Mathematisches Forschungsinstitut Oberwolfach

Report No. 44/2015

DOI: 10.4171/OWR/2015/44

Arbeitsgemeinschaft: Mathematical Quasicrystals

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4 October – 9 October 2015

Abstract. This introductory workshop encouraged participants to read important recent works in the topology, geometry and dynamics of highly regular (but aperiodic) discrete sets in Euclidean spaces, and their corresponding tiling spaces. These sets have been recently under intensive investigation by researchers in topology, mathematical physics, dynamics, diophantine approximation, and discrete mathematics, and various different perspectives were emphasized.

Mathematics Subject Classification (2010): 52C23.

Introduction by the Organisers

The mathematical study of aperiodic tilings and aperiodic discrete sets in $\mathbb{R}^d$ began with the discovery in the 1960s by Wang and Berger, followed by Robinson and Penrose in the 1970s, of a finite set of tiles which tile $\mathbb{R}^2$ aperiodically. These studies later received physical motivation with the discovery in 1982 by Schechtman et al. of materials which do not have crystalline structures. These structures were called quasicrystals by Levine and Steinhardt.

In recent years much intensive work has been devoted to investigating the large scale geometry of discrete subsets of $\mathbb{R}^d$. The sets which are studied are typically not periodic, but share some of the properties of periodic sets, corresponding to various weakenings of the notion of periodicity. As a consequence they have come to be studied under the loosely defined term quasicrystals. The study of such sets has a long history in different mathematical disciplines, such as dynamics (in connection with cross-sections for continuous group actions, virtual subgroups), mathematical physics (quasicrystals and almost periodic structures, questions of diffraction), operator algebras and K-theory (cohomology theories for pattern spaces, solenoids and laminated spaces), geometric group theory (quasi-isometries and coarse isometries, almost subgroups), and geometric combinatorics (packing and covering questions). Although researchers from different disciplines are interested in different questions, there are many connections between work being done by different groups of people. The goal of this meeting is to provide an introduction to some of the main examples and questions surrounding mathematical quasicrystals, making it possible to bridge the cultural gaps between people studying the same objects from different points of view.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Arbeitsgemeinschaft: Mathematical Quasicrystals

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Abstracts

Examples of tilings and cut and project sets

Uwe Grimm

In my talk, which was mainly based on material from the first half of the monograph [1], I introduced a number of notions and properties of point sets and tilings in Euclidean space. In particular, local derivability (LD) of one such pattern from another was introduced, and the equivalence relation mutual local derivability (MLD) was explained. This equivalence allows us to switch between point set and tiling descriptions (or more general descriptions) of the same underlying structure, which is frequently used without explicit reference.

The notions for point sets introduced in my talk are largely based on those used in an influential article by Lagarias [2]; see also references therein. Local finiteness and finite local complexity (FLC) were introduced, and Delone sets as well as Meyer sets were defined. The action of translations is exploited to introduce the notion of a (continuous) hull for FLC tilings or point sets, defined as the closure of the translation orbit in the local topology (where FLC tilings or point sets are close if they agree on a large ball around the origin, up to a small translation). Symmetries were defined using the equivalence relation of local indistinguishability (LI). In particular, this concerns local inflation deflation symmetry (LIDS), which was explained for the example of the silver mean point set. This point set was then shown to lift naturally into a rectangular planar lattice, using algebraic conjugation, which gives rise to a description of the silver mean point set as a cut and project set (or model set) with a simple interval as the window. The freedom of choosing a scale in internal space was briefly discussed, linking this approach with the more commonly used projection from a square lattice with a canonical choice of the window.

Finally, cut and project schemes for planar tilings with n-fold rotational symmetry were discussed. These cyclotomic model sets in the plane are considered as subsets of the ring of integers in a cyclotomic field, which naturally lift via a diagonal (Minkowski) embedding to higher-dimensional lattices, using the non-trivial automorphisms of the underlying cyclotomic field. The well-known Ammann-Beenker and Penrose tilings are examples of cyclotomic model sets with eightfold and tenfold rotational symmetry, respectively. The model set construction, which originally goes back to Meyer's work in harmonic analysis [3], generalises to arbitrary dimensions and more general choices of internal spaces; see Moody's review article [4] for details and further developments. An important property of regular model sets (which are model sets with windows whose boundaries have zero Haar measure) is the uniform distribution of projected lattice points into the window in internal space. For example, the property makes it possible to express frequencies of local patches and the diffraction intensities of the pure point diffraction measure in terms of integrals over the window.

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Dynamics and ergodic theory, I

Michael Bjoerklund

This lecture will introduce (without proof) standard concepts from ergodic theory and dynamics, including the following: Ergodicity and ergodic theorems for actions of $\mathbb{R}$ and $\mathbb{R}^d$. Possible choices and properties of averaging sets, Følner sequences and van Hove sequences, with examples. Minimality and unique ergodicity. Existence of minimal sets in actions on compact spaces, and existence of invariant measures for actions of amenable groups. For amenable groups acting on compact spaces, unique ergodicity (with a measure of full support) implies minimality. Syndetic sets, sets of visit times for minimal actions.

Dynamics and ergodic theory, II

Maria Rita Iacò

This talk is meant as a discussion on the conjugacy problem in ergodic theory, that is to decide whether given two measure-preserving transformations are conjugate. Measure-preserving transformations are morphisms between measure spaces and conjugacy is an equivalence relation on the set of all measure-preserving transformations. More precisely, two measure-preserving transformations T_1 and T_2, defined on the probability spaces $(X_1, \mathcal{B}_1, m_1)$ and $(X_2, \mathcal{B}_2, m_2)$, respectively, are said to be conjugate if there exists a measure-algebra isomorphism $\phi\colon (\tilde{\mathcal{B}_2}, \tilde{m_2}) \to (\tilde{\mathcal{B}_1}, \tilde{m_1})$ such that $\phi \circ \tilde{T_2}^{-1} = \tilde{T_1}^{-1} \circ \phi$. The definitions of isomorphism and spectral isomorphism for measure-preserving transformations have been given as well and it has been shown that spectral isomorphism is weaker than conjugacy, which is weaker than isomorphism. This was not done just to draw up a list of properties, but to highlight the fact that one method for tackling the conjugacy problem is by looking for isomorphism invariants. In particular, it turns out that the set of eigenvalues of a measure-preserving transformation is an example of set of invariants. Therefore, we introduced the Koopman operator U_T. It is a unitary operator on $L^2(X, \mathcal{B}, m)$, associated to an invertible measure-preserving transformation T on X defined by $(U_T f)(x) = f(Tx)$. The eigenvalues and the corresponding eigenfunctions of U_T are called the eigenvalues and eigenfunctions of T. In this way, we can define the discrete spectrum property of an ergodic measure-preserving T, meaning that the eigenfunctions of U_T induced by T span $L^2(X, \mathcal{B}, m)$. Moreover, the set of eigenvalues forms a subgroup of the unit circle K and it provides a complete invariant set for ergodic measure-preserving transformations with discrete spectrum. This simply means that two ergodic measure-preserving transformations having the same group of eigenvalues are isomorphic, and thus conjugate. Finally, we discussed a canonical example of ergodic measure-preserving transformations with discrete spectrum, namely ergodic rotations. Motivated by this example, we stated the following Representation Theorem and Existence Theorem for ergodic measure-preserving transformations with discrete spectrum.

Theorem 1 (Representation Theorem)**.** *An ergodic measure-preserving transformation T with discrete spectrum on a probability space $(X, \mathcal{B}, m)$ is conjugate to an ergodic rotation on some compact abelian group.*

Theorem 2 (Representation Theorem)**.** *Every subgroup Λ of the unit circle K is the group of eigenvalues of an ergodic measure-preserving transformation T with discrete spectrum.*

Thus, we have a satisfactory solution to the conjugacy problem for this class of transformations. In fact, each conjugacy class of ergodic measure-preserving transformations with discrete spectrum is characterised by a subgroup of K, and each subgroup of K corresponds to a conjugacy class. We refer to [2, 1] as general references.

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Pattern spaces

Franz Gähler

This talk gives an overview on spaces of aperiodically ordered patterns and tilings, as well as on their associated dynamical systems. Background material can be found in the monographs [1, 2]. After defining what kind of patterns we want to consider (mainly locally finite Delone sets and tilings), some important properties and equivalence concepts are introduced. A pattern has *finite local complexity* (FLC), if up to translations it contains only finitely many different bounded patches. A pattern is *repetitive*, if for any bounded patch in the pattern, the set of its translates is relatively dense. Finally, two patterns are called *locally indistinguishable* (LI), if any bounded patch from one occurs also in the other, and vice versa. With this equivalence relation, patterns can be divided into *LI classes*, consisting of patterns with exactly the same bounded configurations.

It is then natural to turn LI classes and other sets of patterns into *pattern spaces*. For a given pattern, we define its *hull* as the closure (in a suitable topology) of the set of all its translates. In the *local topology*, two patterns are ε-close, if they agree exactly in a ball of radius $1/\varepsilon$, up to an ε-small rigid translation. For FLC patterns, this is the most convenient topology. It makes the hull a compact space, on which the translations act continuously. The hull, together with the translation action, thus becomes a topological dynamical system. This dynamical system is minimal if and only if the starting pattern (and thus any pattern in the hull) is repetitive. In this case, the hull consists exactly of the LI class of any of its member tilings. For non-FLC patterns, the hull is not compact in the local topology. In that case, a different topology can be used, which allows locally varying ε-small translations or motions in order make the two patterns match.

If a pattern P can be derived from another pattern P' by some local rule, this derivation rule can be extended to a map between the two LI classes. As it intertwines the respective translation actions, we obtain in fact a factor map between the two dynamical systems. If such local derivability (LD) holds in both directions, the two LI classes are called *mutually locally derivable* (MLD), and the two dynamical systems are topologically conjugate (though not every topological conjugacy is of this type). Factor maps arising from LD give important information. This is illustrated with a factor map from hull of the Tübingen Triangle Tiling to that of the Penrose tiling, which is a 5–1 covering map, and by a factor map from the hull of octagonal Ammann-Beenker tiling with matching rule decoration to that of the undecorated Ammann-Beenker tiling, which 1–1 almost everywhere, but not everywhere. The set where it fails to be 1–1 determines the difference in topology between the two tiling spaces.

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Introduction to Čech cohomology

Antoine Julien

This talk is intended to provide a beginners introduction to simplicial and Čech cohomology. First, we introduce the notions of simplicial complex, Δ-complex and cellular complex. In all these cases, one wants to express a d-dimensional topological space as a union of basic oriented cells (simplices or balls) of dimension $0, \ldots, d$ such that each k-dimensional cell has a boundary which consists in $(k-1)$-dimensional cells. The difference between simplicial, Δ- and cellular complexes lies in what is considered a basic cell, and how can k-cells be glued on lower-dimensional cells, but the underlying ideas are similar. A 1-dimensional cellular complex, for example, is an oriented graph.

Once a space is given a structure of (say) simplicial complex, one can see it as a combinatorial object. It is then possible to define the space k-cochains for all k: a k-cochain is a function which attributes a value (in $\mathbb{Z}$) to each k-cell. There is a natural notion of *coboundary* of a cochain: if ϕ is a k-cochain, $\delta\phi$ is a $(k+1)$-cochain. It is dual to the notion of boundary of a cell. A straightforward computation shows that $\delta \circ \delta = 0$.

A basic question which motivates cohomology is "given a k-cochain ϕ, does the equation

$$\phi = \delta\alpha$$

have a solution"? Since $\delta \circ \delta = 0$, it is clear that if $\delta\phi \neq 0$, the equation above can't have a solution (this is a *local obstruction*). However, the condition $\delta\phi = 0$ is not always sufficient: there could be a *global obstructions*. Cohomology measures how many such global obstructions there can be:

$$H^k(X, \mathbb{Z}) = \big\{\phi \ k\text{-cochain} \ : \ \delta\phi = 0\big\} \Big/ \big\{\delta\alpha \ : \ \alpha \text{ a } (k-1)\text{-cochain}\big\}.$$

In particular, if the cohomology groups are 0, the condition $\delta\phi = 0$ is both necessary and sufficient for the equation $\phi = \delta\alpha$ to have a solution α.

It is a good exercise to compute the cohomology of some classical spaces: for example the 2-sphere can be seen as a complex made of three vertices, three edges (glued to form a triangle), and two 2-simplices glued on this triangle (each forming one hemisphere). Each of the cochain groups is free Abelian, and the coboundary maps can be expressed as matrices in a chosen basis. The computation of cohomology becomes an exercise of linear algebra.

Cohomology satisfies several important properties. First, it only depends on the space and not on the simplicial or cellular decomposition. Then, a continuous map $X \to Y$ between topological spaces induces a map $H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$ between the cohomology groups. As a consequence, two homeomorphic spaces have isomorphic cohomology groups: cohomology is an invariant.

To conclude, the use of Čech cohomology groups (noted $\check{H}^k(X, \mathbb{Z})$) can be motivated by two of their properties. First, they are isomorphic to the simplicial cohomology groups for "nice" spaces (in particular for finite simplicial or cellular complexes). Then, the Čech cohomology groups of an inverse limit are the direct limit of the Čech cohomology groups. This property is quite relevant to tilings, since tiling spaces are inverse limits of finite cellular complexes.

As an illustration, we compute that if X is the dyadic solenoid (*i.e.* the stationary inverse limit of circles under the map $z \mapsto z^2$), its first Čech cohomology group is $\check{H}^1(X, \mathbb{Z}) = \mathbb{Z}[1/2]$. This is done by computing the simplicial cohomology of a circle and the map induced in cohomology by $z \mapsto z^2$, and by using the two properties of Čech cohomology above.

The general goal of this talk is to provide enough background to follow cohomology computations, rather than give a full and formal presentation of the topic. Recommended references for this material are [1, Chapters 2 and 3] and [2, Chapter 3].

References

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Gähler and Anderson-Putnam Complexes

Scott Schmieding

Associated to the hull of a pattern are certain cohomology groups, which have proven very effective as both topological invariants and as messengers of certain information about the pattern. While many versions of these cohomology groups exist, a fundamental one is the Cech cohomology of the topological hull of the pattern. An effective method to approach Cech cohomology in general is through inverse limits, and we first briefly review the concept of inverse limit spaces and some basic properties. We describe the connection between inverse limit spaces and Cech cohomology, with the standard dyadic solenoid serving as a guiding example. In general, the method of writing a space as an inverse limit of CW complexes is particularly effective for calculation the Cech cohomology, and with this in mind we then give two presentations of the hull associated to a tiling of R^d as an inverse limit of certain CW complexes built from the tiling. The first, called the Gaehler complex, works for very general tilings, but is difficult to compute with. The second, the Anderson-Putnam complex, works only for tilings which arise from a substitution system, i.e. for tilings which come from an inflation and subdivision rule. The Anderson-Putnam construction however is much more amenable to calculation, and we give two examples, the period-doubling substitution in dimension one, and the half-hex substitution in dimension two, to indicate how the construction can be used to obtain important topological information about the associated hulls.

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Pattern equivariant cohomology

James Walton

This talk served as an introduction to pattern equivariant cohomology, a tool designed to provide intuitive, geometric descriptions of the Čech cohomology of a tiling space. Let T be a tiling of $\mathbb{R}^d$ with finite local complexity. The tiling space Ω_T of T (defined in the talk *Pattern spaces*) is a naturally defined moduli space of tilings which are locally indistinguishable from T. For T aperiodic this space is not easily visualised, not easily described in any combinatorial fashion (such as how, say, a regular cell complex is) and many classical topological invariants, such as the homotopy groups or singular (co)homology groups, fail to provide any useful information about Ω_T. However, topological invariants such as K-theory or the Čech cohomology $\check{H}^\bullet(\Omega_T)$ *do* provide useful information about Ω_T.

Although the above considerations naturally lead us to invariants such as $\check{H}^\bullet(\Omega_T)$, it is not immediately clear how abstract topological invariants of an abstract topological space relate back to our original tiling T! It would be preferable to define cohomology directly in terms of T, in a way which yields isomorphic cohomology to $\check{H}^\bullet(\Omega_T)$ so as to retain the powerful abstract foundation of these groups whilst simultaneously providing a more intuitive viewpoint on them. This is precisely what pattern equivariant (PE) cohomology achieves.

The two main approaches to PE cohomology are through PE forms (see [2]) and PE cellular cochains (see [5]); both were covered in the talk. For the former, one begins with the de Rham cochain complex $C^\bullet_{\mathrm{dR}}(\mathbb{R}^d)$ of cochain groups $C^k_{\mathrm{dR}}(\mathbb{R}^d)$ given by the vector spaces of smooth k-forms of $\mathbb{R}^d$ and coboundary given by the exterior derivative. Loosely, a form is called *pattern equivariant* (with respect to T) if it is the same locally at any two points $x, y \in \mathbb{R}^d$ for which $T - x$ and $T - y$ agree to a sufficiently large radius about the origin. Restricting to PE forms provides a sub-cochain complex $C^\bullet_{\mathrm{dR}}(T)$ of $C^\bullet_{\mathrm{dR}}(\mathbb{R}^d)$ whose cohomology $H^\bullet_{\mathrm{dR}}(T)$ we call the *pattern equivariant cohomology of T*. Kellendonk and Putnam [3, 2] proved the following:

Theorem. *$\check{H}^\bullet(\Omega_T;\mathbb{R}) \cong H^\bullet_{\mathrm{dR}}(T)$.*

If T is a cellular tiling, then we may repeat the above construction using cellular cochains instead of forms, defining the PE cohomology $H^\bullet_{\mathrm{cell}}(T;G)$, where G may be taken to be any discrete Abelian group. Sadun [5] proved the following:

Theorem. *$\check{H}^\bullet(\Omega_T;G) \cong H^\bullet_{\mathrm{cell}}(T;G)$.*

The above theorems provide us with a powerful way of visualising elements of the Čech cohomology of a tiling space. Moreover, PE cohomology is a vital tool in describing further structures on the cohomology, notably trace maps, order structure and Ruelle-Sullivan currents, as well as asymptotically negligible cochains, tools with applications to gap-labelling (see talk *Gap labelling theorem*, and references therein), shape deformations [1] and bounded discrepancy [4].

The proof of the above was discussed in the talk. It turns out to be remarkably simple, hinging on the following two fundamental properties of Čech cohomology (given in the talk *Introduction to Čech cohomology*), along with the description of Ω_T as a certain inverse limit of CW complexes (discussed in the talk *Gähler and Anderson–Putnam complexes*):

- (Č1) The Čech cohomology functor is naturally isomorphic to the singular cohomology functor on the category of topological spaces homotopy equivalent to CW complexes, and thus to cellular cohomology on spaces endowed with a CW decomposition.
- (Č2) For an inverse system $\Gamma_0 \xleftarrow{f_1} \Gamma_1 \xleftarrow{f_2} \cdots$ of compact, Hausdorff spaces, there exists an isomorphism $\check{H}^\bullet(\varprojlim(\Gamma_i, f_i)) \cong \varinjlim(\check{H}^\bullet(\Gamma_i), f_i^*)$.

(Gä) $\Omega_T \cong \varprojlim(\Gamma_i, f_i)$, where the Γ_i are the Gähler complexes of T and the f_i are the corresponding forgetful maps.

Some examples were discussed, deriving the generators of $C^\bullet_{\mathrm{dR}}(T)$ and $C^\bullet_{\mathrm{cell}}(T)$ for some simple tilings T such as periodic tilings and the one dimensional (non-repetitive) symbolic tiling $\cdots aaabaaa \cdots$. The generators of a more interesting tiling, the Fibonacci tiling, were also given, which may be found from the standard Anderson–Putnam approach to calculation of the cohomology groups of a substitution tiling.

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Cohomology for cut and project pattern spaces

Henna Koivusalo

This talk is devoted to establishing sufficient and necessary conditions under which the cohomology of a pattern space associated to a cut and project Delone set is finitely generated. The material will mainly come from [1] and [2, Sections 1-4]. The case for codimension 1 cut and project sets will be worked out in detail. Given a totally irrational d-subspace E of $\mathbb{R}^k$, a complementary $k-d$-dimensional subspace F and a window $W \subset F$, a cut and project set Y is a discrete subset of E, obtained by projecting all the integer lattice points in the acceptance strip $E+W$ to E along F. The set Y is termed nonsingular when no lattice point lands on the boundary of the acceptance strip. We will begin by taking a close look at this definition and the tiling space corresponding to a nonsingular cut and project set. Turns out that in the case where the codimension $k-d=1$ and the window is an interval, the tiling space can be analyzed as an inverse limit of 'a k-torus with rips'. There are either 1 or 2 rips in the torus, and the number only depends on whether or not the endpoints of the window are on the same Γ-orbit, where Γ is the integer lattice in $\mathbb{R}^k$, acting on F after projection. Through this description we will deduce that the cohomology of the tiling space of a cut and project set is as for the once or twice punctured torus. At the end we will develop the geometric ideas to higher codimensions, and state necessary and sufficient conditions for the cohomology being finitely generated. This, again, has to do with the behaviour of the boundary of the window under the Γ-action.

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Pattern complexity

Lorenzo Sadun

One measure of the complexity of a tiling is the number $C(R)$ of patterns of a given size R that appear somewhere in that tiling. There are a number of choices to be made in defining the complexity function $C(R)$, with slightly different definitions for tilings with and without finite local complexity (FLC), but they all give the same asymptotic growth rate. Antoine Julien proved that the growth rate is a homeomorphism invariant. In this talk, I'll go over the different constructions, especially for cut-and-project tilings, discuss how this relates to the decomposition of the "window" into acceptance domains, and connect this with Diophantine gap problems.

Perfectly ordered quasicrystals, I

Sigrid Grepstad

In this talk we discuss the interplay between order and aperiodicity in Delone sets. As quantitative measures of the complexity of a Delone set X we introduce the repetitivity and patch counting functions, $M_X(T)$ and $N_X(T)$, associated to X. This is followed by a discussion of the group of periods of X, together with the Period Conjecture by Lagarias and Pleasants. It is shown that if $M_X(T) < T/3$ for any value of T, then X is a ideal crystal. Finally, we define what it means for X to be linearly or densely repetitive, and explain Lenzs proof that any aperiodic linearly repetitive Delone set is also densely repetitive.

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Perfectly ordered quasicrystals, II

David Damanik

This talk explains why uniform patch frequencies exist in linearly (and densely) repetitive Delone sets. Following [1], the proof of this statement is obtained via a uniform ergodic theorem for almost additive functions on box-shaped patches. Applications to diffraction and the address map are discussed briefly.

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Quasicrystals and the Poisson summation formula

Jose Aliste-Prieto

Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of *Schwartz functions*, i.e., the space of rapidly decreasing C^∞-functions, equipped with the metric topology. Given a Schwartz function φ, its *Fourier transform* is defined by

$$\widehat{\varphi}(t) = \int_{\mathbb{R}^n} \varphi(x) e^{-2i\pi\langle t,x\rangle} dx, \qquad t \in \mathbb{R}^n.$$

Theorem 1 (Poisson's summation formula for functions)**.** *For all ϕ in $\mathcal{S}(\mathbb{R}^n)$ we have*

$$\sum_{m\in\mathbb{Z}^n} \phi(m) = \sum_{m\in\mathbb{Z}^n} \widehat{\phi}(m)$$

We sketch of this theorem, following [1, Chapter 8]. Consider the Fourier series $g(x) = \sum_{m\in\mathbb{Z}^n} c_m e^{2i\pi\langle m,x\rangle}$ of the function $g(x) := \sum_{\ell\in\mathbb{Z}^n} \phi(x+\ell)$. A simple computation shows that $c_m = \widehat{\phi}(m)$ for all m in $\mathbb{Z}^n$. We now have $\sum_{m\in\mathbb{Z}^m} \widehat{\phi}(m) = \sum_{m\in\mathbb{Z}^n} c_m = g(0) = \sum_{m\in\mathbb{Z}^m} \phi(m)$, from which the conclusion follows. The same ideas can be used to proved a more general Poisson's summation formula for lattices. Let L be a lattice. That is,

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \cdots \oplus \mathbb{Z}b_n$$

, where $B = (b_1|b_2|\cdots|b_n)$ is an invertible $n\times n$ matrix. Consider also L^* the dual lattice, which is defined by $L^* = \{x \in \mathbb{R}^n \mid \langle x,l\rangle \in \mathbb{Z} \text{ for all } l \in L\}$.

Theorem 2. *For every Schwartz function ϕ, we have*

$$\sum_{m\in L} \phi(m) = \frac{1}{\det(B)} \sum_{m\in L^*} \widehat{\phi}(m)$$

The Poisson's summation formula has immediate consequences for computing Fourier transforms of distributions supported on lattices. Recall that a **tempered distribution** is a continuous linear functional $\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, its value at φ is denoted $\langle \alpha, \varphi \rangle$. The Fourier Transform $\widehat{\alpha}$ of a tempered distribution α is defined by

$$\langle \widehat{\alpha}, \varphi \rangle := \langle \alpha, \widehat{\varphi} \rangle, \quad \text{for all } \varphi.$$

The δ_x distribution is the distribution defined by $\langle \delta_x, \varphi \rangle = \varphi(x)$ for all φ in $\S(\mathbb{R}^n)$.

Theorem 3 (Poisson's summation Formula for Dirac Combs)**.**

$$\widehat{\sum_{m \in \mathbb{Z}^n} \delta_m} = \sum_{m \in \mathbb{Z}^n} \delta_m$$

The a similar result also holds for lattices. The remainder of the talk consists in explaining the proof of the following theorem.

Theorem 4 (Lev-Olevskii [3])**.** *Let μ be a complex* tempered *measure on $\mathbb{R}^n$ supported on u.d. Λ*

$$\mu = \sum_{\lambda \in \Lambda} \mu(\lambda)\delta_\lambda, \quad \mu(\lambda) \neq 0, \quad d(\Lambda) > 0$$

s.t. $\widehat{\mu}$ is also a measure supported on a u.d. set S:

$$\widehat{\mu} = \sum_{s \in S} \widehat{\mu}(s)\delta_s, \quad \widehat{\mu}(s) \neq 0, \quad d(S) > 0$$

($n > 1$ we need to assume μ is positive). Then Λ is contained in finite union of translates of a lattice.

Remark 1. *Cordoba* [2] *proved a similar result for $n > 1$ if μ is complex and $\mu(\lambda)$ takes finitely many values and $\sum_{s \in Q} |\widehat{\mu}(s)| < C$ for any unit cube Q.*

Much of the work in Lev-Olevskii paper consists in showing that μ is supported on a Meyer set. The methods use depend on the dimension. In dimension 1, one can use Fourier analysis to show that a gap in the spectrum gives certain restrictions to the density of the support. In higher dimension, similar results hold but depend on subtler interpolation and sampling theory. In the talk, we suppose that Λ is Meyer set and sketch the proof of the following result.

Theorem 5. *Let μ be a complex* tempered *measure on $\mathbb{R}^n$ supported on a Meyer set Λ*

$$\mu = \sum_{\lambda \in \Lambda} \mu(\lambda)\delta_\lambda, \quad \mu(\lambda) \neq 0, \quad d(\Lambda) > 0$$

s.t. $\widehat{\mu}$ is also a measure supported on a u.d. set S:

$$\widehat{\mu} = \sum_{s \in S} \widehat{\mu}(s)\delta_s, \quad \widehat{\mu}(s) \neq 0, \quad d(S) > 0$$

Then Λ is contained in finite union of translates of a lattice.

The proof of the last theorem uses the following characterization of Meyer sets.

Theorem 6 (Meyer)**.** *Let Λ be a point set. Then Λ is a Meyer set if and only if there exists a model set $M = M(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega)$ and a finite set F such that*

$$\Lambda \subset M + F \quad \text{and} \quad \pi(\Gamma) \cap \mathbb{Z}[F] = \{0\}.$$

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Diffraction, I

Felix Pogorzelski

In 1982, the Technion physicist Dan Shechtman verified the existence of physical quasicrystals via diffraction experiments - an observation for which he was awarded the Nobel prize for chemistry in 2011. In the past decades, mathematical diffraction theory evolved into a beautiful and rich research topic combining various disciplines such as functional analysis, fourier analysis and ergodic theory. By introducing autocorrelation measures for Dirac combs over general Delone sets, Hof [H95] provided a major cornerstone for casting physical diffraction experiments into rigorous mathematical models. This talk aimed at providing the notational framework and basic results used in the talk Diffraction II. The specific goals were two-fold.

- In a first part, the notion of autocorrelation for Delone sets was explained in terms of dynamical systems.
- For abelian groups, the diffraction measure was defined via the Fourier transform of the autocorrelation measure.

In the sequel, let G be a locally compact, second countable, abelian group along with a Delone set $\Lambda \subset G$. Since a considerable part of the presented results does not rely on the fact that G is abelian, the group multiplication is denoted by $x \cdot y$ in this note.

Taking the closure set of translates $g\Lambda$ in the Chabauty topology defined over all closed subsets of G, one obtains a compact space X_Λ. Assuming further that Λ is of finite local complexity (FLC), one can verifty that the topology on X_Λ coincides with the so-called local mean topology. In the latter structure, two points in X_Λ are close if after a small translation, both sets coincide within a large compact set. It is not hard to see that G acts bi-continuously on X_Λ which yields a topological dynamical system $G \curvearrowright X_\Lambda$. In many situations, i.e. if G is amenable, one finds G-invariant, ergodic measures on X_Λ. For each such measure ν_Λ, one obtains a measure dynamical system $G \curvearrowright (X_\Lambda, \nu_\Lambda)$.

Definition 1 (Siegel transform)**.** *The* S*iegel transform for* Λ *is given by the map*

$$S : C_c(G) \to C(X_\Lambda) : Sf(P) := \sum_{x \in P} f(x).$$

This map is well-defined due to uniform continuity of compactly supported functions on G.

The proof of the following proposition was presented in the talk.

Proposition 1. *Let a measure dynamical system $G \curvearrowright (X_\Lambda, \nu_\Lambda)$ as above be given.*

- *There is a unique Radon measure $\eta_\Lambda = \eta_\Lambda(\nu_\Lambda)$ on G such that*

$$\eta_\Lambda(f_1^* * f_2) = \langle \overline{Sf_1}, Sf_2 \rangle_{L^2(X_\Lambda, \nu_\Lambda)}$$

for all $f_1, f_2 \in C_c(G)$.

- *There exists a unique constant $h_{\Lambda,\nu} > 0$ (called the* Siegel constant*) such that for all $f \in C_c(G)$, one obtains*

$$\int_{X_\Lambda} Sf \, d\nu = h_{\Lambda,\nu} \cdot \int_G f \, dg.$$

The measure η_Λ obtained by the above proposition is called the *autocorrelation measure* for Λ and ν. This approach is new: unlike to the classical situation, this notion does not require an ergodic convergence theorem to hold true. Hence, it is well-defined for general locally compact, second countable groups. This advantage is heavily exploited in [BHP15], where a non-commutative spherical diffraction theory is developed for cut-and-project-schemes over Gelfand pairs. The following theorem shows that in the uniquely ergodic situation, the proposed notion for the autocorrelation coincides with the classical definition and therefore is well justified.

Theorem 1.1 (cf. e.g. [BL04])**.** *Let $G \curvearrowright (X_\Lambda, \nu_\Lambda)$ be uniquely ergodic. Then for all $P \in X_\Lambda$, the weak-$*$-convergence*

$$\eta_\Lambda = \lim_{n \to \infty} |F_n|^{-1} \sum_{x \in P \cap F_n} \sum_{y \in P \cap F_n} \delta_{x^{-1}y}$$

holds true for every van-Hove sequence (F_n) in G.

The talk was concluded with the definition of the diffraction measure for a Delone set. Denote by the $\widehat{G}$ the dual group of G. For $f \in C_c(G)$, write $\hat{f}$ for the Fourier transform of f, defined on $\widehat{G}$.

Definition 2 (Diffraction measure)**.** *Let $G \curvearrowright (X_\Lambda, \nu_\Lambda)$ be as above and denote the autocorrelation for Λ and ν by η_Λ. Then, the* diffraction measure *of η_Λ is the unique Radon measure $\hat{\eta}_\Lambda$ on $\widehat{G}$ such that*

$$\eta_\Lambda(f_1^* * f_2) = \langle \overline{\hat{f}}_1, \hat{f}_2 \rangle_{L^2(\hat{G}, \hat{\eta}_\Lambda)}$$

for all $f_1, f_2 \in C_c(G)$.

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Diffraction, II

Tobias Hartnick

In this talk we illustrated the definition of diffraction as introduced in the previous talks by computing the diffraction measure for regular model sets. The goal was to provide a modern proof for a classical formula of Meyer [3].

We started out by recalling the construction of a regular model set P_0 in a locally compact abelian group G obtained from a regular window W_0 in an internal group H and a lattice Γ in $G \times H$. We then collected some basic topological properties of the hull $X = X_{P_0}$ of P_0. In particular we pointed out that the Chabauty-Fell topology of the hull can be identified with the local topology, which allows for a computationally convenient description of the topology. We used this identification to construct a canonical transversal $\mathcal{T} \subset X$. Namely,

$$\mathcal{T} := \{P \in X \mid P \subset \pi_G(\Gamma)\}.$$

The main part of the talk was then devoted to the outline of a new proof (taken from [2]) of Schlottmann's generalized torus parametrization [4] of a regular model set. We constructed a continuous G-equivariant surjection

$$\beta : X \to T := (G \times H)/\Gamma$$

from X to the generalized torus $T = (G \times H)/\Gamma$, which we then proved to be one-to-one over a generic (in the sense of Haar measure) subset $T^{\mathrm{ns}} \subset T$ of non-singular parameters. Moreover, for non-singular elements P in the canonical transversal we derived an explicit formula for P in terms of $\beta(P)$.

Using the generalized torus parametrization we immediately obtained minimality and unique ergodicity of the hull. This showed that regular model set are repetitive and have uniform patch frequencies. Also, in view of the previous talk, unique ergodicity allowed us to write the auto-correlation measure of a regular model set in terms of the Siegel transform of the hull.

The aforementioned formula for non-singular elements in the canonical transversal then allowed us to compute this Siegel transform explicitly in terms of the parametrization map β, leading to the following formula for the auto-correlation measure. Here we denote by $\mathcal{P}_\Gamma : C_c(G \times H) \to C(T)$ the operator given by periodization over the lattice Γ and given a function $f \in C_c(G)$ we denote $f^*(g) := \overline{f(g^{-1})}$.

Theorem ([2]). *Let η be the auto-correlation measure of a regular model set $P_0 \subset G$ associated with a lattice Γ in $G \times H$ and a window $W_0 \subset H$, and let $T := (G \times H)/\Gamma$. Then for every $f \in C_c(G)$,*

$$\eta(f^* * f) = \|\mathcal{P}_\Gamma(f \otimes \chi_{W_0})\|^2_{L^2(T)}$$

We then pointed out that this formula remains valid for suitably defined model sets in non-abelian groups. This leads to the non-commutative quasi-crystals of [2], but we did not pursue this direction any further. Instead we used the theorem to provide an explicit formula for the diffraction measure of a regular model set in a locally compact abelian group G, which in essence goes back all the way to the pioneering work of Meyer (see [3] or [1]).

Theorem. *Let $P_0 \subset G$ be a regular model set associated with a lattice Γ in $G \times H$ and a window $W_0 \subset H$. Let $\widehat{G}$, $\widehat{H}$ be the character groups of G and H respectively and let $T := (G \times H)/\Gamma$. Then the diffraction measure $\widehat{\eta}$ of P_0 is given by*

$$\sum_{(\xi,\eta) \in \widehat{T} \subset \widehat{G} \times \widehat{H}} \left|\widehat{\chi_{W_0}}(\eta)\right|^2 \cdot \delta_\xi.$$

In particular, P_0 is pure-point diffractive.

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Bi-Lipschitz equivalence, I

Daniel Coronel

This talk is devoted to the question of the existence of Delone sets which are not bi-Lipschitz equivalent to lattices. After explaining the problem and its history, the solution will be sketched via the reduction of Burago-Kleiner [1] and McMullen [2] to the prescribed Jacobian problem. Then the construction of a concrete Delone set which is not bi-Lipschitz to a lattice will be given, following the treatment of Cortez and Navas [3].

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Bi-Lipschitz equivalence, II

Andrés Navas

I will start by recalling the Hall's marriage lemma to produce injections from one set into another. Then, I will use this to show Mc Mullen's argument: if all positive functions bounded from above and away from zero were Jacobian of bi-Lipschitz homeomorphisms, then all Delone sets would be rectifiable. After that, I will give the Burago-Kleiner's sufficient criterium for a Delone set to be rectifiable. The proof presented will be that of Aliste-Coronel and Gambaudo, which holds in any dimension larger than or equal to 2. Some examples of application (e.g. linearly repetitive Delone sets) will be given. Finally, I will come back to the construction of explicit non-rectifiable Delone sets by Cortez and Navas to explain how all these issues are addressed along the construction. If time allows, I will compare all of this to the case of non-amenable spaces, where Whyte's theorem applies (all Delone subsets are bi-Lipschitz equivalent), and I will provide examples of non-rectifiable Delone sets in certain solvable groups.

Bounded displacement equivalence

Yaar Solomon

A discrete set $X \subseteq \mathbb{R}^d$ is called a *separated net*, or a *Delone set*, if it is both uniformly separated and relatively dense in $\mathbb{R}^d$. That is, there exists constants $R, r > 0$ so that the distance between every two points in X is at least r, and every ball of radius R intersects X. The question whether every Delone set is bi-Lipschitz to a lattice or not goes back to Furstenberg, that posed it in the sixties, and it was posed again by Gromov in his book in 93'. This question was settled by McMullen, and independently by Burago-Kleiner in 98', that showed that the answer is negative using an analytic reformulation of the question.

In this talk we consider a finer equivalence relation on the collection of Delone sets in a metric space, where our discussion will mostly focus on $\mathbb{R}^d$. Discrete sets $X, Y \subseteq \mathbb{R}^d$ are called *bounded displacement equivalent* (BD) if there are constants $b, M > 0$ and a bijection between X and bY that moves each point in X at most M. We will see why BD equivalence implies bi-Lipschitz equivalence for Delone sets. Then we will talk about the main tools that we have to prove BD equivalence, and we will see that any two lattices of the same co-volume are BD equivalent. In particular we will discuss a theorem of Laczkovich, with a partial proof, that gives a complete characterization of discrete sets which are BD to a lattice, in terms of well enough estimates of the discrepancy of measurable sets. At the end we will discuss applications of that theorem to discrete sets that arise from substitution tilings and from cut-and-project constructions.

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Deviation of ergodic averages for self-similar tilings

Valérie Berthé

This talk is mainly based on the paper [4] which highlights the boundary effects that occur in dimension d at least 2, when considering deviation of ergodic averages for the $\mathbb{R}^d$-actions associated with self-similar tilings that are assumed to be primitive, aperiodic, and with finite local complexity (FLC). These boundary effects are expressed in terms of the eigenvalues of the underlying substitution matrix. More precisely, we consider a tile-substitution in $\mathbb{R}^d$ that acts on a set of m tiles: every expanded prototile can be decomposed into a union of tiles (which are all translates of the prototiles) with disjoint interiors. We associate with it a square matrix, the substitution matrix, of size m: S_{ij} counts the number of tiles of type i in the image of a tile of type j. The paper [4] stresses the role played by the eigenvalues of the substitution matrix S that satisfy

$$|\theta| > \theta_1^{\frac{d-1}{d}}.$$

When $d = 1$, this condition yields $|\theta| > 1$: if there are no such eigenvalues, we are in the so-called Pisot case, where the deviation of ergodic averages is known to be bounded for cylinders associated with letters (see e.g. [1]); if $d = 1$, there is no difference between the case $|\theta| > \theta_1^{\frac{d-1}{d}}$ and the case $|\theta| > 1$.

The theorem considered in this this lecture is the following [4, Corollary 4.5]. Let $d \geq 2$. Let $(X_\omega, \mathbb{R}^d, \mu)$ be a non-periodic self-similar tiling dynamical system with FLC. Let $\theta_1, \cdots, \theta_m$ be the (real and complex) eigenvalues of the substitution matrix S, counted with multiplicities and ordered in such a way that $\theta_1 > |\theta_2| \geq \cdots \geq |\theta_m|$. Let s be the size of the largest Jordan block associated with the eigenvalues of absolute value $|\theta_2|$. We consider ergodic averages with respect to a bounded Lipschitz domain Ω. Let Ω_R stand for $R\Omega$. There exists a constant $C > 0$ such that for any cylindrical function with $||f||_1 = 1$, any tiling $\mathcal{T} \in X_\omega$, and $R \geq 2$, then

$$\left| \int_{\Omega_R} f(\mathcal{T} - y)dy - \mathcal{L}^d(\Omega_R) \int_{X_\omega} f d\mu \right| \leq \begin{cases} CR^{d-1} & \text{if } |\theta_2| < |\theta_1|^{\frac{d-1}{d}}, \\ CR^{d-1}(\log R)^s & \text{if } |\theta_2| = |\theta_1|^{\frac{d-1}{d}}, \\ CR^{\alpha}(\log R)^{s-1} & \text{if } |\theta_2| > |\theta_1|^{\frac{d-1}{d}}, \end{cases}$$

with $\alpha = d \log |\theta_2| / \log \theta_1$. Note that $\alpha \in (d-1, d)$.

Hence, if $|\theta_2| < |\theta_1|^{\frac{d-1}{d}}$ the main contribution comes from the boundary of the domain. If $|\theta_2| > |\theta_1|^{\frac{d-1}{d}}$, the main contribution comes from the interior ($\alpha > d-1$).

A Lipschitz domain is an open bounded set that has a $d-1$-rectifiable boundary. What is used here is the property that if A is $d-1$-rectifiable, then for all $b > 0$, there exists C such that

$$\mathcal{L}^d(U(A, r)) \leq Cr, \qquad \text{for all } r \in [0, b),$$

where

$$U(A, r) = \{x \in \mathbb{R}^d \mid \text{dist}(x, A) \leq R\}.$$

Cylindrical functions play the role here of characteristic functions for cylinder sets of tiles: a function f on X_ω is called cylindrical if it is integrable with respect to the unique invariant measure μ and depends only on the tile containing the origin.

The proof makes substantial use of linear algebra in the flavor of [1] which handles the $d = 1$ case, together with the use of families of finitely-additive measures. Similar results can be found in [5] with cohomological methods, and in [2, 3, 6, 7], mainly motivated by questions on bi-Lipschitz equivalence and bounded displacement of separated nets, arising from self-similar tilings, to the lattice.

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Gap labeling theorem

Michael Whittaker

1. Abstract

In this talk I examined the Gap Labelling Theorem, first formulated as the Gap Labelling Conjecture by Bellissard in [3]. The Gap Labelling Conjecture was proved independently by Bellissard-Benedetti-Gambaudo [4], Benameur-Oyono-Oyono [7], and Kaminker-Putnam [10].

In 1981, Moser defined a Schrödinger operator with a Cantor set spectrum. His discovery prompted a flurry of research into these operators, and it was realised that the gaps in the spectrum could be labelled by integers in such a way that the labelling is stable under perturbation of the Schrödinger operator. Bellissard was at the forefront of this research [1], and connected the gap labelling with the K-theory of an associated C^*-algebra in low dimensions [2]. The Gap Labelling Conjecture was first formulated in [3] and the following purely mathematical formulation, of what is now the Gap Labelling Theorem, appeared in [5, 10].

Theorem 1 ([4, 7, 10]). *Let Σ be a Cantor set and let $\Sigma \times \mathbb{Z}^n \to \Sigma$ be a free and minimal action of $\mathbb{Z}^n$ on Σ with invariant measure μ. Let $\mu : C(\Sigma) \to \mathbb{C}$ and $\tau_\mu : C(\Sigma) \rtimes \mathbb{Z}^n \to \mathbb{C}$ be the trace induced by μ. Then*

$$\mu_*(K_0(C(\Sigma))) = \tau_{\mu*}(K_0(C(\Sigma) \rtimes \mathbb{Z}^n)),$$

as subsets of $\mathbb{R}$.

In this talk, I considered the proof given by Kaminker and Putnam [10]. The containment $\tau_{\mu*}(K_0(C(\Sigma) \rtimes \mathbb{Z}^n)) \subseteq \mu_*(K_0(C(\Sigma)))$ is relatively straightforward. The reverse containment occupies the bulk of Kaminker and Putnam's paper. The proof uses several deep results that are succinctly described by the diagram appearing on [10, p.538]. The conclusion of the proof is that there exists an integer N, that only depends on the dimension of $\mathbb{Z}^n$, such that

$$N\big(\tau_{\mu*}\big(K_0\big(C(\Sigma) \rtimes \mathbb{Z}^n\big)\big)\big) \subseteq \mu_*\big(K_0\big(C(\Sigma)\big)\big) \subseteq \tau_{\mu*}\big(K_0\big(C(\Sigma) \rtimes \mathbb{Z}^n\big)\big).$$

The talk concluded with the famous example of the Schrödinger operator $H_\theta \in B(\ell^2(\mathbb{Z}))$ given by

$$(H_\theta \xi)(n) := \xi(n+1) + \xi(n-1) + 2\cos(2\pi n\theta)\xi(n).$$

For θ irrational, the spectrum of H_θ is the same as the spectrum of the operator $u+v+(u+v)^*$ in the irrational rotation algebra A_θ. As the parameter θ varies over the interval the spectrum of this operator gives rise to the magnificent Hofstadter butterfly.

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The Fibonacci Hamiltonian

Xifeng Su

In this talk, I will introduce some models in the classical mechanics (especially solid state physics) and quantum mechanics associated with mathematical quasi-crystals (especially the Fibonacci Hamiltonian operator). The models I will mention in solid state physics are the generalized Frenkel-Kontorova models on the crystals and quasi-crystals. The Frenkel-Kontorova models on the crystal are very well developed. In the one dimension case, we have the celebrated Kolmogorov-Arnold-Moser theory, converse KAM theory and Aubry-Mather theory for the existence and non-existence of equilibrium solutions and ground states and their stability. In higher dimension, one can introduce discrete weak KAM theory, which is similar to the Frenkel-Kontorova theory developed by Aubry and Le Daeron or Chou and Griffiths in dimension one. Moreover, the existence of the discrete weak KAM solutions is related to the additive eigenvalue problem in ergodic optimization. However, to the best to my knowledge, for the quasi-crystals models, there are less known and I will introduce some models for the Fibonacci quasi-crystals and quasi-periodic media.

For the quantum mechanics models, I will talk about the spectrum and spectral characteristic of the Schrodinger operators given by the crystals and quasi-crystals. In particular, I will concentrate on the Fibonacci Hamiltonian operator, which is given by the Fibonacci quasi-crystals. In this case, the trace map plays an important role and one can see the deep relations among the spectral characteristics, i.e. the upper transport exponents, the dimension of the spectrum, the dimension of the density of states measure and the optimal Holder exponent of the integrated density of states.

I will survey on the recent results on all these types of models as possible. In particular, there are some dynamics behind these models.

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Danzer problem

Andreas Thom

This talk is about the current status of the Danzer problem, a classical problem in elementary geometry [1]. The problem will be stated, and a history of partial results given: negative results of [2, 3] and positive results (also [2, 3]). The dynamical approach ([3, Prop. 3.1]) will be explained and the relation to the classification of minimal sets for the action of the affine group in the space of closed subsets of $\mathbb{R}^d$ (see [3, §7.3]) will be explained. A sketch of proof of [3, Thm. 1.2] will be given.

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Dense forests

Michael Kelly

The notion of a Dense forest will be defined following [1]. The construction of a Delone dense forest will be given, following [2, §4]. Questions of optimizing the rate in the definition of dense forest will be stated, and results will be surveyed, for both uniformly discrete dense forests, and dense forests whose asymptotic growth is $O(T^d)$.

References

[1] Christopher J. Bishop, *A set containing rectifiable arcs QC-locally but not QC-globally*, Pure Appl. Math. Q. **7** (2011), no. 1, 121–138. MR 2900167

[2] Y. Solomon and B. Weiss, *Dense forests and Danzer sets*, ArXiv e-prints (2014).

Space of cut and project sets

Nicolas Bedaride

Consider a Delone set in $\mathbb{R}^d$ given by the vertices of a cut and project tiling. Consider the balls of radius ρ centered at the points of the Delone set. Denote their union by K_ρ. Then for $(q,v) \in \mathbb{R}^d \times S_1^{d-1}$ we define the free path length as $\tau_1(q,v) = \inf\{t \in [0,1], q+tv \in K_\rho\}$. In a first paper the authors prove:

Theorem 1. *Fix a lattice $\mathcal{P}$ and the initial position ρ. There exists a non increasing continuous function $F_{\mathcal{P}}$ such that for any Borel probability measure Λ on $T^1(\mathbb{R}^d)$, and for every $\xi > 0$ we have*

$$\lim_{\rho \to 0} \Lambda(\{(q,v) \in \mathcal{K}_\rho, \rho^{d-1}\tau_1(q,v,\rho) \geq \xi\}) = F_{\mathcal{P}}(\xi)$$

The limit does not depend on Λ : any probability measure on $\mathbb{R}^d \times S_1^{d-1}$ which is continuous with respect to Lebesgue measure.

The theorem presented is the following generalization:

Theorem 2. *Fix a regular cut and project set $\mathcal{P}$ and the initial position ρ. There exists a non increasing continuous function $F_{\mathcal{P}}$ such that for any Borel probability measure Λ on $T^1(\mathbb{R}^d)$, and for every $\xi > 0$ we have*

$$\lim_{\rho \to 0} \Lambda(\{(q,v) \in \mathcal{K}_\rho, \rho^{d-1}\tau_1(q,v,\rho) \geq \xi\}) = F_{\mathcal{P}}(\xi).$$

In order to prove this, the authors introduce the action of the affine group $ASL_d(\mathbb{R})$ on the Delone set, and use Ratner's theorem in order to compute the limit. It allows them to obtain the expression of $F_{\mathcal{P}}(\xi)$

$$F_{\mathcal{P}}(\xi) = \mu_g(\{\mathcal{R} \in \mathcal{Q}_g \mid \mathfrak{Z}(\xi) \cap \mathcal{R} = \emptyset\})$$

where $\mathcal{Q}_g$ is the space of quasi-crystals and μ_g is a measure on $\mathcal{Q}_g$.

References

[1] Marklof-Strömbergsson, *Free path lengths in quasicrystals*, Communications in Mathematical Physics **330** (2014) 723–755

Siegel summation for cut and project sets

Daniel El-Baz

The Siegel integration formula is a technique introduced by Siegel [1] in connection with problems in the geometry of numbers, developing ideas of Minkowski. It relates integrals on $\mathbb{R}^d$ with integrals on the space of unimodular d-dimensional lattices, with respect to the natural measure induced by Haar measure on $SL_d(\mathbb{R})$, and can be used as part of a probabilistic method to prove the existence of lattices with certain properties (notably, as in [1], lattices whose shortest nonzero vector is long). This approach was axiomatized by Veech [2] to general spaces of point-sets in $\mathbb{R}^d$ and recently extended to the space of cut and project sets by Marklof and Strömbergsson [3]. In this talk the method will be introduced in the abstract setting (following Veech) and the results of Marklof and Strömbergsson will be presented.

References

[1] Carl Ludwig Siegel, *A mean value theorem in geometry of numbers*, Ann. of Math. (2) **46** (1945), 340–347. MR 0012093 (6,257b)

[2] William A. Veech, *Siegel measures*, Ann. of Math. (2) **148** (1998), no. 3, 895–944. MR 1670061 (2000k:37028)

[3] Jens Marklof and Andreas Strömbergsson, *Free path lengths in quasicrystals*, Comm. Math. Phys. **330** (2014), no. 2, 723–755. MR 3223485

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