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# Dynamische Systeme

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Abstract. This workshop continued the biannual series at Oberwolfach on Dynamical Systems that started as the "Moser-Zehnder meeting" in 1981. The main themes of the workshop are the new results and developments in the area of dynamical systems, in particular in Hamiltonian systems and symplectic geometry related to Hamiltonian dynamics.

*Mathematics Subject Classification (2010):* 37, 53D.

## Introduction by the Organisers

The workshop was organized by H. Eliasson (Paris), H. Hofer (Princeton) and J.-C. Yoccoz (Paris). It was attended by more than 50 participants from 13 countries and displayed a good mixture of young, mid-career and senior people. The workshop covered a large area of dynamical systems centered around classical Hamiltonian dynamics: symplectic dynamics and geometry; billiards; Hamiltonian PDE's; dynamics of vector fields and mappings on manifolds; Hamilton-Jacobi theory and weak KAM; celestial mechanics; circle diffeomorphisms; diffusion. Also other parts of dynamics were represented.

J. Fish presented a new result showing that Hamiltonian flows on compact hypersurfaces in 4-space are not minimal. This answers a question raised by M. Herman in his 1998 ICM address. The result constitutes a significant progress on a problem raised by Gottschalk in 1958 concerning the existence of a minimal flow on the three-sphere.

L. Polterovich reported on work finding robust obstructions to represent a Hamiltonian diffeomorphism as a k-th power using a Floer-theoretic version of persistence modules and described applications to the geometry and dynamics of Hamiltonian diffeomorphisms.

D. Christofaro-Gardiner discussed an important relationship between the Reeb dynamics on a three-dimensional closed manifold equipped with a contact form and volume considerations coming from Seiberg-Witten theory. In particular there always have to be at least two periodic orbits.

N. Roettgen displayed examples of Reeb vector fields in higher dimensions, where the existence of trapped orbits does not implies the existence of periodic orbits unlike the three-dimensional case, where it was established by Eliashberg and Hofer. P. Albers talked about Hofer-Zehnder capacities and S. Hohloch about semi-toric integrable Hamiltonian systems.

D. Peralta-Salas presented, in a beautiful talk, new stationary solutions of the 3D Euler equation and studied their dynamics by KAM-theory. These solutions possess linearly stable periodic orbits surrounded by invariant tori (vortex tubes). M. Berti talked about quasi-periodic solutions for water wave equations, and Z. Zhao discussed ballistic motion in lattice Schrödinger equations. The talk of S. Kuksin focused on wave turbulence.

V. Baladi presented in a masterful manner the proof of exponential decay for Sinai billiards, the conclusion of a long search that has led to the creation of important tools in the functional-analytic approach to hyperbolic dynamics. G. Forni focused on non-rational polygonal billiards pointing out many important open questions. He presented a criterion for the ergodicity of such billiards related to the Cheeger constant of the phase space equipped with a renormalized metric.

A. Katok asked a natural and intriguing question about Lyapunov exponents of volume-preserving diffeomorphisms. P. Berger presented new results on the Newhouse phenomenon, and K. Kuperberg discussed the Seifert and the Modified Seifert conjecture.

M. Zavidovique presented a convergence result of viscosity solutions of Hamiltonian – Jacobi equations which lies at the basis of weak KAM-theory. Transposing to the context of optimal transport in compact metric spaces, he demonstrated how elementary arguments may lead to deep results.

M. Guardia, A.Knauf and Y. Long presented new results in celestial mechanics. V. Kaloshin displayed a new phenomenon of stochastic Arnold diffusion, and R. Krikorian discussed recent results on almost linearization of circle diffeomorphism.

J. Bochi proved results on linear representations using dominated splitting for co-cycles. A. Gorodetski presented new results on sums of Cantor sets. M. Levi discussed high-frequency vibrations in mechanical models and S. Katok Fuchsian groups and coding of geodesics.

The meeting was held in an informal and stimulating atmosphere. The weather was excellent and many participants attended the traditional walk to St. Roman under the leading of Sergei Tabachnikov.

*Acknowledgement:* The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

## Workshop: Dynamische Systeme

## Table of Contents

Joel W. Fish (joint with Helmut Hofer)
*Feral pseudoholomorphic curves and minimal sets* .................... 1941

Pierre Berger
*Generic family with robustly infinitely many sinks* .................... 1941

Daniel Peralta-Salas
*Existence of knotted and linked invariant tori in the stationary Euler equations* .................... 1942

Jairo Bochi (joint with Rafael Potrie, Andrés Sambarino)
*Linear representations and dominated splittings* .................... 1945

Viviane Baladi (joint with Mark Demers, Carlangelo Liverani)
*Exponential decay of correlations for Sinai billiard flows* .............. 1948

Leonid Polterovich (joint with Egor Shelukhin)
*Hamiltonian diffeomorphisms and persistence modules* ................ 1949

Giovanni Forni
*On the ergodicity of billiards in non-rational polygons* ................ 1950

Vadim Kaloshin (joint with Oriol Castejon, Marcel Guardia, Jianlu Zhang, Ke Zhang)
*Stochastic Arnold diffusion of deterministic systems* ................. 1953

Peter Albers (joint with Alexandru Oancea, Urs Frauenfelder)
*Homology with twisted coefficients and vanishing of the Hofer-Zehnder capacity* .................... 1953

Nena Röttgen (joint with Hansjörg Geiges, Kai Zehmisch)
*Trapped Reeb orbits do not imply periodic ones* ...................... 1955

Anton Gorodetski
*Sums of Cantor sets and convolutions of singular measures* ........... 1957

Zhiyan Zhao
*Ballistic motion in one-dimensional lattice Schrödinger equation* ...... 1958

Massimiliano Berti (joint with Riccardo Montalto)
*KAM for Water Waves* .................... 1960

Sonja Hohloch (joint with Silvia Sabatini, Daniele Sepe, Margaret Symington)
*From semi-toric systems to Hamiltonian* $\mathbb{S}^1$*-actions and back* .......... 1964

Mark Levi
*Geometry of high frequency vibrations* ... 1966

Marcel Guardia (joint with Pau Martin, Tere M. Seara, Lara Sabbagh)
*Oscillatory orbits in the restricted planar three body problem* ... 1967

Daniel Cristofaro-Gardiner (joint with Michael Hutchings, Vinicius Ramos)
*Volume in Seiberg-Witten theory and the existence of two Reeb orbits* ... 1968

Maxime Zavidovique (joint with Andrea Davini, Albert Fathi, Renato Iturriaga)
*Convergence in the vanishing discount method for Hamilton–Jacobi equations* ... 1969

Svetlana Katok (joint with Ilie Ugarcovici)
*Reduction theory for Fuchsian groups and coding of geodesics* ... 1972

Sergei B. Kuksin
*Wave turbulence: the conjecture, approaches and rigorous results* ... 1975

Raphaël Krikorian
*On the almost reducibility of circle diffeomorphisms* ... 1978

Yiming Long
*Determining the linear stability of periodic orbits of the N-body problem via index theory* ... 1979

Andreas Knauf (joint with Jacques Féjoz, Richard Montgomery)
*Train Correspondences for Celestial Bodies* ... 1981

# Abstracts

## Feral pseudoholomorphic curves and minimal sets

Joel W. Fish
(joint work with Helmut Hofer)

From collaboration with H. Hofer, we discuss some recent results in which we use a new class of pseudoholomorphic curve, called a "feral pseudoholomorphic curve," to study certain divergence free flows in dimension three. In particular, we show that if $H$ is a smooth and proper Hamiltonian on $\mathbb{R}^4$, then no energy level of $H$ is minimal. The idea of the proof is to embed $\mathbb{R}^4$, containing the energy level, as $\mathbb{C}^2$ inside $\mathbb{C}P^2$ and then then make use of a result of Gromov which guarantees the existence of a unique degree-one embedded pseudoholomorphic sphere passing through a pair of distinct points of our choosing. One then makes use of a neck-stretching type construction coming from Symplectic Field Theory, and proves a compactness result for such curves. The key difficulty is that since the hypersurfaces are not contact (or more precisely, they are not stable Hamiltonian), the curves in question will sometimes develop infinite Hofer energy. This obstacle is surmounted, and the result is a new class of curve: the aforementioned feral pseudoholomorphic curve, which is proper, has finite topology (finite number of connected components, finite genus, etc), and has finite $\omega$-energy, but infinite Hofer-energy. Although the ends of such curves need not limit to the usual closed orbit – indeed, the ends may be quite complicated – they nevertheless limit to a closed invariant set. This result, together with some intersection arguments, establish that the limit set is not the entire energy level, which in turn completes the proof.

## Generic family with robustly infinitely many sinks

Pierre Berger

Given a manifold $M$ of dimension $n \geq 2$ and $k \geq 0$, we show the following:

**Theorem.** For every $\infty > d \geq 2$, there exist a nonempty open subset $U \subset C^d(\mathbb{R}^k, C^d(M,M))$ of $C^d$-families of $C^d$-dynamics of $M$, and a Baire generic set $\mathcal{R} \subset U$, so that the following property holds:

($\star$) for every $(f_a)_{a\in\mathbb{R}^k} \in \mathcal{R}$, for every $\|a\| < 1$, the dynamics $f_a$ has infinitely many sinks.

**Theorem.** When the dimension $n$ of $M$ is at least 3, property ($\star$) holds true for a generic family of diffeomorphisms in a nonempty open subset $U \subset C^d(\mathbb{R}^k, \mathrm{Diff}^d(M))$.

**Remark.** For $\infty \geq r > d \geq 0$, property $(\star)$ holds true for a generic family of dynamics in a nonempty open subset $U \subset C^d(\mathbb{R}^k, C^d(M,M))$.

This result generalizes Newhouse Theorem and it is a counter-example to a conjecture of Pugh and Shub [1].

The proof is done by introducing two new concepts:

- the homoclinic paratangency,
- the parablender.

A family $(f_a)_a$ has a homoclinic $C^d$-paratangency at the parameter $a_0$ for a saddle point $(P_a)_a$, if $P_{a_0}$ has a quadratic homoclinic tangency and $(f_a)_a$ is an unfolding of this tangency $d$-times degenerated.

This means that after a small perturbation with respect to $\|a-a_0\|^d$, the saddle point $P_a$ of $f_a$ has a homoclinic tangency.

A blender is a basic set created by Bonatti and Diaz [3] with special properties. In particular it enables to construct persistent homoclinic tangencies.

A $C^d$-parablender is a family of basic sets which enables one to construct persistent homoclinic $C^d$-paratangencies.

The proof of the above theorem is done by generalizing the construction of Diaz-Pujals-Nogueira [2] from diffeomorphism to families of diffeomorphisms.

## References

[1] C. Pugh and M. Shub, *Stable ergodicity and partial hyperbolicity*, International Conference on Dynamical Systems (Montevideo, 1995), Pitman Res. Notes Math. Ser., **362**, (1996), 182–187.

[2] L. J. Díaz, A. Nogueira, A. and E. R. Pujals, *Heterodimensional tangencies*, Nonlinearity **19 11** (2006), 2543–2566.

[3] C. Bonatti and L. J. Díaz, *Persistent nonhyperbolic transitive diffeomorphisms*, Ann. of Math. (2), **143 2** (1996) , 357–396.

[4] P. Berger, *Generic family with robustly infinitely many sinks*, ArXiv e-prints, **1411.6441** (2014), 1–41.

## Existence of knotted and linked invariant tori in the stationary Euler equations

Daniel Peralta-Salas

Let $(M,g)$ be a smooth Riemannian 3-manifold. The dynamics of an ideal fluid flow in $(M,g)$ is modeled by the Euler equations

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla p\,, \qquad \operatorname{div} u = 0\,,$$

where $u(x,t)$ is the velocity field of the fluid, which is a time-dependent vector field on $M$, and $p(x,t)$ is the pressure function, which is defined by these equations up to a constant. Here $\nabla_u$ denotes the covariant derivative along $u$ and the operators $\nabla$ and div are computed with respect to the metric $g$.

A solution $u$ to the Euler equations is called *stationary* when it does not depend on time. It describes an equilibrium configuration of the fluid. It is easy to check that a $C^1$ vector field $u(x)$ is a stationary solution of the Euler equations if and only if it satisfies the system of PDEs:

$$u \times \omega = \nabla B \,, \qquad \operatorname{div} u = 0 \,,$$

where

$$B := p + \frac{1}{2}|u|^2$$

is the Bernoulli function, $\omega := \operatorname{curl} u$ is the vorticity and $\times$ denotes the vector product on the 3-manifold. Let us recall that the curl of a vector field $u$ is another vector field given by the unique solution to the equation $i_{\operatorname{curl} u}\, \mu = d\alpha$, where $\mu$ is the volume form associated to the metric $g$ and $\alpha := i_u g$ is the dual 1-form of $u$. We notice that the stationary Euler equations can be written in terms of differential forms as

$$i_u d\alpha = -dB \,, \qquad \operatorname{div} u = 0 \,.$$

The goal of this talk is to address the problem of the existence of knotted structures in stationary solutions of the Euler equations. More precisely, let us recall that the integral curves of the velocity field $u$ are called stream lines (they represent the fluid particle paths), and the integral curves of the vorticity are called *vortex lines*. Moreover, a solid torus in $M$ that is bounded by an invariant torus of the vorticity is a *vortex tube*. The analysis of knotted and linked stream lines, vortex lines and vortex tubes for solutions to the Euler equations has attracted considerable attention since the foundational works of Helmholtz and Kelvin in the 19th century. In fact these structures have been recently realized experimentally by Irvine and Kleckner at Chicago [6].

In [2, 3] we proved the existence of stationary solutions of the Euler equations in the Euclidean space $\mathbb{R}^3$ with a prescribed set of knotted and linked vortex lines and vortex tubes. A popular exposition of these results can be found in [4]. A key feature of our approach is that it does not consider arbitrary solutions to the stationary Euler equations, but a particular class known as Beltrami fields. These are the solutions of the equation

$$\operatorname{curl} u = \lambda u$$

in $\mathbb{R}^3$ for some nonzero constant $\lambda$. This immediately implies that $u$ is an analytic, divergence-free vector field. Obviously the stream lines of a Beltrami field are the same as its vortex lines, so henceforth we will only refer to the latter. Notice that the celebrated Arnold's theorem on the structure of the stationary solutions to the Euler equations [1] does not apply to Beltrami fields because their associated Bernoulli function is identically constant, so in particular Beltrami fields do not need to be integrable. The main results in [2, 3] can be summarized as follows:

**Theorem 1.** *Let $\mathcal{S}$ be a finite collection of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes in $\mathbb{R}^3$. Then for any nonzero real constant $\lambda$ one can deform $\mathcal{S}$ with a diffeomorphism $\Phi$ of $\mathbb{R}^3$ so that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of a Beltrami field $u$, which satisfies $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$ and decays at infinity as $|D^j u(x)| < C_j/|x|$.*

A few remarks on this result are in order. First, if the set $\mathcal{S}$ consists just of closed curves, it is allowed to have infinitely many components [2], in which case the resulting Beltrami field $u$ does not decay at infinity. Second, the diffeomorphism $\Phi$ is connected with the identity but it is not generally close to it, in fact each vortex tube in $\Phi(\mathcal{S})$ is *thin* in the sense that its width is much smaller than its length [3]. Third, the set $\Phi(\mathcal{S})$ is structurally stable under volume-preserving perturbations of the field $u$, in fact the vortex lines are elliptic and the invariant tori satisfy a KAM nondegeneracy twist condition.

The proof of this result combines ideas from partial differential equations and dynamical systems. It is based on the construction of a robust local Beltrami field in a neighborhood of $\mathcal{S}$ using elliptic PDE estimates and KAM theory, which is then approximated by a global Beltrami field in $\mathbb{R}^3$ using a Runge type global approximation theorem with controlled behavior at infinity. The same methods allow us to prove analogous results for any open Riemannian 3-manifold.

These techniques do not work for compact manifolds, where the complement of the set $\mathcal{S}$ is precompact, so we cannot apply the Runge type global approximation theorem obtained in [2, 3]. In fact, as is well known, this is not just a technical issue, but a fundamental obstruction in any Runge type theorem. This invalidates the whole strategy followed in [2, 3] and therefore new tools are needed to address the problem in compact manifolds.

Recently [5], we have been able to overcome this difficulty showing that there are Beltrami fields in the round sphere $\mathbb{S}^3$ and in the flat torus $\mathbb{T}^3$ with vortex lines and vortex tubes of any link type. For the ease of notation, we shall write $\mathbb{M}^3$ to denote either $\mathbb{T}^3$ or $\mathbb{S}^3$:

**Theorem 2.** *Let $\mathcal{S}$ be a finite collection of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes in $\mathbb{M}^3$. In the case of the torus, we assume that $\mathcal{S}$ is contained in a contractible subset. Then for any sufficiently large odd integer $\lambda$ there exist a Beltrami field $u$, satisfying $\operatorname{curl} u = \lambda u$, and a diffeomorphism $\Phi$ of $\mathbb{M}^3$ such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of $u$.*

The proof of this result relies on semiclassical methods that allow us to understand the behavior of Beltrami fields with eigenvalue $\lambda \to \infty$ in balls of $\mathbb{M}^3$ of radius $\lambda^{-1} \to 0$. This involves an interplay between rigid and flexible properties of high-energy Beltrami fields. Rigidity appears because the curl operator in any closed 3-manifold has a discrete spectrum. However, since large eigenvalues of the curl operator in the torus or in the sphere have increasingly high multiplicities, there is some flexibility in the problem that allows us to prove that for any Beltrami field $v$ in $\mathbb{R}^3$ there exists a Beltrami field $u$ in $\mathbb{M}^3$ whose dynamics in a ball of radius $\lambda^{-1}$ is very close to the dynamics of $v$ in the unit ball. For this reason, the proof does not work in a general Riemannian 3-manifold. However, the control that we have over the diffeomorphism $\Phi$ allows us to prove an analog of Theorem 2 for quotients of the sphere by finite groups of isometries (lens spaces).

### References

[1] V.I. Arnold, B. Khesin, *Topological Methods in Hydrodynamics*, Springer, New York, 1999.
[2] A. Enciso, D. Peralta-Salas, *Knots and links in steady solutions of the Euler equation*, Ann. of Math. **175** (2012), 345–367.
[3] A. Enciso, D. Peralta-Salas, *Existence of knotted vortex tubes in steady Euler flows*, Acta Math. **214** (2015), 61–134.
[4] A. Enciso, D. Peralta-Salas, *Knotted vortex lines and vortex tubes in stationary fluid flows*, EMS Newsletter **96** (2015), 26–33.
[5] A. Enciso, D. Peralta-Salas, F. Torres de Lizaur, *Knotted structures in high-energy Beltrami fields on the torus and the sphere*, Preprint 2015 (Arxiv 1505.01605).
[6] D. Kleckner, W. Irvine, *Creation and dynamics of knotted vortices*, Nature Phys. **9** (2013), 253–258.


## Linear representations and dominated splittings


Jairo Bochi
(joint work with Rafael Potrie, Andrés Sambarino)


Anosov representations appear as a generalization of convex cocompact subgroups in pinched negative curvature to higher rank Lie groups. They were introduced by Labourie [9] in the study of the Hitchin component and later extended by Guichard and Wienhard [6]. Other characterizations of this concept have been recently found [5, 7, 8]. In this talk we adopt the most elementary of these characterizations as our starting point, and then explain how it relates to other structures. The concept of *dominated splittings* and its relation to singular values [2] is central to our approach.

1.1. **Exponential gap.** Let $\Gamma$ be a finitely generated group. Let $|\gamma|$ denote the word length of an element $\gamma \in \Gamma$ with respect to a fixed set of generators. Let $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a linear representation, and let $p \in \{1, 2, \dots, d-1\}$. We say that $\rho$ has a *exponential gap of index p* if there exist constants $c$, $\lambda > 0$ such that for all $\gamma \in \Gamma$,

$$\frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))} < ce^{-\lambda|\gamma|},$$

where $\sigma_1(A) \geq \cdots \geq \sigma_d(A)$ denote the singular values of a matrix $A \in \mathrm{GL}(d, \mathbb{R})$. The simplest non-trivial examples are faithful representations of the free group on two generators in $\mathrm{SL}(2, \mathbb{R})$ constructed using the classical "ping-pong" argument.

1.2. **Gromov-hyperbolicity theorem.** Recall that a geodesic metric space is *Gromov hyperbolic* if there exists $\delta \geq 0$ such that every geodesic triangle is $\delta$-slim, in the sense that each side of the triangle is contained in the $\delta$-neighborhood of the other two sides. A finitely generated group is called *Gromov hyperbolic* if so is its Cayley graph (for some and therefore any choice of the set of generators). since their Cayley graphs are trees.

Kapovich, Leeb, and Porti [7, 8] have obtained the following result: *If a finitely generated group* $\Gamma$ *admits a linear representation* $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ *with an exponential gap of some index* $p$ *then* $\Gamma$ *is Gromov hyperbolic.* We will sketch another, more elementary proof of this fact in §1.5 below.

1.3. **Dominated splittings.** Consider an action of $\mathbb{R}$ on a compact metric space $X$, i.e. a flow $\{\phi^t\colon X \to X\}_{t\in\mathbb{R}}$. Suppose that $E$ is a vector bundle over $X$, and that $\{\psi^t\colon E \to E\}_{t\in\mathbb{R}}$ is a flow of automorphisms of $E$ that fibers over the flow $\{\phi^t\}$. We endow $E$ with a Riemannian metric, the particular choice of it being irrelevant.

Suppose that $E = E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ is a continuous, $\{\psi^t\}$-invariant splitting. This splitting is called *dominated* if there exist constants $c$, $\lambda > 0$ such that for all $x \in X$, all unit vectors $v \in E^{\mathrm{cu}}_x$, $w \in E^{\mathrm{cs}}_x$, and all $t \geq 0$, we have

$$\frac{\|\psi^t_x(w)\|}{\|\psi^t_x(v)\|} < ce^{-\lambda t}\,.$$

We say that the bundle $E^{\mathrm{cu}}$ *dominates* the bundle $E^{\mathrm{cs}}$. The *index* of the dominated splitting is the dimension of $E^{\mathrm{cu}}$. Analogous definitions make sense if the time $t$ takes values in $\mathbb{Z}$. The concept of dominated splitting is widely used in dynamics of diffeomorphisms and flows (in which case $E$ is the tangent bundle and $\{\phi^t\}$ is the derivative).

Existence of a dominated splitting is equivalent to the presence of an exponential gap between singular values. More precisely, the following characterization was proved in [2] (extending the $\mathrm{SL}(2,\mathbb{R})$ case first considered in [11]): *a linear flow* $\{\psi^t\colon E \to E\}_{t\in\mathbb{R}}$ *has a dominated splitting of index* $p$ *if and only if there exist constants* $c$, $\lambda > 0$ *such that for all* $x \in X$ *and* $t \geq 0$,

$$\frac{\sigma_{p+1}(\psi^t_x)}{\sigma_p(\psi^t_x)} < ce^{-\lambda t}\,.$$

The proof in [2] relies in Oseledets theorem and is short. The dominating bundle, for example, is obtained as:

$$E^{\mathrm{cu}}_x := \lim_{t\to+\infty} U_p\big(\psi^t_{\phi^{-t}(x)}\big)\,,$$

where, if $L$ is a linear map such that $\sigma_p(L) > \sigma_{p+1}(L)$, we denote by $U_p(L)$ the $p$-dimensional space most contracted by $L^{-1}$.

1.4. **Anosov representations.** Using the aforementioned result from [2], one checks that *a representation* $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ *has an exponential gap of index* $p$ *if and only if a certain natural linear flow has a dominated splitting of index* $p$. The latter condition is equivalent to the representation being Anosov in the sense of Labourie [9] and Guichard and Wienhard [6]. Thus we reobtain some of the results of [5, 7, 8].

Let us explain what this "natural linear flow" is in the case that $\Gamma$ is the fundamental group of a compact hyperbolic surface $\Sigma$, and in particular acts on the hyperbolic plane $\mathbb{H}$ in such a way that $\mathbb{H}/\Gamma = \Sigma$. Let $U\Sigma$ (resp. $U\mathbb{H}$) be the unit tangent bundle of $\Sigma$ (resp. $\mathbb{H}$), and let $\{\phi^t\}$ (resp. $\{\tilde{\phi}^t\}$) be the corresponding geodesic flow. So $\{\tilde{\phi}^t\}$ fibers over $\{\phi^t\}$. Let us lift the flow $\{\tilde{\phi}^t\}$ to a linear flow $\{\tilde{\psi}^t\}$ on the trivial vector bundle $\tilde{E} := U\mathbb{H} \times \mathbb{R}^d$ by acting as identity on the fibers. Since the group $\Gamma$ already acts on $U\mathbb{H}$, we can use the representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ to define an action of $\Gamma$ on $\tilde{E}$ by $\gamma(u,v) := (\gamma u, \rho(\gamma)v)$. Taking quotient by this action we obtain a new vector bundle $E$, together with a linear flow $\{\psi^t\colon E \to E\}$ that fibers over the geodesic flow $\{\phi^t\colon U\Sigma \to U\Sigma\}$. This is the "natural linear flow" we were looking for. For general Gromov hyperbolic groups $\Gamma$ (which are the only ones we need to consider), the construction is similar, but $\{\phi^t\}$ needs to be replaced by the Gromov–Mineyev "abstract geodesic flow".

1.5. **Proof of the Gromov-hyperbolicity theorem.** Bowditch [3] proved that *if a finitely generated group* $\Gamma$ *acts on a perfect metric space* $M$ *in such a way that the induced action on* $M^{(3)} := \{(x_1,x_2,x_3) \in M^3;\ x_i \neq x_j\}$ *is properly discontinuous and compact then* $\Gamma$ *is Gromov-hyperbolic.* Using notation $U_p$ as in §1.2, we let

$$M := \bigcap_{n=n_0}^{\infty} \overline{\{U_p(\rho(\gamma));\ |\gamma| \geq n\}} \subset \mathrm{Grass}_p(\mathbb{R}^d)$$

Except in the case that $\Gamma$ is virtually $\mathbb{Z}$, the set $M$ is a perfect space. It essentially follows from [2] that $M$ satisfies the other assumptions of Bowditch's criterion, enabling us to conclude.

1.6. **Other remarks.** A Gromov hyperbolic groups gives rise to a "geodesic automaton", a structure similar to a subshift of finite type that specifies the geodesics on the the Cayley graph. Using it we can characterize Anosov representations in terms of a "ping-pong" condition, similar to the multicone theorems of [1, 2].

Our descriptions allow us to reobtain some of the analyticity results of [4].

The results we have discussed can be extended further by replacing the linear group $\mathrm{GL}(d,\mathbb{R})$ by a general non-compact real-algebraic semi-simple Lie group $G$. These extensions follow basically from the existence of appropriate linear representations of $G$ as constructed by Tits [10].

## References


[1] A. Avila, J. Bochi, J.-C. Yoccoz, *Uniformly hyperbolic finite-valued* $\mathrm{SL}(2,\mathbb{R})$*-cocycles*, Comment. Math. Helv. **85** (2010), 813–884.

[2] J. Bochi, N. Gourmelon, *Some characterizations of domination*, Math. Z. **263** (2009), 221–231.

[3] B.H. Bowditch, *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc. 11 (1998), 643–667.

[4] M. Bridgeman, R. Canary, F. Labourie, A. Sambarino, *The pressure metric for Anosov representations*, Geom. Funct. Anal. DOI: 10.1007/s00039-015-0333-8

[5] F. Guéritaud, O. Guichard, F. Kassel, A. Wienhard, *Anosov representations and proper actions*, arXiv:1502.03811

[6] O. Guichard, A. Wienhard, *Anosov representations: domains of discontinuity and applications.* Invent. Math. **190** (2012), 357–438.

[7] M. Kapovich, B. Leeb, J. Porti, *Morse actions of discrete groups on symmetric space*, arXiv:1403.7671

[8] M. Kapovich, B. Leeb, J. Porti, *A Morse Lemma for quasigeodesics in symmetric spaces and euclidean buildings*, arXiv:1411.4176

[9] F. Labourie, *Anosov flows, surface groups and curves in projective space.* Invent. Math. **165** (2006), 51–114.

[10] J. Tits, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque*, J. Reine Angew. Math. **247** (1971), 196–220.

[11] J.-C. Yoccoz, *Some questions and remarks about* $\mathrm{SL}(2,\mathbb{R})$ *cocycles*, Modern dynamical systems and applications, 447–458, Cambridge Univ. Press, Cambridge, 2004.


## Exponential decay of correlations for Sinai billiard flows


Viviane Baladi

(joint work with Mark Demers, Carlangelo Liverani)


In a joint work [1] with *M. Demers* and *C. Liverani*, I have recently proved *exponential decay of correlations for two-dimensional Sinai billiard flows with finite horizon* (and Hölder observables) by combining and strengthening the tools recently made available: In 1998, Young [7] obtained exponential decay of correlations for the finite-horizon discrete-time Sinai billiard on the 2-torus. The continuous-time billiard is much more difficult to handle and the best known result was Chernov's 2007 stretched exponential upper bound [3]. The recent papers [4] and [2] allowing discontinuities constituted important steps. However, neither the two-dimensional piecewise hyperbolic map paper of Demers and Liverani [4] nor the piecewise hyperbolic contact flow model in my paper [2] with Liverani allowed for derivatives to blow up along the boundaries of the domains of regularity. In 2011, inspired by [4], M. Demers and H.-K. Zhang [5] obtained a new "functional" proof (no Markov partition, no tower) of Young's result exponential decay result for discrete-time Sinai billiards. *The winning strategy was to combine the techniques in* [2] *with the work of Demers–Zhang* [5] *to obtain exponential decay for the 2d finite horizon Sinai billiard flow* [1]. The key tool to obtain the spectral gap is the construction (inspired from [6], [2]) of approximate stable/unstable foliations (in the kernel of the contact form) at a fixed scale.

### References


[1] V. Baladi, M. Demers, and C. Liverani *Exponential decay of correlations for finite horizon billiard flows,* Preprint 2015 `arXiv:1506.02836.`

[2] V. Baladi and C. Liverani, *Exponential decay of correlations for piecewise cone hyperbolic contact flows,* Comm. Math. Phys. **314** (2012), 689–773.

[3] N.I. Chernov, *A stretched exponential bound on time correlations for billiard flows,* Journal of Statistical Physics **127** (2007), 21–50.

[4] M. Demers and C. Liverani, *Stability of statistical properties in two-dimensional piecewise hyperbolic maps,* Trans. Amer. Math. Soc. **360** (2008), 4777–4814.

[5] M. Demers and H.K. Zhang, *Spectral analysis of the transfer operator for the Lorentz gas,* J. Modern Dynamics **5** (2011), 665–709.

[6] C. Liverani, *On contact Anosov flows,* Ann. of Math. **159** (2004), 1275–1312.

[7] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity,* Ann. of Math. **147** (1998), 585–650.

## Hamiltonian diffeomorphisms and persistence modules

Leonid Polterovich
(joint work with Egor Shelukhin)

Let $d$ be the Hofer metric on the group $\mathrm{Ham}(M,\omega)$ of $C^\infty$-smooth Hamiltonian diffeomorphisms of a closed symplectic manifold $(M,\omega)$. For an integer $k \geq 2$ write $\mathrm{Powers}_k = \{\phi = \psi^k | \ \psi \in \mathrm{Ham}\}$ for the set of Hamiltonian diffeomorphisms admitting a root of order $k$ and denote

$$\mathrm{powers}_k := \sup_{\phi \in \mathrm{Ham}} d(\phi, \mathrm{Powers}_k) \ .$$

The main result of the talk is as follows [3]:

**Theorem 1.** *Let $\Sigma$ be a closed oriented surface of genus $\geq 4$ equipped with an area form $\sigma$, and $k \geq 2$ an integer. Then for every closed symplectically aspherical manifold $(M,\omega)$*

$$\mathrm{powers}_k(\Sigma \times M, \sigma \oplus \omega) = +\infty \ .$$

Since every autonomous diffeomorphism admits a root of any order, Theorem 1 yields existence of Hamiltonian diffeomorphisms lying arbitrarily far (in the sense of Hofer's metric) from the set of all autonomous Hamiltonian diffeomorphisms. Additionally, our methods yield a result of a dynamical flavor providing a symplectic take on Palis' dictum "Vector fields generate few diffeomorphisms" [2]: for symplectically aspherical manifolds the subset of non-autonomous Hamiltonian diffeomorphisms contains a $C^\infty$-dense Hofer-open subset.

General lower bounds on the distance from a given Hamiltonian diffeomorhism to $\mathrm{Powers}_k$ come from the theory of persistence modules [4], a rapidly developing area lying on the borderline of algebraic topology and topological data analysis. We apply it in the context of filtered Floer homology enhanced with a special periodic automorphism. The latter is induced by the natural action (by conjugation) of a Hamiltonian diffeomorphism $\phi$ on the Floer homology of its power $\phi^k$.

Specific examples of Hamiltonian diffeomorphisms $\phi_\lambda$ for which the distance $d(\phi_\lambda, \mathrm{Powers}_k)$ becomes arbitrarily large as $\lambda \to +\infty$ come from chaotic dynamics. We choose $\phi_\lambda$ of the form $\psi_\lambda \times \mathbf{1}$, where $\psi_\lambda$ is a (slightly modified) *egg-beater map* (see [1]) which appeared in the modeling of mixing in duct flows.

### References

[1] Franjione, J. G., Ottino, J. M., *Symmetry concepts for the geometric analysis of mixing flows,* Philosophical Transactions of the Royal Society of London, Series A: Physical and Engineering Sciences, **338**, no. 1650, (1992), 301–323.

[2] Palis, J., *Vector fields generate few diffeomorphisms,* Bull. Amer. Math. Soc. **80** (1974), 503–505.

[3] Polterovich, L., Shelukhin, E., *Autonomous Hamiltonian flows, Hofer's geometry and persistence modules,* Preprint arXiv:1412.8277, 2014.

[4] Weinberger, S., *What is...persistent homology?,* Notices Amer. Math. Soc. **58** (2011), no. 1, 36–39.

## On the ergodicity of billiards in non-rational polygons

Giovanni Forni

In this talk we present a geometric criterion for the ergodicity of the billiard flow of a non-rational polygons and more generally of the geodesic flow on flat surfaces with conical singularities (and non-trivial holonomy).

Given a polygon $P$, the associated billiard flow is a discontinuous, almost everywhere defined, contact flow on $P \times S^1$, defined as follows. For every $(x, v) \in \text{int}(P) \times S^1$, the orbit follows the straight line through the point $x \in P$ in the direction of the unit tangent vector $v \in S^1$ and it is then continued at $\partial P$ by the law of elastic reflection (angle of incidence equal to angle of reflection). In other terms, we look at the motion of a point mass moving freely on the polygonal table with elastic collisions at the boundary. Alternatively, we can imagine to put a mirror at every edge of the polygon and look at trajectories of light rays. It is a standard step, to avoid dealing with the boundary of the polygon, to construct a closed flat surface whose geodesic flow is essentially equivalent to the billiard flow. This is accomplished by taking a double of $P$ to get a topological sphere which inherits a flat metric with cone singularities from $P$. For these closed flat surfaces, we are interested in the dynamical properties of the geodesic flow.

There is a well-known dichotomy between the *rational* and the *non-rational* cases. A polygon $P$ is called rational if the group generated by reflections with respect to axes parallel to its sides is finite. For instance, regular polygons are rational. This condition implies that the angles of the polygon are rational multiples of $\pi$, and the two conditions are equivalent if the polygon has connected boundary. Otherwise the polygon is called non-rational.

In the rational case, by passing to a finite branched cover of the corresponding flat surface, it is possible to pass to a flat surface with *trivial holonomy.* This means that the parallel transport map along any loop is the identity map on the fiber of the unit tangent bundle. Under these conditions there exists an atlas whose coordinate changes are translations, hence such a flat surface is called a *translation surface.* On a translation surface it is possible to choose a horizontal (and a vertical) parallel vector field and define an angle function on the unit tangent bundle on the complement of the cone points. The angle function is invariant under the geodesic flow and its level sets are surfaces of the same genus as the base surface. A system of this type can be integrable in the standard sense (if the invariant surfaces are tori) or pseudo-integrable.

The main question we want to address concerns *ergodicity* of the geodesic flow. It is clear that ergodicity never holds for translation surfaces as the geodesic flow is integrable or pseudo-integrable. The question in that case then becomes whether the flow is ergodic on the invariant surfaces. In a celebrated paper, S. Kerckhoff, H. Masur and J. Smillie [2] were able to prove, building on earlier work of Masur (and also W. Veech), that for *all* translation surfaces (in particular for those coming from rational billiards) the geodesic flow is uniquely ergodic on almost all invariant surfaces. In the same paper [2] they derived from this result, by an approximation argument, that there exists a dense $G_\delta$ set of ergodic polygons in any number of edges. Ya. Vorobets [4] later proved an effective version of unique ergodicity on invariant surfaces in the rational case and from that derived ergodicity in the non-rational case, under an explicit condition on fast approximation of the non-rational billiard table by rational ones. His result requires an extremely fast, super-exponential rate of approximation and, as a consequence, ergodicity is proved for a set of polygons of zero Lebesgue measure. It is therefore an open problem whether there exists a set of *full Lebesgue measure* of ergodic non-rational polygons in any number of edges. This problem appears, together with other fundamental questions on the dynamics of billiards in non-rational polygons, as Problem 3 in A. Katok's list of *Five Most Resistant Problems of Dynamics*.

We will state below our contribution to this problem: a criterion for the ergodicity of the geodesic flow on flat surfaces, which generalizes Masur's (unique) ergodicity criterion for translation surfaces. In fact, in his thesis R. Treviño has derived from our methods a sharp effective version of Masur's criterion. Masur's criterion is based on renormalization ideas. For translation flows renormalization is related to the so-called *Teichmüller flow* on the space of Abelian holomorphic differentials and to its discrete times symbolic realizations, that is, the Rauzy-Veech induction and its accelerations (Zorich, Marmi-Moussa-Yoccoz, Avila).

The unit tangent bundle $S(M)$ of a flat surface $M$, with cone singularities at a finite set $C \subset M$, has a local homogeneous structure over $M \setminus C$. In fact, let $X$ be the generator of the geodesic flow, let $Y$ be the generator of the perpendicular geodesic flow and let $\Theta$ the generator of the rotation in the fiber of $S(M \setminus C)$ over $M \setminus C$. We recall that the perpendicular geodesic flow time-$t$ map is defined on the unit tangent bundle as the composition of a rotation by a right angle in the positive direction, followed by a geodesic flow time-$t$ map, followed by a rotation by a right angle in the negative direction. By the theory of Riemannian geometry of surfaces, the frame $\{X, Y, \Theta\}$ satisfies the following commutation relation:

$$[X, Y] = 0 \quad [\Theta, X] = Y\,, \quad [\Theta, Y] = -X\,.$$

In particular, the distribution $\{X, Y\}$ is integrable by Frobenius theorem and its tangent foliation $\mathcal{H}$ is called the *holonomy foliation*. The leaves of the holonomy foliation are translation surfaces and are closed surfaces (with conical singularities) if and only if the surface $M$ itself is a translation surface. F. Valdez [3] has proved that if the surface $M$ comes from a non-rational billiard, then each holonomy leaf is a *Loch Ness Monster* (an infinite genus non-compact translation surface with a single end). We remark that any two leaves of the holonomy foliation can be identified by a rotation, hence they are isomorphic as translation surfaces.

We introduce the following deformation of the the local homogenous structure:

$$X_s := e^s X\,, \quad Y_s := e^{-s} Y\,, \quad \Theta_s := e^{-2s}\Theta\,. \tag{1}$$

We remark that the above deformation is not given by automorphisms of the (solvable) Lie algebra generated by the frame $\{X, Y, \Theta\}$, in fact

$$[X_s, Y_s] = 0\,, \quad [\Theta_s, X_s] = Y_s\,, \quad [\Theta_s, Y_s] = -e^{-4s} X_s\,.$$

By the above commutation formula the solvable Lie structure given by the frame $\{X_s, Y_s, \Theta_s\}$ degenerates to a nilpotent (Heisenberg) Lie structure. In particular, the above deformation cannot induce a recurrent dynamics on a moduli space.

Let $R_s$ denote the Riemannian metric on $S(M \setminus C)$ defined by the condition that the frame $\{X_s, Y_s, \Theta_s\}$ is orthonormal. Our ergodicity criterion roughly states that if the degeneration of the metric $R_s$, as measured by the *Cheeger constant*, is under control, then the geodesic flow is ergodic. We recall the definition of the Cheeger constant, introduced by Cheeger [1] as a lower bound for the first non-trivial eigenvalue of the Laplace-Beltrami operator. For any Riemannian metric $R$ on a manifold $M$, the Cheeger constant $h_R(\Omega)$ of the open subset $\Omega \subset M$ for the metric $R$ is the non-negative real number

$$h_R(\Omega) := \inf_{\Sigma} \frac{\mathrm{Area}_R(\Sigma \cap \Omega)}{\min\left(\mathrm{vol}_R(\Omega'_\Sigma), \mathrm{vol}_R(\Omega''_\Sigma)\right)}\,.$$

In the above formula the infimum is taken over all separating surfaces $\Sigma$ and $\Omega'_\Sigma$, $\Omega''_\Sigma$ denote the components of the complement $\Omega \setminus \Sigma$.

For all $s > 0$ and for all $d > 0$ let $V_s(d)$ denote the set of points which are at distance at most $d > 0$ from the boundary of the manifold $S(M \setminus C)$ with respect to the deformed metric $R_s$ on the open manifold $S(M \setminus C)$.

**Theorem 1.** *Assume that there exists a set $\mathcal{P} \subset \mathbb{R}^+$ of positive lower density and a real number $d_0 \in (0,1)$ such that the following holds. For any $s \in \mathcal{P}$ and for any $d \in (0, d_0)$ there exists a connected open set $\Omega_s(d) \subset S_r \setminus V_s(d)$ such that*

$$\begin{aligned} &(i)\ \liminf_{s\in\mathcal{P}} \mathrm{h}_{R_s}(\Omega_s(d)) > 0, \quad \textit{for all } d \in (0, d_0)\,, \textit{and} \\ &(ii)\ \lim_{d\to 0^+} \mathrm{vol}_{R_0}(\Omega_s(d)) = 1, \quad \textit{uniformly with respect to } s \in \mathcal{P}\,. \end{aligned} \tag{2}$$

*It follows that the geodesic flow is ergodic.*

Conjecturally, it should be possible to derive from the above theorem the ergodicity of the geodesic flow under a full measure (Diophantine) condition on the cone angles or other full measure condition.

## References


[1] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, in *Problems in analysis*, 195–199. Princeton University Press, Princeton, NJ, USA, 1970.

[2] S. Kerckhoff and H. Masur and J. Smillie, *Ergodicity of billiard flows and quadratic differentials*, Ann. of Math. **124** (2)(1986), 293–311.

[3] F. Valdez, *Infinite genus surfaces and irrational polygonal billiards*, Geom. Dedicata **143** (2009), 143–154.

[4] Ya. Vorobets, *Ergodicity of billiards in polygons*, Mat. Sb. **188** (1997), 65–112.

## Stochastic Arnold diffusion of deterministic systems

Vadim Kaloshin

(joint work with Oriol Castejon, Marcel Guardia, Jianlu Zhang, Ke Zhang)

In the talk we report on the study of stochastic behavior for a nearly integrable system given by the product of the pendulum and the rotator perturbed with a small coupling between the two. This example was introduced by Arnold. For an open class of trigonometric perturbations we prove presence of stochasic diffusive behavior. Namely, for each such a system $H_\epsilon$ there is a probability measure $\mu_{H_\epsilon}$ such that the projection of its pushfoward under the flow of $H_\epsilon$ onto action of the rotor $I$ in the time scale $\epsilon^{-2}\log 1/\epsilon$ weakly converges to a stochastic diffusion process

$$x_0 + \int_0^t a(x_s)ds + \int_0^t \sigma(x_s)dw_s,$$

where $a(\cdot)$ and $\sigma(\cdot)$ are smooth functions and $w_s$ is the white noise.

This gives an indication that for Arnold's example for a class of orbits inside of a stochastic layer indeed there is a stochastic diffusive behavior.

This result is based on a several preprints:

— one, joint with M. Guardia and J. Zhang we derive a higher order term in a separatrix map introduced by Zaslavski and studied by Treschev,

— another, joint with J. Zhang and K. Zhang, we construct a normally hyperbolic invariant lamination and compute the restricted system, which has a form of a skew-product,

— another, joint with O. Castejon, we study a class of skew-products and show that weak convergence to a stochastic diffusion process.

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## Homology with twisted coefficients and vanishing of the Hofer-Zehnder capacity

Peter Albers

(joint work with Alexandru Oancea, Urs Frauenfelder)

We explore the relationship between some symplectic invariants of cotangent bundles (symplectic homology, Hofer-Zehnder capacity) and local systems of coefficients on the free loop space $\Lambda M$ over a smooth manifold $M$. For simplicity we assume that $M$ is closed, that is, compact and without boundary.

**Theorem 1.** *Let $M$ be a manifold such that the image of the Hurewicz map*

$$\pi_2(M) \to H_2(M;\mathbb{Z})$$

*is nonzero. Then there exists a local system of coefficients $\mathcal{L}$ on $\Lambda M$ with nonzero fiber, which is trivial on the constant loops and such that*

$$H_\cdot(\Lambda M;\mathcal{L}) = 0.$$

*The same conclusion holds more generally if there exists a prime number $\ell \geq 2$ such that the map $H^2(G;\mathbb{Z}/\ell) \to H^2(M;\mathbb{Z}/\ell)$ is not surjective, or the map $H^3(G;\mathbb{Z}/\ell) \to H^3(M;\mathbb{Z}/\ell)$ is not injective, with $G = \pi_1(M)$.*

As a sample application of dynamical character we prove finiteness of the $\pi_1$-sensitive Hofer-Zehnder capacity of cotangent bundles $T^*M$ where $M$ is as in the Theorem.

We next describe two Hofer-Zehnder capacities on $W := T^*M$ equipped with the standard symplectic form $\sum dp_i \wedge dq_i$. A beautiful account can be found in the book [5] by Hofer-Zehnder. We need the following terminology. A smooth function $H\colon W \to [0,\infty)$ with compact support in the interior of $W$ is called *simple*, if it satisfies the following two conditions

- **(i):** There exists a nonempty open subset $U \subset W$ such that $H|_U = \max H$,
- **(ii):** The only critical values of $H$ are 0 and $\max H$.

In particular, since $H$ is nonnegative it follows that the oscillation of $H$ equals

$$\operatorname{osc}(H) := \max H - \min H = \max H.$$

A simple function $H$ is called HZ-*admissible* if the Hamiltonian vector field of $H$ has no nonconstant periodic orbits of period less than or equal to 1, and a simple function is called HZ$^0$-*admissible* if the same is true for contractible periodic orbits. The two Hofer-Zehnder capacities of the Liouville domain $W$ are defined as

$$c_{HZ}(W) := \sup\{\max H : H \text{ HZ-admissible}\},$$

$$c^0_{HZ}(W) := \sup\{\max H : H \text{ HZ}^0\text{-admissible}\}.$$

$c_{HZ}(W)$ is the 'original' Hofer-Zehnder capacity whereas $c^0_{HZ}(W)$ is called $\pi_1$-sensitive Hofer-Zehnder capacity. We have the following obvious inequality

$$c_{HZ}(W) \leq c^0_{HZ}(W).$$

Finiteness of the Hofer-Zehnder capacities has the following striking dynamical consequence, see [5, Chapter 4, Theorem 4] for a proof.

**Theorem 2** (Hofer-Zehnder). *Assume that $H\colon W \to \mathbb{R}$ has the property that $H|_{\partial W} = 1$ and there exists $\epsilon_0 \in (0,1)$ such that every $\epsilon \in (-\epsilon_0, \epsilon_0)$ is a regular value of $H$. If $c_{HZ}(W) < \infty$, then for almost every $\epsilon \in (-\epsilon_0, \epsilon_0)$ there exists a periodic orbit of the Hamiltonian vector field $X_H$ on the level set $H^{-1}(\epsilon)$. If $c^0_{HZ}(W) < \infty$, then almost every level set of $H$ carries a periodic orbit which is contractible in $W$.*

We now use the following facts from symplectic geometry. We know ([3, 1, 2]) that $SH^0_\cdot(T^*M;\mathcal{L}\otimes\eta) \simeq H_\cdot(\Lambda M;\mathcal{L})$, where $\eta$ is a $\mathbb{Z}/2$-local system defined by transgressing the second Stiefel-Whitney class of $M$. More precisely, the holonomy of $\eta$ along a loop $\gamma : S^1 \to \Lambda M$ equals $-1$ precisely when $w_2(M)$ evaluates nontrivially on the homology class of the torus $T_\gamma : S^1 \times S^1 \to M$ given by $T_\gamma(t,\theta) = \gamma(t)(\theta)$. Note that $\eta$ restricts to a trivial local system on the space of constant loops since $T_\gamma$ represents the zero homology class whenever $\gamma$ takes values in $M$. Our assumptions on $M$ imply that there exists a local system $\mathcal{L}$ which is trivial on constant loops and such that $H_\cdot(\Lambda M;\mathcal{L}) = 0$, and therefore $SH^0_\cdot(T^*M;\mathcal{L}\otimes\eta) = 0$. In particular, for every manifold $M$ for which $\pi_2(M) \to H_2(M;\mathbb{Z})$ is non-zero we find a local system $\mathcal{L}$ which is trivial on constant loops such that $SH^0_\cdot(T^*M;\mathcal{L}\otimes\eta) = 0$. Now we can apply Irie's result [6, Corollary 3.5] to obtain the following conclusion.

**Corollary 3.** *If $\pi_2(M) \to H_2(M;\mathbb{Z})$ is non-zero both Hofer-Zehnder capacities $c_{HZ}(T^*M)$ and $c^0_{HZ}(T^*M)$ are finite and the almost existence theorem holds true on $T^*M$.*

**Remark 4.** *We point out that $c^0_{HZ}(T^*M)$ is not necessarily finite for every closed manifold $M$. An example is the torus $M := T^n$. The geodesic flow on $T^n$ equipped with its flat metric has no contractible closed orbits. Therefore the $\pi_1$-sensitive Hofer-Zehnder capacity $c^0_{HZ}(T^*T^n)$ has to be $\infty$ in this case.*

## References


[1] A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math., **59**(2):254–316, 2006.

[2] A. Abbondandolo and M. Schwarz, *Corrigendum: On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math., **67**(4):670–691, 2014.

[3] M. Abouzaid, *Symplectic cohomology and Viterbo's theorem*, arXiv:1312.3354, 2013.

[4] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[5] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 1994.

[6] K. Irie, *Hofer-Zehnder capacity of unit disk cotangent bundles and the loop product.*, J. Eur. Math. Soc. (JEMS), **16**(11):2477–2497, 2014.


# Trapped Reeb orbits do not imply periodic ones


Nena Röttgen

(joint work with Hansjörg Geiges, Kai Zehmisch)


We present examples showing that dynamics of Reeb flows in dimension five and higher can be different from those in dimension three. More explicitly, Eliashberg and Hofer, [4], proved for every contact form on $\mathbb{R}^3$ that induces the standard contact structure and coincides with the standard contact form outside a compact set, the following holds: If the associated Reeb flow has no periodic orbit then it induces coordinates in which the contact form is standard (and hence the flow is induced by the $z$-coordinate vector field with respect to canonical notation).

In contrast, we construct an example of a contact form on $\mathbb{R}^{2n+1}$ with $n \geq 2$ inducing the standard contact structure and coinciding with the standard contact form $dz + \frac{1}{2}\sum_{i=1}^{n}(x_i\,dy_i - y_i\,dx_i)$ outside a compact set whose only compact invariant set is an $n$-dimensional torus, [5]. This precludes the generalization of Eliashberg and Hofer's result to higher dimensions, thus contradicting a conjecture by Hofer.

Furthermore the existence of a compact invariant set together with the boundary conditions imply the existence of a trapped orbit, i.e. an orbit which comes from infinity and is bounded in positive time. We emphasize the connection between this property and the Weinstein conjecture: Consider a compact cylinder over a disc in the $(\bar{x}, \bar{y})-$hyperplane that includes the points where the contact form is nonstandard. Then by restriction of the Reeb flow to this cylinder we obtain a (noncomplete) flow that has all properties of a Wilson plug, [9], except for a certain symmetry property. Therefore we call it a semi-contact-plug. Inserting Wilson type plugs in a given flow yields a very effective method to produce vector fields without periodic orbits, see e.g. [8], [7] for counterexamples to Seifert's conjecture, or [6] for Hamiltonian energy surfaces without periodic orbits. In contrast inserting the semi-contact-plug in a Reeb flow can be used to kill existing closed orbits but the lack of symmetry allows new periodic orbits to develop at the same time. We know that the creation of new orbits cannot be avoided in general since the Weinstein conjecture has been proven in some cases, e.g. for (PS-)overtwisted contact structures, [1].

Now we describe more explicitly the Reeb vector field $R$ that we obtain in our examples: Considering polar coordinates $(r_i, \varphi_i)$ on the $(x_i, y_i)$-planes, the compact invariant set is given by $T = \{r_1 = \cdots = r_n = 1,\ z = 0\}$. On $T$ the Reeb vector field equals $\sum_i \alpha_i \partial_{\varphi_i}$ for well chosen rationally independent $\alpha_1, \ldots, \alpha_n$. Away from $T$ we obtain $dz(R) > 0$. Furthermore the above boundary conditions require $R = \partial_z$ outside a larger compact invariant set. It is easy to see that a Reeb vector field with these properties induces the claimed result. The construction relies then on the use of basic facts about contact Hamiltonians.

The geometric ansatz in our approach was motivated by a construction by Victor Bangert and the author of a Riemannian metric on $\mathbb{R}^n$ where $n \geq 4$ that coincides with the euclidean metric outside a compact set and such that the induced geodesic flow has a compact invariant 2-torus but no periodic orbit, [2]. Note that also in this setting the situation changes dramatically in low dimension: Birkhoff proved for a two-dimensional Riemannian disc $D$ with locally convex boundary that the existence of a geodesic $c : \mathbb{R} \to D$ induces the existence of a closed geodesic in $D$, [3].

## References

[1] P. Albers and H. Hofer, *On the Weinstein conjecture in higher dimensions*, Comment. Math. Helv. **84** (2009), 429–436.

[2] V. Bangert and N. Röttgen, *Isoperimetric inequalities for minimal submanifolds in Riemannian manifolds: a counterexample in higher codimension*, Calc. Var. Partial Differential Equations **45** (2012), no. 3-4, 455–466.

[3] George D. Birkhoff, *Dynamical systems*, American Mathematical Society Colloquium Publications, Vol. VIII, American Mathematical Society, Providence, R.I., 1927.

[4] Y. Eliashberg, H. Hofer. *A Hamiltonian characterization of the three-ball.*, Differential Integral Equations **7** (1994), 1303–1324.

[5] Hansjörg Geiges, Nena Röttgen, and Kai Zehmisch. *Trapped Reeb orbits do not imply periodic ones.*, Invent. Math., **198** (2014) 211–217.

[6] Viktor L. Ginzburg. *An embedding* $S^{2n-1} \to \mathbf{R}^{2n}$, $2n-1 \geq 7$, *whose Hamiltonian flow has no periodic trajectories*, Internat. Math. Res. Notices, **2** (1995) 83–97.

[7] K. Kuperberg. *A smooth counterexample to the Seifert conjecture*, Ann. of Math. (2), **140** (1994) 723–732.

[8] P. A. Schweitzer. *Counterexamples to the Seifert conjecture and opening closed leaves of foliations*, Ann. of Math. (2), **100** (1974) 386–400.
[9] F. W. Wilson, Jr. *On the minimal sets of non-singular vector fields*, Ann. of Math. (2), **84** (1966) 529–536.

## Sums of Cantor sets and convolutions of singular measures

Anton Gorodetski

Questions on the structure and properties of sums of Cantor sets appear naturally in dynamical systems, number theory, harmonic analysis, and spectral theory, e.g. see [3] for references and more details. J. Palis asked whether it is true (at least generically) that the arithmetic sum of dynamically defined Cantor sets either has measure zero, or contains an interval. This claim is currently known as the "Palis' Conjecture". The conjecture was answered affirmatively in [5] for generic dynamically defined Cantor sets. For sums of generic *affine* Cantor sets Palis' Conjecture is still open.

In [4] we prove, in particular, the following statement.

**Theorem 1.** *Generically (for Lebesgue almost all admissible sets of the parameters) the sum of two affine Cantor sets has positive Lebesgue measure provided the sum of their Hausdorff dimensions is greater than one.*

Notice that the set of pairs of affine Cantor sets with given combinatorics is a finite parameter family. Moreover, the requirement that the Cantor sets under consideration are affine can in fact be dropped, and a similar statement holds for just one parameter families of sumsets, as was shown in [3]. Namely, the following statement holds.

**Theorem 2.** *Let* $\{C_\lambda\}$ *be a family of dynamically defined Cantor sets of class* $C^2$ *(i.e.,* $C_\lambda = C(\Phi_\lambda)$*, where* $\Phi_\lambda$ *is an expansion of class* $C^2$ *both in* $x \in \mathbb{R}$ *and in* $\lambda \in J = (\lambda_0, \lambda_1)$*) such that* $\frac{d}{d\lambda} \dim_{\mathrm{H}} C_\lambda \neq 0$ *for* $\lambda \in J$*. Let* $K \subset \mathbb{R}$ *be a compact set such that*

$$\dim_{\mathrm{H}} C_\lambda + \dim_{\mathrm{H}} K > 1 \quad \textit{for all} \quad \lambda \in J.$$

*Then* $\mathrm{Leb}(C_\lambda + K) > 0$ *for a.e.* $\lambda \in J$*.*

Since spectrum of Fibonacci Hamiltonian is a dynamically defined Cantor set for all positive coupling constants [2], this result can be applied to better understand the structure of Square Fibonacci Hamiltonian, a model that was suggested by physicists as a toy version of two-dimensional quasicrystal, see [3] and references therein.

The proofs of both results use the fact that a support of a convolution of two measures is equal to the sumset of their supports, together with the results on absolute continuity of some singular measures obtained in [1].

### References

[1] D. Damanik, A. Gorodetski, B. Solomyak, Absolutely Continuous Convolutions of Singular Measures and an Application to the Square Fibonacci Hamiltonian, *Duke Mathematical Journal*, **164** (2015), 1603–1640.
[2] D. Damanik, A. Gorodetski, W. Yessen, The Fibonacci Hamiltonian, arXiv:1403.7823.
[3] D. Damanik, A. Gorodetski, Sums of regular Cantor sets of large dimension and the Square Fibonacci Hamiltonian, preprint.
[4] A. Gorodetski, S. Northrup, On Sums of Nearly Affine Cantor Sets, preprint.
[5] C. Moreira, J.-C. Yoccoz, Stable intersections of regular Cantor sets with large Hausdorff dimensions, *Ann. of Math.* **154** (2001), 45–96.


## Ballistic motion in one-dimensional lattice Schrödinger equation

Zhiyan Zhao

### 1. Introduction

Consider the lattice Schrödinger equation, which has the form

$$\mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V_n q_n, \quad n \in \mathbb{Z}.$$

Here, the potential $\{V_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is independent of time. The problem is related the growth rate of the diffusion norm: $\|q(t)\|_D := \left(\sum_{n\in\mathbb{Z}} n^2 |q_n(t)|^2\right)^{1/2}$. If we interpret $n$ as an index of the Fourier series, then this norm is the equivalent of $H^1-$norm.

We suppose that $q(0) \neq 0$ and $\|q(0)\|_D < \infty$. For this system, we have known the $l^2-$conservation law, which means the $l^2-$norm is a constant independent of time. So the initial condition implies the concentration on $q_n$ with $|n|$ not so large, and $\|q(t)\|_D$ measures the propagation into $q_n$, $|n| \gg 1$.

The property of the linear equation is determined by the spectral property of the corresponding linear operator:

$$H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad (Hq)_n = -(q_{n+1} + q_{n-1}) + V_n q_n.$$

For this self-adjoint operator on the Hilbert space, we have the decomposition for the spectrum: $\sigma(H) = \sigma_{pp} \cup \sigma_{sc} \cup \sigma_{ac}$.

In particular, in the case that the spectrum is only pure point, there is a well-studied notion which is called "dynamical localization". It means the absence of transport in a disordered medium.

**Definition 1.** *The operator $H$ exhibits* **dynamical localization** *if for any $q(0)$ with exponentially decaying, the solution of equation $\mathrm{i}\dot{q} = Hq$ satisfies that*

$$\sup_t \sum_{n\in\mathbb{Z}} |n|^s |q_n(t)|^2 < \infty, \quad \forall s > 0.$$

In general, for the Schrödinger operator (at least in one-dimension), if the potential $\{V_n\}_n$ is a sequence disordered, dynamical localization has been proven in many cases of pure point spectrum.

**Theorem 2.** *$H$ has dynamical localization, if*

- *(Anderson model)* [6, 9]*:$\{V_n\}_n$ is a family of random variables i.i.d. (independently identically distributed), with probability* 1
- *(Harper model)* [3, 10]*: $V_n = \lambda \cos 2\pi(\theta + n\alpha)$, $\alpha$ Diophantine, $|\lambda| > 2$, a.e. $\theta \in \mathbb{T}$.*
- *(Maryland model)*[2, 4]*: $V_n = \tan \pi(\theta + n\alpha)$, $\alpha$ Diophantine, a.e. $\theta \in \mathbb{T}$.*

In contrast, if the potential is well ordered (for example, periodic or quasi-periodic but sufficiently small), normally there is the absolutely continuous spectrum. Correspondingly, there should be something related to the diffusion norm. At first, we were inspired by a numerical result (Hiramoto-Abe [13]). They said that for the Harper model,

$$V_n = \lambda \cos 2\pi n\alpha, \quad \alpha \text{ is the inverse of the golden mean}, \quad q_n(0) = \delta_{n,0},$$

if $|\lambda| < \lambda_c$, $\|q(t)\|_D \sim t$(the diffusion norm will grow linearly with the time). For the physicist, this is called “ballistic regime”. Damanik-Lukic-Yessen [5] have recently shown the “ballistic motion” for the periodic Schrödinger equation, as the periodic Schrödinger operator is a well-known example of purely absolutely continuous spectrum.

## 2. Ballistic motion for quasi-periodic Schrödinger equation

Consider the quasi-periodic Schrödinger equation

$$\mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V(\theta + n\omega)q_n, \quad n \in \mathbb{Z}. \tag{1}$$

Here $V : \mathbb{T}^d \to \mathbb{R}$ is analytic in a complex neighborhood $\{z \in \mathbb{C}^d : |\mathrm{Im} z| < r \leq 1\}$ and $\omega \in DC(\gamma, \tau) \subset \mathbb{R}^d$.

**Theorem 3.** [1, 8, 11] *There exists $\varepsilon_* = \varepsilon_*(\gamma, \tau, r)$ such that, if $|V|_r < \varepsilon_*$ then for all $\theta \in \mathbb{T}^d$, the spectrum of $H$ is purely absolutely continuous.*

**Theorem 4.** *If $|V|_r < \varepsilon_*$, then for any $\theta \in \mathbb{T}^d$, there exist two constants depending on $|V|_r$, $\theta$ and $q(0)$, with $0 < C_1 \leq C_2 < 2\|q(0)\|_{\ell^2(\mathbb{Z})}$ and $C_2 - C_1 \to 0$ as $|V|_r \to 0$, such that $C_1 \leq t^{-1}\|q(t)\|_D \leq C_2$ as $t \to \infty$.*

**The rough idea:** Let $\psi(E) = (\psi_n(E))$, $E \in \sigma(H)$ be a eigenvector generalized of $H$. For $g(E,t) = \sum_n q_n(t)\psi_n(E)$, we have $\mathrm{i}\partial_t g(E,t) = E \cdot g(E,t)$, so $g(E,t) = e^{-\mathrm{i}Et} g(E,0)$. Then, if $\psi_n(E)$ is well derived et well estimated, we can get

$$\sum_n q_n(t)\psi_n'(E) = \partial_E g(E,t) \sim t. \tag{2}$$

If $\psi_n'(E)$ has the form $n \cdot (\cdots)$ ($n$ multiplies by something bounded), for example, as $(e^{\mathrm{i}n\theta})' = n \cdot (\mathrm{i}e^{\mathrm{i}n\theta})$ for the Fourier transform, then, with some suitable norm,

$$\left\| \sum_n q_n(t)\psi_n'(E) \right\|^2 \sim \sum_{n\in\mathbb{Z}} n^2 |q_n(t)|^2. \tag{3}$$

By combining (2) and (3), we can realise the proof.

For constructing such $\psi$, we need to apply the previous works on the reducibility of Schrödinger co-cycle [8, 11] and the property of its rotation number [7, 12, 14].

## References


[1] Avila, A.: The absolutely continuous spectrum of the almost Mathieu operator. Available in the website of author.

[2] Bellissard, J., Lima, R., Scoppola, E.: Localization in $\nu$-dimensional incommensurate structures. Commun. Math. Phys. **88**, 465–477(1983).

[3] Bourgain, J.: Anderson localization for quasi-periodic lattice Schrödinger operators on $\mathbb{Z}^d$, $d$ arbitrary. Geom. Funct. Anal. **17**, 682–706(2007).

[4] Craig, W.: Pure point spectrum for discrete almost periodic Schrödinger operators. Commun. Math. Phys. **88**, 113–131(1983).

[5] Damanik, D., Lukic, M., Yessen, W.; Quantum dynamics of periodic and limit-periodic Jacobi and block Jacobi matrices with applications to some quantum many body problems. Commun. Math. Phys. **337(3)**, 1535–1561(2015).

[6] Damanik, D., Stollmann, P.: Multi-scale analysis implies strong dynamical localization. Geom. Funct. Anal. **11**, 11–29(2001).

[7] Deift, P., Simon, B.: Almost periodic Schrödinger operators. $III$. The absolutely continuous spectrum in one dimension. Commun. Math. Phys. **90**, 389–411(1983).

[8] Eliasson, L. H.: Floquet solutions for the $1-$dimensional quasi-periodic Schrödinger equation. Commun. Math. Phys. **146**, 447–482(1992).

[9] Germinet, F., De Bièvre, S.: Dynamical localization for discrete and continuous random Schrödinger operators. Commun. Math. Phys. **194**, 323–341(1998).

[10] Germinet, F., Jitomirskaya, S. Ya.: Strong dynamical localization for the almost Mathieu model. Rev. Math. Phys. **13**, 755–765(2001).

[11] Hadj Amor, S.: Hölder continuity of the rotation number for quasi-periodic co-cycles in $SL(2,\mathbb{R})$. Commun. Math. Phys. **287(2)**, 565–588(2009).

[12] Herman, M.-R.: Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2. Commun. Math. Helv. **58(3)**, 453–502(1983).

[13] Hiramoto, H., Abe, S.: Dynamics of an Electron in Quasiperiodic Systems. II. Harper's Model. J. Phys. Soc. Jpn. **57**, 1365–1371(1988).

[14] Johnson, R., Moser, J.: The rotation number for almost periodic potentials. Commun. Math. Phys. **84**, 403–438(1982).


## KAM for Water Waves


Massimiliano Berti

(joint work with Riccardo Montalto)


We look for the existence of non trivial, small amplitude, *quasi-periodic* in time, *linearly stable*, gravity-capillary standing water waves of a 2-d perfect, incompressible, irrotational fluid with infinite depth, periodic boundary conditions, and which occupies the free boundary region

$$\mathcal{D}_\eta := \big\{ (x,y) \in \mathbb{T} \times \mathbb{R} \,:\, y < \eta(t,x)\,, \quad \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}) \big\}\,.$$

More precisely we look for quasi-periodic in time solutions of the system

$$
\begin{cases}
\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = \kappa \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} & \text{at } y = \eta(x) \\
\Delta\Phi = 0 & \text{in } \mathcal{D}_\eta \\
\nabla\Phi \to 0 & \text{as } y \to -\infty \\
\partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } \quad y = \eta(x)
\end{cases}
\tag{1}
$$

where $g = 1$ is the acceleration of gravity, $\kappa \in [\kappa_1, \kappa_2]$, $\kappa_1 > 0$, is the surface tension coefficient and $\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}$ is the mean curvature of the free surface. The unknowns of the problem are the free surface $y = \eta(x)$ and the velocity potential $\Phi : \mathcal{D}_\eta \to \mathbb{R}$, i.e. the irrotational velocity field $v = (\Phi_x, \Phi_y)$ of the fluid. The first equation in (1) is the Bernoulli condition according to which the jump of pressure across the free surface is proportional to the mean curvature. The last equation in (1) expresses that the velocity of the free surface coincides with the one of the fluid particles.

Following Zakharov [7] and Craig-Sulem [4], the evolution problem (1) may be written as an infinite dimensional Hamiltonian system. At each time $t \in \mathbb{R}$ the profile $\eta(t, x)$ of the fluid and the value $\psi(t, x) = \Phi(t, x, \eta(t, x))$ of the velocity potential $\Phi$ restricted to the free boundary uniquely determine the velocity potential $\Phi$ in the whole $\mathcal{D}_\eta$, solving (at each $t$) the elliptic problem

$$
\Delta\Phi = 0 \text{ in } \mathcal{D}_\eta,\ \Phi(x + 2\pi, y) = \Phi(x, y)\,,\ \Phi|_{y=\eta} = \psi,\ \nabla\Phi(x, y) \to 0 \text{ as } y \to -\infty\,.
$$

As proved in [7], [4], system (1) is then equivalent to

$$
\begin{cases}
\partial_t \eta = G(\eta)\psi, \\
\partial_t \psi + \eta + \dfrac{1}{2}\psi_x^2 - \dfrac{1}{2}\dfrac{\big(G(\eta)\psi + \eta_x\psi_x\big)^2}{1+\eta_x^2} = \kappa \dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}
\end{cases}
\tag{2}
$$

where $G(\eta)$ is the so-called Dirichlet–Neumann operator

$$
G(\eta)\psi(x) := (\partial_y \Phi)(x, \eta(x)) - \eta_x(x) \cdot (\partial_x \Phi)(x, \eta(x))\,.
$$

definite, actually its Kernel The Dirichlet-Neumann operator $G(\eta)$ is *pseudo-differential* with principal symbol $|D_x|$. The equations (2) are a quasi-linear Hamiltonian system $\partial_t \eta = \delta_\psi H(\eta, \psi)$, $\partial_t \psi = -\delta_\eta H(\eta, \psi)$ where $\delta H$ denotes the $L^2$-gradient and the Hamiltonian

$$
H(\eta, \psi) := \frac{1}{2}(\psi, G(\eta)\psi)_{L^2(\mathbb{T}_x)} + \int_{\mathbb{T}} \frac{\eta^2}{2}\, dx + \kappa \int_{\mathbb{T}} \sqrt{1+\eta_x^2}\, dx
$$

is the sum of kinetic energy, potential energy and area surface integral.

The water-waves system (2) exhibits several symmetries. First of all, the mass $\int_{\mathbb{T}} \eta\, dx$ is a prime integral of (2) and the subspace $\int_{\mathbb{T}} \eta\, dx = \int_{\mathbb{T}} \psi\, dx = 0$ is invariant under the evolution of (2). We restrict to this subspace. In addition, the subspace of functions which are even in $x$, $\eta(x) = \eta(-x)$, $\psi(x) = \psi(-x)$, is invariant under (2). Thus we restrict $(\eta, \psi)$ to the phase space of $2\pi$-periodic functions which admit the Fourier expansion

$$\eta(x) = \sum\nolimits_{j\geq 1} \eta_j \cos(jx)\,, \quad \psi(x) = \sum\nolimits_{j\geq 1} \psi_j \cos(jx)\,. \tag{3}$$

In this case also the velocity potential $\Phi(x, y)$ is even and $2\pi$-periodic in $x$ and so the $x$-component of the velocity field $v = (\Phi_x, \Phi_y)$ vanishes at $x = k\pi$, $\forall k \in \mathbb{Z}$. Hence there is no flux of fluid through the lines $x = k\pi$, $k \in \mathbb{Z}$, and a solution of (2) satisfying (3) describes the motion of a liquid confined between two walls.

The capillary water waves system is reversible, namely the Hamiltonian $H(\eta, \psi) = H(\eta, -\psi)$ is even in $\psi$, and it is natural to look for solutions of (2) satisfying

$$\eta(-t, x) = \eta(t, x)\,,\ \psi(-t, x) = -\psi(t, x)\ , \quad \forall t, x \in \mathbb{R}\,. \tag{4}$$

Time periodic/quasi-periodic solutions of (2) satisfying (3) and (4) are called gravity-capillary *standing water waves.*

This is a small divisors problem. Existence of small amplitude time-periodic pure gravity (without surface tension) standing water wave solutions has been proved by Iooss, Plotnikov, Toland in [5], and in [6] in finite depth. Existence of time-periodic gravity-capillary standing wave solutions of (2) has been recently proved by Alazard-Baldi [1].

In order to study small amplitude solutions we first consider the linearized water waves system (2) at the equilibrium $(\eta, \psi) = (0, 0)$, namely by

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi + \eta = \kappa \eta_{xx} \end{cases} \tag{5}$$

where $G(0) = |D_x|$ is the Dirichlet-Neumann operator for the flat surface $\eta = 0$.

Fix an arbitrary finite subset $\mathbb{S}^+ \subset \mathbb{N}^+ := \{1, 2, \ldots\}$ (tangential sites) and consider the linear standing wave solutions of (5)

$$\eta(t, x) = \sum_{j\in\mathbb{S}^+} \sqrt{\xi_j} \cos(\omega_j t) \cos(jx),\ \ \psi(t, x) = -\sum_{j\in\mathbb{S}^+} \sqrt{\xi_j} j^{-1} \omega_j \sin(\omega_j t) \cos(jx)\,,$$

with linear frequencies of oscillations

$$\omega_j := \omega_j(\kappa) := \sqrt{j(1 + \kappa j^2)}\,, \quad j \geq 1\,. \tag{6}$$

The goal is to prove that they can be continued to solutions of (2) for most values of the surface tension $\kappa$. Let $\nu := |\mathbb{S}^+|$ denote the cardinality of $\mathbb{S}^+$. The following result is presented in [2].

**Theorem 1.** *For every choice of finitely many tangential sites $\mathbb{S}^+ \subset \mathbb{N}^+$, there exists $\bar{s} > s_0 := \left[\frac{\nu+1}{2}\right] + 1$, $\varepsilon_0 \in (0, 1)$ such that for every $|\xi| \leq \varepsilon_0^2$, $\xi := (\xi_j)_{j\in\mathbb{S}^+}$, there exists a Cantor like set $\mathcal{G} \subset [\kappa_1, \kappa_2]$ with asymptotically full measure as $\xi \to 0$, i.e.*

$$\lim_{\xi\to 0} |\mathcal{G}| = \kappa_2 - \kappa_1\,,$$

*such that, for any surface tension coefficient $\kappa \in \mathcal{G}$, the capillary-gravity system (2) has a time quasi-periodic standing wave solution $u(\tilde{\omega} t, x) = (\eta(\tilde{\omega} t, x), \psi(\tilde{\omega} t, x))$*

*of the form*

$$\eta(\tilde{\omega}t, x) = \sum_{j \in \mathbb{S}^+} \sqrt{\xi_j} \cos(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|}),$$
$$\psi(\tilde{\omega}t, x) = -\sum_{j \in \mathbb{S}^+} \sqrt{\xi_j} j^{-1} \omega_j \sin(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|})$$

*with a diophantine frequency vector* $\tilde{\omega} := \tilde{\omega}(\kappa, \xi) \in \mathbb{R}^\nu$ *satisfying* $\tilde{\omega}_j - \omega_j(\kappa) \to 0$, $j \in \mathbb{S}^+$, *as* $\xi \to 0$. *The solution* $(\eta, \psi)(\varphi, x) \in H^{\bar{s}}(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R}^2)$. *In addition these quasi-periodic solutions are linearly stable.*

The fact that the quasi-periodic solutions exist for a proper subset of parameters $\kappa$ is clearly not a technical issue, because the gravity-capillary water-waves equations (2) are expected to be non-integrable.

Theorem 1 is proved by a Nash-Moser implicit function theorem. A key step is the reducibility to constant coefficients of the quasi-periodic system obtained linearizing (2) at a quasi-periodic approximate solution. It turns out that the Floquet exponents of the quasi-periodic solution are purely imaginary $\mathrm{i}\mu_j^\infty$ and have an asymptotic expansion like

$$\mu_j^\infty := \lambda_3^\infty \sqrt{j(1+\kappa j^2)} + \lambda_1^\infty j^{\frac{1}{2}} + r_j \in \mathbb{R}\,, \ j \in \mathbb{N} \setminus \mathbb{S}^+\,,$$

with $\lambda_3^\infty = 1 + O(\varepsilon^a) \in \mathbb{R}$, $\lambda_1^\infty = O(\varepsilon^a)$, $\sup_{j \in \mathbb{N} \setminus \mathbb{S}^+} |r_j^\infty| = O(\varepsilon^a)$ for some $a > 0$. The measure estimates for the Cantor like set $\mathcal{G}$ are obtained generalizing ideas of degenerate KAM theory for PDEs, i.e. [3], using that the linear frequencies $\omega_j(\kappa)$ defined in (6) are analytic and non degenerate in $\kappa$.

## References


[1] Alazard T., Baldi P. *Gravity capillary standing water waves* Arch. Rat. Mech. Anal, 217, 3, 741-830, 2015.

[2] Berti M., Montalto R. Quasi-periodic standing wave solutions of gravity-capillary water waves, preprint.

[3] Bambusi D., Berti M., Magistrelli E. Degenerate KAM theory for partial differential equations, *Journal Diff. Equations*, 250, 8, 3379-3397, 2011.

[4] Craig W., Sulem C. Numerical simulation of gravity waves. *J. Comput. Phys.*, 108(1):73-83, 1993.

[5] Iooss G., Plotnikov P., Toland J. Standing waves on an infinitely deep perfect fluid under gravity. *Arch. Rat. Mech. Anal.*, 177(3):367–478, 2005.

[6] Plotnikov P., Toland J. Nash-Moser theory for standing water waves. *Arch. Rat. Mech. Anal.*, 159(1):1–83, 2001.

[7] Zakharov V. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9(2):190–194, 1968.

## From semi-toric systems to Hamiltonian $\mathbb{S}^1$-actions and back

Sonja Hohloch
(joint work with Silvia Sabatini, Daniele Sepe, Margaret Symington)

This talk presents the transition from so-called semi-toric Hamiltonian systems to Hamiltonian $\mathbb{S}^1$-actions and back.

More precisely, let $(M, \omega)$ be a closed, connected, 4-dimensional symplectic manifold. $\Phi = (J, H) : (M, \omega) \to \mathbb{R}^2$ is a *completely integrable Hamiltonian system* if **(i)** the Hamiltonian vectorfields $X^J$ and $X^H$ are almost everywhere linearly independent and **(ii)** $\{J, H\} := \omega(X^J, X^H) = 0$ everywhere, i.e. the flows $\varphi_s^J$ and $\varphi_t^H$ commute and thus induce an action $\mathbb{R}^2 \times M \to M$ via $(s,t).x := \varphi_s^J(\varphi_t^H(x))$.

Eliasson [El] and Miranda & Zung [MZ] proved a local normal form around so-called *non-degenerate* singular points and orbits of completely integrable systems. If we do not admit hyperbolic components, the only non-degenerate singular points possible on 4-dimensional manifolds are of *elliptic-elliptic*, *elliptic-regular* or *focus-focus type.*

A very special class of completely integrable systems are *toric* manifolds where $\Phi$ induces an $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$-action. Delzant proved that effective toric manifolds $(M, \omega, \Phi)$ are classified by $\Phi(M)$ which is a 'very nice' convex polytope, called *Delzant polytope.*

*Semi-toric systems* $(M, \omega, \Phi = (J, H))$ are completely integrable systems such that **(i)** $J$ induces an effective Hamiltonian $\mathbb{S}^1$-action and **(ii)** all singular points are non-degenerate and admit no hyperbolic components. These systems induce an $\mathbb{S}^1 \times \mathbb{R}$-action and sit therefore 'between' toric systems and general integrable systems. Semi-toric systems were introduced by Vũ Ngọc and classified by Pelayo & Vũ Ngọc [PV09, PV11] by means of 5 invariants. The first 2 invariants are the number of focus-focus singular points and a 'straightened version' of the 'curved' image $\Phi(M)$, called (an equivalence class of) a 'semi-toric polygone'. The remaining 3 invariants describe the local behaviour of the system near the focus-focus points, their heights in the semi-toric polytope and how the local behaviour is 'patched together' to yield the whole system.

If we 'forget' the second integral $H$ in toric and semi-toric systems, we are left with the Hamiltonian $\mathbb{S}^1$-action induced by $J$. Effective Hamiltonian $\mathbb{S}^1$-actions $J : (M, \omega) \to \mathbb{R}$ have been classified by Karshon [Ka] by means of a labeled, directed graph.

The relations between toric, semi-toric, general integrable systems and Hamiltonian $\mathbb{S}^1$-actions are displayed in Figure 1.

Our work Hohloch & Sabatini & Sepe [HSS] answers the question which invariants in Pelayo & Vũ Ngọc's classification of semi-toric systems $(M, \omega, \Phi = (J, H))$ are needed in order to recover the Karshon graph of the underlying Hamiltonian $\mathbb{S}^1$-action $(M, \omega, J)$.

**Theorem 1** (Hohloch & Sabatini & Sepe [HSS])**.** *Among the 5 Pelayo & Vũ Ngọc invariants, only the first 2 are needed to recover the Karshon graph of the underlying Hamiltonian* $\mathbb{S}^1$*-space.*

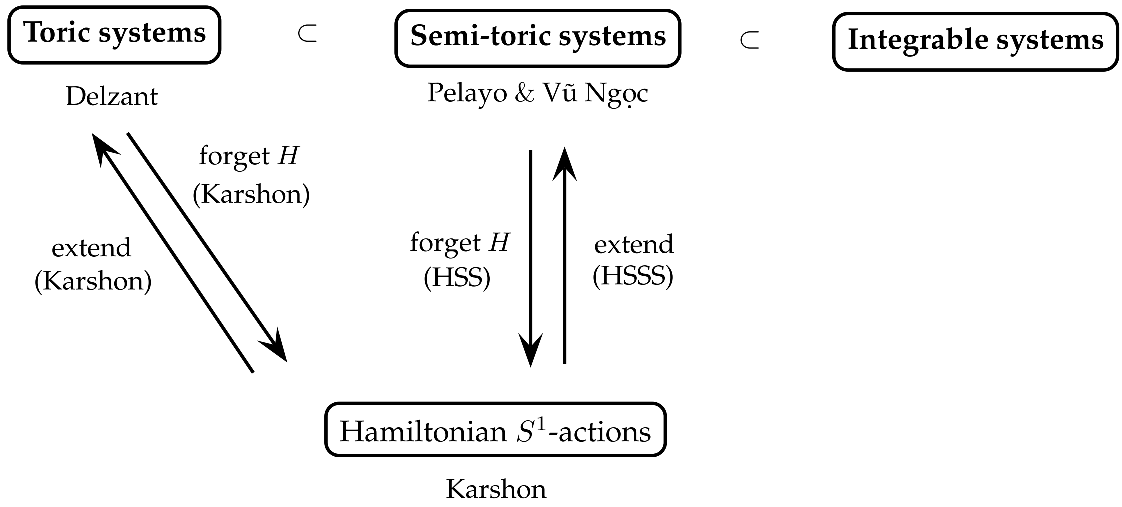


FIGURE 1. Overview.

An effective Hamiltonian $\mathbb{S}^1$-action $(M, \omega, J)$ is *extendable* if it extends to an effective toric manifold $(M, \omega, \Phi = (J, H))$. A semi-toric system is *adaptable* if its underlying Hamiltonian $\mathbb{S}^1$-action is extendable.

**Theorem 1** (Hohloch & Sabatini & Sepe [HSS]). *A semi-toric system is adaptable if and only if its equivalence class of semi-toric polygons contains a Delzant polygon.*

Karshon [Ka] states explicit conditions under which a Hamiltonian $\mathbb{S}^1$-action extends to a toric one. Our corresponding result is

**Theorem 2** (Hohloch & Sabatini & Sepe & Symington [HSSS]). *Let $(M, J, \omega)$ be a Hamiltonian $\mathbb{S}^1$-action satisfying **(i)** fixed surfaces (if any) have genus zero and **(ii)** each level set of $J$ intersects at most two so-called $\mathbb{Z}_k$-spheres. Then there exists a smooth $H : M \to \mathbb{R}$ such that $(M, \omega, \Phi = (J, H))$ is a semi-toric system.*

## REFERENCES


[De] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment.* (French) [Periodic Hamiltonians and convex images of the momentum mapping], Bull. Soc. Math. France **116** (1988), no. 3, 315–339.

[El] L.H. Eliasson, *Hamiltonian systems with Poisson commuting integrals*, PhD thesis, University of Stockholm, 1984.

[HSS] S. Hohloch, S. Sabatini, D. Sepe, *From compact semi-toric systems to Hamiltonian $\mathbb{S}^1$-actions*, Discr. Cont. Dyn. Sys., Series A **35** (2015), no. 1, 247–281.

[HSSS] S. Hohloch, S. Sabatini, D. Sepe, M. Symington, *From Hamiltonian $\mathbb{S}^1$-actions to compact semi-toric systems*, in preparation.

[Ka] Y. Karshon, *Periodic Hamiltonian flows on four dimensional manifolds*, Mem. Amer. Math. Soc., 672 (1999).

[MZ] E. Miranda, N.T. Zung, *Equivariant normal form for non-degenerate singular orbits of integrable Hamiltonian systems*, Ann. Sci. Éc. Norm. Sup., (4) **37** (2004), no. 6, 81–93.

[PV09] Á. Pelayo, S. Vũ Ngọc, *Semitoric integrable systems on symplectic 4-manifolds*, Invent. Math. **177** (2009), no. 3, 571–597.

[PV11] Á. Pelayo, S. Vũ Ngọc, *Constructing integrable systems of semitoric type*, Acta Math., **206** (2011), no. 1, 93–125.

## Geometry of high frequency vibrations

Mark Levi

This talk described some old and some new results on various counterintuitive phenomena in dynamics associated with high frequency vibrations, and connections with non–holonomic systems, Riemannian geometry and the stationary Schrödinger equation. Perhaps the simplest such phenomenon is the stabilization of an inverted pendulum in the upside–down position when the pendulum's pivot is vibrated with sufficiently high frequency. The surprising phenomenon is sometimes referred to as the Kapitsa effect [1], although it has been demonstrated by Stephenson in 1908 [2]; it played the key role in the discovery of the Paul trap [3], for which W. Paul was awarded the Nobel prize in physics in 1989. It was observed later [4] that the phenomenon, which in his Nobel paper Paul explained by a computation, has a geometrical explanation, which in its simplest version states, roughly speaking, that in the limit of high frequency vibration the pendulum is approximated by a nonholonomic system: a two dimensional "bike", i.e. a moving segment one of whose ends traces out a prescribed path, while the other end moves so that its velocity is aligned with a segment. This purely geometrical system turns out to be a singular limit (for high frequency) of the pendulum with its pivot guided along a prescribed curve. On the other hand, the same "bike" turns out to be equivalent to the stationary Schrödingier equation [5]. This equivalence, outlined in the talk, establishes a correspondence between the paths of the "front wheel" of the "bike" on the one hand, and the potentials of the stationary Schrödingier equation on the other. A recent extension of this result was also mentioned: to a system with more degrees of freedom, e.g.

a multiple pendulum, subjected to rapid oscillatory forcing, one can associate a direction field in the Riemannian configuration manifold; and the effective potential of the averaged system is given in terms of curvature of curves in this direction field. In a related direction, a recently discovered observation [6] was mentioned: rapid rotation of a potential has the effect of producing a Coriolis–like force in an *inertial frame* – apparently the first known example where such force is not due to the rotation of the frame.

### References

[1] P. L. Kapitsa, *Dynamical stability of a pendulum when its point of suspension vibrates.*, Collected Papers by P.L. Kapitsa, Volume II, Pergamon Press, London, pages 714–725, 1965.

[2] Stephenson, *On a new type of dynamical stability.*, Manchester Memoirs **52** (1908), 1-10.

[3] W. Paul, *Electromagnetic traps for charged and neutral particles.*, Revs. Mod. Phys. **62** (1990), 531–540.

[4] M. Levi, *Computing certain invariants of topological spaces of dimension three*, Topology **32** (1990), 100–120.
[5] M. Levi, *Schrödinger's equation and "bike tracks"–a connection*, arXiv:1405.174v1 (2014), 1–10.
[6] O. Kirillov, M. Levi, *Rotating saddle trap as Foucault's pendulum*, Am. J. of Physics, to appear.

## Oscillatory orbits in the restricted planar three body problem

Marcel Guardia
(joint work with Pau Martin, Tere M. Seara, Lara Sabbagh)

Consider the planar three body problem, that is the motion of three bodies in the plane under the Newtonian gravitational force. The restricted planar three body problem is a simplified version of this model, where one of the bodies is assumed to have zero mass and, consequently, its movement does not affect the other two, *the primaries*. This implies that the orbits of the primaries are governed by the classical Kepler's laws. Let us assume that the primaries perform circular orbits (the restricted planar circular three body problem, RPC3BP) or elliptic orbits (the restricted planar elliptic three body problem, RPE3BP). Since the RPC3BP has a rotational symmetry, it has a first integral which is usually called the Jacobi constant.

Normalizing the total mass of the system to be one, the RPC3BP depends on one parameter $\mu$, which measures the quotient between the masses of the two primaries and therefore satisfies $\mu \in [0, 1/2]$. In the RPE3BP we have also as a parameter the eccentricity $e_0 \in (0, 1)$ of the ellipses of the primaries.

The purpose of [1, 2] is to prove the existence of oscillatory orbits for the RPC3BP and the RPE3BP, that is, orbits that leave every bounded region but return infinitely often to some fixed bounded region. Namely, if $q \in \mathbb{R}^2$ is the position of the body of zero mass, orbits such that

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \text{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

When $\mu = 0$, there is only one primary. In this case, the motion of the massless body satisfies Kepler's laws. In particular, oscillatory motions cannot exist.

In 1980, J. Llibre and C. Simó [3] proved the existence of oscillatory motions for the RP3BP provided the ratio between the masses of the primaries $\mu$ is small enough.

In [1] we prove the following statement and thus we generalize the results in [3] to any value of mass ratio.

**Theorem 1.** *Fix any $\mu \in (0, 1/2]$. Then, there an orbit $(q(t), p(t))$ of the RPC3BP which satisfies*

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

To obtain such motions, we restrict ourselves to the level sets of the Jacobi constant. We show that, for any value of the mass ratio and for large values of the Jacobi constant, there exist transversal intersections between the stable and unstable manifolds of infinity in these level sets. These transversal intersections guarantee the existence of a symbolic dynamics that creates the oscillatory orbits. The main achievement is to prove the existence of these orbits without assuming the mass ratio $\mu$ small. When $\mu$ is not small, this transversality cannot be checked by means of classical perturbation theory.

In [2], we extend this result to the RPE3BP provided the primaries perform almost circular orbits.

**Theorem 2.** *Fix any $\mu \in (0, 1/2]$. There exists $e_0^*(\mu) > 0$ such that for any $e_0 \in (0, e_0^*(\mu))$ there exists an orbit $(q(t), p(t))$ of the RPE3BP which satisfies*

$$\limsup_{t\to+\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t\to+\infty} \|q\| < +\infty.$$


### References

[1] M. Guardia, P. Martín, and T. Seara, *Oscillatory motions for the restricted planar circular three body problem*, Inventiones Mathematicae DOI:10.1007/s00222-015-0594-8 (2015).

[2] M. Guardia, P. Martin, L. Sabbagh, T. Seara, *Oscillatory orbits in the restricted elliptic planar three body problem*, Available at *http://arxiv.org/abs/1506.04551*.

[3] J. Llibre, C. Simó, *Oscillatory solutions in the planar restricted three-body problem*, Math. Ann., 248(2):153–184, 1980.


## Volume in Seiberg-Witten theory and the existence of two Reeb orbits


Daniel Cristofaro-Gardiner

(joint work with Michael Hutchings, Vinicius Ramos)


The Weinstein conjecture states that on a closed manifold, every "Reeb" vector field has at least one closed orbit. Recently, Taubes [5] used Seiberg-Witten theory to prove the three-dimensional version of this conjecture. In this talk, we explain how to show that in dimension three, every Reeb vector field has at least two closed orbits [1].

Our proof also uses Seiberg-Witten theory. In fact, to prove the result, we establish a relationship between the lengths of certain sets of closed Reeb orbits and the volume of the manifold [2]. This statement is of potentially independent interest; for example, it was recently used in [4] to prove that for a suitably generic Reeb vector field in dimension 3, the union of the closed periodic orbits forms a dense set. To translate Seiberg-Witten theory into the dynamical setting, we use a tool called "embedded contact homology" (ECH). This is a kind of Floer homology whose generators are certain equivalence classes of sets of closed orbits of the Reeb vector field. Taubes has shown that ECH is canonically isomorphic to the manifold's Seiberg-Witten Floer cohomology [6].

While our result is sharp, in the sense that examples exist in dimension 3 with exactly two Reeb orbits, much remains unknown about this kind of question. For example, we currently know no examples of Reeb vector fields with finitely many closed orbits on any manifold that is not a lens space. It may well be the case that no such examples exist. We conjecture more modestly that a Reeb vector field on a manifold that is not a lens space has at least 3 closed orbits. A partial version of this is shown in [3].

## References


[1] D. Cristofaro-Gardiner and M. Hutchings, *From one Reeb orbit to two*, Journal of Differential Geometry, to appear.

[2] D. Cristofaro-Gardiner, M. Hutchings, and V. Ramos, *The asymptotics of ECH capacities*, Inventiones Mathematicae **199** (2015), 187–214.

[3] M. Hutchings and C. Taubes, *The Weinstein conjecture for stable Hamiltonian structures*, Geometry and topology **13** (2009), 901–941.

[4] K. Irie, *Dense existence of periodic Reeb orbits and ECH spectral invariants*, arXiv:1508.07542.

[5] C. Taubes, *The Seiberg-Witten equations and the Weinstein conjecture*, Geometry and Topology **11** (2007), 2117–2202.

[6] C. Taubes, *Embedded contact homology and Seiberg-Witten Floer cohomology I-V*, Geometry and Topology **14** (2010), 2497–3000.


# Convergence in the vanishing discount method for Hamilton–Jacobi equations


Maxime Zavidovique

(joint work with Andrea Davini, Albert Fathi, Renato Iturriaga)


The results presented build up on previous results [6] and were since extended in various settings [6, 8, 1, 4].

Let $M$ be a closed compact manifold and $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, convex and coercive with respect to the momentum variable $p$. It is then known that if $\lambda > 0$ there exists a unique viscosity solution $u_\lambda$ to the discounted Hamilton–Jacobi equation $\lambda u(x) + H(x, D_x u) = 0$, $x \in M$. We prove the following:

**Theorem 1.** *There exists a unique constant $c(H)$ for which the functions $u_\lambda + c(H)/\lambda$ uniformly converge to a function $u_0$ (as $\lambda \to 0$) which then solves the stationary undiscounted equation $H(x, D_x u_0) = c(H)$.*

Such a function $u_0$ is called a weak KAM solution. In a way, this result closes a loop in the history of weak KAM theory. Let us describe the setting and explain why.

## 1. History of the problem

In 1987, Lions, Papanicolaou and Varadhan issue a (never published) preprint [7] on the Homogenization of Hamilton–Jacobi equations. They study the following equation with unknown $u^\varepsilon : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$:

$$\frac{\partial u^\varepsilon}{\partial t} + H\big(\frac{x}{\varepsilon}, D_x u^\varepsilon\big) = 0$$

with initial condition $u^\varepsilon(0,x) = u_0(x)$. In the above, $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, which is 1-periodic in the first variable $x$ (meaning it is the lift of a function on $\mathbb{T}^N \times \mathbb{R}^N$) and uniformly coercive with respect to the second variable $p$. The initial condition $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded uniformly continuous function on $\mathbb{R}^N$. It is then known there exists a unique continuous solution[1] $u^\varepsilon : [0,+\infty) \times \mathbb{R}^N$ to the above equation.

**Theorem 2** (Lions, Papanicolaou, Varadhan)**.** *The functions $u^\varepsilon$ uniformly converge to a function $u^0$ which solves a new Hamilton–Jacobi equation $\frac{\partial u^0}{\partial t} + \overline{H}(D_x u^0) = 0$, with same initial condition. The effective Hamiltonian, $\overline{H} : \mathbb{R}^N \to \mathbb{R}$, is characterized as follows: for any $P \in \mathbb{R}^N$, $\overline{H}(P)$ is the only constant for which the following equation admits a 1- periodic solution:*

$$H(x, P + D_x u) = \overline{H}(P). \tag{1}$$

Solutions to (1) were later on independently introduced by Fathi as weak KAM solutions. In order to prove such a constant exists, they use an ergodic perturbation and solve $\lambda u + H(x, P + D_x u) = 0$, where $\lambda > 0$ is a parameter that will be sent to 0. It is known that such an equation admits a unique periodic solution $u_\lambda$. Moreover, because of the coercivity of $H$, the family $u_\lambda$ is equi–Lipschitz. Therefore, the functions $\hat{u}_\lambda = u_\lambda - \min u_\lambda$ admit converging subsequences. They then prove that up to extracting, the $\lambda u_\lambda$ uniformly converge to a constant $-\overline{H}(P)$ and $\hat{u}_\lambda$ converges to a function $u$ which then solves (1).

Our theorem is that under the extra condition of convexity, no extraction is needed.

## 2. A formula for the solutions of the discounted equation

Our proof of Theorem 1 relies on an explicit formula for $u_\lambda$ when $H$ is convex in $p$. Let us introduce the Lagrangian function: $L : TM \to \mathbb{R} \cup \{+\infty\}$ by

$$\forall (x,v) \in TM, \quad L(x,v) = \sup_{p \in T_x M} p(v) - H(x,p).$$

[1] All solutions subsolutions or supersolutions will be implicitly continuous and in the viscosity sense and the terms will be omitted from now on.

Then the following holds for all $t > 0$ and $\lambda > 0$ and $x \in M$,

$$\begin{aligned} u_\lambda(x) &= \inf_\gamma e^{-\lambda t} u_\lambda\big(u_\lambda(\gamma(-t)\big) + \int_{-t}^0 e^{\lambda s} L\big(\gamma(s), \dot\gamma(s)\big)\mathrm{d}s \\ &= \inf_\gamma \int_{-\infty}^0 e^{\lambda s} L\big(\gamma(s), \dot\gamma(s)\big)\mathrm{d}s. \end{aligned}$$

Where the infima are taken amongst absolutely continuous curves such that $\gamma(0) = x$.

Note that the function $u_0$ given by Theorem 1 will then verify a similar relation for all $t > 0$:

$$\forall x \in M, \quad u_0(x) = \inf_\gamma u_0\big(\gamma(-t)\big) + \int_{-t}^0 \big[L\big(\gamma(s), \dot\gamma(s)\big) + c(H)\big]\mathrm{d}s. \tag{2}$$

This is Fathi's original characterization of weak KAM solutions. The $(\inf, +)$ convolution it involves is called Lax-Oleinik semi–group.

## 3. The discrete setting

At this point, we may introduce a discrete analogue of the previous problem. The philosophy is that the Lagrangian represents a cost to pay to move infinitesimally in a direction $v$. This is replaced by discretizing the time variable and introducing a cost function which evaluates the cost to go between two points in time 1.

Let $(X, d)$ be a compact metric space, and $c : X \times X \to \mathbb{R}$ be a continuous function. The Lax–Oleinik operator, acting on continuous functions $u : X \to \mathbb{R}$ is defined by $u \mapsto \mathcal{T}u$:

$$\forall x \in X, \quad \mathcal{T}u(x) = \inf_{y \in X} u(y) + c(y, x).$$

It is easily verified that $\mathcal{T}$ has values in (equi–)continuous functions, is 1-Lipschitz for the sup–norm, is order preserving and commutes with addition of constants. Therefore

**Proposition 3** (weak KAM)**.** *There exists a unique constant $c_0$ such that there is a continuous function $u : X \to \mathbb{R}$ verifying $u = \mathcal{T}u + c_0$.*

The discounted operators are defined as follows: given a constant $\mu \in (0, 1)$ (which may be seen as $e^{-\lambda}$ in the continuous setting) $\mathcal{T}_\mu$ acts on continuous functions by $\mathcal{T}_\mu u = \mathcal{T}(\mu u)$. This operator is now $\mu$–Lipschitz for the sup–norm, hence it admits a unique fixed point which may be computed taking the limit of iterates starting with any function $u$, for instance the 0 function. A computation gives that this unique fixed point is given by the formula

$$\forall x \in X, \quad u_\mu(x) = \inf_{(x_n)_{n \leqslant 0}} \sum_{n=-\infty}^{-1} \mu^{n+1} c(x_n, x_{n+1}),$$

where the infimum is taken on all sequences such that $x_0 = x$.

Our second theorem is then:

**Theorem 4.** *There exists a function $u_0$ such that $u_\mu + c_0/(1-\mu)$ converges to $u_0$ as $\mu \to 1$. Moreover, $u_0$ verifies $u_0 = \mathcal{T} u_0 + c_0$.*

The characterization of $u_0$ and the proof of the convergence heavily rely on the notion of closed minimizing measures, that is probability measures $m$ on $X \times X$ which have the same projection on both factors and minimize the quantity $\int c(x,y)\mathrm{d}m(x,y)$. They are discrete analogues of Mather measures in the classical Hamiltonian setting. Similar ideas previously appeared in [5] and were recovered independently in the present works..


## References

[1] E.S. AL–Aidarous, E.O. Alzahrani, H. Ishii and A.M.M. Younas, *A convergence result for the ergodic problem for Hamilton–Jacobi equations with Neumann type boundary conditions*, preprint (2014).

[2] A. Davini, A. Fathi, R. Iturriaga and M. Zavidovique, *Convergence of the solutions of the discounted equation*, preprint (2014).

[3] A. Davini, A. Fathi, R. Iturriaga and M. Zavidovique, *Convergence of the solutions of the discounted Hamilton–Jacobi equation: the discrete case*, in preparation (2015).

[4] A. Davini and M. Zavidovique, *Convergence of the solutions of the discounted equation for weakly coupled systems of Hamilton–Jacobi equations*, in preparation (2015).

[5] D. Gomes, *Generalized Mather problem and selection principles for viscosity solutions and Mather measures*, Adv. Calc. Var. **1** (2008), 291–307.

[6] R. Iturriaga and H. Sanchez–Morgado, *Limit of the infinite horizon discounted Hamilton-Jacobi equation*, Discrete Contin. Dyn. Syst. Ser. B **15** (2011), 623–635.

[7] P.–L. Lions, G. Papanicolaou and S. Varadhan , *Homogenization of Hamilton-Jacobi equation*, unpublished preprint (1987).

[8] H. Mitake and H.V. Tran, *Selection problems for a discounted degenerate viscous Hamilton-Jacobi equation*, preprint (2014).


## Reduction theory for Fuchsian groups and coding of geodesics

Svetlana Katok

(joint work with Ilie Ugarcovici)

Let $\Gamma$ be a Fuchsian group of the first kind acting on upper half-plane $\mathcal{H}$ (or the unit disk $\mathbb{D}$), and $\mathbb{S}$ be the Euclidean boundary, $\mathbb{S} = \partial(\mathcal{H}) = \mathbb{P}^1(\mathbb{R})(\partial(\mathbb{D}) = S^1)$. Let $\mathcal{F}$ be a fundamental domain for $\Gamma$ with an even number $K$ of sides identified by the set of generators $A = \{\gamma_1, \dots, \gamma_K\}$ of $\Gamma$, and $\rho : \mathbb{S} \to A$ be a (surjective) map locally constant on $\mathbb{S} \setminus J$, where $J = \{x_0, x_1, \dots x_K = x_0\}$ is the set of jumps.

A *boundary map* $f : \mathbb{S} \to \mathbb{S}$ is defined by $f(x) = \rho(x)x$. It is a piecewise fractional-linear map whose set of discontinuities is $J$. Let $F : \mathbb{S}\times\mathbb{S}\setminus\Delta \to \mathbb{S}\times\mathbb{S}\setminus\Delta$ be given by

$$F(u,w) = (\rho(w)u, \rho(w)w).$$

This is a *(natural) extension* of $f$, and if we identify $(u,w) \in \mathbb{S}\times\mathbb{S}\setminus\Delta$ with an oriented geodesic from $u$ to $w$, we can think of $F$ as a map on geodesics $(u,w)$ which we will also call a *reduction map*.

In 2007 Don Zagier proposed the following conjecture:

**Zagier's Reduction Theory Conjecture for Fuchsian groups (RTC).** For every Fuchsian group $\Gamma$ there exist $\mathcal{F}, A$ as above, and an open set of $J's$ in $\mathbb{R}^K$ such that

1. (1) The map $F$ possesses a global attractor set $D = \cap_{n=0}^{\infty} F^n(\mathbb{S} \times \mathbb{S} \setminus \Delta)$ consisting of two (or one, in degenerate cases) connected components each having *finite rectangular structure*, i.e. bounded by non-decreasing step-functions with a finite number of steps, on which $F$ is essentially bijective.
2. (2) Every point $(u, w) \in \mathbb{S} \times \mathbb{S}$ $(u \neq w)$ is mapped to $D$ after finitely many iterations of $F$.

We solved this conjecture completely in two leading cases: for surface groups in a paper in preparation [5], and for $SL(2,\mathbb{Z})$ in [4].

In both cases (2) was proved for almost every point $(u, w) \in \mathbb{S} \times \mathbb{S}$ $(u \neq w)$ and the exceptions were analyzed.

For a surface group $\Gamma$ such that $\Gamma\backslash\mathbb{D}$ is a compact surface of genus $g$ we choose the fundamental region $\mathcal{F}$ to be a regular $K = 8g - 4$-gon centered at the origin whose sides are orthogonal geodesics segments with a particular identification by the generators of $\Gamma$. They satisfy the *extension condition*: the geodesic extensions of these segments never intersect the interior of the tiling sets $\gamma\mathcal{F}$, $\gamma \in \Gamma$.

Let $[P_i, Q_{i+1}]$ be the oriented geodesic extending the side $i$ of $\mathcal{F}$, $J = \bar{P} = \{P_i\}_{1\le i\le K}$ be the Bowen-Series partition [2], $f_{\bar{P}} : \mathbb{S} \to \mathbb{S}$ be defined by

$$f_{\bar{P}}(x) = \gamma_i(x) \quad \text{if } x \in [P_i, P_{i+1})\,,$$

and $F_{\bar{P}}$ be the corresponding reduction map. Adler and Flato [1] proved that $F_{\bar{P}}$ has an invariant domain $D_{\bar{P}} \subset \mathbb{S} \times \mathbb{S}$ with finite rectangular structure.

partition $\{P_1, Q_1, P_2, Q_2, \ldots, P_K, Q_K\}$ is expanding and satisfy Rényi's distortion estimates, hence it admits a unique finite invariant ergodic measure equivalent to Lebesgue measure. We now perturb the set of jumps by letting $J = \bar{A} = \{A_i\}_{1\le i\le K}$, $A_i \in [P_i, Q_i]$, and define $f_{\bar{A}} : \mathbb{S} \to \mathbb{S}$ by

$$f_{\bar{A}}(x) = \gamma_i(x) \quad \text{if } x \in [A_i, A_{i+1})\,.$$

$F_{\bar{A}}$ will be the corresponding reduction map. some non-negative integers $m_i, k_i$ The following property is crucial for the proof of the RTC:

**Theorem 1.** *Each partition point $A_i$, $1 \le i \le 8g - 4$ satisfies the cycle property, i.e. for some non-negative integers $m_i, k_i$*

$$f_{\bar{A}}^{m_i}(\gamma_i A_i) = f_{\bar{A}}^{k_i}(\gamma_{i-1} A_i).$$

We prove RTC for the open set of points $\bar{A} = \{A_i\}_{1\le i\le K}$ that satisfy the *short cycle property*, i.e. the cycle property with $m_i = k_i = 1$. Here are the steps in the proof:

1. (1) We prove that $F_{\bar{P}}$ satisfies properties of the RTC and find $D_{\bar{P}}$ explicitly.
2. (2) We construct $D_{\bar{A}}$ by a *quilting technique* first described by I. Smeets in her thesis [7], modifying the known domain $D_{\bar{P}}$ by *adding and deleting*.
3. (3) We prove bijectivity of $F_{\bar{A}}$ on $D_{\bar{A}}$.
4. (4) The proof of the global trapping property also uses quilting construction.

For the modular group the boundary maps are related to a family of $(a,b)$-continued fractions. In this case the fundamental domain is standard, $A = \{T, S, T^{-1}\}$, where $Tx = x+1$, $Sx = -\frac{1}{x}$ are generators of $SL(2,\mathbb{Z})$, $\mathbb{S} = \mathbb{R} \cup \infty$, $J = \{-\infty, a, b, +\infty\}$, and the parameter set $\mathcal{P} = \{a \le 0 \le b,\ b-a \ge 1,\ -ab \le 1\}$ is the maximal possible. The reduction theory in this case was described in [4].

**Theorem 2.** *There is an explicit one-dimensional Lebesgue measure zero, uncountable set $\mathcal{E}$ on the diagonal boundary $b = a+1$ of $\mathcal{P}$ such that*

*(1) for all $(a,b) \in \mathcal{P} \setminus \mathcal{E}$ the map $F_{a,b}$ has an attractor $D_{a,b}$ satisfying property (1) of the RTC;*

*(2) for an open and dense set in $\mathcal{P} \setminus \mathcal{E}$ property (2), and hence the RTC, holds. For the rest of $\mathcal{P} \setminus \mathcal{E}$ property (2) holds for almost every point $(u,w) \in \bar{\mathbb{R}}^2$, $u \neq w$.*

It was recently proved in [8] that the Hausdorff dimension of the exceptional set $\mathcal{E}$ is equal to zero.

Reduction theory is applied to coding of geodesics on surfaces of constant negative curvature. In [3] we described how to use the RTC and the attractor of the natural extension map $F_{a,b}$ for representing the geodesic flow on $SM$ as a special flow over a cross-section of "reduced" geodesics - the symbolic system of coding sequences parametrized by the attractor. In special cases, when an $(a,b)$-expansion has a so-called "dual", the coding sequences are obtained by juxtaposition of the boundary expansions of the fixed points, and the set of coding sequences is a countable sofic shift. The construction of the cross-section parametrized by the attractor generalizes for Fuchsian groups, and the geodesic flow is represented as a special flow over a symbolic system of coding sequences on alphabet $A$, the set of generators of $\Gamma$. The notion of "dual" also generalizes.

## References


[1] R. Adler, L. Flatto, *Geodesic flows, interval maps, and symbolic dynamics*, Bull. Amer. Math. Soc. **25** (1991), no. 2, 229–334.

[2] R. Bowen, C. Series, *Markov maps associated with Fuchsian groups*, Inst. Hautes Études Sci. Publ. Math. No. 50 (1979), 153–170.

[3] S. Katok and I. Ugarcovici, *Applications of $(a,b)$-continued fraction transformations*, Ergod. Th. & Dynam. Sys., **32** (2012), 755–777.

[4] S. Katok, I. Ugarcovici, *Structure of attractors for $(a,b)$-continued fraction transformations*, Journal of Modern Dynamics, **4** (2010), 637–691.

[5] S. Katok, I. Ugarcovici, *Fuchsian groups and boundary maps*, in preparation

[6] C. Series, *Symbolic dynamics for geodesic flows*, Acta Math. **146** (1981), 103–128.

[7] I. Smeets, *On continued fraction algorithms*, PhD thesis, Leiden, 2010

[8] C. Carminati, S. Isola and G. Tiozzo, *Continued fractions with $SL(2,\mathbb{Z})$-branches: combinatorics and entropy*, arXiv:1312.6845.

## Wave turbulence: the conjecture, approaches and rigorous results

Sergei B. Kuksin

Consider the following nonlinear PDE on the torus $\mathbb{T}^d_L = \mathbb{R}^d/(L\mathbb{Z}^d)$, $L \geq 1$:

$$\dot{u} + i\Delta u + \varepsilon\rho i\,|u|^2 u = -\nu(-\Delta u + 1)^p u + \sqrt{\nu}\,\langle\text{random force}\rangle\,. \tag{1}$$

Here $u = u(t,x)$ ($t \geq 0,\ x \in \mathbb{T}^d$) is an unknown complex function, the parameters $\varepsilon, \nu, \rho$ satisfy $0 < \varepsilon \leq 1$, $0 \leq \nu \leq 1$, $\rho \geq 1$, and $p$ is an integer, "not too small in terms of $d$" (e.g. if $d \leq 3$, then it suffices to assume that $p \geq 1$). The random force is smooth in $x$ and is a while noise in $t$; it is specified below. We regard (1) as a dynamical system in a suitable function space of complex functions on $\mathbb{T}^d$. This is a random dynamical system if $\nu > 0$.

The wave turbulence (WT) studies solutions of this equation for large $t$ when $\varepsilon, \nu \ll 1$ and $L \gg 1$ ($\rho$ and the random force for a while are assumed to be constant). Classically $\nu = 0$, then (1) becomes the defocusing NLS equation; see [1, 4]. The stochastic model $(1)_{\nu>0}$ was suggested in [3], also see [2] and [4]. Relation between the small parameters $\varepsilon$ and $\nu$ is crucial. If $\nu > 0$ is "much smaller than $\varepsilon$", then the stochastic model becomes similar to the deterministic case, while if $\nu$ is "much bigger than $\varepsilon$", then (1) becomes similar to the non-interesting linear stochastic system $(1)_{\varepsilon=0}$. Natural relation between $\nu$ and $\varepsilon$ is

$$\nu = \varepsilon^2 \tag{2}$$

(cf. the references above and [5, 6]). We assume it everywhere below when talking about eq. (1) with positive $\nu$.

In the classical setting (when $\nu = 0$) the WT is concerned with the behaviour of solutions for (1) with "typical initial data" when

$$t \to \infty\,, \quad \varepsilon \to 0\,, \quad L \to \infty \tag{3}$$

(the relation between the three parameters in unclear and has to be specified). Usually people, working in WT, decompose solutions $u$ to Fourier series

$$u(t,x) = \sum_{k \in \mathbb{Z}^d_L} v_k(t) e^{2\pi i k\cdot x}, \qquad \mathbb{Z}^d_L = L^{-1}\mathbb{Z}^d\,.$$

One of their prime interests is the behaviour under the limit (3) of the averaged actions

$$n_k(t) = \tfrac{1}{2}\big\langle |v_k|^2(t)\big\rangle, \qquad k \in \mathbb{Z}^d_L\,,$$

where $\langle\,\cdot\,\rangle$ signifies a suitable averaging. When $L \to \infty$, the function $n_k(t) = n^L_k(t)$ on the lattice $\mathbb{Z}^d_L$ asymptotically becomes a function $n^0_k(t)$ on $\mathbb{R}^d$. The main conjecture of the WT, made in 1960's (and going back to an earlier work of R. Peierls on the heat conduction in crystals) says that, under a suitable scaling of time $\tau = \varepsilon^{a_1} t$ and of the constant $\rho = L^{a_2}$, where $a_1, a_2 > 0$, the limit $n^0_k(\tau)$, $k \in \mathbb{R}^d$, exists and satisfies the *wave kinetic equation*

$$\frac{d}{d\tau} n_k(\tau) = \text{Const}\int_{\Gamma_k \subset \mathbb{R}^{3d}} n_{k_1} n_{k_2} n_{k_3} n_k \Big(\frac{1}{n_k} + \frac{1}{n_{k_3}} - \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}}\Big) dk_1 dk_2 dk_3\,, \tag{4}$$
$$\Gamma_k = \{(k_1,k_2,k_3) : k_1 + k_2 = k_3 + k,\ |k_1|^2 + |k_2|^2 = |k_3|^2 + |k|^2\}\,.$$

The celebrated Zakharov ansatz (see [1, 4]) applies to this equation and implies that it has autonomous solutions of the form $n_k(\tau) = |k|^\gamma$, $\gamma < 0$, intensively discussed in the physical literature as the *Kolmogorov-Zakharov energy spectra.*

Progress in the rigorous study of the deterministic equation $(1)_{\nu=0}$ was achieved in two very different works [7, 8], but it seems that both approaches do not allow to derive rigorously the kinetic equation (4).

In our talk we report recent progress in deriving a wave kinetic equation for the limiting behaviour of averaged actions for solutions of the stochastic equation (1), (2). Let us pass in this equation to the slow time $\tau = \nu t = \varepsilon^2 t$ and denote $\lambda_k = (|k|^2 + 1)^p$. Then the equation, written in terms of the Fourier coefficients $v_k$, reeds

$$\frac{d}{d\tau} v_k(\tau) + i|k|^2\nu^{-1} v_k = -i\rho \sum_{k_1+k_2=k_3+k} v_{k_1} v_{k_2} \bar{v}_{k_3} - \lambda_k v_k + b_k \frac{d}{d\tau}\beta_k(\tau)\,. \tag{5}$$

Here $\{\beta_k, k \in \mathbb{Z}^d_L\}$ are independent standard complex Wiener processes and the numbers $b_k$ are real non-zero, fast converging to zero when $|k| \to \infty$ (the random force in eq. (5) specifies that in (1)). This equation is well posed and mixing. The latter means that in a suitable function space of complex sequences $\{v_k, k \in \mathbb{Z}^d_L\}$ there exists a unique measure $\mu_{\nu,L}$, called a *stationary measure* for the equation, such that the law $\mathcal{D}(v(\tau))$ of any solution $v(\tau)$ for (5) weakly converges to $\mu_{\nu,L}$ when $\tau \to \infty$; see in [5]. So if $t$ is much bigger than $L$ and $\nu^{-1}$, then the problem of studying the behaviour of solutions for (5) under the limit (3) may be recast as the problem of studying the measure $\mu_{\nu,L}$ when $L \to \infty$ and $\nu \to 0$.

Consider the following $\nu$-independent *effective equation* for (5):

$$\frac{d}{d\tau} a_k = -i\rho \sum_{\substack{k_1+k_2=k_3+k \\ |k_1|^2+|k_2|^2=|k_3|^2+|k|^2}} a_{k_1} a_{k_2} \bar{a}_{k_3} - \lambda_k a_k + b_k \frac{d}{d\tau}\beta_k(\tau)\,. \tag{6}$$

This equation also is well posed and mixing, see [5]. Denote by $m_L$ its unique stationary measure. It is proved in [5] that eq. (6) comprises asymptotical properties of solutions for (5) as $\nu \to 0$. Namely, that

i) $\mu_{\nu,L} \rightharpoonup m_L$ as $\nu \to 0$;

ii) if $v^{\nu,L}(\tau)$ is a solution for (5) with some $\nu$-independent initial data at $\tau = 0$ and $a^L(\tau)$ is a solution for (6) with the same initial data, then

$$\mathcal{D}\Big(\tfrac{1}{2}|v_k^{\nu,L}(\tau)|^2\Big) \rightharpoonup \mathcal{D}\Big(\tfrac{1}{2}|a_k^L(\tau)|^2\Big) \quad \text{as } \nu \to 0 \quad \text{for} \quad 0 \le \tau \le T\,,$$

for each fixed $T > 0$ and every $k \in \mathbb{Z}^d_L$.

Denote $n_k^L(\tau) = \frac{1}{2}\,\mathbb{E}|a_k^L(\tau)|^2$ and choose in (6)

$$\rho = L^{1/2}\rho_0, \qquad b_k = L^{-d/2} b_k^0\,. \tag{7}$$

Using certain heuristic tools from the arsenal of WT, in [6] we proved, on the physical level of rigour, that under the limit $L \to \infty$ the function $\mathbb{Z}^d_L \ni k \mapsto$

$n_k^L(\tau)$ weakly converges to a function $\mathbb{R}^d \ni k \mapsto n_k(\tau)$, which is a solution of the damped/driven wave kinetic equation

$$\frac{d}{d\tau} n_k(\tau) = -2\lambda_k n_k + (b_k^0)^2 + \int_{\Gamma_k} \frac{f_k(k_1, k_2, k_3)}{\lambda_k + \lambda_{k_1} + \lambda_{k_2} + \lambda_{k_3}} n_{k_1} n_{k_2} n_{k_3} n_k \times \Big( \frac{1}{n_k} + \frac{1}{n_{k_3}} - \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} \Big) dk_1 dk_2 dk_3 \,. \tag{8}$$

Here the surface $\Gamma_k \subset \mathbb{R}^{3d}$ is the same as in (4), and the function $f_k$ is constructed in terms of $\Gamma_k$. It is positive, smooth outside the origin, and such that $C^{-1} \leq f \leq C$ for all $k, k_1, k_2, k_3$ and a suitable constant $C$. Moreover, the Zakharov ansatz applies to (8) under a certain natural limit and allows to construct its homogeneous solutions of the Kolmogorov-Zakharov form. Based on this heuristics result and the rigorous assertions i), ii) above, we conjectured in [6] that, under the double limit $\lim_{L\to\infty} \lim_{\nu\to 0}$, the function $\mathbb{Z}_L^d \ni k \mapsto n_k^{\nu,L}(\tau) = \frac{1}{2}\mathbb{E}|v_k^{\nu,L}(\tau)|^2$ weakly converges to a function $n_k^0(\tau)$, $k \in \mathbb{R}^d$, which is a solution of eq. (8). The main goal of this talk is to announce the following result, which is a modification of that conjecture:

**Theorem 1.** *(SK, a work under preparation). Let $\mathbb{R}^d \ni k \mapsto v_k^0$ be a smooth function with compact support, let $v_k^{\nu,L}(\tau)$ be a solution of (5), (7) such that $v_k^{\nu,L}(0) = v_k^0$ for $k \in \mathbb{Z}_L^d$, and let $n_k^{\nu,L} = \frac{1}{2}\mathbb{E}|v_k^{\nu,L}|^2$. Let $\nu \to 0$ and $L \to \infty$ in such a way that $\nu L \to 0$ sufficiently fast. Then there exists $\tau_0 > 0$ such that*

$$\lim_{\nu\to 0,\, L\to\infty} n_k^{\nu,L}(\tau) = n_k^0(\tau) \quad \textit{for } 0 \leq \tau \leq \tau_0 \,,$$

*where $n_k^0(\tau)$, $k \in \mathbb{R}^d$, is a solution for (8), and the limit holds with respect to a suitable weak convergence of functions.*

## References


[1] V. Zakharov, V. L'vov, G. Falkovich, *Kolmogorov Spectra of Turbulence*, Springer 1992.

[2] J. Cardy, G. Falkovich, K. Gawedzki, *Non-equilibrium Statistical Mechanics and Turbulence*, CUP 2008.

[3] V. Zakharov, V. L'vov, *Statistical description of nonlinear wave fields*, Radiophys. Quan. Electronics **18** (1975), 1084-1097.

[4] S. Nazarenko, *Wave Turbulence*, Springer 2011.

[5] S. Kuksin and A. Maiocchi, *Resonant averaging for weakly nonlinear stochastic Schrödinger equations*, preprint (2014).

[6] S. Kuksin, A. Maiocchi, *Derivation of a wave kinetic equation from the resonant-averaged stochastic NLS equation*, Physica D **309** (2015), 65-70.

[7] J. Lukkarinen, H. Spohn, *Weakly nonlinear Schrödinger equation with random initial data*, Invent. Math. **183** (2011), 79-188.

[8] E. Faou, P. Germain, Z. Hani, *The weakly nonlinear large box limit of the 2D cubic nonlinear Schrödinger equation*, preprint (2013).

## On the almost reducibility of circle diffeomorphisms

Raphaël Krikorian

Herman-Yoccoz theorem asserts that a smooth orientation preserving diffeomorphism of the circle with a diophantine rotation number $\alpha$ is smoothly conjugated to $R_\alpha$, the rigid rotation with angle $\alpha$; when $\alpha$ is Liouville this is no longer true. On the other hand Yoccoz [3] proved that whenever $\alpha$ is irrational the closure in the smooth topology of the set $O_\alpha$ of diffeomorphisms of the circle smoothly conjugated to $R_\alpha$ is equal to the set $F_\alpha$ of smooth orientation preserving diffeomorphisms of the circle with rotation number $\alpha$. In other words, one can always approximate in the smooth topology a diffeomorphism with rotation number $\alpha$ by a conjugate of the rigid rotation $R_\alpha$. The almost reducibility problem is in some sense a dual notion: a smooth orientation preserving diffeomorphism of the circle $f$ with an irrational rotation number $\alpha$ is said to be smoothly *almost reducible* if one can find a sequence of smooth conjugations $h_n$ of the circle such that $h_n^{-1} \circ f \circ h_n$ converges in the smooth topology to $R_\alpha$. Mostapha Benhenda [1] proved that for a class of (but not all) Liouville numbers almost reducibility holds. The aim of the talk is to prove the following result which doesn't require assumptions on the rotation number: assume $f$ is in $F_\alpha$ ($\alpha$ irrational) and that it is *Hölder conjugated* to the rotation $R_\alpha$; then $f$ is smoothly almost reducible. Notice that Matsumoto [2] proved that, whenever $\alpha$ is Liouville, in $F_\alpha$ the set of diffeomorphisms for which the invariant measure has zero dimensional Hausdorff measure is a generic set in the smooth topology; as a consequence, the assumption needed in our theorem only holds for a meager set in $F_\alpha$. We also address the almost reducibility problem for analytic diffeomorphisms of the circle. We prove that if $f$ is an analytic diffeomorphism in $F_\alpha$ such that the real axis is not accumulated by periodic orbits then it is (smoothly) almost reducible. In the analytic case our approach is based on the fundamental work of Yoccoz on the renormalization of analytic diffeomorphisms of the circle [4] and on approximation of diffeomorphisms close to the identity by vector fields. The smooth situation can be dealt with by using in addition asymptotically holomorphic extensions of smooth diffeomorphisms.

### References


[1] M. Benhenda, Mostapha, *Estimates of the linearization of circle diffeomorphisms*, Bull. Soc. Math. France **142** (2014), no. 4, 673–718.

[2] S. Matsumoto, *A generic-dimensional property of the invariant measures for circle diffeomorphisms*, J. Mod. Dyn. **7** (2013), no. 4, 553–563.

[3] J.-C. Yoccoz, *Centralisateurs et conjugaison différentiable des difféomorphismes du cercle, Petits diviseurs en dimension 1.* Astrisque no. 231 (1995), 89–242.

[4] J.-C. Yoccoz, *Analytic linearization of circle diffeomorphisms*, Lect. Notes in Math. **1784**, 125–173, Springer

## Determining the linear stability of periodic orbits of the N-body problem via index theory

Yiming Long

Periodic solution orbits of Hamiltonian systems is a traditional and active topic in Hamiltonian dynamics. A useful tool to study the linear stability of these orbits is the new analytical method introduced by X. Hu, S. Sun and the author in [3] of 2014 via the index iteration theory for symplectic paths. In the talk at MFO, I gave a brief introduction on this new method based on the index iteration theory and a survey on the linear stability of certain periodic solutions of N-body problems, as well as other related problems including some recent new results using this new method and index iteration theory.

In 1772, Lagrange ([9]) discovered some celebrated homographic periodic solutions, now named after him, to the planar three-body problem, namely the three bodies form an equilateral triangle at any instant of the motion and at the same time each body travels along a specific Keplerian elliptic orbit with the same eccentricity about the center of masses of the system. It is now well known that the linear stability of Lagrangian elliptic equilateral triangle homographic solutions in the classical planar three-body problem depends on the mass parameter $\beta = 27(m_1m_2 + m_2m_3 + m_3m_1)/(m_1 + m_2 + m_3)^2 \in [0, 9]$ and the eccentricity $e \in [0, 1)$ (cf. [14]).

The stability problem of such solutions has been deeply studied since one hundred years ago. We refer readers to Gascheau ([2], 1843) and Routh ([17], 1875) for the case $e = 0$, and Danby ([1], 1964), Roberts ([16], 2002) for the case of sufficiently small $e \geq 0$ by perturbation techniques. In 2005, Meyer and Schmidt (cf. [14]) used heavily the central configuration nature of the elliptic Lagrangian orbits and decomposed the fundamental solution of the elliptic Lagrangian orbit into two parts symplectically, one of which is the same as that of the Keplerian solution and the other is the essential part for the stability. They also did the stability analysis by normal form theory for small enough $e \geq 0$. In 2004-2006, Martínez, Samà and Simó ([11],[12], [13]) studied the stability problem when $e > 0$ is small enough by using normal form theory, and $e < 1$ and close to 1 enough by using blow-up technique in general homogeneous potential. They further gave a much more complete bifurcation diagram numerically and a beautiful figure was drawn there for the full $(\beta, e)$ rectangle $[0, 9] \times [0, 1)$. But we are not aware of any existing analytical methods which work for this linear stability problem on the full $(\beta, e)$ range $[0, 9] \times [0, 1)$ before 2010 aside from perturbation methods for $e > 0$ small enough, blow-up techniques for $e$ sufficiently close to 1, and numerical studies.

In [8] of 2010, Hu and Sun studied the linear stability of Lagrangian elliptic solutions $z_{\beta,e}$ by Maslov-type index theory, proved $2 \leq i_1(Z^2_{\beta,e}) \leq 4$ by assuming $z_{\beta,e}$ being non-degenerate, and then classified the linear stability when $i_1(z^2_{\beta,e}) = 4, 3, 2$.

In order to completely understand this problem globally and analytically, in the joint paper [3] of Hu, Long and Sun published in Arch. Rat. Mech. Anal.

of 2014, a new rigorous analytical method was established in order to study the linear stability of these solutions in terms of the two parameters in the full $(\beta, e)$ range $[0, 9] \times [0, 1)$ via the $\omega$-index theory of symplectic paths for $\omega$ belonging to the unit circle of the complex plane, and the theory of linear operators.

After establishing the $\omega$-index decreasing property of the solutions in $\beta$ for fixed $e \in [0, 1)$, for the Lagrangian equilateral triangle homographic solutions, we proved the non-degeneracy of $z_{\beta,e}$ for every $(\beta, e) \in (0, 9] \times [0, 1)$, and the existence of three curves located from left to right in the rectangle $[0, 9] \times [0, 1)$, among which two are $-1$ degeneracy curves and the third one is the right envelop curve of the $\omega$-degeneracy curves, and show that the linear stability pattern of such elliptic Lagrangian solutions changes if and only if the parameter $(\beta, e)$ passes through each of these three curves. This answers the linear stability problem of the Lagrangian solutions completely.

Then this lecture I gave also a brief survey on recent results after Hu-Long-Sun's work [3] in other related linear stability problems of the N-body problems including the Euler collinear solutions. We refer readers to the papers [4], [5], [6], [15], [18], [19] for further studies. We believe that this new method can be applied to get more understanding of the linear stability on periodic solutions in the N-body problems.

This talk is based on recent joint works with Xijun HU, Shanzhong SUN, and Qinglong ZHOU, and joint works of Xijun HU, Yuwei OU, Shanzhong Sun and Penghui WANG respectively.

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## References

[1] J. M. A. Danby, The stability of the triangular Lagrangian point in the general problem of three bodies. *Astron. J.* 69. (1964) 294–296.

[2] M. Gascheau, Examen d'une classe d'équations différentielles et application à un cas particulier du problème des trois corps. *Comptes Rend. Acad. Sciences.* 16. (1843) 393–394.

[3] X. Hu, Y. Long and S. Sun, Linear stability of elliptic Lagrangian solutions of the planar three-body problem via index theory. *Rat. Mech. Anal.* 213 (2014) 993-1045.

[4] X. Hu and Y. Ou, An estimation for the hyperbolic region of elliptic Lagrangian solutions in the planar three-body problem. *Regular and Chaotic Dynamics* 18 (6) (2013) 732–741.

[5] X. Hu and Y. Ou, Collision index and stability of elliptic relative equilibrium in planar n-body problem. Preprint (2015).

[6] X. Hu, Y. Ou and P. Wang, Trace formula for linear Hamiltonian systems with its applications to elliptic Lagrangian solutions. *Arch. Ration. Mech. Anal.* 216 (1)(2015) 313–357.

[7] X. Hu, S. Sun, Index and stability of symmetric periodic orbits in Hamiltonian systems with its application to figure-eight orbit. *Commun. Math. Phys.* 290. (2009) 737–777.

[8] X. Hu, S. Sun, Morse index and stability of elliptic Lagrangian solutions in the planar three-body problem. *Advances in Math.* 223. (2010) 98-119.

[9] J. L. Lagrange, Essai sur le problème des trois corps. Chapitre II. Œuvres Tome 6, Gauthier-Villars, Paris. (1772) 272–292.

[10] Y. Long, Index Theory for Symplectic Paths with Applications. Progress in Math. 207, Birkhäuser. Basel. 2002
[11] R. Martínez, A. Samà, C. Simó, Stability of homographic solutions of the planar three-body problem with homogeneous potentials. in International conference on Differential equations. Hasselt, 2003, eds Dumortier, Broer, Mawhin, Vanderbauwhede and Verduyn Lunel, World Scientific, (2004) 1005–1010.
[12] R. Martínez, A. Samà, C. Simó, Stability diagram for 4D linear periodic systems with applications to homographic solutions. *J. Diff. Equa.* 226. (2006) 619-651.
[13] R. Martínez, A. Samà, C. Simó, Analysis of the stability of a family of singular-limit linear periodic systems in $R^4$. Applications. *J. Diff. Equa.* 226. (2006) 652-686.
[14] K. R. Meyer, D. S. Schmidt, Elliptic relative equilibria in the N-body problem. *J. Diff. Equa.* 214. (2005) 256–298.
[15] Y. Ou, Hyperbolicity of elliptic Lagrangian orbits in planer three body problem. *Sci. China Math.* 57 (7) (2014) 1539–1544.
[16] G. E. Roberts, Linear stability of the elliptic Lagrangian triangle solutions in the three-body problem. *J. Diff. Equa.* 182. (2002) 191–218.
[17] E. J. Routh, On Laplace's three particles with a supplement on the stability or their motion. *Proc. London Math. Soc.* 6. (1875) 86-97.
[18] Q. Zhou and Y. Long, Equivalence of linear stabilities of elliptic triangle solutions of the planar charged and classical three-body problems. *J. Diff. Equa.* 258 (2015), 3851–3879.
[19] Q. Zhou and Y. Long, Maslov-type indices and linear stability of elliptic Euler solutions of the three-body problem. Preprint (submitted). (2015).

## Train Correspondences for Celestial Bodies

Andreas Knauf
(joint work with Jacques Féjoz, Richard Montgomery)

**The main result:** The configuration space of $n \in \mathbb{N}$ particles in $d \in \mathbb{N}$ dimensions is

$$\widehat{M} := \mathbb{R}^{dn}_q \setminus \Delta, \text{ with } \Delta := \{q = (q_1, \ldots, q_n) \in \mathbb{R}^{dn}_q \mid q_i = q_j \text{ for some } i \neq j\}.$$

We consider the Hamiltonian function

$$\widehat{H} : T^*\widehat{M} \to \mathbb{R} \quad , \quad \widehat{H}(p,q) := K(p) + V(q), \tag{1}$$

with kinetic energy $K(p) := \sum_{i=1}^{n} \frac{\|p_i\|^2}{2m_i}$ with the masses $m_i > 0$, and potential $V(q) := \sum_{1 \leq i < j \leq n} \frac{I_{i,j}}{\|q_i - q_j\|}$. For celestial mechanics $I_{i,j} := -m_i m_j$, whereas in the electrostatic case $I_{i,j} = Z_i Z_j$ for the charges $Z_i \in \mathbb{R} \setminus \{0\}$.

More specifically we are interested in *scattering solutions* $q_{x_0} : \mathbb{R} \to \widehat{M}$ of the Hamiltonian equations with initial data $x_0$, for which the *asymptotic velocities*

$$\overline{v}^{\pm}(x_0) := \lim_{t \to \pm\infty} \frac{q(t, x_0)}{t} \ \in \ \mathbb{R}^{dn}_v$$

exist, and for which $\overline{v}^{\pm}_i(x_0) \neq \overline{v}^{\pm}_j(x_0)$ for $1 \leq i < j \leq n$.

For comparison, we use a kinematic model of $n$ particles on $\mathbb{R}^d$ that move with constant velocity unless some of them collide. Then the total momentum and the total kinetic energy of the colliding particles are preserved during collision. So it is a billiard with pointlike particles. But even allowing only binary collisions, the kinematic model defines a dynamical system only for $d = 1$ (that is, particles on the line with backscattering). However, it basically follows from [BFK] that the number of collisions is finite, with a bound depending only on $d$, $n$ and the mass ratios.

We show that, given a transversal solution of the kinematic model, there is a one-parameter family of solutions of the $n$-body problem with parameter $\varepsilon > 0$ and interactions $I_{i,j}$ multiplied by $\varepsilon$ that converges to that solution as $\varepsilon \searrow 0$. By homogeneity of the kinetic energy $K$ and of the potential $V$ this implies that for any given total energy $h > 0$ there exists a one-parameter family of solutions for (1) that arises by appropriate rescaling of this family.

We now describe some ideas involved in the proof and further results.

**Transversality:** We only consider kinematic solutions for which

(1) all collisions are binary
(2) the initial velocities of all particles are different (same for final velocities)
(3) forward scattering (that is, collision without scattering) does not occur.
(4) No particle is allowed to go through without any deflection.

**Kinematic solutions:** The initial values for $n$ non-colliding free particles on $\mathbb{R}^d$ form an open dense subset of $T^*\mathbb{R}^{nd}$. Likewise, if we combinatorically prescribe the succession of binary collisions (a *train connection*), then the set of transversal kinematic solutions for this train connection is always (if non-void) a $2nd$-dimensional real-analytic manifold.

**Collisions:** Whereas the kinematic model involves binary collisions, for the $n$–body problem collisions only can occur in case of backscattering. However, even if we would exclude kinematic solutions with backscattering, we would need a way to control *near*-collisions.

Many schemes to regularize the Kepler problem ($n = 2$) have been devised. On the other hand, it is known since [MG] that collisions of three or more bodies are not regularizable. We show by generalizing a result in [Kn02] that for all $d$ and $n$ single binary collisions can be regularized by symplectic phase space extension of $\left(T^*\widehat{M}, dq \wedge dp, \widehat{H}\right)$ in a real-analytic way, without change of the time parameter or introducing rest points.

Simultaneous binary collisions are known to lead to a loss of smoothness [MS2], but to be continuously regularizable [EB1, EB2]. We show them to be $C^1$, which is important for applying variational techniques.

**Topology:** In [Kn99, KM08] a topological degree was introduced in potential scattering and shown to equal 1 respectively -1 for the Keplerian or Coulomb case. Here we implicitly use the non-triviality of that degree to show the existence of the one-parameter family of solutions approximating the kinematic solution.

**Scattering and convergence:** Classical scattering theory developed later than quantum scattering theory [DG], and many basic questions are still not answered. We develop some of this scattering theory for the case of long ranged pair potentials. In particular, we found an explicit phase space region so that for initial conditions $x_0$ in this region all asymptotic velocities $\overline{v}_i^+(x_0)$ are different, and is entered by every such orbit. In this case we show for short ranged pair potentials smoothness of the so-called Møller transformations. These compare the true motion with free motion.

Keplerian potentials are not short-ranged, and already Kepler hyperbolae are not asymptotic in time to *any* parameterized straight line. So for our $V$ Møller transformations w.r.t. free motion do not converge. We use a time dependent *Dollard* Hamiltonian, a tool borrowed from quantum mechanics, to obtain smooth control of the scattering solutions at spatial infinity.

Convergence to the kinematic solutions is locally uniform in time. It is not globally uniform in time because $V$ is not short-ranged.

The talk refers to an article that is not yet published.

## References


[BFK] D. Burago, S. Ferleger, S. Kononenko, *Uniform estimates on the number of collisions in semi-dispersing billiards*, Annals of Mathematics **147** (1998), 695–708

[DG] J. Dereziński, C. Gérard, *Scattering Theory of Classical and Quantum N-Particle Systems*, Texts and Monographs in Physics. Berlin: Springer 1997

[EB1] M. ElBialy, *The Flow of the N–Body Problem near a Simultaneous-Binary-Collision Singularity and Integrals of Motion on the Collision Manifold*, Archive for Rational Mechanics and Analysis **134** (1996), 303–340

[EB2] M. ElBialy, *Collective branch regularization of simultaneous binary collisions in the 3D N–body problem*, J. Math. Physics **50** (2009), 052702

[Kn99] A. Knauf, *Qualitative Aspects of Classical Potential Scattering*, Regular and Chaotic Dynamics, **4** (1999), 1–20

[Kn02] A. Knauf, *The n-Centre Problem of Celestial Mechanics for Large Energies*, Journal of the European Mathematical Society **4** (2002), 1–114

[KM08] A. Knauf, M. Krapf, *The non-trapping degree of scattering*, Nonlinearity **21** (2008), 2023–2041

[MS2] R. Martínez, C. Simó, *The degree of differentiability of the regularization of simultaneous binary collisions in some N–body problems*, Nonlinearity **13** (2000), 2107–2130

[MG] R. McGehee, *Triple collision in the collinear three-body problem*, Invent. Math. **27** (1974), 191–227


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