}(TM)$$

Here $\Lambda^{11}(TM)$ are the skew-symmetric endomorphisms commuting with J, or equivalently the J-invariant 2-forms. This space is isomorphic to the Lie algebra of $U(3)$. Moreover the space $\Lambda^{(2,0)+(0,2)}(TM)$ of skew-symmetric endomorphisms anti-commuting with J, or equivalently 2-forms anti-commuting with J, is isomorphic to TM, via the isomorphism $\xi \mapsto \Psi^+_\xi = \xi \lrcorner \Psi^+$. Finally we have the following decomposition of the space of 3-forms

$$\Lambda^3(TM) = \Lambda^{(3,0)+(0,3)}(TM) \oplus [\Lambda^1(TM) \wedge \omega \oplus \Lambda_0^{(2,1)+(1,2)}(TM)]$$

The first summand is spanned by Ψ^+ and Ψ^-. The summand $\Lambda_0^{(2,1)+(1,2)}(TM)$, of primitive 3-forms of type $(2,1)+(1,2)$, is isomorphic to $Sym^-(TM)$ under the isomorphism $S \mapsto S_*\Psi^+ := \sum S(e_i) \wedge e_i \lrcorner \Psi^*$, for some ortho-normal frame $\{e_i\}$.

Let $\dot g, \dot J, \dot\omega, \dot\Psi^\pm$ denote the derivative of the $g_t, J_t, \omega_t, \Psi_t^\pm$ at $t = 0$. Then corresponding to the decompositions introduced above we have $\dot g = g((h+S)\cdot,\cdot)$ with $h \in Sym^+(TM)$ and $S \in Sym^-(TM)$, $\dot J = J \circ S + \Psi^+_\xi$ for some $\xi \in TM$ and $\dot\omega = \phi + \Psi^+_\xi$ for $\phi \in \Lambda^{11}(TM)$, with $\phi(X,Y) = g(h \circ J\cdot,\cdot)$. Moreover we have the splitting

$$\dot\Psi^+ = \lambda\Psi^+ + \mu\Psi^- - \frac{1}{2}S_*\Psi^+ - \xi \wedge \omega$$

and

$$\dot\Psi^- = -\mu\Psi^+ + \lambda\Psi^- - \frac{1}{2}S_*\Psi^+ - J\xi \wedge \omega$$

where μ is some function and $\lambda = \frac{1}{4}tr(h) = \frac{1}{4}tr(\dot g)$. Hence infinitesimal deformations of $SU(3)$-structures are parametrized by a vector field ξ, a symmetric endomorphism S, anti-commuting with J, a J-invariant (11)-form ϕ and a function μ. Assuming that we have a family of nearly Kähler structures we obtain the additional equations:

$$d\dot\omega = 3\dot\Psi^+ \qquad \text{and} \qquad d\dot\Psi^- = -4\dot\omega \wedge \omega\ .$$

As usual we only consider deformations transversal to the action of the diffeomorphism group and we consider a family of metrics with a fixed volume. This gives the further equations $\delta\dot g = 0$ and $tr_g(\dot g) = 0$, which immediately leads to $\lambda = 0$. Using all these equations we obtain

$$\dot\Psi^+ = -\tfrac{1}{2}S_*\Psi^+, \quad \dot\Psi^- = -\tfrac{1}{2}S_*\Psi^-, \quad \dot\omega = \phi \in \Omega_0^{11}(M)\ .$$

It follows that ϕ is a primitive, co-closed $(1,1)$-form with $\Delta\phi = 12\phi$ and that such forms completely describe infinitesimal deformations of nearly Kähler structures. Indeed given such a form $\phi \in \Omega_0^{11}(M)$ with $d^*\phi = 0$ and $\Delta\phi = 12\phi$. We can show that $d\phi \in \Omega_0^{(12)+(21)}(M)$. Thus there exists a uniquely determined endomorphism $S \in Sym^-(TM)$ with $d\phi = -\frac{3}{2}S_*\Psi^+$. Then $\lambda = \mu = 0, \xi = 0, \dot\omega = \phi, \dot\Psi^\pm = -\frac{1}{2}S_*\Psi^\pm$ is a nearly Kähler deformation satisfying the set of linearized structure equations. Let V be the space of infinitesimal deformations of nearly Kähler structures, ie.

$$V = \{\phi \in \Omega^{11}_0(M) \,|\, d^*\phi = 0,\ \Delta\phi = 12\phi\}\ .$$

We want to show how to compute the space V for the homogeneous examples. All these examples are 3-symmetric spaces $M = G/K$ with a compact group G. The metric is given as $g = -\frac{1}{12}B$, where B is the Killing form of G. Hence these spaces are in particular natural reductive.

Let $\pi : K \to Aut(E)$ be any complex representation of K and let $EM := G \times_\pi E$ be the associated vector bundle then one has the following isomorphism of G-representations

$$L^2(EM) \cong \overline{\oplus}_{\gamma\in\hat{G}} V_\gamma \otimes Hom_K(V_\gamma, E)\ .$$

Here $\hat{G}$ is the set of isomorphism classes of irreducible G-representations and V_γ denotes the irreducible G-representation with highest weight γ. The group G acts on the space $L^2(EM)$ of square-integrable sections of EM via the left-regular representation.

We introduce the Hermitian Laplace operator $\bar\Delta$ defined as $\bar\Delta = \bar\nabla^*\bar\nabla + q(\bar R)$, where $q(\bar R)$ is a certain endomorphism of EM linear in the curvature $\bar R$ of the connection $\bar\nabla$.

It is well-known that 3-symmetric spaces have a canonically defined almost-complex structure and it is a remarkable fact that in the case of a naturally reductive metric the canonical hermitian connection coincides with the canonical homogeneous connection. In consequence the action the operator $\bar\Delta$ on sections of EM is given by the action of the Casimir operator Cas^G_π under the identification above. Let $\{X_i\}$ be a ortho-normal basis of the Lie algebra $\mathfrak{g}$. Then $Cas^G_\pi := \sum_i \pi_*(X_i)^2$ and it is well known that $Cas^G_{V_\gamma} = -<\gamma, \gamma + 2\rho>$, where ρ is half the sum of positive roots of $\mathfrak{g}$. This allows to compute spectrum of the Hermitian Laplace operator on sections of any homogeneous vector bundle EM. To compute the space of infinitesimal nearly Kähler deformations we further need the surprising fact that Δ and $\bar\Delta$ coincide on co-closed, primitive (11)-forms.

The method introduced above allows to compute the $\bar\Delta$ eigenspace $\Omega^{11}_0(12)$ of primitive (11)-forms for the eigenvalue 12. But some of the eigenforms could be non co-closed. Indeed let $f \in \Omega^0(12)$ be any Δ-eigenfunction for the eigenvalue 12 and let ξ be any Killing vector field. Then the projections of $d\xi$ and of $dJdf$ to the space of primitive (11)-forms are non-vanishing, non-coclosed $\bar\Delta$ eigenforms for the eigenvalue 12. It follows that

$$\dim V \le \dim \Omega^{11}_0(12) - \dim Iso(M, G) - \dim \Omega^0(12)\ .$$

Using the inequality and the computation of the spectrum of $\bar\Delta$ it follows that there are no infinitesimal deformations on the nearly Kähler spaces $S^6, \mathbb{C}P^3, S^3 \times S^3$ and that $\dim V \le 8$ for the flag manifold F_3.

Finally an explicit calculation shows that F_3 has a 8-dimensional space of infinitesimal nearly Kähler deformations. It is isomorphic to the Lie algebra of $SU(3)$. In fact, let h_1, h_2, h_3 be the standard basis of the Lie algebra of the maximal torus in $U(3)$ and let $e_1, \ldots, e_6$ be the usual real root vectors of $su(3)$. Then

$\omega := e_{12} - e_{34} + e_{56}$ and $\Psi^+ := e_{136} + e_{246} + e_{235} - e_{145}$ define left-invariant forms which project to the flag manifold $F_3 = U(3)/T^3$ and define the standard nearly Kähler structure of F_3. For any $\xi \in su(3)$ let X be the right-invariant vector field defined by ξ. Then the functions $v_i := g(X, h_i), i = 1, 2, 3$, where g is minus the Killing form of $U(3)$, project to F_3 and it can be shown by direct calculations that

$$\phi := v_1 e_{56} - v_2 e_{36} + v_3 e_{12}$$

is a co-closed, primitive $(1,1)$-form satisfying $\Delta\phi = 12\phi$. Hence ϕ defines an infinitesimal nearly Kähler deformation.

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Evolution of G_2-structures

Frederik Witt

(joint work with Hartmut Weiß)

Following [7] and [8] we discuss the evolution of G_2-structures along the negative gradient flow of a natural energy functional.

The Dirichlet functional. We assume throughout that M is compact, oriented and spin. Then Ω_+, the subset of G_2-forms inside $\Omega^3(M)$ which are compatible with the given orientation, is non-empty. The quantities associated with the choice of $\omega \in \Omega_+$ such as a Riemannian metric will be written as g_ω etc.. We define the *Dirichlet functional* by

$$\mathcal{D} : \Omega_+ \to \mathbb{R}, \quad \omega \mapsto \tfrac{1}{2}\int_M (|d\omega|^2_{g_\omega} + |\delta^\omega \omega|^2_{g_\omega})\mathrm{vol}_{g_\omega},$$

where δ^ω denotes the formal adjoint of d. This functional is invariant under $\mathit{Diff}(M)_+$, the orientation preserving diffeomorphisms. Since Ω_+ is open, $\mathcal{D}$ can be differentiated. The only critical points are absolute minimisers, that is, the torsion-free forms characterised by $d\omega = 0$ and $\delta^\omega\omega = 0$.

A natural way of deforming G_2-structures is to consider their evolution under the negative gradient flow of $\mathcal{D}$ to which we refer as the *Dirichlet flow*. The $\mathit{Diff}(M)_+$-invariance gives rise to a non-trivial kernel of the principal symbol of the linearised gradient $D_\omega \mathrm{grad}\,\mathcal{D}$. However, the symbol is non-negative with kernel tangent to the $\mathit{Diff}(M)_+$-orbits. Along the lines of deTurck's trick for Ricci flow [1], we define a geometrical perturbation $P_{\tilde\omega}$ of $-\mathrm{grad}\,\mathcal{D}$ depending on a fixed form $\tilde\omega \in \Omega_+$. This new operator is strongly elliptic so that standard parabolic theory applies to give

Theorem 3 (Short-time existence and uniqueness). *Given a* G_2*-form* ω_0 *in* Ω_+ *there exists* $\epsilon > 0$ *and a smooth family* $\omega_t \in \Omega_+$ *for* $t \in [0, \epsilon]$ *such that*

$$\frac{\partial}{\partial t}\omega = -\mathrm{grad}\,\mathcal{D}(\omega), \quad \omega(0) = \omega_0.$$

Furthermore, for any two solutions ω_t *and* $\tilde{\omega}_t$ *we have* $\omega_t = \tilde{\omega}_t$ *whenever defined.*

The moduli space. Next let $\tilde{\omega}$ be a torsion-free G_2-form, and assume for simplicity that M satisfies in addition $H^1(M, \mathbb{R}) = 0$. Then $P_{\tilde{\omega}}(\omega) = 0$ if and only if ω is torsion-free and ω is perpendicular to the tangent space of the orbit of $\tilde{\omega}$ under $\mathit{Diff}(M)_0$, the diffeomorphisms isotopic to the identity. Put differently, $P_{\tilde{\omega}}^{-1}(0)$ defines a *slice* near $\tilde{\omega}$ for the $\mathit{Diff}(M)_0$-action on $\mathcal{X} = \{\omega \in C^\infty(\Lambda^3_+ M) \,|\, d\omega = 0,\, \delta^\omega \omega = 0\}$. Hence the G_2-analogon of Teichmüller space $\mathcal{M}_{G_2} = \mathcal{X}/\mathit{Diff}(M)_0$ is a smooth manifold. Further, a Hodge theoretic argument shows that $T_{\tilde{\omega}} P_{\tilde{\omega}}^{-1}(0)$ is isomorphic with $H^3(M, \mathbb{R})$. In this way we recover the

Theorem 4 (Joyce [5]). *The moduli space of torsion-free* G_2*-structures* $\mathcal{M}_{G_2}$ *is a smooth manifold of dimension* b_3.

Stability. Let us now examine the question of long-time existence and convergence. One can show that if ω_t exists for all times then $\lim_{t\to\infty} \mathcal{D}(\omega_t) = 0$, but ω_t might not converge (cf. [8, Corollary 2.3] and the discussion thereafter). On the other hand, consider a so-called G_2-*soliton* which is a G_2-form Ω such that

$$-\mathrm{grad}\,\mathcal{D}(\omega) = \mu\omega + \mathcal{L}_X \omega$$

for $\mu \in \mathbb{R}$ and $X \in \Gamma(TM)$. Examples are provided by so-called *weak holonomy* or *nearly parallel* G_2-manifolds [2, 3] which are characterised by $d\omega = c \star_{g_\omega} \omega$ for a constant $c \neq 0$. One can show that if Ω is not torsion-free, then a G_2-soliton is necessarily a shrinker, i.e. $\mathcal{L}_X \Omega = 0$ and $\mu < 0$. The Dirichlet flow starting in such a soliton necessarily dies in finite time since the total volume of (M, g_{ω_t}) shrinks to zero. However, in the vicinity of a torsion-free form, we can show:

Theorem 5 (Stability). *Let* $\tilde{\omega} \in \Omega_+$ *be a torsion-free* G_2*-form. For initial conditions sufficiently* C^∞*-close to* $\tilde{\omega}$ *the Dirichlet flow exists for all times and converges modulo diffeomorphisms to a torsion-free* G_2*-form.*

The key properties of the flow we use here are "linear stability", i.e. $D^2_{\tilde{\omega}}\mathcal{D} \geq 0$ and the smoothness of the moduli space. Unlike for similar stability theorems for Ricci flow (cf. [6]) these properties hold automatically and need not to be imposed. A main ingredient for longtime existence is uniform existence of the Dirichlet flow on $[0, 1]$ for starting points close to $\tilde{\omega}$. This is done by an implicit function theorem argument in the vein of [4]. Convergence modulo diffeomorphisms comes from the analysis of the perturbed flow $\partial_t \omega = P_{\tilde{\omega}} \omega_t$ and parabolic regularity.

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The asymptotic geometry of the Higgs bundle moduli space over a Riemann surface

Jan Swoboda
(joint work with Rafe Mazzeo, Hartmut Weiß, and Frederik Witt)

The moduli space of Higgs bundles, introduced by Hitchin [5] and Simpson [10], is a well investigated object in algebraic geometry and topology. In our talk we discussed our recent results in [7, 8] which concern aspects of its large scale Riemannian geometry. Hitchin showed that there exists a natural hyperkähler metric on the smooth locus of the moduli space; in many cases the moduli space has no singularities and the metric is complete. However, its asymptotics are still not well understood. We explained our results concerning the degeneration profile of points in the moduli space representing configurations with large Higgs field.

There are several reasons to study this metric carefully. The first is to understand the L^2-cohomology of this space. Hausel proved [3] that the image of the compactly supported cohomology in the ordinary cohomology vanishes, leading him to conjecture that the L^2-cohomology of the Higgs bundle moduli space must vanish. This was made in analogy with Sen's conjecture about the L^2-cohomology of the monopole moduli spaces [9]. Hitchin proved a rather general result [6] showing that under conditions satisfied in both these cases, the L^2-cohomology vanishes outside the middle degree. Hausel's conjecture remains open. Following, for example, the approach of [4], an understanding of this middle-degree cohomology relies on some finer knowledge of the metric structure at infinity.

However, this is part of a much broader picture concerning hyperkähler metrics on algebraic completely integrable systems. Indeed, the work of Gaiotto, Moore and Neitzke [1, 2] hints at an asymptotic development of this hyperkähler metric g, where the leading term is a so-called semiflat metric and the correction terms decay at increasingly fast exponential rates. The exponents and coefficients of these correction terms are described in terms of expressions coming from a wall-crossing formalism, but these are unfortunately a priori divergent. Clarifying this circle of ideas is a high priority.

The goal of this talk was to discuss the main results of [7], which constructs a dense open subset near infinity in the moduli space of Hitchin's self-duality equations. The degeneration behavior of generic solutions is captured by the notion of **limiting configurations**. These constitute a family of singular solutions to the self-duality equations (1) below which give a geometric realization of the elements of the top stratum in the compactification of the moduli space. As a second result, we presented a desingularization theorem for limiting configurations.

Let Σ be a closed Riemann surface, i.e. a compact (orientable) surface endowed with a complex structure. We assume that the genus γ of Σ is at least 2. We also fix a complex vector bundle $E \to \Sigma$ of rank $r = r(E)$ and degree $d = d(E)$. The pair (r, d) determines E as a smooth bundle. We furthermore fix a hermitian metric h on E and consider the system of nonlinear PDEs, called Hitchin's **self-duality equations**

$$\text{(1)} \qquad \begin{cases} \bar{\partial}_A \Phi = 0, \\ F_A^\perp + [\Phi \wedge \Phi^*] = 0, \end{cases}$$

where A is a unitary connection on (E, h) inducing a fixed connection on the determinant of E, $F_A^\perp$ denotes the pure-trace part of its curvature, and $\Phi \in \Omega^{1,0}(\Sigma, \mathrm{End}_0(E))$ is a so-called **Higgs field**. Clearly, solutions (A, Φ) of (1) are invariant under special unitary gauge transformations $g \in \mathcal{G}(E, h)$. To explain our results, we specialize to the case $r = 2$ and odd degree d and define the moduli space

$$\mathcal{M}_d = \{(A, \Phi) \mid (1)\}/\mathcal{G}(E, h).$$

It is a noncompact smooth manifold of dimension $12(\gamma - 1)$. The map

$$\det \colon \mathcal{M}_d \to H^0(\Sigma, K_\Sigma^2), \quad [(A, \Phi)] \mapsto \det \Phi$$

to the vector space $H^0(\Sigma, K_\Sigma^2)$ of holomorphic quadratic differentials on Σ is proper. It gives rise to the so-called **Hitchin fibration**, the typical fibre being a half-dimensional complex torus. To motivate the kind of behaviour one should expect of solutions with large Higgs field, we replace Φ in the second equation of (1) by $t\Phi$, where $t > 0$ is a large parameter. The limit of solutions where $t \to \infty$ is described by the decoupled **limiting equations**

$$\text{(2)} \qquad \begin{cases} \bar{\partial}_A \Phi = 0, \\ F_A^\perp = [\Phi \wedge \Phi^*] = 0. \end{cases}$$

Theorem 6 (Limiting configurations)**.** *For a holomorphic quadratic differential $q \in H^0(\Sigma, K_\Sigma^2)$ with only simple zeroes , let $\Sigma^\times = \Sigma \setminus q^{-1}(0)$. Then there exists a $6(\gamma - 1)$-torus of solutions (A, Φ) of (2) such that $\det \Phi = q$, (A, Φ) is smooth on the punctured surface $\Sigma^\times$, and A has a pole of order 1 in the points of $q^{-1}(0)$. After a modification by a unitary gauge transformation if necessary, (A, Φ) equals in a neighborhood of $q^{-1}(0)$ the singular model solution*

$$\text{(3)} \qquad A_\infty^{\mathrm{fid}} = \frac{1}{8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right), \quad \Phi_\infty^{\mathrm{fid}} = \begin{pmatrix} 0 & |z|^{\frac{1}{2}} \\ \frac{z}{|z|^{\frac{1}{2}}} & 0 \end{pmatrix} dz.$$

To explain our next result, we point out that the singular local solution in (3) can be desingularized; i.e. there exists a family of smooth solutions $(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}})$ on the unit disk $\mathbb{D} \subseteq \mathbb{C}$ which converges to $(A_\infty^{\mathrm{fid}}, \Phi_\infty^{\mathrm{fid}})$ as $t \to \infty$. Rescaling $\rho = \frac{8}{3}t|z|^{\frac{2}{3}}$, it admits a rather explicit description in terms of a single function $\psi(\rho)\colon [0,\infty) \to \mathbb{R}$ which arises as solution to a Painlevé III equation. By gluing in these special solutions $(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}})$ to a limiting configuration (A, Φ) we obtain smooth solutions to Eq. (1) with "large" Higgs fields.

Theorem 7 (Glueing)**.** *For each limiting configuration (A_∞, Φ_∞) as in Theorem 6 and sufficiently large parameter $t > 1$, there exists a solution $(A_t, t\Phi_t)$ of Eq.* (1) *such that*

$$(A_t, \Phi_t) \to (A_\infty, \Phi_\infty)$$

at an exponential rate, locally uniformly on $\Sigma^\times$. Conversely, any solution $(A, t\Phi)$ of (1) *such that* $\det \Phi$ *has only simple zeroes is of this form, provided $t > 1$ is sufficiently large.*

We furthermore explained an extension of this last theorem to the case of a general holomorphic quadratic differential q using a related family of model solutions as obtained in [8]. Finally, it was described how these results may help in understanding asymptotic properties of the hyperkähler metric on $\mathcal{M}_d$.

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