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Mini-Workshop: New interactions between homotopical algebra and quantum field theory

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Abstract. Recent developments in quantum field theory strongly call for techniques from homotopical algebra to develop the mathematical foundations of quantum gauge theories. This mini-workshop brought together experts working at the interface between topological field theory, quantum field theory and homotopical algebra with the goal of triggering major advances towards understanding quantum gauge theory. This was achieved via a fruitful exchange of ideas and technologies across different research communities and encouraging a comparison between recent approaches to homotopical quantum field theory.

Mathematics Subject Classification (2010): 18Gxx, 81Txx.

Introduction by the Organisers

Understanding the mathematical foundations of quantum gauge theories, such as Chern-Simons, Dijkgraaf-Witten and Yang-Mills theories, and in particular their descent properties requires a combination of traditional frameworks for quantum field theory with techniques coming from homotopical algebra. Several approaches towards homotopical generalizations of quantum field theory have been developed quite recently, however they are all starting from very different perspectives on quantum field theory. This makes the comparison between these approaches particularly challenging. The primary goal of this mini-workshop was to bring together experts at the interface between topological field theory, quantum field theory and homotopical algebra to encourage a comparison and exchange of techniques between such different approaches, which will eventually result in major progress towards establishing the mathematical foundations of quantum gauge theory.

The 16 participants of this mini-workshop represented different areas of mathematics and mathematical physics, ranging from homotopy theory to quantum field theory and topological field theory. Each participant contributed either with an introductory lecture (90 min) or with a research seminar (60 min). The introductory lectures covered the main subjects involved at this mini-workshop, including homotopical algebra (Richter), higher mathematical structures (Schreiber), locally covariant quantum field theory (Fredenhagen) and the Batalin-Vilkovisky formalism (Cattaneo). The purpose of the introductory lectures was to bridge the gap between the very diverse mathematical backgrounds represented in the audience, thus setting the basis for the research seminars, as well as for a fruitful scientific discussion. The research seminars were mostly situated at the overlap between two or more of the topics addressed by the introductory lectures, combining homotopy theory, topological field theories, BV-quantization, locally covariant quantum field theory, factorization algebras, higher geometric prequantization and higher structures thereof. The structure of the mini-workshop, combining introductory lectures and research seminars, was very well-received by the participants as it helped to set a common ground for sharing ideas between mathematicians coming from very different backgrounds. In particular, it strongly fostered stimulating discussions between experts from different areas, but with a common goal, namely understanding the mathematical foundations of quantum gauge theory.

Also on behalf of the participants, we would like to address our most sincere gratitude to the MFO. Besides giving us the opportunity to present the state of research in the field, the atmosphere in Oberwolfach tremendously helped in successfully achieving the main goal of this mini-workshop, namely to establish new connections between different approaches to the combination of quantum field theory and homotopical algebra. We are confident that these new interactions will soon trigger new collaborations as well as crucial breakthroughs towards understanding the mathematical foundations of quantum gauge theory.

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Table of Contents

Abstracts

An Introduction to Locally Covariant Quantum Field Theory

Klaus Fredenhagen

Quantum Field Theory is, according to Rudolf Haag [1], the incorporation of the principle of locality into quantum physics. Due to crucial nonlocal phenomena of quantum physics, in particular entanglement, locality can be only implemented on the level of observables, whereas states necessarily have nonlocal features. The observables are elements of an algebra of Hilbert space operators, typically unital C*-algebras. States are defined as linear functionals which assume positive values on positive operators and are 1 at the unit. Their values are interpreted as expectation values of the corresponding observable. The principle of locality is incorporated by associating to a region of spacetime the algebra of all observables measurable within this region. This association $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ is subject to the following axioms (Haag-Kastler-Axioms [2]):

Isotony: To any inclusion $\mathcal{O}_1 \subset \mathcal{O}_2$ there exists an injective homomorphism

$$i_{\mathcal{O}_2\mathcal{O}_1} : \mathfrak{A}(\mathcal{O}_1) \to \mathfrak{A}(\mathcal{O}_2) \quad \text{such that} \quad i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_1} \ .$$

Locality: If two subregions $\mathcal{O}_1.\mathcal{O}_2$ of $\mathcal{O}$ are spacelike separated, then the associated algebras commute,

$$[i_{\mathcal{O}\mathcal{O}_1}(\mathfrak{A}(\mathcal{O}_1)), i_{\mathcal{O}\mathcal{O}_2}(\mathfrak{A}(\mathcal{O}_2))] = \{0\} \ .$$

Timeslice: If a subregion $\mathcal{O}_1$of $\mathcal{O}$ contains a Cauchy surface of $\mathcal{O}$ then the homomorphism $i_{\mathcal{O}\mathcal{O}_1}$ is an isomorphism.

Covariance: The group G of spacetime isometries, which preserve orientation and time orientation, is represented by isomorphisms $\alpha_g^{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(g\mathcal{O})$ such that

$$\alpha_g^{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{g\mathcal{O}_2 g\mathcal{O}_1} \circ \alpha_g^{\mathcal{O}_1} \quad \text{and} \quad \alpha_{g_1 g_2}^{\mathcal{O}} = \alpha_{g_1}^{g_2\mathcal{O}} \circ \alpha_{g_2}^{\mathcal{O}} \ .$$

On Minkowski space, this set of axioms has been investigated in much detail. In a first step one can construct the algebra of all local observables as the inductive limit of all local algebras $\mathfrak{A}(\mathcal{O})$. One then studies the representations of this algebra. A subclass was completely classified. These are representations which are, after restriction to the spacelike complement of a bounded region, unitarily equivalent to a distinguished representation (the vacuum representation). It turns out that this subclass of representations has a monoidal structure (often called fusion) which makes it, in more than 2 spacetime dimensions, equivalent to the representation category of a certain compact group with a distinguished element k with $k^2 = e$ [3, 4]. This group can be identified with the group of global internal symmetries of the system, and k refers to the alternative between Bose and Fermi statistics. Moreover, provided some of these representations describe particles (in the sense of eigenstates of the mass operator), one obtains the outgoing as well as the incoming multiparticle states with the correct statistics [5]. In 2 dimensions one can perform an analogous analysis, but here more general structures can occur (quantum groups, braid group statistics etc.)[6, 7, 8].

In order to extend the analysis to theories describing also gravity one tries, as an intermediate step, to formulate quantum field theory on generic Lorentzian manifolds. If these manifolds are globally hyperbolic, the Haag-Kastler axioms can easily be formulated. The axiom of covariance, however, becomes empty in case the spacetime has no nontrivial symmetries. This restricts the usefulness of the axioms a lot. In particular, in eliminating the ultraviolet divergences of perturbation theory, one does not have a mean to compare the choices of renormalization conditions at different points of spacetime [9].

This problem was in principle solved by a generalization of the Haag-Kastler Axioms, called Locally Covariant Quantum Field Theory [10]. Instead of looking only at subregions of a given spacetime, one considers the category of all spacetimes which satisfy some general conditions, in particular global hyperbolicity, with structure (i.e. metric, orientation, time orientation, causal relations) preserving embeddings as morphisms. Quantum field theory is then defined as a functor $\mathfrak{A}$ into the category of unital C*-algebras with injective homomorphisms as morphisms, which satisfies the axioms of locality and timeslice. Note that after restriction of the functor to the subregions of a fixed spacetime one obtains the Haag-Kastler axioms. In particular, the axioms of isotony and covariance are both implied by the covariance of the functor.

This new formulation of quantum field theory immediately allows to define further *natural* structures, where the physical intuition of naturality nicely coincides with the mathematical concept of natural transformations. So two theories are equivalent if the the corresponding functors are naturally isomorphic. Quantum fields are natural transformations between the functor of test function spaces and the quantum field theory functor (combined with the appropriate forgetful functor). Moreover the described ambiguity of renormalization is reduced to the choice of universal constants by requiring that renormalization has to be natural in the appropriate sense [11].

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Homotopical algebra and homotopy colimits

Birgit Richter

Homotopical algebra was introduced by Quillen in 1967 [4] when he defined the concept of a (closed) *model category*. This is a flexible setting for *doing homotopy theory*: One has to specify three classes of maps in a complete and cocomplete category $\mathcal{C}$, the *weak equivalences* (we), *cofibrations* (cof) and *fibrations* (fib) that satisfy several compatibility axioms: The identity morphisms are in each of the classes, they are closed under composition and retracts. The weak equivalences satisfy the 2-out-of-3 property: if g and f are composable morphisms in $\mathcal{C}$, and if two out of $f, g, g \circ f$ are weak equivalences, then so is the third. Every morphism f in $\mathcal{C}$ can be factored as $f = q \circ i$ with i in cof and q in fib $\cap$ we and also as $f = p \circ j$ with j in cof $\cap$ we and p in fib. The morphisms in cof have the left lifting property with respect to morphisms in fib $\cap$ we and the morphisms in cof $\cap$ we have the left lifting property with respect to morphisms in fib. If a map from the initial object in $\mathcal{C}$ to some object C is in cof, then C is *cofibrant.* Dually, if the unique map from an object X to the terminal object is a fibration, then X is *fibrant.*

For the category of non-negatively graded chain complexes, Ch, of R-modules (for some associative ring R) there are several model category structures. In the projective model structure a chain map is a weak equivalence, if it is a quasi-isomorphism, it is a fibration if its components f_n are surjective for all $n \geq 1$ and it is a cofibration if the f_n are monomorphism with projective cokernel for all $n \geq 0$.

For chain complexes the notion of chain homotopies can be expressed in terms of the cylinder chain complex. Two chain maps $f, g\colon C_* \to D_*$ are chain homotopic if and only if they factor over the cylinder cyl_{C_*} with $(\mathrm{cyl}_{C_*})_n = C_n \oplus C_{n-1} \oplus C_n$, i. e., if there is a chain map $H\colon \mathrm{cyl}_{C_*} \to D_*$ with $H \circ i_0 = f$ and $H \circ i_1 = g$ where i_0, i_1 are the two canonical inclusions of C_* into cyl_{C_*}.

In a general model category this concept is generalized to cylinder objects which give a notion of left homotopy. There is a dual notion of right homotopy defined in terms of path objects. If one restricts to cofibrant and fibrant objects, then this gives a well-defined notion of homotopy. The homotopy category of a model category, Ho$\mathcal{C}$, is the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are homotopy classes of maps between fibrant-cofibrant replacements.

In Ch all objects are fibrant, but only chain complexes that are degreewise projective are cofibrant. A cofibrant replacement of an R-module M viewed as a chain complex concentrated in degree zero is precisely a projective resolution of M.

Colimits are in general not homotopy invariant: The pushout of the diagram $\{\{*\} \leftarrow \mathbb{S}^n \rightarrow \{*\}\}$ of topological spaces (with $\mathbb{S}^n$ denoting the n-dimensional unit sphere) is a one-point space. If you replace $\{*\}$ by the weakly equivalent $(n+1)$-ball, $\mathbb{D}^{n+1}$, then you obtain $\mathbb{S}^{n+1}$. Thus replacing a pushout diagram by one where the nodes are weakly equivalent to the original ones, changes the homotopy type of the pushout. A concept that avoids these phenomena is the *homotopy colimit* [2, 1, 3]. If you fix a small category $\mathcal{D}$ and a model category $\mathcal{C}$ then in good cases the category of functors from $\mathcal{D}$ to $\mathcal{C}$, $\mathcal{C}^{\mathcal{D}}$, possesses a model category structure [3, 2]. A homotopy colimit can then be defined as the colimit of the cofibrant replacement of the diagram in $\mathcal{C}^{\mathcal{D}}$. As the model structures on $\mathcal{C}^{\mathcal{D}}$ are in general quite involved, it is desirable to have explicit models for homotopy colimits: In topological spaces the double mapping cylinder and the mapping telescope are explicit model of a homotopy pushout and a homotopy colimit for a sequential diagram. Beatriz Rodríguez-González established concrete criteria for the existence of explicit models for homotopy colimits [5]. In particular, any diagram $F \in \mathsf{Ch}^{\mathcal{D}}$ possesses such a model as the totalization of a double complex constructed out of the nerve of the category $\mathcal{D}$ and the functor F.

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Homotopical locally covariant quantum field theory I

Alexander Schenkel

(joint work with Marco Benini and Richard J. Szabo)

In the first part of my talk I reviewed the problems one faces when one formulates quantum gauge theories in the framework of LCQFT. Being a description in terms of gauge invariant quantum observables, the functor $\mathfrak{A} : \mathsf{Loc} \rightarrow \mathsf{Alg}$ of a quantum gauge theory violates the axioms of LCQFT that are relating local and global properties of the theory, see e.g. [1]. More precisely, it violates the isotony axiom and any kind of descent axiom such as additivity or a cosheaf property. This implies that ordinary LCQFT is insufficient to describe quantum gauge theories.

In order to solve this problem, we started developing a novel and more powerful framework, called *homotopical LCQFT*, that combines LCQFT with homotopical algebra [2, 3]. A homotopical LCQFT is modeled by a functor $\mathfrak{A} : \mathsf{Loc} \to \mathsf{cAlg}$ from the category of spacetimes to the model category of cosimplicial algebras that satisfies homotopically meaningful generalizations of the axioms of LCQFT. The choice of cosimplicial algebras as the target model category is motivated by the following observation: The field configuration 'space' of a gauge theory on a spacetime M is a smooth simplicial set (i.e. ∞-stack)

$$\mathfrak{F}(M) = \Big(\mathfrak{F}_0(M) \Longleftarrow \mathfrak{F}_1(M) \Lleftarrow \mathfrak{F}_2(M) \Lleftarrow \cdots \Big) .$$

Functions on this smooth simplicial set are described by the cosimplicial algebra

$$C^\infty(\mathfrak{F}(M)) = \Big(C^\infty(\mathfrak{F}_0(M)) \Longrightarrow C^\infty(\mathfrak{F}_1(M)) \Rrightarrow C^\infty(\mathfrak{F}_2(M)) \Rrightarrow \cdots \Big) ,$$

and our cosimplicial algebra of quantum observables $\mathfrak{A}(M)$ should be obtained from deformation quantization of $C^\infty(\mathfrak{F}(M))$.

Even though the axiomatic framework of homotopical LCQFT is not yet developed in full detail, we obtained some promising results towards homotopical descent in toy-models of gauge theories [2] and the structure of the up-to-homotopy LCQFT axioms [3]. These aspects were discussed in M. Benini's talk and I refer to his contribution to these reports for more details.

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An introduction to BV

Alberto S. Cattaneo

(joint work with Pavel Mnëv, Nicolai Reshetikhin)

The BV formalism is a general method for setting the computation of functional integrals in the presence of symmetries. I discussed the general mathematical framework of this formalism and a recent development for the case of manifolds with boundary, including the quantum version. I presented the examples of abelian and non abelian BF theories and showed how the ensuing quantum theories are compatible with cutting and gluing.

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Quantization via twisted generalized cohomology

Urs Schreiber

Recently there has been much progress in identifying mathematical axioms for quantum field theories, taking into account more of the *local* structure than had been considered in the past. This concerns quantum, hence *quantized* field theories. What has received less attention is the development of the corresponding higher local geometric structures on the side of classical field theories, as well as the higher local refinement of the process of quantization that should take the latter to the former.

For more review of the problem, exposition of the following partial solution, and for references to the literature, see [3, 4]. This here is to advertize some first progress in this direction, as worked out in two master theses that I had advised [1, 2], see [3, 5] for the wrap-up:

In certain good situations (which are satisfied in particular for Chern-Simons theory) ordinary geometric quantization is in fact equivalent to push-forward of the pre-quantum line bundle in G-equivariant K-theory, where G is a given group of prequantized Hamiltonian symmetries acting leaf-wise on a given Poisson manifold. This is in refinement to an observation that goes back to Bott and is often known as Spin^c-quantization.

This is noteworthy for the following reason: The traditional prescription for geometric quantization – in terms of polarized sections of a prequantum line bundle with symplectic curvature – is problematic when generalizing to higher dimensional local field theory, where the prequantum line bundle becomes a higher pre-quantum gerbe. On the other hand, the concept of push-forward in twisted generalized cohomology is intrinsically homotopy theoretic and as such naturally lends itself to such a generalization.

Second, the geometric quantization of any (compact) Poisson manifold this way may naturally be understood as the non-perturbative boundary quantization of the 2d-Poisson sigma model with that Poisson manifold as target space. This may be regarded as a non-perturbative refinement of the famous perturbative result by Kontsevich and Cattaneo-Felder.

Finally, this construction is such that it has an evident generalization to higher dimensional field theories, provided the correct higher pre-quantum data: on the moduli stack of boundary fields of some $d+1$-dimensional (topological) field theory we need a pre-quantum $(d-1)$-gerbe, and then we need a "superposition principle" embodied in a choice of some E_∞-ring spectrum E (playing the role of the ground field and replacing the complex numbers in traditional quantum mechanics) together with a homomorphism $\mathbf{B}^{d-1}U(1) \to \mathrm{GL}_1(E)$ from the "∞-group of phases" to the ∞-group of units of E.

One finds that in this perspective all the analytic subtleties of quantum theory are packaged into the choice of ring spectrum E. For instance the reason that the K-theory spectrum $E = \mathrm{KU}$ knows about quantum mechanics is ultimately due to the fact that C^*-algebras and Hilbert bimodules present the category of KU-modules, via KK-theory. In [5] I discuss evidence that one plausible choice of ring spectrum for quantizing 2-dimensional conformal field theory as the boundary theory of 3-dimensional Chern-Simons theory is $E = \mathrm{tmf}$, the spectrum of topological modular forms. Again, as the name suggests, this captures just the kind of analytical data that controls 2d CFTs.

The main open questions remaining in this approach of higher geometric quantization via twisted generalized cohomology theory are 1) the identification of the ring spectra that quantize a given type of d-dimensional field theory, and 2) the refinement of the whole prescription from single boundaries to an n-functor on higher codimension singularities.

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2-Hilbert spaces from sections of prequantum bundle gerbes

Severin Bunk
(joint work with Christian Sämann and Richard J. Szabo)

Geometric quantisation is based on the space of sections of a hermitean line bundle whose field strength is required to provide a geometric realisation of a symplectic form with integer periods. This space of sections comes endowed with an inner product given by evaluating the hermitean metric of the line bundle on a pair of sections and integrating the resulting function over the base M. However, several mathematically and physically relevant geometries are not symplectic, but come with higher-degree analogues of symplectic forms instead. Examples are the canonical 3-forms on compact simple Lie groups, and special instances of the H-field in string theory. The question for a different quantisation scheme adapted to this situation arises naturally. A geometric realisation of a 3-form with integer periods is a gerbe with connection, or a bundle gerbe with connection as introduced by Murray [3]. This is to replace the hermitean line bundle with connection in geometric quantisation.

The collection of bundle gerbes with connection on a given manifold M has been shown to form a symmetric monoidal 2-category $\mathsf{BGrb}^\nabla(M)$ [4]. Similar to line bundles, the monoidal unit in $\mathsf{BGrb}^\nabla(M)$ deserves to be called the trivial bundle gerbe $\mathcal{I}_0$ with the trivial connection, and we can define a category of sections $\Gamma(M,\mathcal{G}) := \mathsf{BGrb}^\nabla(M)(\mathcal{I}_0,\mathcal{G})$ for any bundle gerbe $\mathcal{G}$ on M. The sections of a bundle gerbe then form a category as one would expect from the point of view that gerbes should categorify line bundles.

We have shown that this category of sections carries a monoidal structure, stemming from the direct sum of hermitean vector bundles, and that it forms a semisimple Abelian category (although with infinitely many simple objects) enriched in the category Hilb of finite-dimensional Hilbert spaces. Furthermore, sections form a module category over the rig category of *higher functions* $\mathsf{BGrb}^\nabla(M)(\mathcal{I}_0,\mathcal{I}_0)$, which is equivalent to the rig category $\mathsf{HVBdl}^\nabla(M)$ of hermitean vector bundles with connection on M and parallel morphisms. Finally, $\Gamma(M,\mathcal{G})$ can be endowed with a higher bundle metric, which is a sesquilinear functor mapping pairs of sections to higher functions, i.e. $\mathfrak{h}_\mathcal{G} : \Gamma(M,\mathcal{G})^{\mathrm{op}}\times\Gamma(M,\mathcal{G}) \to \mathsf{HVBdl}^\nabla(M)$.

A 2-Hilbert space basically is a Kapranov-Voevodsky 2-vector space [5] taken over Hilb with an inner product functor. In order to turn the sections of $\mathcal{G}$ into a module category over Hilb, we use the embedding $\mathsf{Hilb} \hookrightarrow \mathsf{HVBdl}^\nabla(M)$, where V is mapped to the trivial vector bundle with fibre V and with the trivial connection. This is in complete analogy with how $\mathbb{C}$ sits in $C^\infty(M)$ as constant functions. A non-degenerate, sesquilinear, Hilb-valued inner product of two sections of $\mathcal{G}$ is given by first evaluating the higher bundle metric on a pair of sections and then taking the space of parallel sections of the resulting hermitean vector bundle with connection. This makes the category $\Gamma(M,\mathcal{G})$ into a 2-Hilbert space.

The above construction has been worked out in [1], and moreover it has been shown there that the structures on the morphism categories in $\mathsf{BGrb}^\nabla(M)$ are completely compatible with transgression of gerbes. The results concerning 2-Hilbert spaces have been presented non-technically in [2], where additionally several examples of the abstract formalism have been worked out explicitly.

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Abelian and non-abelian *BF* theory on cobordisms endowed with cellular decomposition

Pavel Mnev
(joint work with Alberto S. Cattaneo, Nicolai Reshetikhin)

We present an example of a topological field theory living on cobordisms endowed with CW decomposition (this example corresponds to the so-called BF theory in its abelian and non-abelian variants). Its partition function satisfies is constructed by a finite-dimensional integral (replacing the functional integral of quantum field theory) and satisfies the following properties.

- The partition function satisfies Batalin-Vilkovisky quantum master equation (modified by a boundary term).
- Change of gauge-fixing choices changes the partition function by a homotopy (in an appropriate sense).
- Partition function satisfies a version of Segal's gluing axiom w.r.t. concatenation of cobordisms.
- Partition function is compatible with cellular aggregations (inverses of subdivisions of the cellular decomposition of the cobordism).

In non-abelian case, the action functional of the theory is constructed out of local unimodular L_∞ algebras on cells. The partition function is invariant under simple-homotopy equivalence and carries the information about the Reidemeister torsion, together with certain information pertaining to formal geometry of the moduli space of local systems (and, in simply connected case, contains the complete information on the rational homotopy type of the cobordism). Also, the partiton function contains a mod 16 complex phase depending solely on the (twisted) Betti numbers of the cobordism. This theory provides a combinatorial example of the BV-BFV programme for quantization of field theories on manifolds with boundary in cohomological formalism [1, 2] (see [3] for a short overview), with partition functions given by finite-dimensional BV pushforwards. This is a joint work [4] with Alberto S. Cattaneo and Nicolai Reshetikhin and is an extension of previous works of P.M. [5, 6].

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Operads

Ulrich Krähmer

Operads offer a formalism for defining types of algebraic structures such as commutative, Lie, Poisson or - most relevant for this workshop - Gerstenhaber, Batalin-Vilkovisky and Beilinson-Drinfel'd algebras. The operad itself consists of the vector spaces $\mathcal{O}(n)$ of all the operations

$$A^{\otimes n} \to A$$

that are present in the free algebra of the given type, together with the partial composition maps

$$\circ_i \colon \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(m+n-1)$$

obtained by inserting the output of one operation as i-th input into another, and - if one considers symmetric operads - the natural action by the permutation group S_n on $\mathcal{P}(n)$.

In the above we tacitly assumed we are working with algebraic structures on vector spaces, but one can consider symmetric operads in any symmetric monoidal category, e.g. topological spaces.

In this survey talk we begin by introducing operads and the necessary preliminary notiuons and then focus for a while on examples.

One important advantage of using the language of operads is that symmetric monoidal functors allow one to relate algebraic structures in different categories. A good example is the little discs operad whose n-ary operations are the configurations of n little discs, i.e. the continuous embeddings of $D \times \cdots \times D$ into D, where $D \subset \mathbb{C}$ is the unit disc. Applying the singular chains functor yields an operad in chain complexes, taking homology an operad in graded vector spaces. The latter turns out to define Gerstenhaber algebras, and a pivotal result in homological algebra, the by now proven Deligne conjecture, asserts that the Gerstenhbaer algebra structure on the Hochschild cohomology of an associative algebra lifts in fact to the structure of an algebra over the singular chains on the little discs on the Hochschild cochain complex itself.

After discussing these topics in some detail, I will finish with a brief discussion of Hopf algebroids, an algebraic structure that allows one to unify various generalisations of the setting of the Deligne conjecture such as Poisson algebras.

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An introduction to factorization algebras towards topological field theories

Claudia Scheimbauer

First introduced by Lurie [6] and Costello-Gwilliam [2] in the topological setting, factorization algebras are algebraic structures designed to encode the structure of the observables of a quantum field theory (henceforth QFT).

Definition. *A* factorization algebra $\mathcal{F}$ on M *is an algebra over the colored operad with open sets in M as colors and*

$$\mathcal{P}reFact_M(U_1,\dots,U_n;V) = \begin{cases} \{*\} & \text{if } U_1 \amalg \dots \amalg U_n \subseteq V; \\ \emptyset & \text{otherwise,} \end{cases}$$

satisfying multiplicativity, i.e. $\mathcal{F}(U)\otimes\mathcal{F}(V) \xrightarrow{\simeq} \mathcal{F}(U \amalg V)$, and descent for Weiss covers.

One can think of them as a multiplicative, non-commutative version of cosheaves and they turn out to be a tool useful for describing algebraic structures such as bimodules between algebras, centralizers, universal enveloping algebras, and E_n-algebras. Moreover, they are very similar to the structures appearing in (perturbative) Algebraic Quantum Field Theory, as we also saw in the other talks in this workshop.

A source of examples of "topological" factorization algebras, which reflect topological field theories, is factorization homology, also called topological chiral homology [7, 1]: these factorization algebras are fully local, i.e. are determined by their value at a small disk, [4]. This value has the structure of an E_n-algebra, which is an algebra in $\mathcal{S}$ for the little cubes operad in dimension n. Examples include n-fold loop spaces, associative algebras up to homotopy for $n = 1$, i.e. A_∞-algebras, and braided monoidal categories for $n = 2$.

After giving a detailed introduction to factorization algebras and factorization homology, we discussed joint work with Calaque [3], namely, how topological factorization algebras give examples of fully extended functorial topological field theories in the sense of Lurie [7] and their twisted cousins following Stolz-Teichner [8, 5]. Finally, we briefly mentioned ongoing joint work with Gwilliam on first simple examples of fully extended twisted field theories which include the information of the state space of the QFT.

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Linear Batalin-Vilkovisky quantization as a functor of ∞-categories

Rune Haugseng
(joint work with Owen Gwilliam)

The most fundamental example of quantization assigns to the vector space $\mathbb{R}^{2n}$ the *Weyl algebra*, which is the associative algebra on $2n$ generators $p_1, \dots, p_n, q_1, \dots, q_n$ with relations $[p_i, p_j] = 0 = [q_i, q_j]$ and $[p_i, q_j] = \delta_{ij}$. Here $\mathbb{R}^{2n}$ should be thought of as the cotangent bundle of $\mathbb{R}^n$, equipped with its standard symplectic structure, which is the arena for classical mechanics on $\mathbb{R}^n$. This assignment can be formulated as a functor, known as *Weyl quantization*, from symplectic vector spaces (or more generally vector spaces equipped with a skew-symmetric pairing) to associative algebras, which naturally breaks up into three steps:

(1) To a vector space V with skew-symmetric pairing ω we associate its *Heisenberg Lie algebra* $\mathrm{Heis}(V, \omega)$, which is the direct sum $V \oplus \mathbb{R}\hbar$ equipped with the Lie bracket where
$$[x, y] = \omega(x, y)\hbar$$
for x, y in V, and all other brackets are zero.
(2) To the Lie algebra $\mathrm{Heis}(V, \omega)$, we assign its universal enveloping algebra $U\mathrm{Heis}(V, \omega)$.
(3) If we now set $\hbar$ to 1, we get the Weyl algebra, i.e.
$$\mathrm{Weyl}(V, \omega) := U\mathrm{Heis}(V, \omega)/(\hbar = 1).$$

On the other hand, if we set $\hbar = 0$ we get $\mathrm{Sym}(V)$, which we can equip with the Poisson bracket
$$\{x, y\} = \lim_{\hbar \to 0} [x, y]/\hbar = \omega(x, y),$$
for $x, y \in V$; this is an algebraic version of the Poisson algebra of classical observables. The universal enveloping algebra $U\mathrm{Heis}(V, \omega)$ can thus be viewed as a deformation quantization of the Poisson algebra $\mathrm{Sym}(V)$. This procedure is at the core of all approaches to "free theories," and hence the base case for the more challenging and more interesting interacting theories.

In our paper [1] the main object of study is a *derived* and *shifted* version of this procedure — *derived* in the sense that we replace vector spaces by cochain complexes and *shifted* in the sense that we consider pairings of degree 1. We call this *linear BV quantization* as it produces the simplest possible examples of

Batalin-Vilkovisky (BV) quantization; this is a homological approach to quantization of field theories introduced by Batalin and Vilkovisky as a generalization of the BRST formalism, in an effort to deal with complicated field theories such as supergravity.

In the BV formalism, the classical observables form a 1-shifted Poisson algebra, and the quantum observables are an E_0-*algebra*, i.e. just a pointed cochain complex. Our construction gives a functorial quantization of cochain complexes with shifted pairings to E_0-algebras, using shifted versions of the Heisenberg Lie algebra and the universal enveloping algebra; using a variety of homotopical machinery, we implement these as symmetric monoidal functors of ∞-categories.

However, the enveloping algebra is no longer an associative algebra, but rather a *BD-algebra*. A BD-algebra (for Beilinson-Drinfeld) is a differential graded module (M, d) over $k[\hbar]$ equipped with an $\hbar$-linear unital graded-commutative product of degree zero and an $\hbar$-linear shifted Poisson bracket of degree one such that

$$\mathrm{d}(\alpha\beta) = \mathrm{d}(\alpha)\beta + (-1)^{\alpha}\alpha\,\mathrm{d}(\beta) + \hbar\{\alpha, \beta\}$$

for any α, β in M. Starting with a BD-algebra M, by setting $\hbar = 0$ we obtain a shifted Poisson algebra, which we interpret as the *dequantization* of M. On the other hand, if we set $\hbar$ to 1, i.e. we pass to the quotient $M/(\hbar - 1)$, then the differential is not a derivation and so, up to quasi-isomorphism, the only remaining algebraic structure is the unit. That is, the reduction $M/(\hbar - 1)$ is essentially just a pointed A-module.

Our construction has a number of pleasant properties:

- Due to the naturality of the construction, it induces a "higher BV quantization" from E_n-algebras in cochain complexes with 1-shifted pairings, which we expect to be equivalent to cochain complexes with $(1-n)$-shifted pairings, to E_n-algebras. In particular, for $n = 1$ we expect to recover Weyl quantization.
- Everything we do works not just over the base field k, but also over an arbitrary commutative differential graded k-algebra. It then follows quite formally from Lurie's descent theorem for ∞-categories of modules that there is a natural extension of our functor to derived algebraic geometry.
- Over Artinian differential graded algebras, and more generally over formal moduli problems, our functor behaves like a determinant. More precisely, in these cases the BV-quantization of a perfect complex with a non-degenerate pairing is invertible.

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Dictionary between the pAQFT and factorization algebras approaches to renormalization

Kasia Rejzner
(joint work with Owen Gwilliam)

In my talk I gave an account of how the BV formalism is used in constructing models in perturbative algebraic QFT (pAQFT) [BDF09, FR12, Rej16]. Next, I presented a dictionary between this approach and the one advocated by Costello and Gwilliam, which uses factorisation algebras [Cos11, CG11]. The main difference between the two approaches is that the former works in Lorentzian signature of spacetime (making use of the causal structure), while the latter in the Euclidean signature. A natural ground to compare the two is the pAQFT construction of time ordered products $\mathcal{T}_n$. Vacuum expectation values of these are then related to Euclidean Green functions constructed using factorisation algebras language. For simplicity, on the pAQFT side, I discussed only regular observables.

Table 1: Dictionary between the approaches.

Fredenhagen-Rejzner	**Costello-Gwilliam**
$M = (\mathbb{R}^4, \eta)$, $\eta = \mathrm{diag}(1,-1,-1,-1)$	$M = (\mathbb{R}^4, \mathbb{1})$
The space of field configurations: $\mathcal{E} = \mathcal{C}^\infty(M,\mathbb{R})$	
$T\mathcal{E} = \mathcal{E} \times \mathcal{E}_c$, if $\mathcal{E}$ is equipped with the Whitney topology; here $\mathcal{E}_c \doteq \mathcal{C}^\infty_c(M,\mathbb{R})$	$U \subset M$, $T_c\mathcal{E}(U) = \mathcal{E}(U) \times \mathcal{E}_c(U)$
Regular functionals: $\mathcal{F}_{\mathrm{reg}}$	smooth/smeared observables $\mathrm{Sym}(\mathcal{E}_c^!)$
Solutions to filed equations: zero locus of a 1-form dS on $\mathcal{E}$	
$dS \in \Gamma(T^*\mathcal{E})$, where $T^*\mathcal{E} = \mathcal{E} \times \mathcal{E}_c'$	$dS \in \Gamma(T_c\mathcal{E})$
Free field equation	
$dS(\varphi) = (\Box + m^2)\varphi = 0$	$dS(\varphi) = (\Delta + m^2)\varphi = 0$
Polyvector fields	
$\mathcal{PV}_{\mathrm{reg}}(\mathcal{O})$ regular functionals on $T^*[-1]\mathcal{E}$, ([Rej16, 3.4])	$PV_c(\mathcal{E}(U))$ as in [CG11]
Classical observables	
$\mathrm{Obs}^{cl}_{\mathrm{reg}}(\mathcal{O}) = (\mathcal{PV}_{\mathrm{reg}}(\mathcal{O}), \delta_S)$, where $\delta_S \doteq -\iota_{dS}$ (insertion of dS)	$\mathrm{Obs}^{cl}(U) = PV_c(\mathcal{E}(U))$,with the differential ι_{dS}
Feynman propagator satisfies: $-(\Box + m^2) \circ G^{\mathrm{F}} = -G^{\mathrm{F}} \circ (\Box + m^2) = i\delta$	G satisfies $(\Delta + m^2) \circ G = \delta$
Wick (normal) ordering operator	
$\mathcal{T} = e^{\frac{i\hbar}{2}\mathcal{D}_{\mathrm{F}}}$, where $\mathcal{D}_{\mathrm{F}} = \left\langle G^{\mathrm{F}}, \frac{\delta^2}{\delta\varphi^2} \right\rangle$	$W = e^{\hbar\partial_G}$, where ∂_G is essentially contraction with G
Quantum observables	
$\mathrm{Obs}^q_{\mathrm{reg}}(\mathcal{O}) \doteq (\mathcal{PV}_{\mathrm{reg}}(\mathcal{O})[[\hbar]], \hat{s}_0, \triangle)$ $\hat{s}_0 = \delta_{S_0} - i\hbar\triangle$ $\mathrm{Obs}^q_{\mathrm{reg}} = \mathrm{Obs}^{cl}_{\mathrm{reg}}[[\hbar]]$ as vector spaces Commutative, associative product $\cdot$ on $\mathrm{Obs}^q_{\mathrm{reg}}$	$\mathrm{Obs}^q = (\mathrm{Obs}^{cl}[[\hbar]], d = d_1 + d_2)$ $\mathrm{Obs}^q = \mathrm{Obs}^{cl}[[\hbar]]$ as vector spaces Factorisation product on Obs^q
Continued on next page	

Table 1 – continued from previous page

Fredenhagen-Rejzner	Costello-Gwilliam
We have a map $\mathcal{T}^{-1}: \mathrm{Obs}^{cl}_{\mathrm{reg}}(\mathcal{O})[[\hbar]] \to \mathrm{Obs}^{q}_{\mathrm{reg}}(\mathcal{O})$ that intertwines the differentials, and induces a new product on $\mathrm{Obs}^{cl}_{\mathrm{reg}}[[\hbar]]$: $F \cdot_{\mathcal{T}} G = \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$	There is a co-chain isomorphism $W_U : \mathrm{Obs}^{cl}(U)[[\hbar]] \to \mathrm{Obs}^{q}(U)$ it deforms the factorisation product as follows: $\alpha * \beta = e^{-\hbar\partial_G}\left(e^{\hbar\partial_G}\alpha \cdot e^{\hbar\partial_G}\beta\right)$
$\mathcal{T}_n(\Phi(f_1), \dots, \Phi(f_n))(0)$ is the n-point Green's function.	Euclidean Green's functions (Schwinger functions)

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Modular categories from cobordism categories

Bruce Bartlett

(joint work with Chris Schommer-Pries, Chris Douglas and Jamie Vicary)

An n-dimensional oriented topological quantum field theory (the author suggests the simpler terminology of *bordism representation*) in the Atiyah-Segal sense is a symmetric monoidal functor

$$Z : (\mathrm{Bord}^{\mathrm{or}}_{n-1,n}, \sqcup) \to (\mathrm{Vect}, \otimes) \tag{1}$$

from the n-dimensional oriented bordism category, whose objects are closed $(n-1)$-manifolds and whose morphisms are oriented n-dimensional cobordisms, to the category of vector spaces.

It is well-known that two-dimensional oriented bordism representations are classified by commutative Frobenius algebras (the image of the circle under the functor Z).

It has long been expected that three-dimensional oriented bordism representations should be classified by modular categories (a finitely semisimple linear category equipped with a nondegenerate braiding and a compatible twist), but making this precise has proved difficult. The seminal book of Bakalov and Kirillov [1] uses the technique of surgery and gets quite far in this regard, but encounters some difficulties (they need to supply the underlying category 'by hand' as well as demand that it is rigid as a monoidal category; there is an issue with the anomaly; and there are not natural functors in both directions establishing an equivalence).

We resolve these difficulties and complete the classification by adopting a 2-categorical approach and using Cerf theory. Namely, we consider 'three-dimensional bordism representation' to mean a symmetric monoidal 2-functor

$$Z : (\mathrm{Bord}^{\mathrm{or}}_{1,2,3}, \sqcup) \to (\mathrm{Cat}_{\mathbb{C}}, \boxtimes) \tag{2}$$

from the symmetric monoidal 3-dimensional oriented bordism *bicategory* $\mathrm{Bord}^{\mathrm{or}}_{1,2,3}$ (objects are 1-manifolds, morphisms are 2-dimensional cobordisms, 2-morphisms are 3-dimensional cobordisms) to the symmetric monoidal bicategory $\mathrm{Cat}_{\mathbb{C}}$ (objects are Cauchy complete $\mathbb{C}$-linear categories, morphisms are linear functors, 2-morphisms are natural transformations).

We have three main results. The first is that, using higher-dimensional Cerf theory, we obtain a simple finite presentation $\mathcal{O}$ for the symmetric monoidal bicategory $\mathrm{Bord}^{\mathrm{or}}_{1,2,3}$. In other words, $\mathrm{Bord}^{\mathrm{or}}_{1,2,3}$ is "the free symmetric monoidal bicategory $F(\mathcal{O})$ on a bunch of generating objects, 1-morphisms and 2-morphisms, with some relations, all specified by the presentation $\mathcal{O}$". The main point here is that the *topology* of 3-dimensional cobordisms translates naturally into the language of *higher categories with duals* — an important principle encoded in the Cobordism Hypothesis of Baez and Dolan [2].

Our second result is that, dropping a relation ('anomaly-freeness') from the presentation $\mathcal{O}$, we obtain a simple presentation $\mathcal{M}$ for a certain *central extension by* $\mathbb{Z}$ of the oriented bordism bicategory. Bordism representations arising from physics such as Chern-Simons theory are actually representatons of *this* bicategory (they are anomalous when considered as representations of the oriented bordism bicategory).

Our third result is that representations of $F(\mathcal{M})$ correspond precisely to modular categories equipped with a choice of sign, while representations of $F(\mathcal{O})$ correspond preicsely to anomaly-free modular categories. In this way, and in this sense, we complete the classification of 3-dimensional TQFTs.

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Tureav-Viro theories based on non-semisimple spherical categories

Christoph Schweigert
(joint work with Jürgen Fuchs and Gregor Schaumann)

We presented results on 3-2-1-extended oriented topological field theories of Turaev-Viro type in the presence of topological defects and physical boundaries. Besides motivations from physics, we also argued that the isomorphism that relates the Brauer-Picard group of a fusion category $\mathcal{A}$ to the group of braided equivalences of its Drinfeld center $Z(\mathcal{A})$ [1] and the equivariance properties of the generalized Frobenius-Schur indicator [6] should find natural explanations in this framework.

To a 1-cell, such an extended TFT assigns a $\mathbb{C}$-linear category, and to a 2-cell a linear functor. We explained that it should be possible to construct these parts of a TFT by taking a finite spherical tensor category as an input datum, without necessarily requiring it to be semisimple. General considerations for three-dimensional TFTs [4] imply that surface defects are labeled by bimodule categories. The category of Wilson lines embedded into a surface defect can be described as the category of modules over a natural monad on the bimodule categories that label the two-dimensional strata adjacent to the Wilson line. The resulting categories can be expressed in terms of relative Deligne products of bimodule categories and of a category-valued trace [2]. For the specific subclass of Dijkgraaf-Witten theories, this construction matches results obtained via a gauge theoretic construction [5] that involves a generalization of relative bundles.

The functor that the TFT associates to a surface is called a block functor. We explained that block functors should be left exact and outlined the ideas of a construction of block functors that involves two steps. In a first step, yielding functors that we call pre-blocks, the state sum inherent to Turaev-Viro models is implemented via coends; the second step involves a projection. We showed that an Eilenberg-Watts calculus for finite categories [3] allows one to find convenient and illustrative expressions for pre-blocks and and block functors.

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A Cohomological Perspective on AQFT

Eli Hawkins

The space of antisymmetric multivector fields on a smooth manifold carries two algebraic structures. The exterior product is an associative and graded commutative product of degree 0. The Schouten-Nijenhuis bracket is a graded Lie bracket of degree -1. The bracket is a biderivation of the product. These structures together constitute a *Gerstenhaber algebra* structure.

Given an associative algebra, A, the degree q component of the Hochschild complex is the space of q-multilinear maps $C^q(A,A) = \mathrm{Hom}(A^{\otimes q}, A)$. The complex is a differential graded Lie algebra, and deformations of the associative product $\mathsf{m} \in C^2(A,A)$ are characterized by the Maurer-Cartan equation. The Hochschild cohomology is a Gerstenhaber algebra.

In particular, for a smooth manifold, the Hochschild cohomology of the algebra of smooth functions is the Gerstenhaber algebra of multivector fields. An infinitesimal deformation of the algebra is characterized by a degree 2 cohomology class satisfying the Maurer-Cartan equation, and that is simply a Poisson bivector field.

An algebraic quantum field theory can be described as a functor $\mathfrak{A} : \mathfrak{X} \to \mathsf{Alg}$ from a small category to the category of associative algebras (a *diagram of algebras*). Hochschild cohomology extends to such a structure. The Hochschild bicomplex of $\mathfrak{A}$ is constructed by using the simplicial structure of the nerve of $\mathfrak{X}$. This is not a differential graded Lie algebra, but an L_∞-algebra, because the Maurer-Cartan equation is not quadratic.

Deformations of $\mathfrak{A}$ are described by a truncated Hochschild bicomplex (dropping the degree $(0,*)$ part). From this perspective, an infinitesimal deformation of $\mathfrak{A}$ is given by a degree 2 cohomology class, but there are 2 fundamental types of deformation. The transition from classical to quantum field theory comes from degree $(2,0)$. The transition from free to interacting field theory comes from degree $(1,1)$.

In the untruncated Hochschild bicomplex, degree $(0,2)$ provides another kind of deformation. This leads me to define a *skew diagram of algebras* and thus a generalization of AQFT. Automorphisms in the category of skew diagrams are a sort of generalized symmetry of an AQFT.

The simplest way of constructing a Poisson bivector field is as the product of 2 commuting vector fields. A corresponding strict deformation quantization can be constructed by the methods of Rieffel [1] using the abelian group action generated by the vector fields.

The Gerstenhaber algebra structure on the Hochschild cohomology, $H^\bullet(\mathfrak{A},\mathfrak{A})$ suggests that an interaction can be constructed as a product of classes in $H^1(\mathfrak{A},\mathfrak{A})$. In this way, an interacting AQFT might be constructed using a group of skew automorphisms of a free AQFT.

The details are presented in [2].

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Homotopical locally covariant quantum field theory II

Marco Benini
(joint work with Alexander Schenkel and Richard J. Szabo)

Following the introduction to homotopical locally covariant quantum field theory presented by Alexander Schenkel, we presented the first instances of this structure, focusing in particular on the local-to-global property and on the causality and time-slice axioms up to homotopy.

Concerning the local-to-global axiom, we showed that global observables for the Abelian gauge theory of principal $U(1)$-bundles equipped with connection can be reconstructed from local ones [2]. Our approach is motivated by the observation that stacks are equivalent to homotopy sheaves of groupoids [6], where the homotopy sheaf condition is formulated in terms of a suitable homotopy limit. Dually, the local-to-global process is performed forming the homotopy colimit of a diagram in chain complexes describing linear observables associated to contractible regions of a fixed manifold. It turns out that the resulting global observables distinguish principal $U(1)$-bundles with connection (up to isomorphism).

Homotopical versions of the causality and time-slice axioms arise instead from quantum field theories defined on categories fibered in groupoids over the category of spacetimes. Examples arise from quantum field theory functors that assign observable algebras to spacetimes equipped with additional structure (forming a fibered category over the category of spacetimes), e.g. spacetimes equipped with a spin structure or with a bundle and possibly a connective structure. It turns out that forming the homotopy right Kan extension of a quantum field theory functor defined on a category fibered in groupoids over spacetimes along its "projection" functor (that only remembers the underlying spacetime) provides a new functor that resembles the behaviour of a gauge theory. In particular, the causality and time-slice axioms of locally covariant quantum field theory (LCQFT) [4] hold only up to homotopy, providing examples of the homotopy LCQFT axioms.

A comprehensive analysis of these structures requires the implementation of higher homotopies taking care of a number of coherencies. This indicates the need for an operadic approach that effectively keeps track of the necessary coherency relations. To this aim, we proposed a novel approach based on encoding the structure of a LCQFT into a coloured operad (later called LCQFT operad): the colours correspond to spacetimes, while the operations arise from spacetime embeddings and algebraic multiplication and are subject to relations encoding the LCQFT axioms. As a result, the LCQFT operad "interpolates" between associativity and commutativity, according to the causal relations between spacetime embeddings.

Generalizing the factorization algebra approach of [5] to the Lorentzian framework, the proposed operadic approach paves the way to completely novel constructions in the realm of LCQFT, which are based on forming free algebras over the LCQFT operad and quotients thereof, and provides a natural candidate to model the homotopical extension of LCQFT, namely the coloured operad which is obtained by a cofibrant resolution [3] of the LCQFT operad.

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Generalised abelian gauge theories

Richard J. Szabo

The configuration space of gauge fields in Maxwell theory on an oriented manifold M is given by the differential cohomology group $\hat{H}^2(M)$, a differential refinement of the degree two integer cohomology $H^2(M,\mathbb{Z})$; this is the group of isomorphism classes of $U(1)$-bundle/connection pairs (P,∇) on M, and the characteristic class map to integer cohomology is just the Chern class $c_1(P)$. The coupling of gauge fields to charged particles is described by the holonomy of the connection along the worldline γ of the particle. This group generalises in various ways to describe configuration spaces of generalised (or 'higher') abelian gauge theories of interest in string theory, see e.g. [1]. In particular, the differential cohomology groups $\hat{H}^k(M)$ in varying degrees describe higher-form fields (e.g. B-fields and C-fields) coupled to elementary charged degrees of freedom (e.g. fundamental strings and membranes). Quantization of these gauge theories in this context was originally carried out in [2] in the case of ultrastatic spacetimes $M = \mathbb{R}\times\Sigma$, where the quantum theory was shown to naturally exhibit the S-duality symmetry of the classical theory. It was subsequently generalised to arbitrary globally hyperbolic spacetimes in [3] within the context of locally covariant quantum field theory, and the implementation of quantum S-duality in this framework was provided in [4], including a fully covariant quantum field theory of the elusive self-dual field in any dimension.

In string theory, there are other higher-form fields which couple to extended degrees of freedom other than the fundamental ones mentioned above; generally, the configuration spaces of such gauge theories are described by generalised differential cohomology theories [1] which are differential refinements of generalised cohomology theories. In particular, the self-dual Ramond-Ramond field, which couples to D-branes, is described by differential (complex) K-theory. In this talk we explained how the quantisation of these abelian gauge theories on ultrastatic spacetimes can be carried out using the formalism of [2]; the key feature is the differential extension of the Adams operation in K-theory which together with the intersection pairing enable the construction of a non-degenerate skew-symmetric pairing between differential K-theory classes. An open problem in this regard is to find a suitable model for differential K-theory which enables the fully covariant quantisation of [3] to be applied. There is also a cochain model for differential K-theory which describes the groupoid of Ramond-Ramond fields, and presumably should enable a full global characterization of Ramond-Ramond fields and their corresponding observables using homotopical methods along the lines of [5]. The quantisation can also be generalised to string orbifolds, whose corresponding differential K-theory was first constructed in [6], by using the model of [7].

References

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