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Arbeitsgemeinschaft: Diophantine Approximation, Fractal Geometry and Dynamics

Organised by
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Abstract. Recently conjectures by Schmidt and by Davenport were solved in papers by Dzmitry Badziahin, Andrew Pollington and Sanju Velani, and by Victor Beresnevich. The methods which they used are the source for a growing number of works in Diophantine approximation, fractal geometry and flows on homogeneous spaces, and their full power is still far from being well understood. The goal of this workshop was to introduce those methods to a broader audience, and to allow for an opportunity to further their development.

Mathematics Subject Classification (2010): 11J83, 37A17.

Introduction by the Organisers

The workshop *Diophantine approximation, fractal geometry and dynamics*, organised by Victor Beresnevich (York) and Sanju Velani (York) was well attended with over 50 participants and a nice blend of experts with backgrounds in metric number theory and flows on homogeneous spaces, as well as researchers from adjacent fields, graduate students and postdocs. The workshop was well planned, aimed both at newcomers, starting with a broad introduction to metric Diophantine approximation, and at experts, including informal discussions on the details of the latest discoveries.

In 1983 W. M. Schmidt formulated a conjecture about the existence of points in the Euclidean plane that are simultaneously badly approximable with weights $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$, that is $\mathbf{Bad}(\frac{1}{3}, \frac{2}{3}) \cap \mathbf{Bad}(\frac{2}{3}, \frac{1}{3}) \neq \emptyset$. Should this intersection be empty, it would immediately prove a well known conjecture of Littlewood from the 1930s. Weighted badly approximable points are characterised by (non-)proximity by rational points in a metric 'skewed' by some weights of approximation, say, i and j where $i, j \geq 0$, $i + j = 1$, which are 'attached' to coordinate directions.

The central goal of this workshop was to expose its participants to recent breathtaking development regarding Schmidt's conjecture stemming from its proof by Badziahin, Pollington and Velani in 2011. We delved into the details of the proof of the conjecture given in a subsequent work of Jinpeng An (2013) and also discussed the solution to Davenport's problem, which boils down to the study of the intersections of $\mathbf{Bad}(i, j)$ with planar curves. More generally we studied the recent work of Beresnevich (2015) on badly approximable points on manifolds.

The workshop started with a discussion of the broader area of metric Diophantine approximation: theorems of Khinchine and Jarník, Ubiquity and Mass Transference, the notions of Dirichlet improvable and singular points, dynamical aspects of Diophantine approximation and the landmark results of Kleinbock and Margulis. It then continued with a detailed account of the main ideas which led to the recent achievements described above. There also has been a burst in the development of new techniques, namely variants of Schmidt's game and Generalised Cantor sets constructions, which were also studied in some depth.

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Arbeitsgemeinschaft: Diophantine Approximation, Fractal Geometry and Dynamics

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Abstracts

1-Dimensional Diophantine Approximation

Agamemnon Zafeiropoulos

In this talk we present some basic results in Diophantine approximation in the real line. The most fundamental is Dirichlet's theorem:

Theorem 1. *(Dirichlet): Let $x \in \mathbb{R}$, $Q \in \mathbb{N}$. There exist $p \in \mathbb{Z}$, $1 \le q \le Q$ such that*

$$\left| x - \frac{p}{q} \right| \le \frac{1}{qQ}.$$

As an immediate corollary to Dirichlet's theorem we get that for all irrationals x there exist infinitely many rationals p/q such that $|x - p/q| < 1/q^2$. In other words we have a rate of approximation satisfied for all reals and it makes sense to ask if this is the best possible for all reals. We define the set of badly approximable numbers to be the set of irrationals for which this bound is the best possible, i.e.

$$\mathbf{Bad} = \{x \in (0,1) : \text{there exists } c > 0 \text{ such that } \; q\|qx\| \ge c \text{ for all } q = 1, 2, \dots\},$$

where $\|\cdot\|$ denotes the distance to the integers. We try to estimate the size of this set in terms of the Lebesgue measure following a different way than the original proof by Khinchine. First we present Khinchine's theorem:

Theorem 2. *Let $\psi : \mathbb{N} \to (0, \infty)$ be any monotonic function. Consider the set*

$$W(\psi) = \left\{ x \in (0,1) : \left| x - \frac{p}{q} \right| \le \frac{\psi(q)}{q} \;\; \textit{for infinitely many} \;\; p/q \in \mathbb{Q} \right\}.$$

Then

$$|W(\psi)| = \begin{cases} 0, & \textit{if } \sum_{q=1}^{\infty} \psi(q) < \infty \\ 1, & \textit{if } \sum_{q=1}^{\infty} \psi(q) = \infty. \end{cases}$$

This gives the Lebesgue measure of any set of irrationals satisfying a specific rate of approximation. For the proof of Khinchine's theorem we introduce the notion of ubiquitous systems.

Let $I_0 \subseteq \mathbb{R}$ be an interval, $\mathcal{R} = (R_\alpha)_{\alpha \in \mathcal{J}}$ be a countable family of points, called the resonant points, $\beta : \mathcal{J} \to (0, \infty)$ be a function which assigns a weight β_α to each resonant point R_α, $u = (u_n)_{n=1}^{\infty}$ be a positive and increasing sequence with $u_n \to \infty$, $\rho : (0, \infty) \to (0, \infty)$ be a function with $\lim_{t \to +\infty} \rho(t) = 0$, called the ubiquitous function. Assume that for all $n = 1, 2, \dots$ the set $J^u(n) = \{\alpha \in \mathcal{J} : \beta_\alpha \le u_n\}$ is finite.

We say that the system $(\mathcal{R}, \beta)$ is locally ubiquitous in I_0 relative to (ρ, u) if there exists an absolute constant $\kappa > 0$ such that for any subinterval $I \subseteq I_0$ the set $\Delta^u(\rho, n) = \bigcup_{\alpha \in J^u(n)} B(R_\alpha, \rho(u_n))$ satisfies

$$|\Delta^u(\rho, n) \cap I| \ge \kappa |I|.$$

The main theorem regarding the apparatus of ubiquity is the following:

Theorem 3. *Suppose* $(\mathcal{R}, \beta)$ *is locally ubiquitous in* I_0 *relative to* (ρ, u) *and* $\Psi : (0, \infty) \to (0, \infty)$ *is a function such that* $\Psi(u_{n+1}) \leq \lambda \Psi(u_n), n \geq n_0$ *for some constant* $0 < \lambda < 1$. *Consider the set*

$$\Lambda(\mathcal{R}, \beta, \Psi) = \{x \in I_0 : |x - R_\alpha| < \Psi(\beta_\alpha) \text{ for infinitely many } \alpha \in \mathcal{J}\}.$$

Then $|\Lambda(\mathcal{R}, \beta, \Psi)| = |I_0|$, *if* $\sum_{n=1}^{\infty} \frac{\Psi(u_n)}{\rho(u_n)} = +\infty$.

This is the theorem we use to prove Khinchine's theorem, after showing that the rationals are ubiquitous in the unit interval. In turn Khinchine's theorem implies that $|\,\mathbf{Bad}\,| = 1$; it suffices to observe that the complement of **Bad** contains the set $W(\psi_0)$, where $\psi_0(q) = \frac{1}{q \log q}, q = 2, 3, \ldots$

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Jarník, Besicovitch and Mass Transference

Demi Allen

In Diophantine Approximation sets of interest often satisfy elegant "zero-one" laws, a famous example being Khintchine's Theorem (discussed in the talk *Background: one-dimensional Diophantine approximation*). While these zero-one laws often provide simple criteria for determining whether the Lebesgue measure of a set is zero or one they do have a drawback. Such zero-one laws involve "exceptional" sets of Lebesgue measure zero and provide no further information which allows us to distinguish these sets even though, intuitively, one might not expect the sizes of such sets to be the same.

With this motivation in mind, we recall the definitions of Hausdorff measure and Hausdorff dimension (see, for example, [4]) and explain why we might be interested in using these to study sets of interest in Diophantine approximation. We then mention some relevant results of Jarník and Besicovitch (see, for example, [1, 2] and references therein).

Recall that given a function $\psi : \mathbb{N} \to \mathbb{R}^+$ the set of ψ-approximable numbers in the unit interval, $\mathbb{I}$, is

$$\mathcal{A}(\psi) = \left\{ x \in \mathbb{I} : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

For $\tau > 1$, we will also use the notation $\mathcal{A}(\tau) = \mathcal{A}(q \to q^{-\tau})$.

Jarník and Besicovitch both independently proved:

Jarník-Besicovitch Theorem. *Let* $\tau > 1$. *Then*

$$\dim_H(\mathcal{A}(\tau)) = \frac{2}{\tau + 1}.$$

For $\tau > 1$ all we can infer from Khintchine's Theorem is that $|\mathcal{A}(\tau)| = 0$ whereas the Jarník-Besicovitch Theorem allows us to further distinguish these sets.

In a further study, Jarník actually proved the following more general result which can be viewed as the Hausdorff measure analogue of Khintchine's Theorem.

Jarník's Theorem. *For* $s \in (0, 1)$ *and* $\psi : \mathbb{N} \to \mathbb{R}^+$,

$$\mathcal{H}^s(\mathcal{A}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi(q)^s < \infty, \\ \\ \infty & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi(q)^s = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

We show that, somewhat surprisingly, it actually turns out that Khintchine's Theorem implies (the seemingly more general) Jarník's Theorem.

That this is the case is a consequence of the Mass Transference Principle [3] due to Beresnevich and Velani - a remarkable result which allows us to transfer Lebesgue measure statements for lim sup sets arising from a sequence of balls in $\mathbb{R}^k$ to Hausdorff measure statements.

Before we state this result we recall that if E_i is a sequence of sets

$$\limsup_{i\to\infty} E_i = \{x : x \in E_i \text{ for infinitely many } i\} = \bigcap_{N=1}^{\infty} \bigcup_{i \geq N} E_i.$$

Also, if $B = B(x, r)$ is a ball in $\mathbb{R}^k$ and $s > 0$ let

$$B^s = B(x, r^{\frac{s}{k}}).$$

We then have the following:

Mass Transference Principle (Beresnevich, Velani [3] - 2006)**.** *Let* $\{B_i\}_{i\in\mathbb{N}}$ *be a sequence of balls in* $\mathbb{R}^k$ *with* $r(B_i) \to 0$ *as* $i \to \infty$, *let* $s > 0$, *and let* Ω *be a ball in* $\mathbb{R}^k$. *Suppose that*

$$|\Omega \cap \limsup_{i\to\infty} B_i^s| = |\Omega|.$$

Then, for any ball B *in* Ω,

$$\mathcal{H}^s\big(B \cap \limsup_{i\to\infty} B_i\big) = \mathcal{H}^s(B)\ .$$

Remark. The above statement is a simplified version of the statement proved in [3]. For simplicity, here we restrict ourselves to using Hausdorff s-measure whereas a more general statement in terms of Hausdorff f-measures is proved in [3].

Two, perhaps unexpected, consequences of the Mass Transference Principle are:

Corollary 1. *Khintchine's Theorem implies Jarník's Theorem.*

Corollary 2. *Dirichlet's Theorem implies the Jarník-Besicovitch Theorem.*

In addition to some general discussion of the Mass Transference Principle we show how it may be used to obtain Corollaries 1 and 2. Finally, we conclude by giving an outline of the proof of the Mass Transference Principle.

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Bad and Dirichlet in Higher Dimensions

Anne-Maria Ernvall-Hytönen

The aim of the talk was to give an overview of the basic concepts in the theory of badly approximable points in higher dimensions. The main sources are the books by Cassels [4] and Schmidt [6], and the expository paper [3] by Beresnevich, Ramírez and Velani.

Minkowski's theorem for systems of linear forms is a crucial tool in proving the basic results in the higher dimensional Dirichlet theory. Minkowski's theorem is the following:

Theorem 1. *Let $\beta_{i,j} \in \mathbb{R}$, where $1 \leq i, j \leq k$, and let $C_1, \dots, C_k > 0$. If*

$$|\det(\beta_{i,j})_{1\leq i,j\leq k}| \leq \prod_{i=1}^{k} C_i,$$

then there exists a non-zero integer point $\overline{x} = (x_1, \dots, x_k)$ such that

$$\begin{cases} |x_1\beta_{i,1} + \cdots + x_k\beta_{i,k}| < C_i & (1 \leq i \leq k-1) \\ |x_1\beta_{k,1} + \cdots + x_k\beta_{k,k}| \leq C_k \end{cases}$$

The proof of this relies on the Minkowski's convex body theorem stating that if B is a convex subset of $\mathbb{R}^n$, which is symmetric about the origin (i.e., if $x \in B$ then $-x \in B$), and if the volume of B is greater than 2^n (or equal to 2^n in case B is compact), then B contains a non-zero integer point.

Assume that $0 < i_1, \dots i_n < 1$ satisfy $\sum_{j=1}^{n} i_j = 1$, and let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Applying Minkowski's linear forms theorem to the forms with coefficients $\beta_{j,j} = -1$ when $1 \leq j \leq n$. $\beta_{n+1,n+1} = 1$ and $\beta_{j,n+1} = \alpha_j$, and with parameters $C_j = N^{-i_j}$ when $1 \leq j \leq n$ and $C_{n+1} = N$, we obtain that there are integers $(p_1, \dots p_n, q)$ such that the inequalities $|q\alpha_j - p_j| < N^{-i_j}$ for $1 \leq j \leq n$ and $|q| \leq N$ are satisfied. Hence, the points $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be simultaneously approximated: For any $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $N \in \mathbb{N}$, there exists $q \in \mathbb{Z}$ such that

$$\max\left\{||\alpha_1 q||^{1/i_1}, \ldots, ||\alpha_n q||^{1/i_n}\right\} < N^{-1}$$

and $1 \leq q \leq N$. In particular, there are infinitely many integers $q > 0$ such that

$$\max\left\{||\alpha_1 q||^{1/i_1}, \ldots, ||\alpha_n q||^{1/i_n}\right\} < q^{-1}$$

The question that arises is: Can we replace the q^{-1} on the right side of the inequality by something smaller, say εq^{-1}? This leads to the definition of badly approximable points:

Definition 2. *Assume* $0 < i_1, ..., i_n < 1$ *and* $\sum_{j=1}^n i_j = 1$. *The set* $\mathbf{Bad}(i_1, ..., i_n)$ *consists of badly approximable points, namely, points* $(\alpha_1, ..., \alpha_n)$ *such that there exists a constant* $c(\alpha_1, ..., \alpha_n)$ *so that*

$$\max\left\{||q\alpha_1||^{1/i_1}, \ldots, ||q\alpha_n||^{1/i_n}\right\} > c(\alpha_1, \ldots, \alpha_n)q^{-1}.$$

for all $q \in \mathbb{N}$.

For all n-tuples $(i_1, ..., i_n)$ with $0 < i_1, ..., i_n$ and $\sum_{j=1}^n i_j$, the set $\mathbf{Bad}(i_1, ..., i_n)$ is non-empty.

The claim

$$\mathbf{Bad}(i_1, j_1) \cap \mathbf{Bad}(i_2, j_2) \neq \emptyset$$

whenever $0 < i_1, i_2, j_1, j_2 < 1$ and $i_1 + j_1 = i_2 + j_2 = 1$ is known as Schmidt's conjecture, although originally Schmidt conjectured the result for $i_1 = j_2 = \frac{1}{3}$ and $i_2 = j_1 = \frac{2}{3}$. Even this weaker formulation resisted attacks until Badziahin, Pollington and Velani [1] proved a much stronger statement:

Theorem 3. *Let* $(i_1, j_i), \ldots, (i_d, j_d)$ *be a finite number of pairs satisfying the conditions* $0 < i_t, j_t < 1$ *and* $i_t + j_t = 1$ *for* $1 \leq t \leq d$. *Then*

$$\dim\left(\cap_{t=1}^d \mathbf{Bad}(i_t, j_t)\right) = 2.$$

If Schmidt's conjecture had not been true, Littlewood's conjecture, which states that

$$\lim\inf_{q \to \infty} q||qx||||qy|| = 0$$

for any $(x, y) \in \mathbb{R}^2$, would have been true. Indeed, if $(x, y) \in \mathbb{R}^2$ and $0 < i, j < 1$, $i + j = 1$ satisfy $(x, y) \notin \mathbf{Bad}(i, j)$, then for any constant c there exists an integer q such that

$$\max\left(||qx||^{1/i}, ||qy||^{1/j}\right) < cq^{-1}.$$

By multiplying one gets that $\liminf_{q\to\infty} q||qx||||qy|| = 0$. However, Littlewood's conjecture is still wide open.

Let us now briefly look at linear forms. Using Minkowski's theorem for systems of linear forms with $\beta_{1,j} = \alpha_j$ when $1 \leq j \leq n$, $\beta_{1,n+1} = -1$ and $\beta_{j+1,j} = 1$ for $1 \leq j \leq n$, and with $C_1 = N^{-n}$ and $C_j = N$ when $2 \leq j \leq n+1$ we obtain that for any $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and for any real $N > 1$, there exist $q_1, \ldots, q_n, p \in \mathbb{Z}$ such that

$$|q_1\alpha_1 + \cdots + q_n\alpha_n - p| < N^{-n}$$

and $1 \leq \max_{1\leq j\leq n}[q_j| \leq N$. More generally, for the system of linear forms

$$\mathcal{L}(\overline{x}) = (L_1(\overline{x}), \ldots, L_n(\overline{x}))$$

consisting of linear forms

$$L_i(\overline{x}) = \alpha_{i1}x_1 + \cdots \alpha_{im}x_m$$

with $\overline{x} = (x_1, \ldots, x_m)$, there exists an integer point $(\overline{x}, \overline{y}) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ with $\overline{x} \neq \overline{0}$ such that

$$\max|\mathcal{L}(\overline{x}) - \overline{y}|^n < C_{m,n}\frac{1}{\max|x_i|}$$

where

$$C_{m,n} = \frac{m^m n^n}{(m+n)^{m+n}} \cdot \frac{(m+n)!}{m!n!}.$$

The system of linear forms $\mathcal{L}(\overline{x}) = (L_1(\overline{x}), \ldots L_n(\overline{x}))$ is called badly approximable if the constant $C_{m,n}$ above cannot be replaced by an arbitrarily small constant, namely, if there is a constant $\gamma(L_1, \ldots, L_n)$ such that

$$(\max|x_i|^m)\,(\max|\mathcal{L}(\overline{x}) - \overline{y}|^n) > \gamma$$

for every integer point $(\overline{x}, \overline{y})$, $\overline{x} \neq \overline{0}$. The system of linear forms corresponding to a matrix is badly approximable if and only if so is the system of linear forms coming from the transpose of the matrix [5]. An elegant special case of this principle is the following theorem [2]:

Theorem 4. *Let the numbers $i_1, \ldots, i_n$ satisfy the conditions $0 < i_1, \ldots, i_n < 1$ and $\sum_{j=1}^n i_j$, and let $\overline{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then the following statements are equivalent:*

(1) $\overline{y} \in \mathbf{Bad}(i_1, \ldots, i_n)$

(2) There exists $c > 0$such that for any $Q \geq 1$, the only integer solution $(q, p_1, \ldots, p_n)$ to the system

$$|q| < Q, \quad |qy_j - p_j| < (cQ)^{i_j} \quad (1 \leq j \leq n)$$

is $q = p_1 = \cdots = p_n = 0$.

(3) There exists $c > 0$ such that for any $H \geq 1$ the only integer solution $(a_0, a_1, \ldots, a_n)$ to the system

$$|a_0 + a_1y_1 + \cdots + a_ny_n| < cH^{-1}, \quad |a_j| < H^{i_j} \quad (1 \leq j \leq n)$$

is $a_0 = a_1 = \cdots = a_n = 0$.

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Diophantine approximation and dynamics

Nattalie Tamam

The primary goal of this talk was to present the correspondence between approximation properties and the behavior of certain orbits in the space of unimodular lattices. We introduced the definition of unimodular lattices. Then presented examples of such correspondence.

1. Lattices and Mahler's Compactness Criterion

Definition 1. $\Lambda \subset \mathbb{R}^n$ is a **lattice** if we can find linearly independent vectors $v_1, \dots v_n \in \mathbb{R}^n$ such that

$$\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n.$$

The set $\{v_1, \dots, v_n\}$ is the **generating set** of the lattice Λ. Lattices with covolume (the n-dimensional volume of the parallelepiped created by the generating set) 1 are called **unimodular** and $\mathcal{L}_n$ denote the set of all **unimodular lattices** in $\mathbb{R}^n$.

The group $\mathrm{SL}_n(\mathbb{R})$ acts on $\mathcal{L}_n$ by

$$g(\mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n) = \mathbb{Z}gv_1 \oplus \cdots \oplus \mathbb{Z}gv_n \quad \text{for } g \in \mathrm{SL}_n(\mathbb{R}).$$

This group action is transitive, in particular any lattice in $\mathcal{L}_n$ is of the form $g\mathbb{Z}^n$ for some $g \in \mathrm{SL}_n(\mathbb{R})$. Since $\mathrm{SL}_n(\mathbb{Z})$ is the stabilizer of $\mathbb{Z}^n$, we get

$$\mathcal{L}_n = \mathrm{SL}_n(\mathbb{R})\,\mathbb{Z}^n \cong \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}) \tag{1}$$

In order to discuss the behavior of orbits in $\mathcal{L}_n$ we need a topology. We may use the quotient topology which arise from (1) and get the following.

Let $|\cdot|$ be the sup-norm and denote $\delta(\Lambda) = \inf_{0 \neq v \in \Lambda} |v|$.

Theorem 2 (Mahler's compactness, [C, p. 137])**.** *A subset $X \subset \mathcal{L}_n$ is precompact if and only if there exists $r > 0$ such that all $\Lambda \in X$ satisfy*

$$\delta(\Lambda) \geq r.$$

2. The Correspondence

2.1. Dani's Correspondence.

Definition 3. Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$, $0 < \epsilon < 1$.

- A is **badly approximable** if there exists a constant $c > 0$ such that for every $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$
$$|A\mathbf{q} + \mathbf{p}|^m \cdot |\mathbf{q}|^n > c.$$
- A is **ϵ-Dirichlet improvable** if there exists Q_0 such that for all $Q > Q_0$ there exist $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n$ such that
$$0 < |\mathbf{q}|^n \leq \epsilon Q \text{ and } |\mathbf{p}+A\mathbf{q}|^m < \epsilon Q^{-1}.$$
- A is **singular** if it is ϵ-Dirichlet improvable for all $0 < \epsilon < 1$.

For $A \in \mathrm{M}_{m,n}(\mathbb{R})$ let
$$\Lambda_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \mathbb{Z}^{m+n} \quad \text{and} \quad g_t = \operatorname{diag}\left(e^{\frac{t}{m}}, \ldots, e^{\frac{t}{m}}, e^{-\frac{t}{n}}, \ldots, e^{-\frac{t}{n}}\right).$$

Theorem 4 ([D])**.** *Let $A \in M_{m,n}(\mathbb{R})$.*

- *A is badly approximable if and only if the trajectory $\{g_t\Lambda_A : t \geq 0\}$ is bounded.*
- *A is singular if and only if the trajectory $\{g_t\Lambda_A : t \geq 0\}$ is divergent.*

Let
$$K_\epsilon = \{g \in \mathcal{L}_{m+n} : \delta(g) \geq \epsilon\}.$$
Applying the proof of theorem 4, one can deduce:

Theorem 5. *$A \in M_{m,n}(\mathbb{R})$ is ϵ-Dirichlet improvable if and only if there exists $t_0 > 0$ such that $\{g_t\Lambda_A : t \geq 0\} \cap K_\epsilon = \emptyset$.*

2.2. Kleinbock's Generalization.

Let $\mathbf{r} \in \mathbb{R}_+^m$, $\mathbf{s} \in \mathbb{R}_+^n$ such that $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j = 1$, the 1-parameter subgroup of $\mathrm{SL}_{m+n}(\mathbb{R})$
$$g_t = \operatorname{diag}\left(e^{r_1 t}, \ldots, e^{r_m t}, e^{-s_1 t}, \ldots, e^{-s_n t}\right)$$
What can we say about the elements $L \in \mathrm{SL}_n(\mathbb{R})$ such that $\{g_t L\mathbb{Z}^{m+n} : t \in \mathbb{R}\}$ is bounded?

Badly approximable is a notion which is inspired by Dirichlet's theorem:

Corollary 6. *For any $A \in M_{m,n}(\mathbb{R})$, there exist infinitely many $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that*
$$|\mathbf{p} + A\mathbf{q}|^m |\mathbf{q}|^n < 1.$$

More generally, one can use the following corollary of the **Minkowski's convex body theorem**:

Corollary 7. *For any $L = (L_1, \ldots, L_{m+n}) \in SL_{m+n}(\mathbb{R})$, there exist infinitely many $\mathbf{x} \in \mathbb{Z}^{m+n} \setminus \{0\}$ such that*
$$\max\left(|L_1\mathbf{x}|^{\frac{1}{r_1}}, \ldots, |L_m\mathbf{x}|^{\frac{1}{r_m}}\right) \cdot \max\left(|L_{m+1}\mathbf{x}|^{\frac{1}{s_1}}, \ldots, |L_{m+n}\mathbf{x}|^{\frac{1}{s_n}}\right) < 1. \tag{2}$$

The terms in the left hand side of 2 are a generalization of a norm called **quasinorm** defined as follows. For a k-tuple $\mathbf{w} = (w_1, \ldots, w_k)$, $w_i > 0$, $\sum_{i=1}^{k} w_i = 1$ define the **w-quasinorm** $|\cdot|_{\mathbf{w}}$ on $\mathbb{R}^k$ by$|\mathbf{x}|_{\mathbf{w}} = \max_{1\le i\le k} |x_i|^{\frac{1}{w_i}}$.

Definition 8 ([K]). A matrix $L \in G$ is called $(\mathbf{r},\mathbf{s})$**-loose** if

$$\inf_{\mathbf{x}\in\mathbb{Z}^{m+n}\setminus\{0\}} |(L_1\mathbf{x}, \ldots, L_m\mathbf{x})|_{\mathbf{r}} \, |(L_{m+1}\mathbf{x}, \ldots, L_{m+n}\mathbf{x})|_{\mathbf{s}} > 0.$$

Theorem 9 ([K]). *$L \in SL_{m+n}(\mathbb{R})$ is $(\mathbf{r},\mathbf{s})$-loose if and only if the trajectory $\{g_t L\mathbb{Z}^{m+n} : t \in \mathbb{R}\}$ is bounded.*

2.3. **Littlewood's Conjecture.** for $x \in \mathbb{R}$ let $\|x\|$ denote the distance from x to the nearest integer. **The Littlewood's conjecture** concerns simultaneous approximation of two numbers x, y by rationals. It asserts that:

$$\liminf_{n\ge 1} n \cdot \|nx\| \cdot \|ny\| = 0 \tag{3}$$

for all x, y. I.e. x, y may be simultaneously approximated, moderately well, by rationals with the same denominator.

For any $(x, y) \in \mathbb{R}^2$, define (A^+ is a semigroup)

$$\Lambda_{(x,y)} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^3, \qquad A^+ = \left\{ \begin{pmatrix} e^r & & \\ & e^s & \\ & & e^{-r-s} \end{pmatrix} : \text{ for } r, s \ge 0 \right\}.$$

Theorem 10. *The tuple (x, y) satisfies (3) if and only if the orbit $A^+\Lambda_{(x,y)}$ is unbounded.*

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Quantitative nondivergence in the space of lattices

Arijit Ganguly

In the landmark work ([KM]), D. Y. Kleinbock and G. A. Margulis opened a new horizon to metric Diophantine approximation while proving a longstanding conjecture of V. Sprindžuk. Their approach is based on a correspondence between approximation properties of vectors in $\mathbb{R}^n$ and orbit properties of certain flows on the homogeneous space $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SL}(n+1,\mathbb{Z})$. This approach also proves several related hypotheses of Baker and Sprindžuk formulated in 1970s. The core of the proof is a theorem, providing a quantitative estimate, which generalizes and sharpens earlier results on non-divergence of unipotent flows on the space of lattices. That "Quantitative nondivergence estimate" is the topic of discussion in this talk.

We begin with, to prepare the required background, the notion of (C, α)-good functions, exterior algebra on $\mathbb{R}^n$ and how the 'norm' of a discrete subgroup of $\mathbb{R}^n$ is defined using exterior products. Then we state the main quantitative nondivergence result.

Afterwards I have given a sketch, mentioning each major steps, how the just discussed nondivergence estimate applies to resolve the conjecture of V. Sprindžuk. This completes the discussion.

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Schmidt (α, β) Games

Yotam Smilansky

Schmidt (α, β) games are used as a technique to show that sets which may be small in the sense of measure or category, behave in fact like very large sets in many other senses, such as Hausdorff dimension and rigidness under countable intersection.

1. The Game

Given $0 < \alpha, \beta < 1$, Schmidt's (α, β) game is played by two players, *Alice* and *Bob*. They take turns and specify a nested sequence of closed balls in a complete metric space (X, d). If $B \subset X$ is a ball, we denote by $\rho(B)$ the radius of the B, and by $A \overset{c}{\subset} B$ a subset of B for which A is a ball of radius $\rho(A) = c \cdot \rho(B)$.

1.1. **Playing the game.** We can now play the game: Fix a target set $S \subset X$.

(1) First, *Bob* chooses a ball B_0 of radius $\rho(B_0) = r_0$.
(2) On his nth turn, *Alice* chooses a ball $A_n \overset{\alpha}{\subset} B_{n-1}$, so $\rho(A_n) = r_0 \alpha (\alpha\beta)^{n-1}$.
(3) On her nth turn, *Bob* chooses a ball $B_n \overset{\beta}{\subset} A_n$, so $\rho(B_n) = r_0 (\alpha\beta)^n$.
(4) This produces a decreasing sequence of closed balls

$$B_0 \overset{\alpha}{\supset} A_1 \overset{\beta}{\supset} B_1 \overset{\alpha}{\supset} A_2 \overset{\beta}{\supset} ...$$

and since the space is complete there exist a single point

$$x_\infty \in \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=0}^{\infty} B_n.$$

(5) If $x_\infty \in S$ then *Alice* wins, otherwise *Bob* wins.

2. Strategy and Winning sets

A strategy for *Alice* is a guide for choosing his nth move, given the moves already played by *Bob*. More formally, it is a sequence of functions f_n such that for any $n \in \mathbb{N}$ and any legal choices made by *Bob* until the nth turn $B_0, ..., B_{n-1}$, the function f_n defines a ball $A_n \overset{\alpha}{\subset} B_{n-1}$. So if *Alice* is playing according to the strategy he must choose $A_n = f_n (B_0, ..., B_{n-1})$. A winning strategy for *Alice* is a strategy which guarantees a win for *Alice*, that is, no matter what moves *Bob* plays, if *Alice* uses the strategy then $x_\infty \in S$.

(1) A set $S \subset X$ is called (α, β)-winning if there is a winning strategy for *Alice* in the (α, β) game with a target set S.
(2) A set $S \subset X$ is called α-winning if for every $0 < \beta < 1$ the set S is (α, β)-winning.
(3) A set $S \subset X$ is called winning if there exists $0 < \alpha < 1$ for which S is α-winning.

3. Elementary Properties of (α, β)-winning and α-winning Sets

The following properties are easily proven using elementary methods and translating strategies to show that a set which is winning in one game is in fact winning in various other games. They tell us what can be expected of a winning set S, and about the possible patterns we can expect to see in the (α, β) space for a given set S.

(1) If S is (α, β)-*winning* for some α, β then S is dense in X.
(2) If S is (α, β)-*winning* for α, β such that $2\alpha \geq 1 + \alpha\beta$ (α is big, β is small) then $S = X$ (only one winning set).
(3) The only α-*winning* set with $\alpha > \frac{1}{2}$ is X itself.
(4) For α, β such that $2\beta \geq 1 + \alpha\beta$ (β is big, α is small) , every dense set S is (α, β)-*winning* ("many" winning sets).
(5) If S is (α, β)-*winning* for some α, β and $\alpha\beta = \alpha'\beta'$ with $\alpha' \leq \alpha$ then S is also (α', β')-*winning.*
(6) For $0 < \alpha' < \alpha < 1$ every α-*winning* set is α'-*winning.*
(7) If S is (α, β)-*winning* then it is $\left(\alpha (\beta\alpha)^k, \beta\right)$-*winning* for any $k \in \mathbb{N}$.
(8) If S is (α, β)-*winning* and $\alpha'\beta' = (\alpha\beta)^k$ for $k \in \mathbb{N}$ and $\beta' \geq \beta$ then S is (α', β')-*winning.*
(9) If S is (α, β)-*winning* for every $0 < \beta < \varepsilon$ then it is α-*winning.*

4. The Main Properties

The following results tell us something about the size of winning sets:

Theorem 1. *The intersection of countably many $\alpha-$winning sets is $\alpha-$winning.*

The idea is to play simultaneous chess, that is use the following decomposition of the natural numbers to a countable union of independent arithmetical progressions

$$\mathbb{N} \;=\; \bigcup_{k=1} P_k = \{1,3,5,7,9,...\} \cup \{2,6,10,14,...\} \cup \{4,12,20,28,...\} \cup ...$$

and for any given β to define a strategy for *Alice* and the target set $\bigcap S_n$ using his winning strategies on S_k with β_k, which we are free to choose since S_k are all α-winning.

Theorem 2. *Let (X, d_X), (Y, d_Y) be two complete metric spaces and let $f : X \to Y$ be a c-biLipschitz mapping. Let $S \subset X$ be α-winning, then $f(S)$ is $\frac{\alpha}{c^2}$-winning.*

In this case we translate our strategies back and forth, using the biLipschitz property to show that balls of the required radii can be chosen at each turn, for example if *Bob* picks $\tilde{B_0} \subset Y$ of radius $\tilde{r_0}$, then $f^{-1}\left(\tilde{B_0}\right) \subset X$ contains a ball B_0 of radius $r_0 = \frac{\tilde{r_0}}{c}$ and *Alice* can choose a ball $A_1 \subset B_0$ of radius $\frac{\tilde{r_0}}{c}\alpha$ in the (α, β) game on X according to his winning strategy.

Theorem 3. *Assume S is (α, β)-winning and t, m are integers such that given a winning strategy and a ball B_n, after t steps in the game, there exist m different choices for B_{n+t} of pairwise disjoint interiors. then*

$$\dim S \geq \frac{1}{t}\frac{\log m}{|\log \alpha\beta|}.$$

If S is a winning set then S is of full dimension.

We use a mass distribution argument and the definition of the Hausdorff dimension.

5. An Example of a Winning Set - $\mathbf{Bad}_d$ is Winning

We show a very interesting example of a winning set, which is the set of badly approximable vectors in $\mathbb{R}^n$. More precisely we prove:

Theorem 4. $\mathbf{Bad}_d$ *is α-winning for all $0 < \alpha < \frac{1}{2}$ in $\mathbb{R}^d$, where*

$$\mathbf{Bad}_d = \left\{ \mathbf{x} \in \mathbb{R}^d \;:\; \exists c > 0 : \forall \mathbf{p} \in \mathbb{Z}^d, q \in \mathbb{N} : \left| x - \frac{\mathbf{p}}{q} \right| \geq \frac{c}{q^{1+\frac{1}{d}}} \right\}.$$

This is done using the structure of the rational vectors of bounded denominators, which is evident from the so called simplex lemma which is discussed in another talk. The idea is to show that *Alice* can play in such a way that forces *Bob* to play such that for every $n \in \mathbb{N}$, $\mathbf{x} \in B_n$ and $\frac{\mathbf{p}}{q}$ with $1 \leq q < R^n$

$$\left| \mathbf{x} - \frac{\mathbf{p}}{q} \right| \geq \frac{c}{q^{1+\frac{1}{d}}}$$

for suitable c, R. Then, if $\mathbf{x}_\infty \in \cap B_n$, the above holds for every positive integer q and $\mathbf{x}_\infty$ is badly approximable. A very similar proof shows not only that $\mathbf{Bad}_d$ is α-winning for all $\alpha < \frac{1}{2}$, but also its image under any non singular linear transformation is α-winning.

$\mathbf{Bad}(\mathbf{r})$ has full Hausdorff dimension

Ofir David

Dirichlet's theorem implies that every $x \in \mathbb{R}$ has a lot of "good" rational approximations, namely there are infinitely many solutions to

$$\left|x - \frac{p}{q}\right| \leq \frac{c}{q^2},\ p, q \in \mathbb{Z},\ q \neq 0$$

where $c = 1$. A similar result holds in higher dimension - given $\mathbf{r} = (r_1, \ldots, r_d)$ such that $r_i \geq 0$ for all $1 \leq i \leq d$ and $\sum_1^d r_i = 1$, for any vector $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ there are infinitely many solutions to

$$\left|x_i - \frac{p_i}{q}\right| \leq \frac{c}{q^{1+r_i}}, \quad p_i, q \in \mathbb{Z},\ q \neq 0$$

where $c = 1$. We say that a vector $\mathbf{x}$ is *well approximable* with respect to $\mathbf{r}$ if we can take $c > 0$ to be as small as we want and still have a solution to the inequalities above. The vectors for which this is impossible are called *badly approximable vectors* and we denote them by $\mathbf{Bad}(\mathbf{r})$.

A generalization of Khintchine's theorem shows that $\mathbf{Bad}(\mathbf{r})$ has zero Lebesgue measure. The main goal of this lecture is to show that while it has zero measure, the set $\mathbf{Bad}(\mathbf{r})$ is large in the sense that it has full Hausdorff dimension. This proof is based on the paper [3] by Pollington and Velani.

In order to show full Hausdorff dimension, we construct Cantor-like sets contained in $\mathbf{Bad}(\mathbf{r})$ which have large Hausdorff dimension. These sets are constructed recursively such that the condition $\exists i\ \left|x_i - \frac{p_i}{q}\right| > \frac{c}{q^{1+r_i}}$ of being badly approximable, i.e. $\mathbf{x}$ is not close to rational vectors, is satisfied for all rational vectors $\frac{\mathbf{p}}{q}$ with bounded denominator q, and the bound increases in each step of the recursion. The main idea in this construction is the Simplex Lemma which states that for a fixed R, the rational points with denominator bounded by R in any small enough d-dimensional box (as a function of R), lie on a $d-1$-dimensional hyperplane (and hence most of the vectors in that box will be "far" from these rational points).

Once we have this construction we will prove the following generalization:

(1) Inhomogeneous settings (Beresnevich and Velani) [1]: Given $\bar{r} \in \mathbb{R}^d$ as above and a fixed vector $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_d) \in \mathbb{R}^d$, we denote by $\mathbf{Bad}(\boldsymbol{\lambda}, \mathbf{r})$ the vectors $\mathbf{x} \in \mathbb{R}^d$ for which there exists a constant $c = c(\mathbf{x})$ such that

$$\left|x_i - \frac{p_i + \lambda_i}{q}\right| > \frac{c}{q^{1+r_i}}, \quad \forall p_i, q \in \mathbb{Z},\ q \neq 0.$$

Similar to the construction in the homogeneous settings, we show that $\mathbf{Bad}\,(\boldsymbol{\lambda}, \mathbf{r})$ had full Hausdorff dimension.

(2) $\mathbf{Bad}\,(\mathbf{r})$ is almost winning (Kleinbock and Weiss) [2]: While $\mathbf{Bad}\,(\mathbf{r})$ is winning in the one dimensional case, the standard proof generalizes in higher dimensions only to the symmetric case, namely that $\mathbf{Bad}(\frac{1}{d}, ..., \frac{1}{d})$ is winning. On the other hand, once we understand the reason why it has full Hausdorff dimension, we can define a modified Schmidt game under which $\mathbf{Bad}(\mathbf{r})$ is winning. These modified games have many of the properties of the standard Schmidt games, and in particular they imply the full Hausdorff dimension property.

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Bad(*i*,*j*) wins on a fibre: Rooted trees and the strategy

Faustin Adiceam

Given real numbers $i, j \geq 0$ such that $i + j = 1$, denote by $\mathbf{Bad}(i, j)$ the set of badly approximable vectors in dimension two with respect to the weights (i, j); that is,

$$\mathbf{Bad}(i,j) \;:=\; \left\{(x,y) \in \mathbb{R}^2 \;:\; \inf_{q\geq 1} \max\{q^i||qx||,\, q^j||qy||\} > 0\right\}.$$

Here and in what follows, $||x||$ denotes the distance from a real number x to the set of integers.

In 1983, W. Schmidt [1] formulated the claim that $\mathbf{Bad}\left(\frac{1}{3}, \frac{2}{3}\right) \cap \mathbf{Bad}\left(\frac{2}{3}, \frac{1}{3}\right) \neq \emptyset$. A more general version of this claim has become known as *Schmidt's Conjecture* :

Conjecture (Schmidt's Conjecture, 1983)**.** *For any pair of weights* (i_1, j_1) *and* (i_2, j_2),

$$\boldsymbol{Bad}(i_1, j_1) \cap \boldsymbol{Bad}(i_2, j_2) \neq \emptyset.$$

The motivation behind this conjecture stems from the fact that a counter–example to it would immediately provide a counter–example to the Littlewood conjecture, which states that

$$\liminf_{q\to\infty} q \cdot ||qx|| \cdot ||qy|| \;=\; 0$$

for all $x, y \in \mathbb{R}$.

In 2011, Badziahin, Pollington and Velani [2] established the validity of Schmidt's Conjecture in a stronger form. In order to state it, given weights (i, j) and a real number θ, denote by

$$\mathbf{Bad}(i, j, \theta) \; := \; \{y \in \mathbb{R} \; : \; (\theta, y) \in \mathbf{Bad}(i, j)\}$$

the set of weighted badly approximable vectors lying on the fibre $\{x = \theta\}$.

Theorem 1 (Badziahin, Pollington & Velani, 2011)**.** *Let* $((i_n, j_n))_{n\geq 1}$ *be a sequence of weights. Set*

$$i \; := \; \sup_{n\geq 1} i_n$$

and assume that

$$\liminf_{n\to\infty} \min\{i_n, j_n\} > 0. \tag{1}$$

Then, for any $\theta \in \mathbb{R}$ *such that*

$$\inf_{q\geq 1} q^{1/i} \cdot ||q\theta|| > 0, \tag{2}$$

the set $\cap_{n=1}^{+\infty} \boldsymbol{Bad}(i_n, j_n, \theta)$ *is thick in* $\mathbb{R}$.

By *thick*, we mean that the intersection of the set under consideration with any non–empty open set has full Hausdorff dimension.

The relevance of Assumption (2) in the theorem above is justfied in [2] by noticing that, whenever it does not hold, $\mathbf{Bad}(i, j, \theta) = \emptyset$ (here, one has obviously set $j = 1 - i$). Assumption (1) appears however to be of a purely technical nature and the authors of [2] make the comment that the result should hold without this condition. In 2013, J.An [3] confirmed this fact by proving the following result :

Theorem 2 (An, 2013)**.** *Let* (i, j) *be weights and let* $\theta \in \mathbb{R}$ *satisfying Assumption (2). Then, the set* $\boldsymbol{Bad}(i, j, \theta)$ *is* $1/2$*–winning for Schmidt's game.*

As a winning set is thick and as a countable intersection of winning sets is also winning, a corollary of this theorem is that the Badziahin–Pollington–Velani Theorem indeed holds without Assumption (1), and with replacing Assumption (2) by the weaker necessary condition:

$$\inf_{q\in\mathbb{N}} q^{1/i_n} ||q\theta|| > 0, \; \text{ for every } n \geq 1.$$

The initial approach undertaken by Badziahin, Pollington and Velani did not make use of Schmidt games but made use of a so–called dual formulation of the condition that a vector (x, y) lies in $\mathbf{Bad}(i, j)$. An's strategy, although following key ideas from this first proof, deals directly with the inequality appearing in the definition of the set $\mathbf{Bad}(i, j)$. It relies on a construction of nested intervals, which construction is formulated in terms of rooted trees.

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Bad(s,t) wins on a fiber: Partitioning of intervals

David Simmons

In this talk I sketched the proof of a theorem of Jinpeng An [1]: for all $s, t \geq 0$ with $s + t = 1$ and for all $\theta \in \mathbb{R}$ such that

$$\inf_{q \in \mathbb{N}} q^{1/s} \|q\theta\| > 0,$$

the set

$$\mathbf{Bad}(s,t;\theta) := \left\{ y \in \mathbb{R} : \inf_{q \in \mathbb{N}} \max(q^s \|q\theta\|, q^t \|qy\|) > 0 \right\}$$

is winning for Schmidt's game. The proof involves looking at the quantitative set $\mathbf{Bad}_c(s,t;\theta)$ as the complement of the union of the "obstacles" $\Delta_c(P)$ ($P \in \mathcal{P}$), where

$$\mathcal{P} = \{P = (\tfrac{p}{q}, \tfrac{r}{q}) \in \mathbb{Q}^2 : q^s |q\theta - p| \leq c\}$$
$$\Delta_c(\tfrac{p}{q}, \tfrac{r}{q}) = \{y \in \mathbb{R} : q^t |qy - r| \leq c\}$$

and $c > 0$ is a fixed small number. The question is then how to play Schmidt's game to avoid these obstacles. It is helpful to draw an analogy to the proof of the classical Jarník–Schmidt theorem on the full dimension of the classical set of badly approximable vectors. The classical proof proceeds by grouping obstacles into "windows" and then applying the Simplex Lemma to show that all obstacles in a given window can be avoided simultaneously. In An's setup, it is not immediately obvious how to group the obstacles, nor is it obvious how to show that all obstacles in a given window can be avoided simultaneously.

Rather than describing the proof of An's theorem in detail, I instead gave some motivation for the method of grouping obstacles that An uses in his paper. The motivation takes an "Ansatz" form in which we assume that we have a grouping, and ask what properties it should have. If we assume that we have a grouping, then the most relevant characteristics of any given obstacle are

- its "group size", or the size of the union of the obstacles in the group the obstacle is a part of, and
- its "visibility", or the length threshold below which the obstacle and its group become "visible" for the purposes of Schmidt's game.

It is possible to precisely define these characteristics in terms of the geometry of the situation. If we then assume that we do not have access to a grouping but we have access to these two characteristics, then we can group obstacles according to these two characteristics. If we get back the grouping that we started with, then we count it as a success. It turns out that An's construction does exactly this.

It turns out that many problems in Diophantine approximation yield this same basic structure upon analysis, and in general, the structure always yields a set which is winning for Schmidt's game. There are at least two different ways to show this. Due to time constraints I presented the shorter method of [1], which is essentially as follows: the strategy in Schmidt's game is just to avoid as many groups of obstacles as possible, with each group being "weighted" in proportion to how big it is. Only groups which are visible by the current length threshold are eligible for being avoided. An induction argument shows that the total weight of the groups remains bounded by a uniform constant (relative to the current length threshold) throughout the entire game, which implies that all obstacles are eventually avoided.

The other method, which is the one appearing in [1], is to introduce the concept of "rooted trees" to change Schmidt's game into a combinatorial game. Then the question of whether or not a set is winning can be reduced to the question of whether or not certain infinite rooted trees have infinite intersection. It is possible to prove by induction a lower bound on the cardinality of the intersection at any given finite stage, which implies that the trees have infinite intersection and thus that $\mathbf{Bad}(s,t;\theta)$ is winning. However, this proof is more technical and also does not give an explicit strategy for winning Schmidt's game.

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Playing Schmidt's game on fractals

Luca Marchese

We consider the set $\mathbf{Bad}\left(\frac{1}{d},\dots,\frac{1}{d}\right) \subset \mathbb{R}^d$ of badly approximable vectors in $\mathbb{R}^d$ and we explain Fishman's result in [1], which establishes that any image of such set under an affine non-singular map $f:\mathbb{R}^d \to \mathbb{R}^d$ is winning for the Schmidt's game played on any fractal which is the support of an "absolutely friendly" measure. Friendly measures on $\mathbb{R}^d$ were introduced by Kleinbock, Lindenstrauss and Weiss in [3], and the term absolutely friendly was first used by Pollington and Velani [6]. It follows immediately that the same is true for any countable intersection of such images, by a classical property of winning sets established by Schmidt in [7].

We also explain that a set which is winning on the support of an Ahlfors regular measure intersects the support in a set of full dimension, which was also proved in [1] by Fishman, following the lines of the argument working in the euclidian space, first given by Schmidt in [7].

Moreover we describe a category of measures for which both results apply, namely self-similar measures arising from systems of contracting similitudes which satisfy the open set condition and irreducibility. Ahlfors regularity was established in [2] by Hutchinson assuming the open set condition. Then in [3] Kleinbock, Lindenstrauss and Weiss proved that such measures are absolutely friendly if irreducibility is also satisfied.

Finally, in dimension two, we discuss without proofs similar results for the set $\mathbf{Bad}(i,j)$ of badly approximable vectors with weights. According to a result of Nesharim-Weiss established in [5], any such set intersects the vertical line $\{\alpha\}\times\mathbb{R}$ passing through a badly approximable horizontal coordinate in a set which is winning for a new game introduced by McMullen in [4] and is known as the absolute game. The great flexibility and stability of this second game enables to deduce that, for the same α, any countable intersection of sets $\mathbf{Bad}(i_n,j_n)$ intersects $\{\alpha\}\times\mathbb{K}$ in a set of full dimension, where $\mathbb{K}\subset\mathbb{R}$ is the middle third cantor set.

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Bad(*i*,*j*) is winning

Lifan Guan

The aim of my talk is to give a brief sketch of the colored rooted tree argument An used in the proof of the following theorem:

Theorem 1 ([1]). $\mathbf{Bad}(i,j)$ *is α_0-winning for some $\alpha_0>0$ independent of weight (i,j).*

Basics on trees. We are going to define subtrees of type (I) and type (II) here. To begin, we shall recall some basic notation. A rooted tree is a connected graph $\mathcal{T}$ without cycles and with a distinguished vertex τ_0. We identify a tree $\mathcal{T}$ with its set of vertices. Any vertex $\tau\in\mathcal{T}$ is connected to τ_0 by a unique path, and the length of the path is called the *height* of τ. Then the set $\mathcal{T}$ is decomposed into subsets $\mathcal{T}_n$ indexed by the height. A vertex τ' is called a *successor* of τ if τ lies on the path connecting τ' and τ_0, and the height of τ' equals the height of τ plus one. The set of successors of τ is denoted as $\mathcal{T}_{\mathrm{suc}}(\tau)$. Say a tree $\mathcal{T}$ is N*-regular* if $\#\mathcal{T}_{\mathrm{suc}}(\tau) = N$ for all $\tau \in \mathcal{T}$. Let $D|N$. A D*-colored* N*-regular* tree is a N-regular rooted tree $\mathcal{T}$ equipped with a map $\gamma : \mathcal{T} \to \{1, \dots, D\}$ such that $\#\mathcal{T}_{\mathrm{suc}}(\tau) \cap \gamma^{-1}(i) = N/D$ for all τ and all i. The set $\#\mathcal{T}_{\mathrm{suc}}(\tau) \cap \gamma^{-1}(i)$ is also denoted simply as $\#\mathcal{T}^{(i)}_{\mathrm{suc}}(\tau)$.

Let $\mathcal{T}$ be an D-colored N-regular tree and let $\mathcal{S}$ a subtree of $\mathcal{T}$.

- $\mathcal{S}$ is of *type (I)* if for any $\tau \in \mathcal{S}$ and $1 \le i \le D$, we have $\#\mathcal{S}_{\mathrm{suc}}(\tau)^{(i)} = 1$.
- $\mathcal{S}$ is of *type (II)* if for any $\tau \in \mathcal{S}$, there exists $1 \le i(\tau) \le D$ such that $\mathcal{S}_{\mathrm{suc}}(\tau) = \mathcal{T}_{\mathrm{suc}}(\tau)^{(i(\tau))}$.

Parameterize squares using trees. Let B be a square in the plane with its edges parallel to the coordinate lines. The side length $\ell(B)$ is denoted simply as l. Let $m \in \mathbb{N}$, $R \in \mathbb{R}$ be two positive numbers with $R \ge 2m$. A map Φ from a m^2-colored $m^2[R/m]^2$-regular tree $\mathcal{T}$ to the set of sub-squares of B is called *admissible* if it satisfies the following conditions:

- For any τ, edges of $\Phi(\tau)$ are parallel to coordinate lines.
- For any $n \ge 0$ and $\tau \in \mathcal{T}_n$, we have $\ell(\Phi(\tau)) = lR^{-n}$.
- If τ' is a successor of τ, then $\Phi(\tau) \subset \Phi(\tau')$.
- For any $n \ge 1$ and $\tau \in \mathcal{T}_{n-1}$, the interiors of the squares $\{\Phi(\tau') : \tau' \in \mathcal{T}_{\mathrm{suc}}(\tau)\}$ are mutually disjoint, the union $\bigcup_{\tau' \in \mathcal{T}_{\mathrm{suc}}(\tau)} \Phi(\tau')$ is a square of side length $m[R/m]lR^{-n}$, and for any $1 \le i \le [R/m]^2$, the union $\bigcup_{\tau' \in \mathcal{T}_{\mathrm{suc}}(\tau)^{(i)}} \Phi(\tau')$ is a square of side length mlR^{-n}.

Type (I) trees and winning. To begin, we quickly review the definition of Schmidt's (α, β)-game. Let $\alpha.\beta \in (0, 1)$ be two real numbers. Suppose two players, say Alice and Bob, take turns choosing a nested sequence of closed squares

$$B_0 \supset A_0 \supset B_1 \supset A_1 \supset B_2 \supset A_2 \supset \dots$$

in $\mathbb{R}^2$ satisfying $\ell(A_i) = \alpha\ell(B_i)$ and $\ell(B_{i+1}) = \beta\ell(A_i)$. Say a set X is (α, β)-*winning* if Alice can make sure that the outcome point $x_\infty = \cap_{i=0}^{\infty} A_i$ belongs to X. And say a set X is α*-winning* if it is (α, β)-winning for all $\beta \in (0, 1)$.

Let $B \subset \mathbb{R}^2$ a closed square. Say a set X is (α, β, B)*-winning* if Alice can make sure that the outcome point x_∞ belongs to X whenever Bob chooses B as his B_0. The following observation follows directly from these definitions.

(1)

$$X \text{ is } (\alpha, \beta, B)\text{-winning for all } \beta \in (0, 1) \text{ and all square } B \iff X \text{ is } \alpha\text{-winning.}$$

In view of (1), if we want to prove a subset X is α_0-winning for $\alpha_0 = (2m)^{-1}$ with m a large enough integer, it suffices to show that X is (α_0, β, B)-winning for all $\beta \in (0, 1)$ and all square B. Hence we may fix a $\beta \in (0, 1)$ and a square B from now on.

Let $R = 2m\beta^{-1}$ and Φ be an admissible map from a m^2-colored $m^2[R/m]^2$-regular tree $\mathcal{T}$ to the set of sub-squares of B. Then we have

Lemma 2. Let $\mathcal{S}$ be a subtree of type (I). Then the limit set of $\mathcal{S}$ defined as

$$\lim \mathcal{S} = \bigcap_{n\geq 1} \bigcup_{\tau\in\mathcal{S}_n} \Phi(\tau)$$

is (α_0, β, B)-winning.

Lemma 3. Let $\mathcal{S}$ be a subtree that $\#(\mathcal{S}\cap\mathcal{S}') = \infty$ for any subtree $\mathcal{S}'$ of type (II), then $\mathcal{S}$ contains a subtree of type (I).

In short, Lemma 2 says that if a set X contains the limit set of a type (I) tree, then it is (α_0, β, B)-winning, and Lemma 3 says that we can use trees of type (II) to give a sufficient condition for a subtree to be of type (I).

$\mathbf{Bad}(i, j)$ is winning. To begin, we quickly review the definition of $\mathbf{Bad}(i, j)$. For any $\epsilon > 0$, set

$$\mathbf{Bad}_\epsilon(i, j) = \mathbb{R}^2 \setminus \bigcup_{P\in\mathbb{Q}^2} \Delta_\epsilon(P), \text{ and } \mathbf{Bad}(i, j) = \bigcup_{\epsilon>0} \mathbf{Bad}_\epsilon(i, j),$$

where for $P = (p/q, r/q)$ written in reduced form,

$$\Delta_\epsilon(P) = \left\{(x, y) \in \mathbb{R}^2 : \left|x - \frac{p}{q}\right| < \frac{\epsilon}{q^{1+i}}, \left|y - \frac{r}{q}\right| < \frac{\epsilon}{q^{1+j}}\right\}.$$

In view of (1), Theorem 1 can be deduced from the following

Lemma 4. For any $\beta \in (0, 1)$ and any square B, there exists $\epsilon = \epsilon(\beta, B) > 0$ such that $\mathbf{Bad}_\epsilon(i, j)$ is (α_0, β, B)-winning.

The set $\mathbb{Q}^2$ is decomposed into a union of disjoint subsets $\mathcal{V}_n (n \geq 1)$ using a carefully chosen height function. Then based on this decomposition, for any $\epsilon > 0$, we can define a subtree $\mathcal{S}^\epsilon \subset \mathcal{T}$ by

(2) $\mathcal{S}^\epsilon_n = \{\tau \in \mathcal{T}_n : \tau \in \mathcal{T}_{suc}(\tau') \text{ with } \tau' \in \mathcal{S}^\epsilon_{n-1} \text{ and } \Phi(\tau)\cap\Delta_\epsilon(P) = \emptyset, \forall P \in \mathcal{V}_n\}.$

In view of Lemma 2 and Lemma 3, Lemma 4 can be deduced from the following

Lemma 5. For any $\beta \in (0, 1)$ and any square B, there exists $\epsilon = \epsilon(\beta, B) > 0$ such that if $\mathcal{S}^\epsilon$ is defined as in (2), then $\#(\mathcal{S}^\epsilon \cap \mathcal{S}') = \infty$ for any subtree $\mathcal{S}'$ of type (II).

In conclusion, Theorem 1 is reduced to vertices-counting problems on some particular colored rooted trees.

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Strong, Absolute, and Hyperplane Winning

Yu Yasufuku

In this talk, we discussed several recent variants of Schmidt's game and highlight the inter-relations and differences.

First, we introduced McMullen's strong and absolute games [3]. Given $\alpha, \beta \in (0,1)$, the (α,β)-**strong game** is played by Alice and Bob taking turns choosing nested closed balls

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset B_3 \supset \cdots$$

in $\mathbb{R}^n$, so that their radii satisfy

$$r(A_i) \geq \alpha \cdot r(B_i), \qquad r(B_{i+1}) \geq \beta \cdot r(A_i) \tag{1}$$

for all $i \geq 1$. Note that in the Schmidt's game, we assumed

$$r(A_i) = \alpha \cdot r(B_i), \qquad r(B_{i+1}) = \beta \cdot r(A_i)$$

in place of (1). We say a set E is an (α,β)-**strong winning set** if Alice has a strategy so that $\cap A_i$ meets E, an α-**strong winning set** if it is (α,β)-strong winning for all $\beta \in (0,1)$, and **strong winning set** if there exists an $\alpha \in (0,1)$ such that it is α-strong winning. A strong winning set is obviously a winning set.

McMullen proves that strong winning sets are stable under quasi-symmetric homeomorphisms, in particular by bi-Lipschitz maps. He further explicitly constructs a set which is winning but not strong winning, by showing that it is winning but its images under some quasi-symmetric homeomorphisms are not.

In the β-**absolute game**, the balls are no longer assumed to be nested. Instead, closed balls satisfy

$$B_1 \supset B_1 \backslash A_1 \supset B_2 \supset B_2 \backslash A_2 \supset B_3 \supset \cdots$$

and

$$r(A_i) \leq \beta \cdot r(B_i), \qquad r(B_{i+1}) \geq \beta \cdot r(B_i).$$

If A_i is of radius $\beta \cdot r(B_i)$ and is at the center of B_i, then the largest ball that Bob can choose for B_{i+1} has radius $\frac{1-\beta}{2} \cdot r(B_i)$, so we only consider $\beta \in (0, \frac{1}{3})$ for the absolute game. We say E is β-**absolute winning** if Alice has a strategy so that $\cap B_i$ meets E, and **absolute winning** if it is absolute β-winning for all $\beta \in (0, \frac{1}{3})$. One can easily show that a β-absolute winning set is $(\frac{1-\beta}{2}, \frac{\beta}{(1-\beta)/2})$-strong winning. Since $\frac{1-\beta}{2}$ goes upward to $\frac{1}{2}$ and $\frac{\beta}{(1-\beta)/2}$ goes downward to 0 as β goes to 0, it follows that an absolute winning set is strong winning. We can also show that the countable intersection of strong-winning (resp. absolute winning) sets is strong winning (resp. absolute winning), by the same argument as showing the same property for winning sets.

Theorem 1. *The set of badly approximable numbers is an absolute winning set.*

In fact, McMullen [3] proves more generally that what he calls a Diophantine set is absolute winning. A **Diophantine set** is

$$D(\Gamma) = \{x \in \mathbb{R}^n : \pi(\gamma_x(t)) \text{ remains bounded as } t \to +\infty\},$$

where Γ is a full lattice inside the group of isometries of the hyperbolic upper half space $\mathbb{H}^{n+1}$ such that Γ has a cusp at ∞, $\pi : \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}/\Gamma$, and γ_x is the "vertical geodesic" to x, that is, $\gamma_x(t) = (x, e^{-t})$. The main ingredient in the proof of this result is the structure theory of finite-volume hyperbolic manifolds.

The absolute winning sets are much too restrictive in certain situations, since Alice can only remove just one of Bob's possible moves. For example, the set of badly approximable vectors in $\mathbb{R}^d$ for $d \geq 2$ is *not* an absolute winning set. Indeed, for any constant C and $(x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$, $(d-1)$-dimensional Dirichlet's theorem shows that there exist $p_2, \ldots, p_d \in \mathbb{Z}$ and a sufficiently large $q \in \mathbb{N}$ such that

$$\max\left(\left|1 - \frac{q}{q}\right|, \left|x_2 - \frac{p_2}{q}\right|, \ldots, \left|x_d - \frac{p_d}{q}\right|\right) = \max\left(\left|x_2 - \frac{p_2}{q}\right|, \ldots, \left|x_d - \frac{p_d}{q}\right|\right) < \frac{1}{q^{1+\frac{1}{d-1}}} < \frac{C}{q^{1+\frac{1}{d}}}.$$

This means that $(1, x_2, \ldots, x_d)$ is never badly approximable, so Alice loses if Bob chooses the centers of B_i's to lie on $1 \times \mathbb{R}^{d-1}$.

This example leads to the **hyperplane absolute game**, introduced by Broderick, Fishman, Kleinbock, Reich, and Weiss [1]. More generally, in the k-dimensional β-absolute game, Alice chooses as A_i an ϵ_i-neighborhood of a k-dimensional affine subspace (a translation of a k-dimensional linear subspace of $\mathbb{R}^d$), where ϵ_i is less than or equal to $\beta \cdot r(B_i)$. Bob then has to choose B_{i+1} to sit inside $B_i \backslash A_i$. A set is a **k-dimensional β-absolute winning** if Alice has a strategy so that $\cap B_i$ meets this set, and it is **k-dimensional absolute winning** if it is k-dimensional β-absolute winning for all $\beta \in (0, \frac{1}{3})$. In particular, a set is called **hyperplane absolute winning** if it is $(d-1)$-dimensional absolute winning.

The k-dimensional absolute winning sets are $(k+1)$-dimensional absolute winning sets, and 0-dimensional absolute winning sets are precisely absolute winning sets. Further, a hyperplane absolute winning set is α-strong winning for $\alpha \in (0, \frac{1}{2})$. We also have:

Theorem 2. *The set of badly approximable vectors in $\mathbb{R}^d$ is a hyperplane absolute winning set.*

This is a straight-forward application of a Schmidt-Davenport type lemma. Another key feature of a hyperplane absolute winning set is the following:

Theorem 3. *A hyperplane absolute winning set is stable under $\mathscr{C}^1$-diffeomorphisms.*

In fact, [1] shows much more generally that

$$\bigcap_{i=1}^{\infty} f_i^{-1}(E) \cap K$$

has positive Hausdorff dimension, where E is hyperplane absolute winning, $f_i : U \to \mathbb{R}^d$ is a $\mathscr{C}^1$-diffeomorphism from an open set U of $\mathbb{R}^d$ onto its image, and K is what they call **hyperplane-diffuse set** with $K \cap U \neq \emptyset$.

As a generalization of Theorem 2, Nesharim and Simmons [4] prove:

Theorem 4. $\mathbf{Bad}(s,t)$ *is a hyperplane absolute winning set.*

The proof of this result goes through **hyperplane potential games** in which Alice gets to remove a suitable countable union of tubular neighborhoods of affine hyperplanes at each stage. The hyperplane potential winning sets are proved to be equivalent to hyperplane absolute winning sets in [2].

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Potential winning and weighted Bad in higher dimensions

Victoria Zhuravleva

The main purpose of this talk is to discuss recent results concerning Schmidt's problem about the intersections of the sets of weighted badly approximable points. Particularly, we discuss results that were obtained using a new modification of Schmidt's (α,β)-game

For $d \in \mathbb{N}$ we consider a set of *weights*

$$\mathcal{R}_d := \{\mathbf{r} = (r_1,\dots,r_d) : r_i \geq 0, \sum_{i=1}^{d} r_i = 1\}.$$

Set for any $c > 0$

$$\mathbf{Bad}_c(\mathbf{r}) = \{(x_1,\dots,x_d) : \inf_{q\in\mathbb{N}} \max_{1\leq i\leq d} q^{r_i}||qx_i|| \geq c\},$$

where $||\cdot||$ means the distance of a real number to its nearest integer.

The set

$$\mathbf{Bad}(\mathbf{r}) = \bigcup_{c>0} \mathbf{Bad}_c(\mathbf{r})$$

is called the set of $\mathbf{r}$-*badly approximable vectors* in $\mathbb{R}^d$.

Two following problems have remained open in dimensions $d \geq 3$.

Conjecture 1. *For any $d \geq 2$, let $\mathcal{S}$ be a countable subset of $\mathcal{R}_d$. Then*

$$\dim_H \bigcap_{r\in S} \mathbf{Bad}(\mathbf{r}) = d.$$

One of the ways to obtain this result is to prove the following:

Conjecture 2. *For any $d \geq 2$, $\mathbf{r} \in \mathcal{R}_d, \mathbf{Bad}(\mathbf{r})$ is α-winning in the sense of (α, β)-Schmidt's game.*

In this talk we discuss results related to Conjecture 2.

In [5] Schmidt proved that $\mathbf{Bad}\left(\frac{1}{d}, \dots, \frac{1}{d}\right)$ is $\frac{1}{2}$-winning for any $d \in \mathbb{N}$. In this paper you can also find the definition of (α, β)-game and the explanation why winning sets have full Hausdorff dimension.

In [1] An showed that for any $\mathbf{r} \in \mathcal{R}_2$, $\mathbf{Bad}(\mathbf{r})$ is $(24\sqrt{2})^{-1}$-winning. He invented a special partition lemma. Nesharim and Simmons in [4] used this lemma to show that for any $r \in \mathcal{R}_2$, $\mathbf{Bad}(\mathbf{r})$ is $\frac{1}{2}$-winning. Also they used a new modification of (α, β)-Schmidt's game.

This new game is called *hyperplane potential game*. It was introduced by Fishman, Simmons and Urbanski in [2]. The main difference from the previous games is that Alice is allowed to delete not only one but countably many objects (here we have hyperplane neighbourhoods).

Guan and Yu in [3] used this game and modified An's partition to prove that for $d \geq 2$ the set of badly approximable vectors is hyperplane potential winning, but only for some special weights. Instead of $\mathcal{R}_d$ they take weights from the set

$$\mathcal{R}'_d := \{\mathbf{r} = (r_1, \dots, r_d) \in \mathcal{R}_d : \#\{i : r_i = \max_{1 \leq j \leq d} r_j\} \geq d - 1\}.$$

Their proof consists of the following steps:

- Constructing a partition of all allowed balls
- Attaching a hyperplane to any rational point
- Dividing all rational points into groups according to their heights and denominators
- Proving that if two balls belong to the same part of this partition then they belong to one hyperplane
- According to this partition giving Alice a strategy to win

This proof demands one special condition called admissibility which doesn't allow authors to give a proof in general case.

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Generalised Cantor sets

Simon Baker

Within Diophantine approximation we often say that an element of Euclidean space is badly approximable if it fails to satisfy some system of inequalities. We emphasise that badly approximable is meant in a general sense and should not be interpreted as the classical notion of badly approximable. Given a notion of badly approximable we denote the set of badly approximable vectors by **Bad**. Two problems we are particularly interested in are:

(1) Determining whether **Bad** has full Hausdorff dimension.
(2) Given a countable collection of bad sets $\{\mathbf{Bad}_i\}_{i=1}^{\infty}$, can one determine the size of the set

$$\bigcap_{i=1}^{\infty} \mathbf{Bad}_i \, .$$

For some notions of badly approximable one can show that **Bad** is winning in the sense of Schmidt games. It is a consequence of the winning property that **Bad** is of full Hausdorff dimension, and moreover the countable intersection also has full Hausdorff dimension. For more general notions of badly approximable it is not clear whether **Bad** is winning and we have to resort to other techniques. In my lecture I spoke about generalised Cantor sets which are one such technique.

A traditional Cantor set is generated by firstly taking some cube. We have some splitting procedure that splits this cube into some finite collection of subcubes all of the same size. We then have a removal rule which states that a certain configuration of subcubes is removed from this collection. We repeat this procedure with what remains and so on to infinity. The Cantor set associated to this construction is the intersection of all of these collections. A generalised Cantor set construction differs from a traditional Cantor set construction in three ways. Firstly, the splitting procedure is allowed to depend on what stage of the construction we are in. Secondly, the removal rule is no longer local, i.e., whether a subcube remains after step (n+1) depends not only on the level n cube it is contained in, but also on the subcubes of level $(n-1)$, level $(n-2)$,... level 1 it is contained in. Thirdly, the removal rule states that at most a certain number of cubes are removed. It doesn't guarantee any are removed. What is more, we don't know the configuration of those cubes that are being removed. This general approach means that a generalised Cantor set is not uniquely determined by its splitting procedure and removal rule.

Generalised Cantor sets were introduced in [1] by Badziahin and Velani where they were used to determine the Hausdorff dimension of some **Bad** set that naturally arises from the mixed Littlewood conjecture. In this paper it is also shown that if **Bad** contains a sufficiently large generalised Cantor set then it is of full Hausdorff dimension. Thus generalised Cantor sets can be used as a tool for proving full dimension results for general **Bad** sets. Often they allow us to solve problem (1) above.

To make progress with problem (2) one can again use the theory of generalised Cantor sets. Loosely speaking, given a countable collection of bad sets $\{\mathbf{Bad}_i\}_{i=1}^{\infty}$, they can be shown to have a large nonempty intersection if they each contain a generalised Cantor set for which we can either put strong restrictions on the number of subcubes that can be thrown away at each step in our construction, or strong restrictions on the number of cubes that appear in the splitting part of our construction. This approach was successfully applied firstly by Beresnevich in [3], and then by Badziahin and Harrap in [2]. In particular, Beresnevich introduced the notion of a Cantor rich set and Badziahin and Harrap introduced the notion of a Cantor winning set. These properties effectively quantify what is outlined above. Both of these properties allow one to prove that the countable intersection of certain bad sets is not only nonempty but also of full Hausdorff dimension. Note that the approach of Badziahin and Harrap applies to general metric spaces that satisfy some minor technical conditions.

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What could be 'badly approximable' in multiplicative sense?

Or Landesberg

In the classical setup of Diophantine approximation the set of badly approximable numbers is defined as:

$$\mathbf{Bad} = \{\alpha \mid \liminf_{q\to\infty} q \cdot \|q\alpha\| > 0\}$$

Although this set is of zero Lebesgue measure it is quite large in the sense that it has full Hausdorff dimension. In the multiplicative setup one may consider the set

$$Mad^0 = \{(\alpha,\beta) \in \mathbb{R}^2 \mid \liminf_{q\to\infty} q \cdot \|q\alpha\| \cdot \|q\beta\| > 0\}$$

as a candidate for the set of 'badly approximable' points in $\mathbb{R}^2$. This might not be the best choice as this set is conjectured to be empty (Littlewood's conjecture) and was proven in [5] to have zero Hausdorff dimension.
Similarly in the p-adic multiplicative setup, the set:

$$Mad_p^0 = \{\alpha \in \mathbb{R} \mid \liminf_{q\to\infty} q \cdot |q|_p \cdot \|q\alpha\| > 0\}$$

is also conjectured to be empty, where $|q|_p$ is the p-adic norm of q, and was proven in [6] to have zero Hausdorff dimension.
Consider the sets:

$$Mad^\lambda = \{(\alpha,\beta) \in \mathbb{R}^2 \mid \liminf_{q\to\infty} q \cdot (\log q)^\lambda \cdot \|q\alpha\| \cdot \|q\beta\| > 0\}$$

A Khinchine type theorem of Gallagher implies $Leb(Mad^\lambda \cap [0,1])$ is zero for all $\lambda \leq 2$ and one otherwise. Badziahin and velani conjectured in [1] that Mad^λ has full Hausdorff dimension for all $\lambda \geq 1$ and is empty otherwise. Accordingly it was suggested that Mad^1 is the proper analogue for 'badly approximable' in the multiplicative sense. A similar conjecture was formulated for the family Mad_p^λ in the p-adic setup.

In this talk we survey relevant results regarding the multiplicative and p-adic multiplicative setups (e.g. [2],[3],[4],[7],[8],[9]) and describe the analogues of **Bad** as suggested in [1]. We then sketch a proof from [1], using generalized Cantor sets, of the fact that:

$$\{(\alpha,\beta) \in \mathbb{R}^2 \mid \liminf_{q\to\infty} q \cdot \log q \cdot \log\log q \cdot |q|_p \cdot \|q\alpha\| > 0\}$$

is of full Hausdorff dimension. Denoting:

$$Mad_p^\lambda = \{\alpha \in \mathbb{R} \mid \liminf_{q\to\infty} q \cdot (\log q)^\lambda \cdot |q|_p \cdot \|q\alpha\| > 0\}$$

this theorem implies:

$$\dim Mad_p^\lambda = 1 \qquad \forall \lambda > 1$$

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Davenport's problem: an overview

Hanna Husakova

The aim of this talk is to describe Davenport's problem and to make an overview of the results on the problem obtained for curves and manifolds. In [1] Davenport proved the following result:

Theorem 1. *Given a finite collection $F = \{F_i, 1 \leq i \leq r\}$ of continuously differentiable maps $F_i : \mathbb{R}^m \to \mathbb{R}^{n_i}$ such that for some point $\mathbf{x}_0 \in \mathbb{R}^m$ the Jacobians of F_i at $\mathbf{x}_0$ has rank n_i, $1 \leq i \leq r$. Then the set of points $\mathbf{x} \in \mathbb{R}^m$ such that $F_i(\mathbf{x}) \in \mathbf{Bad}\left(\frac{1}{n_i}, \dots, \frac{1}{n_i}\right)$ for all $F_i \in F$ is uncountable.*

It is clear that the Jacobian condition which is a necessary for Davenport's proof implies that $m \geq n_i$, $1 \leq i \leq r$ and therefore Davenport writes that the much more difficult problem arises when the number of independent parameters is less than the dimension of simultaneous approximation. In particular, given the some planar curve $\mathcal{C}$ we have the following question:

Problem 1 (Davenport)**.** Is the set $\mathcal{C} \cap \mathbf{Bad}(i, j)$ uncountable?

The first result on this problem has been obtained in case of vertical lines [2, 3]. Let L_θ denote the vertical line parallel to y-axis passing through the point $(\theta, 0)$. It is easily verified that for any $\theta \in \mathbb{R}$ satisfying $\liminf_{q \to \infty} q^{1/i} \|q\theta\| = 0$ we have that $L_\theta \cap \mathbf{Bad}(i, j) = \emptyset$, which means that the answer to the question mentioned above is negative. Otherwise we obtain $\dim(L_\theta \cap \mathbf{Bad}(i, j)) = 1$ and the answer is positive.

The general case of straight lines in $\mathbb{R}^2$ is somewhat less understood. Given real $a, b \in \mathbb{R}$ let $L_{a,b}$ denote the line defined by the equation $f(x) = ax + b$. The following result were proved in the papers [4, 3]: For any $a, b\mathbb{R}$ such that there exists $\epsilon > 0$ satisfying

$$\liminf_{q \to \infty} q^{1/\sigma - \epsilon} \max\{\|qa\|, \|qb\|\} > 0, \tag{1}$$

where $\sigma := \min\{i, j\}$, we have $\dim(\mathbf{Bad}(i, j) \cap L_{a,b}) = 1$. Whether or not condition (1) may be relaxed to $\epsilon = 0$ remains an open problem.

It was only a special case of vertical lines. Now assume that $\mathcal{C}_f$ is given as a graph for some smooth function f defined on an interval $I \subset \mathbb{R}$. This case was considered in papers [4, 3]. Since there exist lines which don't contain points from the set $\mathbf{Bad}(i, j)$ we need some additional restriction on the curvature of $\mathcal{C}_f$.

Definition 1. We will say that $\mathcal{C}_f$ is a $C^{(2)}$ *non-degenerate planar curve* if f is two times continuously differentiable on the interval I and there exists at least one point $x_0 \in I$ such that $f''(x_0) \neq 0$.

Then for any $C^{(2)}$ non-degenerate planar curve $\mathcal{C}_f$ we have [4, 3]:

$$\dim(\mathbf{Bad}(i, j) \cap \mathcal{C}_f) = 1.$$

In particular, the answer to the Davenport's question in this case is positive.

That was results on the problem obtained in two-dimensional case. The original results of Davenport were formulated in case of arbitrary dimensions, namely for a finite collection $F_i = (f_{i,1}, \dots, f_{i,m}) : \mathbb{R}^m \to \mathbb{R}^{n_i}$ of continuously differentiable maps. The Jacobian condition implies that $m \geq n_i$ and the problem arises when $m < n_i$ and F_i lies on a sub-manifold of $\mathbb{R}^{n_i}$. Hence, Davenport's problem boils down to investigate badly approximable points restricted to sub-manifolds $M \subset \mathbb{R}^n$ of Euclidean spaces.

Let $F = (f_1, \dots, f_n) : B \to \mathbb{R}^n$ be an analytic map defined on a ball $B \subset \mathbb{R}^m$.

Definition 2. The map F will be called *non-degenerate* if the functions $1, f_1, \dots, f_n$ are linearly independent over $\mathbb{R}$.

Given an integer $n \geq 2$, $F_n(B)$ will denote a family of maps F with common domain B. Every map $F \in F_n(B)$ defines some non-degenerate manifold $M := F(B)$ of dimension m embedded in $\mathbb{R}^n$. Define also the collection R_n of weights of approximation $(r_1, \dots, r_n)$, $r_i \geq 0$, $\sum_{i=1}^{n} r_i = 1$.

Now we are ready to formulate the recent result of Beresnevich [5] which gives the solution of Davenport's problem in case of arbitrary dimension.

Theorem 2. *Let $m, n \in \mathbb{N}$, $1 \leq m \leq n$, B be an open ball in $\mathbb{R}^m$ and $F_n(B)$ be a finite family of analytic non-degenerate maps. Let W be a countable subset of R_n such that* $\inf\{\min\{r_i : r_i > 0\} : (r_1, \dots, r_n) \in W\} > 0$. *Then we have*

$$\dim \left(\bigcap_{F \in F_n(B)} \bigcap_{r \in W} F^{-1}\, \mathbf{Bad}(r_1, \dots, r_n) \right) = m.$$

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Badly approximable points on manifolds, I

Felipe A. Ramírez

We discuss the main result of Victor Beresnevich's paper *Badly approximable points on manifolds* [1]. Namely, suppose $n \in \mathbb{N}$, $n \geq 2$, $I \subset \mathbb{R}$ is an open interval, and $\mathcal{F}_n(I)$ is a finite family of maps $I \to \mathbb{R}^n$, all of which are nondegenerate at some common point $x_0 \in I$. Let W be a finite or countable subset of

$$\mathcal{R}_n := \{\mathbf{r} = (r_1, \dots, r_n) : r_i \geq 0, r_1 + \dots + r_n = 1\},$$

and suppose that $\inf\{\tau(\mathbf{r}) : \mathbf{r} \in W\} > 0$, where $\tau(\mathbf{r})$ is the minimal non-zero entry of $\mathbf{r}$. Then

$$\dim \bigcap_{\mathbf{f} \in \mathcal{F}_n(I)} \bigcap_{\mathbf{r} \in W} \mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r})) = 1. \tag{1}$$

This result, which answers a problem of Davenport [2], has many consequences. The first is a higher-dimensional version, where we replace I by a ball $B \subset \mathbb{R}^m$.

In this case, the intersection corresponding to (1) has dimension m. (For this higher-dimensional version, one requires that the maps $\mathbf{f}$ are analytic as well as nondegenerate, but this is only because of the "slicing" and "fibering" argument used to deduce it. It is expected that the result remains true without analyticity.) If we consider the function $\mathbf{f}$ as parametrizing an analytic nongenerate m-dimensional submanifold M of $\mathbb{R}^n$, then the result states that the set of badly approximable points lying on M has dimension m. If $m = n$, the case of the inclusion map $B \hookrightarrow \mathbb{R}^n$ answers (newly) Schmidt's Conjecture about intersections of sets of badly approximable points of different weights.

The proof of (1) proceeds by showing that the set

$$\bigcap_{\mathbf{f}\in\mathcal{F}_n(I)} \bigcap_{\mathbf{r}\in W} \mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r}))$$

is *Cantor rich*. Roughly speaking, this means that the set contains an abundance of generalized Cantor sets having dimension arbitrarily close to 1. One of the nice features of Cantor richness is that a countable intersection of Cantor rich sets is also Cantor rich, which reduces the problem to treating $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r}))$ for a single choice of $\mathbf{f}, \mathbf{r}$. One constructs generalized Cantor sets in $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{r}))$ by repeatedly partitioning the interval into a number of equal sub-intervals, and at each step discarding some of those sub-intervals. The discarded sub-intervals are those that intersect so-called *dangerous intervals*. These are intervals where "bad approximability" may be violated for a given set of possible denominators. That is, they are intervals of points x where $|\mathbf{q}\cdot\mathbf{f}(x) - p|$ can be made small, for some $\mathbf{q} \in \mathbb{Z}^n$ in some $|\cdot|_\infty$-ball, by choosing $p \in \mathbb{Z}$. (Here, we think of the "dual" definition of bad approximablity.) Since we would like to construct generalized Cantor sets of large dimension, the challenge is to bound the number and length of these dangerous intervals. For this, it is beneficial to have a good control over the sizes of derivatives of $\mathbf{f}$, since these will tell us how many zeros $\mathbf{q}\cdot\mathbf{f}(x) - p$ has (which contributes to the number of dangerous intervals), and for how long the expression remains small near a zero (which contributes to the lengths of dangerous intervals). Such control on derivatives is afforded by nondegeneracy of $\mathbf{f}$.

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Badly approximable points on manifolds, II

Oleg German

The talk continues the story Felipe Ramírez began. The aim is to show the main ideas the proof of Beresnevich's theorem is based upon. The strategy is to construct a Cantor rich Cantor type set with parameter R responsible for the number of equal subintervals each interval under consideration is divided into. In the process, some of those subinterval get discarded, namely those that have a nonempty intersection with so called *dangerous intervals*.

Given a map $\mathbf{f}(x) = (f_1(x), \dots, f_n(x))$ nondegenerate at x_0 and an n-tuple of weights $\mathbf{r} = (r_1, \dots, r_n)$, $r_i \geqslant 0$, $r_1 + \dots + r_n = 1$, upon setting $\mathbf{F}(x) = (-1, \mathbf{f}(x))$ we have to deal with intervals arising from the systems similar to

$$\begin{cases} |\mathbf{a} \cdot \mathbf{F}(x)| \ll b^{-t} \\ |a_i| < b^{r_i t}, \ i = 1, \dots, n, \end{cases} \tag{1}$$

where b is an appropriately chosen base and $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1}$. For each fixed x the above system determines a one-parametric family of parallelepipeds centered at the origin, and we want those parallelepipeds to contain no nonzero integer points. If this is the case, x gives us a badly approximable point. Otherwise, x is *dangerous* and should be avoided. Controlling the whole parallelepiped is difficult, so we cut it into slices with the derivative linear form $\mathbf{F}'(x)$. This gives us two types of slices: the ones separated from the origin obtained by completing (1) with

$$b^{\gamma t-(1+\gamma)l} \leqslant |\mathbf{a} \cdot \mathbf{F}'(x)| < b^{\gamma t-(1+\gamma)(l-1)}, \quad 0 \leqslant l \leqslant l_t,$$

where $\gamma = \max(r_1, \dots, r_n)$, and the corresponding inner slice which is contained in a larger domain obtained by completing (1) with

$$|\mathbf{a} \cdot \mathbf{F}'(x)| \ll b^{(\gamma-\varepsilon)t}.$$

Then, for each nonzero $\mathbf{a} \in \mathbb{Z}^{n+1}$ the measure of the set consisting of x which have $\mathbf{a}$ in their l-th slice can be estimated from above. In case of the inner slice such an estimate is obtained by applying a deep result of Bernik, Kleinbok, Margulis. As for the slices separated from the origin, it can be shown that if I_p is an interval from level p, then the set $S(I_p)$ of integer points $\mathbf{a}$ contained in the l-th slice of some points of I_p spans a sublattice of $\mathbb{Z}^{n+1}$ of lower rank. This allows applying Blichfeldt's theorem to show that the cardinality of $S(I_p)$ is rather small, which eventually leads us to the desired estimates.

Problem session

Sam Chow

Problem 1 (V. Beresnevich) Prove that for any nondegenerate submanifold $M \subseteq \mathbb{R}^n$ and any $\mathbf{r} \in \mathbb{R}^n_{\geq 0}$ with $\sum_i r_i = 1$ we have

$$\dim(\mathbf{Bad}(\mathbf{r}) \cap M) = \dim M.$$

One might wish to read [16]. The result is known for curves, and if M is analytic then fibering is successful. Without this assumption one may need to develop a direct approach.

Problem 2 (V. Beresnevich) Now consider the case $n = 2$. Let $i, j \geq 0$ with $i + j = 1$. Let L be a line in $\mathbb{R}^2$. Prove that

$$\dim(\mathbf{Bad}(i,j) \cap L) = 1$$

if and only if $\mathbf{Bad}(i,j) \cap L = \emptyset$. This is known for C^2 curves. It's also known if the line has rational slope.

Problem 3 (V. Beresnevich) Given any countable collection of weights $W \subseteq \mathcal{R}_n$, and any nondegenerate analytic manifold $M \subseteq \mathbb{R}^n$, prove that

$$\cap_{\mathbf{r} \in W} \mathbf{Bad}(\mathbf{r}) \cap M \neq \emptyset.$$

This is known if we impose an additional technical condition. It's also known in the case $n = 2$. One could hope for more; winning would be ideal.

Problem 4 (V. Beresnevich) Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, and let $\mathbf{r} \in \mathbb{R}^n_{\geq 0}$ with $\sum_i r_i = 1$. Write $\mathbf{Bad}(\mathbf{r}; \boldsymbol{\theta})$ for the set of $\mathbf{y} \in \mathbb{R}^n$ such that there exists $C = C(\mathbf{y}, \boldsymbol{\theta})$ for which

$$\max_{1 \leq i \leq n} \|q y_i - \theta_i\|^{1/r_i} \geq C q^{-1} \qquad (q \in \mathbb{N}).$$

Let M be a nondegenerate analytic manifold in $\mathbb{R}^n$. Prove that

$$\mathbf{Bad}(\mathbf{r}; \boldsymbol{\theta}) \cap M \neq \emptyset.$$

The set is likely to be of full dimension. The result is known for $n = 2$, using the dual form of approximation (ref: Schmidt's conjecture).

Problem 5 (V. Beresnevich) Write B_n^* for the set of $x \in \mathbb{R}$ such that

$$|x - \alpha| \geq c_{x,n} H(\alpha)^{-n-1} \qquad (\alpha \in \mathbb{C} : \deg\ \alpha = n).$$

Prove that $\cap_{n=1}^{\infty} B_n^* \neq \emptyset$. This is known for $\cap_{n=1}^{N} B_n^*$. In a way, Problem 3 implies Problem 5 — see [7, §2.3]. One can also consider the problem with $\alpha \in \mathbb{R}$ instead of $\alpha \in \mathbb{C}$.

Problem 6 (V. Beresnevich) Let $d, n \in \mathbb{N}$ with $d \leq n$. Write $B_{n,d}^*$ for the set of $\mathbf{x} \in \mathbb{R}^d$ such that

$$\max_{1 \leq i \leq d} |x_i - \alpha_i| \geq c_{\mathbf{x},n} H(\alpha_1)^{-(n+1)/d}$$

for all conjugate algebraic numbers $\alpha_1, \ldots, \alpha_d$ of degree at most n. Prove that $B_{n,d}^*$ is uncountable. We believe that the problem of showing that $B_{n,d}^*$ is nonempty is also open. It's known that the measure is zero.

Problem 7 (S. Chow) Recall the dream theorem for manifolds:

Conjecture. *Let M be a nondegenerate submanifold of $\mathbb{R}^n$, and let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a monotonic function. Then*

$$|M \cap W(n, \psi)|_M = \begin{cases} 0, & \text{if } \sum_q \psi(q)^n < \infty \\ \text{FULL}, & \text{if } \sum_q \psi(q)^n = \infty. \end{cases}$$

The result is known when $n = 2$. The divergence case is known when M is analytic. Under an additional curvature hypothesis, the convergence case is known [9] for $d \geq 2$. The convergence case has recently been solved [8] under a mild assumption on M whenever $d > (n+1)/2$. Prove the convergence case for curves in $\mathbb{R}^3$, under suitable conditions on M. Can the hypothesis $d > (n+1)/2$ from [8] be slightly relaxed?

Problem 8 (D. Simmons) For $M \subseteq \mathbb{R}^d$ a nondegenerate submanifold, consider the *intrinsic Dirichlet exponent*

$$\omega(M) = \inf_{x \in M} \limsup_{p/q \in M} \frac{-\log|x - p/q|}{\log q}.$$

Let $\omega(k,d)$ be the supremum of $\omega(M)$ over nondegenerate submanifolds $M \subseteq \mathbb{R}^d$ of dimension k. From [12], we have

$$\omega(d,d) = \frac{d+1}{d}, \quad \omega(d-1,d) = 1,$$
$$\omega\left(k, \binom{k+n-1}{k-1} - 1\right) = \frac{k+1}{kn},$$

and also an explicit upper bound for $\omega(k,d)$. Can the upper bound be improved? Can $\omega(k,d)$ be computed exactly for more values of k and d? What if we consider $\omega'(M)$, wherein $\inf_{x\in M}$ is replaced by the essential infimum? Is it true that $\omega'(k,d) = \omega(k,d)$?

Problem 9 (E. Nesharim) For subsets of the reals, we know that ε-Cantor winning sets are M-Cantor rich whenever $M > 4^{1/(1-\varepsilon)}$. Is there a converse? Can the result above be strengthened?

Problem 10 (E. Nesharim) Let $\theta_1, \theta_2, \ldots \in \mathbf{F}_q$. Show that if $\ell \geq 0$ then there exist $h, k \geq 1$ such that

$$\operatorname{rank}\begin{pmatrix} \theta_{k+1} & \cdots & \theta_{k+h} \\ \vdots & & \vdots \\ \theta_{k+h+\ell} & \cdots & \theta_{k+2h+\ell-1} \end{pmatrix} < h.$$

This is mixed Littlewood in the function field setting.

Problem 11 (E. Zorin) Write S_p for the set of counterexamples to the p-adic Littlewood conjecture. Let $\mathcal{H}$ be the Hausdorff measure determined by the dimension function $f : r \mapsto (\log|\log r|)^{-1}$, i.e. take the infimum of

$$\sum_i f(r(B_i))$$

over ρ-covers of your set, then take ρ to zero. It's known that if S_p is nonempty then $\mathcal{H}_f(S_p) > 0$. Prove that if S_p is nonempty then $\mathcal{H}_g(S_p) > 0$, where $g(r) = |\log r|^{-1}$. Read [10], and use it to prove that $H_g < \infty$. Then do it in the classical Littlewood setting.

Problem 12 (S. Baker) Let $\phi_1, \dots, \phi_N$ be contractions of $\mathbb{R}^n$. Let F be the *attractor*, which is the unique set defined by

$$F = \cup_i \phi_i(F).$$

Pick $x \in F$, and an approximating function

$$\psi : \cup_{k=1}^{\infty} \{1, 2, \dots, n\}^k \to \mathbb{R}_{\geq 0}.$$

We ask when the set

$$W(x, \psi) = \Big\{ y \in F : y \in B(\phi_{s_1} \circ \dots \circ \phi_{s_k}, \psi(s_1, \dots, s_k)) \text{ i.o.} \Big\}$$

is large; ideally we'd like to classify the ψ for which this holds. It's known that

$$\dim_H F \leq \min(s, n),$$

where s is the similarity dimension of the iterated function system. How about a Khintchine theory when this inequality is an equality? (This is known for the Cantor set.) One may wish to consult [5, 6, 17].

Problem 13 (D. Simmons) We count rationals in the Cantor set C, up to given height. This has an application to the convergence case of the Khintchine theory of intrinsic diophantine approximation in the Cantor set — see [13]. Define

$$N_C(Q) = \#\{p/q \in C : q \leq Q\}.$$

The contribution from when the denominator is a power of 3 is at least 2^k, so

$$N_C(3^K) \geq 2^k.$$

A heuristic provided in [13] tells us not to expect many more rationals in total. Prove that

$$N_C(Q) \ll_\varepsilon Q^{\frac{\log 2}{\log 3} + \varepsilon}.$$

We know that at least a logarithmic factor must be present. To give a feel for this problem, recall that $x \in C$ if and only if the base 3 expansion uses only 0s and 2s, and that $x \in \mathbb{Q}$ if and only if this expansion is preperiodic. For example

$$0.[a_1 \dots a_k]_3 = \frac{[a_1 \dots a_k]_3}{3^k - 1} \in C \cap \mathbb{Q},$$

where the notation refers to the base 3 expansion.

Problem 14 (E. Nesharim) Is the quantity

$$\inf_{i+j=1} \inf_{x,y} \sup_{\alpha,\beta} \inf_q \max(q^i \|qx - \alpha\|, q^j \|qy - \beta\|)$$

positive? We can also ask the question with $\liminf_{q \to \infty}$ in place of $\inf_q$. In dimension 1, Khintchine [15] showed that if $x \in \mathbb{R}$ then there exists $\alpha \in \mathbb{R}$ such that

$$\inf_q q \|qx - \alpha\| > 0.$$

This can be made uniform: on the right hand side we could replace 0 by an absolute constant c. Formally, we have

$$c = \inf_x \sup_\alpha \inf_q q \|qx - \alpha\|.$$

It's known that
$$1/10 < c < 1/7;$$
the best lower bound is by Godwin [14] in 1953. Determine the value of c.

Problem 15 (S. Velani, via S. Baker) We know [2, 4] that the sets
$$\left\{(\alpha,\beta) : \liminf_{q\to\infty}(q\log q\log\log q)\|q\alpha\|\cdot\|q\beta\| > 0\right\}$$
and
$$\left\{\alpha : \liminf_{q\to\infty}(q\log q\log\log q)|q|_p\|q\alpha\| > 0\right\}$$
have full Hausdorff dimension. Can the $\log\log q$ be dropped? Are the sets winning? How about intersecting with a planar curve?

Problem 16 (S. Velani, via S. Baker) For all α,β, we know from Hurwitz's theorem that
$$\liminf_{q\to\infty} q\|q\alpha\|\cdot\|q\beta\| \le \frac{1}{2\sqrt{5}}.$$
What improvement can we make on the RHS? One may attempt this using continued fractions. Of course we expect this quantity to be zero (Littlewood conjecture). Editor: Note that Badziahin [3] obtains 1/19.

Problem 17 (S. Velani, via S. Baker) Let $\gamma\in\mathbb{R}$, and let $\psi:\mathbb{N}\to\mathbb{R}_{\ge 0}$. Put
$$W_\gamma(\psi) = \{\alpha\in[0,1] : \|q\alpha-\gamma\| \le \psi(q) \text{ i.o.}\}$$
and
$$W'_\gamma(\psi) = \{\alpha\in[0,1] : |q\alpha-\gamma-p| \le \psi(q) \text{ i.o. with } (p,q)=1\}$$
When $\gamma = 0$ we have the homogeneous 0-1 laws of Cassels and Gallagher. Prove it for arbitrary γ. (The scenario in which $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ is fixed and the problem is metric in γ was solved by Kurzweil.)

Problem 18 (A. Pollington, via S. Chow) Recall the Duffin–Schaeffer conjecture, which states that if

$$\sum_q \frac{\psi(q)\varphi(q)}{q} = \infty \tag{1}$$

then $W'(\psi)$ has full Lebesgue measure. Erdős [11] established this under the assumption that $\psi(q) = q^{-1}$ on its support. Vaaler [18] subsequently extended Erdős's work to handle $\psi(q) \ll q^{-1}$. A recent paper by Aistleitner [1] settles DS whenever

$$\sum_{n=2^{2^h}+1}^{2^{2^{h+1}}} \frac{\psi(q)\phi(q)}{q} \ll \frac{1}{h}. \tag{2}$$

If (1) holds but (2) does not then the support of ψ must be particularly uneven. Using this, can one prove DS when $\psi(q) = q^{-1/2}$ on its support? One may wish to consult [1, Lemma 3].

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