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## Classical Algebraic Geometry

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ABSTRACT. Progress in algebraic geometry often comes through the introduction of new tools and ideas to tackle the classical problems the development of the field. Examples include new invariants that capture some aspect of geometry in a novel way, such as the derived category, and the extension of the class of geometric objects considered to allow constructions not previously possible, such as the transition from varieties to schemes or from schemes to stacks. Many famous old problems and outstanding conjectures have been resolved in this way over the last 50 years. While the new theories are sometimes studied for their own sake, they are in the end best understood in the context of the classical questions they illuminate. The goal of the workshop was to study new developments in algebraic geometry, with a view toward their application to the classical problems.

*Mathematics Subject Classification (2010):* 14-XX.

### Introduction by the Organisers

The workshop *Classical Algebraic Geometry* held June 12–18, 2016 at the “Mathematisches Forschungsinstitut Oberwolfach” was organized by Olivier Debarre (ENS), David Eisenbud (Berkeley), Gavril Farkas (Berlin), and Ravi Vakil (Stanford). There were 17 one-hour talks with a maximum of four talks a day, and an evening session of short presentations allowing young participants to introduce their current work (and themselves). The schedule deliberately left plenty of room for informal discussion and work in smaller groups.

The extended abstracts give a detailed account of the broad variety of topics of the meeting, often classical questions in algebraic geometry approached with

modern methods. (It should be noted that six of the lectures were given by recent Ph.D.'s.) We focus on a representative sample here:

- **Degrees of Irrationality (Lazarsfeld)** Rob Lazarsfeld presented some exciting new ideas about "Degrees of Irrationality", covering joint work of his with Lawrence Ein and a young woman who is currently a postdoc, Brooke Ullery, and inspired by earlier work of Bastianelli, Cortini and De Poi (BCP).

The ideas excited many of the participants. As an introduction, consider the case of smooth projective curves over an algebraically closed field. They are usually classified by their genus  $g$ , the case  $g = 0$  being the projective line, the only rational curve. One may think that the curves become "less rational" as the genus grows, and this is the scheme that has been taken over into most of higher-dimensional birational geometry. But there is another measure: the "gonality" of the curve, which is the lowest degree of a map to the projective line. For example, genus 1 curves ("elliptic curves") all admit degree 2 maps to the projective line, and, in general, curves that admit degree 2 maps to the projective line are called "hyperelliptic" and the gonality of a curve of genus  $g \geq 2$  can be anything from 2 to  $\lceil g/2 \rceil + 1$ . When  $C$  is a smooth plane curve of degree  $d \geq 3$ , the gonality is  $d - 1$ , with projection from a point of  $C$  being the lowest degree map to the line.

This idea was extended to smooth surfaces in  $\mathbf{P}^3$  by Hisao Yoshihara in 1994, and further progress was made by Bastianelli, Cortini and De Poi who improved the results for surfaces and extended the idea to higher dimensions.

The current best result, proven by Ein, Lazarsfeld and Ullery with an improvement by Bastianelli, Cortini and De Poi: The conjecture is true for very general hypersurfaces if  $d \geq 2n + 1$ .

- **Gushel–Mukai Varieties (Perry)** Gushel–Mukai varieties are prime Fano complex varieties of degree 10 and coindex 3. They were classified by Gushel and Mukai, who proved that most of them are quadratic sections of a linear section of the Grassmannian  $G(2, \mathbf{C}^5)$ . Their study very much parallels that of cubic hypersurfaces, with several important differences: for example, their period mappings are not injective. However, in dimension 4, the two are very similar: there is an attached hyperkähler fourfold (the variety of lines for cubic fourfolds, a "double EPW sextic" for Gushel–Mukai fourfolds), and in both cases, some fourfolds are rational and one expects the very general one to be irrational. In addition, some Gushel–Mukai fourfolds are birationally isomorphic to cubic fourfolds.

Alex Perry presented his work in collaboration with Alexander Kuznetsov on another common feature of the two families of varieties: their derived categories admit a canonical semi-orthogonal decomposition, one term of which "looks like" the derived category of a K3 surface in even dimensions, and of an Enriques surface in odd dimensions. The central

conjecture is that for both cubic and Gushel–Mukai fourfolds, the rationality of the fourfold should imply that this mysterious category is that of an actual K3 surface (for a very general fourfold, this category is not the derived category of any variety).

- Hilbert’s 17th Problem in low degree (Benoist) Finally, a young speaker with a spectacular result was Olivier Benoist. Recall that part of Hilbert’s 17th problem asked whether every polynomial in  $n$  variables over the real numbers that takes on only non-negative values can be written as a sum of squares of rational functions; and this was proven by Emil Artin, not long afterwards. However, the proof was ineffective, and it was not until much later that Pfister was able to prove that the number of squares needed was at most  $2^n$ . It is conjectured that this is best possible, but the conjecture is only known for  $n = 1, 2!$

On the other hand, the conjecture is not in general optimal for low degree polynomials; for example any non-negative quadratic form is a sum of at most  $n$  squares. It is expected that  $2^n$  is best possible for polynomials of degree  $2n + 2$  and higher, but very few other cases are known.

In his talk, the first of the workshop, Benoist proved that  $2^n - 1$  squares suffice for polynomials of degree  $\leq 2n$ . The proof employs some of the deepest ideas in algebraic geometry, and both the ideas and the presentation were extremely impressive.

The young participants’ presentations by Ben Bakker, Christian Bopp, Lionel Darondeau, Victor González-Alonso, Frank Gounelas, Jérémy Guéré, Ariyan Javanpeykar, Zhiyuan Li, Wenhao Ou, Anand Patel, Alexander Pavlov, Alex Perry, and Arnav Tripathy covered a similarly wide spread of topics, from the structural and formal to the specific and geometric. As with previous years’ young participants, we expect these researchers to quickly establish themselves as leaders in their areas.

*Acknowledgement:* The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.



## Workshop: Classical Algebraic Geometry

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## Abstracts

### On Hilbert's 17th problem in low degree

OLIVIER BENOIST

A polynomial  $f \in \mathbb{R}[X_1, \dots, X_n]$  is said to be  $\geq 0$  if  $f(x_1, \dots, x_n) \geq 0$  for every  $(x_1, \dots, x_n) \in \mathbb{R}^n$ . The topic of Hilbert's 17th problem is to try to explain the positivity of  $f$  by writing it as a sum of squares. It was understood by Hilbert that there is no hope to write  $f$  as a sum of squares of polynomials in general, and that one should consider sums of squares of rational functions instead. This question was settled by Artin [1]:

**Theorem 1** (Artin). *A polynomial  $f \in \mathbb{R}[X_1, \dots, X_n]$  that is  $\geq 0$  is a sum of squares in  $\mathbb{R}(X_1, \dots, X_n)$ .*

This result was later improved by Pfister [8], who realized that the number of squares needed only depends on the number of variables:

**Theorem 2** (Pfister). *A polynomial  $f \in \mathbb{R}[X_1, \dots, X_n]$  that is  $\geq 0$  is a sum of  $2^n$  squares in  $\mathbb{R}(X_1, \dots, X_n)$ .*

Proving that Pfister's result is optimal (i.e. showing that it is not possible to improve on the bound  $2^n$ ) may be the most important related open problem [9, §4 Problem 1]. If  $n = 1$ , it is obviously optimal, because  $X_1^2 + 1$  is not a square.

In two variables, it is also known that Pfister's result is optimal: Cassels, Ellison and Pfister [4] have shown that the polynomial  $1 + X_1^2 X_2^4 + X_1^4 X_2^2 - 3X_1^2 X_2^2$  is  $\geq 0$ , but not a sum of 3 squares in  $\mathbb{R}(X_1, X_2)$ .

When  $n \geq 3$ , this question is completely open.

We explore another direction: is it possible to improve on Pfister's result, when the degree  $d$  of  $f$  is low? Two results were previously known. One is very easy: if  $f \in \mathbb{R}[X_1, \dots, X_n]$  is  $\geq 0$  of degree 2, diagonalization of quadratic forms shows that it is a sum of  $n + 1$  squares. The other is due to Hilbert [7]: a degree 4 polynomial in  $\mathbb{R}[X_1, X_2]$  that is  $\geq 0$  is a sum of 3 squares.

Our main result [2, Theorem 0.1] generalizes this last theorem in more variables:

**Theorem 3.** *Let  $n \geq 2$ . A polynomial  $f \in \mathbb{R}[X_1, \dots, X_n]$  of degree  $d \leq 2n$  that is  $\geq 0$  is a sum of  $2^n - 1$  squares in  $\mathbb{R}(X_1, \dots, X_n)$ , with possible exceptions if  $n \geq 7$  is odd and  $d = 2n$ .*

The particular case with 3 variables is new when  $d = 4$  or  $d = 6$ :

**Corollary 1.** *A polynomial  $f \in \mathbb{R}[X_1, X_2, X_3]$  that is  $\geq 0$  and of degree  $\leq 6$  is a sum of 7 squares in  $\mathbb{R}(X_1, X_2, X_3)$ .*

Another reason why Theorem 3 is interesting lies in the expectation that the bound  $d \leq 2n$  on the degree is the best possible, and that from  $d \geq 2n + 2$  on, there should exist polynomials achieving Pfister's bound.

The geometric proof of Theorem 3 uses an algebraic variety  $X$  naturally associated to  $f$ , extending to a higher number of variables arguments that have been used by Colliot-Thélène [5] when  $n = 2$ . Let  $F \in \mathbb{R}[X_0, \dots, X_n]$  be the homogenization of  $f$ , and introduce  $X := \{Y^2 + F(X_0, \dots, X_n) = 0\}$ : a real algebraic variety that is a double cover of  $\mathbb{P}_{\mathbb{R}}^n$  ramified over the hypersurface  $\{F = 0\}$ .

The first step of the proof is to reformulate Theorem 3 into a geometric statement about  $X$ . Using the work of Pfister on multiplicative quadratic forms and Voevodsky's proof of the Milnor conjecture, one proves that  $f$  being a sum of  $2^n - 1$  squares is equivalent to the cohomology class  $\{-1\}^n \in H^n(X, \mathbb{Z}/2\mathbb{Z})$  being of coniveau 1, that is vanishing on a non-empty Zariski open subset of  $X$ .

A key idea is to work with cohomology with integral coefficients (say, 2-adic cohomology) instead of the mod 2 coefficients that come out of the theory of quadratic forms. The class  $\{-1\} \in H^1(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$  lifts uniquely to a class  $\omega \in H^1(\mathbb{R}, \mathbb{Z}_2(1))$ , and it turns out that  $f$  being a sum of  $2^n - 1$  squares is also equivalent to  $\omega^n \in H^n(X, \mathbb{Z}_2(n))$  being of coniveau 1. Equivalently, we need to show that  $\omega^n$  vanishes in the unramified cohomology group  $H_{\text{nr}}^n(X, \mathbb{Z}_2(n))$ .

The main tool we use to prove it is Bloch-Ogus theory [3].

Suppose first that  $X$  is smooth. Then, an important point in the analysis is the vanishing of the unramified cohomology group  $H_{\text{nr}}^n(X_{\mathbb{C}}, \mathbb{Z}_2)$  [6, Proposition 3.3]. More precisely, that it has no torsion is a consequence of the Milnor conjecture (this is the argument for which it is crucial to work with integral coefficients) and it is torsion by decomposition of the diagonal (this uses that  $X_{\mathbb{C}}$  is rationally connected for  $d \leq 2n$ : it is the only place where this degree hypothesis is used).

Together with explicit computations for the 2-adic cohomology of  $X$ , that  $H_{\text{nr}}^n(X_{\mathbb{C}}, \mathbb{Z}_2) = 0$  allows us to obtain the required vanishing of  $\omega^n \in H_{\text{nr}}^n(X, \mathbb{Z}_2(n))$ .

Finally, to deal with the case where  $X$  is singular, we reduce to the case where  $X$  is smooth using a degeneration argument. To implement this argument, it is necessary to run the whole proof over an arbitrary real closed field, and not only over the field  $\mathbb{R}$  of real numbers.

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## On the geometry of Fano 4-folds with large second Betti number

CINZIA CASAGRANDE

This is a report on the author's ongoing project to study the geometry of Fano 4-folds with large second Betti number.

Let  $X$  be a smooth, complex Fano variety. We recall that the second Betti number of  $X$  coincides with the Picard number  $\rho_X$  of  $X$ . Up to dimension 3, Fano varieties are classified; in particular, we have the following.

**Theorem 1** (Mori-Mukai, see [5], §7.1). *Let  $X$  be a Fano 3-fold. If  $\rho_X \geq 6$ , then  $X \cong S \times \mathbb{P}^1$ , where  $S$  is a del Pezzo surface.*

Our goal is to prove an analog of this theorem in dimension 4; we are going to explain partial results and a strategy in this direction. Besides the 4-dimensional case, a motivation for this project is also to better understand the case of higher dimensional Fano varieties.

It is possible to give an explicit bound on the Picard number of Fano 4-folds, such as  $\rho_X \leq 3^8 5^{304} 23^4 26$  (see [3, Rem. 3.1]). On the other hand, to the author's knowledge, all known examples of Fano 4-folds which are not products of surfaces have  $\rho \leq 9$ , and products of del Pezzo surfaces have  $\rho \leq 18$ .

To study this problem, we use techniques from birational geometry and the Minimal Model Program (MMP), and we also rely on a useful result on the Picard number of prime divisors in  $X$ . Before stating it, we need to introduce some notation.

Let  $\mathcal{N}_1(X)$  be the vector space of 1-cycles in  $X$ , with real coefficients, up to numerical equivalence, so that  $\dim \mathcal{N}_1(X) = \rho_X$ . For any prime divisor  $D \subset X$ , set  $\mathcal{N}_1(D, X) := i_* \mathcal{N}_1(D)$ , where  $i: D \hookrightarrow X$  is the inclusion. Thus  $\mathcal{N}_1(D, X)$  is the linear span of classes of curves in  $D$ , and  $\dim \mathcal{N}_1(D, X) \leq \rho_D$ . We have the following.

**Theorem 2** ([1], Th. 1 and Cor. 1.3, [3], Th. 1.2). *Let  $X$  be a smooth Fano variety. If there exists  $D \subset X$  with  $\text{codim} \mathcal{N}_1(D, X) \geq 4$ , then  $X \cong S \times Y$ , where  $S$  is a del Pezzo surface.*

*Suppose now that  $\dim X = 4$ , and that  $X$  is not a product of del Pezzo surfaces. If there exists  $D \subset X$  with  $\text{codim} \mathcal{N}_1(D, X) = 3$ , then  $\rho_X \leq 6$ . If there exists  $D \subset X$  with  $\text{codim} \mathcal{N}_1(D, X) = 2$ , then  $\rho_X \leq 12$ .*

Theorem 2 can be applied together with techniques from the MMP: the strategy is to use birational geometry in order to find a special divisor  $D$  with small  $\dim \mathcal{N}_1(D, X)$ , and then either bound the Picard number of  $X$ , or show that  $X$  is a product.

Let  $f: X \rightarrow Y$  be an elementary contraction of  $X$ , that is: a surjective morphism with connected fibers, with  $Y$  normal and projective, and  $\rho_X - \rho_Y = 1$ . We say that  $f$  is of type  $(a, b)$  if  $a = \dim \text{Exc}(f)$  and  $b = \dim f(\text{Exc}(f))$ . The following is a consequence of Theorem 2.

**Theorem 3** ([1], Cor. 1.5 and 1.6). *Let  $X$  be a Fano 4-fold. Suppose that  $X$  has an elementary contraction which is either of fiber type, or divisorial of type  $(3, 0)$  or  $(3, 1)$ . Then either  $X \cong S_1 \times S_2$  with  $S_1 = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_1$  (so that  $\rho_X \leq 11$ ), or  $\rho_X \leq 6$ .*

Thanks to this result, we are reduced to study Fano 4-folds  $X$  such that every elementary contraction of  $X$  is either small, or divisorial of type  $(3, 2)$ .

Let now  $\varphi: X \dashrightarrow X'$  be a birational map given by a sequence of flips; we say that  $\varphi: X \dashrightarrow X'$  is a *small  $\mathbb{Q}$ -factorial modification* (SQM) of  $X$ . One can show that  $X'$  is again smooth, and that every non-small elementary contraction of  $X'$  is  $K$ -negative (see [2, Rem. 3.6]). Notice that we can always find a SQM  $\varphi: X \dashrightarrow X'$  such that  $X'$  has an elementary contraction which is either of fiber type, or divisorial. In the fiber type case, we have the following.

**Theorem 4** ([2], Th. 1.1). *Let  $X$  be a Fano 4-fold. If there exists a SQM  $\varphi: X \dashrightarrow X'$  such that  $X'$  has an elementary contraction of fiber type, then  $\rho_X \leq 11$ .*

Concerning the divisorial case, we have a description of the possible elementary divisorial contractions occurring in  $X'$  when  $\rho_X \geq 7$ , as follows.

**Theorem 5** ([2, 4]). *Let  $X$  be a Fano 4-fold with  $\rho_X \geq 7$ . Then there exist a SQM  $\varphi: X \dashrightarrow X'$ , and an elementary divisorial contraction  $f: X' \rightarrow Y$ , such that  $Y$  is Fano, and one of the following occurs:*

- (1)  $Y$  is smooth,  $f$  is the blow-up of a point, and  $\rho_X \leq 12$ ;
- (2)  $Y$  is smooth, and  $f$  is the blow-up of a smooth curve;
- (3)  $\text{Exc}(f)$  is an irreducible quadric, with normal bundle  $\mathcal{O}(-1)$ , and it is contracted to an isolated terminal and locally factorial singularity;
- (4)  $X = X'$ ,  $f$  is of type  $(3, 2)$ , and  $Y$  has at most isolated, locally factorial, ordinary double points.

Case (1) is studied in [4]. In order to bound  $\rho_X$ , one has to study the remaining cases; we expect (4) to be the hardest.

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## Birational geometry of the moduli space of quartic surfaces

KIERAN G. O'GRADY

(joint work with Radu Laza)

An important problem in algebraic geometry is to construct a geometric compactification for the moduli space of polarized degree  $d$   $K3$  surfaces  $\mathcal{K}_d$ . By Global Torelli,  $\mathcal{K}_d$  is isomorphic to a locally symmetric variety, and hence it has natural compactifications, such as Baily-Borel compactification  $\mathcal{K}_d^*$ , Mumford's toroidal compactifications, and more generally Looijenga's semitoric compactifications. However, a priori, none of these compactifications have geometric meaning. In order to attach some geometric meaning to them, it is natural to compare these compactifications, especially the Baily-Borel one, with GIT compactifications. Moreover, an understanding of the birational relationship between Baily-Borel and GIT compactifications leads to deep results about the period map (e.g. see [10], [7]), and to results about the structure of the GIT quotient. The simplest instance of such comparison results is the isomorphism

$$(\mathfrak{H}/\mathrm{SL}(2, \mathbb{Z}))^* \cong |\mathcal{O}_{\mathbb{P}^2}(3)| // \mathrm{SL}(3) (\cong \mathbb{P}^1)$$

between the compactified  $j$ -line and the GIT moduli space of plane cubic curves. In a vast generalization of this fact, Looijenga ([5, 6]) has given a comparison framework that applies to locally symmetric varieties associated to type  $IV$  or  $I_{1,n}$  Hermitian symmetric domains. This framework was successfully applied in the case of moduli of degree 2  $K3$  surfaces ([4], [10]), cubic fourfolds ([7], [3]), and a few other related examples (e.g. cubic threefolds, del Pezzo surfaces, etc.). The nominal purpose of this paper is to investigate the similar looking cases of degree 4  $K3$  surfaces ([11]) and double EPW sextics ([8, 9]). While attempting to study these cases in detail, we uncovered a rich and intriguing picture.

The starting point of our investigation are two limitations in Looijenga's construction. First of all, a certain technical assumption for Looijenga's construction is false for quartic  $K3$ s, while, in contrast, for the degree 2 case this assumption is satisfied. Namely, for arithmetic reasons, the combinatorics of the hyperplane arrangement involved in Looijenga's construction [6] is much simpler for degree 2  $K3$  surfaces (and similarly cubic fourfolds) than for degree 4  $K3$  surfaces. Secondly, and more seriously, there exists a plethora of GIT models. In the low degree cases considered here and in the literature, there might be a "natural" choice for GIT, but this is misleading (see [1] for a hint of what would happen already in degree 6). The solution that we propose to handle these two issues is to give flexibility to Looijenga's construction by considering a continuous variation of models. More precisely, we recall that for a locally symmetric variety  $\mathcal{F} = \mathcal{D}/\Gamma$ , Baily-Borel have shown that the eponymous compactification  $\mathcal{F}^*$  is the Proj of the ring of automorphic functions, i.e.  $\mathcal{F}^* = \mathrm{Proj}R(\mathcal{F}, \lambda)$ , where  $\lambda$  is the Hodge bundle. Looijenga's deep insight was to observe that in certain situations of geometric interest, a certain GIT quotient  $\overline{\mathcal{M}}$  is nothing but the Proj of the ring of meromorphic automorphic forms with poles on a (geometrically meaningful) Heegner (or Noether–Lefschetz) divisor  $\Delta$ , and thus  $\overline{\mathcal{M}} = \mathrm{Proj}R(\mathcal{F}, \lambda + \Delta)$ . Furthermore, Looijenga has shown

that under a certain assumption on  $\Delta$  (which fails for quartics),  $\text{Proj}R(\mathcal{F}, \lambda + \Delta)$  has an explicit combinatorial/arithmetic description. Our approach is to continuously interpolate between the two models by controlling the order of poles for the meromorphic automorphic function, i.e. to consider  $\text{Proj}R(\mathcal{F}, \lambda + \beta\Delta)$  where  $\beta \in [0, 1]$ . This allows to understand the case of quartics and more importantly to capture more GIT quotients.

The main result of the paper is to predict the behavior of the models  $\mathcal{F}(\beta) := \text{Proj}R(\mathcal{F}, \lambda + \beta\Delta)$ , where  $\mathcal{F}$  is the period space for quartic  $K3$  surfaces, and  $\beta$  varies in  $[0, 1] \cap \mathbb{Q}$ . As explained below, while these predictions are only conjectural, we have strong evidence that they are accurate, and they represent a significant improvement (both qualitative and quantitative) on what was previously known.

**Prediction 1.** *The variation of models  $\text{Proj}R(\mathcal{F}, \lambda + \beta\Delta)$  interpolating between Baily-Borel ( $\beta = 0$ ) and GIT for quartic surfaces ( $\beta = 1$ ) undergoes birational transformations (flips, except the two boundary cases) at the following critical values for  $\beta$ :*

$$\beta \in \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{9}, 0 \right\}.$$

Furthermore, the centers of the flips for  $\beta < \frac{1}{5}$  correspond to  $T_{3,3,4}$ ,  $T_{2,4,5}$ , and  $T_{2,3,7}$  marked  $K3$  surfaces (loci inside the moduli  $\mathcal{F}$  of quartic  $K3$  surface) and those loci are flipped to the loci of quartics with  $E_{14}$ ,  $E_{13}$ , an  $E_{12}$  singularities respectively in the GIT model (see Shah [10]).

There is a complete geometric (part conjectural, part provable) matching between Shah's strata in the GIT quotient  $\overline{\mathcal{M}}$  (e.g. the  $E_r$  strata mentioned,  $r = 12, 13, 14$ ) and the strata resulting from the predicted flips above. Furthermore, the extremal cases can be inferred from the work of Shah and Looijenga. Specifically, the morphism  $\mathcal{F}(\epsilon) \rightarrow \mathcal{F}(0) \cong \mathcal{F}^*$  is Looijenga's  $\mathbb{Q}$ -factorialization of the (closure of the) Hegner divisor  $\Delta$  as discussed in [6] (in particular,  $\mathcal{F}(\epsilon)$  is a semi-toric compactification in the sense of Looijenga). At the other extreme, the map  $\mathcal{F}(1 - \epsilon) \rightarrow \mathcal{F}(1) \cong \overline{\mathcal{M}}$  is the divisorial contraction (of the strict transforms of) hyperelliptic and unigonal divisors in the moduli of quartic  $K3$ s to the GIT polystable points corresponding to the double quadric and to the tangent developable to the twisted cubic respectively. Alternatively,  $\mathcal{F}(1 - \epsilon) \rightarrow \mathcal{F}(1) \cong \overline{\mathcal{M}}$  is a Kirwan type blow-up of the GIT quotient  $\overline{\mathcal{M}}$  at those two special points (as discussed in [11, Sect. 3 and 4]). By way of comparison, in the case of degree 2  $K3$  surfaces, there is no intermediate flip and thus the two extremal cases suffice to compare GIT and Baily-Borel compactification (see [10] and [4]). Even for cubic fourfolds ([2, 3], [7]), there is only one intermediate flip, and thus a much simpler picture.

Actually, our methods give predictions for the behaviour of certain log canonical models of locally symmetric varieties associated to  $D$  lattices. More precisely, we let  $\mathcal{F}(N)$  be the  $N$ -dimensional Type IV locally symmetric variety corresponding to the lattice  $\Lambda_N := U^2 \oplus D_{N-2}$  (so that  $\dim \mathcal{F}(N) = N$ ), and an arithmetic group  $\Gamma_N$ , which is intermediate between the orthogonal group  $O^+(\Lambda_N)$ , and

the stable orthogonal subgroup  $\tilde{O}^+(\Lambda_N)$ . Then  $\mathcal{F}(19)$  is the period space for quartic  $K3$  surfaces (previously denoted by  $\mathcal{F}$ ),  $\mathcal{F}(20)$  is the period space for double EPW sextics modulo the duality involution (or EPW-cubes), and  $\mathcal{F}(18)$  is the period space for hyperelliptic quartic surfaces. The salient point in the  $D$  tower is that  $\mathcal{F}(N-1)$  is isomorphic to a natural “hyperelliptic” Heegner divisor  $H_h(N)$  of  $\mathcal{F}(N)$ , and this leads to an inductive behavior. From the perspective of comparing to GIT we are led to study the variation of models under  $\lambda(N) + \beta\Delta(N)$ , where  $\lambda(N)$  is the automorphic (or Hodge) orbiline bundle on  $\mathcal{F}(N)$ , and  $\Delta(N)$  is  $H_h(N)/2$  except when  $N \equiv 3, 4 \pmod{8}$ , in which case  $\Delta(N) = (H_h(N) + H_u(N))/2$  where  $H_u(N)$  is the “unigonal” divisor. Thus we have predictions for the behaviour of the period map for double EPW sextics, similar to those stated above for quartic surfaces.

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## A spectral sequence for stratified spaces and a conjecture of Vakil–Wood

DAN PETERSEN

**A spectral sequence associated to a stratification.** Let  $X$  be a topological space satisfying some mild point-set hypotheses, and let  $X = \bigcup_{\alpha \in P} S_\alpha$  be a stratification; that is, the set  $P$  of strata is finite, each stratum  $S_\alpha$  is locally closed, and the closure of a stratum is a union of strata. Define a partial order on  $P$  by  $\alpha \leq \beta$  if  $\overline{S}_\alpha \supseteq S_\beta$ .

Let  $\chi_c(-)$  denote the compactly supported Euler characteristic of a space. Since  $\chi_c(Y \setminus Z) = \chi_c(Y) - \chi_c(Z)$  for a closed subspace  $Z \subset Y$ , one has

$$(1) \quad \chi_c(\overline{S}_\alpha) = \sum_{\alpha \leq \beta} \chi_c(S_\beta)$$

for all  $\alpha \in P$ . By the Möbius inversion formula for the poset  $P$ , this relation is equivalent to

$$(2) \quad \chi_c(S_\alpha) = \sum_{\alpha \leq \beta} \mu_P(\alpha, \beta) \cdot \chi_c(\overline{S}_\beta)$$

where  $\mu_P$  is the Möbius function of the poset.

One may ask if these relationships between Euler characteristics can be promoted (“categorified”) to relationships between the actual cohomology groups with compact support. To do this for Equation (1), choose any strictly decreasing function  $\sigma: P \rightarrow \mathbb{Z}$ . Such a function defines a filtration of  $\overline{S}_\alpha$  by closed subspaces, and the corresponding spectral sequence in compactly supported cohomology reads

$$(3) \quad E_1^{pq} = \bigoplus_{\substack{\alpha \leq \beta \\ \sigma(\beta) = -p}} H_c^{p+q}(S_\beta, \mathbb{Z}) \implies H_c^{p+q}(\overline{S}_\alpha, \mathbb{Z})$$

By equating the Euler characteristics of the  $E_1$  and  $E_\infty$  pages of this spectral sequence one recovers Equation (1).

It is then natural to ask whether also the dual Equation (2) admits a similar interpretation. We point out that the quantity  $\mu_P(\alpha, \beta)$  is *also* an Euler characteristic, by Philip Hall’s theorem: the Möbius function  $\mu_P(\alpha, \beta)$  equals the reduced Euler characteristic of  $N(\alpha, \beta)$ , by which we mean the nerve of the poset  $(\alpha, \beta)$ , where  $(\alpha, \beta)$  denotes an open interval in  $P$ . So one can expect such a categorification to also involve the reduced cohomology groups of the poset. The first main result gives such a construction. It is convenient to now let  $\sigma: P \rightarrow \mathbb{Z}$  be a strictly *increasing* function.

**Theorem A.** [5] *There exists a spectral sequence*

$$E_1^{pq} = \bigoplus_{\substack{\alpha \leq \beta \\ \sigma(\beta) = p}} \bigoplus_{i+j+2=p+q} H_c^j(\overline{S}_\beta, \widetilde{H}^i(N(\alpha, \beta), \mathbb{Z})) \implies H_c^{p+q}(S_\alpha, \mathbb{Z}).$$

Taking Euler characteristics of both sides recovers Equation (2). One might expect such a construction to be known already, but to my knowledge the result is new. If  $X$  is algebraic, then this is a spectral sequence of Galois representations/mixed Hodge structures.

Many familiar spectral sequences arise as special cases. For instance, if  $X$  is a complex manifold and  $D$  is a strict normal crossing divisor, then the induced stratification of  $X$  gives rise to a spectral sequence computing the compact support cohomology of  $X \setminus D$ . For  $X$  projective, this is the Poincaré dual of the spectral sequence used by Deligne to construct the mixed Hodge structure on a smooth noncompact complex algebraic variety. As another example, if  $X$  is a euclidean

space stratified by an arrangement of affine subspaces, then the closed strata have trivial topology, and the  $E_1$  page only involves the cohomology groups of the poset of strata. It is not very hard to prove that there are no differentials in this case, so one obtains formulas for the Betti numbers of the complement of the subspace arrangement in terms of the combinatorics of the arrangement. This recovers theorems of Goresky–MacPherson and Björner–Ekedahl.

The spectral sequence can also be fruitfully applied to configuration spaces. Let  $M$  be a space, and let  $X = M^n$ . We stratify  $X$  according to how points collide, so that the closed strata are all homeomorphic to cartesian powers  $M^k$ ,  $k = 1, \dots, n$ . The poset is in this case the *partition lattice*, whose cohomology groups have been calculated explicitly by Stanley, Joyal, Klyachko, Hanlon, ... Thus we get a spectral sequence calculating the compact support cohomology of the *configuration space*  $F(M, n)$  of  $n$  distinct ordered points on  $M$ . This spectral sequence was first constructed by Getzler [2]; when  $M$  is an oriented manifold, the spectral sequence is Poincaré dual to one constructed earlier by Totaro [6] (and Cohen–Taylor before him). The  $E_1$  page and its differential can be expressed directly in terms of the compact support cohomology of  $M$  and its cup product. The higher differentials are given by Massey products, which is not obvious from previous constructions of the spectral sequence.

**Representation stability for configuration spaces.** There is a long history of homological stability theorems for configuration spaces, starting with the following 1975 result of McDuff [4]:

**Theorem:** (McDuff) *Let  $M$  be a connected oriented noncompact manifold of dimension  $\geq 2$ . For any fixed  $k \geq 0$ , the cohomology groups  $H^k(F(M, n)/S_n, \mathbb{Z})$  and  $H^k(F(M, n+1)/S_{n+1}, \mathbb{Z})$  are isomorphic for  $n \gg 0$ .*

If one considers instead the spaces  $F(M, n)$  of *ordered* configurations, then it's easy to see that the Betti numbers become unbounded in general as  $n$  grows. A relatively recent insight of Church and Farb is that in this case, *representation stability* holds. The definition, informally stated, is that a sequence  $V_n$  of representations of the symmetric groups  $S_n$  (say over  $\mathbb{Q}$ ) is *representation stable* if the following holds: when each  $V_n$  is decomposed into irreducible representations of  $S_n$ , corresponding to Young diagrams with  $n$  boxes, then  $V_{n+1}$  is obtained from  $V_n$  by adding a box to the first row of each Young diagram, if  $n$  is sufficiently large. Thus  $V_{n+1}$  and  $V_n$  are not isomorphic, but it is still true that  $V_{n+1}$  is determined by  $V_n$  in a predictable manner. Church proved the following theorem [1]:

**Theorem:** (Church) *Let  $M$  be a connected oriented manifold of dimension  $\geq 2$ . For any fixed  $k \geq 0$ , the cohomology groups  $H^k(F(M, n), \mathbb{Q})$  are representation stable.*

Taking  $S_n$ -invariants recovers a rational version of McDuff's theorem, also for compact  $M$ , even though integral homological stability fails in general when  $M$  is compact.

Church’s proof uses the spectral sequence of Totaro. The spectral sequence of Theorem A exists in much greater generality and can be used to give a significant generalization of Church’s theorem. Let  $M$  be a space, and let  $\mathcal{A}$  be a finite collection of closed subspaces  $A_i \subset M^{n_i}$ . We define a configuration space  $F_{\mathcal{A}}(M, n)$ , parametrizing  $n$  ordered points on  $M$  “avoiding all  $\mathcal{A}$ -configurations”. For instance, if  $\mathcal{A}$  consists only of the diagonal inside  $M^2$ , then  $F_{\mathcal{A}}(M, n)$  is the usual configuration space of distinct ordered points on  $M$ .

**Theorem B.** [5] *Suppose that there exists  $d \geq 2$  such that  $H_c^d(M, \mathbb{Q}) \cong \mathbb{Q}$  and cohomology vanishes above this degree. Under “mild” assumptions on the collection  $\mathcal{A}$ , the cohomology groups  $H_c^{k+nd}(F_{\mathcal{A}}(M, n), \mathbb{Q})$  are representation stable.*

For example, if  $M$  is an irreducible algebraic variety, then an *arbitrary* collection  $\mathcal{A}$  of closed subvarieties will satisfy the “mild” assumptions of the theorem. If  $M$  is an oriented manifold of dimension  $d$ , then this result recovers Church’s theorem by Poincaré duality; the degree shift in the theorem is necessary if we wish to recover Church’s result. I want to emphasize how few assumptions are made about the space  $M$  — to my knowledge, all previous results about homological stability for configuration spaces have restricted their attention to *manifolds*.

Vakil and Wood [7] introduced certain configuration spaces  $\overline{w}_{\lambda}^c(M)$ , depending on a partition  $\lambda$ . For a suitable choice of  $\mathcal{A}$ , one has  $F_{\mathcal{A}}(M, n)/S_n = \overline{w}_{\lambda}^c(M)$ , so Theorem B implies in particular a homological stability theorem for the spaces  $\overline{w}_{\lambda}^c(M)$  as  $n \rightarrow \infty$ , which gives a proof of [7, Conjecture F]. This conjecture has previously been proven by Kupers–Miller–Tran [3] when  $M$  is a smooth oriented manifold. However, we obtain the stronger assertion of representation stability, and by working with compact support cohomology we need very few assumptions on the space  $M$ .

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**Measures of irrationality for hypersurfaces of large degree**

ROBERT LAZARSFELD

(joint work with Lawrence Ein and Brooke Ullery)

Consider a smooth complex hypersurface

$$X = X_d \subseteq \mathbb{P}^{n+1}$$

of degree  $d$  and dimension  $n$ . When  $d$  is small compared to  $n$ , then it is a very interesting and subtle question — which has seen a great deal of recent interest and progress – to determine whether or not  $X$  is rational or unirational. On the other hand, when  $d \geq n + 2$ , then it is elementary that  $X$  is irrational. We are interested in trying to understand “how irrational” is such a hypersurface when  $d$  is large.

In the case of an irreducible projective curve  $C$ , the natural measure of irrationality is the gonality of  $C$ . Recall that the *gonality*  $\text{gon}(C)$  the least degree of a branched covering

$$C' \longrightarrow \mathbb{P}^1,$$

where  $C'$  is the normalization of  $C$ . Thus

$$\text{gon}(C) = 1 \iff C \approx_{\text{birat}} \mathbb{P}^1,$$

and it is profitable in general to view  $\text{gon}(C)$  as measuring the failure of  $C$  to be rational.

Several authors have proposed and studied analogous measures of irrationality for an irreducible complex projective variety  $X$  of arbitrary dimension  $n$ . We will be principally concerned here with three of these – the *degree of irrationality*, the *connecting gonality*, and the *covering gonality* of  $X$  – defined as follows:

$$\text{irr}(X) = \min \left\{ \delta > 0 \mid \begin{array}{l} \exists \text{ degree } \delta \text{ rational covering} \\ X \dashrightarrow \mathbb{P}^n \end{array} \right\};$$

$$\text{conn. gon}(X) = \min \left\{ c > 0 \mid \begin{array}{l} \text{General points } x, y \in X \text{ can be connected} \\ \text{by an irreducible curve } C \subseteq X \text{ with} \\ \text{gon}(C) = c. \end{array} \right\};$$

$$\text{cov. gon}(X) = \min \left\{ c > 0 \mid \begin{array}{l} \text{Given a general point } x \in X, \exists \text{ an} \\ \text{irreducible curve } C \subseteq X \text{ through } x \text{ with} \\ \text{gon}(C) = c. \end{array} \right\}.$$

(Note that the curves  $C$  computing the connecting and covering gonalities are allowed to be singular.) Thus

$$\begin{aligned} \text{irr}(X) = 1 &\iff X \text{ is rational,} \\ \text{conn. gon}(X) = 1 &\iff X \text{ is rationally connected,} \\ \text{cov. gon}(X) = 1 &\iff X \text{ is uniruled,} \end{aligned}$$

and in general one has the inequalities

$$(1) \quad \text{cov. gon}(X) \leq \text{conn. gon}(X) \leq \text{irr}(X).$$

The integer  $\text{irr}(X)$  is perhaps the most natural generalization of the gonality of a curve, but  $\text{cov. gon}(X)$  often seems to be easier to control.

The question then is what one can say about these integers in the case of smooth hypersurfaces  $X_d \subseteq \mathbb{P}^{n+1}$  of large degree. Bastianelli, Cortini and De Poi [1] proved that when  $d \geq n + 3$  then

$$d - n \leq \text{irr}(X) \leq d - 1.$$

They proved that if  $X$  is a very general surface of degree  $d \geq 5$  or threefold of degree  $d \geq 7$ , then in fact

$$\text{irr}(X) = d - 1.$$

They conjectured that (\*) remains true in all dimensions  $n$  provided that  $d \geq 2n + 1$ .

I discussed two main results:

**Theorem 1.** *For any smooth  $X_d \subseteq \mathbb{P}^{n+1}$  as above,*

$$\text{cov. gon}(X) \geq d - n.$$

**Theorem 2.** *In all dimensions  $n$ , if  $X$  is very general of degree  $d > \frac{5}{2}n$ , then*

$$\text{irr}(X) = d - 1.$$

In an appendix to [2], Bastianelli and De Poi subsequently showed that in fact the conclusion holds in the full conjectured range  $d \geq 2n + 1$ .

The main idea is that all of the measures of irrationality are controlled by the birational positivity of the canonical bundle of  $X$ . Theorem 1 is extremely elementary, while the proof of Theorem 2 builds on the nice arguments of [1] involving order one congruences of lines.

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**Unirationality of moduli spaces of special cubic fourfolds  
and K3 surfaces**

LEV BORISOV, FOLLOWING HOWARD NUER

This talk was a report on the work of Howard Nuer, who was a PhD student advised by the speaker.

One of the extensively studied families of algebraic varieties are the cubic fourfolds in  $\mathbb{P}^5$ , i.e. solution spaces  $X$  of homogeneous degree three equations in 6 complex variables. Up to linear changes of coordinates, these are parametrized by a moduli space of dimension 20. A generic member of this 20-dimensional family is unirational, but is expected to be non-rational, which is a long-standing open problem in algebraic geometry.

It is known that for a very general  $X$

$$H^{2,2}(X) \cap H^4(X, \mathbb{Z}) = \mathbb{Z}H^2$$

where  $H$  is the restriction of the hyperplane class of  $\mathbb{P}^4$  to  $X$ . The 20-dimensional family of  $X$  contains a countable number of irreducible 19-dimensional subfamilies  $C_d$  indexed by positive integers  $d > 6$  which satisfy  $d \equiv 0, 2 \pmod{6}$ . Specifically, these are families of  $X$  where the

$$H^{2,2}(X) \cap H^4(X, \mathbb{Z})$$

contains a copy of  $\mathbb{Z}^2 = \mathbb{Z}H^2 + \mathbb{Z}A$  with some additional class  $A$  such that the discriminant of the intersection form of  $X$  restricted to  $\mathbb{Z}H^2 + \mathbb{Z}A$  is equal to  $d$ .

These moduli spaces  $C_d$  have been investigated by Hassett in [3]. For example,  $C_8$  may be described as the locus of cubic fourfolds  $X$  that contain a plane  $\mathbb{P}^2 \subset \mathbb{P}^5$ . The paper [2] gives analogous description of  $C_d$  for  $d \leq 38$  by requiring  $X$  to contain a blowup of  $\mathbb{P}^2$  at a certain number of points in general position, embedded into  $\mathbb{P}^5$  by the full linear system of dimension 6. Moreover, for  $d = 44$ , the space  $C_{44}$  can be described as the locus of  $X$  that contain an Enriques surface  $S$  in its Fano embedding.

One consequence of the construction is that moduli spaces  $C_d$  for  $d \leq 38$  and  $d = 44$  are unirational, where in  $d = 44$  case one needs to use the result of Verra, [5]. The case  $d = 42$  has been recently studied by Lai in [1] by using singular rational scrolls. As an interesting observation, for  $d = 26$  Nuer's paper implies that the moduli space of K3 surfaces with polarization of degree 26 is unirational, which has been previously suspected, but not proved.

As  $d$  grows, the moduli spaces  $C_d$  eventually become of general type. Specifically, this is known for  $d > 200$  and many smaller values of  $d$ , see [4]. The precise range of unimodular  $C_d$  is presently unknown, but it is reasonable to expect that it extends beyond  $d = 44$ .

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## Rational curves on hypersurfaces

ERIC RIEDL

(joint work with David Yang)

One way to understand a variety is to study the curves on that variety. Rational curves, being particularly simple curves, are particularly useful in understanding a variety, and there are many open questions about the rational curves on a variety. Hypersurfaces are a particularly simple class of varieties, and form a natural class of examples for studying rational curves on varieties.

Let  $X \subset \mathbb{P}^n$  be a very general hypersurface of degree  $d$ . Let  $R_e(X)$  be the space of degree  $e$  integral rational curves on  $X$ . Then a simple dimension count leads to the following natural guess for the dimension of  $R_e(X)$ , where a negative dimension means that  $R_e(X)$  is empty.

**Conjecture 1.** *If  $X \subset \mathbb{P}^n$  is a very general degree  $d$  hypersurface, then  $\dim R_e(X) = e(n - d + 1) + n - 4$ , for  $(n, d) \neq (3, 4)$ .*

This question has been studied by many different people. In the general type range  $d \geq n + 2$ , work of Clemens [2], Ein [4], Voisin [8], Pacienza [7], and Clemens and Ran [3] proves Conjecture 1 for  $d \geq 2n - 3$  and  $n \geq 6$ . We improve this result.

**Theorem 1.** *Conjecture 1 holds for  $d \geq \frac{3n+1}{2}$ . That is, if  $X \subset \mathbb{P}^n$  is a very general degree  $d$  hypersurface with  $2n - 3 \geq d \geq \frac{3n+1}{2}$ , then  $X$  contains lines but no other rational curves.*

In the Calabi-Yau range, the question appears to be very difficult. The result is false for the very general quartic K3 surface in  $\mathbb{P}^n$  by a result known to Mumford: a very general quartic in  $\mathbb{P}^3$  will contain rational curves. These rational curves have been a subject of much interest, see for example [6].

In the Fano range, previous work of Harris, Roth and Starr [5], and Beheshti and Kumar [1] had proven Conjecture 1 for  $d \leq \frac{2n+2}{3}$ . We improve this result as well, settling all but the  $d = n$  and  $d = n - 1$  cases in the Fano range.

**Theorem 2.** *Conjecture 1 holds for  $d \leq n - 2$ . In fact, for  $d \leq n - 2$  and  $X \subset \mathbb{P}^n$  a general hypersurface, the Kontsevich space  $\mathcal{M}_{0,0}(X, e)$  (and hence,  $R_e(X)$ ) is irreducible of the expected dimension for all  $e$ .*

In the Fano range, all of the known results are proven using Bend-and-Break, which requires having a large family of rational curves that can be broken up. There are two crucial new ideas in the proof of Theorem 2. The first is to work in families of hypersurface, using all of the rational curves from all of the hypersurfaces in the family to apply Bend-and-Break. However, this requires working with singular hypersurfaces, which requires extending the proof techniques from [5] to singular hypersurfaces.

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**The  $F$ -splitting ratio of a seminormal affine toric variety**

MILENA HERING

(joint work with Kevin Tucker)

The Frobenius morphism is a useful tool for studying classical questions in algebraic geometry. The existence of a global splitting of the Frobenius morphism on a projective algebraic variety has strong consequences, for example, the vanishing of all higher cohomology of every ample line bundle. The existence of splittings that are compatible with the diagonal implies that every ample line bundle is projectively normal, and similar criteria exist for the section ring of every line bundle to be Koszul. One application is that homogeneous coordinate rings of Schubert varieties in flag varieties are Koszul. This theory has been developed by Ramanathan, Mehta, and Inamdar.

In commutative algebra, the Frobenius morphism is also used to study local properties of rings. In this report, we focus on the  $F$ -signature, or its more refined cousin, the  $F$ -splitting ratio. They are measures of the singularities of a ring of characteristic  $p$  defined using the Frobenius endomorphism. We review the computation of the  $F$ -signature of a normal semigroup ring, and compute the  $F$ -splitting ratio of a seminormal semigroup ring.

Let  $R$  be a reduced Noetherian local or graded ring of prime characteristic  $p$  with perfect residue field. The powers of Frobenius act on such a ring, and we let  $F_*^e R$  be the  $R$ -module whose underlying set is  $R$  with module structure given by  $r \star s = r^{p^e} s$ . We assume that  $R$  is  $F$ -finite, i.e., that  $F_*^e R$  is module finite over  $R$ . Then  $R = R^{a_e} \oplus M_e$ , where  $M_e$  has no free direct summands.

Tucker proved in [6] that the limit

$$s(R) := \lim_{e \rightarrow \infty} \frac{a_e}{p^{ed}}$$

exists. It is called the  $F$ -signature of  $R$  and was originally defined by Huneke and Leuschke. It is a measure of the singularities of  $R$ . For example  $s(R) = 1$  if and only if  $R$  is regular. And if  $s(R) > 0$ , then  $R$  is normal and Cohen-Macaulay. Moreover, if  $R$  is the invariant ring of a small finite group  $G$  acting on a regular local ring, then  $s(R) = \frac{1}{|G|}$ . Moreover, if  $s(R) > 0$  and  $R$  is a Henselian domain, then the reciprocal of the  $F$ -splitting ratio is an upper bound for the size of the local étale fundamental group of the punctured spectrum, [7], mirroring a result in characteristic zero that the étale local fundamental group of a germ of a KLT singularity is finite [8].

The  $F$ -signature of an affine toric ring was computed Bruns, Singh, and v. Korff [3, 5, 4]. To see how it works, we need to introduce a bit of notation. Let  $M \cong \mathbb{Z}^n$  be a lattice and let  $C \subset M_{\mathbb{R}}$  be a rational polyhedral cone. Then  $S = M \cap C$  is a finitely generated normal semigroup. We denote by  $k[S]$  the associated semigroup ring. It is a normal ring. Let  $\sigma \subset N_{\mathbb{R}}$  be the dual cone to  $C$ . Then  $\sigma = \langle v_1, \dots, v_r \rangle$ , where  $v_i$  are the primitive generators of the rays of  $\sigma$ . To  $S$  we associate a polytope  $P_S := \{u \in M \mid 0 \leq \langle u, v_i \rangle \leq 1 \text{ for } 1 \leq i \leq r\}$ . Then  $s(k[S])$  is the lattice Volume of  $P_S$ .

When  $s(R) = 0$ , we can define a refined version of the  $F$ -signature, called the  $F$ -splitting ratio. Indeed, Tucker, based on work of Aberbach and Enescu [1], shows that for  $R$  as above with  $R$   $F$ -split, (i.e.,  $a_1 \geq 1$ ), there exists a positive integer  $\delta$ , called the *splitting dimension*, such that the limit

$$r(R) := \lim_{e \rightarrow \infty} \frac{a_e}{p^{e\delta}}$$

exists. Moreover, this limit is positive [2].

We now consider a semigroup ring that is not necessarily normal. If  $k[S]$  is  $F$ -split, then  $k[S]$  must be a seminormal ring. So for our purposes it suffices to consider seminormal rings. There is a combinatorial condition on  $S$  that determines whether  $k[S]$  is seminormal. Let  $C = \mathbb{R}_{\geq 0} S$ , be the cone generated by  $S$ . Then  $S$  is seminormal if for every face  $D$  of  $C$ , there is a sublattice  $M_D$  of  $M$  such that  $S \cap \text{int}(D) = M_D \cap \text{int}(D)$ . One can then show that  $k[S]$  is seminormal if and only if  $S$  is seminormal. Moreover,  $k[S]$  is  $F$ -split if and only if, for every face  $D$  of  $C$ ,  $p$  does not divide the index  $[M : M_D]$ . We call a face  $D$  *relatively unsaturated* (RUF) if

$$M_D \subsetneq \bigcap_{D \not\leq D'} M_{D'}.$$

We show that for a  $F$ -split seminormal affine semigroup ring  $k[S]$ , the splitting dimension is given by  $\delta = \dim \bigcap_{DRUF} D$  and the  $F$ -splitting ratio by

$$r(k[S]) = \text{Vol} \left( P_{\overline{S}} \cap \left( \bigcap_{DRUF} D \right) \right),$$

where the volume is taken with respect to the lattice  $\bigcap_{DRUF} M_D$ .

We also compute  $\text{Hom}(R, F_*^e R)$ , even in the case when  $k[S]$  is not  $F$ -split, and we use this to compute the test ideal.

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## Level structures on abelian varieties and the conjectures of Lang and Vojta

DAN ABRAMOVICH

(joint work with Anthony Várilly-Alvarado, Keerthi Madapusi Pera)

Before Mazur proved in 1977 his theorem on the possible torsion groups on elliptic curve over  $\mathbb{Q}$ , see [5], Manin had shown in 1969 that for a fixed number field  $K$  and prime  $p$ , the  $p$ -primary torsion of elliptic curves over  $K$  was uniformly bounded, [4].

In this lecture I discussed results of a similar nature, conditional on the conjectures of Lang and Vojta, about level structures on abelian varieties of higher dimensions.

Lang’s conjecture says that on a positive dimensional variety  $X$  of general type over a number field  $K$ , the set of rational points  $X(K)$  is not Zariski dense. Vojta’s conjecture is a more delicate conjecture relating heights and certain counting functions.

**Theorem 1** (A-VA,[1, Theorem 1.1]). *Assume Lang’s conjecture holds true. Fix a number field  $K$ , a dimension  $g$ , and a prime  $p$ . There is a number  $N$  such that for any  $r$  such that  $p^r > N$  no abelian variety of dimension  $g$  over  $K$  has full level  $p^r$ .*

**Theorem 2** (A-VA,MP, [2]). *Assume Vojta’s conjecture holds true. Fix  $K$  and  $g$  as above. There is a number  $N$  such that for any prime  $p > N$  no abelian variety of dimension  $g$  over  $K$  has full level  $p$ .*

In these theorems, a full level- $m$  on an abelian variety is an isomorphism of  $(\mathbb{Z}/m\mathbb{Z})^g \times \mu_m^g \simeq A[m]$ , the group of  $m$ -torsion points of  $A$ .

The classical algebraic geometry result underlying this is the following: fix  $g$ , and let  $\mathcal{A}_g$  be the moduli space of principally polarized abelian varieties of dimension  $g$ , and  $\mathcal{A}_g^{[m]}$  the space principally polarized abelian varieties with full level- $m$ . For a subvariety  $X$  of  $\mathcal{A}_g$  we denote by  $X_m$  its inverse image in  $\mathcal{A}_g^{[m]}$ .

**Theorem 3** (A-VA,[1, Theorem 1.3]). *There is an integer  $m_X$  such that if  $m > m_X$  then  $X_m$  is of general type.*

This theorem relies on a theorem of Zuo [6, Theorem 0.1], and the fact that the covering  $\mathcal{A}_g^{[m]} \rightarrow \mathcal{A}_g$  is highly ramified at infinity.

We note that during the week of the meeting the following stronger result appeared on the archive:

**Theorem 4** (Y. Brunebarbe, [3, Theorem 1.6]). *There is  $m_0$  such that if  $m > m_0$  every subvariety of  $\mathcal{A}_g^{[m]}$  is of general type.*

Either of these two results can be used to prove Theorem 1, by working on  $\mathcal{A}_g^{[p^r]}$ , which is a scheme as soon as  $p^r > 3$ . On the other hand, Theorem 2 requires working directly with points of the stack  $\mathcal{A}_g$  (coming from points of  $\mathcal{A}_g^{[p]}$  for high  $p$ ). This relies on a close study of the boundary of  $\mathcal{A}_g^{[m]}$  provided by Keerthi Madapusi Pera, and on our careful application of Vojta’s conjecture to algebraic stacks.

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**Towards Beauville's splitting principle and Motivic hyperkähler resolution conjecture for generalised Kummer varieties**

LIE FU

(joint work with Charles Vial, Zhiyu Tian)

Given a smooth projective variety  $M$  endowed with an action of a finite group  $G$ , following Jarvis-Kaufmann-Kimura [3] and Fantechi-Göttsche [2], we define the orbifold motive (or Chen-Ruan motive) of the quotient stack  $[M/G]$  as an algebra object in the category of Chow motives. Inspired by Ruan [4], one can formulate a motivic version of his Cohomological Hyperkähler Resolution Conjecture (CHRC). I present this conjecture in two situations associated to an abelian surface  $A$  and a positive integer  $n$ . Case(A) concerns Hilbert schemes of points of  $A$ : the Chow motive of  $A^{[n]}$  is isomorphic as algebra objects, up to a suitable sign change, to the orbifold motive to the quotient stack  $[A^n/\mathfrak{S}_n]$ . Case (B) for generalised Kummer varieties: the Chow motive of the generalised Kummer variety  $K_n(A)$  is isomorphic as algebra objects, up to a suitable sign change, to the orbifold motive of the quotient stack  $[A_0^{n+1}/\mathfrak{S}_{n+1}]$ , where  $A_0^{n+1}$  is the kernel abelian variety of the summation map  $A^{n+1} \rightarrow A$ . In particular, these result give complete descriptions of the Chow motive algebras (resp. Chow rings) of  $A^{[n]}$  and  $K_n(A)$  in terms of the first Chow motive of  $A$  (resp.  $CH^*(A)$  the Chow ring of  $A$ ). As an application, we provide multiplicative Chow-Künneth decompositions for Hilbert schemes of abelian surfaces and for generalised Kummer varieties. In particular, we have a multiplicative direct sum decomposition of their Chow rings with rational coefficients, which are supposed to be the splitting of the conjectural Bloch-Beilinson-Murre filtration. The existence of such a splitting for holomorphic symplectic varieties is conjectured by Beauville [1].

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## Complex varieties with infinite Chow groups modulo 2

BURT TOTARO

The talk discussed the following result [7]:

**Theorem 1.** *Let  $X$  be a very general principally polarized complex abelian 3-fold. Then the Chow group  $CH^2(X)$  of codimension-2 cycles on  $X$  is infinite modulo every prime number  $l$ .*

In particular, the theorem shows for the first time that there are complex varieties whose Chow groups modulo 2 are infinite. This also gives the first complex varieties  $X$  for which the Witt group  $W(X)$  of bundles with quadratic forms is infinite. The approach combines several methods in the theory of algebraic cycles into a machine that should be applicable more widely.

Rosenschon and Srinivas had proved the theorem for  $l$  at least some (unknown)  $l_0$  [5]. Schoen gave the first examples of complex varieties with infinite Chow groups modulo a prime number in 2000 [6]. In particular, for the product  $X$  of 3 copies of the Fermat elliptic curve  $x^3 + y^3 = z^3$  in  $\mathbf{P}_{\mathbf{Q}}^2$ , Schoen showed that  $CH^2(X_{\mathbf{Q}})/l = CH^2(X_{\mathbf{C}})/l$  is infinite for all primes  $l \equiv 1 \pmod{3}$ .

By taking products of  $X$  with other varieties, we find that the  $l$ -torsion subgroup  $CH^i(X)[l]$  and the “cotorsion”  $CH^i(X)/l$  can be infinite in all dimensions, except for the known exceptions:

**Corollary 1.** *Given integers  $0 \leq i \leq n$  and any prime number  $l$ , there is a smooth complex projective variety of dimension  $n$  with  $CH^i(X)[l]$  infinite, and also one with  $CH^i(X)/l$  infinite, except in the following cases, which are always finite:  $CH^0(X)[l]$ ,  $CH^0(X)/l$ ,  $CH^1(X)[l]$ ,  $CH^1(X)/l$ ,  $CH^2(X)[l]$ ,  $CH^n(X)[l]$ , and  $CH^n(X)/l$ .*

The hard cases in that list are the finiteness of  $CH^2(X)[l]$ , by Bloch and Merkurjev-Suslin [3], and the finiteness of  $CH^n(X)[l] = CH_0(X)[l]$ , by Roitman.

The theorem is proved using a special feature of dimension 3: a general principally polarized abelian 3-fold  $X$  is the Jacobian of a curve  $C$  of genus 3. As a result, there is an “interesting” 1-cycle on  $X$ , the *Ceresa cycle*  $C - C^-$ . (After choosing a point  $p \in C(\mathbf{C})$ , the curve  $C$  embeds in its Jacobian  $X$ . We write  $C^-$  for the image of  $C \subset X$  under the automorphism  $-1: X \rightarrow X$ .) The Ceresa cycle maps to zero in the homology of  $X$ ,  $H_2(X, \mathbf{Z})$ . Ceresa showed that for a very general curve  $C$  of any genus  $g \geq 3$ , writing  $X$  for the Jacobian  $J(C)$ , the Ceresa cycle is nonzero in  $CH_1(X)/(\text{alg. eq.}) \otimes \mathbf{Q}$  [2]. Thus the *Griffiths group* of 1-cycles on  $X$  (cycles algebraically equivalent to zero modulo cycles homologically equivalent to zero) is nonzero, in fact not torsion.

Nori went further, showing that for a very general principally polarized abelian 3-fold  $X$  over  $\mathbf{C}$ , the Griffiths group of 1-cycles tensor  $\mathbf{Q}$  has infinite dimension as a  $\mathbf{Q}$ -vector space [4]. The point is that  $X$  is isogenous to infinitely many non-isomorphic principally polarized abelian 3-folds  $X_1, X_2, \dots$ . By the special feature of dimension 3, these other abelian 3-folds are also Jacobians of curves  $C_1, C_2, \dots$  of dimension 3. So we have a Ceresa cycle on each  $X_i$ . Pulling it back by the

isogeny  $X \rightarrow X_i$  gives infinitely many 1-cycles on  $X$ , and Nori showed that these cycles span an infinite-dimensional  $\mathbf{Q}$ -linear subspace of the Griffiths group tensor  $\mathbf{Q}$ .

Finally, for every prime number  $l$ , we want to show that these infinitely many 1-cycles on  $X$  are in fact independent in  $CH^1(X)/l^r$  for some  $r \geq 1$ . That implies that  $CH^1(X)/l^r$  is infinite, and hence that  $CH^1(X)/l$  is infinite.

The idea is to use an  $l$ -adic Abel-Jacobi map for 1-cycles on  $X$ , analogous to the complex Abel-Jacobi map introduced by Griffiths. A crucial step uses ideas from the work of Bloch-Kato and Bloch-Esnault on  $p$ -adic Hodge theory [1]. Namely, in order to show that the coniveau filtration is nontrivial modulo  $l$  on a complex abelian 3-fold  $X$ , one has to reduce  $X$  to an abelian 3-fold  $Y$  in characteristic  $l$ , and use that  $H^0(Y, \Omega^3)$  is not zero. Hodge theory for complex varieties would give information about the coniveau filtration tensor  $\mathbf{Q}$ , but that is not enough in order to study Chow groups modulo a prime number  $l$ .

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### Extremal varieties of general type in all dimensions

BANGERE PURNAPRAJNA

(joint work with Jungkai Chen and Francisco J. Gallego)

Relations among fundamental invariants play an important role in algebraic geometry. For a minimal surface of general type, it is a well known result of Noether that  $K_S^2 \geq 2p_g - 4$ . The surfaces for which  $K_S^2 = 2p_g - 4$  have been studied by Horikawa in his well known work (see [Hor 1].) For higher dimensions, it is known that a  $n$ -dimensional minimal Gorenstein variety of general type whose image under the canonical map is of maximal dimension satisfies  $K_X^n \geq 2(p_g - n)$ . We investigate the very interesting extremal situation  $K_X^n = 2(p_g - n)$ .

**Definition 1.** *A minimal complex projective variety with at worst Gorenstein terminal singularities is said to be a Horikawa variety if it satisfies the following two conditions:*

- (1)  $\dim \varphi_1(X) = n$ ;
- (2)  $K_X^n = 2(p_g(X) - n)$

The classification of Horikawa varieties involves showing the canonical linear system is base point free and they are indeed canonical covers of varieties of minimal degree. The canonical covers of varieties of minimal degree have a significant presence in the geometry of algebraic surfaces and higher dimensional varieties of general type. They occur in a variety of contexts such as determination of very ampleness of linear series, ring generation, deformation theory and construction of varieties with given invariants (see [GP1] [GP2], [G], [GGP2], [GGP3], [Hor 1]). One of the most natural contexts where the canonical covers of minimal degree varieties occur is at the boundary of the geography of surfaces of general type. Our results and those in [Ko] and [Fu], show that this is true for all higher dimensional varieties of general type as well on what can be called the “Noether faces”.

We study various geometric and topological aspects of these extremal varieties. Some very interesting work in this direction has been done by Kobayashi and Fujita. We further carry out the studies. We show that Horikawa varieties are regular. If  $X$  is a Horikawa variety of any dimension, we prove that the general deformation of the canonical morphism, which is generically finite of degree 2, is again canonical and of degree 2. In [GGP2], it is shown that the general deformation of canonical morphism that induces smooth double covers, not necessarily of minimal degree varieties, again remains a double cover. It is not at all the case that a general deformation of canonical morphism of degree 2 onto a rational variety always remains degree 2. In fact, in [GGP1] it is explicitly shown that such is not the case. There are implications of these results to the moduli space of varieties of general type and to a question of Enriques concerning the construction of canonical surfaces in projective space. We prove a structure theorem for a large class of Horikawa varieties. If  $X$  is a Horikawa variety of dimension  $n$  and if the image under the canonical morphism  $Y$  is smooth, then  $X$  is pluriregular. In most cases it possesses a fibration, then fibers  $F$  which are of general type have  $p_g(F) = n$ . Coming to topics in topology of Horikawa varieties, we show that if Horikawa varieties have a finite canonical map, then they are simply connected. More generally, we show that they have finite abelian fundamental group.

It is indeed compelling to note the beauty of the analogy with lower dimensional results, even though methods are different. The analogies are striking, despite the existence of much worse singularities. In view of these recent results, smooth Horikawa varieties can be considered as analogues of genus two curves, especially from the point of view of deformations and moduli. Kobayashi’s results on base point freeness of  $K_X$  (reproved by us with some new variations), the implications of the classification of Horikawa varieties, its regularity and its deformations in this article, show interesting consequences for the moduli space of even threefolds of general type with given  $(K^3, p_g, c_3)$  and  $K_X^3 = 2p_g - 6$ .

Horikawa had shown that for a minimal surface of general type  $X$  on the Noether line,  $p_g(X) \leq 6$  if the image of the canonical morphism was singular. We have a striking analogy of Horikawa’s result in all dimensions greater than 2. Namely, if

$X$  is a Horikawa variety of dimension  $n$  such that the image of the canonical map is singular, then  $p_g(X) \leq (n + 4)$ .

The question of projective normality of pluricanonical systems of a surface of general type has a long history dating back to Kodaira and Bombieri. We show that if  $X$  is a Horikawa variety of dimension  $n$  with an optimal condition on its geometric genus, then  $|mK_X|$  embeds  $X$  as a projectively normal variety if and only if  $m \geq n + 1$ . The standard methods involving Castelnuovo-Mumford regularity, and vanishing theorems do not yield this result. We do in fact show a general statement that follows from the standard methods. But to get the optimal statements, one has to use the essential structure of Horikawa varieties proved in this article and find different methods to prove the result.

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### A probabilistic approach to Noether normalization

DANIEL ERMAN

(joint work with David J. Bruce)

Given a projective variety  $X \subseteq \mathbb{P}^r$  over an infinite field, any generic collection of  $k$  polynomials of degree  $d$  will be a (partial) system of parameters, in the sense that the vanishing locus will have codimension  $k$  on  $X$ . We compute the corresponding probabilities over finite fields, relating this to the numerics of subvarieties in  $X$ . We then apply these probabilistic results to prove the existence of uniform Noether normalizations for projective families over  $\mathbb{Z}$ .

More precisely: let  $X \subseteq \mathbb{P}_{\mathbb{Z}}^r$  be a closed subscheme with homogeneous coordinate ring  $R$ . Write  $X_p$  for the fiber of  $X$  over the prime  $p \in \mathbb{Z}$ . If  $\dim X_p = n$  for all

$p$ , then for some  $d$ , there exist polynomials  $f_0, f_1, \dots, f_n \in R_d$  inducing a finite morphism  $\pi : X \rightarrow \mathbb{P}_{\mathbb{Z}}^n$ .

**Remarks:**

- (1) A similar result holds with  $\mathbb{Z}$  replaced by the polynomial ring over a finite field.
- (2) Over a field, we can choose a vector  $(f_0, \dots, f_n) \in R_d^{n+1}$ , and there is a codimension one condition of choices given by the Chow form which describes the set of vectors that fails to yield a finite map. Thus, working over  $\mathbb{Z}$  instead of over a field, there is a codimension 1 bad locus to avoid, but the base is 1-dimensional. So it will not be easy to find a choice of  $f_0, \dots, f_n$  that work for all fibers simultaneously, and in fact, such choices arise with density zero.
- (3) In contrast with Noether normalization over a field, where the affine and projective cases are often treated simultaneously, the family  $X$  must be projective over  $\mathbb{Z}$ . For example,  $\mathbb{Z}[x]/(3x^2 - 5x) \cong \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{3}]$  is finite type, but not finite, over  $\mathbb{Z}$ .
- (4) The result can fail if we replace  $\mathbb{Z}$  by  $\mathbb{Q}[t]$  or  $\mathbb{Z}[t]$ .

Previous analogues include: [1] and [2] which prove something similar in the case where  $X$  is a normal projective curve over  $\mathbb{Z}$  or more general Dedekind domains; [4], which shows that Noether normalizations of semigroup rings always exist over  $\mathbb{Z}$ ; and [3, Theorem 14.4], which implies that given a family over any base, one can find a Noether normalization over an open subset of the base. A different analogue is Poonen's Bertini theorem over  $\mathbb{Z}$  [5, Theorem 5.1], which is based on similar techniques, but is contingent on the abc conjecture.

**Example 1.** On  $\mathbb{P}_{\mathbb{Z}}^1$  the forms  $f_0 = ax^2 + bxy + cy^2$  and  $f_1 = dx^2 + exy + fy^2$  will determine a finite map  $\pi : \mathbb{P}_{\mathbb{Z}}^1 \rightarrow \mathbb{P}_{\mathbb{Z}}^1$  if and only if the determinant

$$\det \begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ d & e & f & 0 \\ 0 & d & e & f \end{pmatrix} = \pm 1.$$

The above determinant is the resultant of these two forms, and if they are divisible by some prime  $p$ , then the map  $(x, y) \mapsto (f_0(x, y), f_1(x, y))$  will have a base point over  $\mathbb{F}_p$ .

**Example 2.** Let

$$X = [1 : 4] \cup [3 : 5] \cup [4 : 5] = V((4x - y)(5x - 3y)(5x - 4y)) \subseteq \mathbb{P}_{\mathbb{Z}}^1.$$

The fibers are 0-dimensional, so a finite map  $X \rightarrow \mathbb{P}_{\mathbb{Z}}^0$  will be determined by a single polynomial  $f_0$  that restricts to a unit on all of the points simultaneously. No linear form will work. In fact, there exists an  $f_0(x, y)$  restricting to unit on  $X$  if and only if  $\deg f_0$  is divisible by 60.

The proof of our main result is based on a computation of the probability that randomly chosen elements of degree  $d$  form a (partial) system of parameters, over

a finite field. A key new idea in this computation is an adaptation Poonen’s closed point sieve [5], instead sieving over higher dimensional varieties; this computes the desired probability via a zeta function type enumeration of subvarieties of a specified dimension and degree. In each fiber over  $\mathbb{Z}$ , the error in the sieve is bounded using geometric results about the locus of partial systems of parameters, while the global error bound over  $\mathbb{Z}$  relies on a uniform convergence over  $\mathbb{Z}$  obtained via uniform bounds on Hilbert functions.

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## Derived categories of Gushel–Mukai varieties

ALEXANDER PERRY

(joint work with Alexander Kuznetsov)

**Background.** We work over an algebraically closed field  $\mathbf{k}$  of characteristic 0. Let  $V_5$  be a 5-dimensional vector space. A *Gushel–Mukai (GM) variety* is a smooth  $n$ -dimensional intersection

$$X = \text{CGr}(2, V_5) \cap \mathbf{P}^{n+4} \cap Q, \quad 2 \leq n \leq 6,$$

where  $\text{CGr}(2, V_5) \subset \mathbf{P}^{10}$  is the cone over the Grassmannian  $\text{Gr}(2, V_5) \subset \mathbf{P}(\wedge^2 V_5)$  in its Plücker embedding,  $\mathbf{P}^{n+4} \subset \mathbf{P}^{10}$  is a linear subspace, and  $Q \subset \mathbf{P}^{n+4}$  is a quadric hypersurface. Results of Gushel [3] and Mukai [7], recently revisited and generalized in [2], show that this class of varieties coincides with the class of all smooth Fano varieties of Picard number 1, coindex 3, and degree 10, together with the Brill–Noether general polarized K3 surfaces of degree 10.

In dimension four, GM varieties behave very similarly to cubic fourfolds. For instance, both types of fourfolds are unirational, but conjecturally irrational if very general. Further, for fourfolds of either type, there are certain Noether–Lefschetz loci where the “non-special cohomology” is isomorphic to (a Tate twist of) the primitive cohomology of a polarized K3 surface (see [4], [1]). In fact, this condition is conjecturally necessary for rationality.

For a cubic fourfold  $X'$ , rationality is also connected to the structure of the bounded derived category of coherent sheaves on  $X'$ , denoted  $\text{D}^b(X')$ . Namely, Kuznetsov [5] showed there is a semiorthogonal decomposition

$$\text{D}^b(X') = \langle \mathcal{A}_{X'}, \mathcal{O}_{X'}, \mathcal{O}_{X'}(1), \mathcal{O}_{X'}(2) \rangle$$

where  $\mathcal{A}_{X'}$  is a “K3 category”, i.e. has Serre functor  $S_{\mathcal{A}_{X'}} = [2]$  given by the shift-by-2 functor. Kuznetsov conjectured that if  $X'$  is rational, then  $\mathcal{A}_{X'}$  is equivalent to the derived category of a K3 surface. Further, he proved that this condition holds for the known families of rational cubic fourfolds, but does not hold for a very general cubic.

**Results.** Our work extends the parallel between GM and cubic fourfolds to the level of derived categories. In fact, for any GM variety  $X$  (not necessarily a fourfold) we defined a special subcategory of  $D^b(X)$  as follows. Projection from the vertex of  $\text{CGr}(2, V_5)$  gives a morphism  $f: X \rightarrow \text{Gr}(2, V_5)$ , and pulling back the ample generator of the Picard group and the rank 2 tautological subbundle on  $\text{Gr}(2, V_5)$  gives bundles  $\mathcal{O}_X(1)$  and  $\mathcal{U}_X$  on  $X$ . We showed there is a semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{U}_X^\vee, \mathcal{O}_X(1), \mathcal{U}_X^\vee(1), \dots, \mathcal{O}_X(n-3), \mathcal{U}_X^\vee(n-3) \rangle,$$

where  $n = \dim(X)$ . The following result gathers some of the basic properties of the category  $\mathcal{A}_X$ .

**Theorem 1** ([6]). *Let  $X$  be an  $n$ -dimensional GM variety.*

- (1) *The Serre functor of  $\mathcal{A}_X$  is given by  $S_{\mathcal{A}_X} = [2]$  if  $n$  is even, and by  $S_{\mathcal{A}_X} = \sigma \circ [2]$  for an involutive autoequivalence  $\sigma$  of  $\mathcal{A}_X$  if  $n$  is odd.*
- (2) *The Hochschild homology of  $\mathcal{A}_X$  is given by*

$$\text{HH}_\bullet(\mathcal{A}_X) \cong \begin{cases} \mathbf{k}[2] \oplus \mathbf{k}^{22}[0] \oplus \mathbf{k}[-2] & \text{if } n \text{ is even,} \\ \mathbf{k}^{10}[1] \oplus \mathbf{k}^2[0] \oplus \mathbf{k}^{10}[-1] & \text{if } n \text{ is odd.} \end{cases}$$

- (3) *Suppose either that  $n \geq 4$  is even and  $X$  is very general, or that  $n$  is odd. Then  $\mathcal{A}_X$  is not equivalent to the derived category of any variety.*

Part (1) of the theorem shows that, in terms of its Serre functor,  $\mathcal{A}_X$  behaves like the derived category of a K3 or Enriques surface according to whether  $n$  is even or odd. Part (2) shows that this analogy persists at the level of Hochschild homology if  $n$  is even, but breaks down if  $n$  is odd (the Hochschild homology agrees with that of a K3 surface if  $n$  is even, but with that of a genus 10 curve if  $n$  is odd). This discrepancy when  $n$  is odd implies part (3) for such  $n$ , whereas the result for even  $n$  follows by relating it to the existence of algebraic cycles on  $X$  and using the period map.

In dimension four, however, we proved that  $\mathcal{A}_X$  does sometimes coincide with the derived category of a K3 surface. To explain this, we need some terminology. A GM variety  $X$  is called *ordinary* if the linear space  $\mathbf{P}^{n+4}$  appearing in the intersection defining  $X$  does not contain the vertex of  $\text{CGr}(2, V_5)$ . Equivalently,  $X$  can be expressed as  $X = \text{Gr}(2, V_5) \cap Q$ , where  $Q \subset \mathbf{P}^{n+4}$  is a quadric hypersurface inside a linear subspace  $\mathbf{P}^{n+4} \subset \mathbf{P}(\wedge^2 V_5)$ .

**Theorem 2** ([6]). *Let  $X$  be an ordinary GM fourfold which can be expressed as  $X = \text{Gr}(2, V_5) \cap Q$  for a rank 6 quadric  $Q \subset \mathbf{P}^8 \subset \mathbf{P}(\wedge^2 V_5)$ . Then there is a K3 surface  $Y$  such that  $\mathcal{A}_X \simeq D^b(Y)$ .*

The fourfolds in the theorem form a 23-dimensional family (a divisor in moduli), and can be characterized as those ordinary GM fourfolds containing a quintic del Pezzo surface. The K3 surface  $Y$  is actually a GM surface, given explicitly by  $Y = \text{Gr}(2, V_5^\vee) \cap Q^\vee$ , where  $Q^\vee$  is the quadric in  $\mathbf{P}^6 = \mathbf{P}(\ker(Q)^\perp) \subset \mathbf{P}(\wedge^2 V_5^\vee)$  projectively dual to  $Q \subset \mathbf{P}(\wedge^2 V_5)$ . In fact, Theorem 2 verifies a special case of a duality conjecture that we formulated, which gives equivalences between the categories  $\mathcal{A}_X$  for GM varieties of possibly different dimensions. Further, we note that fourfolds as in the theorem are rational. Hence the result can be considered as evidence for the GM analogue of Kuznetsov's rationality conjecture for cubic fourfolds.

Our final result directly connects the K3 categories of GM and cubic fourfolds.

**Theorem 3** ([6]). *Let  $X$  be a generic GM fourfold containing a plane of the form  $\text{Gr}(2, V_3) \subset \text{Gr}(2, V_5)$  for a 3-dimensional subspace  $V_3 \subset V_5$ . Then there is a cubic fourfold  $X'$  such that  $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ .*

The GM fourfolds in the theorem form a 21-dimensional family (codimension 3 in moduli). Projection from the plane  $\text{Gr}(2, V_3)$  maps the GM fourfold  $X$  birationally onto the cubic fourfold  $X'$ , and we use the structure of this birational isomorphism to establish the result. We remark that if  $X$  is a very general fourfold satisfying the assumption of the theorem, then  $\mathcal{A}_X$  is *not* equivalent to the derived category of a K3 surface; thus Theorem 3 is of an essentially different nature than Theorem 2.

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## Global Torelli theorem for cubic fourfolds via matrix factorizations and derived categories of K3 surfaces

DANIEL HUYBRECHTS

### 1. INTRODUCTION

**1.1.** Lets start by recalling the two classical Global Torelli theorems.

i) Ruggiero Torelli: Two smooth complex projective **curves**  $C, C'$  are isomorphic if and only if there exists a Hodge isometry  $H^1(C, \mathbb{Z}) \cong H^1(C', \mathbb{Z})$ .

Note that the existence of just an isomorphism of Hodge structures (not necessarily compatible with the intersection product) is equivalent to the existence of an isomorphism  $\text{Jac}(C) \cong \text{Jac}(C')$  between their Jacobians. The latter leads to  $[S^n C] = [S^n C']$  in  $K_0(\text{Var})$  for  $n$  large enough.

ii) Ilya Pjatecki-Sapiro & Igor Safarevic; Dan Burns & Michael Rapoport: Two complex **K3 surfaces**  $S, S'$  are isomorphic if and only if there exists a Hodge isometry  $H^2(S, \mathbb{Z}) \cong H^2(S', \mathbb{Z})$ .

At this moment, it is not clear what the existence of just an isomorphism of Hodge structures (not respecting the intersection pairing) is saying about  $S$  and  $S'$ . Of course, the Hodge conjecture predicts that every such isomorphism is induced by an algebraic class. This has recently been proved by Buskin [2] in the CM case and the algebraic cycles he constructs do carry geometric information about the relation between  $S$  and  $S'$ .

More recently, a Global Torelli theorem for higher-dimensional analogues of K3 surfaces has been proved.

iii) Misha Verbitsky: Two compact **hyperkähler manifolds**  $Y, Y'$  are isomorphic if and only if there exists a Hodge isometry  $H^2(Y, \mathbb{Z}) \cong H^2(Y', \mathbb{Z})$  which is a monodromy operator. A polarized version was proved by Markman. See [9, 16, 20].

**1.2.** A similar question has been asked for hypersurfaces. More precisely, assume  $X, X' \subset \mathbb{P}^{n+1}$  are smooth hypersurfaces of degree  $d$ . Are  $X$  and  $X'$  isomorphic if there exists a Hodge isometry  $H^n(X, \mathbb{Z})_{\text{prim}} \cong H^n(X', \mathbb{Z})_{\text{prim}}$  between their primitive middle cohomologies?

The answer has to be negative in general, e.g. cubic surfaces  $X, X' \subset \mathbb{P}^3$  have all the same Hodge structure  $H^2(X, \mathbb{Z})$  which is of type  $(1, 1)$ , but for many cases of  $(d, n)$  and generic hypersurfaces this has indeed been proved by Ron Donagi [5] (with improvements by Donagi–Green [6] and Cox–Green [4]):

Two **generic hypersurfaces**  $X, X' \subset \mathbb{P}^{n+1}$  of degree  $d$  are isomorphic if and only if there exists a Hodge isometry  $H^n(X, \mathbb{Z})_{\text{prim}} \cong H^n(X', \mathbb{Z})_{\text{prim}}$ , except possibly for the values  $(d, n) = (3, 2), (4, 4n)$  or when  $d \mid (n + 2)$ .

Very roughly, the idea is to use the period map to deduce an isomorphism between the Jacobi rings  $R_X \cong R_{X'}$  (Donagi's symmetrizer lemma plays a crucial role here) and then use a Mather–Yau type argument to conclude. The Jacobi ring  $R_X$  of a hypersurface  $X$  defined by an equation  $W = 0$  is the graded ring  $R_X = R_W = \mathbb{C}[x_0, \dots, x_{n+1}]/(\partial_i W)$ . As it turns out, this ring determines the isomorphism type of  $X$ .

**1.3.** The exceptions in Donagi’s theorem fall into two classes: For  $d = n + 2$  the hypersurfaces are Calabi–Yau varieties, whereas in all other cases they are Fano.

Of course, for  $(d, n) = (3, 1)$  (elliptic curves) and  $(d, n) = (4, 2)$  (quartic K3 surfaces), the Global Torelli theorem still holds. The case  $(d, n) = (5, 3)$  of quintic threefolds  $X \subset \mathbb{P}^4$  was settled in [21].

The first Fano case (beyond cubic surfaces) is  $(d, n) = (3, 4)$ , the case of **cubic fourfolds**. This case was settled by Claire Voisin [22, 23]: Two smooth cubics  $X, X' \subset \mathbb{P}^5$  (not necessarily generic) are isomorphic if and only if there exists a Hodge isometry  $H^4(X, \mathbb{Z}) \cong H^4(X', \mathbb{Z})$  respecting  $c_1(\mathcal{O}(1))^2$ . Her approach does not use the Jacobi ring and the Mather–Yau theorem, but uses the Global Torelli theorem for K3 surfaces that occur naturally for cubic fourfolds containing a plane.

An alternative proof, replacing cubics containing a plane by cubics with an ordinary double point, has been given by Eduard Looijenga in [15]. See also related work by Radu Laza [14].

Yet another proof was recently given by François Charles [3]. He uses the Hodge isometry (up to sign)  $H^4(X, \mathbb{Z})_{\text{prim}} \cong H^2(F(X), \mathbb{Z})_{\text{prim}}$  (due to Beauville–Donagi) between a smooth cubic fourfold  $X$  and its Fano variety of lines  $F(X)$ . Markman’s polarized version of Verbitsky’s Global Torelli theorem applied to the hyperkähler fourfold  $F(X)$  then yields  $F(X) \cong F(X')$  and together with arguments from classical algebraic geometry (Kleiman) this immediately yields  $X \cong X'$ .

The aim of the talk was to report on work in progress (joint with J. Rennemo) which aims at a new proof of the Global Torelli theorem for cubic fourfolds that makes use of the Mather–Yau result, as in Donagi’s proof. It relies on categories of graded matrix factorizations and the curious relation to deformations of the derived category  $D^b(S)$  of coherent sheaves on a K3 surface  $S$  observed by Kuznetsov [13].

## 2. DERIVED CATEGORIES

Consider a smooth cubic fourfold  $X \subset \mathbb{P}^5$  and its bounded derived category  $D^b(X)$  of coherent sheaves. According to results of Bondal and Orlov, two smooth cubics  $X, X' \subset \mathbb{P}^5$  are isomorphic if and only if there exists an exact  $\mathbb{C}$ -linear equivalence  $D^b(X) \cong D^b(X')$ .

The situation changes drastically when  $D^b(X)$  is replaced by its associated K3 category  $\mathcal{A}_X$ , which Kuznetsov introduced as the full triangulated subcategory of  $D^b(X)$  right orthogonal to the three line bundles  $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$ . This category has a Serre functor which is given by the shift  $E \mapsto E[2]$ . Moreover, for certain cubics  $X$  there does exist a K3 surface  $S$  such that  $\mathcal{A}_X \cong D^b(S)$ .

The category  $\mathcal{A}_X$  encodes essential information about the cubic and one could wonder if it still determines  $X$ . However, situation should rather be compared to the case of quartics  $S, S' \subset \mathbb{P}^3$ , where examples are known with  $D^b(S) \cong D^b(S')$ , but  $S \not\cong S'$ .

For K3 surfaces the relation between  $S$  and its derived category  $D^b(S)$  is completely understood, mostly due to work of Shigeru Mukai and Dmitri Orlov. We know that there exists a  $\mathbb{C}$ -linear equivalence  $D^b(S) \cong D^b(S')$  if and only if there

exists a Hodge isometry  $\tilde{H}(S, \mathbb{Z}) \cong \tilde{H}(S', \mathbb{Z})$  (which by definition is the full cohomology of the K3 surface with  $H^0$  and  $H^4$  to be of type  $(1, 1)$ ), cf. [17, 18]. More geometrically, two K3 surfaces are derived equivalent if and only if they are isomorphic or one is a fine moduli space of stable bundles on the other [10]

The beginning of a parallel theory for the K3 categories  $\mathcal{A}_X$  of cubic fourfolds  $X \subset \mathbb{P}^5$  has been worked out in [1, 11]. In particular, Addington and Thomas introduced a weight two Hodge structure  $\tilde{H}(\mathcal{A}_X, \mathbb{Z})$  associated with any smooth cubic as the topological K-group  $K_{\text{top}}(\mathcal{A})$  (defined as the orthogonal complement of the three classes  $[\mathcal{O}_X], [\mathcal{O}_X(1)], [\mathcal{O}_X(2)] \in K_{\text{top}}(X)$ ) endowed with the pull-back via the Chern character of the Hodge structure on  $H^4(X, \mathbb{Z})$ . They also showed that if  $\mathcal{A}_X \cong \mathbb{D}^b(S)$ , then  $\tilde{H}(\mathcal{A}_X, \mathbb{Z}) \cong \tilde{H}(S, \mathbb{Z})$ .

Note that  $H^4(X, \mathbb{Z})_{\text{prim}}$  (with a global sign change) is isomorphic to a primitive sub-Hodge structure of  $\tilde{H}(\mathcal{A}_X, \mathbb{Z})$  with orthogonal complement isomorphic to the  $A_2$ -lattice (which is purely of type  $(1, 1)$ ).

As for K3 surfaces, the Hodge theoretic approach allows one to show that at most finitely many isomorphism classes of smooth cubics  $X$  realize the same category  $\mathcal{A}_X$ .

### 3. SPECIAL CUBICS

Consider the moduli space  $\mathcal{C}$  of isomorphism classes of smooth cubics. It is a (mildly singular) quasi-projective variety of dimension 20 which contains infinitely many divisors  $\mathcal{C}_d \subset \mathcal{C}$  of special cubics (the union of them is analytically dense). They are characterized by the condition that for the very general cubic  $X \in \mathcal{C}_d$  the algebraic part  $H^{2,2}(X, \mathbb{Z})$  is of rank two and discriminant  $d$ .

These divisors have been first studied by Hassett in [8]. In particular, he proved:

i)  $\mathcal{C}_d \neq \emptyset$  if and only if  $d \equiv 0, 2 \pmod{6}$  and  $d > 6$ .

ii) For a cubic  $X$ , there exists a Hodge isometry  $\tilde{H}(\mathcal{A}_X, \mathbb{Z}) \cong \tilde{H}(S, \mathbb{Z})$  for some K3 surface  $S$  if and only if  $X \in \mathcal{C}_d$  with  $d$  even and  $d/2$  is not divisible by 9 or by any prime  $p \equiv 2 \pmod{3}$ .

iii) Addington and Thomas showed that ii) is compatible with a conjecture by Kuznetsov: At least for generic  $X \in \mathcal{C}_d$  as in ii) there does exist a K3 surface with  $\mathcal{A}_X \cong \mathbb{D}^b(S)$ .

Twisted versions of the above results, crucial for our purpose, have been established in [11]:

iv) There exists a Hodge isometry  $\tilde{H}(\mathcal{A}_X, \mathbb{Z}) \cong \tilde{H}(S, \alpha, \mathbb{Z})$  for some twisted K3 surface  $(S, \alpha \in \text{Br}(S))$  if and only if  $X \in \mathcal{C}_d$  with  $d$  even and in the prime factorization of  $d/2$  all primes  $p \equiv 2 \pmod{3}$  occur with even power.

v) For generic  $X \in \mathcal{C}_d$  as in iv) there exists a twisted K3 surface with  $\mathcal{A}_X \cong \mathbb{D}^b(S, \alpha)$ .

An important observation in the present context is the following:

vi) The set of cubics  $X \in \mathcal{C}$  contained in  $\mathcal{C}_d$  with  $d$  as in iv), but without any  $(-2)$ -classes in  $\tilde{H}^{1,1}(\mathcal{A}_x, \mathbb{Z})$  is dense in  $\mathcal{C}$ .

This can be combined with the main result of [12] (which can be understood as a proof of Bridgeland’s conjecture in the generic twisted case): For a twisted K3 surface  $(S, \alpha)$  without  $(-2)$ -classes in  $\tilde{H}(S, \alpha, \mathbb{Z})$  the action of the group of exact linear autoequivalences  $\text{Aut}(\mathbb{D}^b(S, \alpha))/\mathbb{Z} \cdot [2]$  (modulo double shift) on the cohomology  $\tilde{H}(S, \alpha, \mathbb{Z})$  is faithful.

#### 4. MATRIX FACTORIZATIONS

If the cubic  $X \subset \mathbb{P}^5$  is described by a cubic polynomial  $W \in R := \mathbb{C}[x_0, \dots, x_5]$ , then  $\mathcal{A}_X$  is equivalent to the category of graded matrix factorizations  $\text{MF}(W, \mathbb{Z})$ , cf. [19].

Objects in  $\text{MF}(W, \mathbb{Z})$  are pairs  $(\alpha: K \rightarrow L, \beta: L \rightarrow K(3))$  of morphisms of finitely generated free graded  $R$ -modules  $K$  and  $L$  with  $\beta \circ \alpha = \alpha \circ \beta = W$ . Morphisms are homotopy classes of (periodic) morphisms of (periodic) complexes. This category has a natural triangulated structure with the shift functor given by  $(\alpha: K \rightarrow L, \beta: L \rightarrow K(3))[1] = (-\beta: L \rightarrow K(3), -\alpha: K(3) \rightarrow L(3))$ .

But  $\text{MF}(W, \mathbb{Z})$  also admits a natural exact autoequivalence (the grade shift functor) defined by  $(\alpha: K \rightarrow L, \beta: L \rightarrow K(3))(1) = (\alpha(1): K(1) \rightarrow L(1), \beta: L(1) \rightarrow K(4))$ . Obviously,  $(3) \cong [2]$ . In terms of the K3 category  $\mathcal{A}_X \subset \mathbb{D}^b(S)$ , the autoequivalence can also be described as the composition of the inclusion  $\mathcal{A}_X \hookrightarrow \mathbb{D}^b(S)$ , the tensor product  $\otimes \mathcal{O}_X(1)$ , and the left adjoint of the inclusion  $\mathbb{D}^b(X) \rightarrow \mathcal{A}_X$ .

**Theorem** (joint work in progress with J. Rennemo) *If an exact Fourier–Mukai autoequivalence  $\Phi: \mathcal{A}_X \cong \mathcal{A}_{X'}$  between the K3 categories of two smooth cubics  $X, X' \subset \mathbb{P}^5$  commutes with the grade shift functor, i.e.  $(1)_{X'} \circ \Phi \cong \Phi \circ (1)_X$ , then there exists an isomorphism of graded rings  $R_X \cong R_{X'}$ .*

The naive idea is that under the assumptions  $\Phi$  should induce an equivalence between the quotients  $\mathcal{A}_X/(1)_X \cong \mathcal{A}_{X'}/(1)$  which should be thought of as categories of ungraded matrix factorizations  $\text{MF}(W) \cong \text{MF}(W')$ . Now use Dyckerhoff’s result [7] computing the Hochschild cohomology of  $\text{MF}(W)$  as  $HH(\text{MF}(W)) \cong R_W$  (ungraded). There are various technical issues that have to be addressed. Among others: What really is meant by the quotient? In order to apply the Mather–Yau theorem, one needs a graded isomorphism. Hochschild cohomology may depend on the enhancement.

In our approach we let  $R$  act by multiplication. More precisely, if  $f \in R$  is of degree  $d$ , then multiplication defines a functor transform  $f: \text{id} \rightarrow (d)$ . It is easy to see that the derivatives  $\partial_i W$  act trivially on objects.

In order to reduce to the situation of the theorem, one uses deformation theory and the density of cubics with faithful (up to double shift) cohomological action of  $\text{Aut}(\mathcal{A}_X)$ . This turns the question into a Hodge theoretic problem.

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## The Lang-Vojta conjecture and smooth hypersurfaces over number fields

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(joint work with Daniel Loughran)

Let  $K$  be a number field and let  $S$  be a finite set of finite places of  $K$ . Let  $B = \text{Spec}(\mathcal{O}_K[S^{-1}])$  be the spectrum of the ring of  $S$ -integers of  $K$ .

By a classical result of Hermite, for all integers  $d \geq 1$ , the set of isomorphism classes of finite étale morphisms  $X \rightarrow B$  of degree  $d$  is finite. Inspired by Hermite's theorem, Shafarevich conjectured that, for all  $g \geq 2$ , the set of isomorphism classes of smooth proper curves of genus  $g$  over  $B$  is finite [6]. Shafarevich's conjecture

for smooth proper curves over  $B$  was proven by Faltings in his paper on Mordell’s conjecture [4].

The starting point of this project is a similar finiteness result for Fano varieties of dimension at most two. Indeed, in [5] Scholl proves that the set of isomorphism classes of smooth del Pezzo surfaces over  $B$  is finite. In other words, the set of isomorphism classes of smooth Fano varieties over  $B$  of dimension at most two is finite.

Our first result extends Scholl’s finiteness result to Fano threefolds, under suitable assumptions. To state our result, recall that the geometric index  $r(X)$  of a Fano threefold  $X$  over a field  $k$  is the largest integer  $r \geq 1$  such that  $-K_{X_{\bar{k}}}$  is divisible by  $r$  in  $\text{Pic}(X_{\bar{k}})$ .

**Theorem 1.** *The set of isomorphism classes of smooth Fano threefolds  $X$  over  $B$  with geometric Picard rank  $\rho(X) = 1$  and geometric index  $r(X) \geq 2$  is finite.*

Our next result however shows that Scholl’s result does not generalize in a naive sense to all Fano threefolds.

**Theorem 2.** *Assume that 58 is invertible on  $B$ . The set of isomorphism classes of smooth Fano threefolds with  $\rho(X) = 2$  and  $r(X) = 1$  over  $B$  is infinite.*

It is natural to ask whether the set of  $B$ -isomorphism classes of smooth Fano threefolds over  $B$  with Picard rank one finite. We do not know the answer to this question. For example, is the set of  $B$ -isomorphism classes of smooth Gushel-Mukai Fano threefolds over  $B$  finite?

The hyperbolicity of the stack of Fano threefolds should be closely related to the finiteness of the set of integral points. Indeed, the Lang-Vojta conjecture implies that, if  $\mathcal{X}$  is a finite type scheme over  $\mathbb{Z}$  such that  $\mathcal{X}_{\mathbb{C}}$  is a smooth quasi-projective scheme whose integral closed subvarieties are of log-general type, then  $\mathcal{X}(B)$  is finite.

This brings us to our next result.

**Theorem 3.** *Assume the Lang-Vojta conjecture. If  $d \geq 2$  and  $n \geq 1$ , then the set of  $B$ -isomorphism classes of smooth hypersurfaces of degree  $d$  in  $\mathbb{P}_B^{n+1}$  is finite.*

To prove Theorem 3 we show that the complex algebraic stack  $\mathcal{C}_{(d;n),\mathbb{C}}$  of smooth hypersurfaces of degree  $d$  in  $\mathbb{P}^{n+1}$  is uniformisable by a hyperbolic variety over  $\mathbb{C}$ ; see [1]. That is, there is a smooth affine scheme  $U$  over  $\mathbb{C}$  such that the following statements hold.

- (1) The scheme  $U$  is Brody hyperbolic and all its integral closed subvarieties are of log-general type.
- (2) There is a finite étale morphism  $U \rightarrow \mathcal{C}_{(d;n),\mathbb{C}}$ .

We can explicitly construct  $U$  as follows. For  $\ell \geq 1$  an integer, let  $\mathcal{C}_{(d;n)}^{\ell}$  be the stack of smooth hypersurfaces of degree  $d$  in  $\mathbb{P}^{n+1}$  with a “level  $\ell$  structure”; see [2]. This is an algebraic stack of finite type over  $\mathbb{Z}[1/\ell]$ . One can show that, for all  $\ell \geq 3$ , the stack  $U := \mathcal{C}_{(d;n),\mathbb{C}}^{\ell}$  and the forgetful morphism  $U \rightarrow \mathcal{C}_{(d;n),\mathbb{C}}$  have the desired properties.

If we refer to the statement that “the set of smooth hypersurfaces of degree  $d$  in  $\mathbb{P}_B^{n+1}$  is finite” as the Shafarevich conjecture for smooth hypersurfaces, then Theorem 3 says that the Lang-Vojta conjecture *implies* the Shafarevich conjecture for smooth hypersurfaces.

A first step towards proving the Shafarevich conjecture for smooth hypersurfaces was made in [1].

**Theorem 4.** *The set of  $B$ -isomorphism classes of smooth cubic hypersurfaces in  $\mathbb{P}_B^4$  is finite.*

Furthermore, we obtain a finiteness result for smooth sextics in  $\mathbb{P}^3$ ; see [3].

**Theorem 5.** *The set of  $B$ -isomorphism classes of smooth sextic hypersurfaces in  $\mathbb{P}_B^3$  is finite.*

To prove Theorem 5 we use that a double cover  $Y$  of  $\mathbb{P}_{\mathbb{C}}^3$  ramified precisely along a smooth sextic  $X$  is a smooth Fano threefold with Picard rank one, index one, degree two, and third Betti number 104. We then exploit arithmetic properties of the intermediate Jacobian of  $Y$  and the fact that infinitesimal Torelli holds for  $Y$ ; see [3].

One ingredient in our proof of the Shafarevich conjecture for smooth cubic threefolds is an arithmetic analogue of the well-known global Torelli theorem for smooth cubic threefolds (see [1]), as proven by Clemens-Griffiths over algebraically closed fields of characteristic zero.

**Theorem 6.** *Let  $k$  be a field of characteristic zero. Let  $X$  and  $Y$  be smooth cubic threefolds over  $k$ . If  $\text{Jac}(X)$  and  $\text{Jac}(Y)$  are  $k$ -isomorphic as principally polarized abelian varieties, then  $X$  and  $Y$  are  $k$ -isomorphic.*

To prove Theorem 6, we show that the automorphism group of a smooth cubic threefold  $X$  over  $\mathbb{C}$  acts faithfully on its middle cohomology  $H^3(X(\mathbb{C}), \mathbb{C})$ .

Finally, we have obtained analogous results for complete intersections. Again, as a by-product of our methods, we obtain the following extension of Debarre’s global Torelli theorem for intersections of two quadrics.

**Theorem 7.** *Let  $k$  be a field of characteristic zero and let  $n \geq 2$  be an integer. Let  $X$  and  $Y$  be smooth complete intersections of three quadrics over  $k$  in  $\mathbb{P}_k^{2n}$ . If  $\text{Jac}(X)$  and  $\text{Jac}(Y)$  are  $k$ -isomorphic as principally polarized abelian varieties, then  $X$  is  $k$ -isomorphic to  $Y$ .*

We emphasize that Theorem 7 does not follow formally from Debarre’s result and requires showing that the intermediate Jacobian is representable by schemes (as a morphism from the stack of smooth complete intersections of three quadrics in  $\mathbb{P}^{2n}$  to the stack of principally polarized abelian schemes).

Note that the representability of the intermediate Jacobian by schemes fails for smooth cubic curves (as translation by a 3-torsion point on a smooth cubic curve acts trivially on its Jacobian). Also, the representability of the intermediate Jacobian by schemes fails for odd-dimensional smooth complete intersections of two quadrics in  $\mathbb{P}^{2n+1}$ .

In particular, the analogue of Donagi's global Torelli theorem for odd-dimensional intersections of two quadrics can fail over non-algebraically closed fields. Let  $k$  be a field of characteristic zero and recall that Donagi proved the following. Let  $X$  and  $Y$  be smooth complete intersections of two quadrics over  $k$  in  $\mathbb{P}_k^{2n+1}$ . If  $k$  is algebraically closed and the intermediate Jacobian  $\text{Jac}(X)$  of  $X$  is  $k$ -isomorphic to the intermediate Jacobian  $\text{Jac}(Y)$  of  $Y$  as a principally polarized abelian variety, then  $X$  is  $k$ -isomorphic to  $Y$ .

However, we show that there exist infinitely many pairwise non- $\mathbb{Q}$ -isomorphic smooth complete intersections of two quadrics  $X_1, X_2, \dots$  in  $\mathbb{P}_{\mathbb{Q}}^{2n+1}$  such that their intermediate Jacobians  $\text{Jac}(X_1), \text{Jac}(X_2), \dots$  are  $\mathbb{Q}$ -isomorphic.

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### Rationality and irrationality in families

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(joint work with Nicolas Addington, Alena Pirutka, Yuri Tschinkel, Anthony Várilly-Alvarado)

Is rationality a deformation invariant of smooth complex projective varieties? Precisely, given an irreducible complex variety  $B$  and a smooth projective morphism  $\phi : \mathcal{X} \rightarrow B$ , can there be both rational and irrational fibers of  $\phi$ ? When  $\dim(\mathcal{X}/B) = 1$  the fibers are rational precisely when they have genus zero, which is deformation invariant. In dimension two, a fiber  $\mathcal{X}_b = \phi^{-1}(b)$  is rational if and only if  $h^1(\mathcal{O}_{\mathcal{X}_b}) = h^0(\omega_{\mathcal{X}_b}^2) = 0$ , i.e., if the irregularity and second plurigenus vanish; these are deformation invariants. When  $\dim(\mathcal{X}/B) = 3$  it remains open whether rationality is deformation invariant. Our main result addresses the case of fourfolds:

**Theorem 1** (H–Pirutka–Tschinkel [2]). *There exist an irreducible variety  $B$  and family of smooth projective complex fourfolds  $\phi : \mathcal{X} \rightarrow B$  with both rational and irrational fibers.*

The fibers are hypersurfaces  $X \subset \mathbb{P}^2 \times \mathbb{P}^3$  of bidegree  $(2, 2)$ . Projection onto the first factor gives a quadric surface bundle  $\pi : X \rightarrow \mathbb{P}^2$ ; degenerate fibers are parametrized by an octic plane curve  $D \subset \mathbb{P}^2$ .

The construction of rational examples uses elementary properties of quadric surfaces. Such an  $X$  is rational whenever the generic fiber is rational over  $\mathcal{C}(\mathbb{P}^2)$ ; this holds when  $\pi$  admits a rational section. However, by a classical theorem of Springer such sections exist whenever  $\pi$  admits a multisection  $M \subset X$  of odd degree over  $\mathbb{P}^2$ . The integral Hodge conjecture holds in this case, so it suffices to exhibit a Hodge class

$$M \in H^4(X, \mathbb{Z}) \cap H^2(\Omega_X^2)$$

meeting the fibers in odd degree.

Hodge theory tells us when such a condition holds. Let

$$B \subset \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(2, 2))) \simeq \mathbb{P}^{59}$$

denote the open subset parametrizing smooth hypersurfaces and  $\phi : \mathcal{X} \rightarrow B$  the universal family. Since  $h^{3,1}(X) = 3$ , the locus in  $B$  where a given  $M \in H^4(X, \mathbb{Z})$  is algebraic is a codimension  $\leq 3$  subvariety  $B_M \subset B$ . Thus there is a countably infinite collection of subvarieties  $\cup B_M \subset B$  parametrizing rational examples. (This argument builds on techniques developed by Voisin.)

Irrational fibers are found using the technique of ‘Decomposition of the Diagonal’, as developed by Voisin, Colliot-Thélène, Pirutka, Totaro, etc. Any rational (or stably rational) variety  $Y$  admits a decomposition of the diagonal

$$\Delta_Y \equiv y \times Y + Z$$

where  $y \in Y$  and  $Z$  is a cycle supported on  $Y \times E$  for a closed  $E \subsetneq Y$ .

First, we produce an example

$$X_0 = \{yzs^2 + xzt^2 + xyu^2 + F(x, y, z)v^2 = 0\} \subset \mathbb{P}_{[x,y,z]}^2 \times \mathbb{P}_{[s,t,u,v]}^3$$

$$F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$$

such that the unramified cohomology  $H_{unr}^2(X_0) \neq 0$ . This is computed by an algorithm of Pirutka [3] that takes into account the geometry of the degeneracy curve

$$D_0 = \{x^2y^2z^2F(x, y, z) = 0\} \subset \mathbb{P}^2.$$

It follows that  $X_0$  lacks a decomposition of the diagonal. Assuming the singularities of  $X_0$  are mild—verifying this is the most involved step of the proof—any  $X \rightsquigarrow X_0$  also lacks a decomposition of diagonal and thus is irrational. Thus very general  $b \in B$  correspond to irrational  $\mathcal{X}_b \subset \mathbb{P}^2 \times \mathbb{P}^3$ .

The rationality of cubic fourfolds  $X \subset \mathbb{P}^5$  has been intensively studied. Their moduli space  $\mathcal{C}$  has dimension twenty. We have a number of rational examples:

- Cubic fourfolds  $X$  containing a plane  $P$  admit quadric surface bundles  $\pi : \text{Bl}_P(X) \rightarrow \mathbb{P}^2$ ; these form a divisor  $\mathcal{C}_8 \subset \mathcal{C}$ . As before,  $X$  is rational when  $\pi$  admits an odd-degree multisection. This yields a countably infinite union of codimension-two subvarieties in  $\mathcal{C}$  parametrizing rational examples.
- Cubic fourfolds  $X$  containing a quintic del Pezzo surface are always rational; these are the Pfaffian cubic fourfolds studied by Beauville and Donagi and form a divisor  $\mathcal{C}_{14} \subset \mathcal{C}$ .

We obtain a new class of examples:

**Theorem 2** (Addington–H–Tschinkel–Várilly-Alvarado [1]). *There exist smooth cubic fourfolds  $X$  birational to sextic del Pezzo surface bundles  $\varpi : X' \rightarrow \mathbb{P}^2$ ; these form a divisor  $\mathcal{C}_{18} \subset \mathcal{C}$ . Such an  $X$  is rational provided  $\varpi$  admits a multisection of degree prime to three. We obtain a countably infinite union of codimension-two subvarieties in  $\mathcal{C}$  parametrizing new rational examples.*

Unfortunately, the techniques discussed above are not easily adapted to produce irrational cubic fourfolds. The quadric surface bundles arising from cubic fourfolds have sextic degeneracy curve  $D \subset \mathbb{P}^2$ ; Pirutka’s algorithm never gives non-vanishing unramified cohomology. (Auel, Colliot-Thélène, and Parimala have explained this in conceptual terms.) Voisin has shown that cubic fourfolds in  $\mathcal{C}_{18}$  always admit decompositions of the diagonal. No cubic fourfolds have yet been proven to be irrational.

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