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Arithmetic Geometry

Organised by
Gerd Faltings, Bonn
Johan de Jong, New York
Peter Scholze, Bonn

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Abstract. Arithmetic geometry is at the interface between algebraic geometry and number theory, and studies schemes over the ring of integers of number fields, or their p-adic completions. An emphasis of the workshop was on p-adic techniques, but various other aspects including Hodge theory, Arakelov theory and global questions were discussed.

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Introduction by the Organisers

The workshop *Arithmetic Geometry* was well attended by over 50 participants from various backgrounds. It covered a wide range of topics in algebraic geometry and number theory, with some focus on p-adic questions.

Using the theory of perfectoid spaces and related techniques, a number of results have been proved in recent years. At the conference, Caraiani, Gabber, Hansen and Liu reported on such results. In particular, Liu explained general p-adic versions of the Riemann–Hilbert and Simpson correspondences, and Caraiani reported on results on the torsion in the cohomology of Shimura varieties. This involved the geometry of the Hodge–Tate period map, which Hansen extended to a general Shimura variety, using the results reported by Liu. Moreover, Gabber proved degeneration of the Hodge spectral sequence for all proper smooth rigid spaces over nonarchimedean fields of characteristic 0, or even in families, by proving a spreading out result for proper rigid spaces to reduce to a recent result in p-adic Hodge theory.

Another recurring theme was the theory of p-adic families of automorphic forms, and the corresponding Banach representations of p-adic groups, in the framework of the p-adic local Langlands program. This includes the talks of Andreatta, Colmez and Paškūnas.

In the general framework of p-adic cohomologies, we also had a talk of Jannsen on duality in de Rham–Witt cohomology. Another p-adic topic was Zhang's talk about the Poisson equation on Berkovich spaces, which is expected to have important applications to the study of heights of rational points.

Large progress in recent years was made on K3 surfaces over finite fields, where many results including the Tate conjecture and the unirationality conjecture have been proved, as well as new results on derived equivalences of K3 surfaces, as reported by Liedtke, and Olsson.

A striking recent result of Abe is the existence of p-adic companions on curves over finite fields, extending the result of L. Lafforgue, attaching ℓ-adic sheaves to automorphic representations, to p-adic cohomology. At the conference, Kedlaya gave a talk on the existence of companions for higher-dimensional varieties, by reduction to the case of curves.

For applications of Arakelov geometry, it is important to control certain contributions coming from the archimedean places. Here, Wilms presented new results on Faltings's δ-invariant.

Litt has obtained results on the monodromy representations arising from geometry by using anabelian methods and a reduction to characteristic p argument.

Other topics covered were the Grothendieck–Katz p-curvature conjecture by Kisin, the Sansuc formula for general groups by Conrad, Hodge theory and atypical intersections by Klingler, the construction of G_2-local systems on curves by Katz, and generic vanishing theorems by Bhatt.

During the conference, many active discussions took place. In particular, André finished his proof of the direct summand conjecture, but it was unfortunately too soon to report on it at the meeting.

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Workshop: Arithmetic Geometry

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Abstracts

On the delta invariant in Arakelov geometry

Robert Wilms

In this talk, we presented new formulas for Faltings' delta invariant in Arakelov geometry. This invariant plays a crucial role in the Faltings-Noether formula: Let C be a smooth and projective curve of genus $g \geqslant 1$ defined over a number field K and having semi-stable reduction over $B = \operatorname{Spec} \mathcal{O}_K$. Denote by $\pi\colon \mathscr{C} \to B$ its minimal regular model and write $\bar\omega = \bar\omega_{\mathscr{C}/B}$ for the relative dualizing bundle equipped with its Arakelov metric. Then the Faltings-Noether formula [2, 5] states

$$12\widehat{\deg}\det \pi_*\bar\omega = (\bar\omega,\bar\omega) + \sum_{v\in|B|}\delta(\mathscr{C}_{\bar v})\log \mathrm{N}(v) + \sum_{\sigma\colon K\to\mathbb{C}}\delta(C_\sigma) - 4g[K:\mathbb{Q}]\log 2\pi,$$

where $\det \pi_*\bar\omega_{\mathscr{C}/B}$ is equipped with its Faltings metric, $\delta(\mathscr{C}_{\bar v})$ denotes the number of singularities in the geometric fibre at $v \in |B|$ and $\delta(C_\sigma)$ denotes Faltings' delta invariant of the Riemann surface C_σ obtained by the pullback of C induced by any embedding $\sigma\colon K \to \mathbb{C}$.

To state our formulas for the delta invariant, let X be a compact and connected Riemann surface of genus $g \geqslant 1$. We fix a period matrix Ω of X and we write $\operatorname{Jac}(X) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\cdot\mathbb{Z}^g)$ for the Jacobian of X. We identify $\operatorname{Jac}(X) \cong \operatorname{Pic}_{g-1}(X)$, such that the divisor of the theta function $\theta(z) = \sum_{n\in\mathbb{Z}^g} e^{\pi i({}^{\mathrm{t}}n\Omega n + 2{}^{\mathrm{t}}nz)}$ in $\operatorname{Jac}(X)$ coincides with the divisor Θ of line bundles having global sections in $\mathrm{Pic_{g-1}(X)}$. The norm of the theta function $\|\theta\|\colon \operatorname{Pic}_{g-1} \to \mathbb{R}_{\geqslant 0}$ is given by

$$\|\theta\|(z) = (\det \operatorname{Im}\Omega)^{1/4} e^{-\pi {}^{\mathrm{t}}\mathrm{Im} z (\mathrm{Im}\ \Omega)^{-1}\mathrm{Im} z} \cdot |\theta(z)|\,.$$

In [3], de Jong introduced the function $\|\eta\|\colon \Theta \to \mathbb{R}_{\geqslant 0}$, which is given by

$$\|\eta\|(z) = (\det \operatorname{Im}\Omega)^{(g+5)/4} e^{-(g+1)\pi {}^{\mathrm{t}}\mathrm{Im} z (\mathrm{Im}\ \Omega)^{-1}\mathrm{Im} z} \cdot \left|{}^{\mathrm{t}}(\theta_j)(\theta_{jk})^c(\theta_j)(z)\right|,$$

where (θ_j) denotes the vector of the first partial derivatives of θ and $(\theta_{jk})^c$ the cofactor matrix of the second partial derivatives. Further, we write ν for the Kähler form of the Hodge metric on $\operatorname{Jac}(X)$. We define the following invariants

$$H(X) = \int_{\operatorname{Pic}_{g-1}(X)} \log\|\theta\|\frac{\nu^g}{g!}, \quad \Lambda(X) = \int_\Theta \log\|\eta\|\frac{\nu^{g-1}}{g!}, \quad \varphi(X) = \int_{X^2}(\log G)h^2,$$

where $G\colon X^2 \to R_{\geqslant 0}$ denotes the Arakelov–Green function and $h = c_1(\mathcal{O}_{X^2}(\Delta))$ is the first Chern form of the diagonal bundle. The invariant $\varphi(X)$ is called Zhang–Kawazumi invariant and it satisfies $\varphi(X) \geqslant 0$. Moreover, we obtain the bound $H(X) < -\frac{g}{4}\log 2$ by Jensen's inequality.

Theorem ([6]). (i) It holds $\delta(X) = -24H(X) + 2\varphi(X) - 8g\log 2\pi$. In particular, we always have $\delta(X) > -2g\log 2\pi^4$.

(ii) We have the following relations:

$$\delta(X) = 2(g-7)H(X) - 2\Lambda(X) - 4g\log 2\pi,$$
$$\varphi(X) = (g+5)H(X) - \Lambda(X) + 2g\log 2\pi.$$

In particular, the invariants δ and φ have canonical extensions to indecomposable principally polarised complex abelian varieties.

(iii) The Arakelov-Green function satisfies

$$\log G(P,Q) = \int_{\Theta+P-Q} \log\|\theta\| \frac{\nu^{g-1}}{g!} + \tfrac{1}{2g}\varphi(X) - H(X).$$

In particular, we obtain the upper bound

$$\sup_{P,Q\in X} \log G(P,Q) < \tfrac{1}{4g}\max\left(1, \tfrac{2g+1}{12}\right)\delta(X) + g(4+\log g).$$

We remark, that the formula for the Arakelov–Green function is a more explicit version of Bost's formula in [1]. As an application in Arakelov theory, we can deduce the following bound for the heights of Weierstrass points from a formula by de Jong [4, Theorem 4.3].

Corollary. With the notation as above, let W be the divisor of Weierstrass points on C/K. We may assume, that all Weierstrass points are defined over K. Denote by $h_{\mathrm{Ar}}(P) = (\bar{\omega}, \mathcal{O}_{\mathscr{C}}(P))$ the Arakelov height of a point $P \in C(K)$. Then it holds

$$\max_{P\in W} h_{\mathrm{Ar}}(P) \leqslant \sum_{P\in W} h_{\mathrm{Ar}}(P) < (6g^2+4g+2)\widehat{\deg}\det\pi_*\bar{\omega} + 9g^3[K:\mathbb{Q}].$$

Next, we sketch the idea of the proof of the theorem. First, we remark that (ii) and (iii) can be deduced as consequences of (i). The proof of (i) works in two steps. The first step is to prove the formula directly for hyperelliptic Riemann surfaces. Here, the main ingredient is the following decomposition formula for $\log\|\theta\|$:

$$\log\|\theta\|(P_1+\cdots+P_g-Q) = \sum_{j=1}^{g}\log G(P_j,Q) + \sum_{j<k}\log G(P_j,\sigma(P_k)) + S_g(X),$$

where $\sigma : X \to X$ denotes the hyperelliptic involution and $S_g(X)$ is a certain invariant of Riemann surfaces.

In the second step, we can assume $g \geqslant 3$ and it is enough to show that

$$f(X) = \delta(X) + 24H(X) - 2\varphi(X)$$

is constant as a real-valued function on $\mathcal{M}_g$, the moduli space of compact and connected Riemann surfaces. This would follow if we have $\partial\overline{\partial}f(X) = 0$ on $\mathcal{M}_g$, since $\mathcal{M}_g \cong \mathcal{T}_g/\Gamma_g$, where $\mathcal{T}_g$ is the Teichmüller space, which is contractible and Γ_g is the mapping class group, which is perfect. Hence, every pluriharmonic function on $\mathcal{M}_g$ is the real part of a holomorphic function, but there are no non-constant holomorphic functions on $\mathcal{M}_g$, since the boundary of $\mathcal{M}_g$ in its Satake compactification has codimension at least 2.

To prove $\partial\overline{\partial}f(X) = 0$, let $q\colon \mathcal{X}_g \to \mathcal{M}_g[2]$ be the universal family of compact and connected Riemann surfaces of genus g with level 2 structure and Δ the diagonal in the product $q_2\colon \mathcal{X}_g \times_{\mathcal{M}_g[2]} \mathcal{X}_g \to \mathcal{M}_g[2]$. We define the following $(1,1)$ forms on $\mathcal{M}_g[2]$:

$$\omega_{\mathrm{Hdg}} = c_1(\det q_*\Omega_{\mathcal{X}_g/\mathcal{M}_g[2]}),\ e_1^A = \int_q c_1(T_{\mathcal{X}_g/\mathcal{M}_g[2]})^2,\ \int_{q_2} h^3 = \int_{q_2} c_1(\mathcal{O}_{\mathcal{X}_g^2}(\Delta))^3.$$

Now direct computations give $\frac{\partial\overline{\partial}}{\pi i}\varphi = \int_{q_2} h^3 - e_1^A$ and $\frac{\partial\overline{\partial}}{\pi i}\delta = e_1^A - 12\omega_{\mathrm{Hdg}}$.

The hard part of the proof is to express $\frac{\partial\overline{\partial}}{\pi i}H$ in terms of the above forms. For this purpose, one pulls back the function $\log\|\theta\|$ by the map

$$\mathcal{X}_g^{g+1} \to \mathcal{U}_g, \quad [P_1, \dots, P_{g+1}; X] \mapsto [P_1 + \dots + P_g - P_{g+1} - \alpha_X; \mathrm{Jac}(X)],$$

where $\mathcal{U}_g$ denotes the universal family of abelian varieties with level 2 structure and α is a fixed family of theta characteristics on $\mathcal{X}_g$. This yields

$$\frac{\partial\overline{\partial}}{\pi i}H = \tfrac{1}{2}\omega_{\mathrm{Hdg}} - \tfrac{1}{8}e_1^A + \tfrac{(-1)^{g+1}}{(g!)^2}\left(\int_{pr_{g+1}} c_1(\mathcal{L})^{g+1} + g\int_{pr_g} c_1(\mathcal{L}')^g\right),$$

where $\mathcal{L}$ (resp. $\mathcal{L}'$) is a line bundle on $\mathcal{X}_g^{g+1}$ (resp. $\mathcal{X}_g^g$) given by

$$\mathcal{L} = \bigotimes_{j=1}^{g} pr_j^* T_{\mathcal{X}_g/\mathcal{M}_g[2]} \otimes \bigotimes_{j=1}^{g} pr_{j,g+1}^* \mathcal{O}_{\mathcal{X}_g^2}(\Delta)^{\otimes -1} \otimes \bigotimes_{j<k}^{g} pr_{j,k}^* \mathcal{O}_{\mathcal{X}_g^2}(\Delta),$$

$$\left(\text{resp. } \mathcal{L}' = \bigotimes_{j=1}^{g-1} pr_j^* T_{\mathcal{X}_g/\mathcal{M}_g[2]} \otimes \bigotimes_{j<k}^{g-1} pr_{j,k}^* \mathcal{O}_{\mathcal{X}_g^2}(\Delta)\right).$$

Further, we have $\int c_1(\mathcal{L})^{g+1} = c_1\left(\mathcal{L}^{\langle g+1\rangle}\right)$, where the power $\mathcal{L}^{\langle g+1\rangle}$ is taken in the sense of Deligne pairings. Hence, it remains to expand the power $\mathcal{L}^{\langle g+1\rangle}$, which is only a combinatorial problem. This can be solved by associating graphs to the terms in the expansion of $\mathcal{L}^{\langle g+1\rangle}$, which can be classified and counted.

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On the generic part of the cohomology of compact unitary Shimura varieties

Ana Caraiani
(joint work with Peter Scholze)

Let $G/\mathbb{Q}$ be a connected reductive group. Associated to it, we have the symmetric domain $X := G(\mathbb{R})/K_\infty$, where $K_\infty \subset G(\mathbb{R})$ is a maximal compact subgroup. The symmetric domain X still has an action of $G(\mathbb{R})$. Taking $\Gamma \subset G(\mathbb{Q})$ to be a congruence subgroup, we can form the quotient $X_\Gamma := \Gamma\backslash X$, which is a locally symmetric space for G.

Assume that X_Γ is compact and that Γ is torsion-free. Then X_Γ is a real manifold and we can relate its Betti (singular) cohomology $H^*(X_\Gamma, \mathbb{C})$ to automorphic representations of G. More precisely, Matsushima's formula expresses $H^*(X_\Gamma, \mathbb{C})$ in terms of automorphic representations π of G and in terms of the $(\mathfrak{g}, K_\infty)$ cohomology of π_∞, the archimedean component of π. A theorem of Borel and Wallach in [1] implies that, when π_∞ is a *tempered* representation of $G(\mathbb{R})$, it can only contribute to $H^*(X_\Gamma, \mathbb{C})$ in a certain range of degrees $[q_0, q_0 + l_0]$. Here q_0, l_0 are non-negative integers depending on G, defined as $l_0 = \mathrm{rk}G - \mathrm{rk}K_\infty$ and $q_0 = \frac{1}{2}(\mathrm{dim}X - l_0)$. In particular, when X_Γ is a Shimura variety (which implies that $l_0 = 0$), tempered representations only contribute to the cohomology of X_Γ in the middle degree.

In this talk, I described a mod ℓ analogue of the result of Borel and Wallach, concerning $H^*(X_\Gamma, \mathbb{F}_\ell)$ in the case when X_Γ is a compact unitary Shimura variety. One reason for studying the cohomology of locally symmetric spaces with torsion coefficients comes from trying to prove the modularity of Galois representations. A recent insight of Calegari and Geraghty [3] is that understanding torsion as well as characteristic zero classes is crucial for proving modularity lifting theorems beyond the self-dual case for GL_n (this is the so-called *Betti setting* in their work). The Calegari-Geraghty method requires the existence of Galois representations associated to torsion classes in the cohomology of locally symmetric spaces, which has been supplied by Scholze in [6], but also requires these Galois representations to satisfy some form of local-global compatibility and the relevant torsion classes to be concentrated in a range of degrees of length l_0. Theorem 1 below gives a way to access both missing ingredients, with the main restriction being that we require our Shimura varieties to be compact.

Before stating the precise mod ℓ condition that will play the part of temperedness, it is worth noting that for cuspidal automorphic representations of GL_n which are self-dual, temperedness at finite places is closely related to *genericity* and to the fact that such representations contribute to the cohomology of unitary Shimura varieties in the middle degree (see Remark 1.8 of [4] for more details). Thus, we replace temperedness, which is an analytic condition, by a genericity condition, which can be formulated mod ℓ.

Let G be an anisotropic unitary similitude group of dimension n, corresponding to a CM field F with totally real subfield F^+. Assume that F contains an imaginary quadratic field. We can either assume that G comes from a division algebra with center F or (under a mild extra assumption on F) that it is quasi-split at all finite places. The Shimura varieties X_Γ associated to G are compact. Let $\mathbb{T}$ be the spherical Hecke algebra acting on $H^*(X_\Gamma, \mathbb{F}_\ell)$ and let $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal in the support of $H^*(X_\Gamma, \mathbb{F}_\ell)$.

Theorem 1. *(1) There exists a Galois representation*

$$\rho_\mathfrak{m} : G_F \to GL_n(\bar{\mathbb{F}}_\ell)$$

such that the Frobenius eigenvalues of $\rho_\mathfrak{m}$ match the Satake parameters of $\mathfrak{m}$ at the unramified places.

(2) Assume that there exists a rational prime $p \neq \ell$ such that p splits completely in F and such that $\rho_\mathfrak{m}$ is decomposed generic *at p. Then*

$$H^i(X_\Gamma, \mathbb{F}_\ell)_\mathfrak{m} \neq 0$$

implies that $i = q_0 = \dim X$.

We remark that the first part of Theorem 1 is not new. The existence of the Galois representation $\rho_\mathfrak{m}$ follows from [6] and it applies to non-compact as well as compact Shimura varieties. What is new is the method of proof: in both settings, the key step is to lift the mod ℓ class to a characteristic 0 class. The argument in [6] relies on an ℓ-adic tower and analogues of the Hasse invariant to construct the characteristic 0 lift and thus loses track of the properties of the Galois representation $\rho_\mathfrak{m}$ at the prime ℓ, the most delicate part of the local-global compatibility that these representations are expected to satisfy. In Theorem 1, we give a new argument using a p-adic tower, for an auxiliary prime $p \neq \ell$. This gives a strategy for proving local-global compatibility at ℓ.

As for the second part of Theorem 1, Boyer proved a similar result for Shimura varieties of Harris-Taylor type [2] and Lan and Suh proved results of this kind for general Shimura varieties under restrictions on level and weight [5]. We explain the condition that $\rho_\mathfrak{m}$ be *decomposed generic* at p. For any prime $\mathfrak{p}$ of F lying above p, we ask that

(1) $\rho_\mathfrak{m}|_{G_{F_\mathfrak{p}}}$ be unramified;
(2) any two eigenvalues λ, λ' of $\mathrm{Frob}_\mathfrak{p}$ satisfy $\lambda/\lambda' \notin \{1, (\#k(\mathfrak{p}))^{\pm 1}\}$.

This condition implies that any lift of $\rho_\mathfrak{m}$ to characteristic 0, restricted to $G_{F_\mathfrak{p}}$ corresponds under local Langlands to a generic principal series representation of $GL_n(F_\mathfrak{p})$.

The proof of Theorem 1 relies on the geometry of the Hodge-Tate period morphism. The idea behind the Hodge-Tate period morphism is the following: if we set $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$, then p-divisible groups over the ring of integers $\mathcal{O}_{\mathbb{C}_p}$ are classified by pairs (T, W). Here T is the Tate module, a free $\mathbb{Z}_p$-module, and $W \subset T \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ is a sub-vector space, identified with the (one-step) Hodge-Tate filtration. The classification result is due to Scholze and Weinstein [7]. Over an infinite-level, perfectoid Shimura variety, the Tate modules of the abelian varieties it parametrizes are trivialized; by sending an abelian variety to its associated p-divisible group, we obtain a map π_{HT} from the infinite-level Shimura variety to a Grassmannian $\mathcal{F}\ell_{G,\mu}$, which measures the relative position of W in $T \otimes_{\mathbb{Z}_p} \mathbb{C}_p$. This is actually a map of adic spaces, since the above can be made to work in families.

In [4], we use a Leray spectral sequence to reduce the computation of $H^*(X_\Gamma, \mathbb{F}_\ell)$ to understanding the geometry of π_{HT}. Along the way, we prove several foundational results concerning π_{HT}.

Theorem 2. *(1) Shimura varieties of Hodge type admit a $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period morphism*

$$\pi_{HT} : \mathcal{X}_{\Gamma(p^\infty)} \to \mathcal{F}\ell_{G,\mu},$$

which agrees with that of [6] *in the Siegel case and which is functorial in the Shimura datum.*

(2) $\mathcal{F}\ell_{G,\mu}$ has a stratification by locally closed subspaces indexed in terms of a subset $B(G,\mu)$ of the Kottwitz set $B(G)$, which classifies isocrystals with G-structure.

(3) The fibers of the Hodge-Tate period morphism over the stratum $\mathcal{F}\ell^b_{G,\mu}$ for $b \in B(G,\mu)$ are canonical lifts of perfections of Igusa varieties Ig^b.

To complete the proof of Theorem 1, we need two more ingredients. Firstly, we compute the contribution of automorphic forms which are generic principal series at all primes above p to the alternating sum of cohomology groups $[H(\mathrm{Ig}^b, \bar{\mathbb{Q}}_\ell)]$ and show that it is non-zero only when b corresponds to the the closed, μ-ordinary stratum. Secondly, we show that the pushforward $R\pi_{HT,*}\mathbb{F}_\ell|_{\mathcal{F}\ell^b_{G,\mu}}$ is concentrated in one degree, when $\mathcal{F}\ell^b_{G,\mu}$ is a maximal stratum on which this complex is non-zero. This should be thought of as saying that $R\pi_{HT,*}\mathbb{F}_\ell$ is a perverse sheaf, for some notion of perversity on rigid-analytic spaces.

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Moduli of Supersingular K3 Crystals

Christian Liedtke

Supersingular K3 surfaces. A K3 surface X over an algebraically closed field k of positive characteristic p is called *supersingular*, if the following two equivalent conditions are fulfilled

(1) X has Picard rank $\rho(X) = 22$,

(2) the height of the formal Brauer group $\widehat{\mathrm{Br}}(X)$ is infinite.

Concerning the equivalence: (1) $\Rightarrow$ (2) easily follows from the Igusa–Artin–Mazur inequality and the implication (2) $\Rightarrow$ (1) follows from the Tate-conjecture for K3 surface, which is now a theorem of Charles, Kim, Nygaard, Maulik, Madapusi-Pera, and Ogus. Moreover, the discriminant of the Néron–Severi group satisfies

$$\mathrm{disc}\,\mathrm{NS}(X) \;=\; -p^{2\sigma_0} \qquad \text{for some integer} \quad 1 \leqslant \sigma_0 \leqslant 10$$

that is called the *Artin invariant* and that was introduced by Artin [Ar74]. In fact, σ_0 determines the Néron–Severi $\mathrm{NS}(X)$ up to isometry and such lattices are called *supersingular K3 lattices.*

Moving torsor families. If X is a supersingular K3 surface with Artin invariant $\sigma_0 \leqslant 9$ in characteristic p, then it admits a genus-one fibration $X \to \mathbb{P}^1_k$ with a section [Li15]. Associated to this data, there exists a smooth family of supersingular K3 surfaces, a family of *moving torsors*

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k & \to & \mathrm{Spec}\, k[[t]] \end{array}$$

such that specialization induces a short exact sequence

$$0 \;\to\; \mathrm{Pic}(\mathcal{X}_{\overline{\eta}}) \;\to\; \mathrm{Pic}(X) \;\to\; \mathbb{Z}/p\mathbb{Z} \;\to\; 0,$$

whose cokernel is generated by the class of the section of the fibration $X \to \mathbb{P}^1$ from above. In particular, $\sigma_0(\mathcal{X}_{\overline{\eta}}) = \sigma_0(X) + 1$, and thus, this family has non-trivial moduli. Moreover, this family can be algebraized and spread out from $\mathrm{Spec}\, k[[t]]$ to a family of smooth supersingular K3 surfaces over a curve that is of finite type over k. We refer to [Li15] for details.

Moduli of supersingular K3 crystals. The second crystalline cohomology group of a supersingular K3 surface in positive characteristic p is an F-crystal, which is of slope 1 and weight 2. Moreover, Poincaré duality equips it with a perfect pairing. In [Og79], Ogus classified such crystals together with markings by a supersingular K3 lattice N, so called *N-marked supersingular K3 crystals.* In [Og79], he also constructed a moduli space $\mathcal{M}_N \to \mathrm{Spec}\,\mathbb{F}_p$ for such crystals.

Theorem (Ogus)**.** Let N be a supersingular K3 lattice in odd characteristic p. Then, $\mathcal{M}_N$ is smooth, proper, and of dimension $(\sigma_0(N) - 1)$ over $\mathbb{F}_p$, and it has two geometric components.

Given a K3 surface X together with a genus-one fibration $X \to \mathbb{P}^1_k$ and a section, the divisor classes of a fiber and the divisor class of the section span a hyperbolic plane U inside the Néron-Severi lattice $\mathrm{NS}(X)$. This said, the moving torsor families from above manifest themselves on the level of Ogus' moduli spaces $\mathcal{M}_N$ as follows.

Theorem. Let N and N_+ be supersingular K3 lattices in odd characteristic such that $\sigma_0(N_+) = \sigma_0(N) + 1$. Then, a choice of hyperbolic plane $U \subset N$ gives rise to a morphism

$$\mathcal{M}_{N_+} \to \mathcal{M}_N.$$

This morphism is a fibration, whose geometric fibers are rational curves with at worst unibranch singularities ("cusps").

Examples (Ogus)**.** Let N be a supersingular K3 lattice in characteristic $p \geqslant 3$.

(1) If $\sigma_0(N) = 1$, then $\mathcal{M}_N \cong \operatorname{Spec} \mathbb{F}_{p^2}$.
(2) If $\sigma_0(N) = 2$, then $\mathcal{M}_N \cong \mathbb{P}^1_{\mathbb{F}_{p^2}}$.
(3) If $\sigma_0(N) = 3$, then $\mathcal{M}_N \times_{\operatorname{Spec} \mathbb{F}_p} \operatorname{Spec} \overline{\mathbb{F}}_p$ is isomorphic to two disjoint copies of the Fermat surface F_{p+1}.

Shioda [Sh74] showed that the Fermat surface $F_n := \{x_0^n + ... + x_3^n = 0\} \subset \mathbb{P}^3$, $n \geqslant 4$, in characteristic p with $\gcd(n, p) = 1$ is unirational if and only if there exists an integer $\nu \geqslant 1$ such that $p^\nu \equiv -1 \mod n$. In particular, the Fermat surfaces F_{p+1} are unirational in characteristic p. Together with Ogus' examples, we obtain a new proof of Shioda's theorem in the following special case.

Corollary (Shioda)**.** For every prime p, the Fermat surface $F_{p+1} \subset \mathbb{P}^3$ is unirational in characteristic p.

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The Spectral Halo

Fabrizio Andreatta

(joint work with Adrian Iovita, Vincent Pilloni)

Let p be a prime number and let N be a positive integer coprime to p. Let Γ be the congruence subgroup $\Gamma_1(N) \cap \Gamma_0(p)$.

Set $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^*]]$, the Iwasawa algebra for the group $\mathbb{Z}_p^*$. The classical rigid analytic space $\mathcal{W}^{\mathrm{rig}}$ associated to Λ is the so called *weight space* for (elliptic) p-adic modular forms: its $\mathbb{C}_p$-valued points correspond to continuous homomorphisms $\kappa\colon \mathbb{Z}_p^* \to \mathbb{C}_p^*$. For any such Coleman has defined in [Co] an orthonormalizable space $M_\kappa^\dagger(\Gamma)$ of overconvergent modular forms of level Γ and of weight κ. It is endowed with a compact operator U_p. We denote by $\mathcal{P}_\kappa(X) := \det(1 - XU_p|_{M_\kappa^\dagger(\Gamma)})$ the characteristic series of U_p on the space $M_\kappa^\dagger(\Gamma)$.

Coleman has proved that there exists a power series $\mathcal{P}(X) \in \Lambda[\![X]\!]$ with the following remarkable property. For every p-adic weight $\kappa\colon \mathbb{Z}_p^* \to \mathbb{C}_p^*$ the specialization of $\mathcal{P}(X)$ at κ coincides with the characteristic series $\mathcal{P}_\kappa(X)$.

Based on the analysis of the cases $p = 2$, due to Buzzard and Kilford [BK], and $p = 3$, due to Roe [Ro], he made the following conjecture.

Conjecture (Spectral Halo): Set $T := \exp(p) - 1 \in \Lambda$.

(1) There exists an $\mathbb{F}_p((T))$-Banach space of overconvergent modular forms $\overline{M}^\dagger(\Gamma)$ and a compact operator $\overline{U}_p$ whose characteristic power series $\det(1 - X\overline{U}_p|_{\overline{M}^\dagger(\Gamma)})$ is the reduction $\overline{\mathcal{P}}(X)$ of $\mathcal{P}(X)$ modulo p.

(2) Let (n_i, m_i) be the breaking points of the Newton polygon of $\overline{\mathcal{P}}(X)$. Then there exists a positive rational number $r < 1$ such that for all p-adic weights κ lying in the annulus $r < |T| < 1$ of the weight space, the breaking points of the Newton polygon of $\mathcal{P}_\kappa$ are $(n_i, \mathrm{val}_p(\kappa(1+p) - 1)m_i)$.

Remarks: (i) The world "spectral" refers to the spectral curve defined by the zero locus of $\mathcal{P}(X)$ in the affine line over $\mathcal{W}^{\mathrm{rig}}$. Conjecture 2 implies that over $r < |T| < 1$ the spectral curve is the disjoint union of infinitely many components, finite and flat over the weight space (the "halo").

(ii) In [LWX] and in [JN] the authors solve Coleman's conjectures in the quaternionic setting. In this case the underlying Shimura variety is 0-dimensional so that the geometry is very simple. Overconvergent forms are then realized following Buzzard by introducing complicated p-adic coefficients.

(iii) A solution to Conjecture (1) working with modular curves is given in [AIP2] and is the content of the present talk. We are optimistic that also Conjecture (2) will follow. The methods we use are amenable to generalization and in [AIP3] we dealt with the Hilbert case, proving an analogue of Conjecture (1). It is not at all clear, and we think it is an interesting question, what the analogue of Conjecture (2) should be in this context.

The first step is to "complete" $\mathcal{W}^{\mathrm{rig}}$ by considering the analytic adic space $\mathcal{W} := \mathrm{Spa}(\Lambda, \Lambda)^{\mathrm{an}}$ (in the sense of Huber) associated to the (p, T)-adic formal scheme $\mathrm{Spf}(\Lambda)$. It contains the adic space $\mathcal{W}^{\mathrm{rig}}$ associated to the classical rigid analytic weight space introduced by Coleman. The complement consists of the points supported in characteristic p corresponding to the finitely many residue fields of $\Lambda/p\Lambda$; these are the sought for points at infinity.

Let X denote the modular curve over $\mathbb{Z}_p$ of level Γ, as a p-adic formal scheme. Write $E \to X$ for the universal generalized elliptic curve and ω for the sheaf of invariant differentials of E relatively to X. Denote by $\mathfrak{X}$, the partial blow-up of the T-adic formal scheme defined by $X \times \Lambda\langle p/T^p, T/\mathrm{Ha}^{p^3}\rangle$ (here Ha is a local lift of the Hasse invariant and the completion is with respect to the T-adic topology). Let $\mathfrak{X}^{\mathrm{an}}$ be the associated analytic adic space.

Remark: If one replaces T with p the above provides the definition of a tubular neighborhood of the ordinary locus. Here we are considering the T-adic variant of this notion within the product of X and $\mathcal{W}$ (as adic sapces).

The main result of [AIP2] is the following:

Theorem 1. *There exists an invertible sheaf ω^{univ} over $\mathfrak{X}^{\mathrm{an}}$ such that its global sections are endowed with a compact operator U_p.*

Their reduction modulo p coincide with the global sections of the base change of ω^{univ} to the points at infinity and, together with the operator U_p, provide a solution to Conjecture (1).

In [AIP2] we provide a construction in characteristic p and a construction over $\mathcal{W}^{\mathrm{rig}}$, using the characterization of the canonical subgroup in terms of integral p-adic Hodge theory as in [AIP1]. In order to compare the two we use the analysis of the anticanoncial tower provided by Scholze [Sc].

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Rigidity and a Riemann-Hilbert correspondence for p-adic local systems

Ruochuan Liu
(joint work with Xinwen Zhu)

This talk is about to explain a joint work with Xinwen Zhu [5]. Firstly, fix k to be a finite extension of $\mathbb{Q}_p$ as our base field.

De Rham rigidity for p-adic local systems. Our first main result is concerned with de Rham rigidity for p-adic étale local systems.

Theorem 1. *Let X be a geometrically connected rigid analytic variety over k, and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on X_{et}. If there exists a closed point $x \in X$ such that the stalk $\mathbb{L}_{\bar{x}}$ of $\mathbb{L}$ at some geometric point $\bar{x}$ over x, regarded as a p-adic representation of the residue field of x, is de Rham, then the stalk $\mathbb{L}_{\bar{y}}$ is de Rham at every closed point y of X and the Hodge-Tate weights (with multiplicity) of $\mathbb{L}_{\bar{y}}$'s are the same as those of $\mathbb{L}_{\bar{x}}$. Moreover, if X is smooth, then $\mathbb{L}$ is de Rham in the sense of Scholze* [6].

Inspired by the Fontaine–Mazur conjecture, we also prove the following result.

Theorem 2. *Let X be a geometrically connected algebraic variety over a number field E and let $\mathbb{L}$ be a p-adic étale local system on X. If there exists a closed point $x \in X$, such that the stalk $\mathbb{L}_{\bar{x}}$ of $\mathbb{L}$ at some geometric point $\bar{x}$ over x, regarded as a p-adic Galois representation of the residue field of x, is geometric in the sense of Fontaine-Mazur (i.e. it is unramified almost everywhere and is de Rham at p), then the stalk $\mathbb{L}_{\bar{y}}$ at every closed point $y \in X$, regarded as the Galois representation of the residue field of y, is geometric.*

An application to Shimura varieties. Let (G, X) be a Shimura datum. For a (sufficiently small) open compact subgroup $K \subset G(\mathbb{A}_f)$, let

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

be the corresponding Shimura variety. Let V be a $\mathbb{Q}$-rational representation of G, which is trivial on Z_G^s. Here Z_G^s is the largest anisotropic subtorus in the center of G that is $\mathbb{R}$-split. Then it is known that V induces a Betti local system $\mathbb{L}_V$ on $\mathrm{Sh}_K(G, X)$. In addition, the theory of canonical models gives a model of $\mathrm{Sh}_K(G, X)$ (still denoted by the same notation) defined over the reflex field $E \subset \mathbb{C}$, and for a choice of a prime p, a p-adic étale local system $\mathbb{L}_{V,p}$ over $\mathrm{Sh}_K(G, X)$ (cf. [4, §III.6]). Applying Theorem 2 to $\mathrm{Sh}_K(G, X)$ with the set of special points, which is Zariski dense (so in particular nonempty!) in every connected component, we obtain the following theorem.

Theorem 3. *For every closed point x of $\mathrm{Sh}_K(G, X)$, the stalk $(\mathbb{L}_{V,p})_{\bar{x}}$, regarded as a Galois representation of $\mathrm{Gal}(\overline{E}/E(x))$, is geometric in the sense of Fontaine-Mazur.*

A p-adic Riemann-Hilbert correspondence. To prove Theorem 1, by resolution of singularities for rigid analytic varieties, we may assume that X is smooth. It turns out that in this case we may deduce Theorem 1 from a version of p-adic Riemann-Hilbert correspondence.

Let K be a perfectoid field which is the completion of a Galois extension of k (in $\bar{k}$) that contains $k(\mu_{p^\infty})$, and let $\mathrm{Gal}(K/k)$ denote the corresponding Galois group. Let

$$\nu' : X_{\mathrm{proet}}/X_K = (X_K)_{\mathrm{proet}} \to (X_K)_{\mathrm{et}}$$

denote the projection from the pro-étale site of X_K to the étale site of X_K. Let B_{dR} denote the de Rham period ring associated to K. We consider a sheaf of $\mathbb{Q}_p$-algebras $\mathcal{O}_X\hat{\otimes}\mathrm{B}_{\mathrm{dR}}$ on X_K (see [5, §3.1] for the precise definition); it inherits a filtration from the filtration on B_{dR} and a B_{dR}-linear derivation from the derivation on $\mathcal{O}_X$. The ringed space $(X_K, \mathcal{O}_X\hat{\otimes}\mathrm{B}_{\mathrm{dR}})$ is denoted by $\mathcal{X}$, which can be thought of as the "generic fiber of a canonical lifting of X_K to $\mathrm{B}_{\mathrm{dR}}^+$".

Recall that there is the geometric de Rham period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ on X_{proet} constructed by Scholze ([6]). Set

$$\mathcal{RH}(\mathbb{L}) = R\nu'_*(\hat{\mathbb{L}} \otimes \mathcal{O}\mathbb{B}_{\mathrm{dR}}).$$

We have the following theorem, which can be regarded as a first step towards the long sought-after Riemann-Hilbert correspondence on p-adic varieties.

Theorem 4. *(See* [5, Theorem 3.6] *for the full and precise statements.) Let X be a smooth rigid analytic variety over k, and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on X_{et}. Then $R^i\nu'_*(\hat{\mathbb{L}} \otimes \mathcal{O}\mathbb{B}_{\mathrm{dR}}) = 0$ for $i > 0$, and the functor $\mathcal{RH}(\mathbb{L}) = \nu'_*(\hat{\mathbb{L}} \otimes \mathcal{O}\mathbb{B}_{\mathrm{dR}})$ is a tensor functor from the category of $\mathbb{Q}_p$-local systems on X_{et} to the category of vector bundles on $\mathcal{X}$, equipped with a semi-linear action of* $\mathrm{Gal}(K/k)$*, and with a filtration and an integrable connection that satisfy Griffiths transversality. The functor $\mathcal{RH}$ is compatible with pullback along arbitrary morphisms and (under certain conditions) is compatible with pushforward under smooth proper morphisms.*

We regard the above Theorem as a geometric Riemann-Hilbert correspondence for p-adic étale local systems. Now define $D^i_{\mathrm{dR}}(\mathbb{L})$ to be the i-th pushforward of $\mathcal{RH}(\mathbb{L})$ along the natural projection $X_K \to X$. (See [5, §3.2] for the more precise definition.) Then Theorem 1 follows from the following theorem, which in turn follows from Theorem 4 and can be regarded as an arithmetic Riemann-Hilbert correspondence.

Theorem 5. *(See* [5, Theorem 3.7] *for the full and precise statements.) Let X be a smooth rigid analytic variety over k, and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on X_{et}. Then*

(i) D^i_{dR} is a functor from the category of $\mathbb{Q}_p$-local systems on X_{et} to the category of vector bundles on X with an integrable connection as above. If X is a point, D^0_{dR} coincides with the usual D_{dR} functor of Fontaine's.

(ii) For $i > 1$, $D^i_{\mathrm{dR}}(\mathbb{L}) = 0$; and $D^0_{\mathrm{dR}}(\mathbb{L}) \cong D^1_{\mathrm{dR}}(\mathbb{L})$. Moreover, the functors D^i_{dR} commute with arbitrary pullbacks.

(iii) *If X is connected and there exists a classical point x such that $\mathbb{L}_{\bar{x}}$ is de Rham, then there is a decreasing filtration* Fil *on $D^0_{\mathrm{dR}}(\mathbb{L})$ by sub-bundles such that $(D^0_{\mathrm{dR}}(\mathbb{L}), \nabla_{\mathbb{L}}, \mathrm{Fil})$ is the filtered $\mathcal{O}_X$-module with an integrable connection associated to $\mathbb{L}$ in the sense of* [6, Definition 7.4]. *In other words, $\mathbb{L}$ is a de Rham local system in the sense of* [6, Definition 8.3].

Simpson correspondence. To prove Theorem 4, we consider the 0th graded piece $\mathcal{O}\mathbb{C} = \mathrm{gr}^0\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$. Then we are led to study

$$\mathcal{H}(\mathbb{L}) := R\nu'_*(\hat{\mathbb{L}} \otimes \mathcal{O}\mathbb{C}).$$

Note that taking the associated graded of the connection on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ defines a Higgs field on $\mathcal{O}\mathbb{C}$ and therefore a Higgs field $\vartheta_{\mathbb{L}}$ on $\mathcal{H}(\mathbb{L})$. It turns out the functor

$$\mathcal{H} : \mathbb{L} \mapsto (\mathcal{H}(\mathbb{L}), \vartheta_{\mathbb{L}})$$

is nothing but (a special case of) the p-adic Simpson correspondence, which was first proposed by Faltings [3] and recently systematically studied by Abbes-Gros and Tsuji [1, 2]. We note that these works studied more general local systems on $(X_K)_{\mathrm{et}}$ rather than local systems on X_{et}. But in our special case, we give a simpler construction and prove nicer results. For example, we have the following statements.

Theorem 6. *(See* [5, Theorem 2.1] *for the full and precise statements.) Let X be a smooth rigid analytic variety over k, and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on X_{et}. Then $R^i\nu'_*(\hat{\mathbb{L}} \otimes \mathcal{O}\mathbb{C}) =$ for $i > 0$, and $\mathcal{H} = \nu'_*(\hat{\mathbb{L}} \otimes \mathcal{O}\mathbb{C})$ is a tensor functor from the category of $\mathbb{Q}_p$-local systems on X_{et} to the category of nilpotent Higgs bundles on X_K. The functor $\mathcal{H}$ is compatible with pullback along arbitrary morphisms and (under certain conditions) is compatible with pushforward under smooth proper morphisms.*

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p-adic period maps and variations of p-adic Hodge structures

David Hansen

Let X be a connected complex manifold, and let $\mathbf{V} = (V_{\mathbf{Z}}, \mathrm{Fil}^{\bullet} \subset V_{\mathbf{Z}} \otimes_{\underline{Z}} \mathcal{O}_X, \psi)$ be a polarized $\mathbf{Z}$-variation of Hodge structure on X; for example, if $X = \mathcal{X}^{an}$ for some smooth quasiprojective variety $\mathcal{X}/\mathbf{C}$ and $f : \mathcal{Y} \to \mathcal{X}$ is some smooth projective family equipped with a choice of relatively ample line bundle, then the $\mathbf{Z}$-local system given by the primitive part of $R^i f_*^{an} \underline{\mathbf{Z}}_{/\mathrm{torsion}}$ canonically underlies a polarized $\mathbf{Z}$-VHS on X. Writing G for the automorphism group of the (bilinear form associated with the) polarization, there is a natural $G(\mathbf{Z})$-covering $\tilde{X} \to X$ parametrizing trivializations of $V_{\mathbf{Z}}$ compatible with the polarization, and a natural $G(\mathbf{Z})$-equivariant *period morphism* $\pi : \tilde{X} \to \mathcal{F}l$, where $\mathcal{F}l = G(\mathbf{C})/Q$ is a certain generalized flag variety. Much of classical Hodge theory amounts to the study of these period maps and their properties.

What is the p-adic analogue of this story? Until recently this question seemed rather murky; however, the situation changed dramatically with Scholze's discovery of the *Hodge-Tate period map* out of a Siegel Shimura variety with infinite level at p [5]. This construction was then generalized by Caraiani-Scholze (resp. Shen) to arbitrary perfectoid Shimura varieties of Hodge type (resp. abelian type) [1, 6]. In this report we sketch a general framework for constructing p-adic period maps; in particular, we get a uniform and functorial construction of a Hodge-Tate period morphism for *any* Shimura variety,[1] including those with no known interpretation as moduli of motives. Our main guide is the following slogan: the p-adic analogue of a $\mathbf{Z}$- or $\mathbf{Q}$-VHS over a smooth complex manifold is a *de Rham $\mathbf{Q}_p$-local system on a smooth rigid analytic space.*

Given such a $\mathbf{Q}_p$-local system $\mathbf{V}$ on a rigid space X, maybe with some G-structure, we'd like to have some covering space $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to X$ parametrizing "frames of $\mathbf{V}$" analogous to the $G(\mathbf{Z})$-covering from in the archimedean story. One quickly observes that this space must be a bit wild: the map $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X} \to X$ should have infinite (but locally profinite) geometric fibers, and in particular will not be étale. It's unclear if this space should be representable by an adic space in general; our main observation is that if one is willing to work with *diamonds*, then such a space exists.

In order to state these results more precisely, let us quickly fix some notation and terminology. In what follows, we fix a prime p, and use the term "adic space" in reference to (honest) analytic adic spaces over $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$; we also write Perf for the category of perfectoid spaces in characteristic p.

Definition 1. (1) Let $\mathcal{S}$ be any site whose objects U have a "natural underlying topological space $|U|$", and let T be any topological group or ring (or just any topological space). Then we can define a sheaf $\underline{T}$ of groups (or rings) on $\mathcal{C}$ by sheafifying the presheaf $U \mapsto \mathcal{C}(|U|, T)$. If A is a topological ring and $X \in \mathcal{S}$ is any given object, we define an *A-local system on X* as

[1] At the expense of passing to its diamond when the Shimura data is not of abelian type, cf. below.

a sheaf M of $\underline{A}$-modules on $\mathcal{S}_{/X}$ such that $M|_{U_i} \simeq \underline{A}^{n_i}|_{U_i}$ locally on some covering $\{U_i \to X\}$.

(2) (Kedlaya-Liu, Scholze) Let X be any adic space or diamond; then X has a well-behaved pro-étale site X_{proet}, and every $Y \in X_{\text{proet}}$ has an underlying topological space $|Y|$ with a natural map to $|X|$. In particular, given any topological ring A, we have an associated category $A\text{Loc}(X)$ of A-local systems on (the pro-étale site of) X.

Now, given a reductive group $\mathbf{G}/\mathbf{Q}_p$ with $G = \mathbf{G}(\mathbf{Q}_p)$ its (locally profinite) group of $\mathbf{Q}_p$-points, we consider the following category.

Definition 2. Given X any adic space or diamond, a *G-local system* on X is a faithful exact tensor functor $\mathbf{V} : \text{Rep}(\mathbf{G}) \to \mathbf{Q}_p\text{Loc}(X)$ with the following property[2]: for each connected component of X, there is a geometric point $\overline{x} \in X$ lying in that component such that the composite tensor functor

$$\mathbf{V}_{\overline{x}} : \text{Rep}(\mathbf{G}) \to \mathbf{Q}_p\text{Loc}(X) \xrightarrow{\overline{x}^*} \mathbf{Q}_p\text{Loc}(\overline{x}) \cong \text{Vect}_{\mathbf{Q}_p}$$

is tensor- isomorphic to the trivial fiber functor

$$\begin{array}{rcl} \text{Rep}(\mathbf{G}) & \to & \text{Vect}_{\mathbf{Q}_p} \\ (W, \rho) & \mapsto & W. \end{array}$$

These form a category $G\text{Loc}(X)$ in a natural way, and this category turns out to enjoy a number of good properties:

Theorem 3. *(1) If X is any adic space, with associated diamond $X^\diamondsuit$, there is a natural equivalence $G\text{Loc}(X) \cong G\text{Loc}(X^\diamondsuit)$.*

(2) If X is any adic space or diamond and we take $\mathbf{G} = \text{GL}_n$, there is a natural equivalence $G\text{Loc}(X) \cong \mathbf{Q}_p\text{Loc}(X)^{\text{rank}=n}$.

(3) If $\mathcal{D}$ is any diamond and $\mathbf{V}$ is any G-local system on $\mathcal{D}$, then the presheaf with G-action

$$\begin{array}{rcl} \mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}} : \text{Perf}_{/\mathcal{D}} & \to & \text{Sets} \\ (f : T^\diamondsuit \to \mathcal{D}) & \mapsto & \text{Isom}_{G\text{Loc}(T)}(\mathbf{V}_{\text{triv}}, f^*\mathbf{V}) \end{array}$$

defines a pro-étale sheaf on the site $\text{Perf}_{/\mathcal{D}}$. Moreover, the morphism $\mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ is representable, surjective, and pro-étale, and $\mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}}$ is naturally a pro-étale $\underline{G}$-torsor over $\mathcal{D}$. In particular, $\mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}}$ is a diamond.

(4) For any open subgroup $K \subset G$, the sheaf $\mathcal{T}\text{riv}_{\mathbf{V}/\mathcal{D}}/\underline{K}$ of "K-level structures on $\mathbf{V}$" is a diamond, and its structure map to $\mathcal{D}$ is separated, étale, partially proper, and surjective. If $\mathcal{D} = X^\diamondsuit$ is the diamond of a locally

[2]This property turns out to be automatic for some groups (for example, for $\mathbf{G}$ which are $\mathbf{Q}_p$-split with simply connected derived group) but not for all groups. At the time of my talk in the workshop, I didn't properly appreciate the necessity of this condition, and I'm very grateful to several workshop participants, especially B. Bhatt and P. Scholze, for conversations which clarified this point.

Noetherian adic space, then $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^\diamond}/\underline{K}$ *is naturally the diamond of a locally Noetherian adic space over* X*.*

(5) *The association* $\mathbf{V} \mapsto \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}$ *defines an equivalence from the category* $G\mathrm{Loc}(\mathcal{D})$ *to the category* $G\mathrm{Tor}(\mathcal{D})$ *of pro-étale* $\underline{G}$ *torsors over* $\mathcal{D}$*, with an explicit essential inverse.*

We briefly mention some of the ideas. (1) reduces immediately to the tensor equivalence $\mathbf{Q}_p\mathrm{Loc}(X) \cong \mathbf{Q}_p\mathrm{Loc}(X^\diamond)$, which can be deduced from some descent results of Kedlaya-Liu [2, 3]. The proofs of (2-5) are intertwined; aside from various "reduction to GL_n"-type tricks familiar from the theory of Shimura varieties, the most crucial inputs are:

i. The fact (announced by Fargues) that the Kottwitz map is constant on connected components of the stack $\mathrm{Bun}_{\mathbf{G}}$ of $\mathbf{G}$-bundles on the Fargues-Fontaine curve; this allows one to boostrap from knowing that the map $\mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ hits every connected component of the target (which is guaranteed by the definition of a G-local system) to the surjectivity of this map.

ii. The fact that every pro-étale $\underline{G}$-torsor $\tilde{\mathcal{D}}$ over a given diamond $\mathcal{D}$ is a well-behaved diamond; in particular, this guarantees that $\mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}$ is a diamond and has an associated topological space, which is prerequisite to writing down a candidate for the essential inverse mentioned in (5). This fact is deduced in turn from the following result, which may be of independent utility:

Theorem 4. *Let* G *be any locally profinite group, and let* $f : \tilde{\mathcal{F}} \to \mathcal{F}$ *be any* $\underline{G}$*-torsor in pro-étale sheaves on* Perf*. Then the morphism* f *is representable and pro-étale, and* $\tilde{\mathcal{F}}/\underline{K} \to \mathcal{F}$ *is representable and étale for any open subgroup* $K \subset G$*.*

This relies on some technical pro-étale descent results for finite étale and separated étale morphisms to perfectoid spaces, due to Weinstein and to Scholze (respectively).

Returning to the situation of interest, let X be a smooth connected rigid analytic space over a p-adic field E. Given $\mathbf{G}$ as before, we say that a G-local system $\mathbf{V}$ on X is *de Rham* if for some (equivalently, any) faithful representation $(W, \rho) \in \mathrm{Rep}(\mathbf{G})$, the $\mathbf{Q}_p$-local system $\mathbf{V}_{W,\rho}$ is de Rham in the sense of relative p-adic Hodge theory. With these preparations we can state our main result:

Theorem 5. *Maintain the above assumptions on* $\mathbf{V}$ *and* X*. Then:*

(1) *There is a unique geometric conjugacy class of Hodge cocharacters* $\mu : \mathbf{G}_m \to \mathbf{G}$ *such that the weights of* $\rho \circ \mu$ *record the Hodge filtration on the vector bundle* $\mathbf{D}_{\mathrm{dR}}(\mathbf{V}_{W,\rho})$ *for every* $(W, \rho) \in \mathrm{Rep}(\mathbf{G})$*.*

(2) *There is a natural* G*-equivariant Hodge-Tate period morphism*

$$\pi_{\mathrm{HT}} : \mathcal{T}\mathrm{riv}_{\mathbf{V}/X^\diamond} \to \mathscr{F}\!\ell^{\diamond}_{\mathbf{G},\mu}$$

of diamonds over $\mathrm{Spd}\, E$*, where* $\mathscr{F}\!\ell_{\mathbf{G},\mu}$ *is a certain generalized flag variety for* $\mathbf{G}$*.*

Finally, we apply these ideas to Shimura varieties. Let $(\mathbf{G}, X)$ be a Shimura datum, with reflex field E and Hodge cocharacter μ. Let $\{S_K\}_{K \subset \mathbf{G}(\mathbf{A}_f)}$ be the associated tower of smooth quasiprojective Shimura varieties over E. Choose a prime $\mathfrak{p}$ of E over p and a tame level group $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$; then for any open compact subgroup $K_p \subset \mathbf{G}(\mathbf{Q}_p)$, we get a smooth rigid analytic space $\mathcal{S}_{K^pK_p} = (S_{K^pK_p} \times_E E_\mathfrak{p})^{ad}$ over $\mathrm{Spa}\, E_\mathfrak{p}$.

Theorem 6. *There is a natural* $\mathbf{G}(\mathbf{Q}_p)$*-equivariant Hodge-Tate period map*

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p}^\diamondsuit := \lim_{\leftarrow K_p} \mathcal{S}_{K^pK_p}^\diamondsuit \to \mathscr{F}\!\ell_{\mathbf{G},\mu}^\diamondsuit$$

of diamonds over $\mathrm{Spd}\, E_\mathfrak{p}$*, compatible with morphisms of arbitrary Shimura datum. When* $(\mathbf{G}, X)$ *is of Hodge or abelian type, this is the "diamondization" of the Hodge-Tate period morphism constructed by Caraiani-Scholze and Shen.*

The rough idea here is that (under a mild condition on the Shimura datum) the pushout $\widetilde{\mathcal{S}_{K^pK_p}^\diamondsuit} = \mathcal{S}_{K^p}^\diamondsuit \times^{\underline{K_p}} \underline{\mathbf{G}(\mathbf{Q}_p)}$ defines a pro-étale $\underline{\mathbf{G}(\mathbf{Q}_p)}$-torsor over $\mathcal{S}_{K^pK_p}^\diamondsuit$, which gives rise to an associated $\mathbf{G}(\mathbf{Q}_p)$-local system $\mathbf{V}$ over $\mathcal{S}_{K^pK_p}$ by Theorem 3. By a remarkable result of Liu-Zhu [4], $\mathbf{V}$ is de Rham with Hodge cocharacter μ, so Theorem 5 applies.

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Sansuc's Formula and Tate Global Duality (d'après Rosengarten)

Brian Conrad

1. Motivation

I am reporting on work of my PhD student Zev Rosengarten concerning the *Tamagawa number* τ_G of smooth connected affine groups G over global function fields. Let's first review some basics over global fields k. (See [Oes, Ch. I] for details.) A nonzero left-invariant $\omega \in \Omega^{\mathrm{top}}_{G/k}(G)$ (unique up to $k^\times$-scaling) defines a left Haar measure $\mu_{\omega,v}$ on $G(k_v)$ for any place v of k, and $\mu_{c\omega,v} = |c|_v \cdot \mu_{\omega,v}$ for all $c \in k^\times$. Thus, one arrives at a restricted direct product $\prod' \mu_{\omega,v}$ on $G(\mathbf{A}_k) = \prod' G(k_v)$ that is independent of ω except for the problem that this product measure doesn't generally converge (since for a non-empty finite set S of places of k containing the archimedean places and a smooth affine $O_{k,S}$-group $\mathbf{G}$ with generic fiber G, $\mu_{\omega,v}(\mathbf{G}(O_{k_v}))$ is too far from 1 for most $v \notin S$).

Considerations with the character group $\mathrm{X}(G_{k_s})$ as a $\mathrm{Gal}(k_s/k)$-module lead one to the definition of suitable "convergence factors" λ_v to multiply against $\mu_{\omega,v}$

for all place v, yielding a convergent restricted direct product μ_G on $G(\mathbf{A}_k)$ that is a *canonical* left Haar measure. The precise definition of μ_G involves some additional global scalar factors to ensure good behavior with respect to Weil restriction through a finite extension of global fields.

The measure μ_G induces a preferred Haar measure μ_G^1 on the *unimodular* group

$$G(\mathbf{A}_k)^1 = \{g \in G(\mathbf{A}_k) \mid \|\chi(g)\|_k = 1 \text{ for all } \chi \in \mathrm{X}_k(G)\}$$

that contains $G(k)$ as a discrete subgroup. Working with $G(\mathbf{A}_k)^1$ enables one to bypass the problem that $\mathbf{A}_k^\times/k^\times$ has infinite volume, whereas its norm-1 subgroup is compact. The *Tamagawa number* τ_G of G is the volume of $G(\mathbf{A}_k)^1/G(k)$ relative to the quotient measure arising from μ_G^1.

It is non-obvious but true that τ_G is *always* finite. Over number fields this immediately reduces to the case of reductive G, which was settled by Borel [Bo] (resting on his earlier work on reduction theory with Harish-Chandra [BHC]). Over function fields, the finiteness was proved in the semisimple case by Harder [Ha] and in the solvable case by Oesterlé [Oes, IV, 1.3]; the general case is reduced to those cases over finite extensions in k in [Co, Thm. 1.3.1] as an application of the structure theory of pseudo-reductive groups [CGP].

The definition of μ_G is normalized to ensure that τ_G is invariant under Weil restriction through a finite extension of global fields [Oes, Ch. II]. Somewhat deeper, for a short exact sequence $1 \to G' \to G \to G'' \to 1$ of smooth connected affine k-groups there is a formula relating τ_G to $\tau_{G'}$ and $\tau_{G''}$ [Oes, III, §5], but it involves the intervention of several additional arithmetic invariants associated to this exact sequence (and so using this formula in proofs usually requires some effort).

In the study of τ_G for reductive G, two other arithmetic invariants often arise: $\mathrm{Pic}(G)$ and the degree-1 Tate–Shafarevich set $\mathrm{III}^1(k, G)$. These are always finite in the reductive case, by using class field theory (including Tate's duality theorems) and the structure of reductive groups to reduce to the case of simply connected semisimple G, for which $\mathrm{Pic}(G)$ and $\mathrm{III}^1(k, G)$ are (not obviously) trivial.

Ono [Ono] and Voskresenskii [V] proved $\tau_T = \#\mathrm{Pic}(T)/\#\mathrm{III}^1(k, T)$ for k-tori T. Considerations with the Siegel mass formula and various calculations worked out in [We] led to *Weil's Conjecture* that $\tau_G = 1$ whenever G is a simply connected semisimple k-group. (This is "consistent" with the case of tori since for such G both $\mathrm{Pic}(G)$ and $\mathrm{III}^1(k, G)$ are trivial.) Weil's conjecture was proved by Langlands–Lai–Kottwitz over number fields, and only recently by Lurie–Gaitsgory over function fields. Weil's Conjecture and the Ono–Voskresenskii formula for tori are unified by *Sansuc's formula* [S] saying that $\tau_G = \#\mathrm{Pic}(G)/\#\mathrm{III}^1(k, G)$ for reductive G. Over number fields, this formula holds without the reductivity hypothesis since G is (uniquely) an extension of a reductive group by a k-split smooth connected unipotent k-groups. But such an extension structure often *fails* to exist in characteristic $p > 0$ (e.g., $G = \mathrm{R}_{k^{1/p}/k}(\mathrm{SL}_p)/\mu_p$), and $\mathrm{Pic}(G)$ can be infinite in some non-split unipotent cases (e.g., $G = \{y^p = x + cx^p\} \subset \mathbf{G}_a^2$ for $c \in k - k^p$).

2. Results

For a field F and smooth connected affine F-group G, the group $\mathrm{Ext}^1_F(G, \mathrm{GL}_1)$ of (central) extensions of G by GL_1 is naturally isomorphic to the group $\mathrm{Pic}(G)_{\mathrm{prim}}$ of *primitive* classes $[L] \in \mathrm{Pic}(G)$ (i.e., $m_G^*(L) \simeq p_1^*(L) \otimes p_2^*(L)$).

Rosengarten proved several finiteness results beyond the familiar reductive case: (i) if G_{F_s} is F_s-rational then $\mathrm{Pic}(G)$ is finite, (ii) if $G(F)$ is Zariski-dense in G (e.g., F-unirational G) then $\mathrm{Pic}(G) = \mathrm{Pic}(G)_{\mathrm{prim}}$ whenever $\mathrm{Pic}(G)$ is finite, and (iii) if F is a global function field then $\mathrm{Pic}(G)_{\mathrm{prim}}$ is *always* finite (such finiteness fails over every local function field and every imperfect separably closed field). This motivated a general conjecture over function fields, beyond the reductive case:

Conjecture 1 (Rosengarten)**.** $\tau_G = \#\mathrm{Pic}(G)_{\mathrm{prim}}/\#\text{Ш}^1(k, G)$

Using the derived series of G, we obtain a canonical composition series for G each of whose successive quotients is either (a) commutative or (b) pseudo-semisimple (i.e., perfect with no nontrivial smooth connected unipotent normal k-subgroup). It is not evident if cases (a) and (b) imply Rosengarten's conjecture in general, due to the complicated behavior of Tamagawa numbers in short exact sequences. Nonetheless, these are the crucial cases, and Rosengarten's main result is:

Theorem 2. *The conjecture holds for commutative G and pseudo-semisimple G.*

The key points in the proof for pseudo-semisimple G are to use the structure theory of such groups to (i) show $\mathrm{Pic}(G)$ is finite in such cases, and (ii) make a link with Weil's Conjecture over *finite extensions* of k. Central isogenies intervene in this manner, and handling that requires a version of the Poitou–Tate 9-term exact sequence for finite commutative k-group schemes (recently proved in [Čes]).

3. Commutative case

The proof of Rosengarten's conjecture for commutative G requires a Poitou–Tate 9-term exact sequence for arbitrary commutative affine algebraic (i.e., finite type) k-group schemes $\mathcal{G}$. This involves the fppf abelian sheaf $\widehat{\mathcal{G}} := \mathcal{H}om(\mathcal{G}, \mathrm{GL}_1)$ on k-schemes that is never representable when $\mathcal{G}$ has a positive-dimensional unipotent quotient (so $\mathcal{G}$ and $\widehat{\mathcal{G}}$ do not play symmetric roles, in contrast with finite $\mathcal{G}$).

The construction of the 9-term exact sequence rests on a version of Tate local duality for fppf cohomology of $\mathcal{G}$ and $\widehat{\mathcal{G}}$, as well as directly defining perfect pairings

$$\langle \cdot, \cdot \rangle^{\text{Ш}}_{i,\mathcal{G}} : \text{Ш}^i(k, \mathcal{G}) \times \text{Ш}^{3-i}(k, \mathcal{G}) \to \mathbf{Q}/\mathbf{Z}$$

between finite groups for $i = 1, 2$. (The finiteness of these Ш^j's is not hard to prove.) Tate used a Galois-cocycle method to define $\langle \cdot, \cdot \rangle^{\text{Ш}}_{i,\mathcal{G}}$ for finite $\mathcal{G}$ with order not divisible by $\mathrm{char}(k)$. For an fppf-cocycle analogue, one has to prove for any commutative affine algebraic k-group scheme $\mathcal{G}$ that $\check{\mathrm{H}}^2(k, \mathcal{G}) \hookrightarrow \mathrm{H}^2(k, \mathcal{G})$ is an equality (and likewise for $\widehat{\mathcal{G}}$); this is proved by a delicate devissage via exactness of $\mathcal{G} \rightsquigarrow \widehat{\mathcal{G}}$. Such exactness requires proving that the fppf abelian sheaf $\mathcal{E}xt^1_S(\mathcal{G}, \mathrm{GL}_1)$ vanishes for any k-scheme S. (The key point is to prove $\mathrm{Ext}^1_R(\mathbf{G}_a, \mathrm{GL}_1) = 0$ for semiperfect $\mathbf{F}_p$-algebras R.) By a Leray spectral sequence, one then obtains:

Corollary 3. *There is a natural isomorphism* $\mathrm{Ext}^1_k(\mathcal{G}, \mathrm{GL}_1) \simeq \mathrm{H}^1(k, \widehat{\mathcal{G}})$ *for commutative affine algebraic* k*-group schemes* $\mathcal{G}$.

Coming back to τ_G, in the commutative case Oesterlé proved a formula [Oes, IV, 3.2] for τ_G involving contributions from $\text{Ш}^1(k, G)$ and $\text{Ш}^2(k, G)$ as well as from kernels, images, and cokernels arising in the Poitou–Tate 9-term sequence for G. Plugging in the *exactness* of that 9-term sequence, this formula collapses to

$$\#\mathrm{H}^1(k, \widehat{G})/\#\text{Ш}^1(k, G) = \#\mathrm{Ext}^1_k(G, \mathrm{GL}_1)/\#\text{Ш}^1(k, G),$$

which is exactly the desired formula. But how to prove *exactness* of the Poitou–Tate sequence for arbitrary commutative affine algebraic k-group schemes $\mathcal{G}$?

By Tate local duality for $\mathcal{G}$, at 2 of the 9 terms this expresses perfectness of the Ш^j-pairings discussed above. To illustrate the proof of exactness elsewhere, consider the most difficult case: for $\mathcal{G} = \mathbf{G}_a$, prove surjectivity of the natural map $\alpha_{\mathcal{G}} : \mathcal{G}(\mathbf{A}_k) \to \mathrm{H}^2(k, \widehat{\mathcal{G}})^*$ (where $(\cdot)^*$ denotes $\mathbf{Q}/\mathbf{Z}$-dual of a discrete group).

We need to understand $\mathrm{H}^2(F, \widehat{\mathbf{G}_a})$ for fields F of characteristic $p > 0$. Using some spectral sequences of Breen [Br], Rosengarten proved naturally

$$\mathrm{H}^2(F, \widehat{\mathbf{G}_a}) \simeq \mathrm{Ext}^2_F(\mathbf{G}_a, \mathrm{GL}_1) \simeq \mathrm{Br}(\mathbf{G}_{a,F})_{\mathrm{prim}}.$$

The latter is contained in $\mathrm{Br}(\mathbf{G}_{a,F})[p]$ due to the primitivity condition, and for *any* factorial regular affine $\mathbf{F}_p$-scheme Z there is a concrete description (due to Kato) of $\mathrm{Br}(Z)[p]$ in terms of $\Omega^1_{Z/\mathbf{F}_p}$. For $Z = \mathbf{G}_{a,k}$, this yields that $\mathrm{Br}(\mathbf{G}_{a,k})_{\mathrm{prim}}$ is a k-line with an explicit generator, enabling one to prove that when the target of $\alpha_{\mathbf{G}_{a,k}}$ is given the natural compact Hausdorff Pontryagin-dual topology, this map is *continuous with dense image*. But $\alpha_{\mathbf{G}_{a,k}}$ factors through the quotient $\mathbf{A}_k/k$ by class field theory, and that quotient is *compact*, so $\alpha_{\mathbf{G}_{a,k}}$ is surjective!

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Poisson's equations on Berkovich spaces

Shou-Wu Zhang

Let us recall the classical Poisson's equation on a compact Kähler manifold (X, ω) which is the foundation for the Hodge theory. We take a normalization $\int \omega^n = 1$ where $n = \dim X$. Then we have a Poisson's equation:

$$\Delta f = g, \qquad f, g \in C^\infty(X).$$

The uniqueness and existence of this equation are summarized by the following exact sequence:

$$0 \longrightarrow \mathbb{C} \longrightarrow C^\infty(X) \xrightarrow{\Delta} C^\infty(X) \xrightarrow{\int \cdot \omega^n} \mathbb{C} \longrightarrow 0.$$

One important application is the existence of metrics on line bundles on X with harmonic curvature form.

In this talk, we try to formulate a non-archimedean analogue of Poisson's equation on a variety X over an algebraically closed field C with a complete and nontrivial absolute value $|\cdot|$. We first need to define a notion of a curvature form and an operator $\frac{\partial\bar\partial}{\pi i}$. We use the notations in our paper [YZ] on Hodge index theorem with a modification: we change the notion " integrable metrized line bundle" to "dsp metrized line bundles", where dsp is the abbreviation of "difference of semi-positive". On X, we have a vector space $\widehat{\mathrm{Pic}}(X)_{\mathrm{dsp},\mathbb{R}}$ of dsp metrized $\mathbb{R}$-line bundles which includes a subspace $\widehat{\mathrm{Pic}}^0(X)_{\mathbb{R}}$ of flat metrized line bundles. We define the dsp metrized Neron–Severi group by

$$\widehat{\mathrm{NS}}(X)_{\mathbb{R}} := \widehat{\mathrm{Pic}}(X)_{\mathrm{dsp},\mathbb{R}} / \widehat{\mathrm{Pic}}^0(X)_{\mathbb{R}}.$$

For a metrized line bundle $\bar L \in \widehat{\mathrm{Pic}}(X)_{\mathrm{dsp},\mathbb{R}}$, define its first Chern form $c_1(\bar L)$ to be its class in $\widehat{\mathrm{NS}}(X)_{\mathbb{R}}$. We will work on the space $C(X)_{\mathrm{dsp}}$ of dsp functions f on $X(C)$ defined by requiring that the metrized line bundle $\widehat{O}(f) := (O_X, \|1\| = e^{-f})$ is dsp. For a dspfunction f on X, we define the operator

$$\frac{\partial\bar\partial}{\pi i} : C(X)_{\mathrm{dsp}} \longrightarrow \widehat{\mathrm{NS}}(X)_{\mathbb{R}}, \qquad \frac{\partial\bar\partial}{\pi i} f := c_1(\hat{O}(f)).$$

Let $C(X)$ denote the completion of $C(X)_{\mathrm{dsp}}$ with respect to the L^∞-norm. The intersection theory on the models of X over O_C defines to multilinear and continuous pairing:

$$C(X) \times \widehat{\mathrm{NS}}(X)^n \longrightarrow \mathbb{R}, \qquad (f, \bar L_1, \cdots, \bar L_n) \mapsto \int_X f c_1(\bar L_1) \cdots (\bar L_n).$$

By Gubler, $C(X)$ can be naturally identified with the space $C(X^{\mathrm{an}})$ continuous function on the Berkovich space X^{an}. Thus the above pairing define a so called Chambert–Loir measure $c_1(\bar L_1) \cdots c_1(\bar L_n)$ on X^{an}.

Let $\mathrm{Pic}(X)_+$ denote the positive cone in $\widehat{\mathrm{Pic}}(X)_{\mathrm{dsp},\mathbb{R}}$, namely $\mathbb{R}_+$-combinations of ample line bundles with semipositive metrics and let $\widehat{\mathrm{NS}}(X)_+$ denote its image in $\widehat{\mathrm{NS}}(X)_\mathbb{R}$. We take a Kähler form ω on X as an element in $\widehat{\mathrm{NS}}(X)_+$ with a normalization $\int \omega^n = 1$. Now we define a Lapace operator as

$$\Delta : \qquad C(X)_{\mathrm{dsp}} \longrightarrow L^1(X, \omega^n), \qquad \Delta(f) := \frac{\frac{\partial\bar\partial}{\pi i} f \omega^{n-1}}{\omega^n}.$$

Without further restriction to both spaces, it is hard to say anything meaningful about the kernel and the image of this operator. A *key point* of this talk is to put a condition, the so-called *ω-boundedness*, on both sides: We say a form $\alpha \in \widehat{\mathrm{NS}}(X)_\mathbb{R}$ is ω-bounded if there is an $\epsilon > 0$ such that both $\omega \pm \epsilon\alpha \in \widehat{\mathrm{NS}}(X)_+$; and we say a function $f \in C(X)_{\mathrm{dsp}}$ is ω-bounded if $\frac{\partial\bar\partial}{\pi i} f$ is ω-bounded; a measure if called ω-bounded if it is the linear combination of products $\alpha_1 \cdots \alpha_n$ of ω-bounded α_i. Let $\widehat{\mathrm{NS}}(X)_\omega$ denote the space of ω-bounded forms, $L^\infty_\omega(X)$ the space of ω-bounded functions, and $L^1_\omega(X)$ the space of ω-measures. Then we have a restricted Laplacien operator

$$\Delta : \qquad L^\infty_\omega(X) \longrightarrow L^1_\omega(X).$$

Conjecture 1. For a given $g \in L^1_\omega(X)$, the Poisson equation $\Delta f = g$ has a solution $f \in L^\infty_\omega(X)$ if and only if $\int g\omega^n = 0$.

Notice that the uniqueness of the Poisson equation has already been established as a consequence of the local Hodge index theorem in [YZ]. Equivalently, we show that the Laplacian equation $\Delta f = 0$ has only constant solutions. One consequence of the above conjecture is the existence of some canonical metric on any line bundle on X:

Conjecture 2. For any line bundle M on X, there is an dspmetrization $\bar M$ (unique up to scalar multiple) such that the curvature $c_1(\bar M)$ is ω-harmonic in the following sense: $c_1(\bar M)$ is ω-bounded, and satisfies the following equation of measures:

$$c_1(\bar M)\omega^{n-1} = \lambda(M)\omega^n$$

where $\lambda(M)$ is a constant defined by $c_1(M) \cdot [\omega]^n$ with $[\omega]$ the class in $\mathrm{NS}(X)_\mathbb{R}$ under the map $\widehat{\mathrm{NS}}(X)_\mathbb{R} \longrightarrow \mathrm{NS}(X)_\mathbb{R}$.

The following are some results about these conjectures:

(1) Conjecture 1 (and then 2) holds for curves X. In fact in this case, we can solve Poisson's equation using a Green's fucntions $g(x, y)$ for the volume form ω. We can start with a green function $g_0(x, y)$ for any volume form ω_0 (for example one associate to the admissible metrics), and define

$$g(x,y) = g_0(x,y) - \int g_0(x,y)\omega(x) - \int g_0(x,y)\mu(y) + \int g_0(x,y)\omega(x)\omega(y).$$

(2) Conjecture 1 (and then 2) holds for the case ω is a model metric. In fact, in this case $L^\infty_\omega(X)$ and $L^1_\omega(X)$ are both finite dimensional with same dimension, the quadratic form $\langle f, \Delta f\rangle_{L^2}$ is positive definite on $L^\infty_\omega(X)/\mathbb{C}$ by local Hodge index theorem. Thus Δ is bijective.

(3) Conjecture 1 (and then 2) holds for the case residue characteristic of C is 0, and ω is supported on a dual complex, by method of Boucksom–Favre–Jonsson.

(4) Conjecture 2 holds when ω comes from a polarized dynamical system in the sense that there is an endomorphism $f : X \longrightarrow X$ such that $f^*\omega = q\omega$ with q a constant > 1. This follows from the construction of admissible metrics for any line bundle in [YZ].

We would like to give an application of Conjecture 2 to a variety X over a global field K, including the function field $K = k(C)$ for projective curve over another field k with a fixed positive adelic metrized line bundle $\bar{L}$ on X. We write $\omega = c_1(\bar{L})/\deg L^{1/n}$ with $n = \dim X$.

Conjecture 3. Any line bundle M on X has an admissible merization $\bar{M}$ in the sense that at each place v of K the bundle $\bar{M}_v$ has harmonic curvature form $c_1(\bar{M}_v)$, and that

$$\bar{M} \cdot \omega^n = \lambda(M)\omega^{n+1}.$$

Moreover such a metrization is unique up to multiples from $\widehat{Div}(K)$ with degree 0.

If we write $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}} = \widehat{\mathrm{Pic}}(X)_{\mathbb{R}}/\widehat{\mathrm{Pic}}(K)_{\deg=0}$. The above conjecture gives a section to the projection $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{R}}$.

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Spreading-out for families of rigid analytic spaces

Ofer Gabber
(joint work with Brian Conrad)

Let K be a complete rank 1 valued field with ring of integers $\mathcal{O}_K$, A an adic nœtherian ring and $\varphi : A \to \mathcal{O}_K$ an adic morphism. We show that if $g : X \to Y$ is a proper flat morphism between rigid analytic spaces over K then locally on Y a flat formal model of g is the pullback of a proper flat morphism between formal schemes topologically of finite type over A. For this, if S is an affine nœtherian scheme, $T_0 \to S$ affine of finite type and $X_0 \to T_0$ proper flat, we construct a compatible system of versal n-th order deformations of $X_0 \to T_0$ over S. As an application, one can prove that for a proper smooth g and K of characteristic 0, the Hodge to de Rham spectral sequence for g degenerates and the $R^q g_* \Omega^p_{X/Y}$ are locally free. This is reduced to the case where K is a finite extension of $\mathbb{Q}_p$ and Y is a nilpotent thickening of $\mathrm{Sp}\, K$, where the result over K was proved by Scholze and follows for Y by imitating the proof of Deligne over $\mathbb{C}$ using a construction of crystalline cohomology in this case.

Arithmetic Restrictions on Geometric Monodromy

Daniel Litt

Let X be an algebraic variety over $\mathbb{C}$. Let Λ be some coefficient ring, e.g. $\mathbb{Z}, \mathbb{Z}_\ell$, etc. Suppose that

$$\rho : \pi_1(X(\mathbb{C})^{\mathrm{an}}) \to GL_n(\Lambda)$$

is a representation which arises from geometry, e.g. as a monodromy representations associated to a locally constant subquotient of $R^i\pi_*\underline{\Lambda}$, where $\pi : Y \to X$ is a morphism of algebraic varieties (and where the subquotient also arises from geometry). We study the restrictions this geometricity condition places on ρ. Some sample applications of our technique are:

Theorem 1. *Let A be a non-constant Abelian scheme over $\mathbb{C}(t)$, and suppose that $A[\ell]$ is split for some odd prime ℓ. Then A has at least four points of bad reduction.*

This result follows from:

Theorem 2. *Let k be a field with prime subfield $k_0 \subset k$. Let $x_1, \cdots, x_m \subset \mathbb{P}^1_k(k)$, and let $X = \mathbb{P}^1_k \backslash \{x_1, \cdots x_m\}$. Then there exists an explicit N such that if*

$$\rho : \pi_1^{ét}(X_{\bar{k}}) \to GL_n(\mathbb{Z}_\ell)$$

is a representation arising from geometry and trivial mod $\ell^r > N$ for some r, then ρ is unipotent. If ρ is pure, it is in fact trivial. (Here if we let

$$L = k_0 \left(\frac{x_a - x_b}{x_c - x_d} \right)_{a \leqslant b \leqslant c \leqslant d}$$

be the field of moduli of $\{x_1, \cdots, x_m\}$, we have that N only depends on

$$\mathrm{im}(\chi : \mathrm{Gal}(\overline{L}/L) \to \widehat{\mathbb{Z}}),$$

where χ is the cyclotomic character.)

The proof uses "anabelian methods"—in particular, it is a global version of Grothendieck's proof of the local monodromy theorem [4, Appendix], and exploits the action of G_k on $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$, where $\bar{x}$ is a rational basepoint, as in [1]. See [3] for detailed proofs.

The proofs involve several notions of independent interest. Recall that if X is a variety over a field K and

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}) \to GL_n(\mathbb{Z}_\ell)$$

is a continuous representation arising from geometry, there exists a finite-type $\mathbb{Z}$-algebra R and a model $\mathscr{X}/R$ of X such that ρ extends to a representation

$$\tilde{\rho} : \pi_1^{\text{ét}}(\mathscr{X}) \to GL_n(\mathbb{Z}_\ell).$$

Thus to study ρ, it is natural to either

(1) specialize to the case where k is a finite field, by specializing to a well-chosen closed point of $\mathrm{Spec}(R)$, or
(2) pass to the case where k is a p-adic field.

In [3], we pursue both of these methods; in this report, we only describe method (1). Let ℓ be a prime and let $\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})$ be the maximal pro-ℓ quotient of the geometric étale fundamental group of X with basepoint $\bar{x}$ (as defined in [2]). We define the $\mathbb{Z}_\ell$-group algebra of $\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})$ via

$$\mathbb{Z}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]] = \varprojlim_{\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x}) \twoheadrightarrow H} \mathbb{Z}_\ell[H],$$

where the inverse limit is taken over finite quotients of π_1 by open subgroups. We let $\mathscr{I}$ be the augmentation ideal of $\mathbb{Z}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]$ and define the completed group $\mathbb{Q}_\ell$-group algebra of π_1 via

$$\mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]] = \varprojlim_n (\mathbb{Z}_\ell[[\pi_1^{\text{ét},\ell}(X_{\ bark}, \bar{x})]]/\mathscr{I}^n \otimes \mathbb{Q}_\ell).$$

We first show that

Proposition 3. *Let X be a smooth variety with simple normal crossings compactification over a finite field k. Let $x \in X(k)$ be a rational point, and choose an embedding $k \hookrightarrow \bar{k}$; let $\bar{x}$ be the associated geometric point of X. Then the induced G_k-action on*

$$\mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]/\mathscr{I}^n$$

is semisimple. In particular, the $\mathscr{I}$-adic filtration of $\mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]$ splits G_k-equivariantly.

On the other hand, the $\mathscr{I}$-adic filtration of $\mathbb{Z}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]$ does not split G_k-equivariantly. The key to proving Theorems 1 and 2 is to measure and exploit the extent to which this splitting fails.

We first introduce certain *convergent group rings*

$$\mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]^{\leqslant \ell^{-r}} \subset \mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]],$$

defined as follows. Let $\pi_n : \mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]] \to \mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]/\mathscr{I}^n$ be the obvious quotient map, and for $x \in \mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]/\mathscr{I}^n$, we define its valuation

$$v_n(x) := -\inf\{v_\ell(s) \mid s \in \mathbb{Q}_\ell \text{ such that } s \cdot x \in \mathbb{Z}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]\}.$$

Then for fixed $r \in \mathbb{R}_{>0}$, we define

$$\mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]^{\leqslant \ell^{-r}} := \{x \in \mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]] \mid v_n(\pi_n(x)) + rn \to \infty\}.$$

That is, $\mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x})]]^{\leqslant \ell^{-r}}$ consists of those elements in the $\mathscr{I}$-adically completed group ring whose denominators grow at a rate bounded by rn.

The relevant property that these convergent group rings have is the following:

Proposition 4. *Let*

$$\rho : \pi_1^{\text{ét},\ell}(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{Z}_\ell)$$

be a continuous representation which is trivial mod ℓ^r. Then for any $r_0 < r$, there is a unique commutative diagram of ring homomorphisms

$$\begin{array}{ccc} \mathbb{Z}_\ell[[\pi_1^{ét,\ell}(X_{\bar{k}},\bar{x})]] & \xrightarrow{\mathbb{Z}_\ell[[\rho]]} & \mathfrak{gl}_n(\mathbb{Z}_\ell) \\ \downarrow & & \downarrow \\ \mathbb{Q}_\ell[[\pi_1^{ét,\ell}(X_{\bar{k}},\bar{x})]]^{\leqslant \ell^{-r_0}} & \dashrightarrow & \mathfrak{gl}_n(\mathbb{Q}_\ell), \end{array}$$

where the top horizontal arrow is induced by ρ and the vertical arrows are the obvious inclusions.

The dotted arrow in Proposition 4 is Frobenius-equivariant; thus if the infinite-dimensional vector space $\mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}},\bar{x})]]^{\leqslant \ell^{-r_0}}$ admits a set of Frobenius-eigenvectors whose span is dense, the weights of these eigenvectors obstruct the existence of such ρ. Thus the key to the proofs of Theorems 1 and 2 is the following:

Theorem 5. *Let k be a finite field and X/k a smooth variety admitting a simple normal crossings compactification $\overline{X}$. Suppose that X satisfies $(\star)$: $\mathscr{I}^n/\mathscr{I}^{n+1}$ is torsion free for all n, where $\mathscr{I}$ is the augmentation ideal in $\mathbb{Z}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}},\bar{x})]]$. Then if*

(1) $\pi_1^{ét,\ell}(X_{\bar{k}},\bar{x})$ is nilpotent, or
(2) $H^1(X_{\bar{k}},\mathbb{Q}_\ell)$ is pure of weight 2 (e.g. if $\bar{X}$ is simply connected, for example if it is $\mathbb{P}^1$), or
(3) $\dim_{\mathbb{Q}_\ell} H^1(X_{\bar{k}},\mathbb{Q}_\ell) = 2$ and ℓ is sufficiently large, then

there exists an explicit $r > 0$ such that $\mathbb{Q}_\ell[[\pi_1^{ét,\ell}(X_{\bar{k}},\bar{x})]]^{\leqslant \ell^{-r}}$ is admits a set of Frobenius eigenvectors whose span is dense. Moreover the weights occuring in $\mathscr{I}^n$ are at most $-n$.

We conjecture that the conclusions of Theorem 5 essentially always hold:

Conjecture 6. Let X/k be any variety satisfying condition $(\star)$ above, and let ℓ be any prime. Then there exists an $r > 0$ such that $\mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}},\bar{x})]]^{\leqslant \ell^{-r}}$ admits a set of Frobenius eigenvectors whose span is dense.

Equivalently, the Frobenius eigenvectors $\{\gamma_i\}$ in $\mathbb{Q}_\ell[[\pi_1^{\text{ét},\ell}(X_{\bar{k}},\bar{x})]]$ satisfy

$$v_n(\pi_n(\gamma_i)) = O(n).$$

As evidence for this conjecture, we prove:

Theorem 7. *Let X/k be any variety satisfying condition $(\star)$. Then for almost all primes ℓ, the Frobenius eigenvectors $\{\gamma_i\}$ in $\mathbb{Q}_\ell[[\pi_1^{ét,\ell}(X_{\bar{k}},\bar{x})]]$ satisfy $v_n(\pi_n(\gamma_i)) = O(n\log n)$. For the remaining ℓ, one has $v_n(\pi_n(\gamma_i)) = O(n^2)$.*

The proof of this theorem relies a p-adic version of Baker's theorem on linear forms in (p-adic) logarithms, due to Yu [5].

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Cohomologie p-adique de la tour de Drinfeld: le cas de dimension 1

Pierre Colmez

Il s'agit d'un travail en commun avec Gabriel Dospinescu et Wiesława Nizioł dans lequel nous prouvons que la cohomologie étale p-adique des revêtements du demi-plan de Drinfeld encode la correspondance de Langlands locale p-adique pour les représentations de de Rham (il est maintenant classique que la cohomologie étale ℓ-adique, pour $\ell \neq p$, encode la correspondance de Langlands locale classique en même temps que celle de Jacquet-Langlands).

Soient $G = \mathbf{GL}_2(\mathbf{Q}_p)$ et $\check{G} = D^*$, où D est l'algèbre de quaternions sur $\mathbf{Q}_p$. Soit $\mathcal{M}_\infty$ la tour de Drinfeld en niveau infini (non complétée): c'est un revêtement du demi-plan de Drinfeld $\Omega_{\rm Dr} = \mathbf{P}^1 - \mathbf{P}^1(\mathbf{Q}_p)$ de groupe de Galois $\check{G}$. Par ailleurs l'action naturelle de G sur $\Omega_{\rm Dr}$ (par homographies) s'étend à $\mathcal{M}_\infty$. Enfin, $\mathcal{M}_\infty$ est définie sur $\mathbf{Q}_p^{\rm nr}$, mais est munie d'une action du groupe de Weil $\mathrm{W}_{\mathbf{Q}_p}$ de $\mathbf{Q}_p$, et les actions de G, G' et $\mathrm{W}_{\mathbf{Q}_p}$ commutent. Il s'ensuit que la cohomologie géométrique de $\mathcal{M}_\infty$ (i.e. celle de $\mathbf{C}_p \times \mathcal{M}_\infty$) est munie d'une action de $G \times \check{G} \times \mathrm{W}_{\mathbf{Q}_p}$.

Soit V une représentation p-adique de dimension 2 du groupe de Galois absolu $\mathrm{Gal}_{\mathbf{Q}_p}$ de $\mathbf{Q}_p$. Disons que V est *pertinente* si V est de de Rham, à poids de Hodge-Tate 0 et 1, et si la représentation du groupe de Weil-Deligne associée est irréductible. Grâce aux correspondances de Langlands locales p-adique et classique et à la correspondance de Jacquet-Langlands, on peut associer à V les objets suivants:

- $\mathrm{LL}(V)$, représentation lisse, supercuspidale, de G,
- $\Pi(V)$, représentation unitaire admissible de G,
- $\mathrm{JL}(V)$, représentation irréductible de dimension finie de $\check{G}$,

et $\mathrm{LL}(V)$ est l'espace des vecteurs lisses de $\Pi(V)$ et est inclus dans l'espace $\Pi(V)^{\rm an}$ des vecteurs localement analytiques.

Théorème 1. [2] *Soit V une représentation p-adique irréductible de* $\mathrm{Gal}_{\mathbf{Q}_p}$*, de dimension* 2.

(i) *Alors*

$$\mathrm{Hom}_{\mathrm{W}_{\mathbf{Q}_p}}(V, \mathbf{Q}_p \otimes H^1_{\mathrm{et}}(\mathcal{M}_\infty, \mathbf{Z}_p(1))) = \begin{cases} \mathrm{JL}(V)^* \otimes \Pi(V)^* & \textit{si } V \textit{ est pertinente,} \\ 0 & \textit{sinon.} \end{cases}$$

(ii) *Si* V *est pertinente,*

$$\mathrm{Hom}_{\mathrm{W}_{\mathbf{Q}_p}}(V, H^1_{\mathrm{et}}(\mathcal{M}_\infty, \mathbf{Q}_p(1))) = \mathrm{JL}(V)^* \otimes (\Pi(V)^{\mathrm{an}})^*.$$

Le (i) est une conséquence du (ii) et des résultats de [3]. La preuve du (ii) repose sur deux ingrédients: la conjecture de Breuil-Strauch prouvée par Dospinescu et Lebras [5] qui décrit le complexe de de Rham de $\mathcal{M}_\infty$ en termes de la correspondance de Langlands locale p-adique, et le théorème ci-dessous qui décrit la cohomologie étale surconvergente d'un affinoïde de dimension 1 en termes de la cohomologie de Hyodo-Kato surconvergente (définie par Grosse-Klönne). (Le calcul de $H^1_{\mathrm{et}}(\mathcal{M}_\infty, \mathbf{Q}_p(1))$ s'en déduit en écrivant $\mathcal{M}_\infty$ comme une réunion croissante d'affinoïdes (non connexes).)

Soit Y un affinoïde connexe de dimension 1 défini sur une extension finie K de $\mathbf{Q}_p$, et soit $Y^\dagger$ l'espace surconvergent associé (en ayant étendu les scalaires à $\mathbf{C}_p$). Les espaces vectoriels $H^1_{\mathrm{dR}}(Y^\dagger)$ et $H^1_{\mathrm{HK}}(Y^\dagger)$ sont de dimension finie sur $\mathbf{C}_p$ et $\mathbf{Q}_p^{\mathrm{nr}}$ respectivement, et on a un isomorphisme $\iota_{\mathrm{HK}} : \mathbf{C}_p \otimes H^1_{\mathrm{HK}}(Y^\dagger) \cong H^1_{\mathrm{dR}}(Y^\dagger)$.

Théorème 2. *On dispose d'un diagramme commutatif naturel, muni d'une action de* Gal_K*,*

$$\begin{array}{ccccccccc}
C & \longrightarrow & \mathcal{O}_{Y^\dagger} & \xrightarrow{\exp} & H^1_{\mathrm{et}}(Y^\dagger, \mathbf{Q}_p(1)) & \longrightarrow & (\mathbf{B}^+_{\mathrm{st}} \otimes H^1_{\mathrm{HK}}(Y^\dagger))^{\varphi=p, N=0} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathrm{dlog}} & & \downarrow{\scriptstyle \theta\otimes\iota_{\mathrm{HK}}} & & \\
C & \longrightarrow & \mathcal{O}_{Y^\dagger} & \xrightarrow{d} & \Omega^1_{Y^\dagger} & \xrightarrow{\pi_{\mathrm{dR}}} & H^1_{\mathrm{dR}}(Y^\dagger) & \longrightarrow & 0
\end{array}$$

dans lequel les lignes sont exactes, et les flèches verticales injectives.

Il ressort de ce théorème que la cohomologie étale p-adique géométrique d'un affinoïde (surconvergent), bien que très grosse, possède des propriétés de finitude raisonnables: c'est une extension d'un Espace de Banach de Dimension finie par les sections globales d'un faisceau cohérent.

Une démonstration de ce théorème consiste à utiliser l'isomorphisme entre la cohomologie étale et la cohomologie syntomique [4]: cette dernière s'exprime naturellement en utilisant les objets intervenant dans le diagramme. Une autre preuve consiste à calculer la cohomologie étale en termes de symboles; ceci fait apparaître le revêtement universel $\widehat{J}$ du groupe p-divisible de la jacobienne d'une compactification de Y, et on relie $\widehat{J}$ à $(\mathbf{B}^+_{\mathrm{st}} \otimes H^1_{\mathrm{HK}}(Y^\dagger))^{\varphi=p,N=0}$ en utilisant l'intégration p-adique (en particulier les résultats de [1]).

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Étale and crystalline companions

Kiran S. Kedlaya

Let X be a smooth scheme over a finite field of characteristic p. For each prime ℓ, one has a Weil cohomology theory with ℓ-adic coefficients and an associated category of coefficient objects of locally constant rank. For $\ell \neq p$, the cohomology theory is étale cohomology and the coefficient objects are Weil sheaves; for $\ell = p$, the cohomology theory is rigid cohomology and the coefficient objects are overconvergent F-isocrystals. See [6, 7] for more precise definitions.

Theorem 1. *Suppose that $\mathcal{F}$ is a coefficient object on X which is* algebraic*, that is, the characteristic polynomials of Frobenius at each closed point $x \in X$ has coefficients in $\overline{\mathbb{Q}}$. Then these coefficients all belong to a single number field.*

Theorem 2. *Let ℓ' be a prime and fix embeddings of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Q}}_{\ell'}$. Let $\mathcal{F}$ be an algebraic ℓ-adic coefficient object. If $\dim(X) = 1$ or $\ell' \neq p$, then there exists an algebraic ℓ'-adic coefficient object $\mathcal{F}'$ such that for each closed point $x \in X$, the characteristic polynomials of Frobenius on $\mathcal{F}$ and $\mathcal{F}'$ at x coincide (via the chosen embeddings). We say that $\mathcal{F}'$ is a* companion *of $\mathcal{F}$.*

For $\dim(X) = 1$ and $\ell, \ell' \neq p$, Theorem 1 and Theorem 2 are due to L. Lafforgue [8] via the Langlands correspondence for GL_n. For $\dim(X) = 1$ and ℓ, ℓ' arbitrary, the work of Lafforgue was adapted by Abe [1] to obtain Theorem 1 and Theorem 2 as fully stated (including the p-adic cases).

Now suppose $\dim(X)$ is arbitrary. For $\ell \neq p$, Theorem 1 was established by Deligne [4] using one-dimensional companions. This argument is essentially axiomatic, and so extends easily to the case $\ell = p$, as originally observed by Abe–Esnault [2]. Similarly, for $\ell, \ell' \neq p$, Theorem 2 was established by Drinfeld [5] using one-dimensional companions, and this has been extended to the case $\ell = p$, $\ell' \neq p$ by Abe–Esnault.

We are unable to establish the analogue of Theorem 2 for $\ell' = p$; this would include Deligne's conjecture on the existence of *petits camarades cristallines* [3, Conjecture 1.2.10]. Instead, we have the following partial result on the variation of Newton polygons for ℓ-adic coefficients [7].

Theorem 3. *Let $\mathcal{F}$ be an algebraic ℓ-adic coefficient object on X. For each closed point $x \in X$, let $N_x(\mathcal{F})$ be the normalized p-adic Newton polygon of Frobenius on $\mathcal{F}_x$.*

(a) The function $x \mapsto N_x(\mathcal{F})$ extends to an upper semicontinuous function on X.

(b) The function $x \mapsto N_x(\mathcal{F})$ jumps purely in codimension 1.

These statements would follow from the existence of a p-adic companion via the specialization and purity theorems of Grothendieck, Katz, and de Jong–Oort. We instead deduce them by judicious use of one-dimensional companions plus some additional arguments concerning the stability polygons (Harder-Narasimhan polygons) of vector bundles on punctured Riemann surfaces admitting logarithmic connections.

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Patching and the p-adic Langlands correspondence for $\mathbf{GL_2}(\mathbb{Q}_p)$

Vytautas Paškūnas

(joint work with Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Sug Woo Shin)

Let F be a finite extension of $\mathbb{Q}_p$, and let G_F be its absolute Galois group. One would like to have an analog of a local Langlands correspondence for the p-adic representations of G_F. Let E be another finite extension of F, which will be our field of coefficients, with the ring of integers $\mathcal{O}$, a uniformizer ϖ and residue field k. To a continuous representation $r : G_F \to \mathrm{GL}_n(E)$ one would like to a attach (possibly a family of) an admissible unitary E-Banach space representation $\Pi(r)$ of $G := \mathrm{GL}_n(F)$. Such a hypothetical construction should encode the classical Langlands correspondence in the sense that if r is potentially semi-stable with regular Hodge–Tate weights then the subspace of locally algebraic vectors $\Pi(r)^{\mathrm{alg}}$ in $\Pi(r)$ is isomorphic to $\pi_{\mathrm{sm}}(r) \otimes \pi_{\mathrm{alg}}(r)$ as a G-representation, where $\pi_{\mathrm{sm}}(r)$ is a smooth representation of G corresponding to the Weil–Deligne representation attached to r by Fontaine, and $\pi_{\mathrm{alg}}(r)$ is an algebraic representation of G, which encodes the information about the Hodge–Tate weights of r. For example if $F = \mathbb{Q}_p$, $n = 2$ and r is crystalline with Hodge-Tate weights $a < b$, then $\pi_{\rm sm}(r)$ is a smooth unramified principal series representation, whose Satake parameters can be calculated in terms of the trace and determinant of Frobenius on $D_{\rm cris}(r)$, and $\pi_{\rm alg}(r) = \mathrm{Sym}^{b-a-1} E^2 \otimes \det^{m(a,b)}$, where the exponent $m(a,b)$ depends on a choice of normalization.

Such a correspondence has been established in the case of $n = 2$ and $F = \mathbb{Q}_p$ by the works of Breuil, Colmez and others. However, not much is known beyond that case. In [4] we have constructed a candidate for such a correspondence using Taylor–Wiles–Kisin patching method, which is employed to prove modularity lifting theorems for Galois representations. We will describe the end product of the paper [4] now.

Let $\bar{r} : G_F \to \mathrm{GL}_n(k)$ be a continuous representation and let $R_p^{\square}$ be its universal framed deformation ring. Under the assumption that p does not divide $2n$ we construct an $R_\infty[G]$-module M_∞, which is finitely generated as a module over the completed group algebra $R_\infty[[\mathrm{GL}_n(\mathcal{O}_F)]]$, where R_∞ is a complete local noetherian $R_p^{\square}$-algebra with residue field k. If $y \in \operatorname{Spec} R_\infty$ is an E-valued point then $\Pi_y := \mathrm{Hom}_{\mathcal{O}}^{\rm cont}(M_\infty \otimes_{R_\infty, y} \mathcal{O}, E)$ is an admissible unitary E-Banach space representation of G. The composition $R_p^{\square} \to R_\infty \xrightarrow{y} E$ defines an E-valued point $x \in \operatorname{Spec} R_p^{\square}$ and thus a continuous Galois representation $r_x : G_F \to \mathrm{GL}_n(E)$. We expect that the Banach space representation Π_y depends only on x and it should be related to r_x by the hypothetical p-adic Langlands correspondence. We show in [4] that if x lies on an automorphic component of a potentially crystalline deformation ring of $\bar{r}$ then $\Pi_y^{\rm alg} \cong \pi_{\rm sm}(r_x) \otimes \pi_{\rm alg}(r_x)$ as expected. (The Fontaine–Mazur conjecture predicts that every irreducible component of a potentially crystalline ring is automorphic, but it is intrinsic to our method that we would not be able to access these non-automorphic components even if they existed.) However, there are a lot of natural questions regarding our construction that we cannot answer at the moment. For example, it is not clear that Π_y depends only on x, it is not clear that Π_y is non-zero for an arbitrary y, it is not at all clear that M_∞ does not depend on the different choices made during the patching process.

In this talk, we specialize the construction of [4] to the case $F = \mathbb{Q}_p$ and $n = 2$ (so that $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and $K := \mathrm{GL}_2(\mathbb{Z}_p)$ from now on) to confirm our expectation that M_∞ does realize the p-adic Langlands correspondence in this case and does not depend on the choices made. For simplicity let us assume that $\bar{r}$ is irreducible and let R_p be its universal deformation ring. Then $R_p^{\square}$ is formally smooth over R_p. Moreover, in the case, when $F = \mathbb{Q}_p$ and $n = 2$ we may assume that R_∞ is formally smooth over $R_p^{\square}$. The following definition is meant to isolate the key properties of the patched module M_∞.

Definition 1. Let d be a non-negative integer, let $R_\infty := R_p[[x_1, \dots, x_d]]$ and let M be a non-zero $R_\infty[G]$-module. We say that the action of R_∞ on M is *arithmetic* if the following hold:

(AA1) M is a finitely generated module over the completed group algebra $R_\infty[[K]]$;
(AA2) M is projective in the category of pseudo-compact $\mathcal{O}[[K]]$-modules;
(AA3) for each pair of integers $a < b$ the action of R_∞ on

$$M(\sigma^\circ) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M, (\sigma^\circ)^d)^d$$

factors through the action of $R_\infty(\sigma) := R_p(\sigma)[[x_1, \ldots, x_d]]$, where $R_p(\sigma)$ is a quotient of R_p parameterizing representations which are crystalline with Hodge–Tate weights (a, b), σ° is a K-invariant $\mathcal{O}$-lattice in $\sigma := \mathrm{Sym}^{b-a-1} E^2 \otimes \det^{m(a,b)}$, $(*)^d := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(*, \mathcal{O})$ denotes the Schikhof dual.

Moreover, $M(\sigma^\circ)$ is maximal Cohen–Macaulay over $R_\infty(\sigma)$ and the $R_\infty(\sigma)[1/p]$-module $M(\sigma^\circ)[1/p]$ is locally free of rank 1 over its support.
(AA4) for each σ as above and each $y \in \text{m-Spec}\, R_\infty[1/p]$ in the support of $M(\sigma^\circ)$ we have

$$\Pi_y^{\mathrm{alg}} \cong \pi_{\mathrm{sm}}(r_x) \otimes \pi_{\mathrm{alg}}(r_x).$$

The last condition says that M encodes the classical local Langlands correspondence. This motivated us to call such action arithmetic. (In fact in [5] we impose a slightly weaker condition than (AA4).) As already mentioned the patched module M_∞ of [4] carries an arithmetic action of R_∞ for some d.

Theorem 2. *Let M be an $R_\infty[G]$-module with an arithmetic action of R_∞.*

Let π be any irreducible G-subrepresentation of the Pontryagin dual $M^\vee$ of M then π is the representation of G associated to $\bar{r}$ by the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let $\pi \hookrightarrow J$ an injective envelope of π in the category of smooth locally admissible representations of G on $\mathcal{O}$-torsion modules and let P be the Pontryagin dual of J. Then P carries a unique arithmetic action of R_p. Moreover,

$$M \cong P \widehat{\otimes}_{R_p} R_\infty$$

as $R_\infty[G]$-modules.

The theorem shows that M_∞ does not depend on the choices made in the patching process. Its proof does not use (φ, Γ)-modules, which play a key role in Colmez's approach to p-adic Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$. The proof uses the foundational results of Barthel–Livné [1] and Breuil [2] on the classification of irreducible mod p representations of G, the fact that the rings $R_p(\sigma)$ are formally smooth over $\mathcal{O}$, when $1 \leqslant b - a \leqslant p$, the fact, proved by Kisin [7], that $R_p(\sigma)[1/p]$ are regular for all σ, and the fact that the associated Weil–Deligne representation together with Hodge–Tate weights determine a 2-dimensional crystalline representation of $G_{\mathbb{Q}_p}$ up to isomorphism.

The theorem implies that Π_y depends only on the image of y in $\mathrm{Spec}\, R_p$. However, we are still not able to deduce using only our methods that Π_y is non-zero for an arbitrary $y \in \text{m-Spec}\, R_\infty[1/p]$. To prove this we relate the arithmetic action of R_p on P to the results of [8], where an action of R_p on an injective envelope of π in the subcategory of representations with a fixed central character is constructed using Colmez's functor. Then by appealing to the results of [8] we show that Π_y

and r_x correspond to each other under the p-adic Langlands correspondence as defined by Colmez [3]. It follows from the construction of M_∞ that after quotienting out by a certain ideal of R_∞ we obtain a dual of completed cohomology, see [4, Corollary 2.11]. This property combined with the theorem enables us to obtain a new proof of local-global compatibility as in [6].

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Filtered de Rham-Witt complexes and wildly ramified higher class field theory over finite fields

Uwe Jannsen

(joint work with Shuji Saito, Yigeng Zhao)

Let k be a finite field of characteristic $p > 0$, and let X be a smooth projective variety of dimension d over k. Let $D \subset X$ be an effective divisor with simple normal crossings, and let $j : U = X\backslash D \hookrightarrow X$ be the open complement. Then for any prime $\ell \neq p$, each natural number m, and each integer j we have a perfect pairing of finite groups for the étale cohomology groups

$$H^i(U,\mathbb{Z}/l^m(j)) \times H_c^{2d+1-i}(U,\mathbb{Z}/\ell^m(d-j)) \longrightarrow H_c^{2d+1}(U,\mathbb{Z}/\ell^m(d) \cong \mathbb{Z}/\ell^m\,,$$

where $\mathbb{Z}/\ell^m(j)$ denotes the j-th Tate twist of the constant étale sheaf $\mathbb{Z}/\ell^m$, and $H_c^m(U,-)$ is the étale cohomology with compact support. This can be used to describe the quotient $\pi_1^{ab}(U)/\ell^n$ of the abelianized fundamental group π_1^{ab}, and, by passing to the inverse limit, the maximal abelian ℓ-adic quotient of π_1^{ab}. In fact, for $j = 0$ we get isomorphisms for all m

$$H_c^{2d}(U,\mathbb{Z}/\ell^m(d)) \cong H^1(U,\mathbb{Z}/\ell^m)^\vee \cong \pi_1^{ab}(U)/\ell^m\,,$$

and an exact sequence

$$H^{2d-1}(Y,\mathbb{Z}/\ell^m(d)) \to H_c^{2d}(U,\mathbb{Z}/\ell^m(d)) \to H^{2d}(X,\mathbb{Z}/\ell^m(d))$$

which provides a certain description of the middle group.

Now we consider p-coefficients. If D is empty, Milne obtained a perfect duality of finite groups

$$H^i(X, \nu_m^r) \times H^{d+1-i}(X, \nu_m^{d-r}) \to H^{d+1}(X, \nu_m^d) \cong \mathbb{Z}/p^r$$

where $\nu_m^r = \nu_{m,X}^r = W_m\Omega_{X,\log}^r \subset W_m\Omega_X^r$ are Illusie's logarithmic de Rham-Witt sheaves inside the components of the de Rham-Witt sheaves, which can be defined as the image of the $d\log[-]$ map

$$d\log[-] : \mathcal{K}_{r,X}^M/p^m \xrightarrow{\cong} \nu_m^r \subset W_m\Omega_X^r \,, \tag{1}$$

on the Milnor K-sheaf sending a symbol $\{a_1, \ldots, a_r\}$ to $d\log[a_1]_m \wedge \ldots \wedge d\log[a_r]_m$, where $[a]_m = (a, 0, \ldots, 0) \in W_m(\mathcal{O}_X)$ is the Teichmüller representative. One has $\nu_m^0 = \mathbb{Z}/p^m$, and Milne's duality induces isomorphisms

$$H^d(X, \nu_m^d) \cong H^1(X, \mathbb{Z}/p^m)^\vee \cong \pi_1^{ab}(X)/p^m$$

There is no obvious analog of cohomology with compact support for de Rham-Witt sheaves or logarithmic de Rham-Witt sheaves. We propose the following approach. Let $\{D_\lambda\}_{\lambda\in\Lambda}$ be the (smooth) irreducible components of D. For $r \geqslant 0$ and $D = \sum_\lambda n_\lambda D_\lambda$ with $n_\lambda \in \mathbb{N}$ let

$$\nu_{m,X|D}^r = W_m\Omega_{X|D,log}^r \subset j_* W_m\Omega_{U,\log}^r$$

be the étale subsheaf generated étale locally by $d\log[x_1]_m \wedge \ldots d\log[x_r]_m$ with $x_\nu \in \mathcal{O}_U^\times$ for all ν and $x_1 \in 1 + \mathcal{O}_X(-D)$.

Lemma 1. For $D_1 \leqslant D_2$ we have $\nu_{m,X|D_2}^r \subseteq \nu_{m,X|D_1}^r \subseteq \nu_{m,X}^r$.

Moreover we show

Theorem 1. There is an exact sequence

$$0 \to \nu_{m-1,X|[D/p]} \to \nu_{m,X|D} \to \nu_{1,X|D} \to 0 \,,$$

where $[D/p] = \sum_{\lambda\in\Lambda}[n_\lambda/p]D_\lambda$, with $[n_\lambda/p] = min\{n' \in \mathbb{Z} \mid n' \geqslant n/p\}$.

By the isomorphism (1) above this is reduced to (difficult) calculations in Milnor K-theory of local rings (see [1]). Moreover, in analogy to Illusie's exact sequence

$$0 \to \nu_{1,X}^r \to \Omega_X^r \xrightarrow{1-C^{-1}} \Omega_X^r/d\Omega_X^{r-1} \to 0 \tag{2}$$

we prove the following.

Theorem 2. There is an exact sequence

$$0 \to \nu_{1,X|D}^r \to \Omega_{X|D}^r \xrightarrow{1-C^{-1}} \Omega_{X|D}^r/d\Omega_{X|D}^{r-1} \to 0 \,,$$

where $\Omega_{X|D}^r = \Omega_X^r(logD_{red})(-D)$

An important tool for the duality is the introduction of Γ-filtered rings A for a not necessarily totally ordered abelian group Γ, given by collections of subgroups A^γ with $A^{\gamma'} \subset A^\gamma$ for $\gamma \leqslant \gamma'$ (descending filtration!) and $A^\gamma \cdot A^{\gamma'} \subset A^{\gamma+\gamma'}$.

For a Γ-filtered ring A we consider an associated filtration on the Witt rings which is inspired by a filtration introduced by Kato and Brylinski: We say that a Witt vector $a = (a_0, a_1, a_2, \ldots)$ is in $W(A)^\gamma$ if $a_i \in A^{p^i\gamma}$ for all $i \geqslant 0$.

Moreover, with this we define filtered de Rham-Witt sheaves, by using the universal definition of Hesselholt and Madsen [1] for $p \neq 2$, and Costeanu [3] for $p = 2$.

In our situation we start with the descending filtration on $j_*\mathcal{O}_U$, where for a divisor $D = \sum_\lambda n_\lambda D_\lambda$ with $n_\lambda \in \mathbb{Z}$ we define

$$f^D\mathcal{O}_U := \mathcal{O}_X(-D),$$

and the associated filtered de Rham-Witt complex

$$f^D W_m\Omega_U^r \subset j_* W_m\Omega_U^r.$$

They are closely related to the sheaves $\nu^r_{m,X|D}$ (compare Theorem 2). Then we get

Theorem 3. There is a a perfect pairing between a discrete group and a profinite group

$$H^i(U, \nu^r_{m,U}) \times \varprojlim_D H^{d+1-i}(X, \nu^{d-r}_{m,X|D}) \longrightarrow H^{d+1}(X, \nu^d_{m,X}) \cong \mathbb{Z}/p^m\mathbb{Z}$$

where the inverse limit is over the divisors (with multiplicities) D with support in D_{red} (compare Lemma 1).

The proof is in two steps. First of all, the pairings give a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to & \varprojlim_D H^{d+1-i}(X,\nu^{d-r}_{m-1,X|[D/p]}) & \to & \varprojlim_D H^{d+1-i}(X,\nu^{d-r}_{m,X|D}) & \to & \varprojlim_D H^{d+1-i}(X,\nu^{d-r}_{1,X|D}) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^i(U,\nu^r_{m-1,U})^\vee & \longrightarrow & H^i(U,\nu^r_m,U)^\vee & \longrightarrow & H^i(U,\nu^r_1)^\vee & \longrightarrow \cdots,
\end{array}$$

where the first row is induced by Theorem 1, and its exactness is due to the fact that the inverse limit is exact for projective systems of finite groups; and where the second row comes from the classical exact sequence

$$0 \to \nu^r_{m-1,U} \to \nu^r_{m,U} \to \nu^r_{1,U} \to 0.$$

so that its exactness is clear. Using this commutative diagram and induction on m, we reduce our question to the case $m = 1$. For $m = 1$ we noticed that $\Omega^r_U = \varinjlim_D \Omega^r_{X|-D}$, and the exact sequence

$$0 \to \nu^r_{1,U} \to Z\Omega^r_U \xrightarrow{1-C} \Omega^r_U \to 0$$

imply that $\nu^r_{m,U}$ can be written as the direct limit (with respect to D) of the following two term complex

$$\mathcal{F}^\bullet = [Z\Omega^r_{X|-D+D_{red}} \xrightarrow{1-C} \Omega^r_{X|-D+D_{red}}].$$

By Theorem 2 we have $\nu^{d-r}_{1,X|D}$ is isomorphic to the two-term complex

$$\mathcal{G}^\bullet = [\Omega^{d-r}_{X|D} \xrightarrow{F-1} \Omega^{d-r}_{X|D}/d\Omega^{d-r-1}_{X|D}]$$

and finally $\nu^d_{1,X}$ by the two term complex

$$\mathcal{H}^\bullet = [\Omega^d_X \xrightarrow{1-C} \Omega^d_X].$$

Now we use Milne's method of pairings of two term complexes to see that we have a non-degenerate pairing

$$\mathcal{F}^\bullet \times \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet$$

which reduces the pairing to a duality in coherent $\mathcal{O}_X$-sheaves, which also works in étale cohomology, and therefore also gives perfect parings in étale cohomology

$$H^{d+1-i}(X, \nu^{d-r}_{1,X|-D+D_{red}}) \times H^i(X, \nu^r_{1,X|D}) \longrightarrow H^{d+1}(X, \nu^1_{1,X}) \cong \mathbb{Z}/p^m\mathbb{Z}.$$

Passing to the inductive limit over D on the first terms and the inverse limit on D on the right term we obtain the wanted pairing, because the left limit gives $H^i(U, \nu^r_{1,U})$. The two steps above now induces a continuous isomorphism

$$\varprojlim_D H^{d+1-i}(X, \nu^{d-r}_{m,X|D}) \longrightarrow H^i(U, \nu^r_{m,U})^\vee \cong \pi_1^{ab}(U)/p^m$$

which gives a canonical ramification filtration of the abelianized fundamental group on the right. A quotient is ramified of order D if it factors through $H^{d+1-i}(X, \nu^{d-r}_{1,X|D})$, and we can define $\pi_1^{ab}(X, D)/p^m$ as a quotient of $\pi_1^{ab}(U)/p^m$ with ramified of order less or equal to D, which can be thought of as classifying abelian étale coverings of U of degree p^m with ramfication bounded by D.

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Generic vanishing for constructible sheaves, revisited

Bhargav Bhatt

(joint work with Christian Schnell, Peter Scholze)

More than 20 years ago, Green and Lazarsfeld initiated [4, 5] the study of generic vanishing theorems for varieties X equipped with a map to an abelian variety. Roughly, one studies conditions under which $H^i(X, \omega_X \otimes L)$ vanishes for degree 0 line bundles L pulled back from A. The resulting collection of results have been very useful in understanding the geometry of irregular varieties. In recent times, thanks to the work of many people, we have come to understand these results as vanishing results for certain sheaves on an abelian variety. The goal of our talk was to revisit some of these results in the context of constructible sheaves, and provide alternative proof that are more 'topological' or 'motivic' and less 'Hodge theoretic' in nature; ultimately, we seek a completely parallel theory that works in positive characteristic, though we are quite far yet.

Notation 1. If A is a complex torus of dimension g and k is a field of coefficients, we write $\mathrm{Char}(A)$ for the k-linear character variety of A. This is defined as $\mathrm{Spec}(R)$ for $R := k[\pi_1(A)]$, is abstractly isomorphic to $\mathbf{G}_m^{2g}$, and its S-valued points (for any k-algebra S) are naturally in correspondence with rank 1 S-local systems on A; given a point $\chi \in \mathrm{Char}(A)$, write L_χ for the corresponding rank 1 $\kappa(\chi)$-local system over A.

1. Basic generic vanishing

This talk began by explaining a simple proof of the following result:

Theorem 2. *Let A be a complex torus, let k be a field of coefficients, and let M be a perverse sheaf on A. Then there exists a Zariski dense open $U \subset \mathrm{Char}(A)$ such that $H^i(A, M \otimes_k L_\chi) = 0$ for $i \neq 0$ and $\chi \in \mathrm{Char}(A)$.*

When k has characteristic 0 and A is algebraic, this result is due (independently) to Schnell [3] and Kramer-Weissauer [1]. Schnell's proof relies on a result of Simpson in non-abelian Hodge theory, while Kramer-Weissauer employ the Tannakian formalism and the hard Lefschetz theorem. In contrast, the key ingredient in our proof is the simple observation that the universal cover of A is a Stein space.

Proof sketch. The tautological character $\pi_1(A) \to R^*$ defines a rank 1 R-local system L_R on A. Using L_R, there is a "Fourier" transform $\mathcal{F} : D_c^b(A, k) \to D_{perf}(R)$ from the constructible derived category of k-linear sheaves on A to perfect R-complexes defined by the formula

$$M \mapsto \mathcal{F}(M) := R\Gamma(A, M \otimes_k L_R).$$

By standard properties of the Fourier transform and its compatibility with duality, we are reduced to showing that $\mathcal{F}(M) \in D^{\geqslant 0}$ if M is perverse. For this, we simply observe that $L_R \simeq \pi_! k$ for $\pi : V \to A$ the universal cover. Using this fact, and the projection formula, we get

$$\mathcal{F}(M) \simeq R\Gamma_c(V, \pi^* M),$$

which lies in $D^{\geqslant 0}$ since V is a Stein space and $\pi^* M$ is perverse. □

Remark 3. Theorem 2 implies the following result for constructible sheaves: if N is a constructible k-linear sheaf on A, then there exists a Zariski dense open $U \subset \mathrm{Char}(A)$ such that $H^i(A, N \otimes L_\chi) = 0$ for $i > \dim(\mathrm{Supp}(N))$ and $\chi \in U$.

Remark 4. The compatibility with duality shows that $\mathcal{F}(M)$ and its R-linear dual $\mathcal{F}(M)^\vee$ lie in $D^{\geqslant 0}$ for M perverse. This formally implies that $\mathrm{codim}(H^i(\mathcal{F}(M))) \geqslant i$ for all i.

2. Refined codimension estimates

Assume now that k has characteristic 0 and that A is algebraic. Then the codimension estimate from Remark 4 can be strengthened as follows:

Theorem 5 (Schnell). *For a k-linear perverse sheaf M on A, we have* $\mathrm{codim}(H^i(\mathcal{F}(M))) \geqslant 2i$.

Schnell's proof again uses non-abelian Hodge theory. In the talk, we explained an alternative proof using the Hard Lefschetz theorem on A.

Proof sketch. Fix an ample line bundle with first Chern class $c \in H^2(A, k)$. Cupping with c defines functorial maps $K[-i] \xrightarrow{c^i} K[i]$ for any $K \in D(A, k)$. The proof needs the following two observations about these maps.

(1) The association $N \mapsto L_N := L_R \otimes_R N$ defines a functor $D(R) \to D(A, k)$. One first checks this functor is fully faithful; this amounts to A being a $K(\pi, 1)$. It follows that if $N = \kappa(\chi)$ for a point $\chi \in \mathrm{Char}(A)$ of codimension $j < 2i$, then the induced map $L_N[-i] \xrightarrow{c^i} L_N[i]$ is 0 (as N has projective dimension j as an R-module).

(2) Say M is a simple k-linear perverse sheaf. Then the map $M[-i] \xrightarrow{c^i} M[i]$ induces an isomorphism on $H^0(A, -)$ by the Hard Lefschetz theorem.

Now fix χ as in (1) and M as in (2), and consider the map $M \otimes_k L_\chi[-i] \xrightarrow{c^i} M \otimes_k L_\chi[i]$. The induced map on $H^0(A, -)$ is both 0 and an isomorphism by (1) and (2). It follows that $H^i(A, M \otimes_k L\chi) = 0$ if $j < 2i$. This translates to $\mathcal{F}(M) \otimes_R \chi$ having no H^i. By Nakayama, one concludes that $H^i(\mathcal{F}(M))$ is not supported at any such χ, as wanted. □

3. Linearity

Let S be the completion of R at the trivial character, so S identifies with $\widehat{\mathrm{Sym}}(H_1(A, k))$. The last part of the talk focussed on identifying the completed stalk of the Fourier transform of a simple perverse sheaf in terms of an explicit 'linear' complex (due to Popa-Schnell [2], building on Green-Lazarsfeld [5] for trivial coefficients). This result (as well as a twisted variant) is the key ingredient in showing that the supports of the cohomology sheaves of the Fourier transform of perverse sheaves are unions of translates of subtori in $\mathrm{Char}(A)$. The result states:

Theorem 6 (Popa-Schnell). *Let M be a simple perverse sheaf of geometric origin. Then there is a quasi-isomorphism*

$$\mathcal{F}(M) \otimes_R S \simeq \Big(....H^i(A, M) \otimes_k S \to H^{i+1}(A, M)... \Big),$$

with the differential on the right is defined by the map $H^i(A, M) \to H^{i+1}(A, M) \otimes H_1(A, k)$ dual to the cup product.

The proof in [2] uses Saito's theory of mixed Hodge modules. In the talk, we explained an alternative proof of this result. Our proof proceeds by reduction to characteristic p, reducing to $k = \overline{\mathbf{Q}_\ell}$, and using the theory of weights. More precisely, our main ingredient was the formality of the following objects: the E_∞-ring $E := R\Gamma(A, k)$ and the left E-module $R\Gamma(A, M)$. This method yields a canonical quasi-isomorphism as above. However, the twisted variant is not currently accessible by this method.

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Hodge theory and atypical intersections

Bruno Klingler

In this talk we presented a conjecture on the geometry of the Hodge locus of a (graded polarized, admissible) variation of mixed Hodge structure over a complex smooth quasi-projective base, generalizing the Zilber-Pink conjecture for mixed Shimura varieties.

The general context is the study of the following geometric problem. Let k be a field and let $f : \mathcal{X} \longrightarrow S$ be a smooth morphism of quasi-projective k-varieties. Can we describe the locus of points $s \in S$ where the motive $[\mathcal{X}_s]$ of the fiber $\mathcal{X}_s$ is "simpler" than the motive of the fiber at a very general point? Here "simpler" means that the fiber $\mathcal{X}_s$ and its powers contain more algebraic cycles than the very general fiber and its powers. If a Tannakian formalism of k-motives were available, this would be equivalent to saying that the motivic Galois group $\mathrm{GMot}(\mathcal{X}_s)$ is smaller than the motivic Galois group of the very general fiber. We restrict to $k = \mathbb{C}$ and consider the Hodge incarnation of our problem. Let $\mathbb{V} \to S$ be a variation of (graded plarized, admissible) mixed $\mathbb{Z}$-Hodge structures ($\mathbb{Z}$VMHS) over a smooth quasi-projective $\mathbb{C}$-variety S. A typical example of such a gadget is $\mathbb{V} = R^m f_* \mathbb{Z}$ for $f : X \longrightarrow S$ a smooth morphism. Replacing algebraic cycles by Hodge classes and motivated by the Hodge conjecture, one wants to understand the Hodge locus $\mathrm{HL}(S, \mathbb{V}) \subset S$, namely the subset of S of points s for which exceptional Hodge tensors for $\mathbb{V}_{\mathbb{Q},s}$ do occur.

The Tannakian formalism available for Hodge structures is particularly useful for describing $\mathrm{HL}(S, \mathbb{V})$. Recall that for every $s \in S$, the Mumford-Tate group $\mathbf{P}_s(\mathbb{V})$ of the Hodge structure $\mathbb{V}_{\mathbb{Q},s}$ is the Tannakian group of the Tannakian category $\langle \mathbb{V}_{\mathbb{Q},s}^{\otimes} \rangle$ of mixed $\mathbb{Q}$-Hodge structures tensorially generated by $\mathbb{V}_{\mathbb{Q},s}$ and $\mathbb{V}_{\mathbb{Q},s}^{\vee}$. Equivalently, the group $\mathbf{P}_s(\mathbb{V})$ is the stabiliser of the Hodge tensors for $\mathbb{V}_{\mathbb{Q},s}$, i.e.

the Hodge classes in the rational Hodge structures tensorially generated by $\mathbb{V}_{\mathbb{Q},s}$ and its dual. This is a connected $\mathbb{Q}$-algebraic group, which is reductive if $\mathbb{V}_{\mathbb{Q},s}$ is pure, and an extension of the reductive group $\mathbf{P}_s(\mathrm{Gr}^W_\bullet \mathbb{V})$ by a unipotent group in general (where W denotes the weight filtration on $\mathbb{V}$). A point $s \in S$ is said to be Hodge generic for $\mathbb{V}$ if $\mathbf{P}_s(\mathbb{V})$ is maximal. If S is irreducible, two Hodge generic points of S have the same Mumford-Tate group, called the generic Mumford-Tate group $\mathbf{P}_S(\mathbb{V})$ of $(S, \mathbb{V})$. The Hodge locus $\mathrm{HL}(S, \mathbb{V})$ is the subset of points of S which are not Hodge generic.

A fundamental result of Cattani-Deligne-Kaplan [CDK95] (in the pure case, generalized by [BPS10] in the mixed case) states that $\mathrm{HL}(S, \mathbb{V})$ is a countable union of closed irreducible algebraic subvarieties of S, called *special subvarieties* of $(S, \mathbb{V})$. Special subvarieties of dimension zero are called *special points* of $(S, \mathbb{V})$. A special point s whose Mumford-Tate group $\mathbf{P}_s(\mathbb{V})$ is a torus is called a point with complex multiplication (*CM-point*) for $(S, \mathbb{V})$.

We would like to understand the geometry of $\mathrm{HL}(S, \mathbb{V})$. A first precise version of this question would be to describe the Zariski-closure of $\mathrm{HL}(S, \mathbb{V})$. Are there any geometric constraints on the Zariski-closure of $\mathrm{HL}(S, \mathbb{V})$? Can one describe the couples $(S, \mathbb{V})$ such that $\mathrm{HL}(S, \mathbb{V})$ is Zariski-dense in S? Particular cases of this problem have been classically considered by complex algebraic geometers, essentially when $\mathbb{V}$ is pure of weight 1 or 2. For example, when $\mathbb{V}$ is pure of weight 1 (hence we are essentially considering families of Abelian varieties), a typical result (Colombo-Pirola [CiPi90], Izadi [Iz98], Chai [Chai98]) is the following. Let $S \subset \mathcal{A}_g$ be a subvariety of codimension at most g of the moduli space $\mathcal{A}_g$ of principally polarized Abelian varieties of dimension g. Then the set S_k of points $s \in S$ such that the corresponding Abelian variety A_s admits an Abelian subvariety of dimension k is dense (for the Archimedean topology) in S for any integer k between 1 and $g-1$. In particular $\mathrm{HL}(S, \mathbb{V})$ is dense in S (where we denote by $\mathbb{V}$ the $\mathbb{Z}$VHS restriction to S of the Hodge incarnation $R^1 f_* \mathbb{Z}$ of the universal Abelian variety $f : \mathfrak{A}_g \longrightarrow \mathcal{A}_g$ over $\mathcal{A}_g$).

This examples already indicates that special subvarieties for $(S, \mathbb{V})$ are quite common in general. Even if they have a Hodge theoretic significance, they are not special enough to force any global shape for the Zariski-closure of $\mathrm{HL}(S, \mathbb{V})$. The goal of the talk was to define a natural, much better behaved, subset $S_{\mathrm{atyp}}(\mathbb{V}) \subset \mathrm{HL}(S, \mathbb{V})$: the *atypical locus* of $(S, \mathbb{V})$.

The crucial notion for defining the atypical locus $S_{\mathrm{atyp}}(\mathbb{V}) \subset S$ is the notion of Hodge codimension:

Definition 1. (Hodge codimension) Let S be an irreducible quasi-projective variety and $\mathbb{V} \longrightarrow S^{\mathrm{sm}}$ a variation of mixed $\mathbb{Q}$-Hodge structures on the smooth locus S^{sm} of S. Let $\mathbf{P}_S$ be the generic Mumford-Tate group of $(S^{\mathrm{sm}}, \mathbb{V})$ and $\mathfrak{p}_S$ its Lie algebra (endowed with its canonical mixed $\mathbb{Q}$-Hodge structure, of weight $\leqslant 0$).

We define the Hodge codimension of S with respect to V as

$$\text{H-cd}(S, \mathbb{V}) := \dim_{\mathbb{C}}(\mathrm{Gr}_F^{-1}\mathfrak{p}_S) - \mathrm{rk}\,\mathrm{Im}\overline{\nabla} \ ,$$

where $\overline{\nabla} : TS^{\mathrm{sm}} \longrightarrow \mathrm{Gr}_F^{-1} W_0 \mathrm{End}\, \mathbb{V}$ is the Kodaira-Spencer map of $(S, \mathbb{V})$.

Definition 2. (Atypical subvariety) Let S be an irreducible smooth quasi-projective variety and $\mathbb{V} \longrightarrow S$ a variation of mixed $\mathbb{Q}$-Hodge structures on S.

An irreducible subvariety $Y \subset S$ is said to be atypical for $(S, \mathbb{V})$ if

$$\text{H-cd}(Y, \mathbb{V}_{|Y^{\text{sm}}}) < \text{H-cd}(S, \mathbb{V}) \quad . \tag{1}$$

We denote by $S_{\text{atyp}}(\mathbb{V}) \subset S$ the subset of S union of all atypical subvarieties for $(S, \mathbb{V})$.

The locus $S_{\text{atyp}}(\mathbb{V}) \subset S$ is easily shown to be contained in $\text{HL}(S, \mathbb{V})$ and to be a countable union of special subvarieties. The following (equivalent) conjectures predict that $S_{\text{atyp}}(\mathbb{V})$, in contrast with $\text{HL}(S, \mathbb{V})$, has a simple geometry:

Conjecture 3. (Main conjecture, form 1) For any irreducible smooth quasi-projective variety S endowed with a variation of mixed $\mathbb{Q}$-Hodge structures $\mathbb{V} \longrightarrow S$, the subset $S_{\text{atyp}}(\mathbb{V})$ is a finite union of maximal atypical (hence special) subvarieties of S.

Conjecture 4. (Main conjecture, form 2) For any irreducible smooth quasi-projective variety S endowed with a variation of mixed $\mathbb{Q}$-Hodge structures $\mathbb{V} \longrightarrow S$, the subset $S_{\text{atyp}}(\mathbb{V})$ is a strict algebraic subset of S.

Conjecture 5. (Main conjecture, form 3) For any irreducible smooth quasi-projective variety S endowed with a variation of mixed $\mathbb{Q}$-Hodge structures $\mathbb{V} \longrightarrow S$, the subset $S_{\text{atyp}}(\mathbb{V})$ is not Zariski-dense in S.

The restriction of Conjecture 3 to the class of pairs $(S, \mathbb{V})$ where S is a subvariety of a Shimura variety $\text{Sh}^0(S, \mathbb{V})$ and $\mathbb{V}$ is the restriction to S of a standard $\mathbb{Z}$VMHS on $\text{Sh}^0(S, \mathbb{V})$ is the Zilber-Pink conjecture as stated for example by Pila ([Pil16, Conj.2.3]) (this conjecture was formulated by Zilber [Zil02] in the case of multiplicative groups, it is a stronger version of the original Pink Conjecture [Pink05b, Conj.1.1]).

Another particular case of Conjecture 3 is the following generalized André-Oort conjecture:

Conjecture 6 (André-Oort conjecture for $\mathbb{Z}$VHS)**.** Let $\mathbb{V} \longrightarrow S$ be any variation of mixed Hodge structure over a smooth irreducible complex quasi-projective variety S. Suppose that the union of CM-points for $(S, \mathbb{V})$ is Zariski-dense in S. Then there exists a connected mixed Shimura variety $\text{Sh}^0(S, \mathbb{V})$ and a Cartesian diagram

$$\begin{array}{ccc} \mathbb{V} = \Phi^*\mathbb{V}_\rho & \longrightarrow & \mathbb{V}_\rho \\ \downarrow & & \downarrow \\ S & \xrightarrow[\Phi]{} & \text{Sh}^0(S, \mathbb{V}) \end{array} \tag{2}$$

where Φ is a dominant morphism, $\rho : \mathbf{P} \longrightarrow \mathbf{GL}(V)$ is an algebraic representation and $\mathbb{V}_\rho \longrightarrow \text{Sh}^0(S, \mathbb{V})$ is the associated polarized $\mathbb{Z}$VHS on $\text{Sh}^0(S, \mathbb{V})$.

The remaining part of the talk was devoted to explain these conjectures in term of atypical intersection properties in the sense of [Za12] for period maps and state partial results.

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Derived equivalences for K3 surfaces and motives

Martin Olsson

(joint work with Max Lieblich)

Let k be an algebraically closed field of positive characteristic $p \neq 2$. Recall that two K3 surfaces X and Y over k are called *derived equivalent* if there exists a k-linear equivalence $D(X) \to D(Y)$ between their bounded derived categories of coherent sheaves. Let H^* denote either $\mathbb{Q}_\ell$-étale cohomology, for some prime ℓ different than p, or crystalline cohomology, and define (following Mukai) for a $K3$ surface X

$$\widetilde{H}(X) := H^0(X)(-1) \oplus H^2(X) \oplus H^4(X)(1).$$

If H^* is $\mathbb{Q}_\ell$-étale (resp. crystalline) cohomology, then we refer to $\widetilde{H}(X)$ as the *$\mathbb{Q}_\ell$-étale (resp. crystalline) realization of the Mukai motive.* As explained in [1, §2], results of Orlov imply that an equivalence of derived categories $P : D(X) \to D(Y)$ induces an isomorphism

$$\Phi_P^{H^*} : \widetilde{H}(X) \to \widetilde{H}(Y),$$

which in the case when X, Y, and P are all defined over a subfield of k and H^* is $\mathbb{Q}_\ell$-étale cohomology (crystalline cohomology) is compatible with Galois actions (resp. F-isocrystal structure) and the so-called Mukai pairing. Likewise if $A^*(X)_{\mathrm{num},\mathbb{Q}}$ denotes Chow groups modulo numerical equivalence, so

$$A^*(X)_{\mathrm{num},\mathbb{Q}} = \mathbb{Q} \oplus NS(X)_{\mathbb{Q}} \oplus \mathbb{Q},$$

then P induces an isomorphism

$$\Phi_P^{A^*} : A^*(X)_{\mathrm{num},\mathbb{Q}} \to A^*(Y)_{\mathrm{num},\mathbb{Q}}$$

which is compatible with the Mukai pairing. We say that P is *strongly filtered* if $\Phi_P^{A^*}$ preserves the codimension filtration on the Chow groups modulo numerical equivalence, sends $(1, 0, 0)$ to $(1, 0, 0)$, and sends the ample cone of $NS(X)_{\mathbb{Q}}$ to plus or minus the ample cone of $NS(Y)_{\mathbb{Q}}$. The main result of [2] is the following:

Theorem 1 ([2, 1.2])**.** *Let $P : D(X) \to D(Y)$ be a strongly filtered equivalence. Then there exists a unique isomorphism $X \to Y$ inducing the same map as P on the étale and crystalline realizations of the Mukai motive.*

Remark 2. From this result one can recover the derived Torelli theorem [2, 6.1].

Remark 3. The proof uses the Tate conjecture which for K3 surfaces is a theorem of Charles, Maulik, and Pera. Their proofs use analytic techniques, aside from which our arguments are entirely algebraic.

Remark 4. A variant result can also be formulated in characteristic 0.

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D-modules in characteristic *p* and stability

Mark Kisin

(joint work with Hlne Esnault)

Let X be a smooth projective, geometrically connected variety over a number field K, and (E, ∇) a vector bundle with integrable connection on X. Choosing an integral model for $X, (E, \nabla)$ it makes sense to speak of the reduction $X_p, (E_p, \nabla_p)$ of $X, (E, \nabla)$ mod p for almost all primes p

The Grothendieck p-curvature conjecture predicts that if the p-curvature ψ_p of $X_p, (E_p, \nabla_p)$ vanishes for almost all p, then $M = (E, \nabla)$ has finite monodromy. The vanishing of p-curvature means that the action of derivations on E_p given by ∇_p can be extended to an action of operators of the form $\frac{(\partial/\partial x)^p}{p}$ for x a local co-ordinate on X.

We make the stronger assumption, that for almost all p, (E_p, ∇_p) underlies a $\mathcal{D}_{X_p}$ module structure on E_p, and we denote by $MIC^{\mathfrak{D}}(X/K)$, the category of (E, ∇) satisfying this condition. We say that M is isotrivial, if it has finite monodromy.

Then we have the following results.

Theorem 1. *The forgetful functor* $(E, \nabla) \mapsto E$ *is fully faithful on* $MIC^{\mathfrak{D}}(X/K)$. *In particular, if* E *is Nori finite, then* M *is isotrivial.*

Recall that Nori finiteness means that the class of E in the Grothendieck group of vector bundles on X is integral over $\mathbb{Z}$. Our next result is an analogue of Katz's theorem on the Gauss-Manin connection.

Theorem 2. *If* M *in* $MIC^{\mathfrak{D}}(X/K)$ *underlies a polarizable* $\mathbb{Z}$*-variation of Hodge structure, then* M *is isotrivial.*

To prove these results we use arguments involving stability of vector bundles, together with the following theorem, set purely in characteristic p.

Theorem 3. *Let* X_0 *be a smooth projective, geometrically connected scheme over a finite field* k*, and let* M_0 *be a coherent* $\mathcal{D}_{X_0}$*-module on* X_0*. Then* M_0 *is isotrivial.*

Returning to M in $MIC^{\mathfrak{D}}(X/K)$, the last theorem allows us to associate a finite étale group scheme $G_s = G(M_s)$, to the fibre of M_s at s (which depends on the choice $\mathcal{D}_{X_s}$-module structure on M_s). Our final result is

Theorem 4. *Suppose that for infinitely many* s*, the characteristic of* $k(s)$ *does not divided the order of* G_s*. Then* M *is isotrivial.*

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G_2 and some exceptional Witt Vector identities

Nicholas M. Katz

1. The exceptional identities

Fix a prime p, and consider the p-Witt vectors of length 2 as a group scheme over $\mathbb{Z}$. The addtion law is given by

$$(x, a) + (y, b) := (x + y, a + b + (x^p + y^p - (x + y)^p)/p).$$

For an odd prime p, we have

$$(x, 0) + (y, 0) + (-x - y, 0) = (0, (x^p + y^p - (x + y)^p)/p).$$

Let us define, for odd p, the integer polynomial

$$f_p[x, y] := (x^p + y^p - (x + y)^p)/p \in \mathbb{Z}[x, y].$$

For $p = 2$, we have

$$(x, 0) + (y, 0) + (-x - y, 0) = (0, x^2 + xy + y^2),$$

and we define

$$f_2[x, y] := x^2 + xy + y^2 \in \mathbb{Z}[x, y].$$

The exceptional identities we have in mind are

$$f_3 = -xy(x + y), f_5 = f_3 f_2, f_7 = f_3(f_2)^2.$$

2. Basic facts about G_2

We work with algebraic groups over $\mathbb{C}$. Given a prime number p, a theorem of Gabber [Ka-ESDE, 1.6] tells us the possible connected irreducible (in the given p-dimensional representation) Zariski closed subgroups of SL_p. For $p = 2$, the only possibility is SL_2. For p odd and $p \neq 7$, the possibilities are either the image of SL_2 in $Sym^{p-1}(std_2)$, SO_p, or SL_p. For $p = 7$ there is one new possibility, G_2, which sits in

$$\text{image of } \mathrm{SL}_2 \subset \mathrm{G}_2 \subset \mathrm{SO}_7 \subset \mathrm{SL}_7.$$

This new group G_2 can be determined among the four by its third and fourth moments M_3 and M_4. Recall that for a group G (given inside some $GL(V)$), its moments (with respect to the given representation V) are defined by

$$M_n(G) := M_n(G, V) := \dim((\mathrm{V}^{\otimes \mathrm{n}})^{\mathrm{G}}),$$

the dimension of the space of G-invariants. For our four groups, M_3 is successively $1, 1, 0, 0$, and M_4 is successively $7, 4, 3, 2$. In fact, in our application, we will only use M_3. Notice also that $M_3 = 1$ if and only if $M_3 > 0$ for our possible choices.

3. The local systems

Fix a finite field k of odd characteristic p. We have the quadratic character

$$\chi_2 : k^\times \to \pm 1,$$

which we extend to all of k by defining $\chi_2(0) = 0$. Fix a nontrivial additive character

$$\psi : (k, +) \to \mu_p(\mathbf{Q}(\zeta_p)).$$

Given a polynomial $f(x) \in k[x]$ of degree $n \geqslant 2$ which is prime to p, we are interested in the sums

$$-\sum_{x \in k} \chi_2(x)\psi(f(x)).$$

Now fix a prime number $\ell \neq p$ and an embedding of $\mathbb{Q}(\zeta_p)$ into $\overline{\mathbb{Q}_\ell}$. Then this sum is the trace of $Frob_k$ on $H^1_c(\mathbb{A}^1_{\overline{k}}, \mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(f(x))})$. Here $\mathcal{L}_{\chi_2(x)}$ is the Kummer sheaf (extended by 0 across $0 \in \mathbb{A}^1$) and $\mathcal{L}_{\psi(f(x))}$ is the (pullback by f of) the Artin-Schreier sheaf $\mathcal{L}_{\psi(x)}$.

If we consider these sums as we vary f by adding to it a varying linear term,

$$t \mapsto -\sum_{x \in k} \chi_2(x)\psi(f(x) + tx),$$

then we are looking at the traces of a rank n local system on the $\mathbb{A}^1$ of t's, the Fourier Transform

$$FT(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(f(x))}).$$

It is pure of weight one, thanks to Weil. Its description as an FT shows that it is geometrically irreducible. One knows that its I_∞-slopes are

$$\{0, n/(n-1) \text{ repeated n} - 1 \text{ times}\}.$$

When n is odd and f is an odd polynomial (i.e. $f(-x) = -f(x)$), then this FT is orthogonally self dual, and its G_{geom} lies in SO_n. Moreover, after we twist by an explicit Gauss sum, our FT will be pure of weight zero, and we will have $G_{arith} \subset SO_n$.

Here is a general fact about geometrically irreducible local systems $\mathcal{F}$ on $\mathbb{A}^1_k$. If $p > 2n+1$, then $\mathcal{F}$ is Lie-irreducible, meaning that G^0_{geom} acts irreducibly.

4. Looking for local systems whose G_{geom} is G_2

Some years ago, I proved [Ka-ESDE, 9.1.1] that with $f(x) = x^7$, in any characteristic $p \geqslant 17$, the FT had $G_{geom} = G_2$. A question of Rudnick and Waxman made me wonder if there were other odd, degree seven polynomials $f(x)$ for which the FT would have $G_{geom} = G_2$. Using the exceptional identities, it turned out to be a simple matter to show that $M_3 = 1$ for the local system on $\mathbb{A}^2$ with parameters B, t whose trace function is

$$(B,t) \mapsto (1/g) \sum_{x \in k} \chi_2(x)\psi(x^7/7 + 2Bx^5/5 + B^2x^3/3 + tx),$$

g being the aforementioned Gauss sum. Since any specialization of B gives a Lie irreducible local system, our local system on $\mathbb{A}^2$ is a fortiori Lie-irreducible. Moreover, any such specialization has $M_3 \geqslant 1$, so its G^0_{geom} has $M_3 \geqslant 1$. If G^0_{geom} were the image of SL_2 for the specialized local system, then G_{geom} would be this image (because this image group is its own normalizer in SO_7). This is impossible, as in any representation of this image, the slopes would have denominators 1 or 2 or 3 (whereas we have denominator 6). So for the specialized local system, we have $G^0_{geom} = G_2$. As G_2 is its own normalizer in SO_7, we conclude that each specialized local system, with B specialized to $b \in k$, i.e.

$$t \mapsto (1/g) \sum_{x \in k} \chi_2(x)\psi(x^7/7 + 2bx^5/5 + b^2x^3/3 + tx),$$

has $G_{geom} = G_2$. It then follows easily that on (B,t) space, we have $G_{geom} = G_2$.

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Reporter: Andreas Mihatsch

Participants

Prof. Dr. Yves Andre
Département de Mathématiques
École Normale Superieure
45, rue d'Ulm
75230 Paris Cedex 05
FRANCE

Prof. Dr. Fabrizio Andreatta
Dipartimento di Matematica
"Federigo Enriques"
Università di Milano
Via Saldini, 5
20133 Milano
ITALY

Dr. Bhargav Bhatt
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. José Ignacio Burgos Gil
ICMAT, UAM
Instituto de Sciencias Matemáticas
Campus de Cantoblanco
C/Nicolás Cabrera, No. 13-15
28049 Madrid
SPAIN

Dr. Ana Caraiani
Department of Mathematics
Princeton University
710 Fine Hall
Washington Road
Princeton NJ 08544-1000
UNITED STATES

Prof. Dr. Pierre Colmez
Institut de Mathématiques de Jussieu,
CNRS
Université de Paris VI
Theorie des Nombres, Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Brian Conrad
Department of Mathematics
Stanford University
Building 380, Sloan Hall
Stanford, CA 94305
UNITED STATES

Prof. Dr. Johan de Jong
Department of Mathematics
Columbia University
2990 Broadway
New York NY 10027
UNITED STATES

Prof. Dr. Bas Edixhoven
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
2300 RA Leiden
NETHERLANDS

Prof. Dr. Gerd Faltings
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Jean-Marc Fontaine
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Ofer Gabber
I.H.E.S.
Le Bois Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. David Hansen
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Günter Harder
Wegelerstrasse 2
53115 Bonn
GERMANY

Dr. Florian Herzig
Department of Mathematics
University of Toronto
40 St George Street
Toronto, Ont. M5S 2E4
CANADA

Prof. Dr. Lars Hesselholt
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Furo-cho
Nagoya 464-8602
JAPAN

Prof. Dr. Adrian Iovita
Department of Mathematics & Statistics
Concordia University
7141 Sherbrooke St. W.
Montreal Quebec H4B 1R6
CANADA

Prof. Dr. Uwe Jannsen
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Prof. Dr. Nicholas M. Katz
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Prof. Dr. Kiran S. Kedlaya
Department of Mathematics
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
UNITED STATES

Prof. Dr. Mark Kisin
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Bruno Klingler
Equipe Formes Automorphes
Université Paris - Diderot
Croisement Ave. de France / Rue Alice
75013 Paris Cedex
FRANCE

Prof. Dr. Jürg Kramer
Institut für Mathematik
Humboldt-Universität Berlin
10099 Berlin
GERMANY

Prof. Dr. Max Lieblich
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Prof. Dr. Christian Liedtke
Zentrum Mathematik
Technische Universität München
Boltzmannstrasse 3
85748 Garching bei München
GERMANY

Dr. Daniel Litt
Department of Mathematics
Columbia University
2990 Brodway
New York, NY 10027
UNITED STATES

Prof. Dr. Ruochuan Liu
Beijing International Center for
Mathematical Research
No. 5 Yiheyuan Road Haidian District
Beijing 100 871
CHINA

Prof. Dr. Keerthi Madapusi Pera
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Andreas Mihatsch
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Sophie Morel
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Prof. Dr. Wieslawa Niziol
Département de Mathématiques
U.M.P.A.
École Normale Superieure de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Dr. Andrew S. Obus
Department of Mathematics
University of Virginia
Kerchof Hall
Charlottesville, VA 22904-4137
UNITED STATES

Prof. Dr. Martin Olsson
Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley CA 94720-3840
UNITED STATES

Prof. Dr. Vytautas Paskunas
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Prof. Dr. Damian Rössler
Laboratoire Emile Picard
Université Paul Sabatier
118, Route de Narbonne
31000 Toulouse Cedex
FRANCE

Prof. Dr. Peter Schneider
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Peter Scholze
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Yiwei She
Department of Mathematics
Columbia University
2990 Brodway
New York, NY 10027
UNITED STATES

Remy van Dobben de Bruyn
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Dr. Yakov Varshavsky
Einstein Institute of Mathematics
The Hebrew University
Givat Ram
Jerusalem 91904
ISRAEL

Dr. Rafael von Kaenel
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Prof. Dr. Annette Werner
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Straße 8
60325 Frankfurt am Main
GERMANY

Robert Wilms
Institut für Mathematik
Johannes-Gutenberg Universität Mainz
Staudinger Weg 9
55128 Mainz
GERMANY

Prof. Dr. Jean-Pierre Wintenberger
Institut de Mathématiques
Université de Strasbourg
7, rue Rene Descartes
67084 Strasbourg Cedex
FRANCE

Prof. Dr. Shouwu Zhang
Department of Mathematics
Princeton University
Princeton NJ 08544
UNITED STATES