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Differentialgeometrie im Großen

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Abstract. The topics discussed at the meeting were Kaehler geometry, geometric evolution equations, manifolds of nonnegative curvature, metric geometry and geometric representations of groups. The choice of topics reflects current trends in the development of differential geometry.

Mathematics Subject Classification (2010): 53C.

Introduction by the Organisers

The workshop *Differentialgeometrie im Großen* was held June 29- July 3, 2015. The participants were specialists in differential geometry and its neighboring fields, covering a broad spectrum of subareas which are in the focus of current developments.

The lectures during the five days of the meeting were roughly organized according to different thematic themes.

The first day of the meeting, was devoted to differential geometric aspects of Kähler geometry and special holonomy.

The talks of the second day centered around the interplay between differential geometry and geometric analysis, in particular new developments concerning the Ricci flow and geometric measure theory.

On Wednesday (with only three lecture due to the traditional hike), metrics of non-negative curvature were discussed.

The morning lectures of the last two days were mainly devoted to metric geometry and geometric representations of groups. On Thursday afternoon, four young people gave short talks.

The meeting gave a good overview of the current developments, and showed significant progress in the field. The workshop was attended by researchers from around the world, ranging from graduate students to scientific leaders in their areas.

The atmosphere during the meeting was lively and open, and greatly benefited from the ideal environment at Oberwolfach.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Differentialgeometrie im Großen

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Abstracts

Desingularisation of Einstein manifolds

Olivier Biquard

I describe a work contained in the articles [2] and [3]. The first article is about a new obstruction to desingularizing Einstein orbifolds with singularity of the type $\mathbb{C}^2/\mathbb{Z}_2$. The second article is on more precise information on the desingularization in the setting of asymptotically hyperbolic manifolds and on a 'wall crossing formula' arising in that case.

In this report I focus on the second part, since the first part was already described in the Oberwolfach report [1].

The setting is the following. Let M^4 be a compact 4-manifold with boundary X^3. Let x be a defining function of X. Then a Riemannian metric g on M is said to be asymptotically hyperbolic (AH) if near X it has the behaviour

$$g \sim \frac{dx^2 + \gamma}{x^2},$$

where γ is a metric on X. Actually, only the conformal class of γ is well defined and is called the conformal infinity of g.

Dirichlet problem for asymptotically hyperbolic metrics. Given a conformal class $[\gamma]$, find an AH metric g on X such that in addition g is Einstein:

$$\mathrm{Ric}(g) = -3g.$$

This problem, or at least its local version near X, originates to Fefferman-Graham in the 80's as a part of their program to study of conformal metrics.

Now suppose that M contains a 2-sphere of self-intersection -2, and let M_0 be the orbifold obtained from M by contracting the sphere to a point. The resulting point p_0 is an orbifold point, with a singularity of type $\mathbb{C}^2/\mathbb{Z}_2$. Suppose that (M_0, g_0) is an AH Einstein orbifold metric with conformal infinity $[\gamma_0]$ on $X = \partial M_0 = \partial M$. If g_0 is nondegenerate (meaning that the L^2-kernel of the linearization of the Einstein equation vanishes at g_0), then for any nearby γ on X, one can find an orbifold AH Einstein metric $g_0(\gamma)$ which is a solution of the Dirichlet problem on M_0 with conformal infinity $[\gamma]$.

It was shown in [2] that g_0 can be desingularized by a family of AH Einstein metrics on M if g_0 satisfies a curvature condition at the point p_0, namely

$$\det R_+^{g_0}(p_0) = 0,$$

where $R_+^{g_0}$ is the part of the curvature operator acting on selfdual 2-forms (recall that the curvature operator is a symmetric endomorphism of the 2-forms). The desingularized metrics have a varying conformal infinity, and the main result I want to explain is that there is a constraint on which conformal infinities are possible.

Let C be the space of conformal metrics on X, and

$$C_0 = \{[\gamma] \in C, \det R_+^{g_0(\gamma)}(p_0) = 0\}.$$

So C_0 is the space of conformal infinities such that the corresponding Einstein metric can be desingularized. Of course this makes sense only in a neighbourhood of $[\gamma_0]$.

Theorem. Suppose g_0 is generic in the sense that $\mathrm{rk} R_+^{g_0}(p_0) = 2$ (the maximal possible rank for a non invertible endomorphism of the selfdual 2-forms). Then C_0 is a smooth hypersurface of C at $[\gamma_0]$, and all the desingularizations of the Einstein metrics $g_0(\gamma)$ for $[\gamma] \in C_0$ have their conformal infinity on the side of C_0 given by

$$\det R_+^{g_0(\gamma)} > 0.$$

This result fits well with the known example of the 4-ball modulo $\mathbb{Z}_2$, where the orbifold hyperbolic metric has $\det R_+(p_0) < 0$ and its conformal infinity (the round metric on $S^3/\mathbb{Z}_2$) is probably never the conformal infinity of an AH Einstein metric.

More generally, the result can be interpreted as a wall crossing formula for a conjectural degree proposed by Anderson to count the number of solutions to the Dirichlet problem. The number of solutions increases by 1 when one passes from the side $\det R_+^{g_0(\gamma)} < 0$ to the side $\det R_+^{g_0(\gamma)} > 0$.

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A special Lagrangian type equation for line bundles

Adam Jacob
(joint work with Shing-Tung Yau)

In this talk I will introduce the deformed Hermitian-Yang-Mills equation, which was considered in the physics literature [3] and later described by Leung-Yau-Zaslow in a more geometric setting [2]. First, consider a holomorphic line bundle L over a Calabi-Yau manifold X of complex dimension n. The deformed Hermitian-Yang-Mills equation seeks a metric h on L so that the form $(\omega - F)^n$ has constant argument. Here ω is the Kähler form and F is the curvature of the metric h. For example, if X is a Calabi-Yau three-fold, the equation can be written as

$$\frac{\omega^2}{2} \wedge iF - \frac{(iF)^3}{6} = \tan\hat{\theta} \left(\frac{\omega^3}{6} - \omega \wedge \frac{(iF)^2}{2} \right),$$

where $\hat{\theta}$ is a fixed angle which is independent of the metric h. For an alternate formulation, working in coordinates where F is diagonal with respect to ω with eigenvalues λ_i, the deformed Hermitian-Yang-Mills equation takes the form

$$\sum_i \arctan(\lambda_i) = \hat{\theta}. \tag{1}$$

When a solution exists, it defines a special Lagrangian section of the torus fibration in the mirror manifold, given the semiflat setup from SYZ mirror symmetry (here $\hat{\theta}$ is the analogue of the phase of the Lagrangian).

Our first main observation is that this equation makes sense on any compact Kähler manifold, and we are interested in working in this general setting. As a result, we are motivated by mirror symmetry, but never apply it directly, since when the base is not Calabi-Yau, there is no meaningful notion of a mirror special Lagrangian. We have the following natural conjecture:

Conjecture 1. *Let L be a holomorphic line bundle over a compact Kähler manifold X. Then there exists a solution to equation (1) if and only if for all irreducible holomorphic subvarieties $V \subset X$, the following inequality holds*

$$Arg\left(-\int_V e^{-i\omega} ch(L)\right) > -Arg\left(\int_X e^{-i\omega} ch(L)\right). \tag{2}$$

Note that the above integrals are independent of a choice of metric on L, and as a result (2) can be viewed as a geometric stability type condition. Along with S.-T. Yau, in [1] we verified this conjecture in the case that X is a Kähler surface by transforming equation (1) into a complex Monge-Ampère equation. Unfortunately this method does not seem to generalize easily to higher dimensions.

To study existence in higher dimensions, we define a parabolic evolution equation which corresponds to Lagrangian mean curvature flow on the mirror side. We show all metrics which solve (1) minimize a positive functional, for which our flow is the gradient flow, and prove the following convergence result:

Theorem 1 (Jacob-Yau [1]). *Let L be an ample line bundle over a compact Kähler manifold X with non-negative orthogonal bisectional curvature. There exists a natural number k so that $L^{\otimes k}$ admits a solution to (1). Furthermore, this solution is given by a smoothly converging family of metrics along the flow.*

Currently, along with T.C. Collins, we are working on removing the condition restricting the curvature of X and proving convergence under assumptions more closely related to (2). However, the ampleness assumption is more central to our argument, since it ensures concavity of the operator. We have a final regularity result:

Theorem 2 (Jacob-Yau [1]). *If the first derivative of the curvature $|\nabla F|$ is uniformly bounded in C^0, then the flow exists for all time and converges to a solution of (1).*

Extending the analogy to mean curvature flow, ∇F plays the role of the second fundamental form. The above result shows this term either blows up, developing a singularity, or the flow converges.

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New G_2 holonomy cones and exotic nearly Kähler structures on the 6-sphere and on the product of two 3-spheres

Mark Haskins

(joint work with Lorenzo Foscolo)

The main focus of this talk is recent work constructing new nearly Kähler structures on both S^6 and on $S^3 \times S^3$. Our nearly Kähler structures provide the first known complete inhomogeneous nearly Kähler structures in dimension 6 and resolve a longstanding open question in almost-Hermitian geometry. These new nearly Kähler structures give rise to novel Einstein metrics of positive scalar curvature on S^6 and $S^3 \times S^3$. Our motivation for studying nearly Kähler 6-manifolds comes from the study of singular spaces with G_2-holonomy; the cone over a nearly Kähler 6-manifold is a cone with holonomy group G_2. Such G_2-holonomy cones provide the local models for the simplest isolated singularities of G_2-holonomy spaces.

The examples we construct are of cohomogeneity one. The proof of existence of our examples also uses insights gained from considering singular nearly Kähler 6-dimensional spaces (sine cone and attempting to desingularise such spaces to obtain smooth nearly Kähler spaces).

Gauduchon metrics with prescribed volume form

Ben Weinkove

(joint work with Gabor Székelyhidi, Valentino Tosatti)

Let M be a compact complex manifold of complex dimension n. A Hermitian metric $g = (g_{i\bar{j}})$ is Kähler if the associated $(1,1)$ form $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is d-closed. Kähler manifolds include all smooth projective varieties.

Not all complex manifolds admit a Kähler metric. For example, the Hopf manifold

$$M = \mathbb{C}^n \setminus \{0\}/\sim, \quad (z^1, \ldots, z^n) \sim (\alpha z^1, \ldots, \alpha z^n),$$

for a fixed $\alpha \in \mathbb{C}$ with $|\alpha| \neq 0, 1$ is a compact complex manifold diffeomorphic to $S^{2n-1} \times S^1$. It cannot admit a Kähler metric since $H^2(M; \mathbb{R}) = 0$.

On the other hand, Gauduchon [4] showed that if (M, ω_0) is a compact Hermitian manifold then there exists a smooth function f, unique up to the addition of a constant, so that $\omega = e^f \omega_0$ satisfies the condition

$$\partial\bar{\partial}(\omega^{n-1}) = 0.$$

Such a metric ω is called *Gauduchon.* Hence every compact complex manifold admits many Gauduchon metrics.

in 1976, Yau [12] proved the Calabi conjecture, which states that the volume form of a Kähler metric can be prescribed. Namely:

Theorem 1 (Yau [12]). *Let (M, ω) be a compact Kähler manifold. Let $F \in C^\infty(M)$ be given, and assume $\int_M e^F \omega^n = \int_M \omega^n$. Then there exists a unique Kähler metric $\tilde{\omega}$ with $[\tilde{\omega}] = [\omega] \in H^{1,1}(M; \mathbb{R})$ such that*

$$\tilde{\omega}^n = e^F \omega^n.$$

There is an equivalent version of Yau's theorem, stated in terms of the first Chern class $c_1(M)$ of the manifold M. We recall the definition

$$c_1(M) := [\mathrm{Ric}(\omega)] \in H^{1,1}(M; \mathbb{R}), \quad \mathrm{Ric}(\omega) := -\sqrt{-1}\partial\bar{\partial} \log \det g,$$

where we ignore the usual factor of 2π. Yau's Theorem can then be restated as follows. Given any representative Ψ of $c_1(M)$ there exists a unique Kähler metric $\tilde{\omega}$ with $[\tilde{\omega}] = [\omega]$ satisfying

$$\mathrm{Ric}(\tilde{\omega}) = \Psi.$$

To see this equivalence, note that given $\Psi \in c_1(M)$, we can define F by

$$\Psi = \mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}F = -\sqrt{-1}\partial\bar{\partial} \log(e^F \det g), \qquad \int_M e^F \omega^n = \int_M \omega^n,$$

and then the result follows by solving $\tilde{\omega}^n = e^F \omega^n$. The converse direction is similar.

Notice that an immediate consequence of Yau's Theorem is that every compact Kähler manifold M with $c_1(M) = 0$ admits a Ricci-flat Kähler metric in every Kähler class.

In 1984 Gauduchon [5] conjectured that there is a version of Yau's theorem for Gauduchon metrics.

Conjecture 1 (Gauduchon [5]). *Let (M, ω) be a compact Gauduchon manifold. Given $\Psi \in c_1^{\mathrm{BC}}(M)$, there exists $\tilde{\omega}$ Gauduchon with*

$$\mathrm{Ric}(\tilde{\omega}) = \Psi.$$

Here, $c_1^{\mathrm{BC}}(M) := [\mathrm{Ric}(\omega)] \in H^{1,1}_{\mathrm{BC}}(M; \mathbb{R})$, where

$$H^{1,1}_{\mathrm{BC}}(M; \mathbb{R}) := \frac{\{d\text{-closed real } (1,1) \text{ forms}\}}{\{\mathrm{Im}\sqrt{-1}\partial\bar{\partial}\}}$$

and $\mathrm{Ric}(\omega) := -\sqrt{-1}\partial\bar{\partial} \log \det g$ is the *Chern-Ricci form* of the metric g.

We remark that, by the same argument as above, this is equivalent to prescribing the volume form of a Gauduchon metric, up to a scaling. Note that there is no uniqueness statement given in the conjecture.

Our main result is as follows:

Theorem 2 (Székelyhidi-Tosatti-Weinkove [8]). *Let (M,ω) be a compact Gauduchon manifold. Given $\Psi \in c_1^{\mathrm{BC}}(M)$ there exists $\tilde{\omega}$ Gauduchon with $\tilde{\omega}^{n-1} - \omega^{n-1} = \partial\gamma + \overline{\partial\gamma}$ for some $(n-2,n-1)$ form γ, such that*

$$\mathrm{Ric}(\tilde{\omega}) = \Psi.$$

In particular, this establishes the conjecture of Gauduchon. We make a few remarks:

(1) The statement that $\tilde{\omega}^{n-1} - \omega^{n-1} = \partial\gamma + \overline{\partial\gamma}$ is equivalent to saying that $[\tilde{\omega}^{n-1}] = [\omega^{n-1}]$ in the *Aeppli cohomology group* $H_A^{n-1,n-1}(M;\mathbb{R})$.

(2) We have the following precise statement about prescribing volume forms of Gauduchon metrics: given $F \in C^\infty(M)$ there exists $\tilde{\omega}$ Gauduchon with $[\tilde{\omega}^{n-1}] = [\omega^{n-1}]$ in Aeppli cohomology, and $b \in \mathbb{R}$ such that $\tilde{\omega}^n = e^{F+b}\omega^n$.

(3) In $n = 2$ this result follows from solving the complex Monge-Ampère equation $(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^F\omega^n$ which was already carried out by Cherrier [1] (see [9] for this equation with $n > 2$). Note that when $n > 2$, ω being Gauduchon does not imply in general that $\omega + \sqrt{-1}\partial\bar{\partial}u > 0$ is Gauduchon, and so one has to consider a different equation.

(4) A consequence of our result is that $c_1^{\mathrm{BC}}(M) = 0$ if and only if M admits a Gauduchon metric with vanishing Chern-Ricci form.

To prove this theorem, we solve a certain equation of Monge-Ampère type, which we now describe. First note the simple fact that, at each fixed point, the map

$$\omega \mapsto \omega^{n-1},$$

from positive definite $(1,1)$ forms to positive definite $(n-1,n-1)$ forms is a bijection. We have the following:

Theorem 3 (Székelyhidi-Tosatti-Weinkove [8]). *Let (M,ω) be a compact Gauduchon manifold. Given $F \in C^\infty(M)$ there exists a unique pair (u,b) where $u \in C^\infty(M)$ and $b \in \mathbb{R}$ such that if we define $\tilde{\omega}$ by*

$$\tilde{\omega}^{n-1} := \omega^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} + \mathrm{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}\omega^{n-2}) > 0$$

then

$$\tilde{\omega}^n = e^{F+b}\omega^n.$$

Note that we can write this equation as

$$(*) \quad \det(\omega^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} + \mathrm{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}\omega^{n-2})) = e^{(n-1)(F+b)}\det(\omega^{n-1}),$$

where the determinant of an $(n-1,n-1)$ form is defined by applying the Hodge star operator in the obvious way.

To see that Theorem 3 implies Theorem 2 note that $\tilde{\omega}^{n-1}$ from Theorem 3 satisfies

$$\partial\bar{\partial}\tilde{\omega}^{n-1} = 0$$

and hence $\tilde{\omega}$ is indeed Gauduchon. Moreover, note that $\tilde{\omega}^{n-1} - \omega^{n-1} = \partial\gamma + \overline{\partial\gamma}$ for $\gamma = \frac{\sqrt{-1}}{2}\bar{\partial}u \wedge \omega^{n-2}$.

We make a few remarks about Theorem 3. In the case when ω is Kähler, the equation $(*)$ becomes

$$(\dagger) \qquad \det(\omega^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2}) = e^{(n-1)(F+b)} \det(\omega^{n-1}),$$

which was introduced by Fu-Wang-Wu [2], who solved it if ω is Kähler with non-negative orthogonal bisectional curvature [3]. In [10, 11], equation $(\dagger)$ was solved more generally for any ω Hermitian. Note that the equation $(\dagger)$ can be regarded as the Monge-Ampère equation for $(n-1)$-plurisubharmonic functions, in the sense of Harvey-Lawson [6].

Equation $(*)$ was introduced by Popovici (and slightly later, independently, in [11]). In [11] it was shown that the existence of a solution to $(*)$ can be reduced to an *a priori* second order estimate of the form

$$\Delta u \leq C(1 + \sup_M |\nabla u|^2),$$

and this is precisely the estimate that we establish in [8].

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Ricci-flat metrics on A_k singularities

Hans-Joachim Hein
(joint work with Aaron Naber)

1. Introduction

This is a talk about singularities of Kähler-Einstein metrics.

- Unlike in the general Riemannian case, we know how to define and construct weak solutions to the Einstein equations for *Kähler* metrics. The idea is to work at the level of Kähler potentials, solving the complex Monge-Ampère equation in a suitable class of rough Kähler potentials such as $L^\infty \cap \mathrm{PSH}$.
- The price to pay is that we know almost nothing about the metric behavior of the distance function dist_g associated with the Riemannian metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, where $\omega = i\partial\bar{\partial}\phi$ with $\phi \in L^\infty \cap \mathrm{PSH}$ is our weak Kähler-Einstein form.

In this project, which has been in progress since 2012, we were able to resolve this tension in one particular basic example, exhibiting a new phenomenon.

2. The A_1 singularity in $\mathbb{C}^{n+1}$ This is the singular complex hypersurface cut out by $z_0^2 + z_1^2 + \cdots + z_n^2 = 0$ in $\mathbb{C}^{n+1}$. Topologically A_1 is a cone with link $L = A_1 \cap S^{2n+1} = \mathrm{SO}(n+1)/\mathrm{SO}(n-1)$. Notice that L is a smooth manifold, diffeomorphic to $\mathbb{R}P^3$ if $n = 2$ and to $S^2 \times S^3$ if $n = 3$. It is a classical fact that this picture can be metrized in a canonical way: A_1 admits a Ricci-flat Kähler cone metric $\omega = \frac{i}{2}\partial\bar{\partial}r^2$. Here r denotes the metric distance to the origin, and the associated Riemannian metric g can be written as a warped product $g = dr^2 \oplus r^2 g_L$ for a certain homogeneous Einstein metric g_L of positive scalar curvature on L. (g_L is the round metric on $\mathbb{R}P^3$ if $n = 2$, but not a product of round metrics on $S^2 \times S^3$ if $n = 3$.) There is a simple formula for r: $r = |z|^{(n-1)/n}$, where $|z| = (|z_0|^2 + \cdots + |z_n|^2)^{1/2}$. See [8] for all of this.

3. The A_k singularity in $\mathbb{C}^{n+1}$

Let us now destroy the $\mathrm{SO}(n+1)$ symmetry of this example in a minimal way, introducing the A_k singularity $z_0^{k+1} + z_1^2 + \cdots + z_n^2 = 0$ in $\mathbb{C}^{n+1}$. Topologically A_k is still a cone over some smooth closed manifold, hence in particular has only one isolated singular point. The easiest case, $n = 2$, gives some hope for a nice metric picture: if $n = 2$, then A_k is biholomorphic to the quotient space $\mathbb{C}^2/\langle \mathrm{diag}(\zeta, \bar{\zeta})\rangle$, where ζ denotes any primitive $(k+1)$-st root of unity. On this space, we have an obvious *flat* Kähler cone metric pushed down from $\mathbb{C}^2$ for every k. Unfortunately, if $n > 2$, A_k is not isomorphic to a complex orbifold for any k, but we still have an abstract existence theorem for singular Ricci-flat Kähler metrics on A_k.

The key underlying property of this existence theorem is that A_k is a so-called "log-terminal" singularity. This is a fairly restrictive condition, and log-terminal singularities have to some extent been classified in low dimensions. Here we only record one important feature: that there exists a holomorphic volume form Ω on the regular locus $A_k \setminus 0$ such that $\int_U |\Omega \wedge \bar{\Omega}| < \infty$, where $U = A_k \cap \{|z| \leqslant 1\}$.

With this understood, we can now state (a simplified version of) the existence theorem mentioned above: *there exists a potential $\phi \in L^\infty(U) \cap C^\infty(U \setminus 0)$ such that $i\partial\bar{\partial}\phi > 0$ and $(i\partial\bar{\partial}\phi)^n = |\Omega \wedge \bar{\Omega}|$ in the sense of distributions on U.* This follows from the work of many authors; see [7] for the required local version. As a consequence, the Riemannian metric g derived from the Kähler form $\omega = i\partial\bar{\partial}\phi$ on $U \setminus 0$ is Ricci-flat of finite volume, hence [9] of finite diameter, and therefore incomplete. However, we know nothing about the asymptotics of g near $0 \in U$; it is not even clear whether the completion of $(U \setminus 0, \mathrm{dist}_g)$ is homeomorphic to U. By contrast, the link $\partial U = A_k \cap \{|z| = 1\}$ is just a smooth boundary.

4. A nonexistence theorem

The story gets more interesting due to the following nonexistence theorem [4]: *there is no Ricci-flat Kähler cone metric on A_k if $n \geqslant 3$ and $(*)$ $k + 1 \geqslant 2\frac{n-1}{n-2}$.*

5	✓	×	×	×
4	✓	⊗	×	×
3	✓	?	⊗	×
2	✓	✓	✓	✓
n, k	1	2	3	4

✓ : there is a Ricci-flat Kähler cone metric. × (⊗): there is no Ricci-flat Kähler cone metric (and, in addition, $(*)$ is an equality). ?: left open in [4].

Remarks: (1) Strictly speaking, [4] assume that the Hopf vector field $J(r\partial_r)$ of the Ricci-flat Kähler cone structure lies in $\mathfrak{u}(1) \oplus \mathfrak{so}(n)$, where $\mathrm{U}(1)$ acts on $\mathbb{C}^{n+1}$ with weights $(2, k+1, \ldots, k+1)$ and $\mathrm{SO}(n)$ acts naturally on $(z_1, \ldots, z_n)$.

(2) Under the same additional assumption, it was proved in [2] that ? = ×. On the other hand, motivated by our project, [5, 6] showed that ? = ✓ and that this new Ricci-flat Kähler cone metric on $A_2 \subset \mathbb{C}^4$ is in fact $\mathrm{U}(1) \times \mathrm{SO}(3)$-invariant.

5. Main result

Theorem (H-Naber). *If the inequality $(*)$ is strict, then there exists a potential $\phi \in C^{0,\alpha}(U) \cap C^\infty(U \setminus 0)$, invariant under* $\mathrm{U}(1) \times \mathrm{SO}(n)$ *and satisfying $i\partial\bar{\partial}\phi > 0$ and $(i\partial\bar{\partial}\phi)^n = |\Omega \wedge \bar{\Omega}|$ in the sense of distributions on U, such that the Ricci-flat metric g on $U \setminus 0$ derived from the Kähler form $i\partial\bar{\partial}\phi$ has the following geometric property: the completion of $(U \setminus 0, \mathrm{dist}_g)$ is homeomorphic to U, but the Gromov-Hausdorff tangent cone of this completion at its only singular point is isometric to $\mathbb{C} \times A_1$. In particular, the tangent cone has a 2-plane of singularities.*

To see the geometry of this situation, project A_k onto the z_0-coordinate. This yields a fibration of A_k over $\mathbb{C}$ (essentially a slicing by conic sections) whose fiber over $\epsilon \in \mathbb{C}$ is the variety $z_1^2 + \cdots + z_n^2 = -\epsilon^{k+1}$. For $\epsilon = 0$ this is an A_1 singularity in one dimension less, and for $\epsilon \neq 0$, a smooth algebraic manifold diffeomorphic to T^*S^{n-1}. The zero sections of these copies of T^*S^{n-1} in A_k give us a family of $(n-1)$-spheres in A_k, parametrized by $\epsilon \in \mathbb{C}^*$, collapsing to a point as $\epsilon \to 0$.

If the inequality $(*)$ is strict, then the diameters of these $(n-1)$-spheres with respect to g go to zero faster than linearly in $|\epsilon|$. Thus, if we rescale and pass to the tangent cone, these spheres pinch off, creating a 2-plane of singularities.

We prove this theorem by first writing down an explicit approximate solution $\tilde{\phi}$ to the complex Monge-Ampère equation $(i\partial\bar{\partial}\phi)^n = |\Omega \wedge \bar{\Omega}|$ such that $\tilde{g}$ behaves as in the above picture. Then we prove an appropriate implicit function theorem, thereby producing an honest solution ϕ with essentially the same geometry.

To be more precise, our approximate solution is given by $\tilde{\phi} = as + s^c\psi(s^{-\frac{\ell}{2}}r)$ where a is a constant, $s = |z_0|^2$, $r = |z_1|^2 + \cdots + |z_n|^2$, $c = \frac{\ell}{2}\frac{n-2}{n-1} > 1$, $\ell = k+1$, and $\psi(|y_1|^2 + \cdots + |y_n|^2)$ is the potential function of Stenzel's complete Ricci-flat Kähler metric [8] on the smooth affine variety $y_1^2 + \cdots + y_n^2 = 1$.

One important point is that $\tilde{\phi}$ satisfies the Monge-Ampère equation to leading order as $r, s \to 0$, but $\tilde{g}$ has unbounded Ricci curvature, so we cannot hope for ϕ to be more than $C^{2,\alpha}$ or perhaps $C^{3,\alpha}$ close to $\tilde{\phi}$. Another important point: the required $C^{2,\alpha}$ Schauder type estimates for the Laplacian associated with $\tilde{g}$ would definitely fail if we did not restrict to $\mathrm{U}(1) \times \mathrm{SO}(n)$-invariant functions on U.

6. An application

Interestingly, the geometry of our Ricci-flat metrics on $U \subset A_k$ approaches the product geometry on $\mathbb{C} \times A_1$ not only if we fix k and blow up the scale, but also if we fix the scale and let $k \to \infty$. Since we can smooth out and/or resolve the A_k singularity with unbounded topology as $k \to \infty$, and since these smoothings carry Ricci-flat Kähler metrics with sufficient control for us to be able to assert that the topology added in the smoothing has arbitrarily small diameter, we obtain infinite sequences of smooth Ricci-flat Kähler unit balls of complex dimension $n \geqslant 3$ with volume pinched between two positive constants, $\int |\mathrm{Rm}|^2$ uniformly bounded, but $b_n \to \infty$ (and alternatively $b_2 \to \infty$ if $n = 3$). Of course this would be impossible for $n = 2$ even without the L^2 curvature bound, by [1].

7. Concluding remarks

- Our theorem is not "obvious because the first coordinate scales differently" because the first coordinate scales differently for all $k > 1$ if $n = 2$. The condition that $k + 1 > 2\frac{n-1}{n-2}$ has an interpretation in terms of "K-instability" [3].
- It would be nicer to have compact examples without a boundary. We can't imagine any way of achieving this short of proving that *all* weak solutions to the complex Monge-Ampère equation on U (with arbitrary Dirichlet boundary values on ∂U) have the same asymptotics at the origin as our particular examples.
- If $k + 1 < 2\frac{n-1}{n-2}$, then $\tilde{\phi}$ is an almost solution to the Monge-Ampère equation on the complementary region $A_k \setminus U$, with interesting geometry *at infinity.*

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Almost non-negatively curved spin manifolds have vanishing $\hat{A}$-genus.

Burkhard Wilking
(joint work with Esther Cabezas-Rivas)

We give an affirmative answer to a problem posed by John Lott. Recall that a compact manifold M is almost non-negatively curved if there is a sequence of metrics g_i such that $\operatorname{diam}(M, g_i) = 1$, $K_{g_i} \geq -1/i$. John Lott asked whether in the spin case this implies vanishing of the $\hat{A}$-genus. More precisely we show

Theorem 1. *Suppose (M_i, g_i) is a sequence of n-dimensional spin manifolds, satisfying* $\operatorname{diam}(M, g_i) \leq D$, $K_{g_i} \geq 1/i$ *and the Dirac operator $D\!\!\!/_{g_i}$ has a nontrivial kernel. Then M_i is finitely covered by a nilmanifold for all large i.*

Metric cones smoothed out by gradient Ricci expanders

Alix Deruelle

Given a geometric object with singularities, one can ask if there exists a (geometric) flow that smooths it out instantaneously. We focus on metric cones for mainly two reasons : on one hand, this is the simplest geometric singularity one can think of, on the other hand, metric cones are the building blocks of the collapsing theory developed by Cheeger, Colding and Naber, to mention a few. The flow we are interested in is the Ricci flow. Formally speaking, it amounts to solve the following initial value problem.

Let $(C(X), dr^2 + r^2 g_X, o)$ be a metric cone on a smooth compact boundaryless Riemannian manifold (X, g_X). We look for a one parameter family of smooth complete Riemannian manifolds $(M^n, g(t))_{t\in(0,T)}$ where T is a positive number (eventually infinite) satisfying for some $p \in M^n$,

$$\begin{cases} \partial_t g = -2\,\mathrm{Ric}(g(t)) \quad \text{on} \quad M \times (0,T), \\ \\ (M^n, g(t), p) \to (C(X), dr^2 + r^2 g_X, o), \quad \text{as} \quad t \to 0^+, \end{cases}$$

where the convergence to the initial condition is understood in the pointed Gromov-Hausdorff topology and in the C^∞_{loc} topology outside the tip o.

In this short note, we impose further restrictions on the way the Ricci flow smooths out these metric cones : we want $(M^n, g(t))_{t\in(0,T)}$ to be a fixed point of the Ricci flow under the action of homotheties and the group of diffeomorphisms, i.e. a solution that looks like $g(t) = \sigma(t)\phi_t^* g$, where $\sigma(t) \in \mathbb{R}_+^*$ and $\phi_t \in \mathrm{Diff}(M^n)$, $t \in (0,T)$. If the one parameter family of diffeomorphisms $(\phi_t)_{t\in(0,T)}$ is generated by the gradient of a smooth function, we call such a Ricci soliton *gradient*. After a time reparameterization, one can assume that $\sigma(t) = 1 + \epsilon t$, with $\epsilon \in \{-1, 0, 1\}$. This leads us to consider three cases according to the lifetime of the solution : a Ricci soliton with $\epsilon = -1$ (respectively $\epsilon = 0$, $\epsilon = 1$) is called a shrinker, (respectively a steady soliton, an expander). Ricci expanders are the only candidates that can answer our problem. Finally, we give an equivalent and more tractable definition of an expanding gradient Ricci soliton. It consists in the triplet $(M^n, g, \nabla^g f)$ where (M^n, g) is a smooth complete Riemannian manifold and where $f : M^n \to \mathbb{R}$ is a smooth function called the potential function such that the following static equation holds :

$$\mathrm{Ric}(g) + \nabla^{g,2}(-f) = -\frac{g}{2}.$$

In other words, this means that the Bakry-Émery tensor associated to $(g, \nabla^g f)$ is constantly negative.

The main questions are then related to the solvabilty of this kind of Dirichlet problem at infinity. Which metric cones can be smoothed out by gradient Ricci expanders ? Can one deform conical gradient Ricci expanders implicitly ? Are these deformations (globally) unique ?

As shown in [6], the analysis strongly depends on the convergence rate to the asymptotic cone.

Either the convergence is polynomial (generic case) : explicit asymptotically conical Ricci expanders are given by the rotationally symmetric examples due to Bryant [Chap. 1,[5]] coming out of the cones $(C(\mathbb{S}^{n-1}), dr^2 + r^2c^2 g_{\mathbb{S}^{n-1}}, r\partial_r/2)_{c>0}$. In the Kähler setting, similar examples have been built by Cao [4]. This ansatz has been extended by [9] where they produced Kähler Ricci expanders coming out of the cones $(C(\mathbb{S}^{2n-1}/\mathbb{Z}_k), i\partial\bar{\partial}|\cdot|^{2p}/p, r\partial_r/2)$ where $k > n$ (condition equivalent to negative first Chern class) and where p, the angle, is positive different from 1. Implicit deformations of the Bryant examples have been proved to exist by the author [6], [7] :

Theorem 1. *Let* (X, g_X) *be a smooth simply connected compact Riemmanian manifold such that* $\mathrm{Rm}(g_X) \geq 1$.

Then there exists a unique expanding gradient Ricci soliton with nonnegative curvature operator asymptotic to $\big(C(X), dr^2 + r^2 g_X, r\partial_r/2\big)$.

In other terms, theorem 1 gives a classification of conical gradient Ricci expanders with positive curvature operator. The method we use to prove theorem 1 is a continuity method : it shows that the moduli space of such Ricci expanders is connected. Roughly speaking, the main idea comes from the following observation : given a simply connected Riemannian manifold (X, g_X) such that $\mathrm{Rm}(g_X) \geq 1$, one starts the (normalized) Ricci flow and ends up with a one parameter family of metrics $(g(s))_{s\in[0,+\infty]}$ on X of constant volume connecting (X, g_X) to $(X, c^2 g_{\mathbb{S}^{n-1}})$ where $c^{n-1} = \mathrm{vol}(X, g_X)/\,\mathrm{vol}(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$ thanks to the fundamental work of Böhm and Wilking [3]. Therefore, the initial metric cone $(C(X), dr^2 + r^2 g_X)$ is connected to the cone $(C(\mathbb{S}^{n-1}), dr^2 + (cr)^2 g_{\mathbb{S}^{n-1}})$ which is smoothed out by the corresponding Bryant soliton. Then one has to prove the closedness and the openness of the set of such solutions. The openness is the most technical by far, especially because the Fredholm theory for the linearized operator was not available, even in the Euclidean case. Moreover, the use of the Nash-Moser implicit function theorem is required to provide deformations smooth up to the boundary at infinity. Besides, matrix Hamilton Harnack inequalities or adequate entropies not necessarily well-defined in general are essential to get rid of the action of the diffeomorphisms.

Either the convergence is exponential (asymptotically Ricci flat case). The first non flat asymptotically conical Ricci expanders coming out of a Ricci flat cone are the examples due to [9] mentioned above with $p = 1$. [10] provided a more systematic study of Kähler Ricci expanders coming out of Kähler Ricci flat cones. Implicit examples have been built by Siepmann [12], where some of the previous examples are recovered.

It turns out that the uniqueness at infinity should not be true in general, moreover, a continuity method does not seem to be adequate in the asymptotically Ricci flat case. Besides, conical Ricci expanders share many analogies with conformally compact Einstein metrics. In this regard, we benefited from the work of Biquard [2] and Anderson-Herzlich [1] dealing with uniqueness issues at infinity of such metrics. In this context, the obstruction at infinity is given by a symmetric 2-tensor defined on the conformal infinity : it is a global invariant, i.e. is not an invariant depending locally on the metric at the boundary. It turns out that there is a similar obstruction in the setting of conical Ricci expanders : see [8] for the definition of the obstruction tensor at infinity. To conclude, we notice that the main motivation comes from the recent work of Kotschwar-Lu [11] dealing with the uniqueness at infinity of conical Ricci shrinkers : as the asymptotic cone is at the end of the lifetime of such a singularity, there is no obstruction at infinity.

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Weak solutions for the Ricci flow

Robert Haslhofer
(joint work with Aaron Naber)

We introduce a new class of estimates for the Ricci flow, and use them both to characterize solutions of the Ricci flow and to provide a notion of weak solutions for the Ricci flow in the nonsmooth setting.

As a motivation, let us first explain the much easier task of characterizing supersolutions of the Ricci flow. Let $(M, g_t)_{t\in I}$ be a one-parameter family of Riemannian manifolds. We consider the heat equation $(\partial_t - \Delta_{g_t})w = 0$ on our evolving manifolds $(M, g_t)_{t\in I}$. For every $s, T \in I$ with $s \leq T$, and every smooth function u with compact support, we write $P_{sT}u$ for the solution at time T with initial condition u at time s, i.e. $(P_{sT}u)(x) = \int_M u(y)\, H(x, T\,|\, y, s) dV_s(y)$, where $H(x, T\,|\,y, s)$ is the heat kernel with pole at (y, s). We write $d\nu_{(x,T)}(y, s) = H(x, T\,|\,y, s)dV_s(y)$.

Proposition ([1]). *The following are equivalent:*

(1) $\partial_t g_t \geq -2Rc_{g_t}$
(2) $|\nabla P_{sT}u| \leq P_{sT}|\nabla u|$
(3) $|\nabla P_{sT}u|^2 \leq P_{sT}|\nabla u|^2$
(4) $\int_M u^2 \log u^2\, d\nu \leq 4(T-s)\int_M |\nabla u|^2 d\nu$
(5) $\int_M (u-\bar{u})^2\, d\nu \leq 2(T-s)\int_M |\nabla u|^2 d\nu.$

In essence, the proposition follows easily from the parabolic Bochner-formula

$$(\partial_t - \Delta)|\nabla u|^2 = 2\langle \nabla u, \nabla(\partial_t - \Delta)u\rangle - 2|\nabla^2 u|^2 - (\partial_t g + 2\mathrm{Rc})(\nabla u, \nabla u).$$

To characterize solutions of the Ricci flow, and not just supersolutions, we prove infinite-dimensional generalizations of the above estimates. Let $(M, g_t)_{t\in I}$ be a smooth family of Riemannian manifolds. Let $\mathcal{M} = M \times I$ be its space-time with the usual space-time connection, i.e. $\nabla_t Y = \partial_t Y + \frac{1}{2}\partial_t g_t(Y,\cdot)^{\sharp_{g_t}}$. For each $(x,T) \in \mathcal{M}$, we consider the based path space $P_{(x,T)}\mathcal{M}$ consisting of all space-time curves of the form $\{\gamma_\tau = (x_\tau, T-\tau)\}_{\tau\in[0,T]}$, where $\{x_\tau\}_{\tau\in[0,T]}$ is a continuous curve in M with $x_0 = x$. Let $\Gamma_{(x,T)}$ be the Wiener measure of Brownian motion on our evolving family of manifolds based at (x,T), i.e. the probability measure uniquely characterized by the following property. If $e_{\sigma_1,\ldots,\sigma_k} : P_{(x,T)}\mathcal{M} \to M^k$, $\gamma \mapsto (x_{\sigma_1},\ldots,x_{\sigma_k})$, is the evaluation map at $0 \le \sigma_1 \le \ldots \le \sigma_k \le T$ then

$$e_{\sigma_1,\ldots,\sigma_k *} d\Gamma_{(x,T)}(y_1,\ldots,y_k) = d\nu_{(x,T)}(y_1,s_1)\cdots d\nu_{(y_{k-1},s_{k-1})}(y_k,s_k),$$

where $s_i = T - \sigma_i$. Path space can be equipped with two natural notions of gradient, the parallel gradient $\nabla^{\|}$ and the Malliavin gradient $\nabla^{\mathcal{H}}$, see [1]. Our main theorem characterizes solutions of the Ricci flow in terms of certain sharp estimates on path space.

Theorem ([1])**.** *The following are equivalent:*

(1) $\partial_t g_t = -2Rc_{g_t}$

(2) $|\nabla_x \int_{P_T\mathcal{M}} F d\Gamma_{(x,T)}| \le \int_{P_T\mathcal{M}} |\nabla^{\|} F|\, d\Gamma_{(x,T)}$

(3) $\int_{P_T\mathcal{M}} \frac{d[F^\bullet]_\tau}{d\tau} d\Gamma_{(x,T)} \le 2\int_{P_T\mathcal{M}} |\nabla^{\|}_\tau F|^2\, d\Gamma_{(x,T)}$

(4) $\int_{P_T\mathcal{M}} (F^2)^{\tau_2}\log(F^2)^{\tau_2} - (F^2)^{\tau_1}\log(F^2)^{\tau_1} d\Gamma_{(x,T)} \le \int_{P_T\mathcal{M}} \langle F, \mathcal{L}_{\tau_1,\tau_2}F\rangle d\Gamma_{(x,T)}$

(5) $\int_{P_T\mathcal{M}} (F^{\tau_2} - F^{\tau_1})^2\, d\Gamma_{(x,T)} \le 2\int_{P_T\mathcal{M}} \langle F, \mathcal{L}_{\tau_1,\tau_2}F\rangle\, d\Gamma_{(x,T)}$

Here, F^τ denotes the martingale induced by $F \in L^2(P_T\mathcal{M}, \Gamma_{(x,T)})$, and $\mathcal{L}_{\tau_1,\tau_2}$ denotes the $[\tau_1,\tau_2]$-part of the Ornstein-Uhlenbeck operator $\mathcal{L} = \nabla^{\mathcal{H}*}\nabla^{\mathcal{H}}$. The estimates from the theorem are infinite-dimensional generalizations of the estimates from the proposition. In the very special case of 1-point test functions, i.e. test functions of the form $F(\gamma) = u(\gamma(t_0))$ for some $u : M \to \mathbb{R}$, our infinite dimensional estimates reduce to the finite-dimensional estimates from the proposition. Of course, there are many more test functions on path space, and this is one of the reasons why our infinite-dimensional estimates are strong enough to characterize solutions of the Ricci flow, and not just supersolutions.

Finally, let us briefly indicate how the above characterization of solutions of the Ricci flow can be used to provide a notion of weak solutions for the Ricci flow [2]. We consider metric-measure spaces $\mathcal{M}$ equipped with a time function and a linear heat flow. We call $\mathcal{M}$ a weak solution of the Ricci flow if and only if the infinite dimensional gradient estimate $|\nabla_x \int_{P_T\mathcal{M}} F d\Gamma_{(x,T)}| \le \int_{P_T\mathcal{M}} |\nabla^{\|} F|\, d\Gamma_{(x,T)}$ holds. We establish various geometric and analytic estimates for these weak solutions. In particular, one of our applications concerns a question of Perelman about limits of Ricci flows with surgery [4]. Namely, the metric completion of the space-time of Kleiner-Lott [3], which they obtained as a limit of Ricci flows with surgery where the neck radius is sent to zero, is a weak solution in our sense.

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A local regularity theorem for Mean Curvature Flow with triple Edges

Felix Schulze
(joint work with Brian White)

We consider smooth families $\mathcal{M} = \cup_{t\in I} M_t \times \{t\}$ of n-dimensional surface clusters in $\mathbb{R}^{n+k}$. Such a cluster M_t consist of finitely many disjoint, embedded, open hypersurfaces M_t^i, $i = 1, \ldots, N$, such that their closures $\bar{M}_t^i$ are smooth immersed hypersurfaces with corners and $M_t = \bigcup_{i=1}^N \bar{M}_t^i$. We assume that along each $(n-1)$-dimensional face, which we call an edge, three sheets meet under equal angles. Along the lower dimensional faces we assume that the sheets meet modelled on n-dimensional unit density area minimizing cones in R^{n+k}. We call these higher order junctions.

We say that $\mathcal{M} = \cup_{t\in I} M_t \times \{t\}$ solves mean curvature flow, if, given a smooth parametrisation X of the moving cluster, the speed vector satisfies

$$\left(\frac{\partial}{\partial t} X\right)^{\perp} = \vec{H}\,,$$

where $^\perp$ is the projection onto the normal space along each sheet and $\vec{H}$ its mean curvature vector. Along the edges and higher order junctions we require that this holds for each sheet separately.

We denote the backwards parabolic cylinder with radius r, centered at a space-time point $X = (x, t) \in \mathbb{R}^{n+k} \times \mathbb{R}$, by

$$C_r(X) = B_r(x) \times (t - r^2, t)\ .$$

We will write O to denote the origin $(0, 0)$ in space-time.

Theorem 1. *Let $\mathcal{M}^j$ be a sequence of smooth, n-dimensional mean curvature flows with triple edges in $\mathbb{R}^{n+k}$ which converge as Brakke flows to a static union of 3 unit density n-dimensional half-planes in $C_2(O)$. Then the convergence is smooth in $C_1(O)$.*

We also consider the class of integral Brakke flows, which are Y-regular, i.e. any point of Gaussian density one and any point with a tangent flow, which is a static union of 3 unit density half-planes, has a space-time neighborhood in which the flow is smooth. Then the above theorem remains true.

Combining this with Ilmanen's elliptic regularisation scheme we show:

Theorem 2. *Let M_0 be a smooth, compact n-surface cluster in $\mathbb{R}^{n+k}$ without higher order junctions, i.e. a finite union of compact manifolds-with-boundary that meet each other at 120 degree angles along their smooth boundaries. Then there* *exists a* $T > 0$ *and smooth solution to mean curvature flow with triple junctions* $(M_t)_{0<t<T}$ *such that* $M_t \to M_0$ *in* C^1 *and in* C^∞ *away from the triple junctions.*

Furthermore, Theorem 1 implies that the solution exists until either the supremum of the second fundamental form over the cluster blows up, or two triple edges collide.

The corresponding fundamental regularity theorem for smooth mean curvature flow was proven by the second author, [10]. Mean curvature flow with triple edges for curves in codimension one is the network flow. A similar regularity theorem for the network flow was shown by Ilmanen and Neves together with the first author, [7, Theorem 1.3]. For Brakke flows the fundamental regularity theorem is due to Brakke, [1]. More recently, Tonegawa and Wickramasekera have proven the analogous result for 1-dimensional integral Brakke flows close to a static union of three half lines in the plane, [9].

Smooth short time existence for the network flow was first established by Mantegazza, Novaga and Tortorelli [8] using PDE methods, following Bronsard and Reitich [2]. Short time existence of mean curvature flow with triple edges, also in the PDE setting, was considered by Freire [4, 5] in the case of graphical hypersurfaces and by Depner, Garcke and Kohsaka in [3] for special hypersurface clusters. Both the results of Freire and of Depner, Garcke and Kohsaka require as well that no higher order junctions are present.

Outline of proof: As a first step we show that a smooth mean curvature flow with triple edges that is sufficiently close in $C^{2,\alpha}$ to a static union of three unit density half-planes, and converging in C^2, converges also in $C^{2,\alpha}$. We do this by writing the solution as a perturbation of an approximating solution of the heat equation. We use standard Schauder estimates for the heat equation to show first that the perturbation decays in $C^{2,\alpha}$, and use this information, together with the 120 degree condition along the triple edge, again using only standard Schauder estimates for the heat equation, to show that also the approximating solutions of the heat equation converge in $C^{2,\alpha}$. We use this, together with a blow up argument which is analogous to the one used in [10], to prove Theorem 1.

As a second step we extend Theorem 1 to integral Brakke flows, which are smooth in a space-time neighborhood of points with either Gaussian density one, or which have a tangent flow which is a static union of three unit density half-planes. The main ingredient is showing that any static union of three unit density half-planes is, up to rotations, weakly isolated in the space of self-similarly shrinking integral Brakke flows.

Finally, we prove Theorem 2, using Ilmanen's elliptic regularisation scheme, [6], in the setting of flat chains mod 3, together with the previous results. The main ingredient is showing that the Brakke flow constructed via elliptic regularisation has only unit density static planes and static unions of three unit density half spaces as tangent flows at the initial time.

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Filling multiples of embedded cycles

Robert Young

Given a smooth curve T in $\mathbb{R}^N$, there is a minimal surface U with boundary T. If we trace T twice to get a curve $2T$, there is a minimal surface U' with boundary $2T$. One might guess that $U' = 2U$, and, by a theorem of Federer [1], this holds when $N \leq 3$, but a remarkable example of L. C. Young shows that U' and U may be very different. Young [5] constructed a smooth curve T drawn on a nonorientable surface in $\mathbb{R}^4$ such that $\operatorname{area} U' \approx (1 + 1/\pi) \operatorname{area} U$. Morgan [3] and White [4] later found other examples of this phenomenon with different multipliers. A version of Young's example is shown in Figure 1.

One can generalize this to arbitrary dimensions. If T is a d-cycle in $\mathbb{R}^N$, we define $\operatorname{FV}(T)$ to be the minimum mass of an integral $d+1$-chain with boundary T. Young's example can then be generalized to an example of a d-cycle in $\mathbb{R}^{d+3}$ such that $\operatorname{FV}(T) > \operatorname{FV}(2T)/2$.

One might ask whether the ratio $\operatorname{FV}(2T)/\operatorname{FV}(T)$ can be made arbitrarily small. In fact, the following holds:

Theorem 1. *Let $0 < d < N$ be natural numbers. There is a $C > 0$ depending on d and N such that if $T \in C_{d-1}(\mathbb{R}^N; \mathbb{Z})$ is a boundary, then*

$$\operatorname{FV}(T) \leq C \operatorname{FV}(2T).$$

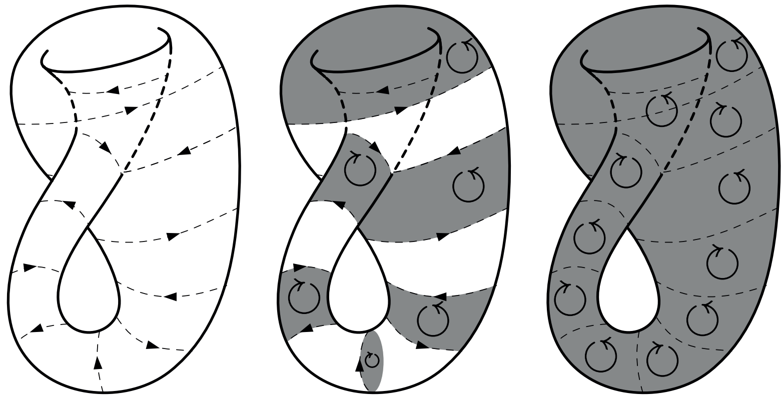

FIGURE 1. Fillings of a 1-cycle on a Klein bottle. The 1-cycle T consists of $2k+1$ loops in alternating directions. In the middle, we fill T with k cylindrical bands and a disc, and on the right, we fill $2T$ with $2k+1$ cylindrical bands with alternating orientations.

This theorem can be reduced to the problem of proving that any mod-2 cellular cycle U in $\mathbb{R}^N$ (for instance, a minimal filling of $2T$) is congruent mod 2 to an integral cycle of comparable mass. That is,

Proposition 1. *There is a $c > 0$, depending on d and N such that for every mod-2 cellular d-cycle U in the unit grid in $\mathbb{R}^N$, there is an integral d-cycle R such that $U \equiv R \pmod 2$ and $\operatorname{mass} R \leq c \operatorname{mass} U$.*

A weaker version of the proposition, showing that there is an R such that $U \equiv R \pmod 2$ and $\operatorname{mass} R \leq c \operatorname{mass} U(\log \operatorname{mass} U)$, can be proved by using the Federer-Fleming Deformation Theorem to construct a sequence of approximations of U, a method similar to those used in [6] and [2].

To remove this factor of $\log \operatorname{mass} U$, we use uniform rectifiability. Uniformly rectifiable sets were developed by David and Semmes as a quantitative version of the notion of rectifiable sets. Recall that a set $E \subset \mathbb{R}^n$ is d-rectifiable if it can be covered by countably many Lipschitz images of $\mathbb{R}^d$. Uniform rectifiability quantifies this by bounding the Lipschitz constants and the number of images necessary to cover E. We introduce a decomposition of cellular cycles in $\mathbb{R}^N$ into sums of cellular cycles supported on uniformly rectifiable sets.

Specifically, we prove:

Theorem 2. *If $A \in C_d(\tau; \mathbb{Z}/2)$ is a d-cycle in the unit grid in $\mathbb{R}^N$, then there are cycles $M_1, \dots, M_k \in C_d(\tau; \mathbb{Z}/2)$ and uniformly rectifiable sets $E_1, \dots, E_k \subset \mathbb{R}^N$ such that*

(1) $A = \sum_i M_i$,
(2) $\operatorname{supp} M_i \subset E_i$,
(3) $\operatorname{mass} M_i \sim |E_i|$, *and*
(4) $\sum_i |E_i| \lesssim \operatorname{mass} A$.

Here, $|\cdot|$ represents d-dimensional Hausdorff measure.

This reduces the proof of Proposition 1 to the case where T is supported on a uniformly rectifiable set. We then prove Proposition 1 by using a corona decomposition to break T into pieces that are close to d-planes in $\mathbb{R}^N$.

Open questions

We can ask a similar question about the relationship between real filling volume and integral filling volume. That is, if $\mathrm{FV}_{\mathbb{R}}(T)$ is the minimum mass of a $d+1$-chain with boundary T and real coefficients, then is $\mathrm{FV}_{\mathbb{R}}(T)/\,\mathrm{FV}(T)$ bounded below?

There are several related questions in geometric measure theory about the relationship between real chains, integral chains, and mod-2 chains, several of which were studied by Almgren. For instance, is the integral flat norm of a chain bounded in terms of its real flat norm? Is every normal mod-2 current equivalent to a normal integral current? Generalizing Theorem 2 to the context of currents rather than cellular cycles might help answer some of these questions.

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Index estimates in geometry

André Neves

(joint work with Fernando Marques)

I spoke about my ongoing work with Fernando Marques from Princeton University.

Second variation arguments have been quite used in Geometry and have been at the core of some classical results such as the Positive Mass Theorem or the Frankel Conjecture. Unfortunately the Almgren–Pitts Min-Max Theory does not provide index estimates for the min-max hypersurfaces.

In an ongoing work, we are now able to show various index estimates for the min-max hypersurfaces. I sketched the proof of those estimates.

Metrics on spheres with all geodesics closed

Marco Radeschi
(joint work with Burkhard Wilking)

The talk concernes the study of Riemannian manifolds with all geodesics closed. Clearly such a condition imposes strong, global restrictions on such a metric, so much so that very few examples of such metrics are known.

The canonical simply connected examples are compact rank one symmetric spaces (also known as CROSSes), i.e. spheres, projective (complex and quaternionic) spaces, and the so called *Cayley plane*, an exceptional example of dimension 16. These examples support a canonical metric, with respect to which all geodesics are closed. To produce non simply connected examples, it is enough to start with a simply connected example ant divide by the isometric action of a finite group acting freely. Lens spaces are such examples, as they can be written a $\mathbb{S}^{2n-1}/\mathbb{Z}_p$ where $\mathbb{S}^{2n-1}$ denotes the unit sphere in $\mathbb{C}^n$ and $\mathbb{Z}_p$ acts by $k\cdot(z_1,\dots z_n)=(\xi^k z_1,\dots \xi^k z_n)$ for some p-th root of unity ξ.

As a matter of fact, if a manifold admits a metric with all geodesics closed, then nothing else can happen, at least at a level of cohomology ring:

Theorem (Bott-Samelson [1] Thm 7.37, Mc Cleary [3])**.** *If (M,g) is a Riemannian manifolds with all geodesics closed, then M is compact with finite fundamental group, and the universal cover $\tilde{M}$ has the integral cohomology ring of a CROSS.*

In order to get this result, one fundamental piece of information is about the period of closed geodesics, which is provided by the following

Theorem (Wadsley [4])**.** *If (M,g) has all geodesics closed, then the prime geodesics have a common least period.*

By prime geodesics we mean here a geodesic which is not simply obtained by iterating a shorter one. One cannot in general expect all prime geodesics to have the same period, as this is not the case for example in the lens spaces described above. An important conjecture of Berger, however, states that this phenomenon cannot occur if the manifold is simply connected:

Conjecture. *If M is a simply connected Riemannian manifold all of whose geodesics are closed, then all the prime geodesics have the same length.*

By a result of Weinstein [5], if an n-dimensional manifold has all prime geodesics closed and of length, say, 2π, then its volume must be an integer multiple of that of the unit round sphere $\mathbb{S}^n$. Thus the conjecture of Berger, together with Weinstein's result, would provide a strong restriction on the geometry of simply connected manifolds with all geodesics closed.

The conjecture of Berger was proved in the case of $M=\mathbb{S}^2$ by Grove and Gromoll. In this talk we show that the conjecture also holds for all spheres of dimension ≥ 4.

Theorem (Wilking, –)**.** *If $(\mathbb{S}^n,g)$, $n>3$, has all geodesics closed, then all prime geodesics have the same length.*

The proof proceeds by studying the energy functional E on the free loop space $\bigwedge M = C^0(\mathbb{S}^1, M)$ of a manifold M with all geodesics closed. Wilking [6] already proved a number of results about the energy functional in such a context, showing for example that this is a Morse-Bott function. This is like a Morse function, with the exception that isolated critical points are replaced with smooth critical submanifolds $C_i \subset \bigwedge M$ and the *space of negative directions* at a critical point is replaced by a *vector bundle of negative directions* $N \to C$ at a critical manifold C. An important step toward the proof of the main theorem consists on the following results, which hold for generic manifolds with all geodesics closed.

Theorem (Wilking, –)**.** *Let M be a Riemannian manifold with all geodesics closed, and let $E : \bigwedge M \to \mathbb{R}$ be the energy functional. Then*

(1) For every critical manifold $C \subset \bigwedge M$ of E, the bundle $N \to C$ of negative directions is orientable.

(2) E is invariant under the $\mathbb{S}^1$ action on $\bigwedge M$ by reparametrization, and moreover it is a perfect Morse-Bott function with respect to rational, $\mathbb{S}^1$-equivariant cohomology.

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On the Hopf conjectures

Manuel Amann
(joint work with Lee Kennard)

The question of whether a given smooth manifold admits a Riemannian metric with positive sectional curvature is nearly as old as the subject of Riemannian geometry itself. The classical Gauss–Bonnet theorem relating the sectional curvature K of a compact surface M to its Euler characteristic, $\chi(M)$, can be considered a first classification theorem of that kind.

So it seems surprising that until today only very few examples of simply-connected closed manifolds admitting positively curved metrics are known; from dimension 25 on this list only comprises the compact rank one symmetric spaces $\mathbb{S}^n$, $\mathbb{C}\mathbf{P}^n$ and $\mathbb{H}\mathbf{P}^n$. At the same time there are not many obstructions to the existence of such a metric.

It was a suggestion by Karsten Grove in the 1990s to study positive curvature in the context of isometric Lie group actions, which seems natural in this context for many reasons. This programme has led to many classification results as for example in [4], [3], [5]—passing from diffeomorphism classifications to cohomology classifications whilst reducing the symmetry assumption.

In this talk we focus on effective isometric actions of a torus on a positively curved manifold M. We assume that the dimension of the torus acting is basically a logarithm of $\dim M$. As a next step we apply this—in the tradition of the classical Gauss-Bonnet theorem—to derive results on the Euler characteristic $\chi(M)$ (cf. [1]).

The Euler characteristic plays a crucial role in a conjecture of Hopf which states that $\chi(M)$ should be positive on an even-dimensional positively curved manifold M and non-negative in non-negative curvature. A second conjecture of Hopf states that the manifold $\mathbb{S}^2 \times \mathbb{S}^2$ should not admit a positively curved Riemannian metric. It is our goal to illustrate how to use (generalisations of) the first conjecture in order to prove generalised versions of the second conjecture—always assuming logarithmic symmetry as above; see [2].

Theorem (Amann–Kennard). *Let M^{2n} be a simply connected, closed manifold with $b_4(M) = 0$. Assume M admits a Riemannian manifold with positive sectional curvature invariant under the action of a torus T with $\dim(T) \geq \log_{4/3}(2n)$. Then we derive that $\chi(M) = \chi(\mathbb{S}^{2n}) = 2$.*

We remark that this result leads to several classification results. For example it implies that if M is a biquotient, it is a diffeomorphism sphere.

Corollary(Amann–Kennard). *Let N^n be a simply connected, closed manifold. The product $N \times N$ does not admit a Riemannian metric with positive sectional curvature and an isometric torus action of rank at least $\log_{4/3}(2n)$.*

Similarly, if n is even and $\chi(N) \neq 2$, the connected sum $N\#N$ does not admit a positively curved metric invariant under a torus action of rank at least $\log_{4/3}(n)$.

The Bott–Grove–Halperin conjecture states that an (almost) non-negatively curved manifold should be (rationally) elliptic, so that in particular its homotopy Euler characteristic $\chi_\pi(M) = \sum_i (-1)^{i+1} \dim \pi_i(M) \otimes \mathbb{Q}$ is defined. We use this in order to suggest and discuss the following generalisation (to odd dimensions) of the Hopf conjecture on the Euler characteristic.

Conjecture. A closed manifold M^n of positive curvature satisfies

$$\chi_\pi(M) = n \mod 2$$

A closed manifold M^n of non-negative curvature satisfies

$$\chi_\pi(M) \equiv n \mod 2$$

(The first part asserts that the homotopy Euler characteristic equals either 1 or 0. Note that the second part of the conjecture merely claims rational ellipticity; the statement on the homotopy Euler characteristic being as redundant as the classical Hopf conjecture in non-negative curvature under the assumption of rational ellipticity.)

We prove the conjecture for rationally elliptic manifolds of positive curvature and the usual logarithmic symmetry assumption, i.e. we show that their homotopy Euler characteristic equals 0 or 1 depending on the parity of the dimension. (Note that this excludes symmetric positive curvature on $\mathbb{S}^k \times \mathbb{S}^l$ with k, l odd, as the simplest example.)

Motivated by the proof of the theorem above–which states that the fixed-point set of a torus has to be a rational sphere—we now apply localisation theorems in equivariant rational cohomology and homotopy theory in order to identify large classes of manifolds which *cannot* carry positive curvature under logarithmic symmetry. (For this we investigate degeneration properties of the Leray–Serre spectral sequence of the Borel construction in the light of rational homotopy techniques.)

Example. *This class comprises $\mathbb{S}^k \times \mathbb{S}^l$ if one of the following holds: Either both k, l are even or odd, or $l > k$ and l is even and k is odd. As a further very concrete example, also $\mathbb{S}^k \times \mathbb{S}^k \times \mathbb{S}^k \# \mathbb{S}^{2k} \times \mathbb{S}^k$ is comprised.*

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Finsler compactifications of symmetric and locally symmetric spaces

Bernhard Leeb
(joint work with Misha Kapovich)

Let $X = G/K$ be a symmetric space of noncompact type, i.e. G is a noncompact semisimple Lie group and K a maximal compact subgroup. We give a differential-geometric interpretation of the maximal Satake compactification of X (see [BJ, Chapter 2]) as a regular *Finsler compactification*

$$\overline{X}^{\bar{\theta}} = X \sqcup \partial_{\infty}^{\bar{\theta}} X$$

obtained by adding points at infinity represented by Finsler horofunctions. These horofunctions arise as limits, modulo additive constants, of distance functions

$$d_x^{\bar{\theta}} = d^{\bar{\theta}}(x, \cdot)$$

where $d^{\bar{\theta}}$ is a certain "polyhedral" G-invariant Finsler distance on X associated with a regular direction type $\bar{\theta}$ in the model spherical chamber σ_{mod}. It turns out that the particular choice of $\bar{\theta}$ is irrelevant.

If one applies the same construction to a G-invariant Riemannian metric on X, one obtains the familiar *visual compactification*

$$\overline{X} = X \sqcup \partial_\infty X.$$

Theorem 1 ([KL]). The Finsler compactification $\overline{X}^{\bar\theta}$ is a compactification of X as a G-space with the following properties:

(i) There are finitely many G-orbits. The G-orbits at infinity are indexed by the faces of the spherical Weyl chamber σ_{mod}.

(ii) The stratification by G-orbits determines a G-invariant manifold–with–corners structure.

(iii) The compactification is homeomorphic to the closed ball. More precisely, there exists a K-equivariant homeomorphism to the closed unit ball in X centered at the fixed point of K with respect to the dual Finsler metric $d^*_{\bar\theta}$.

(iv) It is independent of the regular type $\bar\theta$ in the sense that the identity map id_X extends to a natural homeomorphism of any two such compactifications.

(v) There exists a G-equivariant homeomorphism of manifolds with corners to the maximal Satake compactification $\bar{X}^S_{max}$ which yields a natural correspondence of strata.

The Finsler view point had emerged in several instances during our earlier study [KLP1, KLP2, KLP3] of asymptotic and coarse properties of discrete isometry groups acting on symmetric spaces and euclidean buildings.

Our main application is to discrete subgroups $\Gamma < G$. For subgroups satisfying a certain regularity condition we establish the existence of natural *Finsler bordifications*, as orbifolds with corners, of the associated locally symmetric spaces X/Γ. Furthermore we show that these bordifications are under suitable conditions *compactifications*, including all regular RCA subgroups, equivalently, B-Anosov subgroups.

We recall that the G-orbits in the visual boundary $\partial_\infty X$ are parametrized by the model spherical Weyl chamber, $\partial_\infty X/G \cong \sigma_{mod}$. The *regular* part $\partial_\infty^{reg} X \subset \partial_\infty X$ of the visual boundary consists of the ideal points which project to the interior of σ_{mod}. It is the union of the open (spherical Weyl) chambers at infinity; these are the top-dimensional simplices with respect to the spherical (Tits) building structure on $\partial_\infty X$. There is a natural projection $\partial_\infty^{reg} X \to \partial_{F\ddot{u}} X \cong G/B$ to the space of chambers $\partial_{F\ddot{u}} X$, the so-called Fürstenberg boundary, which is identified with the generalized full flag manifold G/B.

The set $\Lambda(\Gamma) = \overline{\Gamma x} \cap \partial_\infty X$ of accumulation points of an orbit $\Gamma x \subset X$ in the visual boundary is called the *limit set* of Γ. It is independent of the orbit. We call Γ *uniformly regular* if its limit set consists of regular ideal points, $\Lambda(\Gamma) \subset \partial_\infty^{reg} X$. In this case, we call the projection $\Lambda_{ch}(\Gamma) \subset \partial_{F\ddot{u}} X$ of $\Lambda(\Gamma)$ the *chamber limit set* of Γ, cf. [Be]. A limit chamber $\sigma \in \Lambda_{ch}(\Gamma)$ is called *conical* if an(y) orbit Γx has unbounded intersection with a sufficiently large tubular neighborhood of a(ny) euclidean Weyl chamber asymptotic to σ. The chamber limit set is called *conical* if all limit chambers are conical. This condition has been considered in [Al]. We call Γ *antipodal* if any two limit chambers in $\Lambda_{ch}(\Gamma)$ are opposite. Finally, we call the subgroup Γ *RCA* if it is antipodal and has conical chamber limit set.

We recall that if X is a negatively curved symmetric space, then the locally symmetric space X/Γ admits the *visual bordification*

$$X/\Gamma \hookrightarrow (X \cup \Omega(\Gamma))/\Gamma$$

as an orbifold with boundary by attaching at infinity the quotient $\Omega(\Gamma)/\Gamma$ of the domain of discontinuity $\Omega(\Gamma) \subset \partial_\infty X$. Furthermore, the subgroup Γ is *convex cocompact* if and only if this bordification is a compactification.

In our earlier papers [KLP1, KLP2] we introduced and proved the equivalence of several concepts generalizing to higher rank the notion of convex cocompact subgroups of rank 1 Lie groups, among them the notion of RCA subgroups. These properties are equivalent to the concept of *Anosov subgroups* defined in [La, GW], see [KLP2, Theorem 1.7].

Theorem 2 ([KL])**.** Let $\Gamma < G$ be a uniformly regular discrete subgroup.

(i) There exists a Γ-invariant open subset $\Omega^{\bar\theta}(\Gamma) \subset \partial^{\bar\theta}_\infty X$ such that the action

$$\Gamma \curvearrowright X \cup \Omega^{\bar\theta}(\Gamma) \subset \overline{X}^{\bar\theta} \tag{1}$$

is properly discontinuous. As a consequence, the quotient

$$\left(X \cup \Omega^{\bar\theta}(\Gamma)\right)/\Gamma$$

provides a real-analytic bordification of X/Γ as an orbifold with corners.

(ii) If the chamber limit set $\Lambda_{ch}(\Gamma) \subset \partial_{F\ddot{u}} X$ is conical, then the action (1) is also cocompact, and the quotient provides a real-analytic compactification of X/Γ as an orbifold with corners.

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Higgs bundles and applications

Laura P. Schaposnik
(joint work with David Baraglia, Nigel Hitchin)

Higgs bundles were first studied by Nigel Hitchin in 1987, and appeared as solutions of Yang-Mills self-duality equations on a Riemann surface [5]. Classically, a *Higgs bundle* on a compact Riemann surface Σ of genus $g \geq 2$ is a pair (E, Φ) where E is a holomorphic vector bundle on Σ, and Φ, the *Higgs field*, is a holomorphic 1-form in $H^0(\Sigma, \mathrm{End}_0(E) \otimes K_\Sigma)$, for K_Σ the cotangent bundle of Σ and $\mathrm{End}_0(E)$ the traceless endomorphisms of E. Higgs bundles can also be defined for complex groups G_c, and through stability conditions, one can construct their moduli spaces $\mathcal{M}_{G_c}$.

A natural way of studying the moduli space of Higgs bundles is through the *Hitchin fibration*, sending the class of a Higgs bundle (E, Φ) to the coefficients of the characteristic polynomial $\det(xI - \Phi)$. The generic fibre is an abelian variety, which can be seen through line bundles on an algebraic curve S, the *spectral curve* associated to the Higgs field. The *spectral data* is then given by a line bundle on S satisfying certain conditions, and it provides a geometric description of the fibres of the Hitchin fibration. For instance in the case of classical Higgs bundles, the smooth fibres can be seen through spectral data as Jacobian varieties of S.

Higgs bundles and branes. The smooth locus of the moduli space $\mathcal{M}_{G_c}$ of G_c-Higgs bundles on a compact Riemann surface Σ for a a complex reductive Lie group G_c is a hyper-Kähler manifold, so there are natural complex structures I, J, K obeying the same relations as the imaginary quaternions (adopting the notation of [8]). Adopting physicists' language, a Lagrangian submanifold of a symplectic manifold is called an *A-brane* and a complex submanifold a *B-brane*. A submanifold of a hyper-Kähler manifold may be of type A or B with respect to each of the complex or symplectic structures, and thus choosing a triple of structures one may speak of branes of type $(B, B, B), (B, A, A), (A, B, A)$ and (A, A, B).

It is hence natural to construct different families of branes inside the moduli space $\mathcal{M}_{G_c}$, as was first done in [8] (see also [14]). Together with D. Baraglia we introduced a naturally defined triple of commuting real structures i_1, i_2, i_3 on $\mathcal{M}_{G_c}$, and through the *spectral data* gave a detailed picture of their fixed point sets as different types of branes. Given (E, Φ) a G_c-Higgs bundle on Σ, consider pairs $(\overline{\partial}_A, \Phi)$, where $\overline{\partial}_A$ denotes a $\overline{\partial}$-connection on E defining a holomorphic structure, and Φ is a section of $\Omega^{1,0}(\Sigma, \mathrm{ad}(E))$, for $\mathrm{ad}(E)$ the adjoint bundle of E. Through the Cartan involution θ of a real form G of G_c one obtains

$$i_1(\bar{\partial}_A, \Phi) = (\theta(\bar{\partial}_A), -\theta(\Phi)). \tag{1}$$

Moreover, a real structure $f : \Sigma \to \Sigma$ on Σ induces an involution i_2 given by

$$i_2(\bar{\partial}_A, \Phi) = (f^*(\partial_A), f^*(\Phi^*)) = (f^*(\rho(\bar{\partial}_A)), -f^*(\rho(\Phi))). \tag{2}$$

Lastly, by looking at $i_3 = i_1 \circ i_2$, one obtains $i_3(\bar\partial_A, \Phi) = (f^*\sigma(\bar\partial_A), f^*\sigma(\Phi))$. The fixed point sets of i_1, i_2, i_3 are branes of type $(B, A, A), (A, B, A)$ and (A, A, B) respectively. In [3], with D. Baraglia we studied these branes through the associated *spectral data* and described the topological invariants involved using KO, KR and equivariant K-theory. In particular, it was shown that amongst the fixed points of i_1 are solutions to the Hitchin equations with holonomy in G.

Higgs bundles and (A, B, A)-branes. Amongst the fixed points of the involution i_2 are representations of $\pi_1(\Sigma)$ that extend to certain 3-manifolds M whose boundary is Σ. Indeed, consider the space $\overline{\Sigma} = \Sigma \times [-1, 1]$ with involution $\tau(x, t) = (f(x), -t)$, for f the anti-holomorphic involution on Σ giving i_2. The quotient $M = \overline{\Sigma}/\tau$ is a 3-manifold with boundary $\partial M = \Sigma$, and satisfies the following:

Theorem 1 (Baraglia, –)**.** *Let (E, Φ) be a fixed point of i_2 with simple holonomy. Then the associated connection extends over M as a flat connection if and only if the class $[E] \in \tilde K^0_{\mathbb{Z}_2}(\Sigma)$ in reduced equivariant K-theory is trivial.*

Since Langlands duality can be seen in terms of Higgs bundles as a duality between the fibres of the Hitchin fibrations for $\mathcal{M}_{G_c}$ and $\mathcal{M}_{{}^LG_c}$, for ${}^LG_c$ the Langlands dual group of G_c (as was first seen in [4]), it is natural to ask what the duality between branes should be. In [3] we proposed the following:

Conjecture 1 (Baraglia, –)**.** For ${}^L\rho$ the compact structure of ${}^LG_c$, the support of the dual brane of the fixed point set of i_2 is the fixed point set in $\mathcal{M}_{{}^LG_c}$ of

$$ {}^Li_2(\bar\partial_A, \Phi) = (f^*({}^L\rho(\bar\partial_A)), -f^*({}^L\rho(\Phi))). \tag{3}$$

Higgs bundles and (B, A, A)-branes. Higgs bundles can be defined for complex Lie groups G_c, as well as for real forms G of G_c. Moreover, as mentioned in **(A)**, the moduli spaces of real Higgs bundles $\mathcal{M}_G$ lie as (B, A, A) branes inside the moduli spaces $\mathcal{M}_{G_c}$. It is thus natural to ask how $\mathcal{M}_G$ intersects the Hitchin fibration for the complex moduli space $\mathcal{M}_{G_c}$, which is the main subject of [11, 12] (see [10]). Moreover, considering Langlands duality, in [3] we propose the following:

Conjecture 2 (Baraglia, –)**.** The support of the dual brane to the fixed point set of i_1 is the moduli space $\mathcal{M}_{\check H} \subset \mathcal{M}_{{}^LG_c}$ of $\check H$-Higgs bundles for $\check H$ the group associated to the Lie algebra $\check{\mathfrak{h}}$ in [9, Table 1].

On (B, A, A)-branes having finite intersection with smooth fibres. In the case of Higgs bundles for a split real form G of a complex reductive Lie group G_c, from [12] one has the following description of the intersection:

Theorem 2 (–)**.** *The moduli space $\mathcal{M}_G$ as sitting inside $\mathcal{M}_{G_c}$ is given by points of order two in the smooth fibres of the Hitchin fibration $h : \mathcal{M}_{G_c} \to \mathcal{A}_{G_c}$.*

This result is used in [13] to study the moduli space of $SL(2, \mathbb{R})$-Higgs bundles from a combinatorial point of view. The above theorem could also be used when studying L-twisted Higgs bundles (E, Φ) in [1], where the Higgs field is now twisted by any line bundle L obtaining $\Phi : E \to E \otimes L$.

Theorem 3 (Baraglia, –). *The monodromy action of an element $\tilde{s}_\gamma$ in the Hitchin base is the automorphism of $H^1(S,\mathbb{Z})$ induced by a Dehn twist of the spectral curve S around a loop l_γ. Let $c_\gamma \in H^1(S,\mathbb{Z})$ be the Poincaré dual of the homology class of l_γ. Then the monodromy of $\tilde{s}_\gamma$ acts on $H^1(S,\mathbb{Z})$ as a Picard-Lefschetz transformation: $x \mapsto x + \langle c_\gamma, x\rangle c_\gamma$.*

As an application, one can obtain information about some real character varieties $Rep(G)$. For instance one may prove that there are $3.2^{2g}+g-3$ connected components for $Rep(GL(2,\mathbb{R}))$, and recover the number of components of maximal representations $Rep_{2g-2}(Sp(4,\mathbb{R}))$, given by $3.2^{2g}+2g-4$.

On (B,A,A)-branes having no intersection with smooth fibres. For other real forms the brane $\mathcal{M}_G$ may lie interlay inside the singular fibres of the Hitchin fibration for G_c-Higgs bundles. By extending the approach from [12], together with N. Hitchin we showed in [6] that this situation appears naturally when considering Higgs bundles corresponding to flat connections on Σ with holonomy in the real Lie groups $G = SL(m,\mathbb{H})$ and $SO(2n,\mathbb{H})$ (i.e., $SU^*(2m)$ and $SO^*(2n)$ respectively).

Theorem 4 (Hitchin, –). *The fibres of the (B,A,A)-brane in the Hitchin fibration for $SL(m,\mathbb{H})$, $SO(2n,\mathbb{H})$ and $Sp(2m,2m)$-Higgs bundles are not abelian varieties, but are instead moduli spaces of rank 2 bundles on a spectral curve, satisfying certain natural stability conditions.*

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Quasicircles

Yves Benoist
(joint work with Dominique Hulin)

We study the dynamics of the group $G = \mathrm{PSL}(2, \mathbb{C})$ on the set of compact subsets K of the Riemann sphere $\mathbb{S}2 = \mathbb{C} \cup \{\infty\}$ containing at least two points. We prove in [1] that the orbit closure $\overline{GK}$ contains only Jordan curves if and only if K is a quasicircle. We prove in [2] that this G-orbit is closed if and only if K is the limit set of a convex cocompact subgroup of G. We also construct in [2] nonclosed G-orbits whose closure is minimal.

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On $\frac{1}{2}$-PIC 4-manifolds.

Thomas Richard
(joint work with Harish Seshadri)

An oriented riemannian 4-manifold (M^4, g) is said to be $\frac{1}{2}$-PIC if for every $p \in M$ and every *oriented* orthonormal 4-frame (e_1, e_2, e_3, e_4) the curvature tensor R of (M^4, g) at p satisfies :

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0$$

where R_{ijkl} denotes $R(e_i, e_j, e_k, e_l)$.

This is a weakening of the classical PIC (positive isotropic curvature) condition which require the above inequality to hold for *every* 4-frame. Since the foundational work of Micaleff-Moore, the PIC condition is known to have strong topological implications, and in dimension 4 the work of Hamilton and Chen-Zhu has led to a complete classification of compact PIC 4-manifolds using Ricci flow with surgery.

At first sight, $\frac{1}{2}$-PIC 4-manifolds share a great deal of similarities with PIC 4-manifolds :

- $\mathbb{S}^4$, $\mathbb{S}^3 \times \mathbb{R}$ and their metric quotients are $\frac{1}{2}$-PIC.
- A connected sum of $\frac{1}{2}$-PIC 4-manifolds admits a $\frac{1}{2}$-PIC metric.
- The $\frac{1}{2}$-PIC condition is stable under the Ricci flow evolution equation.

However, there are some differences :

- $\overline{\mathbb{CP}}^2$ is $\frac{1}{2}$-PIC whereas it is not PIC.
- Consequently, the second Betti number of a $\frac{1}{2}$-PIC 4-manifold can be non zero whereas it must vanish on even-dimensional PIC manifolds.
- However, a standard Bochner formula can be used to show that $b_2^+(M^4)$ (the dimension of the positive space of the intersection form on $H^2(M^4, \mathbb{R})$) must vanish.

Adapting an argument of Brendle and using the classification by Hitchin of compact half-conformally flat Einstein 4-manifolds allows to show the following rigidity result: a compact Einstien $\frac{1}{2}$-PIC 4-manifold is isometric to $\mathbb{S}^4$ or $\overline{\mathbb{CP}}^2$.

The $\frac{1}{2}$-PIC condition also has a peculiar relation to Ricci flow in 4-dimension: every Ricci flow invariant positivity condition on the curvature of a 4-manifold which is stronger that "positive scalar curvature" is actually stronger than the $\frac{1}{2}$-PIC condition (or the similar condition obtained by reversing the orientation).

A detailed exposition of these result can be found in [1]

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Projective surfaces, holomorphic curves and the SL(3, $\mathbb{R}$)-Hitchin component

Thomas Mettler

A *projective structure* on a smooth surface M is an equivalence class $[\nabla]$ of torsion-free connections on TM. Two such connections ∇ and ∇' are *projectively equivalent* if they share the same unparametrised geodesics. A projective structure $[\nabla]$ is called *flat* if locally $[\nabla]$ is defined by a flat connection. Equivalently, a flat projective structure on M can be defined as a maximal atlas mapping open sets in M into $\mathbb{RP}^2$ such that the transition functions are restrictions of fractional linear transformations. The maximal atlas of a flat projective structure on M gives rise to a *developing map* $\mathrm{dev} : \tilde{M} \to \mathbb{RP}^2$ defined on the universal cover of M, as well as a representation of the fundamental group of M into $\mathrm{PSL}(3,\mathbb{R})$, well defined up to conjugation. A flat projective structure is called *convex* if the developing map is a diffeomorphism onto a convex subset of $\mathbb{RP}^2$. A surface equipped with a projective structure will be called a *projective surface.*

The space $\mathfrak{A}(M)$ of torsion-free connections on TM is an affine space modelled on the space of section of the vector bundle $S^2(T^*M) \otimes TM$. By a classical result of Weyl [6], two torsion-free connection on TM are projectively equivalent if and only if their difference is pure trace. In particular, a projective structure $[\nabla]$ on M may be thought of as an affine subspace $\mathfrak{A}_{[\nabla]}(M) \subset \mathfrak{A}(M)$ which is modelled on the space of 1-forms on M.

A conformal connection ∇ (also called Weyl connection) is a torsion-free connection on TM preserving some conformal structure $[g]$ on M, that is, the parallel transport maps of ∇ are angle-preserving with respect to some $[g]$. As in the case of a projective structure $[\nabla]$, the space of $[g]$-conformal connections is an affine subspace $\mathfrak{A}_{[g]}(M) \subset \mathfrak{A}(M)$ which is modelled on the space of 1-forms on M. It easy to check that two subspaces $\mathfrak{A}_{[g]}(M)$ and $\mathfrak{A}_{[\nabla]}(M)$ intersect in at most one point. More generally, denoting by $\mathfrak{P}(M)$ the space of projective – and by $\mathfrak{C}(M)$ the space of conformal structure on M, we have the following:

Proposition 1. *There exists a* $\mathrm{Diff}(M)$*-equivariant map*

$$\mathfrak{P}(M) \times \mathfrak{C}(M) \to \mathfrak{A}(M) \times \Omega^1(M, \mathrm{End}(TM)), \quad ([\nabla], [g]) \mapsto ({}^{[g]}\nabla, A_{[g]})$$

having the property that $[\nabla]$ *is defined by* ${}^{[g]}\nabla + A_{[g]}$.

In particular, we obtain a $\mathrm{Diff}(M)$-invariant non-negative functional

$$\mathcal{E} : \mathfrak{P}(M) \times \mathfrak{C}(M) \to \mathbb{R}, \quad ([\nabla], [g]) \mapsto \int_M |A_{[g]}|_g^2 \mathrm{d}\mu_g$$

and a global invariant

$$\Upsilon(M, [\nabla]) = \inf_{[g] \in \mathfrak{C}(M)} \mathcal{E}([\nabla], [g]),$$

which measures how "far" $[\nabla]$ is from being defined by a conformal connection. It turns out that $\mathcal{E}$ can – up to a topological constant – be interpreted as a Dirichlet energy. To this end one shows that one can canonically construct an indefinite Kähler-Einstein 3-fold (Y, h, ω) fibering over $(M, [\nabla])$. Furthermore, a conformal structure $[g]$ on M gives rise to a section $\widetilde{[g]}$ of $Y \to M$ so that for M closed we obtain

$$\frac{1}{2} \int_M \mathrm{tr}_g \widetilde{[g]}^* h \, \mu_g = \int_M |A_{[g]}|_g^2 \mu_g - 4\pi\chi(M).$$

If we fix a projective structure $[\nabla]$ and consider $\mathcal{E}_{[\nabla]} := \mathcal{E}([\nabla], \cdot)$ we have:

Theorem 1. *A conformal structure* $[g]$ *on* M *is a critical point of* $\mathcal{E}_{[\nabla]}$ *with respect to compactly supported variations if and only if* $\widetilde{[g]} : M \to (Y, h, \omega)$ *is weakly conformal.*

If the surface M satisfies $\chi(M) < 0$, then $\mathcal{E}_{[\nabla]}$ admits a unique absolute minimiser. The proof of this fact relies on complex geometric methods. Indeed, in [4] it was shown that an oriented projective surface $(M, [\nabla])$ defines a complex surface Z together with a projection to M whose fibres are holomorphically embedded disks. Moreover, a conformal connection in the projective equivalence class corresponds to a section whose image is a holomorphic curve in Z. Locally such sections always exist and hence every affine torsion-free connection on a surface is locally projectively equivalent to a conformal connection. Furthermore every conformal connection on the 2-sphere lies in a complex 5-manifold of conformal connections sharing the same unparametrised geodesics. This is in contrast to the case where M has negative Euler-characteristic. In this case the bundle $Z \to M$ admits at most one section whose image is a holomorphic curve. In particular, a Riemannian metric on M is uniquely determined – up to constant rescaling – by its unparametrised geodesics [3]. This result was previously proved in [2].

One may naturally ask to characterise the projective surfaces that are globally defined by a conformal connection. Whereas this question is wide open in general, there is some evidence that a convex projective structure $[\nabla]$ is defined by a conformal connection if and only if $[\nabla]$ is defined by the Levi-Civita connection of a hyperbolic metric. In fact, finding a minimiser of $\mathcal{E}_{[\nabla]}$ turns out to be related to the parametrisation of the $\mathrm{SL}(3,\mathbb{R})$-Hitchin component in terms of holomorphic cubic differentials.

As a by-product of the above considerations one also obtains a Gauss-Bonnet type identity for closed oriented projective surfaces:

Theorem 2. *Let $(M,[\nabla])$ be a closed oriented projective surface. For any smooth section $s: M \to Y$, we have*

$$\int_M s^* c_1(Y) = 4\chi(M), \tag{1}$$

where $c_1(Y)$ denotes the first Chern-class of Y.

As a corollary one obtains a (slight) strengthening of a result by Benzécri [1] (see also [5])

Corollary 1. *A closed surface M carries a torsion-free connection ∇ on TM whose Ricci curvature is totally skew-symmetric if and only if $\chi(M) = 0$.*

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Minimal entropy for symmetric spaces

Louis Merlin

Introduction

Given a metric space (X,d), together with a Borel measure μ, one defines the *upper* and *lower volume entropies* of (X,d,μ) as

$$h_u = \limsup_{R\to\infty} \frac{\log \mu(B(x_0,R))}{R}, \qquad h_l = \liminf_{R\to\infty} \frac{\log \mu(B(x_0,R))}{R},$$

It is known that these entropies do not depend on the chosen point $x_0 \in X$, furthermore if X is the universal cover of a compact Riemannian manifold, then both entropies coincide and they are equal to the topological entropy associated to the geodesic flow, see [4].

Note that if the metric measure space has polynomial volume growth, that is if $\mu(B(x_0,R)) \leq C \cdot R^d$ for some $d \in \mathrm{R}$ and all $R \geq 1$, then we obviously have

$h_u = 0$, while if $\mu(B(x_0, R)) \sim C \cdot e^{hR}$ for large R, then $h_u = h_l = h$. In particular the volume entropy of the n-dimensional real hyperbolic space is $h = (n-1)$ and that of the complex hyperbolic space is $h = n$ (the complex dimension).

is elementary), yet it has surprisely deep connections with sophisticated geometric invariants such as Gromov's bounded cohomology and simplicial volume, spectral and systolic geometry, the geodesic flow etc.

I propose to describe the so-called minimal entropy problem, give a brief state of the art. I'll be more precise on my own contribution.

The minimal entropy problem

In 1995 Besson, Courtois and Gallot [1] proved a theorem stating that *on a compact locally symmetric Riemannian manifold of negative curvature* (M, g_0), *the symmetric metric* g_0 *minimizes the volume entropy among all Riemannian metrics of same volume on* M *and, if* $\dim(M) \geq 3$, *the metric* g_0 *is the unique entropy minimizer.*

This result answered a conjecture by Gromov and Katok and has a number of deep consequences, including a new proof of Mostow's rigidity theorem and its generalizations by Siu, Corlette and Thurston. The question of extending the above result in now known as the minimal entropy problem.

One of the main ingredient in the proof of the minimal entropy theorem by Besson, Courtois and Gallot is the *method of calibration* which has its origin in the work of Harvey and Lawson on minimal submanifolds. Let me give a very rough sketch of the proof to illustrate some of the main concepts and ideas.

Let (M, g_0) be a compact Riemannian manifold which is locally isometric to symmetric space of strictly negative curvature X (so X is either the usual real hyperbolic space, the complex or quaternionic hyperbolic space or the Cayley hyperbolic plane). In this case the manifold M is isometric to a quotient $\Gamma \backslash X$ with Γ isomorphic to the fundamental group of M. The boundary at infinity of X can be identified with the tangent unit sphere at a base point, and this identification provides $\partial_\infty X$ with a natural measure. Furthermore Γ acts by isometries on $L^2(\partial_\infty X)$ (Lemma 2.2 in [1]). If g is a metric on M, we denote by $\tilde{M}$ the Riemannian universal cover of M (it is diffeomorphic to X). Now Besson, Courtois and Gallot proceed as follows:

(1) To any Riemannian metric g on M, they associate a Γ-equivariant embedding $\Phi_g : \tilde{M} \to L^2(\partial_\infty X)$. Furthermore Φ_g takes its values in the unit sphere $S^\infty \subset L^2(\partial_\infty X)$.
(2) If now Ω is a closed Γ-equivariant n-form on S^∞, then $\Phi_g^*(\Omega)$ is a Γ-invariant n-form on X which defines a volume form on M (still denoted by $\Phi_g^*(\Omega)$). If g_1 and g_2 are two metrics on M, then an equivariant homotopy argument together with Stoke's theorem implies that

$$\int_M \Phi_{g_1}^*(\Omega) = \int_M \Phi_{g_2}^*(\Omega)$$

(3) To any element $\Phi \in S^\infty$ one can associate a "barycenter" $\pi(\Phi) \in X$. Combining this with the previous construction, we have a map $F : (\tilde{M}, g) \to X$, defined by $F(x) = \pi(\Phi_g(x))$ and this maps decreases the volume up to a ratio $\frac{h(g)}{h(g_0)}$.

(4) If ω_0 is the canonical volume form on X and $\Omega_0 = \pi^*(\omega_0)$, then

$$h_0 \cdot \Omega_0(\xi) \leq (4n)^{n/2} \operatorname{vol}_n(\xi)$$

for any n-vector $\xi \in \Lambda^n TS^\infty$, where h_0 is the volume entropy of the canonical metric on X. Furthermore Ω_0 is calibrating the immersion Φ_{g_0} in the sense that for any n-vector ξ tangent to the immersion Φ_{g_0}, the above inequality is an equality [1, Proposition 5.7].

It then follows that

$$\operatorname{Vol}(\Phi_g) \geq \int_M \Phi_g^*(\Omega_0) = \int_M \Phi_{g_0}^*(\Omega_0) = \operatorname{Vol}(\Phi_{g_0}),$$

where the volume $\operatorname{Vol}(\Phi_g)$ of the embedding is by definition the volume of the image of a fundamental domain in X for the metric induced by the map $\Phi_g : X \to L^2$. The first inequality holds for any metric g, the middle equality is explained in item (2) above and the last equality is due to Ω_0 calibrating Φ_{g_0}. It is now not hard to relate $\operatorname{Vol}(\Phi_g)$ to the usual volume and to the entropy of g, which finally gives us

$$h(g)^n \operatorname{Vol}(g) \geqslant h(g_0)^n \operatorname{Vol}(g_0).$$

The heart of the argument is thus to produce an equivariant immersion of X in a function space together with the construction of a calibration Ω_0 to show that the immersion is minimal. The calibration is the form $\pi^*(\omega_0)$ where π is the projection to the center of mass. What breaks down in higher rank symmetric spaces is precisely the construction of the barycenter and new techniques are needed to obtain a calibration.

My Ph.D dissertation brings a contribution to the minimal entropy conjecture. The main result deals with the previously unknown case of compact quotient of products of hyperbolic planes $(\mathbb{H}^2)^n$. Les us state this result.

Theorem 1 ([2]). *Let (M, g_0) be a compact manifold, locally isometric to $(\mathbb{H}^2)^n$. Let g be any other metric on M (no curvature assumption is needed). Then*

$$h(g_0)^{2n} \operatorname{Vol}(g_0) \leqslant h(g)^{2n} \operatorname{Vol}(g).$$

Note that, unlike the quoted works of Besson, Courtois and Gallot, this theorem does not characterize the equality case.

As explained above, the general method is to work out a method of calibration. But several steps in the construction of Besson, Courtois and Gallot are not transposable.

A first difficulty is to choose the appropriate notion of boundary of a symmetric space of higher rank (not negatively curved but only nonpositively curved). Indeed there are several way to compactify a symmetric space which coincide in the rank one case. A natural choice in this situation is to choose the so-called Furstenberg boundary which has, among other nice properties, the quality of being well behaved with respect to product. Indeed the Furstenberg boudary of a product of symmetric space is the product of Furstenberg boundaries of the factors. Hence, the Furstenberg boundary of $\left(\mathbb{H}^2\right)^n$ is the n-torus $\mathbb{T}^n$. Since we only deal with $\left(\mathbb{H}^2\right)^n$-manifolds, we do not need a general definition of the Furstenberg boundary (see however [3]).

A second more serious difficulty is that the barycenter method is no longer efficient. One of the main ideas in this work is to use the following general process to build appropriate differential forms that we can substitute to the barycenter pull backed form.

We use objects from the field of bounded cohomology (see [5] for references on the subject). We start with a bounded function which takes k parameters on $\mathbb{T}^n$

$$c : (\mathbb{T}^n)^k \mapsto \mathbb{R}$$

and we construct a differential form on $\mathrm{L}^2(\mathbb{T}^n)$ by

$$\Omega(c)_\varphi(f_1, \cdots, f_k) = \int_{(\mathbb{T}^n)^{k+1}} c(\theta_0, \cdots, \theta_k)\varphi^2(\theta_0)\varphi f_1(\theta_1) \cdots \varphi f_k(\theta_k) d\theta_0 \cdots d\theta_k.$$

The fact that c is bounded ensures that the integral converges. In order to make a differential form (i.e an alternate form), one has to choose c alternate, that is, for any permutation $\sigma \in \mathfrak{S}_k$ and $(\theta_1, \cdots, \theta_k) \in \mathbb{T}^n$,

$$c(\theta_{\sigma(1)}, \cdots, \theta_{\sigma(k)}) = \mathrm{sign}(\sigma) c(\theta_1, \cdots, \theta_k)$$

This gives us plenty of candidates for being the calibrating form we are looking for. In this construction the flexibility is of course that we can choose any function. The question now becomes : What kind of function c can we choose in order to make $\Omega(c)$ a calibrating form ? I found out some characterizations of c to make the differential form closed and $\mathrm{Isom}(\left(\mathbb{H}^2\right)^n)$ invariant (which are required conditions) and he finally gave a function which can be turned into a calibrating form. More precisely

Proposition 2. *c must define a bounded cohomology class of the quotient manifold M, that is c must be itself* $\mathrm{Isom}(\left(\mathbb{H}^2\right)^n)$*-invariant and must be cohomologically closed.*

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Sweepouts of Riemannian manifolds with Ricci curvature bounded from below

Yevgeny Liokumovich
(joint work with Parker Glynn-Adey, Xin Zhou)

We prove two results about sweepouts of Riemannian manifolds by cycles of controlled mass.

1. Let M be a closed Riemannian manifold of dimension n conformally equivalent to a manifold M_0 with Ricci curvature bounded from below by $-a^2(n-1)$. In [1] we construct a sweepout of M by hypersurfaces of volume at most $Cmax\{1, aVol(M_0)^{1/n}\}Vol(M)^{(n-1)/n}$. This bound is sharp up to a constant. We use this bound and similar bounds for higher parametric families of $(n-1)-$cycles to estimate volumes of minimal hypersurfaces in M. This is a joint work with Parker Glynn-Adey (Toronto).

2. Let M be a 3-manifold with positive Ricci curvature. In [2] we construct a sweepout of M by 1-cycles of length at most $CVol(M)^{1/3}$. This is a joint work with Xin Zhou (MIT).

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Complex deformations of n-Fuchsian representations

David Dumas
(joint work with Andrew Sanders)

An n-Fuchsian representation of the fundamental group $\Pi = \pi_1(S)$ of a compact surface S is a homomorphism $\Pi \to \mathrm{SL}_n\mathbb{R}$ which can be described as a composition of an embedding $\Pi \to \mathrm{SL}_2\mathbb{R}$ as a uniform lattice with the n-dimensional irreducible representation $\mathrm{SL}_2\mathbb{R} \to \mathrm{SL}_n\mathbb{R}$. The connected component of the $\mathrm{SL}_n\mathbb{R}$-character variety of Π containing the n-Fuchsian representations is the *Hitchin component*, a central object of study in recent efforts to generalize classical Teichmüller theory to higher-rank Lie groups.

We study the topology and complex geometry associated with deformations of n-Fuchsian and Hitchin representations into the complex Lie group $\mathrm{SL}_n\mathbb{C}$. Our work focuses on the actions of such representations on the flag variety of $\mathrm{SL}_n\mathbb{C}$, and in particular on the cocompact domains of discontinuity constructed by Guichard-Wienhard [2] and their associated quotient manifolds. We show that for $n \geq 3$ these compact, complex quotient manifolds are non-Kähler and rationally chain-connected, and we compute their cohomology. In the case $n = 3$ (and conjecturally for all n) the manifold is a smooth fiber bundle over the surface S, however this bundle structure cannot be made holomorphic.

We also study the deformation theory of these manifolds and its relation to the $\mathrm{SL}_n\mathbb{C}$-character variety of the surface group. For example, in the n-Fuchsian case the quotient manifold has a natural stratification in which the minimal strata are holomorphically embedded Riemann surfaces diffeomorphic to S. It is natural to ask whether this stratification, or some part of it, persists under the deformation of complex structure induced by a small change in the representation. We establish holomorphic rigidity results showing that the only deformations of representations in which these minimal strata persist to first order are tangent to the image of the $\mathrm{SL}_2\mathbb{C}$-character variety under the n-dimensional irreducible representation (i.e. they are "n-quasi-Fuchsian" to first order). We also classify first-order deformations in which just one minimal stratum persists, identifying them with the SL_n-*opers* of Beilinson-Drinfeld [1].

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Smoothing properties of the Kähler-Ricci flow

Eleonora Di Nezza

(joint work with Lu C., Chalmers University of Technology)

Let X be a compact Kähler manifold of complex dimension n and let $\alpha_0 \in H^{1,1}(X, \mathrm{R})$ be a Kähler class. Fix $\omega_0 \in \alpha_0$ a Kähler form. We say that a family of Kähler metrics $\omega_t := \omega(t)$ solves the Kähler-Ricci flow (KRF for short) starting from ω_0 if

$$(*) \qquad \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$$

and $\omega(0) = \omega_0$.

The Kähler-Ricci flow became one of the major tools in Kähler geometry through the work of many authors starting from Cao [1] who proved that the Kähler-Ricci flow on a compact Kähler manifold with non positive first Chern class $c_1(X) \leq 0$ converges to the unique Kähler-Einstein metric endowed by the manifold.

The existence and uniqueness of the Kähler-Ricci flow starting from any Kähler form is due to Cao [1], Tsuji [7] and Tian-Zhang [6]:

Theorem. *Let $\omega_0 \in \alpha_0$ be a Kähler form. Then there exists a unique family of Kähler metrics $(\omega(t))_{t\in[0,T_{\max})}$ satisfying $(*)$ and $\omega(0) = \omega_0$ where*

$$T_{\max} := \sup\{t > 0 \,|\, \alpha_0 - tc_1(X) > 0\}$$

is the maximal time of existence of the flow.

In relation to the "analytic analogue" of the Minimal Model Program, recently proposed by Song and Tian, one need to start the KRF from a "degenerate" initial data rather than a Kähler form.
Observe that $T_{\max}$ does not depend on the initial data but only on its cohomology class α_0, so at least it makes sense asking whether one can start the flow from any positive closed $(1,1)$-current $T_0 \in \alpha_0$.

In this direction Song and Tian [5] proved that if $T_0 \in \alpha_0$ is a positive $(1,1)$-current with continuous potential, then there exists a unique family of Kähler metrics $(\omega(t))_{t\in(0,T_{\max})}$ satisfying $(*)$ and such that $\omega(t)$ converges to ω_0 uniformly as t goes to zero.

Recently, Guedj and Zeriahi [3] proved that if $T_0 \in \alpha$ is a current with zero Lelong numbers at any point, $\nu(T_0, x) = 0\ \forall x \in X$, then there exists a family of Kähler metrics $(\omega(t))_{t\in(0,T_{\max})}$ satisfying $(*)$ and such that $\omega(t)$ converges to T_0 in the weak sense of currents as t goes to zero.
Observe that the above result insures that starting from any positive current with zero Lelong numbers, the KRF immediately smooths out.

Then, a natural question is: *what does it happen when T_0 has positive Lelong numbers?*
The result that we were able to prove with Lu [2] is the following:

Theorem A. *Let $T_0 \in \alpha_0$ be any positive $(1,1)$-current such that $c(T_0) > \frac{1}{2T_{\max}}$. Then there exists a unique family of positive $(1,1)$-currents, $(\omega(t))_{t\in(0,T_{\max})}$ such that $\omega(t)$ is smooth on the Zariski open subset $X \setminus D_{k(t)}$ and here it solves $(*)$ in the classical sense.*
Moreover, $\omega(t)$ converges to T_0 in the weak sense of currents as t goes to zero.

Here $c(T_0)$ denotes the critical exponent of integrability of T_0.

For any $t \in (0, T_{\max})$, the subset $D_{k(t)}$ in the statement of the Theorem is described as the Lelong superlevel set of T_0:

$$D_{k(t)} := \{x \in X \mid \nu(T_0, x) > k(t)\}$$

where the constant $k(t)$ depends only on t and it is decreasing to 0 as t goes to 0. Note that for any $s > 0$, D_s is an analytic subset of X [4].

We also give a more precise description of the flow. More precisely:

(i) for any $t > 1/2c(T_0)$, ω_t is smooth everywhere, i.e. is a genuine Kähler form on X.
(ii) for short time, precisely when $t < 1/2nc(T_0)$, ω_t has positive Lelong numbers if T_0 has.

In particular, *(ii)* insures that the result in Theorem A is sharp and that Lelong numbers are indeed an obstruction for the smoothing.

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Alexandrov meets Dido

Alexander Lytchak
(joint work with Stefan Wenger)

In the talk I discuss the following theorem:

A proper metric space X satisfies the Euclidean isoperimetric inequality for curves if and only if the space has globally non-positive curvature in the sense of Alexandrov.

Here the first condition is understood in the sense that any closed Lipschitz curve of length l bounds some Lipschitz disc of parametrized Hausdorff area at most $l^2/4\pi$. Our theorem extends to other curvature bounds and non-geodesic quasi-convex spaces.

The if direction of the theorem is a simple consequence of Reshetnyak's majorization theorem. Also the only if direction has been shown by Reshetnyak for 2-dimensional Riemannian manifolds with integral curvature bounds. Essentially, our proof reduces the general case to the case of Riemann surfaces studied by Reshetnyak. The main ingredients of the proof are the solution of the classical Plateau problem in arbitrary proper metric spaces, the local Hoelder regularity of such minimal discs, and the topological and geometric properties of the intrinsic metric on minimal discs. The basic idea is very simple: In order to prove that a triangle in the space is thin, we span it by a minimal disc and prove that this disc comes along with a non-positively curved metric. This provides short curves between sides of the triangles inside the minimal disc.

On the total curvature of geodesics

Anton Petrunin
(joint work with Nina Lebedeva)

We give an upper bound for the total curvature of minimizing geodesics on convex surfaces in the 3-dimensional Euclidean space.

In [1], J. Liberman gave an estimate for the total curvature for geodesics on convex surface in terms of the length of geodesic and the diameter, inradius of the surface. The question whether there is a universal upper bound for minimizing geodesics was asked by D. Burago it was also mentioned in several papers, see [2, 3, 4]

In addition we discuss the following related topics:

- The counterexample of A. Milka and V. Usov (see [5] and [6]) to a related conjecture of A. Pogorelov from [7] on the length of spherical image of minimizing geodesics.
- The construction of corkscrew geodesic given by I. Bárány, K. Kuperberg and T. Zamfirescuin in [4].
- The optimal bound on total curvature of geodesics on the graphs of convex Lipschitz functions and epigraph of arbitrary Lipschitz functions; these results are by V. Usov and D. Berg correspondingly, see [8, 9].

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