Mathematisches Forschungsinstitut Oberwolfach

Report No. 46/2016

DOI: 10.4171/OWR/2016/46

Singularities

Organised by
François Loeser, Paris
András Némethi, Budapest
Duco van Straten, Mainz

25 September – 1 October 2016

Abstract. Singularity theory is a central part of contemporary mathematics. It is concerned with the local and global structure of maps and spaces that occur in algebraic, analytic or differential geometric context. For its study it uses methods from algebra, topology, algebraic geometry and complex analysis.

Mathematics Subject Classification (2010): 14Bxx, 32Sxx, 58Kxx.

Introduction by the Organisers

The workshop *Singularities* that was held in September 2016 stands in a long tradition of workshops on this subject that over the past decades took place at Oberwolfach. It was organized by F. Loeser (Paris), A. Némethi (Budapest) and D. van Straten (Mainz) and was attended by 49 regular participants. A novum was the presence of 5 additional participants to the workshop, that were selected by the organisers from the applicants of the *Heidelberg Laureate Forum*, but which had no specific background in singularity theory. This brought the total number of participants to the workshop up to 54: a very diverse group representing a broad spectrum of interests, age and geographical origin. Two participants were supported as *Oberwolfach Leibniz Graduate Students*, one as *US Junior Oberwolfach Fellow*.

The schedule of the meeting followed the more or less standard format of three morning and two afternoon talks of one hour each. On Tuesday, Wednesday and Thursday evening additional presentations and informal discussions took place, so that a total of 25 talks were presented. Of course, in good Oberwolfach tradition, the wednesday afternoon was kept free for a hike, which on this occasion took us to St. Roman. From the abstracts one gets a good idea of the broad scope of present day singularity theory, and it shows that it is an active field with major open problems.

During the last decades the categorical or non-commutative approach to singularity theory has grown into an important new direction of research. The workshop was opened by a talk by I. Burban that clearified the Krichever correspondence in the higher rank case and used the Fourier-Mukai transform to elucidate its precise relation to torsion free sheaves on the spectral curve. R. Buchweitz talked about a recent result that gives a complete description of graded matrix-factorisations of the polynomial $x^d - y^d$ and more generally of all graded Cohen-Macaulay modules over graded Gorenstein curve singularities. E. Faber reported on work describing non-commutative resolutions of discriminants arising from reflection groups; the resulting picture links the classical McKay correspondence to Chevalley's theorem on quotients by reflection groups.

The theory of singularities of curves and surfaces still occupies a central position in the field, with strong ties to low dimensional topology. W. Neumann reported on the recent progress made in the description of the outer Lipschitz geometry of normal surface singularities. J. Wahl gave an overview of the recent developments on rational homology disk smoothings of normal surface singularities, where now an almost complete picture has emerged. B. Sigurðsson reported on results relating the Seiberg-Witten invariant and the geometric genus for the class of surface singularities described by Newton-nondegenerate polynomials. The talk of S. Rasmussen described applications of Heegard-Floer homology to the link of normal surface singularities, which led to a new characterisation of the class of rational surface singularities. P. Popescu-Pampu introduced the notion of arborescent singularities and gave a nice ultra-metric interpretation of intersection numbers of curves on surfaces.

Open problems in the classical theory of singularities of mappings were presented in the talk of D. Mond. A very basic question relating the deformations of a mapping to the vanishing topology of the image is still wide open, although it has been verified in very non-trivial examples. In his talk, J. Bobadilla described a new attempt for a general proof by introducing a new kind of Jacobi-algebra. Attempts to construct hyperkähler manifolds have led to the study of symplectic singularities. These have very special properties and Y. Namikawa described in his talk a complete classification of a sub-class of them. M. Lehn formulated results and conjectures concerning singularities arising from polar representations and symplectic reductions. S. Gusein-Zade presented joint work with W. Ebeling on orbifold invariants of hypersurface singularities, in particular for finer invariants like the integral structure and Seifert form in the cohomology of the Milnor fibre. L. Goettsche talked on his results on refined curve counting in linear systems on surfaces.

The theory of D-modules plays a fundamental role in the description of the variation of cohomology groups. The local theory of the Gauss-Manin connection was introduced by E. Brieskorn to describe the cohomology of the Milnor fibre and its monodromy. At his talk at the workshop, K. Saito returned to these roots and formulated a general coherence theorem for the direct image of the relative de Rham complex in the non-proper situation. N. Budur presented results on cohomology jump loci that can be seen as a generalisation of Brieskorns approach to the monodromy theorem. The Laplace transformed version of the local Gauss-Manin system describes the behaviour of oscillatory integrals, objects of key importance in many branches of mathematics. G. Compte described a construction of a remarkable algebra of oscillatory functions, closed under integration. A. Dimca presented results and conjectures on the algorithmic calculation of the monodromy of projective curves, which were obtained partly in joint work with M. Saito.

Three evening sessions with an informal character were held at the workshop. On Tuesday, H. Hauser gave three interesting examples related to approximation theorems and resolution of singularities in characteristic p. On Wednesday N. A'Campo gave an *Introduction to Singularities*, aimed especially at the participants from the Heidelberg Laureate Forum, who did not have a specific background in singularity theory. On Thursday J. Rasmussen talked about the spectacular conjectural relation between the HOMFLY-polynomial, Khovanov homology and the Hilbert schemes of curve singularties. All three lectures attracted a large attendance and led to lively discussions.

The remarkable properties of the space of arcs of a singular space are still under intensive study, especially after the spectacular applications of motivic integration were found. Various variants of the Grothendieck group of varieties played a role in the talk of H. Đ. Nguyễn, that aimed at developping a theory of motivic multiple zeta functions. In her talk at the workshop, A. Reguera gave new results on the relation between Mather-discrepancy and the local embedding codimension of arc-space. M. Pe Pereira presented a new approach based on arc-spaces to the old problem of finding the adjacencies between plane curve singularities. V. Batyrev described a new invariant for singular spaces, called the algebraic stringy Euler number, whose invariance is most easily understood in terms of motivic integration. His talk concluded the mathematical program of our workshop.

To summarize, we think the meeting was a great success: old and new conjectures were presented by older and younger participants. Old and new friendships were celebrated, old and new collaborations were started or continued. The organisers thank the Oberwolfach staff for their efficient handling of the boundary conditions, which helped to create the unique Oberwolfach atmosphere.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Singularities

Table of Contents

Abstracts

Fourier–Mukai transform on Weierstrass cubics and commuting differential operators

Igor Burban
(joint work with Alexander Zheglov)

The study of commutative subalgebras of the algebra $\mathfrak{D} = \mathbb{C}[[z]][\partial]$ of ordinary differential operators has a long history, dating back to works of Schur [11] and Burchnall & Chaundy [3]. The next two well–known theorems summarize basic properties of such subalgebras.

Theorem. Let $\mathfrak{B} \subset \mathfrak{D}$ be a commutative subalgebra containing an *elliptic element* (i.e. an element of the form $\partial^n + a_1\partial^{n-1} + \cdots + a_n$ with $n \geq 1$). Then the following statements are true.

(1) All elements of $\mathfrak{B}$ are elliptic (such an algebra $\mathfrak{B}$ itself will be called *elliptic* in what follows).

(2) $\mathfrak{B}$ is a finitely generated integral domain and $\mathrm{kr.dim}(\mathfrak{B}) = 1$. In particular, $X_0 = \mathrm{Spec}(\mathfrak{B})$ is an integral affine curve.

(3) Let $\mathfrak{Q} = \mathrm{Quot}(\mathfrak{B})$ and $r = \mathrm{rk}(\mathfrak{B}) = \gcd\big\{\mathsf{ord}(P)\big|P \in \mathfrak{B}\big\}$ be the rank of $\mathfrak{B}$. Consider the valuation

$$\mathsf{val}_p : \mathfrak{Q} \longrightarrow \mathbb{Z}, \quad \frac{P}{Q} \mapsto \frac{\mathrm{ord}(Q) - \mathrm{ord}(P)}{r}.$$

Then $X = X_0 \cup \{p\}$ is a projective curve (called *spectral curve* of $\mathfrak{B}$) and p is a *smooth* point of X.

Theorem. Let $\mathfrak{B} \subset \mathfrak{D}$ be an elliptic commutative subalgebra of rank r and $F = \mathfrak{D}/x\mathfrak{D} \cong \mathbb{C}[\partial]$. Then the following statements are true.

(1) F is a finitely generated torsion free module over $\mathfrak{B}$. Moreover, $\mathfrak{Q} \otimes_{\mathfrak{B}} F \cong \mathfrak{B}^{\oplus r}$, i.e. $\mathrm{rk}_{\mathfrak{B}}(F) = \mathrm{rk}(\mathfrak{B})$.

(2) For a point $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ of X_0, consider the vector space

$$\mathsf{Sol}\big(\mathfrak{B}, \chi\big) := \Big\{f \in \mathbb{C}[[x]]\Big|P \circ f = \chi(P)f \text{ for all } P \in \mathfrak{B}\Big\}.$$

Then we have a $\mathfrak{B}$–linear map $F \xrightarrow{\eta_\chi} \mathsf{Sol}\big(\mathfrak{B}, \chi\big)^*, \partial^i \mapsto \big(f \mapsto \frac{1}{i!}f^{(i)}(0)\big)$, where $\mathsf{Sol}\big(\mathfrak{B}, \chi\big)^* = \mathsf{Hom}_{\mathbb{C}}\big(\mathsf{Sol}\big(\mathfrak{B}, \chi\big), \mathbb{C}\big)$. Moreover, the induced map

$$\mathfrak{B}/\mathsf{Ker}(\chi) \otimes_{\mathfrak{B}} F \xrightarrow{\bar{\eta}_\chi} \mathsf{Sol}\big(\mathfrak{B}, \chi\big)^*$$

is an isomorphism of $\mathfrak{B}$–modules, i.e. $F\big|_\chi \cong \mathsf{Sol}\big(\mathfrak{B}, \chi\big)^*$ for all $\chi \in X_0$.

(3) There exists a pair $(\mathcal{F}, \varphi)$, where $\mathcal{F}$ is a torsion free sheaf on X and $\varphi : \mathcal{F}\big|_{X_0} \longrightarrow F$ is a $\Gamma(X_0, \mathcal{O}) \cong \mathfrak{B}$–linear map, inducing an isomorphism of $\mathbb{C}$–vector spaces $\Gamma(X, \mathcal{F}) \longrightarrow \langle 1, \partial, \ldots, \partial^{r-1}\rangle$. Moreover, such a pair is unique up to an automorphism of $\mathcal{F}$.

The $\mathfrak{B}$–module F is called *spectral module* of $\mathfrak{B}$, whereas the torsion free sheaf $\mathcal{F}$ on the spectral curve X is called *spectral sheaf* of $\mathfrak{B}$.
The following result is essentially due to Krichever [5, 6]. Singular spectral curves and torsion free sheaves on them were included in the picture by Mumford [9]. See also a work of Mulase [8] for further elaboration.

Theorem [Krichever correspondence]. Consider the following two sets:

$$\mathsf{DiffOp} = \left\{\mathfrak{B} \subset \mathfrak{D} \,\middle|\, \mathfrak{B} \text{ is commutative and elliptic } \right\}$$

and

$$\mathsf{SpecData} = \left\{ (X, p, \mathcal{F}) \,\middle|\, \begin{array}{l} X \text{ is an integral projective curve} \\ p \in X \text{ is a smooth point} \\ \mathcal{F} \text{ is torsion free, } H^1(X, \mathcal{F}) = 0 \\ \Gamma(X, \mathcal{F}) \xrightarrow{\mathsf{ev}_p} \mathcal{F}\big|_p \text{ is an isomorphism} \end{array} \right\}.$$

Then the Krichever map $\mathsf{DiffOp} \xrightarrow{K} \mathsf{SpecData}, \mathfrak{B} \mapsto (X, p, \mathcal{F})$ is surjective. Moreover, its restriction $\mathsf{DiffOp}_1 \xrightarrow{K} \mathsf{SpecData}_1$ on the set of commutative subalgebras $\mathfrak{B} \subset \mathfrak{D}$ of rank one, respectively the set of tuples $(X, p, \mathcal{F})$ with $\mathcal{F}$ of rank one, is essentially a bijection.

All elliptic subalgebras $\mathfrak{B} \subset \mathfrak{D}$ of genus one and rank two were classified by Krichever & Novikov [7] (when the spectral curve is smooth) and by Grünbaum [4] in general. It is a natural problem to describe the spectral sheaf of a such subalgebra. This problem was solved by Previato & Wilson [10] in the case the spectral curve is smooth.

The main tool (and main novelty) in dealing with this problem is the technique of Fourier–Mukai transforms on Weierstraß cubics elaborated by Burban and Kreußler in [1].

Proposition (Burban & Zheglov [2]). Let $\mathfrak{B} = \mathbb{C}[L, P] \subset \mathfrak{D}$ be a genus one and rank two commutative subalgebra. Then the spectral sheaf of $\mathfrak{B}$ is indecomposable and not locally free if and only if $L = \left(\partial^2 + \frac{1}{2}c_2\right)^2 + \left(c_1\partial + \partial c_1\right) + c_0$ with

$$\left\{ \begin{array}{lcl} c_0 & = & -f^2 + \varrho f - \dfrac{\varrho^2}{6} \\ c_1 & = & f' \\ c_2 & = & \dfrac{2\varrho f^3 - \varrho^2 f^2 - f^4 + f''^2 - 2f'f'''}{2f'^2} \end{array} \right.$$

and $P = L_+^{\frac{3}{2}}$. The equation of the spectral curve is $y^2 = 4x^3 - \frac{1}{12}\varrho^4 x + \frac{1}{216}\varrho^6$.

Theorem (Burban & Zheglov [2]). Let $\mathfrak{B} = \mathbb{C}[L, P]$ be a maximal genus one and rank two commutative subalgebra of $\mathfrak{D}$, where $L = \left(\partial^2 + \frac{1}{2}c_2\right)^2 + \left(c_1\partial + \partial c_1\right) + c_0$

and $P = L_+^{\frac{3}{2}}$ with

$$\begin{cases} c_0 &= -f^2 + \varrho_1 f + \varrho_2 \\ c_1 &= f' \\ c_2 &= \dfrac{\varrho_3 - 2\varrho_0 f + 6\varrho_2 f^2 + 2\varrho_1 f^3 - f^4 + f''^2 - 2f'f'''}{2f'^2} \end{cases}$$

Then the following results are true.

(1) The spectral curve X is singular and the spectral sheaf $\mathcal{F}$ is decomposable and not locally free if and only if $\varrho_0 = (3\varrho_2 + \frac{1}{2}\varrho_1^2)\varrho_1$ and $\varrho_3 = -(3\varrho_2 + \frac{1}{2}\varrho_1^2)^2 \neq 0$. In this case, $\mathcal{F} \cong \mathcal{S} \oplus \mathcal{O}\big([q]\big)$, where $q = \big(-2\varrho_2 - \frac{1}{4}\varrho_1^2, -\frac{1}{2}\varrho_1(\varrho_1^2 + 6\varrho_2)\big) \in X$ and $\mathcal{S}$ is the direct image of $\mathcal{O}_{\mathbb{P}^1}$ under the normalization map $\mathbb{P}^1 \to X$.

(2) The spectral curve X is singular and the Fourier–Mukai transform of $\mathcal{F}$ is supported at the singular point of X if and only if $\varrho_0 = \varrho_3 = 0$. The spectral sheaf $\mathcal{F}$ is locally free if and only if $6\varrho_2 + \varrho_1^2 \neq 0$. Moreover, $\mathsf{det}(\mathcal{F}) \cong \mathcal{O}\big(2[q]\big)$ with $q = \big(\frac{1}{4}\varrho_1^2 + \varrho_2, \frac{1}{4}\varrho_1\big(6\varrho_2 + \varrho_1^2\big)\big)$.

References

[1] I. Burban & B. Kreußler, *Fourier–Mukai transforms and semi–stable sheaves on nodal Weierstraß cubics*, J. Reine Angew. Math. **584** (2005), 45–82.

[2] I. Burban & A. Zheglov, *Fourier–Mukai transform on Weierstraß cubics and commuting differential operators*, `arXiv:1602.08694`.

[3] J. Burchnall & T. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. **21** (1923), 420–440.

[4] F. Grünbaum, *Commuting pairs of linear ordinary differential operators of orders four and six*, Phys. D **31** (1988), 424–433.

[5] I. Krichever, *Methods of algebraic geometry in the theory of nonlinear equations*, Uspehi Mat. Nauk **32** (1977), no. 6 (198), 183–208, 287.

[6] I. Krichever, *Commutative rings of ordinary linear differential operators*, Func. Anal. Appl. **12**, no. 3 (1978), 175–185.

[7] I. Krichever & S. Novikov, *Holomorphic bundles over algebraic curves and nonlinear equations*, Russian Math. Surveys, **35**:6 (1980), 47–68.

[8] M. Mulase, *Category of vector bundles on algebraic curves and infinite-dimensional Grassmannians*, Internat. J. Math. **1** (1990), no. 3, 293–342.

[9] D. Mumford, *An algebro–geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related nonlinear equation*, Proceedings of the International Symposium on Algebraic Geometry, 115–153, Kinokuniya Book Store, Tokyo (1978).

[10] E. Previato & G. Wilson, *Differential operators and rank 2 bundles over elliptic curves*, Compositio Math. **81** (1992), 107–119.

[11] I. Schur, *Über vertauschbare lineare Differentialausdrücke*, Sitzungsber. Berl. Math. Ges. **4**, 2–8 (1905).

Graded Matrix Factorizations of $y^d - x^d$

Ragnar-Olaf Buchweitz
(joint work with Osamu Iyama, Kota Yamaura)

Introduction.

1. Recall that if $W \in S$ is an element in a commutative ring S, then a *matrix factorization* of W consists of a pair of square matrices $A, B \in \mathsf{Mat}_{n\times n}(S)$, of size $n \times n$ with $n \geqslant 1$ and entries from S, such that $AB = W\mathbf{1}_n = BA$, where $\mathbf{1}_n$ denotes the identity matrix of the same size.

Set $M = \operatorname{Cok} A$, the $R = S/(W)$–module defined by the matrix factorization. If W is a not a zerodivisor and S is regular, then M is a maximal Cohen–Macaulay (MCM) module over the hypersurface ring R, and each MCM over R can be obtained in this way when S is local by Eisenbud's classical result [Eis80].

2. According to a suggestion by Kontsevich, sectors in Landau–Ginzburg models in String Theory are parametrized by stable (and naturally triangulated) categories $\underline{\mathsf{MF}}(W)$ of matrix factorizations of "potentials" W, typically polynomials or power series in a finite number of variables. Crossing a "defect line" between a sector governed by W and another by W' corresponds then to an exact functor from $\underline{\mathsf{MF}}(W)$ to $\underline{\mathsf{MF}}(W')$ and the relevant functors should correspond to matrix factorizations of $W'(y) - W(x)$, where the variable sets y, x are disjoint. That such *Fourier–Mukai kernels* define indeed such functors goes back to Yoshino [Yos98], see also [DM13].

3. The simplest situation occurs when $W = W' = x^d \in \mathbb{C}[x]$ for some positive integer d. Now each matrix factorization of x^d is a direct sum of the finitely many (matrix) factorizations $(x^i, x^{d-i})_{i=0,\dots,d}$, with $i = 0, d$ yielding zero objects in the stable category. Thus, $\underline{\mathsf{MF}}(x^d)$ is of finite representation type, there are only finitely many indecomposable objects.

4. However, as soon as $d \geqslant 5$, then $y^d - x^d$ is a hypersurface of wild representation type, there are indecomposable (graded) matrix factorizations of arbitrarily large size and the dimension of the families of such factorizations grows exponentially with the size. Mathematical Physicists asked already several years ago how to construct any interesting families of matrix factorizations of this potential and then to study the corresponding crossing of the default line. Note that already the obvious 1×1 (matrix) factorizations here have interesting physical meaning, see [DRCR14].

5. As Lenzing observed when we ran an evening session on this issue at the Casa Matematica Oaxaca (CMO) in October 2015, the situation is (very) similar to what we are used to from linear algebra: Morphisms between vector spaces of, say, dimension d are classified by their rank that can range from 0 to d, while endomorphisms of such a vector space are classified in terms of eigenvalues and Jordan Normal Forms that are not discrete data, at least when the underlying field is, say, $\mathbb{C}$.

Graded Gorenstein Rings of Dimesnion 1.
Our results pertain more generally to one–dimensional, positively graded Gorenstein rings $R = \oplus_{i\geqslant 0} R_i$ with $R_0 = k$ a field. The total graded ring of fractions of R is $\mathcal{K} = \mathcal{N}^{-1}R$, where $\mathcal{N} \subset R$ is the set of homogeneous non-zero-divisors of R.

Because R is Gorenstein, $\mathrm{Ext}^i_R(k, R) = 0$ for $i \neq 1$ and $\mathrm{Ext}^1_R(k, R) = k(-a)[-1]$ as graded R–modules. The occurring integer $a = a(R)$ is the *a–invariant* of R.

Example 6. The ring $R = k[x, y]/(y^2)$ is Gorenstein and with arbitrarily assigned weights or degrees $\deg x, \deg y > 0$ the ring is obviously graded. One has $a(R) = \deg y - \deg x$, whence the a–invariant can take on any integer value.

For the purpose of this abstract we restrict to the case that k is algebraically closed, that R is reduced, and that $\gcd\{i \mid R_i \text{ contains a non-zero-divisor}\} = 1$. In this case, we have the following information.

Lemma 7. *Under the assumptions made, $\mathcal{K} \cong \prod_{j=1}^r k[t_j, t_j^{-1}]$ is a product of graded rings of Laurent polynomials in one variable with $\deg t_j = 1$. The ring $\mathcal{K}_{\geqslant 0} \cong \prod_{j=1}^r k[t_j]$ is then the normalization of R.*

If $a(R) < 0$, then $R \cong k[t]$ is a polynomial ring in one variable with $\deg t = -a = 1$. □

Example 8. For a quasi–homogeneous curve singularity $R \cong k[x, y]/(f(x, y))$, the a–invariant is $a(R) = \deg f - \deg x - \deg y$, thus, for example, for $f = y^d - x^d$ with the standard grading $\deg x = \deg y = 1$, the a–invariant is $d - 2$.

Henceforth assuming $a = a(R) \geqslant 0$, consider the following graded R–modules

- $T_i = R_{\geqslant i}(i) = R(i)_{\geqslant 0}$, for $i = 1, ..., a$, and
- $T_{i+j} = k[t_j]$, for $j = 1, ..., r$, where $\mathcal{K} \cong \prod_{j=1}^r k[t_j, t_j^{-1}]$ as above.

Proposition 9. *The endomorphism ring of degree preserving endomorphisms of $T = \oplus_{i=1}^{a+r} T_i$ in the stable category of graded maximal Cohen–Macaulay (= torsion-free) R–modules is the same as that in the category of all R–modules. In matrix form, it is*

$\mathrm{End}_R(T)$	$R_{\geqslant 1}(1)$	$\cdots$	$R_{\geqslant j}(j)$	$\cdots\cdots$	$R_{\geqslant a}(a)$	$k[t_1]$	$\cdots$	$k[t_r]$
$R_{\geqslant 1}(1)$	k	$\cdots$	0	$\cdots\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$R_{\geqslant i}(i)$	R_{i-1}	$\cdots$	R_{i-j}	$\cdots\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$R_{\geqslant a}(a)$	R_{a-1}	$\cdots$	R_{a-j}	$\cdots\cdots$	k	0	$\cdots$	0
$k[t_1]$	k	$\cdots$	k	$\cdots\cdots$	k	k	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$
$k[t_r]$	k	$\cdots$	k	$\cdots\cdots$	k	0	$\cdots$	k

The strictly lower triangular matrices among these form the Jacobson radical of $\mathrm{End}_R(T)$.

Our main result is then as follows.

Theorem 10. *Let R and T be as above. The stable category of graded maximal Cohen–Macaulay R–modules is exact equivalent to the category of perfect minimal complexes over* $\operatorname{add}(T)$*, such a complex being of the form*

$$0 \to P_b \xrightarrow{\partial} P_{b-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} P_{a+1} \xrightarrow{\partial} P_a \to 0\,,$$

with $b \geqslant a-1$ integers, $P_i \in \operatorname{add} T$, and the components of the morphisms ∂ in the radical of $\operatorname{End}_R(T)$*.*

Moreover, that equivalence is constructive in that it sends the stalk complex $P[-n]$, for $P \in \operatorname{add} T$ and $n \in \mathbb{Z}$ an integer, to the n^{th} R–syzygy of P, and maps mapping cones to mapping cones.

Corollary 11. *The Grothendieck group of the stable category of graded maximal Cohen–Macaulay R–modules is isomorphic to $\mathbb{Z}^{a+r}$.*

Prescribing the "Betti table" of the classes of the P_i, for $i = a, ..., b$, in that Grothendieck group, the corresponding representation space of perfect minimal complexes over $\operatorname{add} T$ is a subvariety of a (large) affine space cut out by the quadratic equations $\partial^2 = 0$.

Graded Matrix Factorizations of $y^d - x^d$. Coming back to the singularity in the title, assuming that the characteristic of k does not divide d, the summands of T have the following simple description:

- $T_i = R_{\geqslant i}(i) = R(i)_{\geqslant 0}$, for $i = 1, ..., d-2$, is the first R–syzygy module of $k[x, y]/(x, y)^i$, with matrix factorization given essentially by the Hilbert–Birch matrix of $(x, y)^i$, and shifted so that the syzygy module is generated in degree zero,
- $T_{i+j} = k[t_j] \cong k[x, y]/(y - \zeta^j x)$, for $j = 1, ..., d$, where ζ is a primitive d^{th} root of unity.

  The corresponding (1×1) matrix factorization is $\left(y - \zeta^j x, \dfrac{y^d - x^d}{y - \zeta^j x)}\right)$.

Moreover, for a matrix factorization (A, B), one has $(A, B)[1] = (-B, -A(d))$ and so $(A, B)[2] = (A, B)(d)$. Further, $R_{i-j} \cong K[x, y]_{i-j}$, for $1 \leqslant i, j \leqslant a = d-2$, is just the vector space of homogeneous polynomials of degree $i - j$ in x and y. The morphisms from $R_{\geqslant i}(i)$ to $k[t_j]$ are scalar multiples of the shift of the natural map $R_{\geqslant i} \hookrightarrow \mathcal{K}_{\geqslant i} \twoheadrightarrow k[x, y]/(y - \zeta^j x)_{\geqslant i} \cong k[t_j]t_j^i$.

With this information it is now possible to write down all graded matrix factorizations of $y^d - x^d$ for any d.

References

[DM13] Tobias Dyckerhoff and Daniel Murfet. Pushing forward matrix factorizations. *Duke Math. J.*, 162(7):1249–1311, 2013.

[DRCR14] Alexei Davydov, Ana Ros Camacho, and Ingo Runkel. $N = 2$ minimal conformal field theories and matrix bifactorisations of x^d. 2014. `arXiv:1409.2144`.

[Eis80] D. Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260(1):35–64, 1980.
[Yos98] Yuji Yoshino. Tensor products of matrix factorizations. *Nagoya Math. J.*, 152:39–56, 1998.

Outer Lipschitz geometry of normal complex surface singularities

Walter D. Neumann
(joint work with Anne Pichon)

A complex germ $(X,0) \subset (\mathbb{C}^N,0)$ has two "natural" metrics: *inner metric* $d_{inn}(x,y)$ measures minimal length of a path from x to y in X and *outer metric* $d_{out}(x,y) := ||x-y||$ in $\mathbb{C}^N$. Up to bilipschitz equivalence these metrics are well defined and independent of embedding.

An attraction of bilipschitz geometry is that it gives discrete classifications, as was first conjectured by Siebenmann and Sullivan [5] in 1997 and proved by Mostowski [2] in 1985. In a 2014 Acta paper [1], Lev Birbrair, Anne Pichon and I gave a complete classification of inner Lipschitz geometry of normal complex surface germs. Then in a subsequent preprint [3] Pichon and I addressed outer geometry, but without giving a full classification.

Nevertheless in [3] we showed that many analytic invariants of a complex normal surface singularity are invariants of its outer geometry, for example, its multiplicity, the minimal ideal cycle of its resolution, the geometry and topology of its polar and discriminant curves for a generic plane projection, etc. In particular, it then followed easily that constant outer Lipschitz geometry of an analytic family of normal surface singularities implies Zariski equisingularity of the family. We also proved the converse.

The new topic of the MFO talk was a complete classification of outer Lipschitz geometry for normal complex surface germs, the full details of which we expect to post on the arXiv soon.

The outer classification starts with a decomposition of the normal surface germ $(X,0)$ into semi-algebraic pieces glued along their boundaries. After embedding $(X,0)$ in some $(\mathbb{C}^n,0)$, for small ϵ the link $L_\epsilon = X \cap S^{2n-1}_\epsilon$ of $(X,0)$ is a 3-manifold, and the resulting decomposition of L_ϵ is a refinement of the one used to classify inner geometry in [1], which itself is a refinement of the JSJ decomposition of L_ϵ. We call the pieces of this decomposition A-pieces and B-pieces. Each of the B-pieces B_i, $i=1,\dots,k$ is labeled by a rational number $q_i \geq 1$ as in [1], representing the exponent of an exponential "rate of shrink" of B_i as $\epsilon \to 0$. We write $B_i = B_i(q_i)$. The other data of the inner classification are also retained; they can be captured as a cohomology class in $H^1(B_i(q_i);\mathbb{Z})$ for each $B_i(q_i)$ with $q_i > 1$. As in [1], the A-pieces in L_ϵ are toral annuli (i.e., homeomorphic to $T^2 \times [0,1]$) which are glued between B pieces, so they have pairs of exponents $1 \leq q < q'$ which are the exponents associated with the adjacent B-pieces (we write $A(q,q')$).

Adding the data of the outer geometry starts by considering a generic projection of $(X,0) \to (\mathbb{C}^2,0)$. In fact, this projection is already used in determining the decomposition of $(X,0)$, using the geometry of the discriminant curve of the projection to first build a decomposition of $(\mathbb{C}^2,0)$ and then lifting the pieces to $(X,0)$ to get to the decomposition of $(X,0)$ (after amalgamating some "trivial pieces"). The map $(X,0) \to (\mathbb{C}^2,0)$ is a branched covering and we show that to understand the outer geometry it suffices to know the "vertical rates" between sheets of the branched cover in addition to the the inner geometry. These vertical rates are again rational numbers ≥ 1, now measuring exponential distance between "vertically aligned points" as $\epsilon \to 0$. We capture the information of these vertical rates by choosing a "test curve" for each B-piece of the decomposition of $(\mathbb{C}^2,0)$; the test curve is the lift to $(X,0)$ of a complex curve in the B-piece. The final data of the classification are trees of rational numbers associated by means of the test curve to each B-piece of the decomposition of $(\mathbb{C}^2,0)$ (the trees are related to the trees of rational numbers used in the usual classification of the topology of plane curves, e.g., the Eggers tree, or the tree described in [4]).

This final step took us a long time to complete. For an $A(q,q')$-piece the vertical rates vary between the two boundaries of the piece, and although we knew a long time ago that this variation was piecewise linear, we were convinced that it should actually be linear. With linearity the vertical data of A-pieces would be determined by the data of the adjacent B-pieces, but piecewise linearity would have necessitated extra data, which was aesthetically unpleasing. Very recently we (mostly Anne Pichon) finally proved the linearity, thus avoiding extra data and completing the classification.

References

[1] L. Birbrair, W.D. Neumann, A. Pichon, *The thick-thin decomposition and bilipschitz classification of normal surface singularities*, Acta Math. **212** (2014), 199–256.

[2] T. Mostowski, *Lipschitz equisingularity*, Dissertationes Math. (Rozprawy Mat.) **243** (1985), 46pp.

[3] W.D. Neumann, A. Pichon, *Lipschitz geometry of complex surfaces: analytic invariants and equisingularity*, http://arxiv.org/abs/1211.4897v3, 2012–1016.

[4] W.D. Neumann, A. Pichon, *Lipschitz geometry of complex curves*, Journal of Singularities **10** (2014), 225–234.

[5] L. Siebenmann, D. Sullivan *On complexes that are Lipschitz manifolds*, Proceedings of the Georgia Topology Conference, Athens, Ga., 1977 (Academic Press, New York-London, 1979), 503–525.

Refined curve counting on Surfaces

Lothar Göttsche

(joint work with V. Shende, F. Block, F. Schoeter, B. Kikwai)

We discuss refined versions of curve counting on surfaces. The most classical way of counting curves are the Severi degrees. Let S be a complex projective surface and L a line bundle on S. Denote $|L| = \mathbb{P}(H^0(L))$ the complete linear system. For δ a nonnegative integer, the *Severi degree* $n_{(S,L),\delta}$ is the number of δ-nodal curves in $|L|$ through $dim|L| - \delta$ general points on S. For $S = \mathbb{P}^2$ the number $n_{d,\delta}$ counts δ-nodal degree d plane curves through $d(d+3)/2 - \delta$ general points.

1. The refined invariants

Let S be a smooth complex surface. Let $\chi_{-y}(X) := \sum_{p,q}(-1)^{p+q}y^p h^{p,q}(X)$ be the χ_{-y}-genus. Let $\mathbb{P}^\delta \subset |L|$ be the sub-linear system of curves through $dim|L| - \delta$ general points, denote $\mathcal{C} \subset S \times \mathbb{P}^\delta$ the universal curve, let $S^{[n]}$ the Hilbert scheme of n points on S and $\mathcal{C}^{[n]} \subset S^{[n]} \times \mathbb{P}^\delta$ the relative Hilbert scheme, parametrising zero dimensional schemes on the curves in $\mathbb{P}^\delta$. Assume that $\mathcal{C}^{[n]}$ is nonsingular of the expected dimension for all n (e.g. L is δ-very ample). In [GS] we define polynomials $N_l^{\mathcal{C}}(y)$ via

$$\sum_{n\ge 0} \chi_{-y}(\mathcal{C}^{[n]})t^n = \sum_{l\ge 0} N_l^{\mathcal{C}}(y)\big((1-t)(1-yt)\big)^{g(L)-l-1}.$$

Here $g(L)$ is the arithmetic genus of curves in $|L|$. The *refined invariant* is $N^{(S,L),\delta}(y) := N_\delta^{\mathcal{C}}(y)/y^\delta$, a symmetric Laurent polynomial in y. As the χ_{-y} genus specialises to the Euler number at $y = 1$, this definition specializes to the formula for the Severi degrees $n_{(S,L),\delta}$ in terms of Euler number of relative Hilbert schemes of points proven in [KST] in the course of their proof of a conjecture of [G] of universal formulas for the Severi degrees. In particular, for L sufficiently ample, $N^{(S,L),\delta}(1) = n_{(S,L),\delta}$, and $N^{(S,L),\delta}(y)$ is given by universal generating functions. For example for K3 surfaces S, we get that $N^{(S,L),L^2/2+1}(y)$ (a refined count of genus 0 curves in $|L|$) is the coefficient of $q^{L^2/2}$ in $\frac{1}{q\prod_{n>0}(1-q^n)^{20}(1-q^ny)^2(1-q^ny^{-1})^2}$.

Below we relate the refined invariants to real and tropical enumerative geometry of curves. First we review the *Welschinger numbers*. Let S be a real algebraic surface, and L a real line bundle on S. Let P be a configuration of $dim|L| - \delta$ real points on S. The *Welschinger number* is $W_{(S,L),\delta}(P) = \sum_C (-1)^{s(C)}$. The count is over real δ-nodal curves in $|L|$ through P, and $s(C)$ is the number of isolated real nodes of C, i.e. the real points of C where two complex branches intersect. These numbers depend in general on the point configuration P.

2. Tropical curve counting and refined Severi degree

Let S be a projective toric surface. Both the Severi degrees and the Welschinger numbers of S can be computed via tropical geometry [Mi], which we will also use to define the refined Severi degrees. For simplicity we restrict to the case $S = \mathbb{P}^2$, but the results hold for all projective toric surfaces. A *plane tropical curve of degree* d is a pair $C = (\Gamma, f)$ with Γ a graph, f a linear immersion to $\mathbb{R}^2$ satisfying.

- the (images of) edges e have rational slope, let $P(e)$ the corresponding primitive integer vector, the edges have a weight $w(e) \in \mathbb{Z}_{>0}$,
- at every vertex v of C the balancing condition holds: $\sum_e w(e)P(e) = 0$, where the sum is over all edges adjacent to v,
- the unbounded edges of $f(\Gamma)$ are d in each direction $(-1,0)$, $(0,-1)$, $(1,1)$.

There are notions of genus $g(C)$ and number of nodes of a tropical curve. Through a configuration P of $d(d+3)/2-\delta$ general points there are finitely many δ-nodal degree d tropical curves, all trivalent as graphs. We count them with multiplicities. We introduce 3 different counts, all based on the same principle. For a vertex v of a tropical curve C introduce a vertex multiplicity $\mu(v)$. The multiplicity of C is then $\mu(C)=\prod_v \mu(v)$, where v runs through all vertices of C. Finally the corresponding count is $\mu_{d,\delta}=\sum_C \mu(C)$, where the sum runs through all δ-nodal degree d curves through P as above.

(1) The *tropical Severi degree* $n^{trop}_{d,\delta}$ is the count corresponding to the Mikhalkin multiplicity $m(v)$ as vertex multiplicity. For this choose two edges e_1, e_2 at the trivalent vertex v. Then $m(v)=w(e_1)w(e_2)|\det(P(e_1),P(e_2))|$. Here $(P(e_1),P(e_2))$ is the 2×2 matrix with columns $P(e_1)$, $P(e_2)$. The other two counts are defined in terms of the Mikhalkin multiplicity.
(2) The *tropical Welschinger number* $W^{trop}_{d,\delta}$ corresponds to the vertex multiplicity $\omega(v)$, which is 0 if $m(v)$ is even and $(-1)^{m(v)/2-1}$ if $m(v)$ is odd.
(3) In [BG] we define the *refined Severi degree* $N_{d,\delta}(y)$ as the count corresponding to the vertex multiplicity $[m(v)]_y$, where the quantum number $[n]_y$ is the Laurent polynomial in $y^{1/2}$ defined by $[n]_y=\frac{y^{n/2}-y^{-n/2}}{y^{1/2}-y^{-1/2}}$. One can show that $N_{d,\delta}(y)$ is a symmetric Laurent polynomial in y. Similarly one defines refined Severi degrees $N_{(S,L),\delta}(y)$ for all projective toric surfaces.

In [Mi] Mikhalkin shows that $n^{trop}_{d,\delta}=n_{d,\delta}$, i.e. the tropical Severi degree is equal to the classical one, and $W^{trop}_{d,\delta}=W_{d,\delta}(P)$ for suitable point configurations P. It is easy to see that $N_{d,\delta}(1)=n^{trop}_{d,\delta}$ and $N_{d,\delta}(-1)=W^{trop}_{d,\delta}$, thus, by the above results of Mikhalkin, the refined Severi degree interpolates between the Severi degree counting complex curves and the Welschinger number counting real curves. In [IM] it is shown that the refined Severi degrees are tropical invariants, i.e. independent of the choice of tropical point configuration. The following conjecture gives the relation to the refined invariants of the first section.

Conjecture 1 *For $d\geq \delta/2+1$ we have $N^{d,\delta}(y)=N_{d,\delta}(y)$. More generally for S nonsingular, projective and toric and L δ-very ample, $N^{(S,L),\delta}(y)=N_{(S,L),\delta}(y)$.*

3. Computations in terms of Heisenberg algebra

The results of this section appliy to a restricted class of pairs toric surface and line bundle (given by h-transversal lattice polygons). These include $\mathbb{P}^2$, many weighted projective planes and the rational ruled surfaces. For simplicity we again restrict to $\mathbb{P}^2$. Let H be the Heisenberg algebra generated by elements $a_n, b_n,\ n\in\mathbb{Z}$ with the commutation relations

$$[a_n,a_m]=0=[b_n,b_m],\quad [a_n,b_m]=[n]_y\delta_{n,-m}.$$

The a_{-n},b_{-n} with $n>0$ are called creation operators, the others annihilation operators. The *Fock space* F is the space of $P((a_{-i})_i,(b_{-j})_j)\mathbf{1}$, where P is a polynomial in the creation operators and $\mathbf{1}$ is called the *vacuum vector*. F is a module under H: concatenate with the element of H from the left, apply the commutation relations and impose that $a_n\mathbf{1} = b_n\mathbf{1} = 0$ for $n \geq 0$. An inner product on F is defined by setting $\langle \mathbf{1}|\mathbf{1}\rangle = 1$ and requiring that a_n is adjoint to a_{-n} and b_n adjoint to b_{-n}. For a partition $\mu = (1^{\mu_1}, 2^{\mu_2}, \ldots)$ define $a_\mu = \prod_i \frac{a_i^{\mu_i}}{\mu_i!}$, $a_{-\mu} = \prod_i \frac{a_{-i}^{\mu_i}}{\mu_i!}$. Denote $\|\mu\| = \sum_i i\mu_i$ the number partitioned by μ.

Theorem 2. *Write $H(t) := \sum_{k>0} b_k b_{-k} + t \sum_{\|\mu\|=\|\nu\|-1} a_\nu a_{-\mu}$, where μ and ν run through the partitions. Then*

$$N_{d,\delta}(y) = \left\langle a_{(1^d)}\mathbf{1} \,\middle|\, \mathrm{Coeff}_{t^d}[H(t)^{d(d+3)/2-\delta}]\mathbf{1}\right\rangle.$$

For the proof we express the refined Severi degrees in terms of floor diagrams, a schematic description of tropical curves, and express, via the commutation relations, the computation in the Heisenberg algebra in terms of Feynman diagrams. Then we show that the floor diagrams are the same as the Feynman diagrams.

4. Further developments

I briefly mention other developments.

(1) With B. Kikwai [GK] we study universal generating functions for the refined Severi degrees $N_{(S,L),\delta}(y)$ also in case the surface S is singular. The formulas will hold when L is sufficiently ample. They are the same as for smooth surfaces, but with correction factors for the singularities, often expressed in terms of modular forms.

(2) There are more general Welschinger invariants where the curves are required to pass through configurations of real points and pairs of complex conjugated points. With F. Schroeter [GSc] we introduce and study new tropical invariants, which interpolate between these more general Welschinger invariants and descendent Gromov-Witten invariants.

(3) In [NPS] Nicaise, Payne and Schroeter give an interpretation of and an approach towards Conjecture 1, relating refined invariants and refined Severi degrees, in terms of nonarchimedian geometry.

(4) In [Mi2] Mikhalkin defines under special assumptions quantum indices of real plane curves, in terms of the signed area of the amoeba of the curve, and relates the corresponding counts of curves to the refined Severi degrees. This suggests that the refined Severi degrees are related to disk invariants.

References

[BG] F. Block and L. Göttsche, *Refined curve counting with tropical Geometry*, Compositio Math. **152** (2016), issue 01, 115–151.

[BG2] F. Block and L. Göttsche, *Fock spaces and refined Severi degrees*, to appear in IMRN, arXiv:1409.4868.

[G] L. Göttsche, *A conjectural generating function for numbers of curves on surfaces*, Comm. Math. Phys. **196** (1998), 523–533.

[GK] L. Göttsche and B. Kikwai, *Refined node polynomials via long edge graphs*, Commun. Number Theory Phys. **10** (2016), 193–234.

[GSc] L. Göttsche and F. Schroeter, *Refined broccoli invariants*, arXiv:1606.09631.

[GS] L. Göttsche and V. Shende, *Refined curve counting on complex surfaces*, Geom. Topol. **18** (2014), 2245–2307

[GS2] L. Göttsche and V. Shende, *The χ_y-genera of relative Hilbert schemes for linear systems on Abelian and K3 surfaces*, Algebraic Geometry **4** (2015), 405 –421.
[IM] I. Itenberg and G. Mikhalkin, *On Block-Goettsche multiplicities for planar tropical curves*, Int. Math. Res. Not. IMRN 2013, no. **23**, 5289–5320.
[KST] M. Kool, V. Shende, and R. P. Thomas, *A short proof of the Göttsche conjecture*, Geom. Topol. **15** (2011), no. 1, 397–406.
[Mi] G. Mikhalkin, *Enumerative tropical algebraic geometry in $\mathbb{R}^2$*, J. Amer. Math. Soc. **18**.2 (2005), 313–377.
[Mi2] G. Mikhalkin, *Quantum indices of real plane curves and refined enumerative geometry*, arXiv :1505.04338.
[NPS] J. Nicaise, S. Payne, F. Schroeter, *Tropical refined curve counting via motivic integration*, arXiv:1603:08424.
[Tz] Y-j. Tzeng, *Proof of the Göttsche-Yau-Zaslow Formula, J. Diff. Geom.* **90**.3 (2012), 439-472.

Mather discrepancy as an embedded dimension in the space of arcs

Ana J. Reguera

The space of arcs X_∞ of a singular variety X over a perfect field k has finiteness properties when we localize at its stable points. This allows to associate invariants of X from its space of arcs. In the talk I have shown some general properties of the stable points, pointing out our interest in computing the dimension of the complete local ring $\widehat{\mathcal{O}_{X_\infty,P_E}}$ when P_E is the stable point defined by a divisorial valuation ν_E on X.

I have also presented our last result, together with H. Mourtada: "Assuming char $k = 0$, we prove that embdim $\widehat{\mathcal{O}_{X_\infty,P_E}} = \widehat{k}_E + 1$ where $\widehat{k}_E$ is the Mather discrepancy of X with respect to ν_E. We also obtain that $\dim \widehat{\mathcal{O}_{X_\infty,P_E}}$ has as lower bound the Mather-Jacobian log-discrepancy of X with respect to ν_E. For X normal and complete intersection, we prove as a consequence that points P_E of codimension one in X_∞ have discrepancy $k_E \leq 0$".

Expressed in terms of cylinders, stable points are precisely the generic points of the irreducible cylinders in X_∞, and our result with H. Mourtada asserts that the embedding dimension of $\widehat{\mathcal{O}_{X_\infty,P_E}}$ is equal to the codimesion as cylinder of N_E, being N_E the closure of P_E in X_∞. But in general we have $\dim \widehat{\mathcal{O}_{X_\infty,P_E}} < \text{embdim } \widehat{\mathcal{O}_{X_\infty,P_E}}$.

References

[1] L. Ein and M. Mustata, *Jet schemes and singularities*, Proc. Symp. Pure Math. 80.2 (2009), 505–546.
[2] H. Mourtada, A.J. Reguera, *Mather discrepancy as an embedded dimension in the space of arcs*, Preprint.
[3] A.J. Reguera, *Towards the singular locus of the space of arcs*, Amer. J. Math. 131, n. 2 (2009), 313–350.

Coherence of the direct image of the relative De Rham complex

Kyoji Saito

We consider a flat holomorphic map $\Phi : Z \to S$ between complex manifolds. We show that the direct images of the relative De Rham complex $\Omega^{\bullet}_{Z/S}$ are coherent on S under the assumptions that the critical set C_Φ of the map is proper over S and that the map satisfies a suitable topological boundary conditions.

The proof uses Forster-Knorr's lemma which was used in a new proof of Grauert proper mapping theorem. More precisely, we cover the manifold Z by relative charts in the sense of Forster-Knorr. Then, on each relative chart, we introduce the complex $\mathcal{K}_\Phi^{\bullet,\star}$ and the complex $\mathcal{H}_\Phi^{\bullet,s}$ which has support only on the critical set C_Φ so that these two complexes together with the relative De Rham complex form an exact triangle. Consider the long exact sequence of direct images of the triangle, where the direct images are expressed by Cech-cohomology groups with respect to the covering of Z by relative charts. Then, the direct images of $\mathcal{K}_\Phi^{\bullet,\star}$ are coherent due to Forster-Knorr lemma and the direct images of $\mathcal{H}_\Phi^{\bullet,s}$ are coherent since C_Φ is proper over S. Therefore, the third term: the direct images of the relative De Rham complex are also coherent.

A characterization of nilpotent orbit closures among symplectic singularities

Yoshinori Namikawa

Let $0 \in V$ be a germ of a normal variety over the complex number $\mathbf{C}$. Denote by m the maximal ideal of $\mathcal{O}_{V,0}$ corresponding to the origin 0. Assume that the smooth locus V_{reg} admits a holomorphic symplectic 2-form ω. Then (V, ω) is a symplectic singularity if ω extends to a holomorphic 2-form on a resolution $\mu : W \to V$. A conjecture of Kaledin says that there would be a $\mathbf{C}^*$-action on V such that m/m^2 has only positive weights and ω is homogeneous symplectic 2-form (if necessary, by replacing the original ω by a suitable one). If the conjecture holds, V can be globalized to an affine variety X. Such a variety is called a conical symplectic variety. More precisely, let X be a normal affine variety which admits a symplectic 2-form ω on the smooth locus X_{reg}. Then

Definition (X, ω) is a *conical symplectic variety* if

(1) the coordinate ring R of X is positively graded $R = \oplus_{i \geq 0} R_i,$, $R_0 = \mathbf{C}$ (that is, X has a good $\mathbf{C}^*$-action).
(2) ω is a homogeneous symplectic 2-form with respect to the $\mathbf{C}^*$-action,
(3) ω extends to a holomorphic 2-form on a resolution Y of X.

The pair $(\tilde{O}, \omega_{KK})$ of the normalization $\tilde{O}$ of a nilpotent orbit closure $\bar{O}$ of a complex semisimple Lie algebra $\mathfrak{g}$ and the Kirillov-Kostant form ω_{KK} becomes a conical symplectic variety. Another type of examples of conical symplectic varieties is constructed by the holomorphic symplectic reduction for a Hamiltonian action of a reductive group G on a complex symplectic manifold (M, ω). For examples, Nakajima quiver varieties and hypertoric varieties are among them. The Brieskorn-Slodowy slices to a nilpotent orbit of a complex semisimple Lie algebra are also conical symplectic varieties. As is well known, a Du Val singularity of type A, D or E arises as such a slice to a subregular nilpotent orbit of the corresponding Lie algebra.

Now let's take a minimal homogeneous generator x_0, ..., x_n of the coordinate ring R of a conical symplectic variety X. Put $a_i := wt(x_i)$, i.e. $x_i \in R_{a_i}$. Then $N := max\{a_0, ..., a_n\}$ is called a *maximal weight* of X.

Recently I proved the following finiteness theorem for conical symplectic varieties (Namikawa, Y: A finiteness theorem on symplectic singularities, Copositio Math. **152**, 1225–1236 (2016)).

Theorem. *For fixed positive integers d and N, there are only finitely many conical symplectic varieties of dimension $2d$ and with maximal weight N.*

The main theorem of the talk is a complete classification of conical symplectic varieties with maximal weight $N = 1$.

Main Theorem. *Let (X, ω) be a conical symplectic variety with maximal weight 1. Then (X, ω) is isomorphic to one of the following*

(1) *an affine space $(\mathbf{C}^{2d}, \omega_{st})$ with a standard symplectic form ω_{st}, or*
(2) *a normal nilpotent orbit closure $(\bar{O}, \omega_{KK})$ of a complex semisimple Lie algebra with the Kirillov-Kostant form ω_{KK}.*

The rough sketch of the proof is as follows. First of all, we prove that $wt(\omega) = 2$ or $wt(\omega) = 1$. In the first case (X, ω) is isomorphic to an affine space $\mathbf{C}^{2d}$ together with the standard symplectic form ω_{st}. In the second case the Poisson bracket has degree -1 and R_1 has a natural Lie algebra structure. Then it is fairly easy to show that X is a coadjoint orbit closure $\bar{O}$ of a complex Lie algebra $\mathfrak{g}$.

The essential part is proving that $\mathfrak{g}$ is semisimple.

If X has a crepant resolution, we can prove that the crepant resolution can be written as the cotangent bundle T^*M of a flag variety M. In this case $\mathfrak{g}$ coincides with the Lie algebra of $Aut(M)$. We can easily check that $Aut(M)$ is semisimple by the fact that M is projective. This is the method we employed in a previous paper (Namikawa, Y.: On the stucture of homogeneous symplectic varieties of complete intersection , Invent. Math. **193**, 159-185 (2013)).

But X generally does not have such a resolution and we need a new method to prove the semisimplicity. This is nothing but the following proposition.

Proposition. *Let $\mathfrak{g}$ be a complex Lie algebra with trivial center whose adjoint group G is a linear algebraic group. Assume that the nilradical of $\mathfrak{g}$ is non-trivial: $\mathfrak{n} \neq 0$. Let O be a coadjoint orbit of $\mathfrak{g}^*$ with the following properties*

(1) *O is preserved by the scalar $\mathbf{C}^*$-action on $\mathfrak{g}^*$;*
(2) *$T_0\bar{O} = \mathfrak{g}^*$, where $T_0\bar{O}$ denotes the tangent space of the closure $\bar{O}$ of O at the origin.*

Then $\bar{O} - O$ contains infinitely many coadjoint orbits; in particular $\bar{O}$ has infinitely many symplectic leaves.

We apply this proposition to our $\mathfrak{g}$ and $X = \bar{O}$ (one can check that $\mathfrak{g}$ has no center and the adjoint group G is a linear algebraic group). If $\mathfrak{g}$ is not semisimple, it must have non-trivial nilradical $\mathfrak{n}$. But, then the proposition shows that $(\bar{O}, \omega_{KK})$ cannot have symplectic singularities. In fact, if $(\bar{O}, \omega_{KK})$ has symplectic singularities, it has only finitely many symplectic leaves by a result of Kaledin.

One can find more details in the preprint:

Namikawa, Y.: A characterization of nilpotent orbit closures among symplectic singularities, arXiv:1603.06105

Orbifold Milnor lattice, orbifold Seifert and intersection forms

Sabir M. Gusein-Zade
(joint work with W. Ebeling)

For a germ of a quasihomogeneous function with an isolated critical point at the origin invariant with respect to an appropriate action of a finite abelian group (an admissible one), H.Fan, T.Jarvis, and Y.Ruan defined the so-called quantum cohomology group. This group is defined in terms of the vanishing cohomology groups of Milnor fibres of restrictions of the function to fixed point sets of elements of the group. The quantum cohomology group is considered as the main object of the so called quantum singularity theory or FJRW-theory. Fan, Jarvis, and Ruan studied some structures on the quantum cohomology group which generalize similar structures in the usual singularity theory. An important role in singularity theory is played by such concepts as the (integral) Milnor lattice, the monodromy operator, the Seifert form and the intersection form. Analogues of these concepts have not yet been considered in the FJRW-theory. We define an orbifold version of the monodromy operator on the quantum (co)homology group and a lattice which is invariant with respect to the orbifold monodromy operator and is considered as an orbifold version of the Milnor lattice. The action of the orbifold monodromy operator on it can be considered as an analogue of the integral monodromy operator. Moreover, we define orbifold versions of the Seifert form and of the intersection form.

To define these concepts we use the language of group rings. An appropriate change of the basis in the group ring allows to give a decomposition of a certain extension of the quantum (co)homology group into parts isomorphic to (co)homology groups of certain suspensions of the restrictions of the function under consideration to fixed point sets. This permits us to define analogues of the Seifert and intersection form on this extension. We show that the intersection of this decomposition with the quantum (co)homology group respects the relations between the monodromy, the Seifert and the intersection form.

Vanishing homology of codimension 1 multi-germs of mappings from n-space to $n+1$-space

David Mond

We consider germs $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ where $|S| < \infty$ with $\mathcal{A}_e$-codimension 1. Infinitely many essentially different such germs exist, in sharp distinction to the case of ICIS, where only the A_1-singularity has $\tau = 1$. All known $\mathcal{A}$-finite germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ in Mather's nice dimensions satisfy the conjectured relation

$$\mu_I \geq \mathcal{A}_e\text{-codim} \tag{1}$$

with equality if f is weighted homogeneous. Here μ_I is the *image Milnor number*, the rank of the middle homology of the image of a stable perturbation f_t, which has the homotopy type of a wedge of n-spheres. When $\mu_I = 1$, the image is a homotopy sphere, but for different germs, the geometric origin of the vanishing cycle is very different, as witness the three Reidemeister moves of elementary knot theory. It can be described in terms of the Image Computing Spectral Sequence, [GorMo93], [Gor95], which calculates the homology of the image of a finite map $f : X \to Y$ from the *alternating homology* $H_*^{\mathrm{Alt}}(D^k(f))$ of the multiple point spaces $D^k(f) \subset X^k$. Here

$$D^k(f) = \mathrm{closure}\{(x_1, \ldots, x_k) \in X^k : x_i \neq x_j \text{ if } i \neq j, f(x_i) = f(x_j) \text{ for all } i, j\},$$

and $H_j^{\mathrm{Alt}}(D^k(f))$ is the j'th homology of the subcomplex of the usual singular chain complex on which the symmetric group S_k, permuting the copies of X, acts by its sign representation. The ICSS has $E^1_{p,q} = H^{\mathrm{Alt}}_{p-q+1}(D^q(f))$ and converges to $H_{p+q}(f(X))$. K. Houston showed in [Hou97] that if $|S| = 1$ then $D^k(f_t)$ has alternating homology only in middle dimension. This implies collapse of the ICSS at E_1, and the formula

$$\mu_I(f) = \sum_{k=2}^{n+1} \mathrm{rank}\, H^{\mathrm{Alt}}_{n-k+1}(D^k(f_t)).$$

So when $\mu_I = 1$, just one of these groups is non-zero.

Conjecture: In this case the non-zero group corresponds to the highest k for which $D^k(f) \neq \emptyset$.

The conjecture is easy to check for map-germs of corank 1, using a normal form and explicit equations for the $D^k(f)$, which are all ICISs. Germs of corank > 1 are harder, since the $D^k(f)$ are no longer ICISs. The simplest germ of corank 2 and $\mathcal{A}_e$-codimension 1, mapping $(\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$, has $\mu_I = 1$, in agreement with (1); it has triple points but no quadruple points. A somewhat complex calculation shows that the non-vanishing alternating homology group is $H_3^{\mathrm{alt}}(D^3(f_t))$, in agreement with the conjecture, see [Mo16]. We observe also that although $D^2(f_t)$ has no *alternating* homology outside its middle dimension, 4, it does have non-trivial homology in dimension 2.

Though not directly relevant to the main theme of the talk, but as evidence for the earlier-mentioned conjecture (1), we mention a recent example found by Ayse Sharland, [AS16],

$$f(x,y,z) = (x^2+yz, xy+z^5+x^2z, x^2y+yz^4+z^7+y^2z, y^2+x^3+z^6),$$

which has $\mu_I = \mathcal{A}_e$–codimension $= 18,967$. Note that this germ is weighted homogenous. Here μ_I is calculated in terms of weights and degrees using a recent formula of T. Ohmoto in [O16] based on Thom polynomial techniques, and the $\mathcal{A}_e$–codimension is calculated by Sharland by means of an isomorphism described in [Mo91],

$$\frac{\theta(f)}{T\mathcal{A}_e f} \simeq \frac{J_h\mathcal{O}_{\mathbb{C}^n,0}}{J_h\mathcal{O}_{D,0}}$$

where $D = h^{-1}(0)$ is the image of f.

References

[AS16] A. Sharland, personal communication, 2016

[Gor95] V.V. Goryunov, Semi-simplicial resolutions and homology of images and discriminants, Proc London Math Soc (3) 70 , 1995, 363–385

[GorMo93] V.V. Goryunov and D.Mond, Vanishing cohomology of singularities of mappings, Comp. Math. 89 No. 1, (1993) 45–80

[Hou97] K. Houston, Local topology of images of analytic maps, Topology 36 (1997), no. 5, 1077–1121

[Mo91] D. Mond, Vanishing cycles for analytic maps, *Singularity Theory and its Applications 1*, Lecture Notes in Math. 1462, Springer Verlag, 1991, 221–234

[Mo16] D. Mond, Disentanglements of corank 2 map-germs: two examples, arXiv:1610.01328, 2016

[O16] T. Ohmoto, Singularities of mappings and characteristic classes, *School on Real and Complex Singularities in São Carlos, 2012*, Adv. Studies in Pure Math. 68, 2016, Math. Soc. of Japan, 191–266

Intergration of subanalytic and oscillatory functions

Georges Comte

(joint work with Raf Cluckers, Dan Miller, Jean-Philippe Rolin, Tamara Servi)

The stability under integration of certain classes of real functions was already considered in [3], [4], [6], [9], [11], but none of these classes allows oscillatory behaviour, let alone stability under Fourier transforms.

I briefly summarize here reference [5], where the question of integration of oscillatory functions is investigated.

Let us recall that a globally subanalytic set X of $\mathbf{R}^n$ is a set definable in the ring language with parameters from $\mathbf{R}$ and symbols for analytic functions restricted to closed balls. A globally subanalytic map is a map with globally subanalytic graph. For sake of brevity, in what follows, subanalytic means globally subanalytic.

The collection of subanalytic sets yields an o-minimal structure, that is a collection of sets containing all real algebraic subsets, stable under basic geometrical operations such as intersection, union, product, projection and complement, and such that a subset of $\mathbf{R}$ of this collection is the union of finitely many intervals (see [7] or [15] for details).

Note that the functions $\sin : \mathbf{R} \to \mathbf{R}$, $\exp : \mathbf{R} \to \mathbf{R}$ and $\log : \mathbf{R} \to \mathbf{R}$ are not subanalytic functions. The reason being that the first one of these functions oscillates: its graph cuts the x-axis in an infinite number of points that cannot define a subanalytic subset of $\mathbf{R}$. For the exponential function, the reason is somewhat less trivial: subanalytic functions are polynomially bounded [1], that is, for any subanalytic function $f : \mathbf{R} \to \mathbf{R}$, there exists an integer ν such that $\forall x \in \mathbf{R}, \quad |f(x)| \leq |x|^\nu$. And finally, since subanalyticity is stable under linear maps, the log function cannot be subanalytic (its graph being obtained from the graph of exp through a symmetry). Nevertheless, the expansion $\mathbf{R}_{\mathrm{an,exp}}$ of the subanalytic structure by the graph of $\exp : \mathbf{R} \to \mathbf{R}$ (or the graph of log) is still o-minimal by [8].

When one deals with integration of subanalytic functions, one can fear that the integration process will provide some perturbations with respect to the geometrical tameness of the subanalytic structure, since this process is far from being a first order logical process.

The results of [6] and [11] proves that the perturbation is enough to destroy the subanalyticity of the integrand we start with, although not enough to rule out o-minimality of the integral. Indeed, for $f : X \times \mathbf{R}^n \to \mathbf{R}$ a subanalytic function defined on the subanalytic set X, the set

$$Int(f) := \{x \in X;\ \int_{\mathbf{R}^n} |f(x,t)|\ dt < \infty\}$$

is subanalytic and

$$Int(f) \ni x \mapsto \int_{\mathbf{R}^n} f(x,t)\ dt,$$

is a function of the algebra

$$\mathcal{C} := \{P(h_1, \cdots, h_r, \log|h_1|, \cdots, \log|h_r|);\ r \in \mathbf{N}, P \in \mathbf{R}[X_1, \cdots, X_{2r}]\},$$

an algebra of functions definable in the o-minimal structure $\mathbf{R}_{\mathrm{an,exp}}$.

Then the natural question of the existence of an algebra of functions, all definable in the same o-minimal structure, and stable under integration, has been solved in [3] and [4]: the algebra $\mathcal{C}$ is such an algebra. Iterated integrations of subanalytic functions does not create new functions after the first step and yield functions definable in an o-minimal structure, namely $\mathbf{R}_{\mathrm{an,exp}}$.

The question treated in [5] allows, from the beginning, oscillatory functions as integrands: what is the smallest $\mathbf{C}$-algebra $\mathcal{E}$ of functions containing all subanalytic functions, their complex exponential e^{ih} (h subanalytic), and that is stable under integration? Of course this algebra has to contain $\mathcal{C}$.

A motivation here for introducing oscillatory functions e^{ih} (with h subanalytic) in the integrand is to initiate the study of the real counterpart of the theory of complex singularities of (real) analytic phases in oscillatory integrals (see [1], [12],

[1] In fact the converse is true: by [13], an o-minimal expansion of the real field is polynomially bounded if and only if it contains the graph of the exponential function.

[16]). One knows, in the analytic case, that exponents appearing in the asymptotic expansion of oscillatory integrals give eigenvalues of the monodromy operator of the (complexification of the) phase. Of course such striking interplay between asymptotic analysis and complex singularity theory seems difficult to transfer to the real setting, but nevertheless we do hope that a well-adapted real version is possible. We give now the following answer to the stability question by fully describing $\mathcal{E}$.

Theorem 1. — *The algebra $\mathcal{E}$ is the $\mathbf{C}$-vector space E generated by the functions of the form $g(x,t)e^{ih(x,t)}\int_{\mathbf{R}} e^{iu}k(x,t,u)\log^{\ell}|u|\ du$, where $g \in \mathcal{C}, h, k$ are subanalytic, and $\ell \in \mathbf{N}$. More accurately, E is a $\mathbf{C}$-algebra and for $f \in E$ defined on some subanalytic set $X \times \mathbf{R}^n$, there exist $i, F \in E$ defined on X such that $Int(f) = i^{-1}(0)$ and such that for any $x \in Int(f)$, $F(x) = \int_{\mathbf{R}^n} f(x,t)\ dt$.*

In particular $\mathcal{E}$ is an algebra of functions containing all subanalytic functions and stable under Fourier transform. Note that this algebra also contains special functions like $x \mapsto e^{-x^2}$ or $\mathrm{Si} : x \mapsto \int_0^x \frac{\sin t}{t}\ dt$.

Moreover $\mathcal{E}$ is closed with respect to point-wise limit: for $f \in \mathcal{E}$, the set $\mathrm{Lim}(f)$ of points x such that $\lim_{y\to\infty} f(x,y)$ exists is a zero level of a function of $\mathcal{E}$ and there exists $l \in \mathcal{E}$ such that for any $x \in \mathrm{Lim}(f)$, $\lim_{y\to\infty} f(x,y) = l(x)$. For any integer number $p \geq 1$, the algebra $\mathcal{E}$ is also closed with respect to the L^p-norm and particular sequences of the form $(f(\cdot,k))_{k\in\mathbf{N}}$, with $f(\cdot,k) \in \mathrm{L}^p(\mathbf{R}^n)$, $f \in \mathcal{E}$: any such sequence converging in L^p converges almost everywhere to a function of $\mathcal{E}$. As an application, for $p = 2$, we get the following result.

Theorem 2. — *The L^2-Fourier transform is an isometry from $\mathcal{E} \cap \mathrm{L}^2$ to itself.*

Theorem 1 and 2 show that integration of functions of $\mathcal{E}$ does not delete the logical and geometrical properties of those functions; analytic tameness and oscillation of functions of $\mathcal{E}$ are well-balanced from integration point of view.

To prove Theorem 1 and 2, starting from the Parusiński and Lion-Rolin subanalytic preparation theorem (see [10], [14]), we prove a preparation theorem for functions of $\mathcal{E}$ that allows to understand the asymptotic behaviour of these functions as well as the asymptotic behaviour of their integrals. Then, using the theory of almost-periodic functions (see [2]) we have to prove that oscillatory terms coming from this preparation cannot cancel. It roughly amounts showing that for $f(t) = \sum_{j=1}^{J} c_j e^{ip_j(t)}$, $c_j \in \mathbf{C} \setminus \{0\}$, $p_j \in \mathbf{R}[t]$, there exists $\varepsilon, \delta > 0$ such that the length of $\{t \in [1,\infty); |f(t)| \geq \varepsilon\}$ is bigger than δ.

References

[1] V. I. Arnol'd, S. M. Guseĭn-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. II*, Monographs in Mathematics, vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1988.

[2] A. S. Besicovitch, *Almost periodic functions*, Dover Publications, Inc., New York, 1955.

[3] R. Cluckers and D. J. Miller, *Stability under integration of sums of products of real globally subanalytic functions and their logarithms*, Duke Math. J. **156** (2011), no. 2, 311–348.
[4] ———, *Loci of integrability, zero loci, and stability under integration for constructible functions on Euclidean space with Lebesgue measure*, Int. Math. Res. Not. IMRN (2012), no. 14, 3182–3191.
[5] R. Cluckers, G. Comte, D. J. Miller, and J.-P. Rolin, T. Servi, *Integration of subanalytic ans oscillatory functions*, preprint arxiv 1601.01850, (2016) .
[6] G. Comte, J.-M. Lion, and J.-P. Rolin, *Nature log-analytique du volume des sous-analytiques*, Illinois J. Math. **44** (2000), no. 4, 884–888.
[7] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), no. 2, 497–540.
[8] L. van den Dries, A. J. Macintyre and D. Marker, *The Elementary Theory of Restricted Analytic Fields with Exponentiation*, Annals of Mathematics **140** (1994), 183–205.
[9] T. Kaiser, *Integration of semialgebraic functions and integrated Nash functions*, Math. Z. **275** (2013), no. 1-2, 349–366.
[10] J.-M. Lion and J.-P. Rolin, *Théorème de préparation pour les fonctions logarithmico-exponentielles*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 3, 859–884.
[11] J.-M. Lion and J.-P. Rolin, *Intégration des fonctions sous-analytiques et volumes des sous-ensembles sous-analytiques*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 3, 755–767.
[12] B. Malgrange, *Intégrales asymptotiques et monodromie*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 405–430 (1975).
[13] C. Miller, *Exponentiation is hard to avoid*, Proc. Amer. Math. Soc. 122, 1 (1994), 257–259.
[14] A. Parusiński, *On the preparation theorem for subanalytic functions*, New developments in singularity theory (Cambridge, 2000), NATO Sci. Ser. II Math. Phys. Chem., vol. 21, Kluwer Acad. Publ., Dordrecht, 2001, pp. 193–215.
[15] A. J. Wilkie, *o-minimal structures*, Astérisque, 326 (2009), Exp. No. 985, vii, 131–142 (2010). Séminaire Bourbaki. Vol. 2007/2008.
[16] A. N. Varchenko, *Newton polyhedra and estimates of oscillatory integrals*, Funct. Anal. Appl. **18** (1976), no. 3, 175–196.

Three (Counter)-Examples

Herwig Hauser

We present three examples, with quite different flavour:

(1) Gabrielov's counterexample to a conjecture of Grothendieck remained mysterious over many years. He constructed in 1971 three analytic functions which admit a formal but no analytic relation, thus disproving the conjecture that the ideal of formal relations is generated by the analytic ones.

We explain the divergence phenomenon by constructing the relations via a generalized division algorithm, applied to echelons. These are a natural extension of the concept of ideals in power series rings, by admitting in linear combinations of the generators only coefficients depending on nested subsets of the variables. The analysis allows us to construct many more counterexamples.

Joint work with Mariemi Alonso, Francisco Castro-Jiménez, Christoph Koutschan.

(2) The Grinberg-Kazhdan-Drinfeld theorem asserts that the formal neighborhood of a non-degenerate arc in an arc scheme is the cartesian product of the

formal neighborhood of a scheme of finite type with countably many copies of the formal disk. Here, non-degenerate means that the arc is not entirely contained in the singular locus of the given scheme.

Bourqui and Sebag have shown by means of an example that this statement is no longer true if one considers arcs inside the singular locus. We prove that this happens always for constant arcs at singular points.

Joint work with Christopher Chiu.

(3) The standard resolution invariant in characteristic zero consists of a string of integers, whose first component is the local order of the defining ideal of the embedded singular scheme, and which is taken with respect to the lexicographic ordering. The invariant is shown to be upper semicontinuous, and blowing up its top locus (points of maximal value) makes the invariant drop.

In positive characteristic, it is well known that the second component of the invariant, the *residual order* – provided that it is defined appropriately in a characteristic free manner – may increase under permissible blowup in the case that the first component remains constant. Moh gave a bound for the maximal increase and there was hope that a suitabl modification of the residual order still allows one to carry out the induction argument, since it was expected that it *decreases in the long run.*

We exhibit an example, constructed by Stefan Perlega from the University of Vienna, of a sequence of permissible blowups along which the residual order tends to infinity. This is NOT a counter-example to resolution, since in the example larger centers could be blown up, thus avoiding the indefinite increase. Nevertheless, the striking phenomenon disproves a theorem of Moh and tells us that, in positive characteristic, we will have to review our standard way to select the centers of the blowups.

Complex surface singularities with rational homology disk smoothings

Jonathan Wahl

Consider a complex normal surface singularity $(V, 0)$ with a smoothing whose Milnor number is 0, i.e., the Milnor fibre has no rational homology. Such a $(V, 0)$ must be a rational singularity, and all cyclic quotient singularities of type $p^2/pq-1$ $(0 < q < p, (p, q) = 1)$ have a unique such smoothing ([5], 9.2). In the 1980's, we discovered three triply-infinite and six singly-infinite families of such singularities, all weighted homogeneous. Later work of Stipsicz, Szabó, Bhupal, and myself ([4], [1]) proved that these were the only weighted homogeneous examples. Our PhD student Jacob Fowler has made substantial progress on remaining questions [2], such as counting the number of distinct smoothings in each case; calculating the fundamental group of the Milnor fibre (it is finite but can be non-abelian); determining the analytic type when there is a modulus in the resolution graph.

We conjectured that these were the only surface singularities, and gave some partial results [6]. A recent preprint of Stipsicz, Park, and Shin [3] offers a proof of this conjecture; the paper is quite difficult and is currently being refereed. We shall discuss the main outstanding questions on these problems, and mention related symplectic/contact geometry issues (such as the relation to the existence of symplectic fillings of the links.)

References

[1] M. Bhupal and A. Stipsicz, Weighted homogeneous singularities and rational homology disk smoothings. Amer. J. Math. **133** (2011), no. 5, 1259–1297.

[2] J. Fowler, Rational homology disk smoothing components of weighted homogeneous surface singularities. Ph.D. Thesis, University of North Carolina, 2013.

[3] H. Park, D. Shin, and A. Stipsicz, Normal complex surface singularities with rational homology disk smoothings. ArXiv:1311.1929.

[4] A. Stipsicz, Z. Szabó, and J. Wahl, Rational blowdowns and smoothings of surface singularities. Journal of Topology **1** (2008), 477–517.

[5] J. Wahl, Smoothings of normal surface singularities. Topology **20** (1981), 219–246.

[6] J. Wahl, On rational homology disk smoothings of valency 4 surface singularities. Geom. Topol. **15** (2011), 1125–1156.

Heegard Floer theory, semigroups, and a new characterization of rational surface singularities

Sarah Dean Rasmussen

The search for relationships between complex analytic structure and topology forms a recurrent theme in the study of complex singularities. This talk addresses two topics, both of which lie in the overlap of analytic structure and topology: a new point of view for singularity semigroups, and a new characterization of rational normal surface singularities.

Near the turn of the millennium, Campillo, Delgado, and Gusein-Zade [1] built on the classical theory of Milnor fiber monodromy of plane curve singularities to show that for $P_C(t)$ the Poincaré series of the ring of germs of functions of an irreducible plane curve singularity $(C,0) \subset (\mathbb{C}^2,0)$, one has $P_C(t) = \Delta_C(t)/(1-t)$, where $\Delta_C(t)$ is the characteristic polynomial of the monodromy of the Milnor fiber of the curve complement $\mathbb{C}^2 \setminus C$. This $\Delta_C(t)$ also coincides with the Alexander polynomial $\Delta(S^3 \setminus K) \in \mathbb{Z}[H_1(S^3 \setminus K)]$ of the link, $S^3 \setminus K$, of the curve complement. As all the coefficients of $P_C(t)$ lie in $\{0,1\}$, $P_C(t)$ can be regarded as the characteristic function of its support $\Gamma \subset \mathbb{Z}_{\geq 0}$. It follows that Γ is a semigroup, called the *singularity semigroup* of C.

I have recently encountered semigroups in the seemingly disparate context of Heegaard Floer homology, which is a gauge-theoretic closed 3-manifold invariant developed by Oszváth and Szabó [11]. Oszváth and Szabó [10], and independently J. Rasmussen [13] have also introduced a more refined invariant called knot Floer homology, which to any knot $K \subset M$ in a closed 3-manifold M, associates a filtered chain complex $HFK(M,K)$ which splits over relative Spin^c-structures.

There is a precise sense in which one can regard the Euler characteristic of the "hat" version of HFK as residing in the group ring of $H_1(M \setminus K; \mathbb{Z})$, so that

$$\chi(\widehat{HFK}(M, K)) = (1 - [\mu])\tau(M \setminus K) \in \mathbb{Z}[H_1(M \setminus K; \mathbb{Z})],$$

where $\mu \in H_1(M \setminus K)$ is the class of the meridian of K, and $[\mu]$ indicates the inclusion of μ into the group ring $\mathbb{Z}[H_1(M \setminus K; \mathbb{Z})]$. Here, τ is a more classical 3-manifold invariant called the Reidemeister-Turaev torsion [16], which, in particular, associates to any compact oriented 3-manifold Y with torus boundary a Laurent series $\tau(Y)$ in an appropriate extension of the group ring $\mathbb{Z}[H_1(Y; \mathbb{Z})]$. For example, for $Y = S^3 \setminus K$, one has $\tau(S^3 \setminus K) = \Delta(S^3 \setminus K)/(1-t)$, where $t = [\mu]$.

For a compact oriented 3-manifold Y with torus boundary, the Reidemeister-Turaev torsion $\tau(Y)$ is a useful tool for studying how $HF(Y(\mu))$ changes as one varies the Dehn filling slope $\mu \in \mathbb{P}(H_1(\partial Y; \mathbb{Z})) \cong \mathbb{Q} \cup \{\infty\}$ for Dehn fillings $\{Y(\mu)\}$ of Y. In joint work [14], J. Rasmussen and I show that for Floer simple Y and a given L-space Dehn filling slope μ_{L}, the Reidemeister-Turaev torsion $\tau(Y)$ determines the "hat" Heegaard Floer homology $\widehat{HF}(Y(\mu))$ for any $\mu \in \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$. Here, the term *Floer simple* indicates that Y has more than one L-space Dehn filling, and an *L-space* is a closed oriented 3-manifold with vanishing reduced Heegaard Floer homology. We also show that for Floer simple Y, the space of L-space Dehn fillings of Y forms an interval $\mathcal{L}(Y) \subset \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$, whose endpoints are given by consecutive elements of the set $\mathbb{P}(\iota_*^{-1}\tilde{\mathcal{D}}^\tau(Y)) \subset \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$, where

$$\tilde{\mathcal{D}}^\tau(Y) := \{x - y | x \notin S[\tau(Y)], y \in S[\tau(Y)]\} \cap H_1(Y; \mathbb{Z})|_{\geq 0}.$$

Here, ι_* is the homomorphism induced on homology by the inclusion $\iota : \partial Y \to Y$ of the boundary, and $S[\tau(Y)]$ indicates the support of $\tau(Y)$ in $H_1(Y; \mathbb{Z})$.

The complement $\Gamma_Y := H_1(Y; \mathbb{Z})|_{\geq 0} \setminus \tilde{\mathcal{D}}^\tau(Y)$ is a semigroup. Moreover, if Y is the link of the complement $\mathbb{C}^2 \setminus C$ of an isolated irreducible planar curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$, then Γ_Y coincides with the singularity semigroup of C.

My claim is that even though Γ_Y can have torsion, it is the correct object to generalize the singularity semigroup of irreducible planar curve singularities. If Y is the link of an end curve complement $X \setminus C$ in an isolated rational or good quasihomogeneous surface singularity, I can show [15] that such Y is Floer simple, and that the Heegaard Floer homology of any Dehn filling of Y is completely determined by Γ_Y. On the other hand, in this case, $\tau(Y)$ is the characteristic function of Γ_Y and descends from the equivariant Poincaré series for $X \setminus C$ or from an appropriate zeta function for $X \setminus C$, (*c.f.* Némethi [9]).

For arbitrary Floer simple Y, J. Rasmussen and I also used the invariant $\tilde{\mathcal{D}}^\tau = H_1(Y; \mathbb{Z})|_{\geq 0} \setminus \Gamma_Y$ to prove a gluing theorem determining, in terms of the intervals $\mathcal{L}(Y_i)$ of L-space Dehn fillings for Floer simple Y_i, whether a union $Y_1 \cup Y_2$ is an L-space [14]. In joint work with Hanselman and Watson [5], who had proved a related gluing theorem [6], we combined an inductive argument with our gluing results to show that a graph manifold is an L-space if and only if it fails to admit a taut foliation. Némethi then exploited this result to prove that an isolated normal surface singularity is rational if and only if its link is an L-space [7].

In independent work [15], I have constructed a finite recursive formula which, for any graph manifold Y with torus boundary, computes the space $\mathcal{L}(Y)$ of L-space Dehn filling slopes of Y in terms of the Seifert data and gluing maps in a JSJ decomposition of Y. In particular, if Y is Floer simple, then one can express Y as a union $Y = \hat{Y} \cup_{\varphi} \coprod_{i=1}^{n_G} Y_i$, with gluing maps $\varphi_i : \partial Y_i \to -\partial_i \hat{Y}$, where each Y_i is a Floer simple graph manifold, and where $\hat{Y} = M_{S^2}(y_1, \ldots, y_n) \setminus \coprod_{i=1}^{n_G+1}(S^1 \times \{p_i\})$ is Seifert fibered over the n_G+1-times punctured two-sphere $S^2 \setminus \coprod_{i=1}^{n_G+1}\{p_i\}$, with Seifert slopes $y_j = \beta_j/\alpha_j$. If we write $y_i^-, y_i^+ \in \mathbb{P}(H_1(\partial_i \hat{Y}))$ for the left-hand and right-hand endpoints of each L-space interval image $\varphi_{i*}^{\mathbb{P}}(\mathcal{L}(Y_i)) \subset \mathbb{P}(H_1(\partial_i \hat{Y})) \cong \mathbb{Q} \cup \{\infty\}$, in terms of the Seifert data basis, then $\mathcal{L}(Y)$ is given by the interval with left-hand and right-hand endpoints y_- and y_+, where

$$y_- = \max_{k \in \mathbb{N}} -\frac{1}{k}\left(1 + \sum_{j=1}^{n} \lfloor k y_j \rfloor + \sum_{i=1}^{n_G} (\lceil k y_i^+ \rceil - 1)\right),$$

$$y_+ = \min_{k \in \mathbb{N}} -\frac{1}{k}\left(-1 + \sum_{j=1}^{n} \lceil k y_j \rceil + \sum_{i=1}^{n_G} (\lfloor k y_i^- \rfloor + 1)\right).$$

In particular, the closed graph manifold $Y(0)$ is an L-space if and only if the L-space interval from y_- to y_+ contains 0. Thus, due to Némethi's result that normal suface singularities have L-space links if and only if they are rational, the above formula provides a new characterization for rational surface singularities.

Note that if $Y(0)$ is the link of an isolated good quasihomogenous surface singularity $(X, 0)$, in which case $n_G = 0$ and $y_- > 0$, then $Y(0)$ is an L-space if and only if $y_+ \geq 0$, or equivalently, if and only if $-1 + \sum_{j=1}^{n} \lceil k y_j \rceil \leq 0$ for all $k \in \mathbb{N}$. This is consistent with the computation $p_a = \sum_{k \in \mathbb{N}} \max(0, -1 + \sum_{j=1}^{n} \lceil k y_j \rceil)$ of the arithmetic genus of X implicit in the models of quasihomogeneous surface singularities described by Pinkham [12] and Dolgachev [3].

References

[1] A. Campillo, F. Delgado, and S. M. Gusein-Zade. The Alexander polynomial of a plane curve singularity via the ring of functions on it. *Duke Math. J.*, 117(1):125–156, 2003.

[2] A. Campillo, F. Delgado, and S. M. Gusein-Zade. An equivariant Poincaré series of filtrations and monodromy zeta functions. *Rev. Mat. Complut.*, 28(2):449–467, 2015.

[3] Igor V. Dolgachev. On the link space of a Gorenstein quasihomogeneous surface singularity. *Math. Ann.*, 265(4):529–540, 1983.

[4] W. Ebeling. Poincaré series and monodromy of the simple and unimodal boundary singularities. *Tr. Mat. Inst. Steklova*, 267(Osobennosti i Prilozheniya):56–64, 2009.

[5] Jonathan Hanselman, Jacob Rasmussen, Sarah Dean Rasmussen, and Liam Watson. Taut foliations on graph manifolds. arXiv:1508.05911, 2015.

[6] Jonathan Hanselman and Liam Watson. A calculus for bordered Floer homology. Preprint, 2015.

[7] András Némethi. Links of rational singularities, L-spaces and LO fundamental groups. arXiv:1510.07128.

[8] András Némethi. On the Ozsváth-Szabó invariant of negative definite plumbed 3-manifolds. *Geom. Topol.*, 9:991–1042, 2005.

[9] András Némethi. Poincaré series associated with surface singularities. In *Singularities I*, volume 474 of *Contemp. Math.*, pages 271–297. Amer. Math. Soc., Providence, RI, 2008.
[10] Peter Ozsváth and Zoltá Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186:58–116, 2004. math.GT/0209056.
[11] Peter. Ozsváth and Zoltán Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311–334, 2004. math.GT/0311496.
[12] H. Pinkham. Normal surface singularities with C^* action. *Math. Ann.*, 227(2):183–193, 1977.
[13] Jacob Rasmussen. Floer homology and knot complements. Harvard University thesis. math.GT/0306378, 2003.
[14] Jacob Rasmussen and Sarah Dean Rasmussen. Floer Simple Manifolds and L-space Intervals. arXiv:1508.05900, 2015.
[15] Sarah Dean Rasmussen. Floer Simple Manifolds and L-space Intervals. arXiv:1511.04413, 2015.
[16] Vladimir Turaev. *Torsions of* 3*-dimensional manifolds*, volume 208 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2002.

Ultrametric spaces of branches on arborescent singularities

Patrick Popescu-Pampu
(joint work with Evelia R. García Barroso and Pedro D. González Pérez)

This work started from our intention to study in which measure a theorem of Płoski from [5] can be generalized from germs of smooth complex analytic surfaces to other normal surface singularities. This theorem may be stated as follows:

Theorem. *Fix a smooth complex analytic surface singularity (S, O). For each pair of branches (that is, germs of irreducible formal curves) A, B drawn on (S, O), consider:*

$$U(A,B) := \begin{cases} \dfrac{m_O(A) \cdot m_O(B)}{A \cdot B}, & \text{if } A \neq B, \\ 0, & \text{if } A = B \end{cases}$$

where m_O denotes the multiplicity at O and $A \cdot B$ denotes the intersection number of A and B at O. Then U is an ultrametric on the set of branches on (S, O).

We discovered that, slightly reformulated purely in terms of intersection numbers, the theorem may be extended to *arborescent* singularities. We call a normal surface singularity *arborescent* if it has a simple normal crossings resolution whose dual graph is a tree – in which case all such resolutions have this property. For instance, the normal surface singularities with rational homology sphere links are precisely the arborescent singularities such that all irreducible exceptional divisors appearing in its resolutions are rational.

Our generalization of Płoski's theorem is:

Theorem. *Fix an arborescent singularity* (S, O) *and a branch* L *on it. For each pair of branches* A, B *on* (S, O) *which are distinct from* L*, consider:*

$$U_L(A,B) := \begin{cases} \dfrac{(L \cdot A) \cdot (L \cdot B)}{A \cdot B}, & \text{if } A \neq B, \\ 0, & \text{if } A = B \end{cases}.$$

Then U_L *is an ultrametric on the set of branches on* (S, O) *distinct from* L*.*

Here we work with Mumford's notion of rational-valued intersection number of Weil divisors drawn on normal surface singularities, introduced in [4].

The previous theorem generalizes Płoski's one. Indeed, each time a finite set $\mathcal{F}$ of branches is fixed on a smooth germ of surface – the simplest kind of arborescent singularity – one may choose a smooth branch L transversal to all of them, in which case the functions U and U_L coincide in restriction to $\mathcal{F}$.

Consider now a finite set $\mathcal{F}$ of branches on an arbitrary arborescent singularity (S, O). The restriction of the ultrametric U_L to $\mathcal{F}$ allows to associate canonically a rooted tree $T_L(\mathcal{F})$ to $\mathcal{F}$. Its set of leaves is $\mathcal{F}$, its set of vertices consists of the closed balls defined by the ultrametric and its root may be seen as the union of $\mathcal{F}$ with an infinitely distant supplementary point. We prove the following interpretation of this rooted tree in terms of dual graphs:

Theorem. *Fix an arborescent singularity* (S, O)*, a branch* L *on it and a finite set* $\mathcal{F}$ *of branches distinct from* L*. Consider an embedded resolution* π *with simple normal crossings of the sum of* L *with the branches of* $\mathcal{F}$*. Denote by* $D_{L,\pi}(\mathcal{F})$ *the union of the geodesics joining the strict transforms of* L *and of the branches of* $\mathcal{F}$ *inside the dual graph of their total transform by* π*, seen as a tree rooted at the strict transform of* L*. Then the rooted trees* $T_L(\mathcal{F})$ *and* $D_{L,\pi}(\mathcal{F})$ *are canonically isomorphic.*

In the special case in which both (S, O) and the branch L are smooth, we recover a theorem of Favre and Jonsson [2].

The tree $D_{L,\pi}(\mathcal{F})$ has also a valuative interpretation, generalizing the one given by Favre and Jonsson [2] for the same case when both S and L are smooth.

As in [2], we work with valuations of the local ring $\mathcal{O}$ of (S, O) with values in $[0, +\infty]$ and which are allowed to take the value $+\infty$ on other elements of $\mathcal{O}$ than the function 0. Their set is naturally partially ordered : $\nu_1 \leq \nu_2$ if and only if $\nu_1(f) \leq \nu_2(f)$ for all $f \in \mathcal{O}$.

Every branch drawn on (S, O) defines a valuation – taking Mumford's intersection number of the principal divisor of each element of $\mathcal{O}$ with the branch. Every irreducible exceptional divisor on a resolution of (S, O) defines also a valuation – taking the order of vanishing of each element of $\mathcal{O}$ along this divisor.

Mumford's notion of intersection product allows moreover to define the value of a valuation of $\mathcal{O}$ on an arbitrary branch L. A valuation ν is called *normalized relative to* L if $\nu(L) = 1$. We prove that:

Theorem. *Fix an arborescent singularity (S, O), a branch L on it and a finite set $\mathcal{F}$ of branches distinct from L. Consider an embedded resolution with simple normal crossings of the sum of L with the branches of $\mathcal{F}$. Then the rooted tree $D_{L,\pi}(\mathcal{F})$ is isomorphic to the Hasse diagram of the poset of valuations defined by the irreducible curves represented by the vertices of $D_{L,\pi}(\mathcal{F})$, once they are normalized relative to L.*

All the theorems of Płoski and Favre-Jonsson which we generalize were proved either by working with Newton-Puiseux series or with sequences of blow-ups. We work instead only with intersection products of relatively nef rational exceptional divisors on a fixed embedded resolution of the sum of L and of the branches of $\mathcal{F}$. Our proofs are based in an essential way on the following fact:

Proposition. *Fix an arborescent singularity (S, O) and a simple normal crossings resolution of it. Denote by $(E_u)_{u\in\mathcal{V}}$ the irreducible components of its exceptional divisor and by $(E_u^*)_{u\in\mathcal{V}}$ the dual exceptional divisors, in the sense that $E_u^* \cdot E_v = \delta_{uv}$ for any $(u, v) \in \mathcal{V}^2$, where δ_{uv} is Kronecker's symbol. If E_w belongs to the segment $[E_u E_v]$ in the dual graph of the resolution, then:*

$$(E_u^* \cdot E_w^*) \cdot (E_v^* \cdot E_w^*) = (E_u^* \cdot E_v^*) \cdot (E_w^* \cdot E_w^*).$$

In turn, this proposition is based on a formula proved by Eisenbud and Neumann in [1], expressing the intersection numbers $(E_u^* \cdot E_w^*)$ in terms of determinants of subtrees of the dual tree of the resolution.

It is interesting to note that the previous proposition may be reinterpreted using spherical geometry. Consider the real vector space freely generated by the divisors $(E_u)_{u\in\mathcal{V}}$, endowed with the negative of the intersection product. It is a euclidean vector space. Look at the unit vectors $(A_u)_{u\in\mathcal{V}}$ which are positively proportional to the vectors $(E_u^*)_{u\in\mathcal{V}}$. Here is the announced interpretation: *if E_w belongs to the segment $[E_u E_v]$ in the dual graph of the resolution, then the spherical triangle with vertices A_u, A_v, A_w is right-angled at the vertex A_w.* Indeed, the equality of the previous proposition may be reformulated as the *spherical Pythagorean equality*:

$$\cos(\angle A_u A_w) \cdot \cos(\angle A_v A_w) = \cos(\angle A_u A_v),$$

which characterizes right-angled triangles in spherical geometry.

Detailed proofs of our results may be found in [3].

References

[1] D. Eisenbud, W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities.* Princeton Univ. Press, 1985.

[2] C. Favre, M. Jonsson, *The valuative tree.* Lecture Notes in Mathematics, **1853**. Springer-Verlag, Berlin, 2004.

[3] E. García Barroso, P. González Pérez, P. Popescu-Pampu, *Ultrametric spaces of branches on arborescent singularities.* ArXiv:1605.02229.

[4] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity.* Inst. Hautes Études Sci. Publ. Math. No. **9** (1961), 5–22.

[5] A. Płoski, *Remarque sur la multiplicité d'intersection des branches planes.* Bulletin of the Polish Academy of Sciences. Mathematics. Volume **33** (1985), 601–605.

Introduction to Singularities

Norbert A'Campo

This evening talk is intended especially for those Young Researchers from the Heidelberg Laureate Forum, september 18th–25th, this year, see *www.heidelberg-laureate-forum.org/event_2016/*, who were invited to attend the Conference on Singularities in Oberwolfach.

We start out with a very general approach to the term *Singularity*.

A singularity is an object in nature that does not fall in the realm of main understanding. Here *main understanding* is very vague. It could mean understanding by the basic laws in Physics, Theorems in Mathematics, ... or by the basic achievements of a theory.

In mathematical nature such an object or rather a feature could be the local behavior near a given point or global behavior of a function, a vector field, a differential form, a space, a representations of a group, etc..

Let us restrict to the case of the local behavior near a point p of differentiable functions that are defined on numerical real or complex spaces $\mathbb{R}^n$ or $\mathbb{C}^n$. So we study locally near $p \in \mathbb{R}^n$ or $p \in \mathbb{C}^n$ differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ with derivatives of any order or holomorphic functions $f : \mathbb{C}^n \to \mathbb{C}$.

The first step is the three term expansion

$$f(p+h) = f(p) + A(h) + \mathrm{Rest}_f(p,h)$$

where $A : \mathbb{R}^n \to \mathbb{R}$ is a linear function and where the third term $\mathrm{Rest}_f(p,h)$ is relatively small compared to h, meaning

$$\lim_{h\to 0} \frac{\mathrm{Rest}_f(p,h)}{||h||} = 0$$

The linear map A is called the *differential* of f at p and is denoted by $(Df)_p$.

Examples of functions on $\mathbb{R}^n$ are the differentiable coordinate functions $x_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, n$, on $\mathbb{R}^2$ also denoted by x, y. A general function f can be expressed using a system of coordinate functions. For example $f = x^5 + y^3 : \mathbb{R}^2 \to \mathbb{R}$. Other examples are $g = x^5 + y^3 + x^2y^2$ and $k = x^4 + y^4 + x^2y^2$.

Functions with least complicated expressions are the coordinate functions. In fact, how complicated a given function f is depends heavily on the system of coordinate functions.

A natural question Q_1 is to ask which functions f can be expressed near a given point p as first coordinate function of a local coordinate system of functions.

The following main theorem gives the answer:

The answer to the question Q_1 is YES if and only if $(Df)_p \neq 0$.

Indeed, let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, $p \in \mathbb{R}^n$ such that the differential $(Df)_p : \mathbb{R}^n \to \mathbb{R}$ at p does not vanish. The kernel $X = (Df)_p^{-1}(0)$ is a linear subspace of dimension $n-1$ in $\mathbb{R}^n$. Let Y be a 1-dimensional subspace such that X and Y span the ambiant space $\mathbb{R}^n$. The inverse mapping theorem of differential mappings tells us that any system of functions $f, x_1, x_2, \cdots, x_{n-1}$ where the functions $x_1, x_2, \cdots, x_{n-1} : \mathbb{R}^n \to \mathbb{R}$ are linear, linearly independent, and vanishing on Y, is locally near p a system of differential coordinate functions.

Consequently, according to the above explanation, the local behavior at a point of a function on numerical space is a singularity if its differential at this point vanishes.

The next natural question Q_2 is as follows. Let a function f on numerical space be singular at a point p, i.e. the above main theorem does not apply. Does some higher order differential quantity attached to f at p by its non-vanishing guarantee the existence of a local coordinate system $y_1, y_2, \cdots, y_n$ near p for which the expression of f is not very complicated.

The next main theorem, the so called Morse lemma, of the theory of singularities of functions gives a YES answer.

The answer to the question Q_2 is YES if the determinant of the Hessian of f at p does not vanish, i.e. $\text{Determinant}((Hf)_p) \neq 0$. *More precisely, if* $\text{Determinant}((Hf)_p) \neq 0$*, there exists a local coordinate system $y_1, y_2, \cdots, y_n$ with $y_1(p) = y_2(p) = \cdots = y_n(p) = 0$ and $f = f(p) \pm y_1^2 \pm y_2^2 \pm \cdots \pm y_n^2$ near p.*

What means *complicated expression?* The above polynomial functions f, g, k on $\mathbb{R}^2$ are singular at 0, none of the two main theorems applies. Fact is that the expression for f has two monomials, expressions for g and k have three. Can one express the functions g or k in some local coordinate system as polynomials with two monomials? The answer is YES for g and NO for k. The theory of Singularities of mappings gives explanations and answers for these kind of questions.

Let us agree to measure how complicated a polynomial is by counting its monomials of total degree ≥ 3 and to call this count the complexity. The above map f is of complexity 2 as all its "cousins" $f = x_1^5 + x_2^3 + x_3^2 + \cdots + x_n^2$, $n > 2$. The maps h and k are of complexity > 2. Maps like $x_1^k + x_2^2 + x_3^2 + \cdots + x_n^2$, $n > 1$, $k \geq 3$, are of complexity 1.

A natural question Q_3 is to classify up to a local coordinate change all singularities of polynomial mappings f on numerical space, complex or real, with $f(0) = 0$, $(Df)_q \neq 0$, $q \neq 0$, locally near 0 and $(Df)_0 = 0$. This is out of reach, but becomes the assertion of a theorem if we moreover assume, that for every deformation $f_t, t \in [0,1]$, with $f_t(0) = 0$, $(Df_t)_0 = 0$ and $(Df_t)_q \neq 0$ for $q \neq 0$, near 0, the complexity of f_t at 0 for some local coordinate system is ≤ 2. These are the so called ADE singularities. Those singularities have a long history dating back to Felix Klein (1884). Underlying geometric objects are the platonic solids with long histories in different civilisations.

The list of ADE singularities, omitting quadratic terms like x_i^2, is:
Type A: x^n, $n \geq 2$.
Type D: $x(y^2 + x^n)$, $n \geq 2$.
Type E: $x^4 + y^3$, $y(y^2 + x^3)$, $x^5 + y^3$.

The natural question Q_3 also gets answered by a theorem if we moreover assume, that for every small deformation $f_t, t \in [0,1]$,, the existence of local coordinates at 0 such that f_t is a polynomial having only the numbers $-1, 0, 1$ as coefficients. Again this characterizes the ADE singularities.

The ADE singularities do not allow a continuous variation of parameters, they are rigid, and show up in many situations dealing with rigid objects of some kind. For example it was a suggestion of Alexandre Grothendieck to search for ADE singularities inside the simple Lie groups. This search was so fruitful that their name changed from Kleinian to ADE singularities.

Not only local behavior matters. A monomial function $f_n = x^n : \mathbb{C} \to \mathbb{C}$ is singular at $0 \in \mathbb{C}$ for $n > 1$ and its Hessian vanishes if $n > 2$. It is natural to deform f_n as a polynomial function $g = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with leading term x^n keeping the following constraints. First we keep a symmetry: $g(p) = (-1)^n g(-p)$. Second, the singularities of g are of Morse type. Third, the value of g at a singular point of g is $+t$ or $-t$, $t \in [0, 1]$. It is Remarkable is that there exists exactly one such deformation family f_n^t with $f_n^0 = x^n$, namely the family with f_n^1 being the Chebyshev polynomial $T_n : \mathbb{C} \to \mathbb{C}$. Chebyshev polynomials are of interest in many mathematical fields due to properties that they share with monomials.

Compositions as mappings of the monomial functions are again monomial: $f_n \circ f_m = f_{nm}$. The same holds for the Chebyshev polynomials: $T_n \circ T_m = T_{nm}$.

The cartesian doubling

$$TT_n = T_n \times T_n : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

$$(p, q) \in \mathbb{C}^2 \mapsto TT_n(p, q) = T_n(p) + T_n(q) \in \mathbb{C}$$

share a factorization property with the real or complex functions $x^n + y^n : \mathbb{C}^2 \to \mathbb{C}$ or $x^n + y^n : \mathbb{R}^2 \to \mathbb{R}$, namely that $x^n + y^n$ factors as product in $n/2$ quadratic functions if n even, in $(n-1)/2$ quadratic and one linear function if n odd.

The cartesian double $TT_{nm}(p, q) = T_n(p) + T_m(q) : \mathbb{R}^2 \to \mathbb{R}$ are Morse functions with three critical values $-2, 0, +2$.

All of the features that one observes in the case of Chebyshev polynomials are initial instances of further developments. For instance, deforming semi-locally or globally functions on numerical space of higher dimension into a function with only Morse singularities has turned out to be an important tool. The functions TT_{nm} provided the building tools for semi-local controlled real morsifications of of real polynomials with an isolated singularity. The property of having a Morse deformation with only two critical values turned out to give one of numerous caracterisations of the ADE singularities. The composition property is an instance of so called matrix factorisations. Chebyshev polynomials have rational coefficients, so the absolut Galois group $G_{\mathbb{Q}}$ of the field $\mathbb{Q}$ acts trivially on Chebyshev polynomials. This is in contrast to the space of generalized Chebyshev polynomials $P : \mathbb{C} \to \mathbb{C}$ with as only critical values $0, 1$ but without restriction on its singularities. After an appropriate coordinate change $z \mapsto az + b$ in the source the coefficients will belong to a number field. The absolute Galois group $G_{\mathbb{Q}}$ acts faithfully on generalized Chebyshev polynomials. The inverse image $P^{-1}([0, 1]) \subset \mathbb{C}$ is a bicollared tree with as many edges as the degree of P. Its isotopy class determines the polynomial P up to a coordinate change $z \mapsto az + b$ in the source. The corresponding tree for a Chebyshev polynomial is a tree with non-terminal vertices all of valency 2. Amazingly, the absolute Galois group $G_{\mathbb{Q}}$ acts now faithfully on the set of isotopy classes of finite planar bicollared trees. Please note that the set of isotopy classes of finite planar bicollared trees has moreover a combinatorial description and hence one gets a very promising combinatorial approach to the absolute Galois group $G_{\mathbb{Q}}$.

Back to the singularity of $x^5 + y^3 + z^2 : \mathbb{C}^3 \to \mathbb{C}$ which Felix Klein has introduced. The vertices, edges and faces of the icosahedron can be drawn on the spaces $P^1(\mathbb{C})$ of complex lines through the origin of $\mathbb{C}^2$ in such a way that the group I of orientation preserving symmetries of the ikosahedron can be realized by holomorphic motions of the Riemann sphere $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. The group I is isomorphic to the alternating group A_5 of order 60 and is important for the study of equations of degree 5. Let $\tilde{I}$ be the subgroup of order 120 in the linear group $\mathrm{SL}(2,\mathbb{C})$ that projects via $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C}) = \mathrm{Aut}(P^1(\mathbb{C})$ to the group I. Felix Klein introduced the in 0 punctured quotient space $\mathbb{C}^2/\tilde{I}$ as the moduli space for the general equation of degree 5 and henceforth needed to study this space. A main step was to identify the quotient space $\mathbb{C}^2/\tilde{I}$ as being the complement of 0 in the 0-level of the function $x^5 + y^3 + z^2$. The first so-called Kleinian singularity at 0 of the complex surface $x^5 + y^3 + z^2 = 0$ in $\mathbb{C}^3$ was born.

The intersection of $x^5 + y^3 + z^2 = 0$ with the unit sphere in $\mathbb{C}^3$ is a 3-manifold $\Sigma_{5,3,2}$, called the Poincaré sphere. It is the counterexample discovered by Henri Poincaré to the homological Poincaré conjecture.

The intersection of the hypersurface $x^5 + y^3 + z^2 + u^2 + v^2 = 0$ with the unit sphere in $\mathbb{C}^5$ is a 7-dimensional manifold $\Sigma_{5,3,2,2,2}$ studied by Egbert Brieskorn, who proved that it is diffeomorphic to the exotic sphere Σ^7, previously discovered by John Milnor. The exotic sphere Σ^7 is homeomorphic to the standard sphere S^7 but not diffeomorphic, so it is a counter example to the differentiable Poincaré conjecture in dimension 7.

We jump to more recent times, omitting many interactions of singularity theory and sciences, and mention that recently a first geometric interpretation of the HOMFLY-PT polynomial was obtained. In 1985 Vaughan Jones made the discovery of a new knot invariant, now called Jones polynomial. Soon after the more classical Alexander polynomial and the Jones polynomial merged into a stronger invariant, the HOMFLY-PT polynomial. The definitions of these invariants have hidden very well the geometric meaning of these new knot invariants, until a first geometric interpretation was obtained 4 years ago by the work of Shende-Oblomkov-Maulik who have proved for the local knots of plane singularities $f(x,y) = 0$ in $\mathbb{C}^2$, that the HOMFLY-PT polynomial can be computed geometrically from the singularity by motivic integration.

The theory of singularities of mappings is very rich in remarkable objects, in results and even more rich in natural questions which are still unanswered. The same holds for the theory of singularities of other mathematical objects. In fact most mathematical, physical, biological and other objects in science develop singularities.

References

[1] V. I. Arnol'd; S. M. Gusein-Zade; A. N. Varchenko. Singularities of Differentiable Maps, Volume I: The Classification of Critical Points, Caustics and Wave Fronts. Birkhäuser (1985).
[2] V. I. Arnol'd; S. M. Gusein-Zade; A. N. Varchenko. Singularities of Differentiable Maps, Volume II: Monodromy and Asymptotics of Integrals. Monographs in Mathematics. Birkhäuser (1988).
[3] John W. Milnor (1968), Singular points of complex hypersurfaces. Annals of Mathematics Studies, No. 61. Princeton, NJ: Princeton University Press; Tokyo: University of Tokyo Press.
[4] Felix Klein, Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom 5ten Grade (1884); English translation by G. G. Morrice, Lectures on the Icosahedron; and the Solution of Equations of the Fifth Degree, (2nd revised edition, New York, 1914)
[5] Davesh Maulik, Stable pairs and the HOMFLY polynomial, arXiv:1210.6323v1 [math.AG].

Symplectic reductions associated to polar representations

Manfred Lehn

(joint work with M. Bulois, C. Lehn, and R. Terpereau)

Polar representations were introduced and studied by Dadok and Kac [3] in an attempt to axiomatise the properties of so-called θ-representations that had previously been studied by Vinberg [10] and that share many properties of the adjoint representation of a reductive group. So let G denote a reductive group with Lie algebra $\mathfrak{g}$ and V a finite dimensional rational representation of G. An element $v \in V$ is called semi-simple if its G-orbit in V is closed; it is regular semi-simple if in addition the dimension of its orbit is maximal among all semi-simple orbits. For such an element v consider the linear subspace $c = \{v' \in V \mid \mathfrak{g}v' \subset \mathfrak{g}v\}$. Dadok and Kac show that c consists of semi-simple elements only and that its dimension is bounded by $\dim(c) \leq \dim(V/\!\!/G)$. A representation (V, G) is called polar if $\dim(c) = \dim(V/\!\!/G)$, in which case c is called a Cartan subspace. Dadok and Kac prove that all Cartan subspaces in a polar representation are conjugate; that the quotient $W := N/H$ of the normaliser subgroup $N = \{g \in G \mid g(c) = c\}$ by the centraliser $H = \{g \in G \mid g|_c = \mathrm{id}_c\}$ is a finite reflection group, the so-called Weyl group of the representation; and that the natural morphism $c \to V$ induces an isomorphism $c/W \to V/\!\!/G$. This generalises the Chevalley isomorphism $\mathfrak{h}/W \to \mathfrak{g}/\!\!/G$ for the action of the Weyl group of reductive Lie group on the Cartan subalgebra $\mathfrak{h}$.

Further examples besides the already mentioned adjoint representations are the following:
1. SL_2 acts on the space of binary quartics $S^4\mathbb{C}^2 = \langle x^4, \ldots, y^4\rangle$. The subspace $c = \langle x^4 + y^4, x^2y^2\rangle$ is a Cartan subspace with Weyl group S_3.
2. SL_3 acts on the space of ternary cubics $S^3\mathbb{C}^3 = \langle x^3, x^2y \ldots, z^3\rangle$. The Hesse pencil $c = \langle x^3 + y^3 + z^3, xyz\rangle$ is a Cartan subspace with Weyl group the binary tetrahedral group T^*.
3. The standard representations $(G_2, \mathbb{C}^7)$ and $(F_4, \mathbb{C}^{26})$ are polar representations with Weyl group $\mathbb{Z}/2$.

4. A large class of examples of polar representations is provided by Vinbergs's θ-representations: Let θ denote an automorphism of order m of a reductive group G, and let $\mathfrak{g} = \bigoplus_{i \bmod m} \mathfrak{g}_i$ be an eigenspace decomposition for the action of θ on the Lie algebra of G. Then $\mathfrak{g}_0$ is a reductive Lie algebra, and each $\mathfrak{g}_i$ is $\mathfrak{g}_0$-representation. Let $G_0 \subset G$ be the connected subgroup corresponding to $\mathfrak{g}_0$. Then $(G_0, \mathfrak{g}_1)$ is a polar representation.

Given a polar representation (G, V) we are interested in the properties of the symplectic reduction $V \oplus V^* /\!/\!/ G$.

A holomorphic 2-form on a smooth complex variety is said to be symplectic if it is closed and non-degenerated, i.e. $d\omega = 0$ and $\omega : T \xrightarrow{\cong} \Omega$. Following Beauville, a complex variety X with a symplectic form ω on its regular part is said to be symplectic, if X is normal and if for any (all) resolution(s) $\pi : X' \to X$ there exists a 2-form ω' on X' with $\omega'|_{\pi^{-1}(X_{\mathrm{reg}})} = \pi^*\omega$. Finally, π is called a symplectic resolution if the extended form ω' is again symplectic. This condition is equivalent to π being crepant. Though there are many examples of symplectic singularities, comparatively few of them admit symplectic resolutions. For finite subgroups $\Gamma \subset \mathrm{Sp}(V, \omega)$ a necessary condition due to Verbitsky, that is far from being sufficient, states that Γ must be generated by symplectic reflections, i.e. elements γ with $\mathrm{codim}(V^\gamma) = 2$. Any finite group $\Gamma \subset \mathrm{GL}(V)$ gives rise to a symplectic action of Γ on the canonical symplectic double $V \oplus V^*$, and for such actions Verbitsky's criterion reduces to Kaledin's criterion that V/Γ must be smooth, that is that Γ be a reflection group. (This is one of the reasons for our interest in polar representations.) Symplectic reflection groups have been classified by A. Cohen, and his list has been skimmed for actions admitting symplectic resolutions by Ginzburg, Kaledin, Bellamy and Schedler. The results are not yet complete but indicate that only very, very few such groups exist.

The symplectic analogue of a finite group quotient for reductive groups $G \subset \mathrm{Sp}(V)$ is the Marsden-Weinstein- or symplectic reduction: The linear action of G on V is always Hamiltonian, i.e. there is G-equivariant morphism $\mu : V \to \mathfrak{g}^*$ such that $d_x\mu(\xi)(A) = \omega_x(\xi, A_x)$ for every tangent vector $\xi \in T_xV$ and every $A \in \mathfrak{g}$, in fact it is given by the quadratic map $\mu_x(A) = \frac{1}{2}\omega(x, Ax)$. The symplectic reduction is defined as $V /\!/\!/ G := \mu^{-1}(0) /\!/ G$. Despite its name it is by no means true that $V /\!/\!/ G$ is a symplectic singularity in the sense of Beauville's definition above, though it always carries a natural Poisson structure.

Starting with a polar representation (G, V), say with Cartan space (W, c), we show that the dual representation V^* is again polar, that c^* naturally embeds into V^* as a Cartan subspace with Weyl group W, and that there is natural morphism $c \oplus c^*/W \to V \oplus V^* /\!/\!/ G$ of Poisson schemes. We conjecture that this map is an isomorphism under the additional hypothesis that (G, V) be visible, which means that fibres of the quotient map $V \to V /\!/ G$ contain only finitely many orbits, if one ignores nilpotent functions in the target, i.e. passes to the underlying reduced space.

This conjecture is known to hold in many cases, among them adjoint representations (Richardson, Hunziger, Joseph), θ-representations of order $m = 2$ (Panyushev, Tevelev), and various special examples (Terpereau). We prove it [1] for visible stable locally free polar representations, where a representation is stable, if the orbits of regular semi-simple elements have maximal dimension among all orbits, and is locally free if the stabilisers of generic elements are finite.

In cases when the isomorphism holds, the question of existence of symplectic resolutions of symplectic reductions is of course reduced to the same problem for finite group quotients. However, in general even the question which symplectic reductions are symplectic singularities in the sense of Beauville, is widely open. For certain types of symplectic reductions arising from quiver varieties normality has been shown by Crawley-Boevey [2], and local factoriality by Kaledin, Lehn and Sorger [5].

References

[1] M. Bulois, C. Lehn, M. Lehn, R. Terpereau: Towards a symplectic version of the Chevalley restriction theorem. arXiv:1609.04573.

[2] W. Crawley-Boevey. Normality of Marsden-Weinstein reductions for representations of quivers. Math. Ann., 325 (2003), 55–79.

[3] J. Dadok and V. Kac: Polar representations. Journal of Algebra, 92:504–524, 1985.

[4] A. Joseph. On a Harish-Chandra homomorphism. Comptes Rendus de l'Académie des Sciences - Series I - Mathematics, 324, Issue 7:759–764, 1997.

[5] D. Kaledin, M. Lehn, Ch. Sorger: Singular symplectic moduli spaces. Invent. Math. 164 (2006), no. 3, 591–614.

[6] D. Panyushev. The Jacobian modules of a representation of a Lie algebra and geometry of commuting varieties. Compos. Math., 94, Issue 2:181–199, 1994.

[7] D. Panyushev. On the irreducibility of commuting varieties associated to involutions of simple Lie algebras. Funct. Anal. Appl., 38, no. 1, 38–44, 2004.

[8] R. Richardson. Commuting varieties of semisimple Lie algebras and algebraic groups. Compos. Math., 38, Issue 3:311–327, 1979.

[9] E. A. Tevelev. On the Chevalley restriction theorem. J. Lie Theory, 10, no.2:323–330, 2000.

[10] E. Vinberg: The Weyl group of a graded Lie algebra. Math. USSR Izvestija, 10:463–495, No.3, 1976.

Motivic nearby cycle of the sum of two regular functions

Hồng Đức Nguyễn

(joint work with Lê, Quý Thường)

We first introduce a new notion of $\circledast$-product of two integrable series with coefficients in distinct Grothendieck rings of algebraic varieties ([5]), preserving the integrability and commuting with the limit map of rational series. Furthermore we show that the $\circledast$-product is associative in the class of motivic multiple zeta functions, which are proved to be integrable and are defined as follows. For a family of regular functions $f_i\colon X_i \to \mathbb{A}^1_k$ with $i = 1, \ldots, r$, we define

$$\zeta_{f_1,\ldots,f_r}(T_1,\ldots,T_r) = \sum\, [\mathcal{D}_{n_1,\ldots,n_r}]\mathbb{L}^{-(\sum \dim X_i)(\sum n_i)}\; T_1^{n_1}\cdots T_r^{n_r},$$

where the sum is taken over the set of elements $(n_1, \dots, n_r)$ in $\mathbb{N}^r_{>0}$ such that $1 \le n_1 < \cdots < n_r$, and $\mathcal{D}_{n_1,\dots,n_r}$ is the set of arcs $\varphi \in \mathcal{L}_{\sum n_i}(X_1 \times \ldots \times X_r)$ such that ord $f_1(\varphi_1) = n_1$ and ord $f_i(\varphi_i) > n_i$ for all $i \ge 2$. This new notation still covers classical motivic zeta functions $Z_{f_1}(T_1)$ defined by Denef-Loeser [1]. We prove a version of the Euler reflexion formula for motivic zeta functions stating that the following identity

$$\zeta_{f_1}(T_1) \circledast \zeta_{f_2}(T_2) = \zeta_{f_1,f_2}(T_1,T_2) + \zeta_{f_2,f_1}(T_2,T_1) + \iota^*\zeta_{f_1\oplus f_2}(T_1T_2)$$

holds in $\mathcal{M}^{\mathbb{G}}_{X_0(f_1)\times X_0(f_2)\times\mathbb{G}}[[T,U]]$, where $f_1 \oplus f_2$ is a regular function on $X_1 \times X_2$ defined as $(x_1, x_2) \mapsto f_1(x_1) + f_2(x_2)$, $X_0(f_1)$ and $X_0(f_1)$ are the zero locus of f_1, f_2, respectively, and ι is the embedding of $X_0(f_1) \times X_0(f_2)$ in the zero locus of $f_1 \oplus f_2$. As an application, taking the limit for the motivic Euler reflexion formula we recover the well known motivic Thom-Sebastiani theorem ([2]).

We also study the motivic nearby cycle of the sum of two regular functions f, g having variables in common, that is, f, g are regular functions on the same variety X. Let $f|_{X_0(g)}$ and $g|_{X_0(f)}$ be the restriction of f (resp. of g) to the zero locus $X_0(g)$ of g (resp. $X_0(f)$ of f). We show that the identity

$$\iota^*\mathcal{S}_{f+g} = \Psi(\mathcal{S}_{f,g}) + \mathcal{S}_{f|_{X_0(g)}} + \mathcal{S}_{g|_{X_0(f)}}$$

holds in $\mathcal{M}^{\mathbb{G}}_{X_0\times\mathbb{G}}$, where $\mathcal{S}_{f,g}$ is the motivic nearby cycle of the map to $\mathbb{A}^2_k$ defined as $x \mapsto (f(x), g(x))$. The convolution map Ψ from $\mathcal{M}^{\mathbb{G}}_{X_0\times\mathbb{G}^2}$ to $\mathcal{M}^{\mathbb{G}}_{X_0\times\mathbb{G}}$, was introduced by Guibert-Loeser-Merle in [3] and generalized in [4] to study the motivic nearby cycles of composition functions.

References

[1] J. Denef and F. Loeser, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), 505–537.

[2] J. Denef and F. Loeser, *Motivic exponential integrals and a motivic Thom-Sebastiani theorem*, Duke Math. J. **99** (1999), no. 2, 285–309.

[3] G. Guibert, F. Loeser and M. Merle, *Iterated vanishing cycles, convolution, and a motivic analogue of the conjecture of Steenbrink*, Duke Math. J. **132** (2006), no. 3, 409–457.

[4] G. Guibert, F. Loeser and M. Merle, *Nearby cycles and composition with a non-degenerate polynomial*, Int. Math. Res. Not. **31** (2005), 1873–1888.

[5] Q.T.Le, H.D.Nguyen, *Euler reflexion formulas for motivic multiple zeta functions*, J. Algebraic Geom.

Computing the Milnor fiber monodromy

Alexandru Dimca

(joint work with Morihiko Saito, Gabriel Sticlaru)

Let $C : f = 0$ be a reduced plane curve of degree $d \ge 3$ in the complex projective plane $\mathbb{P}^2$, defined by a homogeneous polynomial $f \in S = \mathbb{C}[x, y, z]$. Consider the corresponding complement $U = \mathbb{P}^2 \setminus C$, and the global Milnor fiber F defined by $f(x, y, z) = 1$ in $\mathbb{C}^3$ with monodromy action $h : F \to F$, $h(x) = \exp(2\pi i/d) \cdot (x, y, z)$. To determine the eigenvalues of the monodromy operators

$$h^m : H^m(F, \mathbb{C}) \to H^m(F, \mathbb{C}) \tag{1}$$

for $m = 1, 2$ starting from C or f is a rather difficult problem, going back to O. Zariski and attracting an extensive literature, see for instance [1], [2], [3], [10], [11], [12], [13], [16], [6]. When the curve $C : f = 0$ is either free or nearly free, we have presented in [8] an efficient algorithm for listing the eigenvalues of the monodromy operator h^1, which in many cases determines completely the corresponding Alexander polynomial $\Delta_C(t)$.

In this talk we explained an approach working in the general case. This time the results of our computation give not only the dimensions of the eigenspaces $H^m(F, \mathbb{C})_\lambda$ of the Milnor monodromy, but also the dimensions of the graded pieces $Gr_P^p H^m(F, \mathbb{C})_\lambda$, where P denotes the pole order filtration on $H^m(F, \mathbb{C})$, see section 2 below for the definition. More precisely, the algorithm described here gives the following.

(1) the dimensions of the eigenspaces $H^m(F, \mathbb{C})_\lambda$ for $m = 1, 2$, and for any reduced curve $C : f = 0$.
(2) the dimensions of the graded pieces $Gr_P^p H^1(F, \mathbb{C})_\lambda$, for any reduced curve $C : f = 0$. Moreover, we conjecture that the P^p filtration coincides to the Hodge filtration F^p on $H^1(F, \mathbb{C})$.
(3) the dimensions of the graded pieces $Gr_P^p H^2(F, \mathbb{C})_\lambda$, for a reduced curve $C : f = 0$ having only weighted homogeneous singularities. To achieve this efficiently one has to use a recent result by M. Saito. A less efficient approach can be based on the weaker result, obtained in [7].
(4) the dimensions of the graded pieces $Gr_P^p H^2(F, \mathbb{C})_\lambda$, for any reduced curve $C : f = 0$ under the assumption that two basic facts, stated as Conjectures, hold. These conjectures seem to hold in all the examples we have computed so far.

The new information on the pole order filtration P can be applied to describe the set of roots of $b_f(-s)$, where $b_f(s)$ is the Bernstein-Sato polynomial of f, see for details [14], [15]. In fact, using [14, Theorem 2], this comes down to checking whether $Gr_P^p H^2(F, \mathbb{C})_\lambda \neq 0$,.

For full statements of results and proofs see [9]. The computations were made using the computer algebra system Singular [4]. The corresponding codes are available on request.

References

[1] E. Artal Bartolo, Sur les couples de Zariski. J. Algebraic Geom., 3(1994), 223–247.
[2] N. Budur, A. Dimca and M. Saito, First Milnor cohomology of hyperplane arrangements, Contemporary Mathematics 538 (2011), 279-292.
[3] D. C. Cohen, A. I. Suciu, On Milnor fibrations of arrangements, J. London Math. Soc. 51 (1995), no. 2, 105–119.
[4] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann. Singular 4-0-1 — A computer algebra system for polynomial computations, available at http://www.singular.uni-kl.de (2014).
[5] A. Dimca, On the Milnor fibrations of weighted homogeneous polynomials, Compositio Math. 76 (1990), 19-47.

[6] A. Dimca, *Singularities and topology of hypersurfaces*, Universitext, Springer-Verlag, New York, 1992.
[7] A. Dimca, M. Saito, Koszul complexes and spectra of projective hypersurfaces with isolated singularities, arXiv:1212.1081.
[8] A. Dimca, G. Sticlaru, A computational approach to Milnor fiber cohomology, arXiv: 1602.03496, to appear in Forum Mathematicum
[9] A. Dimca, G. Sticlaru, Computing the monodromy and pole order filtration on Milnor fiber cohomology of plane curves, arXiv: 1609.06818.
[10] H. Esnault, Fibre de Milnor d'un cône sur une courbe plane singulière, Invent. Math. 68(1982), 477–496.
[11] A. Libgober, Alexander polynomial of plane algebraic curves and cyclic multiple planes, Duke Math. J. 49 (1982), no. 4, 833–851.
[12] A. Libgober, Development of the theory of Alexander invariants in algebraic geometry. Topology of algebraic varieties and singularities, 3–17, Contemp. Math., 538, Amer. Math. Soc., Providence, RI, 2011.
[13] M. Oka, A survey on Alexander polynomials of plane curves. Singularités Franco-Japonaises, 209–232, Sémin. Congr., 10, Soc. Math. France, Paris, 2005.
[14] M. Saito, Multiplier ideals, b-function, and spectrum of a hypersurface singularity, Compositio Math.143 (2007), 1050–1068.
[15] M. Saito, Hilbert series of graded Milnor algebras and roots of Bernstein-Sato polynomials, arXiv: 1509.06288.
[16] A. Suciu, Hyperplane arrangements and Milnor fibrations, Annales de la Faculté des Sciences de Toulouse 23 (2014), no. 2, 417-481.

The geometric genus and Seiberg–Witten invariants of Newton nondegenerate surface singularities

Baldur Sigurðsson
(joint work with András Némethi)

In this talk, we present results obtained by the autor during his PhD program at Central European University in collaboration with—and under the supervision of—Némethi András. The thesis can be found on the arxiv [14]. We will assume throughout that $(X,0) \subset (\mathbb{C}^3,0)$ is a Newton nondegenerate hypersurface singularity with a rational homology sphere link.

Our first result, published in [11], is a topological identification of the geometric genus using computation sequences. This can be formulated in terms of path lattice cohomology [7] which has a strong relationship with our identification of the Seiberg–Witten invariant, but we will not discuss this here.

The second result identifies the Seiberg–Witten invariant, suitably normalized, with the same topological invariant. In particular, we prove the Seiberg–Witten invariant conjecture of Némethi and Nicolaescu in this case.

1. History

The problem is motivated by several results and conjectures. The following conjecture from 1990, the Casson invariant conjecture, has been proved in several cases, but it is still open.

Conjecture 1.1 (Neumann–Wahl, [12]). *Let $(X,0)$ be an isolated complete intersection singularity with link M and assume that $H_1(M,\mathbb{Z})=0$. Then*

$$\lambda(M)=\frac{1}{8}\sigma(F)$$

where λ is the Casson invariant and $\sigma(F)$ is the signature of the Milnor fiber.

Under the weaker assumption that the link M satisfy $H_1(M,\mathbb{Q})=0$, the Casson invariant can be extended to the Casson–Walker invariant [15]. Furthermore, the signature $\sigma(F)$ can be calculated in terms of a resolution graph and the geometric genus p_g. The conjecture, however, is not generalized by directly replacing the Casson invariant with the Casson–Walker invariant, but the normalized Seiberg–Witten invariant associated with the canonical spin$^{\mathrm{c}}$ structure of the link.

Conjecture 1.2 (Némethi–Nicolaescu, [10]). *Let $(X,0)$ be a surface singularity with link M satisfying $H_1(M,\mathbb{Q})=0$. Then*

$$\mathbf{sw}^0{}_M(\sigma_{\mathrm{can}})-\frac{K^2+s}{8}\geq p_g$$

where K and s are the canonical cycle and the number of irreducible component of the exceptional divisor of a resolution $\tilde{X}\to X$, $\mathbf{sw}^0{}_M(\sigma_{\mathrm{can}})$ is the Seiberg–Witten invariant of the canonical spin$^{\mathrm{c}}$ *structure on M and $p_g=h^1(\tilde{X},\mathcal{O}_{\tilde{X}})$ is the geometric genus. Under suitable analytic conditions on $(X,0)$, equality holds.*

The conjecture has been proved in several instances. Nonetheless, superisolated singularities have been found to provide conterexamples [5]. It is an intriguing question to ask, when the conjecture holds, and why does it fail in this case? We remark that in [11], we use recent developments in low dimensional topology [1] to prove the same topological identification of p_g for Newton nondegenerate singularities and for superisolated singularities. In the former case, we later prove this to coincide with the prediction of conjecture 1.2, whereas in the latter case it is false.

2. Computation sequences

Let $(X,0)$ be a normal surface singularity with link M, assume that M is a rational homology sphere, that is $H_1(M,\mathbb{Q})=0$. Fix a good resolution $(\tilde{X},E)\to(X,0)$ with exceptional divisor $E=\cup_{v\in\mathcal{V}}E_v$. For an effective divisor Z on $\tilde{X}$, supported on E, we define

$$h_Z=\dim_{\mathbb{C}}\frac{H^0(\tilde{X},\mathcal{O}_{\tilde{X}})}{H^0(\tilde{X},\mathcal{O}_{\tilde{X}}(-Z))}.$$

A computation sequence for Z is a sequence $(Z_i)_{i=0}^k$ of divisors on $\tilde{X}$ satisfying $Z_0=0$, $Z_k=Z$, and for each $0\leq i<k$ there is a $v(i)\in\mathcal{V}$ so that $Z_{i+1}=Z_i+E_{v(i)}$. By our assumptions, E_v is a rational curve for all v, and so we have $h^0(E_{v(i)},\mathcal{O}_{E_{v(i)}}(-Z_i))=\max\{0,(-Z_i,E_{v(i)})+1\}$ for all i. The short

exact sequence $0 \to \mathcal{O}_{\tilde{X}}(-Z_{i+1}) \to \mathcal{O}_{\tilde{X}}(-Z_i) \to \mathcal{O}_{E_{v(i)}}(-Z_i) \to 0$ therefore provides the bound

$$h_Z = \sum_{i=0}^{k-1} h_{Z_{i+1}} - h_{Z_i} \le \sum_{i=0}^{k-1} \max\{0, (-Z_i, E_{v(i)}) + 1\}. \tag{1}$$

In the case when $(X, 0)$ is Gorenstein, we have $p_g = h_{Z_K}$, where Z_K is the anti-canonical cycle, that is, the unique divisor supported on E satisfying $(Z_K, E_v) = E_v^2 + 2$ for all $v \in \mathcal{V}$.

Definition 2.1. With the notation as above, assume further that $(X, 0)$ is Gorenstein. The *minimal path lattice cohomology* is the smallest number realized by any computation sequence for Z_K on the right hand side of (1).

3. Statement for Newton nondegenerate singularities

Theorem 3.1. *Assume that $(X, 0) \subset (\mathbb{C}^3, 0)$ is a Newton nondegenerate isolated hypersurface singularity with a rational homology sphere link M. Then there exists a computation sequence $(Z_i)_{i=0}^k$ satisfying*

$$p_g = \sum_{i=0}^{k-1} \max\{0, (-Z_i, E_{v(i)}) + 1\} = \mathbf{sw}^0{}_M(\sigma_{\mathrm{can}}) - \frac{K^2 + s}{8}. \tag{2}$$

Furthermore, this sequence can be calculated directly from a minimal good resolution graph of X.

In the proof, we show that $h_{Z_{i+1}} - h_{Z_i} = \max\{0, (-Z_i, E_{v(i)}) + 1\}$ for all i. The numbers h_Z for any Z supported on E are coefficients of the Hilbert series $H(t) = \sum_l h_l t^l$. A "topological candidate" $Q(t) = \sum_l q_l t^l$ for this highly analytic invariant has been studied by Némethi and László [8, 9, 4]. In particular, q_{Z_K} equals the left hand side of (2). To prove the second equality in (2) we prove that $q_{Z_{i+1}} - q_{Z_i} = \max\{0, (-Z_i, E_{v(i)}) + 1\}$ for all i.

4. Newton diagrams and Oka's algorithm

Assume that $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is a germ defining a hypersurface singularity $(\mathbb{C}^3, 0)$. The concept of Newton diagram and nondegeneracy are described in e.g. [3, 2] We denote by $\Gamma(f)$ the Newton diagram of f and write $\Gamma(f) = \cup_{n \in \mathcal{N}} F_n$, where (F_n) is the family of two dimensional faces of $\Gamma(f)$. In [13], Oka describes a combinatorial algorithm which gives the dual graph G of a resolution in terms of the Newton diagram. In particular, there is a correspondence $n \leftrightarrow E_n$ between the set $\mathcal{N}$ indexing the two dimensional faces of $\Gamma(f)$ and a subset of $\mathcal{V}$. In fact, the dual graph G can be seen as dual to the two-skeleton of the Newton diagram $\Gamma(f)$, taken as a planar graph drawn on the boundary of the Newton polyhedron.

5. Plan of the proof

It is known [6] that p_g equals the number of integral points with positive coordinates under the Newton diagram. To prove theorem 3.1, we use the computation sequence to count these points, in particular, we reprove this statement.

Let P be the set of integral points with nonnegative coordinates, under the shifted diagram $\Gamma(f) - (1,1,1)$. From the result stated above, it is clear that we have $p_g = |P|$. Any face $F_n \subset \Gamma(f)$ has a unique primitive linear support function $\ell_n : \mathbb{R}^3 \to \mathbb{R}$, taking nonnegative values on $\Gamma(f)$. Using a well known formula for Z_K on Oka's resolution graph G [13, 6], we find

$$P = \left\{ p \in \mathbb{Z}^3_{\geq 0} \,\middle|\, \exists n \in \mathcal{N} : \ell_n(p) \leq m_n(Z_K) \right\}$$

where $m_n(Z_K)$ is the multiplicity of Z_K along E_n. Given any computation sequence we can define a partitioning $(P_i)_{i=0}^{k-1}$ as follows (we can in fact assume that $v(i) \in \mathcal{N}$ for all i, if this is not so, the corresponding contribution on the right hand side of (1) can be proved to be 0)

$$P_i = \left\{ p \in \mathbb{Z}^3_{\geq 0} \,\middle|\, \forall n \in \mathcal{N} : \ell_n(p) \geq m_n(Z_i), \quad \ell_{v(i)}(p) = m_n(Z_i) \right\}.$$

The computation sequence (Z_i) is now defined in such a way that for each i, the sets P_i can be realized as the set of integral points in a dilated integral polygon in an affine hyperplane in $\mathbb{R}^3$. An elementary, but nontrivial, formula for counting such points is related with the intersection number $(-Z_i, E_{v(i)})$ to prove that $|P_i| = \max\{0, (-Z_i, E_{v(i)}) + 1\}$. Finally, the monomials $x^p, p \in P_i$ are shown to induce a linearly independent family in the space

$$\frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i))}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1}))}$$

whose dimension is $h_{Z_{i+1}} - h_{Z_i}$. These results show that in this case we have equality in (1) for all i, proving the formula for the geometric genus.

References

[1] Borodzik, Maciej; Livingston, Charles. Heegaard Floer homology and rational cuspidal curves. *Forum Math. Sigma*, 2:23, 2014.

[2] Braun, Gábor; Némethi, András. Invariants of Newton non-degenerate surface singularities. *Compos. Math.*, 143(4):1003–1036, 2007.

[3] Kouchnirenko, A. G. Polyèdres de Newton et nombres de Milnor. *Invent. Math.*, 32:1–31, 1976.

[4] László, Tamás. *Lattice cohomology and Seiberg–Witten invariants of normal surface singularities.* Ph.D. thesis, Central European University, 2013. ArXiv:1310.3682.

[5] Luengo-Velasco, Ignacio; Melle-Hernández, Alejandro; Némethi, András. Links and analytic invariants of superisolated singularities. *J. Algebr. Geom.*, 14(3):543–565, 2005.

[6] Merle, Michel; Teissier, Bernard. Conditions d'adjonction. (D'après Du Val). In *Séminaire sur les singularités des surfaces*, volume 777 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin Heiderberg New York, 1980.

[7] Némethi, András. Lattice cohomology of normal surface singularities. *Publ. Res. Inst. Math. Sci.*, 44(2):507–543, 2008.

[8] Némethi, András. Poincaré series associated with surface singularities. In *Singularities I*, volume 474 of *Contemp. Math.*, 271–297. Amer. Math. Soc., Providence, RI, 2008.

[9] Némethi, András. The cohomology of line bundles of splice-quotient singularities. *Adv. Math.*, 229(4):2503–2524, 2012.
[10] Némethi, András; Nicolaescu, Liviu I. Seiberg–Witten invariants and surface singularities. *Geom. Topol.*, 6:269–328, 2002.
[11] Némethi, András; Sigurðsson, Baldur. The geometric genus of hypersurface singularities. *J. Eur. Math. Soc. (JEMS)*, 18(4):825–851, 2016.
[12] Neumann, Walter D.; Wahl, Jonathan. Casson invariant of links of singularities. *Comment. Math. Helv.*, 65(1):58–78, 1990.
[13] Oka, Mutsuo. On the resolution of the hypersurface singularities. In *Complex analytic singularities*, volume 8 of *Adv. Stud. Pure Math.*, 405–436. North-Holland, Amsterdam, 1987.
[14] Sigurðsson, Baldur. *The geometric genus and Seiberg-Witten invariant of Newton nondegenerate surface singularities.* Ph.D. thesis, Central European University, 2015. ArXiv:1601.02572.
[15] Walker, Kevin. *An extension of Casson's invariant.* Princeton, NJ: Princeton University Press, 1992.

Noncommutative desingularizations of discriminants of reflection groups

Eleonore Faber
(joint work with Ragnar-Olaf Buchweitz, Colin Ingalls)

Let G be a finite subgroup of $\mathrm{GL}(n,K)$ for a field K, whose characteristic does not divide the order of G. For this talk we assume that $K=\mathbb{C}$. The group G acts linearly on a vectorspace $V \cong K^n$, and thus on the ring $S=\mathrm{Sym}_K(V) \cong K[x_1,\ldots,x_n]$. When G is generated by reflections, then the discriminant Δ of the group action of G on S is a hypersurface with a singular locus of codimension 1, in particular, Δ is a so-called free divisor.
In this talk we give a natural construction of a noncommutative resolution of singularities of the coordinate ring of Δ as a quotient of the skew group ring $A=S*G$. The study of such discriminants of finite reflection groups is in various ways complementary to J. McKay's observations about the correspondence between the irreducible representations of finite groups $\Gamma \subseteq \mathrm{SL}(2,K)$ not containing any (pseudo-)reflections and irreducible components of the exceptional divisor of the minimal resolution of singularities of the quotients V/Γ (or, the algebraic version: indecomposable maximal Cohen–Macaulay (MCM)-modules over the invariant ring S^Γ of Γ).

Noncommutative resolutions of a commutative ring R are certain noncommutative R-algebras of finite global dimension which can in turn be viewed as a potential analogue of a resolution of singularities of $\mathrm{Spec}(R)$. A particular highlight is van den Bergh's introduction of noncommutative crepant resolutions (=NCCRs) to interpret Bridgeland's solution to the conjecture by Bondal and Orlov on the derived invariance of flops in 2004. A *noncommutative resolution* of singularities (NCR) of a commutative noetherian ring R (or the scheme $\mathrm{Spec}(R)$) is defined to be an associative R-algebra $A=\mathrm{End}_R M$, where M is a finitely generated R-module of full support and A has finite global dimension. This notion was introduced 2015 by Dao–Iyama–Takahashi–Vial. For A to be *crepant*, that is, a NCCR, one needs additionally that M is torsion-free and A is a non-singular order.

The first examples of noncommutative resolutions of singularities were described by the McKay correspondence. This began in 1979, when J. McKay observed a connection between the representations of finite subgroups Γ of $\mathrm{SL}(2, K)$ and resolutions of the quotient singularities $V/\Gamma = \mathrm{Spec}(S^\Gamma)$, where $S = \mathrm{Sym}_K(V)$ as above. Here a NCR is provided by the skew group ring $A = S * \Gamma \cong \mathrm{End}_{S^\Gamma}(S)$, as noted by M. Auslander [1].

We are interested in the case that G is a complex reflection group, i.e., a finite subgroup of $\mathrm{GL}(n, K) = \mathrm{GL}(V)$ generated by pseudo-reflections. By the theorem of Chevalley–Shephard–Todd the invariant ring S^G under a finite group $G \subseteq \mathrm{GL}(V)$ is isomorphic to a polynomial ring if and only if G is generated by pseudo-reflections. In this case $T := S^G$ is of the form $K[f_1, \dots, f_n]$ where the f_i are polynomials in S. The arrangement of reflecting hyperplanes in V is given by the Jacobian $J = \det(\frac{\partial f_i}{\partial x_j})$. Note that J factors as a product of linear forms whose multiplicities are equal to the order of the corresponding reflection minus 1. Thus, in the case when G is generated by order 2 reflections, J equals the reduced polynomial defining the hyperplane arrangement, which is usually denoted by z. The discriminant Δ is defined by $zJ = \Delta^2$ (or in the real reflection case: $z^2 = \Delta$) and its coordinate ring is $T/(\Delta)$. Geometrically, J defines the reflection arrangement in V and Δ is its image under the natural projection $\pi : V \to V/G$. In the following we assume that G is a real reflection group, that is, generated by order 2 reflections. We construct a noncommutative resolution of $T/(\Delta)$ starting from the skew group ring $A = S * G$. Therefore we also consider the group $\Gamma = G \cap SL(V)$, with invariant ring $R = S^\Gamma$. There is an exact sequence of groups $1 \to \Gamma \to G \to H \to 1$, with quotient $H \cong \mu_2 = \langle \sigma \rangle$. Using ideas from J. Bilodeau [3] and H. Knörrer [8] we can show:

Theorem 1. *With notation introduced in the paragraph above, there is an isomorphism of rings*

$$\mathrm{End}_{R*H}(S * \Gamma) \cong S * G.$$

For convenience set $B = R * H$. Now interpreting B as the path algebra of a quiver with idempotents $e_\pm = \frac{1}{2}(1 \pm \sigma)$ and using the functor

$$\mathrm{mod}(B/Be_-B) \xleftarrow{i^*(-) = - \otimes_B B/Be_-B} \mathrm{mod}(B)$$

from the standard recollement we obtain the following result:

Theorem 2. *The functor i^* induces an equivalence of categories*

$$\mathrm{CM}(B)/\langle e_-B \rangle \simeq \mathrm{CM}(T/(\Delta)),$$

where $\langle e_-B \rangle$ denotes the ideal in the category $\mathrm{CM}(B)$*, the category of MCM-modules over B, generated by the object e_-B.*

Putting all of this together and moreover using results of Stanley about the structure of the ring R as T-module [9], and Auslander–Platzeck–Todorov [2] about global dimension of quotients, we can prove the following

Theorem 3. *Denote by* $e = \frac{1}{|G|}\sum_{g\in G} g \in A$ *the idempotent corresponding to the trivial representation of* G*. Then*

$$\bar{A} := A/AeA \cong \mathrm{End}_{T/(\Delta)}(S/(J)),$$

and $\mathrm{gl.dim}\,\bar{A} < \infty$*. Thus* $\bar{A}$ *yields a NCR of the free divisor* T/Δ*. In particular, if* $\dim T = 1$*, that is,* $T/(\Delta)$ *is an ADE-curve singularity, we obtain that* $S/(J)$ *is a generator of the category of MCM-modules over the discriminant, that is,* $\mathrm{add}(S/(J)) = \mathrm{CM}(T/(\Delta))$*.*
Moreover, the indecomposable projective modules over $\bar{A}$ *are in bijection with the non-trivial representations of* G *and also with certain MCM-modules over the discriminant, namely the* $T/(\Delta)$*-direct summands of* $S/(J)$*.*

Thus we obtain a McKay correspondence for reflection groups G. However, we are still missing a conceptual explanation for the $T/(\Delta)$-direct summands of $S/(J)$: if $T/(\Delta)$ is 1-dimensional, then $S/(J)$ provides a compact description of a representation generator of the discriminant (this was not known before our investigations). But in higher dimensions $T/(\Delta)$ is not of finite MCM type and we can only determine the decomposition of $S/(J)$ in few examples (the normal crossing divisor, as studied in Dao–Faber–Ingalls [5] or the swallowtail, that is, the discriminant of S_4, where we can describe the $T/(\Delta)$-direct summands using Hovinen's classification of graded rank 1 MCM-modules [7]). For the geometric interpretation of the direct summands of $S/(J)$ we wish to obtain a similar correspondence as in [6], or even to realize geometric resolutions as moduli spaces of isomorphism classes of representations of certain algebras as in [4].
On the other hand, we can describe the structure of $\bar{A}$ as an $S/(J)$-module: $\bar{A}$ is isomorphic to the cokernel of the map φ given by left multiplication on G, that is, the matrix of φ corresponds to the multiplication table of the group G. By work of Frobenius and Dedekind in the 1890's it is well-known how this matrix decomposes into blocks. In the near future we plan to flesh out this astounding discovery.
Moreover, looking at the quiver of $\bar{A}$, the exact form of the generating (necessarily quadratic) relations remains mysterious and will be the subject of further work.

References

[1] M. Auslander. Rational singularities and almost split sequences. *Transactions of the AMS*, 293(2):511–531, 1986.

[2] M. Auslander, M. I. Platzeck, and G. Todorov. Homological theory of idempotent ideals. *Trans. Amer. Math. Soc.*, 332(2):667–692, 1992.

[3] J. Bilodeau. Auslander algebras and simple plane curve singularities. In *Representations of algebras and related topics*, volume 45 of *Fields Inst. Commun.*, pages 99–107. Amer. Math. Soc., Providence, RI, 2005.

[4] W. Crawley-Boevey. On the exceptional fibres of Kleinian singularities. *Amer. J. Math.*, 122(5):1027–1037, 2000.

[5] H. Dao, E. Faber, and C. Ingalls. Noncommutative (crepant) desingularizations and the global spectrum of commutative rings. *Algebr. Represent. Theory*, 18(3):633–664, 2015.

[6] G. Gonzalez-Sprinberg and J.-L. Verdier. Construction géométrique de la correspondance de McKay. *Ann. Sci. École Norm. Sup. (4)*, 16(3):409–449 (1984), 1983.

[7] B. Hovinen. Matrix factorizations of the classical determinant. Phd thesis, University of Toronto, 2009.
[8] H. Knörrer. The Cohen–Macaulay modules over simple hypersurface singularities. 1984. Max-Planck Institut für Mathematik und Sonderforschungsbereich theoretische Physik MPI/SFB 84-51.
[9] R. P. Stanley. Relative invariants of finite groups generated by pseudoreflections. *J. Algebra*, 49(1):134–148, 1977.

Sheaves on flag Hilbert schemes and the HOMFLY-PT homology

Jacob Rasmussen
(joint work with Eugene Gorsky and Andrei Negut)

In my talk, I reported on some developments in a conjectural picture relating invariants coming from knot theory with algebraic geometry. This story began with a conjecture of Oblomkov and Shende [10] (since proved by Maulik [8]) relating the punctual Hilbert scheme of a plane curve singularity with the HOMFLY-PT polynomial of its link. To be more precise, let $(X, 0)$ be the germ of a unibranch plane curve singularity with Milnor number μ, and let $K \subset S^3$ be its link. Finally, let $X^{[n]}$ denote the nth Hilbert scheme of X. In its simplest form, the Oblomkov-Shende conjecture states:

Theorem 1 ([8]). $\displaystyle\sum_{i=1}^{\infty} q^{2n}\chi(X^{[n]}) = [(q/a)^{\mu-1}P(K)]_{a=0}.$

where the HOMFLY-PT polynomial $P(K)$ is normalized so that P of the unknot is $(a - a^{-1})/(q - q^{-1})$.

The HOMFLY-PT polynomial has been categorified by Khovanov and Rozansky [7]. The conjecture can naturally be extended to describe this invariant by replacing the Euler characteristic $\chi(X^n)$ with the Poincaré polynomial $\mathcal{P}(X^{[n]})$:

Conjecture 2 ([9]). $\displaystyle\sum_{i=1}^{\infty} q^{2n}\mathcal{P}(X^{[n]}) = [(q/a)^{\mu-1}\mathcal{P}(K)]_{a=0}.$

where $\mathcal{P}(K)$ is the Poincaré polynomial of the unreduced HOMFLY-PT homology $\mathcal{H}(K)$.

A proof of this conjecture seems much difficult than that Theorem 1, since we know much less about $\mathcal{H}(K)$ than we do about $P(K)$. At present, the most feasible means of attacking it seems to be to reformulate it in terms of Hilbert schemes of $\mathbb{C}^2$, an idea which was first suggested by Gorsky in [2]. Coarsely speaking, the goal is to associate to a closed n-strand braid $\overline{\sigma}$ in $S^1 \times D^2$ a $T = \mathbb{C}^* \times \mathbb{C}^*$ equivariant sheaf $\mathcal{F}(\overline{\sigma})$ on $\text{Hilb}^n(\mathbb{C}^2)$ with the property that

$$\mathcal{H}(\overline{\sigma}) = H^*_T(\mathcal{F}(\overline{\sigma}) \otimes \Lambda^*(\mathcal{T}^*)),$$

where $\mathcal{T}$ is the tautological sheaf on $\text{Hilb}^n(\mathbb{C}^2)$. This sheaf should be supported on the subvariety $\text{Hilb}^n(\mathbb{C})$ of ideals whose support lies in $\mathbb{C} \times 0$. If $\overline{\sigma}'$ is the braid obtained by adding a positive full twist to $\overline{\sigma}$, we should have $\mathcal{F}(\overline{\sigma}') = \mathcal{F}(\overline{\sigma}) \otimes \mathcal{O}(1)$, where $\mathcal{O}(1)$ is the ample line bundle on $\mathrm{Hilb}^n(\mathbb{C}^2)$ defined by Haiman [6].

This new formulation has the advantage that it should apply to all knots in S^3, not just algebraic knots. When $\overline{\sigma}$ is a torus knot $T(n,m)$, its relation to Conjecture 2 above is discussed in [4]. When $\overline{\sigma} = T(n, nk+1)$, the sheaf $\mathcal{F}(\overline{\sigma})$ should be $\mathcal{O}_{\mathrm{Hilb}^n(0)} \otimes \mathcal{O}(k)$, where $\mathrm{Hilb}^n(0)$ is the subvariety of ideals supported at the point $(0,0)$. More generally, the sheaf which should correspond to a torus knot $\mathcal{T}(n,m)$ was identified by Gorsky and Negut in [3]. Their approach is to construct a sheaf on the flag Hilbert scheme

$$\mathrm{FHilb}^n(\mathbb{C}^2) = \{I_n \subset I_{n-1} \subset \cdots \subset I_0 = \mathbb{C}[x,y] \,|\, I_k \text{ is an ideal, } \dim I_k/I_{k+1} = 1\}$$

and push it forward to $\mathrm{Hilb}^n(\mathbb{C}^2)$. Although the pushforward may be complicated, the sheaf upstairs on $\mathrm{FHilb}^n(\mathbb{C}^2)$ is relatively easy to describe. There are natural line bundles L_k over $\mathrm{FHilb}^n(\mathbb{C}^2)$ whose fibres are I_k/I_{k+1}. The sheaf constructed by Gorsky and Negut is of the form

$$\mathcal{O}_{\mathrm{FHilb}^n(0)} \otimes \prod_{k=1}^{n-1} L_k^{n_k}$$

where the integers n_k are determined from the pair (n,m).

In [3], we describe a scheme for constructing $\mathcal{F}(\overline{\sigma}) \in D^b(\mathrm{Coh}_T(\mathrm{Hilb}^n(\mathbb{C})))$ for arbitrary $\overline{\sigma}$. The idea is as follows. There is a category SBim_n of bimodules over $R = \mathbb{C}[x_1, \ldots, x_n]$ known as Soergel bimodules, which categorify the Hecke algebra of type A_{n-1}. Given an open n-strand braid σ, its *Rouquier complex* $C(\sigma)$ is a complex over SBim_n. The HOMFLY-PT homology of $\overline{\sigma}$ is obtained by applying the Hochschild homology functor to $C(\sigma)$ to obtain a complex of R-modules, and then taking homology.

Conjecture 3. *There are adjoint functors*

$$\begin{aligned} \iota_* &: K^b(\mathrm{SBim}_n) \to D^b\left(\mathrm{Coh}_T\left(\mathrm{FHilb}_n(\mathbb{C})\right)\right) \\ \iota^* &: D^b\left(\mathrm{Coh}_T\left(\mathrm{FHilb}_n(\mathbb{C})\right)\right) \to K^b(\mathrm{SBim}_n) \end{aligned}$$

which satisfy the additional properties that ι^* is monoidal and

$$\iota^*(L_1 \otimes \cdots \otimes L_{k-1}) = C(FT_k)$$

where FT_k is the full twist on the first k strands.

Assuming that such functors exist, we define $\mathcal{F}(\overline{\sigma}) = \pi_*(\iota_*(C(\sigma)))$, where $\pi : \mathrm{FHilb}^n(\mathbb{C}) \to \mathrm{Hilb}^n(\mathbb{C})$ is the projection. It can be shown that

$$\mathcal{H}(\overline{\sigma}) = H_T^*(\mathcal{F}(\overline{\sigma}) \otimes \Lambda^*(\mathcal{T}^*))$$

as desired. The functors ι_*, ι^* should be constructed inductively on n. In [3] we use Elias and Hogancamp's method of categorical diagonalization [1] to show that Conjecture 3 would follow from another conjecture which can be phrased purely in terms of the categories SBim_n.

References

[1] B. Elias, M. Hogancamp, *Categorical diagonalization*, In preparation.
[2] E. Gorsky, *q, t-Catalan numbers and knot homology*, in *Zeta functions in algebra and geometry*, Contemp. Math. **556**, p. 213–232.
[3] E. Gorsky, A. Negut, *Refined knot invariants and Hilbert schemes*, Journal de mathématiques pures et appliquées **104** (2015), pp. 403–435
[4] E. Gorsky, A. Oblomkov, J. Rasmussen, V. Shende *Torus knots and the rational DAHA*, Duke Math. J. **163** (2014), 2709–2794.
[5] E. Gorsky, A. Negut, J. Rasmussen, *Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology*, arXiv:1608.07308.
[6] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14** (2001), 941-1006.
[7] M. Khovanov, L. Rozansky, *Matrix factorizations and link homology II*, Geom. Topol. **12** (2008), 1387–1425.
[8] D. Maulik, *Stable pairs and the HOMFLY polynomial*, Inv. Math. **204** (2016), 787-831.
[9] A. Oblomkov, J. Rasmussen, V. Shende, *The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link*, arXiv:1201.2115.
[10] A. Oblomkov, V. Shende. *The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link*, Duke Math. J. 161 (2012) 1277–1303.

A Jacobian module for disentanglements and applications to Mond's conjecture

Javier Fernández de Bobadilla
(joint work with J. J. Nuño-Ballesteros, G. Peñafort-Sanchis)

For any hypersurface with isolated singularity $(X, 0)$, we have $\tau(X, 0) \leq \mu(X, 0)$, with equality if $(X, 0)$ is weighted homogeneous. Here, $\tau(X, 0)$ is the Tjurina number, that is, the minimal number of parameters in a versal deformation of $(X, 0)$ and $\mu(X, 0)$ is the Milnor number, which is the number of spheres in the Milnor fibre of $(X, 0)$. If $g\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a function such that $g = 0$ is a reduced equation of $(X, 0)$, then we can compute both numbers in terms of g:

$$\tau(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{J(g) + \langle g \rangle}, \quad \mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{J(g)},$$

where $\mathcal{O}_{n+1}$ is the local ring of holomorphic germs from $(\mathbb{C}^{n+1}, 0)$ to $\mathbb{C}$ and $J(g)$ denotes the Jacobian ideal generated by the partial derivatives of g. Thus, the initial statement about τ and μ becomes evident. The Jacobian algebra deforms flatly over the parameter space of any deformation g_t of g, it is known to encode crucial properties of the vanishing cohomology and its monodromy by its relation with the Brieskorn lattice and it is crucial in the construction of Frobenius manifold structures in the bases of versal unfoldings. See the works of Brieskorn, Varchenko, Steenbrink, Scherk, Hertling and others. Inspired by the previous inequality, D. Mond tried to obtain a result of the same nature in the context of singularities of mappings. He considered a hypersurface $(X, 0)$ given by the image of a map germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, with $S \subset \mathbb{C}^n$ a finite set and which has isolated instability under the action of the Mather group $\mathcal{A}$ of biholomorphisms in the source and the target. The Tjurina number has to be substituted by the $\mathcal{A}_e$-codimension, which is equal to the minimal number of parameters in an $\mathcal{A}$-versal deformation of f. Instead of the Milnor fibre, one considers the disentanglement, that is, the image X_u of a stabilisation f_u of f. Then, X_u has the homotopy type of a wedge of spheres and Mond defined the image Milnor number $\mu_I(f)$ as the number of such spheres. Note that, outside the range of Mather's dimensions, some germs do not admit a stabilisation. Then, he stated the following conjecture:

Conjecture 1. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite map germ, with $(n, n+1)$ nice dimensions. Then,*

$$\mathcal{A}_e\text{-}\operatorname{codim}(f) \leq \mu_I(f),$$

with equality if f is weighted homogeneous.

The conjecture is known to be true for $n = 1, 2$ but it remains open until now for $n \geq 3$. There is a related result for map germs $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $n \geq p$, where one considers Δ the discriminant of f instead of its image and defines the discriminant Milnor number $\mu_\Delta(f)$ in the same way. Damon and Mond showed that if (n, p) are nice dimensions, then $\mathcal{A}_e$-codim$(f) \leq \mu_\Delta(f)$ with equality if f is weighted homogeneous. There are many papers in the literature with related results, partial proofs and examples in which the conjecture has been checked.

Going back to hypersurface singularities g, but now with non-isolated singularities, it is not anymore clear the relation of the Jacobian algebra of g with the vanishing cohomology. Moreover it is apparent in easy examples that the Jacobian algebra does not deform flatly in unfoldings. In fact the possibility of studying the vanishing cohomology via deformations that simplify the critical set (in the same vein that Morsifications do for isolated singularities) does not exist in general. However, for restricted classes of singularities Siersma, Pellikaan, Zaharia, Nemethi, Marco-Buzunáriz and the first author have developed methods that allow to split the vanishing cohomology of a non-isolated singularity in two direct summands according with the geometric properties of a deformation g_u of g which plays the role of a Morsification. The first is a free vector space contributing to the middle dimension cohomology of the Milnor fibre, with as much dimension as the number of Morse points that appear away from the zero set of g_u $(u \neq 0)$, the second is determined by the non-isolated singularities of the zero-set of g_u $(u \neq 0)$.

Given $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, an $\mathcal{A}$-finite map germ, we consider a generic 1-parameter deformation f_u of it (a stabilisation). Let g_u be the equation defining the image of f_u. It turns out that the deformation g_u is suitable to split the vanishing cohomology of g in two direct summands, as explained in the paragraph above, and that the first summand corresponds with the cohomology of the image X_u, whose rank is the image Milnor number. The main novelty of this paper is the definition of an Artinian $\mathcal{O}_{n+1}$ -module $M(f)$, which satisfies

$$\dim_{\mathbb{C}} M(f) = \mathcal{A}_e\text{-}\operatorname{codim}(f) + \dim_{\mathbb{C}}((g) + J(g)/J(g))$$

and, in the nice dimensions, this dimension upper bounds the image Milnor number. Moreover we define a relative version $M_y(F)$ of the module for unfoldings

F of f, and we prove that when we specialise the parameter the relative module specialises to the original $M(f)$.

The first main result of the talk implies that the dimension of $M(f)$ equals the image Milnor number if and only if $M_y(F)$ is flat over the base of the unfolding. We also proved that this is equivalent to the flatness of the Jacobian algebra over the base of the unfolding. Thus, under the flatness condition, $M(f)$ is expected to play the role of the Milnor algebra for isolated singularities, in the sense of encoding the first direct summand of the vanishing cohomology, which is the only one present for isolated singularities. It is very interesting to investigate whether the relation of the vanishing cohomology of isolated singularities with the Jacobian algebra explained admit a generalisation to a relation between the first direct summand of the vanishing cohomology of g and the module $M(f)$.

The second main result says that the flatness of $M_y(F)$ implies Mond's conjecture for f, and it is equivalent to it if f is weighted homogeneous. We also derived the surprising consequence that in order to settle Mond's conjecture in complete generality it is enough to prove it for a series of examples of increasing multiplicity.

The arcspace of $\mathbb{C}^2$ and adjacencies of plane curves.

María Pe Pereira

(joint work with J. Fernández de Bobadilla, P. Popescu-Pampu)

Let (X, O) be a germ of a normal surface singularity. A *model* over X is a proper birational map $\pi : S \longrightarrow X$. A *prime divisor E over O* is a divisor in a model S such that $\pi(E) = O$.

An *arc* in (X, O) centered at O is a parametrization $\alpha(t)$ of curve germ contained in X with $\alpha(0) = 0$. The *Nash set* $\overline{N}_E$ associated with a prime divisor E is the Zariski closure in the space of arcs of (X, O) of the set of arcs whose lifting (to a model where E appears) hit E. A *Nash adjacency* is an inclusion $\overline{N}_F \subset \overline{N}_E$.

Given a normal surface singularity (X, O) and given two different prime divisors E, F appearing on its minimal resolution, Nash conjectured in [11] that *none of the spaces $\overline{N}_E, \overline{N}_F$ is included in the other one.* This conjecture, which was proved in [7], can be generalized in the form of *charaterizing when $\overline{N}_F \subset \overline{N}_E$ for E and F be prime divisors over $O \in X$.* This problem was first stated in [9] for X an irreducible germ of arbitrary dimension.

This problem is wide open even for the smooth germ $(\mathbb{C}^2, O)$ and we focus on this case here. Then, the problem turns to be related to classical adjacency problems of plane curves. We recall that the adjacency problem of plane curves ask about what topological types may happen in a deformation $(f_s)_{s\in(\mathbb{C},0)}$ of plane curves for the special and generic curve.

More precisely, given two divisors over the origin of $\mathbb{C}^2$, the existence of an adjacency $\overline{N}_F \subset \overline{N}_E$ is equivalent to the existence of a holomorphic family of arcs $\alpha_s(t)$ such that $\alpha_0(t)$ lifts transverse to F and $\alpha_s(t)$ lifts transverse to E. This equivalence, which was first proved for the more general case of a surface singularity by Fernández de Bobadilla in [6] using results of Reguera in [17], has an easier proof for $(\mathbb{C}^2, O)$. Moreover, it can be proved that in $\mathbb{C}^2$ we can assume that if such a family exists, it exists one such that the generic curve is a single branch at O. Then, it is clear the relation to the classical adjacency problem: if $\overline{N}_F \subset \overline{N}_E$, then there is a deformation $(f_s)_s$ of curves whose special curve given by $f_0 = 0$ has embedded resolution through F and whose generic curves (given by $f_s = 0$ for $s \neq 0$ small enough) have all embedded resolution through E. Moreover, these deformations are δ-constant deformations.

These deformations f_s are of special type: the generic curves, for $s \neq 0$ small enough, not only share the topological type but also the position of all the free infinitely near points that appear in the blowing up process that resolves the family of generic curves. We say that the family *fixes the free points.* As one can expect, not all adjacencies can be achieved with this restricted type, but it seems that, the amount of them that they are, are not negligible at all.

Let's see now, what can we say about the Nash-adjacency.

The combinatorial type of a pair of divisors is the co mbinatorics of the minimal sequence of blowing ups needed for making both of them appear. Then, the following turns to be true:

Theorem 1. *Let E and F be two prime divisors over the origin. The Nash-adjacency $\overline{N}_F \subset \overline{N}_E$ only depends on the combinatorial type of the pair (E, F).*

The proof can be found in [8] and is based on the work [6].

This theorem has some nice consequences as the following:

Corollary 1. *Let E and F be prime divisors sharing i infinitely near points in the blowing up process for finding the minimal model where they appear. If moreover $\overline{N}_F \subset \overline{N}_E$, then we have the following inclusions:*

$$\bigcup_{F' \equiv_{\geq i} F} \overline{N}_{F'} \subset \bigcap_{E' \equiv_{\geq i} E} \overline{N}_{E'}$$

where $D \equiv_{\geq i} D'$ means that the divisors D and D' have the same combinatorics and they share i or more blowing up points in the process for finding the minimal models where D and D' appear.

We would like to understand the relative position of all these $\overline{N}_E$. As a first step we conjecture a relation, easy to check in easy examples, with the subsets of arcs of the form $\overline{Cont^q D}$. The set $\overline{Cont^q D}$ is the closure of arcs whose lifting meets D with multiplicity greater or equal than q. These sets are the maximal divisorial sets studied in [5] and [9]. In particular, these sets are cylindrical sets (determined at certain level of truncation) as $\overline{N}_E$ are. More precisely, we expect the following to be true (see [8] for more details):

Conjecture 1. *Let p_{i_0} be any free point among the infinitely near points $\{p_i\}_{i \in I}$ which are blown-up in order to obtain the minimal model of a prime divisor E. We conjecture that the set $\bigcap_{E' \equiv_{i_0} E} \overline{N}_{E'}$ coincides with $\overline{Cont^q(E_{p_{i_0}})}$ for certain q where $E_{p_{i_0}}$ is the divisor that appears after blowing up the point p_{i_0}. In case the conjecture is true, it would be interesting to compute q.*

In the case E is a toric divisor and $i_0 = 1$, which means that the generic and special curves have different tangent lines, then $M = \beta_1$. The next easy case would be $i_0 = 1$ and E non toric.

A basic fact, proved in [13] as a generalization of a result in [16], is the implication: is that one has the implication:

$$\overline{N}_F \subset \overline{N}_E \Longrightarrow \nu_E \leq \nu_F$$

where ν_E denotes the divisorial valuation associated to E.

This implication, which we call the *valuative criterion* for Nash adjacency, has been used as a main tool for the investigation of Nash's Conjecture until the works [6], [12] and [7] (see [10], [14] and [15]). However, Ishii [9] showed that the converse implication is not true.

In [8] we clarify the geometric meaning of the inequality $\nu_E \leq \nu_F$ as follows. Moreover, we treat the inequality in a unified way for non prime divisors by defining $\nu_E := \sum_i a_i \nu_{D_i}$ for $E = \sum_i a_i D_i$ with D_i prime. Let $F = \sum_i b_i D_i$ be a divisor over O in a model S over $(\mathbb{C}^2, O)$. We say that f is *associated* to F if $\widetilde{V(f)}$ is a disjoint union of smooth irreducible curve germs and if for every i, b_i of them are transverse to the corresponding D_i at distinct smooth points. Then:

Theorem 2. *Let E and F be divisors above $O \in \mathbb{C}^2$ and let S be the minimal model where $E + F$ appear. The following are equivalent:*

- $\nu_E \leq \nu_F$
- *there exists a deformation $G = (g_s)_s$ with g_0 associated with F in S and g_s associated with E in S, for $s \neq 0$ small enough.*
- *there exists a linear deformation $(g_s = g_0 + sg)_s$ with g_0 associated with F in S and g and g_s associated with E in S, for $s \neq 0$ small enough.*

The equivalence between the last two points was proved in a different language by Alberich and Roe in [1].

In particular this shows that the valuative inequality talks about deformations of functions and doesn't say anything about the δ-constancy of the family. By a classical result of Teissier, the δ-constancy in a deformation of reduced curves is equivalent to the existence of a parametrization in family, that is precisely what a family of arcs is.

A characterization of the inequality $\nu_E \leq \nu_F$ in terms of finitely many inequalities $\nu_E(h) \leq \nu_F(h)$, for functions h only depending on E, is easily achieved in [8]. This gives an algorithmic way to check the inequality in a very effective way, being able for example to find very quickly all the possible adjacencies realizable by families fixing the free points among the ones listed in [2], [4], [18] and [3]. In fact we find two, from Z_{11} and $X_{1,1}$ to E_8, that although would follow easily from the criteria in [2], seem not to be in the previous lists.

Moreover, we can give an algorithm which, given a topological type, gives back all the topological types Arnold adjacent to the given one by a deformation fixing the free points. We expect that this algorithm produces, as in the known examples, a good portion of all the topological types Arnold adjacent to the given one.

See [8] for further details.

Acknowledgements: The author is currently supported by a Juan de la Cierva-Formación Fellowship from the Spanish Ministry of Economy and Competitiveness and partially supported by the ERC Consolidator Grant NMST and by Spanish MICINN project MTM2013-45710-C2-2-P and by the project SEV-2011-0087 Apoyo a Centros y Unidades de Excelencia Severo Ochoa.

References

[1] M. Alberich, J. Roe. *Enriques diagrams and adjacency of planar curve singularities.* Canad. J. Math. **57** (2005), no. 1, 3–16.

[2] V. I. Arnold. *Local Normal Forms of Functions.* Inv. Math. **35** (1976), 87–109.

[3] D. Balkenborg, R. Bauer, F-J. Bilitewski, *Beitrage zur Hierarchie der bimoduaren Singularitaten.* Bonn 1984. Thesis diploma avaliable in internet.

[4] E. Brieskorn. *Die Hierarchie der 1-modularen Singularitäten.* Manuscripta Math. **27** no. 2 (1979), 183–219.

[5] L. Ein, R. Lazarsfeld, M. Mustaţă. *Contact loci in arc spaces.* Compos. Math. **140** (2004), 1229–1244.

[6] J. Fernández de Bobadilla. *Nash problem for surface singularities is a topological problem.* Advances in Math. **230** no. 1 (2012), 131–176.

[7] J. Fernández de Bobadilla, M. Pe Pereira. *The Nash problem for surfaces.* Annals Math. **176** Issue 3 (2012), 2003–2029.

[8] J. Fernández de Bobadilla, M. Pe Pereira, P. Popescu-Pampu *On the generalized Nash Problem for smooth germs and adjacencies of curves singularities.* arXiv:1501.06898v2 [math.AG]

[9] S. Ishii. *Maximal divisorial sets in arc spaces.* Algebraic Geometry in East Asia, Adv. Stud. Pure Math., Hanoi, 2005, vol. **50**, Math. Soc. Japan, Tokyo (2008), 237–249.

[10] M. Lejeune-Jalabert, A. J. Reguera. *Arcs and wedges on sandwiched surface singularities.* Amer. J. Math. **121** no. 6 (1999), 1191–1213.

[11] J. Nash. *Arc structure of singularities.* In *A celebration of John F. Nash Jr.* Duke Math. J. **81** no. 1 (1995), 31–38.

[12] M. Pe Pereira. *Nash problem for quotient surface singularities.* J. London Math. Soc. (2) **87** no. 1 (2013), 177–203.

[13] C. Plénat. *À propos du problème des arcs de Nash.* Ann. Inst. Fourier (Grenoble) **55** no. 3 (2005), 805–823.

[14] C. Plénat; P. Popescu-Pampu. *A class of non-rational surface singularities with bijective Nash map.* Bull. Soc. Math. France **134** no. 3 (2006), 383–394.

[15] C. Plénat; P. Popescu-Pampu. *Families of higher dimensional germs with bijective Nash map.* Kodai Math. J. **31** no. 2 (2008), 199–218.

[16] A. J. Reguera. *Families of arcs on rational surface singularities.* Manuscripta Math. **88** no. 3, (1995), 321–333.

[17] A. J. Reguera. *A curve selection lemma in spaces of arcs and the image of the Nash map.* Compositio Math. **142** (2006), 119–130.

[18] K. Saito. *Einfach-elliptische Singularitaten.* Inv. Math. **23** (1974), 289–325.

Cohomology jump loci

Nero Budur

1. Classical

We consider generalizations of the classical singularity theory package:

Theorem 1.1. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function.*

1. *The eigenvalues of the monodromy of f are roots of unity.*
2. *The size of the Jordan blocks of the monodromy on H_i of the Milnor fiber of f is $\leq i+1$ $(0 \leq i \leq n-1)$.*
3. *(A'Campo) The monodromy zeta function of f is computable from a log resolution.*
4. *(Malgrange, Kashiwara) The set of eigenvalues of the monodromy of f along $f^{-1}(0)$ is the image under $Exp(\_) = \exp(2\pi i \cdot \_)$ of the zeros of the Bernstein-Sato polynomial of f.*

One of the motivations to study classical invariants of singularities comes from the Monodromy Conjecture:

Conjecture 1.2. *(Igusa, Denef-Loeser) Let f be a polynomial in n variables. The poles of the motivic zeta function of f give zeros of the Bernstein-Sato polynomial of f.*

2. Cohomology jump loci

Part (1) of Theorem 1.1 above is generalized by part (c) of the following:

Theorem 2.1. *Let X be either of*

(a) a smooth quasi-projective algebraic variety over $\mathbb{C}$ [2],
(b) a compact Kähler manifold [6],
(c) the complement of $\mathbb{C}$-analytic set in a small ball [3].

Then the cohomology jump loci

$$V^i_k(X) = \{L \in M_B(X) \mid \dim H^i(X, L) \geq k\}$$

are finite unions of torsion-translated subtori of the space of rank one local systems on X,

$$M_B(X) := Hom(H_1(X,\mathbb{Z}), \mathbb{C}^*) \approx (\mathbb{C}^*)^{b_1(X)} \times \text{(finite abelian group)}.$$

$V^i_k(X)$ are affine closed subschemes of $M_B(X)$ and are homotopy invariants of the topological space X. The first two results build on a long list of important partial results and contributions. The compact Kähler manifold case had been conjectured by Beauville in late 1980's. The last part was given a proof independent of all other partial results and pointed out that there seems to be a close-to-universal reason for these structure results: the Riemann-Hilbert correspondence between D-modules and constructible sheaves. This opens up generalizations of the structure results to other types of loci defined by usual (derived) functors. In this direction, we mentioned an ongoing project with Botong Wang.

In a certain sense which can be made precise, classical singularity theory is a 1-parameter diagonal in the space of rank one local systems. The Jordan decomposition of the monodromy actions can be deduced from the scheme-theoretic intersection of this diagonal with the various cohomology jump loci.

From this point of view, one has the following analog of Part (2) of Theorem 1.1:

Theorem 2.2. *([5]) Let $\phi : H_1(X,\mathbb{Z}) \to \mathbb{Z}$ be a surjective homomorphism. Denote by $X^\phi \to X$ the corresponding $\mathbb{Z}$-cover. Let $T_i(X,\phi)$ be the torsion part of $H_i(X^\phi,\mathbb{C})$ as $\mathbb{C}[t,t^{-1}]$-module, where t is induced by the deck action.*

(a) If X is a compact Kähler manifold, then all Jordan blocks for t on $T_i(X,\phi)$ are of size one. That is, $T_i(X,\phi)$ is semi-simple over $\mathbb{C}[t,t^{-1}]$.

(b) If X is a smooth quasi-projective algebraic variety over $\mathbb{C}$ of dimension n, then the size of the Jordan blocks for t on $T_i(X,\phi)$ is $\leq \min\{i+1, 2n-i\}$.

It is very difficult to calculate the cohomology jump loci $V^i_k(X)$. Conjecturally, they are combinatorial for complements of hyperplane arrangements.

3. Cohomology support loci

Part (3) of Theorem 1.1 is generalized by the following:

Theorem 3.1. *([4]) Let $F = (f_1,\dots,f_r) : (\mathbb{C}^n,0) \to (\mathbb{C}^r,0)$ be the germ of a holomorphic map and $f = \prod_i f_i$. Then the locus $V_F \subset (\mathbb{C}^*)^r$ of rank one local systems on $(\mathbb{C}^*)^r$ with non-trivial cohomology on some small ball complement along $f^{-1}(0)$ is a finite union of codimension-one torsion translated subtori, computable from a log-resolution of f.*

V_F can also be obtained as the union of the non-trivial cohomology jump loci of small-ball complements of f along $f^{-1}(0)$. Hence the interesting part of the above result is that all components become hypersurfaces.

In the case all f_i are linear, V_F is easily computable from the combinatorics of the hyperplane arrangement f, see [4].

4. Bernstein-Sato ideals

Part (4) of Theorem 1.1 would be generalized by the following recent conjectures of [1]:

Conjecture 4.1. *Let $F = (f_1,\dots,f_r) : (\mathbb{C}^n,0) \to (\mathbb{C}^r,0)$ be the germ of a holomorphic map and $f = \prod_i f_i$.*

(i) The ideal $B_F \in \mathbb{C}[s_1,\dots,s_r]$ given by $b(s)$ such that

$$b(s) \cdot \prod_i f_i^{s_i} = P \cdot \prod_i f_i^{s_i+1}$$

for some $P \in \mathbb{C}[x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n][s_1, \dots, s_r]$, *is generated by products of polynomials of the type*

$$a_1 s_i + \ldots + a_r s_r + b \quad (a_i \in \mathbb{Q}_{\geq 0}, b \in \mathbb{Q}_{>0}).$$

(ii) The image of the zero locus of B_F *under* $Exp : \mathbb{C}^r \to (\mathbb{C}^*)^r$ *is* V_F.

One inclusion in part (ii) is done, namely $Exp(Zeros(B_F))$ contains V_F, see [1, 4]. There is recent progress by Maisonobe on showing that $Exp(Zeros(B_F))$ is a union of torsion-translated subtori of codimension exactly one of $(\mathbb{C}^*)^r$, and on a geometric computation of the "slopes" of these hypersurfaces.

The Monodromy Conjecture for $F = (f_1, \dots, f_r)$ states that the polar locus of the multi-variable motivic zeta function of F gives hyperplanes lying in the zero locus of the Bernstein-Sato ideal B_F, see [1]. One feature of the multi-variable case, not seeable in the classical case, is that one can also make sense of "half" of the conjecture. This is namely just the statement relating the "slopes" of the hyperplanes of the polar locus with the initial ideal of B_F. This might be more approachable.

References

[1] N. Budur, Bernstein-Sato ideals and local systems. Ann. Inst. Fourier 65 no. 2, 549-603 (2015).

[2] N. Budur, B. Wang, Cohomology jump loci of quasi-projective varieties. Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 1, 227–236.

[3] N. Budur, B. Wang, Local systems on analytic germ complements. arXiv:1508.07867.

[4] N. Budur, Y. Liu, L. Saumell, B. Wang, Cohomology support loci of local systems. arXiv:1511.08013. To appear in Michigan Math. J.

[5] N. Budur, Y. Liu, B. Wang, The monodromy theorem for compact Kähler manifolds and smooth quasi-projective varieties. arXiv:1609.06478.

[6] B. Wang, Torsion points on the cohomology jump loci of compact Kähler manifolds. arXiv:1312.6619. To appear in Math. Res. Lett.

On the algebraic stringy Euler number of a singular variety

Victor Batyrev

(joint work with Giuliano Gagliardi)

If X is a smooth projective algebraic variety over $\mathbb{C}$. The *topological i-th Betti number* $b^i_{\rm top}(X)$ of X is the dimension of the $\mathbb{C}$-space $H^i(X, \mathbb{C})$. The *topological Euler number* of X is the alternating sum $e_{\rm top}(X) := \sum_{i=0}^{2 \dim X} (-1)^i b^i_{\rm top}(X)$.

We denote by $H^{2i}_{\rm alg}(X, \mathbb{C})$ the $\mathbb{C}$-subspace in $H^{2i}(X, \mathbb{C})$ generated by the classes $[Z]$ of algebraic cycles $Z \subseteq X$ of codimension i. The dimension $b^{2i}_{\rm alg}(X)$ of the $\mathbb{C}$-space $H^{2i}_{\rm alg}(X, \mathbb{C})$ we call the $2i$-th *algebraic Betti number* of X. The number

$$e_{\rm alg}(X) := \sum_{i=0}^{\dim X} b^{2i}_{\rm alg}(X)$$

we call the *algebraic Euler number* of a smooth projective variety X. We extend this notion to the case of singular varieties.

Let X be a d-dimensional normal projective variety over $\mathbb{C}$ and let $\rho : Y \to X$ be a desingularization of X such that the exceptional locus of ρ is a union of smooth irreducible divisors $D_1, \ldots, D_s$ with normal crossings. Assume that X is a $\mathbb{Q}$-Gorenstein, i.e. the canononical class K_X is a $\mathbb{Q}$-Cartier divisor, and we can write

$$K_Y = \rho^* K_X + \sum_{i=1}^{s} a_i D_i$$

for some rational numbers a_i. The variety X is said to have at worst log-terminal singularities if $a_i > -1$ for all $1 \leq i \leq s$. We set $I := \{1, \ldots, s\}$ and define $D_\emptyset := Y$, $D_J := \cap_{j \in J} D_j$ for all $\emptyset \neq J \subseteq I$. Then for any subset $J \subseteq I$ the complete intersection D_J is either empty, or a smooth projective subvariety of codimension $|J|$.

We define the *algebraic stringy Euler number* of a singular variety by the formula:

$$e^{\rm str}_{\rm alg}(X) := \sum_{\emptyset \subseteq J \subseteq I} e_{\rm alg}(D_J) \prod_{j \in J} \left(\frac{-a_j}{a_j + 1} \right).$$

Using results of Teh [3] based on the non-Archimedean integration [1], it can be shown that the algebraic stringy Euler number is independent on the choice of the resolution $\rho : Y \to X$.

The algebraic stringy Euler number $e^{\rm str}_{\rm alg}(X)$ is a rational number. Our first conjecture claims:

Conjecture 1. *Let X be a projective algebraic variety with at worst $\mathbb{Q}$-Gorenstein log-terminal singularities. Then $e^{\rm str}_{\rm alg}(X) > 0$.*

The algebraic stringy Euler numbers are expected to be useful in the Mori program.

Definition 2. *A proper birational morphism $f \colon X \to X'$ of two $\mathbb{Q}$-Gorenstein varieties X and X' is called a* divisorial Mori contraction *if f contracts a divisor $D \subseteq X$ and the anticanonical divisor $-K_X$ is f-ample.*

Definition 3. *A birational morphism $g \colon X \dashrightarrow X^+$ of two $\mathbb{Q}$-Gorenstein varieties X and X^+ together with the birational morphisms $f \colon X \to Z$ and $f^+ \colon X^+ \to Z$ in a commutative diagram*

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & X^+ \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^+} & \\ & Z & \end{array}$$

is called a Mori flip *if g is an isomorphism in codimension one, the anticanonical class $-K_X$ is f-ample, and the canonical class K_{X^+} is f^+-ample.*

We conjecture the following strict monotonicity of the algebraic stringy Euler number with respect to the above elementary birational transformations in the Mori program:

Conjecture 4. *Let $f\colon X \to X'$ be a divisorial Mori contraction. Then*
$$e^{\rm str}_{\rm alg}(X) > e^{\rm str}_{\rm alg}(X').$$

Conjecture 5. *Let $g\colon X \dashrightarrow X^+$ be a Mori flip. Then*
$$e^{\rm str}_{\rm alg}(X) > e^{\rm str}_{\rm alg}(X^+).$$

Remark 6. From the viewpoint of Conjectures 1, 4, and 5, a projective algebraic variety X with at worst terminal singularities is a *minimal model* in a given birational class if its algebraic stringy Euler number $e^{\rm str}_{\rm alg}(X)$ has the minimal possible value among all projective algebraic varieties with at worst terminal singularities in the same birational class.

We can prove our conjectures in the equivariant Mori program for arbitrary projective spherical varieties [2].

Theorem 7. *Let G be a connected reductive algebraic group. Conjectures 1, 4 and 5 are true in the G-equivariant Mori program for projective spherical G-varieties.*

This result extends analogous statements in the equivariant Mori program for toric varieties obtained by M. Reid [4].

References

[1] V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities.* Integrable systems and algebraic geometry (Kobe/Kyoto) (1997), 1–31.

[2] Michel Brion, *Variétés sphériques et théorie de Mori*, Duke Math. J., (1993), no. 2., 369–404.

[3] Jyh-Haur Teh, *Motivic integration and projective bundle theorem in morphic cohomology*, Math. Proc. Cambride Philos. Soc. **147** (2009), no. 2, 295–321.

[4] Miles Reid, *Decomposition of toric morphisms*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 395–418.

Reporter: Baldur Sigurðsson

Participants

Prof. Dr. Norbert A'Campo
Departement Mathematik und
Informatik Universität Basel
Spiegelgasse 1
4051 Basel
SWITZERLAND

Dr. Haniya Azam
Department of Mathematics
Syed Babar Ali School of Science and
Engineering
Lahore University of Management
Sciences
D.H.A., Lahore Cantt. 54792
Lahore
PAKISTAN

Prof. Dr. Victor V. Batyrev
Fachbereich V - Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Manuel Blickle
Institut für Mathematik
Johannes-Gutenberg Universität Mainz
55128 Mainz
GERMANY

Prof. Dr. Ragnar-Olaf Buchweitz
Computer and Mathematical Sciences
Dept.
University of Toronto at Scarborough
1265 Military Trail
Toronto, Ont. M1C 1A4
CANADA

Dr. Nero Budur
Departement Wiskunde
Faculteit der Wetenschappen
Katholieke Universiteit Leuven
Celestijnenlaan 200B
3001 Leuven
BELGIUM

Mark Bugden
Mathematical Sciences Institute
Australian National University
Canberra ACT 0200
AUSTRALIA

Prof. Dr. Igor Burban
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Prof. Dr. Georges Comte
Laboratoire de Mathématiques
Université de Savoie - Mont Blanc
73376 Le Bourget du Lac Cedex
FRANCE

Sofia Nerina D'Alesio Souto
Departamento de Matematica - FCEN
Universidad de Buenos Aires
Ciudad Universitaria
Pabellon 1
1428 Buenos Aires
ARGENTINA

Dr. Javier F. de Bobadilla de Olazabal
Basque Center of Mathematical Sciences
Alameda Mazarredo, 14
48009 Bilbao
SPAIN

Prof. Dr. Alexandru Dimca
Laboratoire J.-A. Dieudonne
Université de Nice Sophia Antipolis
Parc Valrose
06108 Nice Cedex 2
FRANCE

Prof. Dr. Wolfgang Ebeling
Institut für Algebraische Geometrie
Leibniz Universität Hannover
30167 Hannover
GERMANY

Dr. Eleonore Faber
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Mauricio D. Garay
23, Avenue Briand
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Victor Goryunov
Department of Mathematical Sciences
University of Liverpool
Peach Street
Liverpool L69 7ZL
UNITED KINGDOM

Dr. Lothar Göttsche
International Centre for Theoretical Physics
(ICTP)
Strada Costiera 11
34100 Trieste
ITALY

Prof. Dr. Dr.h.c. Gert-Martin Greuel
Fachbereich Mathematik
Technische Universität Kaiserslautern
67653 Kaiserslautern
GERMANY

Prof. Dr. Sabir M. Gusein-Zade
Department of Geometry and Topology
Faculty of Mechanics and Mathematics
Moscow State University
Leninskie Gory
Moscow 119 992
RUSSIAN FEDERATION

Prof. Dr. Herwig Hauser
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Prof. Dr. Claus Hertling
Lehrstuhl für Mathematik VI
Universität Mannheim
68131 Mannheim
GERMANY

Dr. Dmitry Kerner
Department of Mathematics
Ben-Gurion University of the Negev
P.O.Box 653
84105 Beer-Sheva
ISRAEL

Tetiana Klymchuk
Department of Algebra and Mathematical Logic
Faculty of Mechanics and Mathematics
Kyiv Taras Shevchenko University
Volodymyrska st., 64
01033 Kiev
UKRAINE

Prof. Dr. Manfred Lehn
Institut für Mathematik
Johannes-Gutenberg Universität Mainz
55099 Mainz
GERMANY

Dr. Ann Lemahieu
Laboratoire J.-A. Dieudonné
Université de Nice Sophia Antipolis
Parc Valrose
06108 Nice Cedex 2
FRANCE

Prof. Dr. Francois Loeser
Institut de Mathématiques de Jussieu
Université de Paris VI, Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Martin Lüdtke
Institut für Mathematik
Goethe-Universität Frankfurt
Postfach 111932
60054 Frankfurt am Main
GERMANY

Prof. Dr. Ignacio Luengo
Facultad de Matematicas
Departamento de Algebra
Universidad Complutense de Madrid
28040 Madrid
SPAIN

Prof. Dr. David Mond
Mathematics Institute
University of Warwick
Zeeman Building
Coventry CV4 7AL
UNITED KINGDOM

Janos Nagy
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Prof. Dr. Yoshinori Namikawa
Department of Mathematics
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. András Némethi
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Prof. Dr. Walter David Neumann
Department of Mathematics
Barnard College
Columbia University
New York, NY 10027
UNITED STATES

Dr. Hong Duc Nguyen
Bissingstrasse 1
21075 Hamburg
GERMANY

Prof. Dr. Tomohiro Okuma
Department of Mathematical Sciences
Yamagata University
1-4-12 Kojirakawa-machi
Yamagata 990-8560
JAPAN

Maria Pe Pereira
Instituto de Ciencias Matematicas
ICMAT - UAM
Campus de Cantoblanco
Nicolas Cabrera, no. 13-15
28049 Madrid
SPAIN

Prof. Dr. Anne Pichon
Institut de Mathématiques de Marseille
Université Aix-Marseille
Case 907
163, Avenue de Luminy
13288 Marseille Cedex 9
FRANCE

Prof. Dr. Patrick Popescu-Pampu
U. F. R. Mathématiques
Université de Lille 1
Bat. M2
59655 Villeneuve d'Ascq. Cedex
FRANCE

Dr. Michel Raibaut
Laboratoire de Mathématiques
Université de Savoie
73376 Le Bourget du Lac Cedex
FRANCE

Dr. Jacob A. Rasmussen
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Sarah Rasmussen
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Ana J. Reguera
Departamento de Algebra, Geometria y
Topologia, Facultad de Ciencias
Universidad de Valladolid
C. Paseo de Belen 7
47011 Valladolid
SPAIN

Prof. Dr. Kyoji Saito
Institute for the Physics and
Mathematics
of the Universe (IPMU)
The University of Tokyo
5-1-5 Kashiwanoha, Kashiwa
Chiba 277-8568
JAPAN

Dr. Morihiko Saito
Research Institute for Mathematical
Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Julien Sebag
Unité de Formation et de Recherche
Mathématiques, I.R.M.A.R.
Université de Rennes 1
263 Avenue du General Leclerc
35042 Rennes Cedex
FRANCE

Dr. Baldur Sigurdsson
Basque Center of Mathematical Sciences
Alameda Mazarredo, 14
48009 Bilbao
SPAIN

Prof. Dr. Jan Stevens
Department of Mathematics
Chalmers University of Technology
412 96 Göteborg
SWEDEN

Prof. Dr. Atsushi Takahashi
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN

Dr. Laszlo Tamas
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Dr. Le Quy Thuong
Department of Mathematics
Vietnam National University
334 Nguyen Trai, Thanh Xuan
Hanoi
VIETNAM

Prof. Dr. Mihai Tibar
Mathématiques
UMR 8524 CNRS
Université de Lille 1
59655 Villeneuve d'Ascq.
FRANCE

Prof. Dr. Duco van Straten
Institut für Mathematik
Johannes-Gutenberg Universität Mainz
55128 Mainz
GERMANY

Prof. Dr. Wim Veys
Department of Mathematics
University of Leuven
Celestijnenlaan 200B
3001 Leuven (Heverlee)
BELGIUM

Prof. Dr. Jonathan M. Wahl
Department of Mathematics
University of North Carolina
Phillips Hall
Chapel Hill NC 27599-3250
UNITED STATES