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Algebraic Groups

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Abstract. Linear algebraic groups is an active research area in contemporary mathematics. It has rich connections to algebraic geometry, representation theory, algebraic combinatorics, number theory, algebraic topology, and differential equations. The foundations of this theory were laid by A. Borel, C. Chevalley, J.-P. Serre, T. A. Springer and J. Tits in the second half of the 20th century. The Oberwolfach workshops on algebraic groups, led by Springer and Tits, played an important role in this effort as a forum for researchers, meeting at approximately 3 year intervals since the 1960s. The present workshop continued this tradition, covering a range of topics, with an emphasis on recent developments in the subject.

Mathematics Subject Classification (2010): 14Lxx, 17Bxx, 20Gxx, 14Mxx.

Introduction by the Organisers

The theory of linear algebraic groups originated in the work of E. Picard in the mid-19th century. Picard assigned a "Galois group" to an ordinary differential equation. This construction was developed into what is now known as "differential Galois theory" by J. F. Ritt and E. R. Kolchin in the 1930s and 40s. Their work was a precursor to the modern theory of algebraic groups, founded by A. Borel, C. Chevalley, J. P. Serre, T. A. Springer and J. Tits in the second half of the 20th century. The Oberwolfach workshops on algebraic groups, originated by Springer and Tits, played an important role in this effort as a forum for researchers, meeting at regular intervals since the 1960s.

The present workshop continued this tradition. There were 53 participants from 10 countries: Australia, Canada, Denmark, France, Germany, Great Britain, Italy, the Netherlands, Russia, Switzerland and the United States. The scientific program consisted of 27 lectures and a problem session. The lectures covered a broad range of topics of current interest, including

- spherical varieties over the complex and real numbers,
- intersection theory of toric and spherical varieties, tropical geometry and Newton-Okounkov theory,
- homogeneous spaces and their twisted forms,
- Hessenberg varieties,
- geometric invariant theory,
- Tamagawa numbers,
- quiver varieties, R matrices and related algebras
- cluster varieties with relations to representation theory
- tilting modules,
- structure theory of finite-dimensional Lie algebras in finite characteristic,
- infinite-dimensional Lie algebras.

Recreational activities during the workshop consisted of the traditional Wednesday afternoon hike and a Thursday night "talent show", featuring classical piano and vocal performances by workshop particupants.

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Workshop: Algebraic Groups

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Abstracts

The isogeny category of commutative algebraic groups

Michel Brion

I gave a talk with the same title at the Oberwolfach workshop on Representation Theory of Quivers and Finite Dimensional Algebras, 19 February to 25 February 2017. Both talks are based on the recent papers [2] and [3]. This abstract complements that of the above workshop by focusing on the algebraic group aspects; by an *algebraic group*, we mean a group scheme of finite type over a fixed field k. All groups are assumed *commutative* unless otherwise specified.

The algebraic groups form an abelian category $\mathcal{C}$. This talk addressed homological properties of $\mathcal{C}$ and some related abelian categories, in particular the *homological dimension*, i.e., the smallest $n = \mathrm{hd}(\mathcal{C})$ such that $\mathrm{Ext}^i_{\mathcal{C}}(G, H) = 0$ for all $i > n$ and all $G, H \in \mathcal{C}$. For an algebraically closed field k, we have $\mathrm{hd}(\mathcal{C}) = 1$ if $\mathrm{char}(k) = 0$ and $\mathrm{hd}(\mathcal{C}) = 2$ if $\mathrm{char}(k) > 0$, by results of Serre and Oort (see [8, 10]). This was generalized by Milne to a perfect field $k:$ then $\mathrm{hd}(\mathcal{C}) = 1+\mathrm{cd}(\Gamma)$ if $\mathrm{char}(k) = 0$, and $\mathrm{hd}(\mathcal{C}) = \max(2, 1 + \mathrm{cd}(\Gamma))$ if $\mathrm{char}(k) > 0$, where Γ denotes the absolute Galois group of k, and $\mathrm{cd}(\Gamma)$ its cohomological dimension (see [7]).

The case of an imperfect field k is largely unknown. By recent work of Conrad, Gabber and Prasad (see [4] and [5]), the structure of pseudo-reductive groups reduces to the commutative case, which is treated there as a black box. Recall that a (possibly non-commutative) algebraic group G is called *pseudo-reductive* if G is smooth, connected and has no non-trivial smooth connected normal unipotent subgroup. The first non-trivial examples of commutative pseudo-reductive groups occur in dimension 2: they are exactly the non-split extensions

$$0 \longrightarrow T \longrightarrow G \longrightarrow U \longrightarrow 0,$$

where T is a k-form of $\mathbb{G}_m$ and U a k-form of $\mathbb{G}_a$ (the latter have been described by Russell in [9]). There is a natural isomorphism $\mathrm{Ext}^1_{\mathcal{C}}(U, \mathbb{G}_m) \cong \mathrm{Pic}(U)^U$ (the group of isomorphism classes of translation-invariant line bundles on U) ; moreover, $\mathrm{Pic}(U) \neq 0$ if U is a non-trivial form of $\mathbb{G}_a$, and $\mathrm{Pic}(U)^U \neq 0$ if in addition k is separably closed (see [11] for these results, and [1] for further developments). Also, very little is known on higher extension groups: a conjectural description of $\mathrm{Ext}^2_{\mathcal{C}}(\mathbb{G}_a, \mathbb{G}_m)$ by generators and relations is proposed in [11] in analogy to the Milnor conjecture, and it is an open question whether $\mathrm{Ext}^i_{\mathcal{C}}(\mathbb{G}_a, \mathbb{G}_m) = 0$ for $i \gg 0$.

Consider the full subcategory $\mathcal{F}$ of $\mathcal{C}$ with objects being the finite group schemes; then $\mathcal{F}$ is a Serre subcategory, i.e., is stable under taking subobjects, quotients and extensions. Thus, we may form the quotient category $\underline{\mathcal{C}} := \mathcal{C}/\mathcal{F}$; it is also obtained from $\mathcal{C}$ by inverting all *isogenies*, i.e., morphisms with finite kernel and cokernel. The *isogeny category* $\underline{\mathcal{C}}$ turns out to be much better behaved than the original category $\mathcal{C}$, as every algebraic group is isogenous to a smooth connected one. It follows (as observed by Serre) that $\underline{\mathcal{C}}$ is artinian and noetherian, whereas

$\underline{\mathcal{C}}$ is artinian but not noetherian. Also, every algebraic group G lies in a unique exact sequence in $\underline{\mathcal{C}}$

$$0 \longrightarrow T \times U \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where T is a torus, U a smooth connected unipotent group, and A an abelian variety; we may further assume that U is *split*, i.e., an iterated extension of copies of $\mathbb{G}_a$. In particular, the simple objects in $\underline{\mathcal{C}}$ are exactly $\mathbb{G}_a$, the simple tori (i.e., those containing no non-trivial subtorus) and the simple abelian varieties.

This defines full subcategories $\underline{\mathcal{T}}$, $\underline{\mathcal{U}}$, $\underline{\mathcal{A}}$ of $\underline{\mathcal{C}}$, which turn out to be Serre subcategories; moreover, every morphism between objects of different subcategories is trivial. The isogeny category $\underline{\mathcal{A}}$ of abelian varieties is semi-simple in view of the Poincaré complete reducibility theorem. Also, the isogeny category $\underline{\mathcal{T}}$ of tori is anti-equivalent to the category of discrete finite-dimensional representations of the absolute Galois group Γ over $\mathbb{Q}$; in particular, $\underline{\mathcal{T}}$ is semi-simple as well.

If $\mathrm{char}(k) = 0$, then the category $\mathcal{U}$ of unipotent algebraic groups is equivalent to that of finite-dimensional k-vector spaces. Thus, $\underline{\mathcal{U}} \cong \mathcal{U}$ is semi-simple, and hence so is the isogeny category $\underline{\mathcal{L}} = \underline{\mathcal{T}} \times \underline{\mathcal{U}}$ of linear algebraic groups. By easy homological arguments, it follows that $\mathrm{hd}(\underline{\mathcal{C}}) = 1$; moreover, the projective objects of $\underline{\mathcal{C}}$ are exactly the linear algebraic groups, and the injective objects, the abelian varieties.

If $\mathrm{char}(k) = p > 0$, then $\underline{\mathcal{C}}$ turns out to be invariant under purely inseparable field extensions; thus, we may assume k perfect. Then it is known that $\underline{\mathcal{C}}$ is equivalent to the category of finitely generated torsion modules over a non-commutative discrete valuation ring, obtained as a localization of the Dieudonné ring (see [6, V.3.6]); it follows that $\mathrm{hd}(\underline{\mathcal{U}}) = 1$. For any algebraic group G, the above exact sequence yields two extensions in $\underline{\mathcal{C}}$

$$0 \longrightarrow U \longrightarrow G \longrightarrow S \longrightarrow 0, \qquad 0 \longrightarrow T \longrightarrow S \longrightarrow A \longrightarrow 0,$$

where S is a *semi-abelian variety*. Moreover, the first extension has a unique splitting, since the multiplication by p defines a nilpotent endomorphism of U and an isomorphism of S in the isogeny category. This yields an equivalence of categories $\underline{\mathcal{C}} \cong \underline{\mathcal{U}} \times \underline{\mathcal{S}}$ with an obvious notation. As an easy consequence, we still have $\mathrm{hd}(\underline{\mathcal{C}}) = 1$; moreover, tori are projective objects of $\underline{\mathcal{C}}$, and abelian varieties are injective. If in addition k is algebraic over $\mathbb{F}_p$, then the second extension also has a unique splitting, as follows from the Weil-Barsotti formula for $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, T)$. Hence we obtain equivalences of categories $\underline{\mathcal{S}} \cong \underline{\mathcal{T}} \times \underline{\mathcal{A}}$ and $\underline{\mathcal{C}} \cong \underline{\mathcal{U}} \times \underline{\mathcal{T}} \times \underline{\mathcal{A}}$.

Returning to an arbitrary field k, since $\underline{\mathcal{C}}$ is artinian and noetherian, every object admits a unique decomposition into a direct sum of indecomposables. These are exactly the algebraic groups G admitting no decomposition $G = G_1 + G_2$, where $G_1, G_2 \subset G$ are algebraic subgroups of positive dimension, and $G_1 \cap G_2$ is finite. The indecomposable linear algebraic groups are easily described: these are are exactly the simple tori, and $\mathbb{G}_a$ if $\mathrm{char}(k) = 0$, resp. the groups of truncated Witt vectors W_n $(n \geq 1)$ if $\mathrm{char}(k) > 0$. In contrast, the classification of possibly non-linear indecomposables is largely open; even the simplest cases involve deep conjectures on the arithmetic of abelian varieties (see [3, Sec. 3.5] for details).

Variants of the isogeny category are also worth exploring, such as the quotient category of $\mathcal{C}$ by the Serre subcategory of infinitesimal group schemes. Work in progress indicates that its homological dimension is given by Milne's formulas.

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Classification of torsors over Laurent polynomials

Philippe Gille

(joint work with Vladimir Chernousov, Arturo Pianzola)

Abstract. I will report on joint work with V. Chernousov and A. Pianzola [4]. Given a linear algebraic group G defined over a field k of characteristic zero, G-torsors over Laurent polynomial rings naturally occur in infinite dimensional Lie theory (e.g., the classification and the proof of conjugacy of Cartan subalgebras of Extended Affine Lie Algebras [3, 5]). We explain that one can associate to such a G-torsor another G-torsor, called its loop form, and how that construction clarifies the classification problem of all G-torsors.

Let k be a field of characteristic zero and let k_s be a separable closure of k. Let G be a linear algebraic group defined over k. We are interested in the classification of G-torsors over the ring $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of Laurent polynomials ($n \geq 1$), that is of principal G–bundles over the torus $(\mathbb{G}_m)^n = \mathrm{Spec}(R_n)$. The G-torsors are classified by the étale cohomology pointed set $H^1(R_n, G)$. Our goal is to compute $H^1(R_n, G)$, far from an easy task. In the case of the linear group GL_d, we have that $H^1(R_n, \mathrm{GL}_d) = 1$; this set classifies finitely generated projective R_n-modules of rank d and those modules are free according to Quillen-Suslin-Swan's theorem. The motivation comes from the example of the algebraic group $\mathrm{Aut}(\mathfrak{g})$ where $\mathfrak{g}$ is a finite dimensional split simple Lie k-algebra. In this case, the yoga of forms shows that the set $H^1(R_n, \mathrm{Aut}(\mathfrak{g}))$ classifies isomorphism classes of R_n-forms of $\mathfrak{g}$, that is of Lie R_n-algebras $\mathcal{L}$ with the property that there exists a flat cover S/R_n such that $\mathcal{L} \otimes_{R_n} S \cong \mathfrak{g} \otimes_k S$.

Examples. (a) If $n = 1$ and $k = \mathbb{C}$, the centreless core of an affine Kac-Moody algebra is a R_1-algebra which is R_1-form of some $\mathfrak{g}$.
(b) More generally, for $n \geq 1$ and $k = \mathbb{C}$, the centreless core of an extended affine Lie algebra is a R_n-algebra which is R_n-form of some $\mathfrak{g}$ provided it is finitely generated over its centroid [1, th. 3.3.1].

In both cases, the R_n-Lie algebras occurring carry a grading and are of some are examples of so-called (multi)loop algebras. For defining the so-called loop G-torsors, we introduce the ring R_n^{sc} which is the universal cover of R_n (in the sense of SGA 1). It plays the role of the Galois closure for a field. We have $R_n = \text{limind}_m k_s[t_1^{\frac{1}{m}}, \dots, t_n^{\frac{1}{m}}]$, where m runs over the positive integers. The fundamental group of R_n is $\pi_1(R_n) = Aut_{R_n}(R_n^{sc})$. It is a profinite group which is isomorphic to the projective limit of the $\mu_m(k_s)^n \times \text{Gal}(k_s/k)$. There is a natural isomorphism

$$(*) \qquad H^1(\pi_1(R_n), G(R_n^{sc})) \cong \ker\Big(H^1(R_n, G) \to H^1(R_n^{sc}, G)\Big)$$

where the right hand side is the (continuous) group non-abelian cohomology set of $\pi_1(R_n)$ with coefficients in the $\pi_1(R_n)$-group $G(R_n^{sc})$. Note that the projection map $\pi_1(R_n) \to \mathcal{G}al(k_s/k)$ gives rise to an action of $\pi_1(R_n)$ on $G(k_s)$.

Definition. A G-torsor E is a *loop* G-torsor if its class belongs to the image of the map

$$H^1(\pi_1(R_n), G(k_s)) \to H^1(\pi_1(R_n), G(R_n^{sc})) \hookrightarrow H^1(R_n, G).$$

We denote by $H^1_{loop}(R_n, G) \subseteq H^1(R_n, G)$ the subset of classes of loop torsors.

Remarks (a) The map $H^1(\pi_1(R_n), G(k_s)) \to H^1(\pi_1(R_n), G(R_n^{sc}))$ has no reason to be injective nor surjective. In case $n = 1$, it is true that all G-torsors are loop but for $n \geq 2$, there are exotic (=not loop) G-torsors for example for $G = \text{PGL}_d$ [8, §3].
(b) The acyclicity theorem states that $H^1(R_n^{sc}, G) = 1$ [9] so that we have a bijection $H^1(\pi_1(R_n), G(R_n^{sc})) \cong H^1(R_n, G)$. In other words, the cohomology set $H^1(R_n, G)$ can be computed by means of Galois cohomology cocycles.

Example. Let q be a regular quadratic form over k and consider the orthogonal group $O(q)$. Then $H^1(R_n, O(q))$ classifies regular quadratic R_n-forms of rank $\dim(q)$. By analogy, we can call loop quadratic forms the quadratic forms whose underlying cohomology class is loop. Loop quadratic forms are those of the following form

$$\bigoplus_{I \subseteq \{1,\dots,n\}}^{\perp} q_I \otimes \langle t_I \rangle$$

where q_I is a regular quadratic k–form and $t_I = \prod_{i \in I} t_i$. In dimension 4 and $n = 2$, there are exotic quadratic forms (Ojanguren-Sridharan's construction), and those are not diagonalizable.

Our main result of [4] states that there is a map $H^1(R_n, G) \to H^1_{loop}(R_n, G)$, $\gamma \mapsto \gamma^{loop}$ such that for each class γ, then γ and γ^{loop} coincide locally for the Zariski topology. Such a map is unique and we note that $\gamma^{loop} = (\gamma^{loop})^{loop}$.

For quadratic forms, this implies that we can associate to a regular R_n-quadratic form q a unique diagonalizable R_n-quadratic form q^{loop} such that q and q^{loop} are locally isometric with respect to the Zariski topology.

Remarks (a) We denote by $F_n = k((t_1))\dots((t_n))$ the field of iterated Laurent series. The retraction is defined by using that the composite map

$$H^1_{loop}(R_n, G) \hookrightarrow H^1(R_n, G) \to H^1(F_n, G)$$

is an isomorphism. This map is then not easy to manipulate and the reason to work with F_n is the use of Bruhat-Tits theory. The crucial step is to show that γ and γ^{loop} coincide rationally, something that is accomplished by using the technique of unramified Galois cohomology developed by Colliot-Thélène and Sansuc [6, §3]. To conclude that γ and γ^{loop} coincide locally for the Zariski topology requires Fedorov-Panin's theorem [7] (former Grothendieck-Serre's injectivity conjecture).

(b) The result shows that the classification of G-torsors requires two steps. The first one is the classification of loop torsors and the second one is to compute the Zariski topology cohomology set $H^1_{Zar}(R_n, {}^E G)$ for each loop G-torsor E where ${}^E G$ stands for the twisted R_n-group scheme.

If k is algebraically closed, the first step has been done completely for $n = 2$ [10] and the second step in classical cases (in rank large enough) by Steinmetz-Ziketch [11]. For $n \geq 3$, there are partial results for step one and not much is known concerning the second question beyond the fact that $H^1_{Zar}(R_n, G) = 1$ [9].

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Classification of reductive real spherical pairs

Henrik Schlichtkrull
(joint work with Friedrich Knop, Bernhard Krötz, Tobias Pecher)

Let G be a connected reductive group over a ground field k, and let $P \subset G$ be a parabolic subgroup, which is minimal with respect to being defined over k. For a subgroup $H \subset G$, also assumed to be defined over k, we say that the homogeneous space G/H is *k-spherical* if it admits an open P-orbit. The notion is well-known for algebraically closed fields, in which case P is a Borel subgroup, but has only recently been systematically studied over non-closed fields (see [2]). When $k = \mathbb{R}$ the pair (G, H) and the pair $(\mathfrak{g}, \mathfrak{h})$ of their Lie algebras are then said to be *reductive real spherical pairs* if H is reductive and G/H is $\mathbb{R}$-spherical. The property depends only on the pair $(\mathfrak{g}, \mathfrak{h})$ and only up to isomorphism.

A pair (G, H) as above is called *absolutely spherical* if it is $\bar{k}$-spherical, and it is an easy lemma that absolute sphericality implies k-sphericality. When G is quasisplit these two notions of sphericality are equivalent, but in general they are not. This is particularly apparent in the extreme case where G is *elementary*, that is, modulo its center it has rank zero over k. In this case $P = G$ and hence every pair (G, H) with $H \subset G$ is k-spherical.

The purpose of the talk was to introduce a recently obtained classification of the reductive real spherical pairs, up to local isomorphism and under a certain assumption of irreducibility which will be explained below. The tables and the proofs are given in two articles, [3] which covers the case where G is simple, and [4] which covers the general case. The work is based on the known classifications for $k = \mathbb{C}$ by Krämer [7] and Brion/Mikityuk [1, 8].

In order to describe the irreducibility assumption the following definition is needed. Let L be a connected real reductive group. For its Lie algebra $\mathfrak{l}$ we write $\mathfrak{l} = \mathfrak{l}_{\mathrm{n}} \oplus \mathfrak{l}_{\mathrm{e}}$ for the decomposition into the sum $\mathfrak{l}_{\mathrm{n}}$ of all non-compact simple ideals and the maximal elementary ideal $\mathfrak{l}_{\mathrm{e}}$. Likewise we write L_{n} and L_{e} for the corresponding normal subgroups of L.

The relevant notion of irreducibility is then defined as follows. The pair $(\mathfrak{g}, \mathfrak{h})$ is said to be *indecomposable* if there exists no non-trivial decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ in ideals such that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{g}_1) \oplus (\mathfrak{h} \cap \mathfrak{g}_2)$, and it is said to be *strictly indecomposable* if $(\mathfrak{g}, \mathfrak{h}_{\mathrm{n}})$ is indecomposable. Our classification concerns the strictly indecomposable pairs.

It follows from the classification that in a strictly indecomposable real spherical pair the Lie algebra $\mathfrak{g}$ can have at most three simple factors, whereas under the more general assumption of indecomposability there would be no upper limit to the possible number of factors of $\mathfrak{g}$.

The main tool for the classification is a reduction, which allows one to detect real sphericality of a pair $(\mathfrak{g}, \mathfrak{h}')$ in the case where $\mathfrak{h}' \subset \mathfrak{h} \subset \mathfrak{g}$ and $(\mathfrak{g}, \mathfrak{h})$ is real spherical. It is based on the following local structure theorem from [5].

Theorem 1. Let (G, H) be an $\mathbb{R}$-spherical pair, and let $P \subset G$ be a minimal parabolic subgroup such that $PH \subset G$ is open. Then there exists a parabolic subgroup $Q \supset P$, defined over $\mathbb{R}$, with a Levi decomposition $Q = L \ltimes U$ such that

(1) the morphism

$$U \times L/L \cap H \to G/H, \qquad (u, l) \mapsto ulH$$

is an isomorphism onto PH/H, and

(2) $L_{\mathrm{n}} \subset H$ (and hence L_{e} acts transitively on $L/L \cap H$).

The subalgebra $\mathfrak{l} \cap \mathfrak{h}$ of $\mathfrak{g}$ is called the *structure algebra* of the pair $(\mathfrak{g}, \mathfrak{h})$. The theorem which allows the reduction then reads as follows.

Theorem 2. Let $\mathfrak{h}' \subset \mathfrak{h} \subset \mathfrak{g}$ be a chain of reductive real Lie algebras, and assume that the pair $(\mathfrak{g}, \mathfrak{h})$ is real spherical with structure algebra $\mathfrak{l} \cap \mathfrak{h}$. Then $(\mathfrak{g}, \mathfrak{h}')$ is real spherical if and only if $\mathfrak{h} = \mathfrak{h}' + \mathfrak{l} \cap \mathfrak{h}$ and $(\mathfrak{l} \cap \mathfrak{h}, \mathfrak{l} \cap \mathfrak{h}')$ is real spherical.

The starting point for the classification is then to investigate the maximal reductive (proper) subalgebras of $\mathfrak{g}$. By using Dynkin's classification of these we show the following result.

Theorem 3. Let $(\mathfrak{g}, \mathfrak{h})$ be a strictly indecomposable real spherical pair for which $\mathfrak{h}$ is a maximal reductive subalgebra of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{h})$ is absolutely spherical.

For each absolutely spherical maximal reductive subalgebra $\mathfrak{h}$ one determines the corresponding structure algebra $\mathfrak{l} \cap \mathfrak{h}$. With this information one proceeds then to determine the real spherical pairs $(\mathfrak{g}, \mathfrak{h}')$ for which $\mathfrak{h}' \subset \mathfrak{h}$.

The interest in real reductive spherical pairs is motivated by some recent developments in harmonic analysis of homogeneous spaces. It appears that the geometric property of being spherical will make it possible to establish a reasonable Plancherel decomposition theory for $L^2(G/H)$. For details see [9] and [6].

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Newton polyhedra, tropical geometry and the ring $\mathcal{R}_n(\Lambda)$

Askold Khovanskii
(joint work with Boris Kazarnovskii)

ABSTRACT. The ring of conditions (see [1]) is an intersection theory for algebraic cycles in a reductive group G with coefficients in Λ. I'll talk about such ring $\mathcal{R}_n(\Lambda)$ for $G = (\mathbb{C}^*)^n$ and $\Lambda = \mathbb{Z}, \mathbb{R}, \mathbb{C}$. It can be reduced to the cohomology rings of toric varieties [1]. Tropical geometry [3], [4] suggests [2] another description of $\mathcal{R}_n(\Lambda)$. I will present $\mathcal{R}_n(\Lambda)$ as an extension of Newton polyhedra theory.

1. The ring $\mathcal{R}_n(\Lambda)$. Two k-dimensional cycles $X_1, X_2 \subset (\mathbb{C}^*)^n$ are *equivalent* $X_1 \sim X_2$ if for any $(n-k)$-dimensional cycle $Y \subset (\mathbb{C}^*)^n$ and for almost any $g \in (\mathbb{C}^*)^n$ we have $< X_1, gY >=< X_2, gY >$ (here $< A, B >$ is the intersection index of A and B). If $X_1 \sim X_2$ and $Y_1 \sim Y_2$ then for almost any $g_1, g_2 \in (\mathbb{C}^*)^n$ we have $X_1 \cap g_1Y_1 \sim X_2 \cap g_2Y_2$. *The product* $X * Y$ of equivalence classes X and Y is the equivalence classes of the intersection $X_1 \cap g_1Y_1$ where X_1 and Y_1 are representatives of X and Y. *The ring of conditions* $\mathcal{R}_n(\Lambda)$ is the ring of the equivalence classes with the multiplication $*$ and with the tautological addition.

2. Tropicalization of $\mathcal{R}_n(\Lambda)$. An Λ-*enriched* k-*fan* is a fan $\mathcal{F} \subset \mathbb{R}^n$ of a toric variety equipped with a *weight function* $c : \mathcal{F}_k \to \Lambda$ defined on the set $\mathcal{F}_k$ of all k-dimensional cones from $\mathcal{F}$. The *support* $|\mathcal{F}|$ of $\mathcal{F}$ is the union of all cones $\sigma_i \in \mathcal{F}_k$ such that $c(\sigma_i) \neq 0$. Two enriched k-fans $\mathcal{F}_1$ and $\mathcal{F}_2$ are *equivalent* if: 1) $|\mathcal{F}_1| = |\mathcal{F}_2|$; 2) the weight functions c_1 and c_2 induce the same weight function on every common subdivision of the fans $\mathcal{F}_1$ and $\mathcal{F}_2$.

2.1. Let $\mathcal{F}$ be an enriched k-fan. For a cone $\sigma_i \in \mathcal{F}_k$ let $L_i^\perp \subset (\mathbb{R}^n)^*$ be the space dual to the span L_i of $\sigma_i \subset \mathbb{R}^n$. Let O be an orientation of σ_i. Denote by $e_i^\perp(O) \in \Lambda^{n-k} L_i^\perp$ the $(n-k)$-vector, such that: 1) the integral volume of $|e_i^\perp(O)|$ in $L_i^\perp$ is equal to one; 2) the orientation of $e_i^\perp(O)$ is induced from the orientation O of σ_i and from the standard orientation of $\mathbb{R}^n$. An enriched k-fan $\mathcal{F}$ satisfies *the balance condition* if for any orientation of any $(k-1)$-dimensional cone $\rho \in F_{k-1}$ the relation $\sum e_i^\perp(O(\rho))c(\sigma_i) = 0$ holds, where c is the weight function and summation is taken over all $\sigma_i \in \mathcal{F}_k$ such that $\rho \subset \partial\sigma_i$ and $O(\rho)$ is such orientation of σ_i that the orientation of $\partial\sigma_i$ agrees with the orientation of ρ.

2.2. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be balanced k- and $(n-k)$-fans. Cones $\sigma_i^1 \in \mathcal{F}_1$, $\sigma_j^2 \in \mathcal{F}_2$ with $\dim \sigma_i^1 = k$, $\dim \sigma_j^2 = n-k$ are *a-admissible* for a vector $a \in \mathbb{R}^n$ if $\sigma_i^1 \cap (\sigma_j^2 + a) \neq \emptyset$. Let $C_{i,j}$ be the index of $\Lambda_i \bigoplus \Lambda_j$ in $\mathbb{Z}^n$ where $\Lambda_i = L_i^1 \cap \mathbb{Z}^n$, $\Lambda_j = L_j^2 \cap \mathbb{Z}^n$ and L_i^1, L_j^2 are linear spaces spanned by σ_i^1, σ_j^2. The *intersection number* $c(0)$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ is $\sum C_{i,j} c_1(\sigma_i^1) c_2(\sigma_j^2)$, where $a \in \mathbb{R}^n$ is a generic vector and the sum is taken over all a-admissible couples σ_i^1, σ_j^2. The *product* $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ is a 0-fan $\mathcal{F} = \{0\}$ with the weight $c(0)$ equal to the intersection number.

2.3. Consider a k-fan $\mathcal{F}_1$ and a m-fan $\mathcal{F}_2$ from the set $T\mathcal{R}_n(\Lambda)$ of all balanced Λ-enriched fans. Let d be $n-(k+m)$. If $d<0$ then $\mathcal{F}_1 \times \mathcal{F}_2 = 0$. If $d=0$ the fan $\mathcal{F}_1 \times \mathcal{F}_2$ is already defined. Let us define the d-fan $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ for $d>0$. Assume that $\mathcal{F}_1$ and and $\mathcal{F}_2$ are subfans of a complete fan $\mathcal{G}$. Then $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ also is a subfan of $\mathcal{G}$. The weight $c(\delta)$ of a cone δ from $\mathcal{G}$ with $\dim \delta = d$ is defined below. Let L be a space spanned by the cone δ and let $(\mathcal{F}_1)_\delta$ and $(\mathcal{F}_2)_\delta$ be the enriched subfans of $\mathcal{F}_1$ and of $\mathcal{F}_2$ consisting of all cones from these fans containing the cone δ. The weight $c(\delta)$ of the cone δ in $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ is equal to the intersection number of the images under the factorization of $(\mathcal{F}_1)_\delta$ and $(\mathcal{F}_2)_\delta$ in the factor space $\mathbb{R}^n/L$.

3. Bergman set. A vector $k \in \mathbb{Z}^n$ is *essential* for a variety $X \subset (\mathbb{C}^*)^n$ if there is a germ of meromorphic map $f : (\mathbb{C}, 0) \to X \subset (\mathbb{C}^*)^n$ where $f(t) = at^d + \dots$, $a \in (\mathbb{C}^*)^n$ and $d \in \mathbb{Z}^n$, such that $k = d$. The *Bergman set* $B(X) \subset \mathbb{R}^n$ of X is the closure of the set of vectors $\lambda k \in \mathbb{R}^n$ where k is essential vector for X and $\lambda \geq 0$.

Theorem 1. If each irreducible component of X has complex dimension m then $B(X)$ is a finite union of rational cones σ in $\mathbb{R}^n$ with $\dim_{\mathbb{R}} \sigma = m$. Moreover $B(X)$ is support of a fan (defined up to subdivision of $B(X)$) of some toric variety.

Toric variety $M \supset (\mathbb{C}^*)^n$ is a *good compactification* for a subvariety $X \subset (\mathbb{C}^*)^n$ with $\dim X = k$ if its closure $\overline{X}$ in M is complete and does not intersect orbits in M whose codimension is bigger than k. The following result is proved in [1].

Theorem 2 (see [1])**.** For any finite set $\mathcal{S}$ of algebraic subvarieties in $(\mathbb{C}^*)^n$ there is a toric variety $M \supset (\mathbb{C}^*)^n$ which provides a good compactification for each subvariety from $\mathcal{S}$. Toric variety M is a good compactification of $X \subset (\mathbb{C}^*)^n$ if and only if the support of its fan contains the Bergman set $B(X)$.

4. For a complete smooth toric variety $M \supset (\mathbb{C}^*)^n$ and for any k-dimensional cycle $X = \sum k_i X_i$ one can defined the cycle $\overline{X}$ in M as $\sum k_i \overline{X}_i$ where $\overline{X}_i$ is the closure in M of $X_i \subset (\mathbb{C}^*)^n$. The cycle $\overline{X}$ defines an element $\rho(\overline{X})$ in $H^{2(n-k)}(M^n, \Lambda)$ whose value on the closure $\overline{O}_i$ of an $(n-k)$-dimensional orbit O_i in M is equal to the intersection index $< \overline{X}, \overline{O}_i >$. A compactification $M \supset (\mathbb{C}^*)^n$ is *good* for a cycle $X = \sum k_i X^i$ in $(\mathbb{C}^*)^n$ if it is good compactification for each X_i.

Theorem 3 (see [1])**.** If a smooth toric compactification M is good for cycles X, Y and Z where $Z = X * Y$, then the product $\rho(X)\rho(Y)$ in the cohomology ring $H^*(M, \Lambda)$ of the elements $\rho(X)$ and $\rho(Y)$ is equal to $\rho(Z)$.

5. Let $\Delta^\perp$ be a fan of a smooth complete projective toric variety M^n_Δ. Let $T\mathcal{R}_n(\Lambda, \Delta)$ be the ring of balanced Λ-enriched fans equal to Λ-linear combination of cones from the fan $\Delta^\perp$. The following theorems 5,6 are proved in [2].

Theorem 4 (see [2])**.** The ring $T\mathcal{R}_n(\Lambda, \Delta)$ is isomorphic to the intersection ring $H_*(M_\Delta, \Lambda)$. The component of $T\mathcal{R}_n(\Lambda, \Delta)$ consisting of k-fans under this isomorphism corresponds to the component $H_{2k}(M_\Delta, \Lambda)$.

Theorem 5 (see [2]). The ring of conditions $\mathcal{R}_n(\Lambda)$ is isomorphic to the tropical ring $T\mathcal{R}_n(\Lambda)$ of all balanced Λ-enriched fans.(The rings $\mathcal{R}_n(\mathbb{Z})$, $\mathcal{R}_n(\mathbb{C})$ have similar descriptions).

6. To a homogeneous polynomial P on a $\mathbb{R}$-linear space $\mathcal{L}$, $\dim \mathcal{L} < \infty$, one can associate the graded commutative ring $A(\mathcal{L}, P)$ (one can produce a similar constructions for homogeneous polynomials on infinite dimensional spaces over any field and for functions analogues to homogeneous polynomials on free abelian groups). Let $D(\mathcal{L})$ be the ring of $\mathbb{R}$-linear differential operators on $\mathcal{L}$ with constant coefficients. Let $I_P \subset D(\mathcal{L})$ be a set defined by the following condition: $L \in I_P \Leftrightarrow L(P) \equiv 0$. It is easy to see that I_P is a homogeneous ideal. By definition the *ring associated to* P is the factor ring $A(\mathcal{L}, P) = D(\mathcal{L})/I_P$. One can to see that: (1) $A(\mathcal{L}, P)$ is a graded ring with homogeneous components $A_0, \dots, A_n$ where $n = \deg P$; (2) $A_0 = \mathbb{R}$; (3) there is a non-degenerate pairing between A_k and A_{n-k} with values in A_0, thus $A_k = A^*_{(n-k)}$ and $A_n \sim \mathbb{R}$.

7. Let L_Δ be the space of formal differences of convex polyhedra whose support functions are linear on each cone from the fan of a smooth projective toric variety M_Δ. Let $n!V$ be the degree n homogeneous polynomial on L_Δ whose value on $\tilde{\Delta} \in L_\Delta$ is equal to the volume of $\tilde{\Delta}$ multiplied by $n!$.

Theorem 6. The intersection ring $H_*(M_\Delta, \mathbb{R})$ is isomorphic (up to a change of the grading) to the ring $A(L_\Delta, n!V)$.

Let $\mathcal{L}_n$, $\dim \mathcal{L}_n = \infty$, be the space of formal differences of convex polyhedra Δ with rational dual fans $\Delta^\perp$. Let $n!V$ be the degree n homogeneous polynomial on $\mathcal{L}_n$ whose value on $\Delta \in \mathcal{L}_n$ is equal to the volume of Δ multiplied by n!.

Theorem 7. The ring $\mathcal{R}_n(\mathbb{R})$ is isomorphic to the ring $A(\mathcal{L}_n, n!V)$. (The rings $\mathcal{R}_n(\mathbb{Z})$, $\mathcal{R}_n(\mathbb{C})$ have similar descriptions).

8. Let $\{\Gamma_i\}$ be a set of n hypersurfaces in $(\mathbb{C}^*)^n$ defined by $P_i = 0$ where P_i are Laurent polynomials with Newton polyhedra Δ_i. Bernstein-Koushnirenko-Khovanskii theorem (BKK theorem) can be stated in the following two ways:

Theorem 8. The intersection number of the hypersurfaces Γ_i in the ring of conditions is equal to the mixed volume of $\Delta_1, \dots, \Delta_n$ multiplied by $n!$.

Let $\mathcal{F}_i$ be $\mathbb{R}$-enriched $(n-1)$-fan dual to Δ_i whose weight function at a cone σ dual to a side $\sigma^\perp$ of Δ_i is equal to the integral length of the $\sigma^\perp$.

Theorem 9. The intersection number of the hypersurfaces Γ_i in the ring of conditions is equal to the intersection number of the $\mathbb{R}$-enriched fans $\mathcal{F}_i$ in the ring $T\mathcal{R}_n$.

Thus theorems 7 and 5 could be considered as generalizations of the BKK theorem. Such generalizations are possible because of the following reason: the cohomology ring of a smooth toric variety is generated by elements of degree two.

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Gröbner theory, tropical geometry and spherical varieties

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(joint work with Christopher Manon)

We discuss some recent developments in spherical tropical geometry and spherical Gröbner theory following [6] and [1]. The idea of developing a tropical geometry from spherical varieties goes back to Gary Kennedy [2]. We would also like to mention [5] which defines a notion of tropicalization for spherical embeddings.

Let $\mathbf{k}$ be an algebraically closed filed of characteristic 0. From the point of view of algebraic geometry, tropical geometry is the study of behavior at infinity of subvarieties of an algebraic torus $(\mathbf{k}^*)^n$. Let $Y \subset (\mathbf{k}^*)^n$ be a subvariety. The behavior at infinity of Y is encoded by a rational polyhedral fan in $\mathbb{Q}^n$ called the *tropical variety* or *tropical fan* of Y. More precisely, let $\mathcal{K} = \mathbf{k}\{\{t\}\}$ denote the field of formal Puiseux series in one variable t (recall that $\mathbf{k}\{\{t\}\}$ is the algebraic closure of the field of formal Laurent series $\mathbf{k}((t))$). The field $\mathbf{k}\{\{t\}\}$ comes with a natural valuation $val : \mathcal{K} \setminus \{0\} \to \mathbb{Q}$ which assigns to a Puiseux series the exponent of its smallest term in t. Let $\text{Trop} : (\mathcal{K}^*)^n \to \mathbb{Q}^n$ be the map defined by

$$\text{Trop}(\gamma) = (val(\gamma_1), \dots, val(\gamma_n)),$$

where $\gamma = (\gamma_1, \dots, \gamma_n) \in (\mathcal{K}^*)^n$. The tropical variety of $Y \subset (\mathbf{k}^*)^n$ is simply $\text{Trop}(Y(\mathcal{K}))$, that is, the image of $Y(\mathcal{K})$ under the above map Trop. It is a basic result in tropical geometry that $\text{Trop}(Y)$ is the support of a rational polyhedral fan in $\mathbb{Q}^n$ (see [4, Section 3.3]).

It is a natural question how one can describe $\text{Trop}(Y)$ if Y is given as a zero set of an ideal I in the Laurent polynomial algebra $\mathbf{k}[x_1^{\pm}, \dots, x_n^{\pm}]$. The answer to this question is the content of the so-called fundamental theorem of tropical geometry. Let $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a Laurent polynomial where $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$. To f one assigns its *tropical polynomial* F which is a piecewise linear function defined by:

$$F(w) = \min\{w \cdot \alpha \mid c_\alpha \neq 0\}.$$

The tropical hypersurface $V(F)$ associated to f is by definition the set of w where the above minimum is attained at least twice. Finally, the *tropical variety* $\text{trop}(I)$ associated to an ideal $I \in K[x_1^{\pm}, \dots, x_n^{\pm}]$ is $\text{trop}(I) = \bigcup_{f \in I} V(F)$. It is also a basic result in tropical geometry that in the above intersection only a finite number of the f suffice. But it is not enough to take a generating set for I.

Now, let $Y \subset (\mathbf{k}^*)^n$ be a subvariety with ideal $I = I(Y) \in \mathbf{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$. The fundamental theorem of tropical geometry asserts that the two sets $\mathrm{Trop}(Y)$ and $\mathrm{trop}(I)$ coincide ([4, Section 3.2]).

The proofs of the above results rely on the Gröbner theory of ideals in a polynomial ring. In particular, the notion of Gröbner fan of a homogeneous ideal.

Now we explain the extension of the above to the setting of spherical varieties. Let G be a connected reductive algebraic group over $\mathbf{k}$ with a maximal torus T and a Borel subgroup B containing T. We let Λ denote the weight lattice of G with Λ^+ the semigroup of dominant weights. A (normal) G-variety X is called *spherical* if the Borel subgroup B has an open orbit in X. It is a generalization of the notion of a toric variety to reductive group actions. Some basic notions associated to X are the following. The weight lattice Λ_X is the sublattice of Λ consisting of weights of B-eigenfunctions for the action of G on the field of rational functions $\mathbf{k}(X)$. Also the valuation cone $\mathcal{V}_X$ is the collection of all G-invariant valuations $v : \mathbf{k}(X) \setminus \{0\} \to \mathbb{Q}$. It can be identified with a simplicial cone in the $\mathbb{Q}$-vector space $\mathrm{Hom}(\Lambda_X, \mathbb{Q})$.

Let G/H be a spherical homogeneous space. For any formal Puiseux curve $\gamma \in G/H(\mathcal{K})$, i.e. a point on G/H defined over $\mathcal{K}$, one can associate a G-invariant valuation $v_\gamma \in \mathcal{V}_{G/H}$ defined as follows. For any $0 \neq f \in \mathbf{k}(X)$ let:

$$v_\gamma(f) = val(g \cdot f_{|\gamma}),$$

for all g in a Zariski open subset $U_f \subset G$. The notion of an invariant valuation associated to a formal curve already appears in the fundamental paper [3]. In [6], the map $\mathrm{Trop} : G/H(\mathcal{K}) \to \mathcal{V}_{G/H}$, $\gamma \mapsto v_\gamma$, is suggested as the spherical tropicalization map.

Theorem 1 (Vogiannou)**.** *Let $Y \subset G/H$ be a subvariety. Let $\mathrm{Trop}(Y)$ denote the image of $Y(\mathcal{K})$ under Trop. Then $\mathrm{Trop}(Y)$ is the support of a rational polyhedral fan in the valuation cone $\mathcal{V}_{G/H}$. Moreover, there is a fan Σ with support $\mathrm{Trop}(Y)$ such that the corresponding spherical embedding gives a tropical compactification of Y.*

In [1, Section 3] a Gröbner theory is developed on the coordinate ring $A = \mathbf{k}[X]$ of an affine spherical variety X. Namely, given a total order $\succ$ on the lattice Λ_X and an ideal $I \subset A$, one defines the initial ideal $\mathrm{in}_\succ(I)$ which lives in the horospherical contraction A_{hc} of A. This gives rise to the notion of a spherical Gröbner basis for I. It is shown that every ideal I has a finite number of initial ideals. Similarly, given a valuation $v \in \mathcal{V}_X$, one defines the initial ideal $\mathrm{in}_v(I)$ which lives in the associated graded algebra $\mathrm{gr}_v(A)$. The algebra $\mathrm{gr}_v(A)$ depends, up to G-algebra isomorphism, only on the face of $\mathcal{V}_X$ on which v lies on. Moreover, if v lies in the interior, then $\mathrm{gr}_v(A)$ is the horospherical contraction A_{hc}. When A is $\mathbb{Z}_{\geq 0}$-graded and I is a homogeneous ideal, the existence of spherical Gröbner fan is proved.

Next, let X_B be the open B-orbit in spherical homogeneous space $X = G/H$. In [1, Section 4.1] it is shown how to define the notion of a spherical tropical variety $\mathrm{trop}(Z)$ for a subvariety $Z \subset X_B$ in terms of the defining ideal of Z. Using the spherical Gröbner theory developed before it is shown that this spherical tropical variety is the support of a rational polyhedral fan in $\mathcal{V}_{G/H}$.

Finally, we have a spherical version of the fundamental theorem of tropical geometry ([1, Section 4.5]).

Theorem 2 (K.-Manon)**.** *Let $Y \subset G/H$ be a subvariety. The following coincide:*

(a) *The set* $\mathrm{trop}(Y) = \bigcup_B \mathrm{trop}(Y \cap X_B)$, *where the union is over all Borel subgroups of G (one shows that it is enough to take the union over a finite collection of Borel subgroups).*

(b) *The set* $\mathrm{Trop}(Y) = \{\mathrm{Trop}(\gamma) \in \mathcal{V}_X \mid \gamma \in Y(\mathcal{K})$ formal Puiseux curve in Y$\}$.

At the end, we address the Archimedean version of the notion of a tropical variety namely that of an amoeba. In the torus case and over $\mathbf{k} = \mathbb{C}$, given $t \in \mathbb{R}$, the map $L_t : (\mathbb{C}^*)^n \to \mathbb{R}^n$ defined by $(z_1, \ldots, z_n) \mapsto (\log_t |z_1|, \ldots, \log_t |z_n|)$ is called the logarithm map. For a subvariety $Y \subset (\mathbb{C}^*)^n$, its image under L_t is called the *amoeba* of Y. It is well-known that as $t \mapsto 0$, the amoeba of Y approaches (in Hausdorff distance) to the tropical variety of Y.

In [1, Section 6] a spherical version of the notion amobea is suggested. It is defined whenever there is an Archimedean Cartan decomposition for the spherical homogeneous space $X = G/H$ over $\mathbf{k} = \mathbb{C}$.

When $X = \mathrm{GL}(n, \mathbb{C})$ regarded as a $(\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}))$-spherical homogeneous space, the notions of spherical logarithm map and spherical tropicalization have familiar linear algebra meanings. For $g \in \mathrm{GL}(n, \mathbb{C})$, the spherical logarithm map sends g to $\log_t$ of its singular values, i.e. the square roots of eigenvalues of $g\bar{g}^t$. For a formal Puiseux curve $g(t) \in \mathrm{GL}(n, \mathcal{K})$, the tropicalization map sends $g(t)$ to its invariant factors. The following is the spherical version of the fact that amoeba approaches the tropical variety in this case. It follows from Hilbert-Courant min-max principle.

Theorem 3 (K.-Manon)**.** *Let $g(t)$ be an $n \times n$ matrix whose entries g_{ij} are Laurent series in t with nonzero radii of convergence. For sufficiently small $t \neq 0$, let $d_1(t) \leq \cdots \leq d_n(t)$ denote the singular values of $g(t)$ ordered increasingly. Also let $v_1 \geq \cdots \geq v_n$ be the invariant factors of $g(t)$ ordered decreasingly. We then have:*

$$\lim_{t \to 0} (\log_t(d_1(t)), \ldots, \log_t(d_n(t))) = (v_1, \ldots, v_n).$$

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Maximal Lie subalgebras of exeptional Lie algebras in good characteristic

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(joint work with David Stewart)

Let G be a simple algebraic group over an algebraically closed field $\mathbf{k}$ and $\mathfrak{g} = \mathrm{Lie}(G)$. In the 1950's, E.B. Dynkin classified the maximal Lie subalgebras of $\mathfrak{g}$ in the case where $\mathrm{char}(\mathbf{k}) = 0$. As a consequence, he obtained a classification of maximal connected subgroups of G. In the case where $\mathrm{char}(\mathbf{k}) = p > 0$, the problem of classifying the maximal connected subgroups of G was solved quite recently in a series of papers by Seitz, Testerman and Liebeck–Seitz. The problem of classifying the maximal Lie subalgebras of $\mathfrak{g}$ in the case where $\mathrm{char}(\mathbf{k}) = p > 0$ is wide open at the moment.

From now on we let G be a simple algebraic group of type G_2, F_4, E_6, E_7 or E_8 over an algebraically closed field $\mathbf{k}$ and assume that $p = \mathrm{char}(\mathbf{k})$ is a good prime for G. This means that $p > 5$ if G is of type E_8 and $p > 3$ in the other cases. Let $\mathcal{O}$ denote the truncated polynomial algebra $\mathbf{k}[X]/(X^p)$. The Witt algebra $W = \mathrm{Der}(\mathcal{O})$ is a free $\mathcal{O}$-module of rank 1 generated by $\mathrm{d}/\mathrm{d}X$. It is a simple Lie algebra of dimension p. If $\mathcal{D}$ is a Lie subalgebra of W which does not stabilise the maximal ideal of the local ring $\mathcal{O}$, then the semidirect product $L_{\mathcal{D}} := (\mathrm{Id}_{\mathfrak{sl}(2)} \otimes W) \ltimes (\mathfrak{sl}(2) \otimes \mathcal{O})$ is a semisimple Lie algebra and $\mathfrak{sl}(2) \otimes \mathcal{O}$ is the unique minimal ideal of $L_{\mathcal{D}}$. We call Lie algebras of this type *exotic semidirect products*.

The aim of the talk is to announce the description of the maximal Lie subalgebras $\mathfrak{m}$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. More specifically, we show that either $\mathfrak{m} = \mathrm{Lie}(M)$ for some maximal connected subgroup M of G or $\mathfrak{m}$ is a maximal Witt subalgebra of $\mathfrak{g}$ or $\mathfrak{m}$ is an exotic semidirect product.

If $\mathfrak{m} = \mathrm{Lie}(M)$ for some maximal connected subgroup M of G, then $\mathfrak{m}$ is known thanks to the aforementioned work of Seitz, Testerman and Liebeck–Seitz. All maximal Witt subalgebras $\mathfrak{m}$ of $\mathfrak{g}$ are G-conjugate and they occur if and only if G is not of type E_6 and $p-1$ equals the Coxeter number of G. We can choose root vectors $e_1, \ldots, e_\ell \in \mathfrak{g}$ associated with a basis of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ of G in such a way that $\mathfrak{m}$ is generated by the regular nilpotent element $e_1 + \cdots + e_\ell$ of $\mathfrak{g}$ and a root vector $e_{-\widetilde{\alpha}}$ corresponding to the lowest root $-\widetilde{\alpha}$ with respect to Π.

We show that there are two conjugacy classes of maximal exotic semidirect products $\mathfrak{m} \cong L_{\mathcal{D}}$ in $\mathfrak{g}$, one in characteristic 5 and one in characteristic 7. Both of them occur when G is a group of type E_7. If $p = 5$ then $\mathcal{D} \cong \mathfrak{sl}(2)$ and if $p = 7$ then $\mathcal{D} = W$. Our arguments rely on the main results of [2], [1] and [3].

As a consequence, we obtain that there are finitely many conjugacy classes of maximal Lie subalgebras of $\mathfrak{g}$. In bad characteristic, there are new examples of non-classical maximal subalgebras [4] and the whole picture remains incomplete. Nevertheless, it is conjecturable that the number of conjugacy classes of maximal Lie subalgebras of exceptional Lie algebras is finite regardless of the characteristic.

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Modular Koszul duality for constructible sheaves on flag varieties

Simon Riche

(joint work with Pramod N. Achar, Shotaro Makisumi and Geordie Williamson)

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1. Representation theory of reductive algebraic groups

Let $\mathbb{K}$ be an algebraically closed field of characteristic p, let G be a connected reductive $\mathbb{K}$-algebraic group, let $B \subset G$ be a Borel subgroup, and let $T \subset B$ be a maximal torus. We consider the category $\mathrm{Rep}(G)$ of finite-dimensional algebraic representations of G. It has a natural structure of highest weight category, with weight poset the dominant weights $\mathbb{X}^+$ with the standard order. The costandard, resp. standard, objects are the induced modules $\mathsf{N}(\lambda) := \mathrm{Ind}_B^G(\lambda)$, resp. the Weyl modules $\mathsf{M}(\lambda) := \big(\mathrm{Ind}_B^G(-w_0\lambda)\big)^*$, where w_0 is the longest element in the Weyl group $W_{\mathrm{f}} := N_G(T)/T$. In particular the simple objects in $\mathrm{Rep}(G)$ are parametrized by dominant weights: to λ we associate the image $\mathsf{L}(\lambda)$ of the unique (up to scalar) nonzero morphism $\mathsf{M}(\lambda) \to \mathsf{N}(\lambda)$. Other objects of interest are the *tilting* objects in $\mathrm{Rep}(G)$, i.e. those which admit both a standard filtration and a costandard filtration. The indecomposable such objects are also parametrized in a natural way by $\mathbb{X}^+$, and we denote by $\mathsf{T}(\lambda)$ the object associated with λ.

In the Grothendieck group $[\mathrm{Rep}(G)]$ we have $[\mathsf{M}(\lambda)] = [\mathsf{N}(\lambda)]$ for any $\lambda \in \mathbb{X}^+$; moreover these classes form a $\mathbb{Z}$-basis. Since the representations $\mathsf{M}(\lambda)$ and $\mathsf{N}(\lambda)$ are relatively well understood (in particular, their character is known), to "describe" an object V in $\mathrm{Rep}(G)$ amounts to expressing the coefficients $\{a_\lambda(M) : \lambda \in \mathbb{X}^+\}$ in the expansion $[V] = \sum_{\lambda \in \mathbb{X}^+} a_\lambda(V) \cdot [\mathsf{M}(\lambda)]$. Therefore, a basic question in this area is the following:

(1) *Describe the integers* $a_\lambda(\mathsf{L}(\mu))$ *and* $a_\lambda(\mathsf{T}(\mu))$ *for* $\lambda, \mu \in \mathbb{X}^+$.

Assume now that $p \geq h$, where h is the Coxeter number of G. Let $W := W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi$ be the affine Weyl group. (Here, Φ is the root system of (G,T).) This group acts on $X^*(T)$ via the "dot action" defined by $(vt_\lambda) \cdot_p \mu := v(\mu + p\lambda + \rho) - \rho$ for $v \in W_{\mathrm{f}}$, $\lambda \in \mathbb{Z}\Phi$ and $\mu \in X^*(T)$, where ρ is the half sum of the positive roots. We set ${}^{\mathrm{f}}W := \{w \in W \mid w \cdot_p 0 \in \mathbb{X}^+\}$. This subset of W does not depend on p, and consists of the elements w which are minimal in $W_{\mathrm{f}}w$ (for the standard Coxeter group structure on W). Classical work of Andersen and Jantzen reduces our basic question (1) to the case when λ, μ are of the form $w \cdot_p 0$ for some $w \in {}^{\mathrm{f}}W$. If moreover $p \geq 2h-2$, further work of Andersen shows that the integers $\{a_{y \cdot_p 0}(\mathsf{L}(w \cdot_p 0)) : y, w \in {}^{\mathrm{f}}W\}$ can be expressed in terms of the integers $\{a_{y \cdot_p 0}(\mathsf{T}(w \cdot_p 0)) : y, w \in {}^{\mathrm{f}}W\}$. Therefore, the main question we consider is:

(2) *Describe the integers* $b_{y,w} := a_{y \cdot_p 0}(\mathsf{T}(w \cdot_p 0))$ *for* $y, w \in {}^{\mathrm{f}}W$.

The main result of [1] is a solution to this question, valid as soon as $p > h$. The answer we give is in terms of the *p-canonical basis* of the Hecke algebra of W.

2. The p-canonical basis

As mentioned above, W has a natural structure of Coxeter group, and we denote by S the corresponding set of simple reflections. The associated *Hecke algebra* is the $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{H}$ with a basis $\{H_w : w \in W\}$ and multiplication determined by the following rules:

- $H_v H_w = H_{vw}$ if $v, w \in W$ and $\ell(vw) = \ell(v) + \ell(w)$;
- $(H_s + v)(H_s - v^{-1}) = 0$ if $s \in S$.

One can construct interesting elements in this algebra out of geometry, as follows. Let $G^\wedge$ be the complex simply-connected semisimple algebraic group whose maximal torus $T^\wedge$ has character group $X_*(T)$, and whose root system is the coroot system of (G, T). Let $B^\wedge \subset G^\wedge$ be the Borel subgroup containing $T^\wedge$ and whose roots are the negative coroots of G. We consider the group $G^\wedge(\mathbb{C}((z)))$ and its Iwahori subgroup I, defined as the inverse image of $B^\wedge$ under the morphism $G^\wedge(\mathbb{C}[[z]]) \to G^\wedge$ sending z to 0, and the *affine flag variety* $\mathsf{Fl} := G^\wedge(\mathbb{C}((z)))/I$. The I-orbits on Fl are parametrized in a natural way by W, which gives rise to the *Bruhat decomposition* $\mathsf{Fl} = \bigsqcup_{w \in W} \mathsf{Fl}_w$. For any field $\mathbb{F}$, we denote by $D^{\mathrm{b}}_{(I)}(\mathsf{Fl}, \mathbb{F})$ the derived category of $\mathbb{F}$-sheaves on Fl which are constructible with respect to this stratification. For any $\mathcal{F}$ in $D^{\mathrm{b}}_{(I)}(\mathsf{Fl}, \mathbb{F})$, we can then consider the element

$$\mathrm{ch}(\mathcal{F}) := \sum_{w \in W} \sum_{k \in \mathbb{Z}} \dim \mathsf{H}^{-\ell(w)-k}(\mathsf{Fl}_w, j_w^* \mathcal{F}) \cdot v^k H_w \quad \in \mathcal{H}.$$

(Here, $j_w : \mathsf{Fl}_w \hookrightarrow \mathsf{Fl}$ is the embedding.)

In particular, consider the *parity sheaves* $\{\mathcal{E}_w : w \in W\}$ (in the sense of Juteau–Mautner–Williamson) with coefficients in $\mathbb{F}$ associated with the stratification $\mathsf{Fl} = \bigsqcup_{w \in W} \mathsf{Fl}_w$. Then if $p = \mathrm{char}(\mathbb{F})$, the *p-canonical basis* $\{{}^p\underline{H}_w : w \in W\}$ is the $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}$ defined by ${}^p\underline{H}_w = \mathrm{ch}(\mathcal{E}_w)$. We also define the *p-Kazhdan–Lusztig polynomials* as the coefficients $\{{}^p h_{y,w} : y, w \in W\}$ in $\mathbb{Z}[v, v^{-1}]$ appearing in the expansion ${}^p\underline{H}_w = \sum_{y \in W} {}^p h_{y,w} \cdot H_y$.

These objects only depend on p (and not on $\mathbb{F}$ itself). For fixed $w \in W$, if $p \gg 0$ the element ${}^p\underline{H}_w$ coincides with the element $\underline{H}_w$ of the Kazhdan–Lusztig basis of $\mathcal{H}$; but for smaller p these elements are more difficult to compute. Note however that there exists another description of this basis, based on a "diagrammatic" category introduced by Elias–Williamson; this description leads to an algorithm for computing this basis which has been implemented by Williamson.

3. The tilting character formula

We now consider the setting of Section 2 when $\mathbb{F} = \mathbb{K}$. The main result of [1] is the following:

Assume that $p > h$. For any $w, y \in {}^{\mathrm{f}}W$ we have $b_{y,w} = \sum_{z \in W_{\mathrm{f}}} (-1)^{\ell(z)} \cdot {}^{p}h_{zy,w}(1)$.

The proof of this formula proceeds in 3 main steps.

(1) In previous work with P. Achar we defined the notion of *mixed perverse $\mathbb{K}$-sheaves* on flag varieties of Kac–Moody groups. These objects form a highest weight category, so that we can consider the indecomposable *tilting* mixed perverse sheaves. In [2] we expressed the coefficients $b_{y,w}$ in terms of the combinatorics of such objects on the *affine Grassmannian* Gr of $G^{\wedge}$, thus reducing problem (2) to the similar problem for these mixed tilting perverse sheaves on Gr.

(2) In [1] we develop a "Koszul duality" formalism for mixed perverse sheaves on flag varieties of Kac–Moody groups, which allows to express the combinatorics of tilting mixed perverse sheaves on Fl in terms of the p-canonical basis of $\mathcal{H}$.

(3) Finally, in [1] again, we show that the pushforward functor associated with the projection morphism $\mathsf{Fl} \to \mathsf{Gr}$ sends every indecomposable tilting mixed perverse sheaf on Fl either to 0 or to an indecomposable tilting mixed perverse sheaf. This allows to express the combinatorics of tilting mixed perverse sheaves on Gr in terms of the p-canonical basis of $\mathcal{H}$, and finally settles the question.

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On tilting characters for $\mathbf{SL_3}$

Geordie Williamson

(joint work with George Lusztig)

Let λ be a partition of n and S_λ the corresponding simple representation of the symmetric group Sym_n over $\mathbb{Q}$. By reducing a Sym_n-stable $\mathbb{Z}$-lattice modulo p we obtain a representation $\overline{S_\lambda}$ of Sym_n over $\mathbb{F}_p$ (the finite field with p elements). We can then try to express $\overline{S_\lambda}$ in the Grothendieck group:

$$[\overline{S_\lambda}] = \sum_{U \in \widehat{\mathbb{F}_p \mathrm{Sym}_n}} m_{U,S_\lambda}[U], \tag{1}$$

where $\widehat{\mathbb{F}_p\mathrm{Sym}_n}$ denotes the set of irreducible representations of $\mathbb{F}_p\mathrm{Sym}_n$. The numbers $m_{U,V}$ are well-defined (i.e. do not depend on the choice of integral lattice) and are called *decomposition numbers*. Despite much work over the last three decades, these numbers are very mysterious. For example, although one has a good description of the set $\widehat{\mathbb{F}_p\mathrm{Sym}_n}$ the dimension of almost all $U \in \widehat{\mathbb{F}_p\mathrm{Sym}_n}$ is unknown.

One fruitul way of attacking this problem is via Schur-Weyl duality. If V is a vector space over $\mathbb{F}_|$ then for any n one has a surjection

$$\mathbb{F}_p\mathrm{Sym}_n \to \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes n}) \tag{2}$$

where the right hand side indicates endomorphisms in the category of algebraic representations of the algebraic group $\mathrm{GL}(V)$. The following is not difficult: Suppose that we can find a decomposition

$$V^{\otimes n} = \bigoplus M_T \otimes T \tag{3}$$

where the sum runs over pairwise non-isomorphic indecomposable modules T, and M_T denotes a multiplicity vector space. Then via (2) each M_T can be made into a $\mathbb{F}_p\mathrm{Sym}_n$-module and is even an irreducible $\mathbb{F}_p\mathrm{Sym}_n$-module.

It was noticed by Donkin that the summands occuring in (3) are of a special form. They are so-called tilting modules. I won't define what this means. Below the reader only needs to know that they are classified by highest weight, just like simple (algebraic) $\mathrm{GL}(V)$-modules.

Because the character of $V^{\otimes n}$ is known, if one knows the characters of tilting modules for $\mathrm{GL}(V)$ then determining the dimensions involved in the decomposition (3) is easy. By the observations of the previous paragraph, this gives the dimensions of many simple Sym_n-modules. This partially explains why the following two problems are equivalent (as was first pointed out by Donkin and Erdmann):

$$\begin{array}{c}\text{determine the characters}\\ \text{of indecomposable tilting}\\ \text{modules for } \mathrm{GL}_m,\end{array} \overset{\sim}{\longleftrightarrow} \begin{array}{c}\text{determine all decomposition}\\ \text{numbers for symmetric groups}\\ \text{for partitions with} \le m\\ \text{rows for all symmetric groups}\end{array} \tag{4}$$

One reason that it is nice to filter the decomposition number problem for symmetric groups by "number of rows" is that it could potentially have a beautiful "generic" answer. That is, we hope that for fixed m and p large there is a combinatorial answer to the equivalent questions above. This is the case for $m = 2$

where one obtains a beautiful "fractal tree":

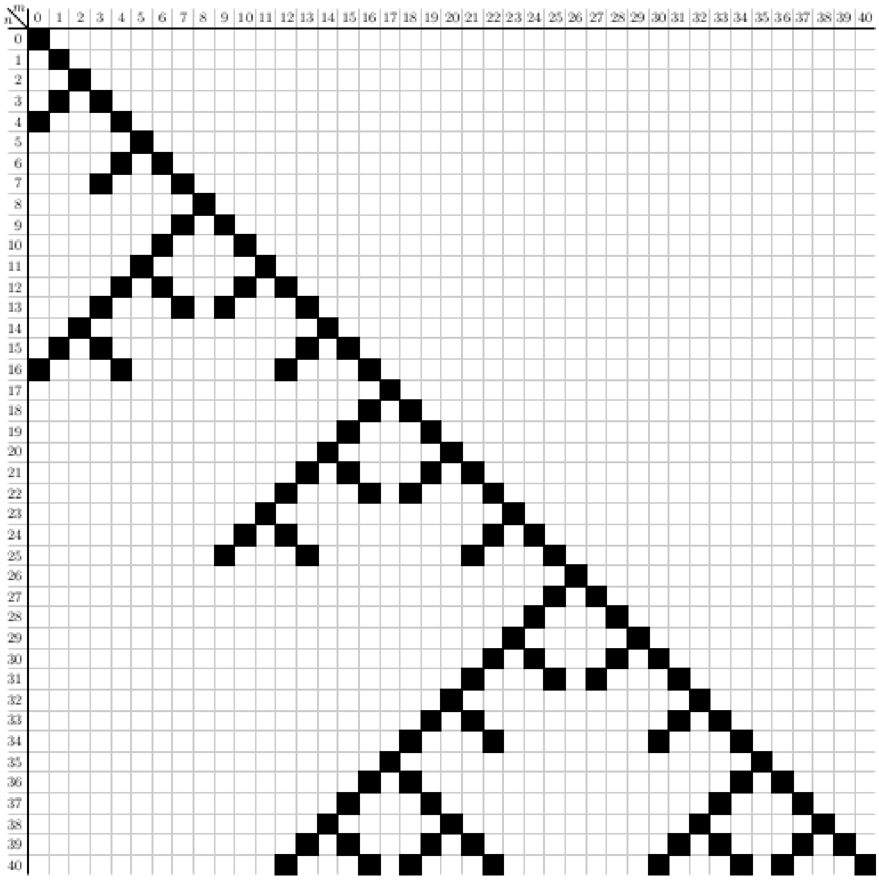

FIGURE 1. The multiplicities of $\Delta(m)$ in $T(n)$ for $p = 3$.

(This picture for $p = 3$ is stolen from [3].)

In addition to this "filtration by number of rows" we hope to gain some ground by filtering the problem according to "generation". Very roughly speaking, in generation g, "p^{g+1} is invertible, but p^g is not". Of course this is nonsense, but it does capture the idea. At this stage we only have a heuristic understanding of these generations, although they can be made rigourous for $g = 0$ (semi-simple world) and $g = 1$ (Hecke algebra / quantum group at a p^{th}-root of unity) and for SL_2 (see [4, 5]).

In general, for any highest weight λ and fixed p (perhaps we need to assume $p \geq h$) we expect a family of characters

$$\Theta^0_\lambda, \Theta^1_\lambda, \Theta^2_\lambda, \dots, \Theta^\infty_\lambda$$

satisfying certain natural conditions (see [6]). The most important is that Θ^∞_λ should agree with the character of the indecomposable tilting module with highest weight λ when p is large. (One cannot expect equality if Lusztig's character formula does not hold. One can hope that one has equality as soon as it does.)

In my talk I outlined a conjecture for Θ^2_λ for $G = SL_3$ and $p \geq 3$. Remarkably, Θ^2_λ appears to be governed by a discrete dynamical system ("billiards"). The reader can view some animations illustrating the conjecture here:

https://youtu.be/Ru0Zys1Vvq4

We came up with these conjectures after staring at calculations of the p-canonical basis in the anti-spherical module performed by a new algorithm [8]. The p-canonical basis determines the characters of tilting modules by a conjecture of Riche and the author [7] , which has been recently confirmed in work with Achar, Makisumi and Riche [1, 2] (see Simon's talk at this meeting). We hope to formulate smilar conjectures in types B_2, A_3 (and perhaps even G_2!) soon.

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Twisted Grassmannians and Torsion in codimension 2 Chow groups

Nicole Lemire

(joint work with Caroline Junkins, Daniel Krashen)

Twisted projective homogeneous varieties for algebraic groups are projective varieties which are isomorphic to a given projective homogeneous variety after extension to the separable closure of the field. For a central simple algebra A of degree n over a field F, the generalised Severi-Brauer variety $\mathrm{SB}(d, A)$ is a twisted form of $\mathrm{Gr}(d, n)$, the Grassmannian of d-dimensional planes in n-dimensional affine space, a projective homogeneous variety for the projective linear group PGL_n. An important special case is the Severi-Brauer variety $\mathrm{SB}(A) = \mathrm{SB}(1, A)$ which is a twisted form of projective space $\mathbb{P}^{n-1} = \mathrm{Gr}(1, n)$. It is well-known that the variety $\mathrm{SB}(d, A)$ has a rational point over an extension K/F if and only if $\mathrm{ind}(A_K)|d$. We can extend this question to ask about other closed subvarieties of $\mathrm{SB}(d, A)$. In [4], we investigate conditions under which generalised Severi-Brauer varieties have rational subvarieties which are forms of given Schubert subvarieties of the associated Grassmannian. We show that a generalised Severi-Brauer variety $\mathrm{SB}(d, A)$ contains a closed subvariety which is a twisted form of a Schubert subvariety of $\mathrm{Gr}(d, n)$ if and only if the index of the algebra divides a certain number arising from the combinatorics of the Schubert cell, using a variation on Fulton's notion of the essential set of a partition [3]. Our results generalise earlier work by Artin and Krashen [1, 7].

Schubert subvarieties of (untwisted) projective homogeneous varieties are particularly of interest as they produce bases for the Grothendieck group and Chow groups of the projective homogeneous variety. The Chow groups of twisted projective homogeneous varieties are a subject of much interest. In particular, the Chow groups of ordinary Severi-Brauer varieties have been much studied and related to important questions about the arithmetic of central simple algebras. Although the Chow groups of dimension 0, codimension 1, and to some extent codimension 2 cycles on Severi-Brauer varieties have been amenable to study, the other groups are in general not very well understood [5, 6, 8]. Even less is understood about algebraic cycles on and Chow groups of generalised Severi-Brauer varieties. Following work of Karpenko for the ordinary Severi-Brauer varieties, we make some computations of the codimension 2 Chow groups for the generalised Severi-Brauer varieties of reduced dimension 2 ideals in certain algebras of small index. We show that the codimension 2 Chow groups of $\mathrm{SB}(2, A)$ are torsion-free for all central simple algebras of index dividing 12. This result uses the explicit descriptions of the Schubert classes obtained in the earlier result together with other geometric constructions to show that the graded pieces of the K-groups with respect to the topological filtration are torsion-free for degree 4 algebras. This allows us to obtain the result for the codimension 2 Chow groups of such varieties. Finally, the theorem follows by an analysis of the motivic decomposition of the Chow motive of $\mathrm{SB}(2, A)$ due to Brosnan [2].

These results suggest natural analogous questions for other twisted projective homogeneous varieties. In upcoming work, we intend to study conditions under which other twisted projective homogeneous varieties have closed subvarieties which are forms of Schubert subvarieties for the corresponding split projective homgeneous variety. We will also study the consequences of such results for the codimension 2 Chow groups of the associated twisted projective homogeneous variety.

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The K-theory of versal flags and cohomological invariants of semisimple linear algebraic groups

Kirill Zainoulline

Let G be a split semisimple linear algebraic group over a field F. Let U/G be a *classifying space* of G in the sense of Totaro, i.e. U is an open G-invariant subset in some representation of G with $U(F) \neq \emptyset$ and $U \to U/G$ is a G-torsor. Consider the generic fiber U' of U over U/G. It is a G-torsor over the quotient field F' of U/G called the *versal* G-torsor. We denote by X the respective flag variety U'/B over F', where B is a Borel subgroup of G, and call it the *versal* flag. The variety X appears in many different contexts, e.g. related to cohomology of homogeneous G-varieties (see [5] for an arbitrary oriented theory; Karpenko [6], [7] for Chow groups; Panin [13] for K-theory) and cohomological invariants of G (see Merkurjev [11] and [4], [12]). It can be viewed as a generic example of the so called *twisted flag variety*.

In the talk we give an explicit presentation of the ring $K_0(X)$ in terms of generators modulo a *finite* number of relations in cases when $G = G^{sc}/\mu_2$, where G^{sc} is the product of simply-connected simple groups of Dynkin types A or C and μ_2 is a central subgroup of order 2.

Observe that for simply-connected G the ring $K_0(X)$ can be identified with $K_0(G/B)$ (e.g., see Panin [13]), and by Chevalley theorems there is a surjective characteristic map $c\colon R(T_{sc}) \to K_0(G/B)$ from the representation ring of the split maximal torus T_{sc} such that the kernel $\ker(c) = I_{sc}^W$ is generated by augmented classes of fundamental representations. So, all relations in $K_0(X)$ correspond to W-orbits of fundamental weights.

If G is not simply-connected (as in the G^{sc}/μ_2-case), then the situation changes dramatically as by [5, Ex.5.4] we have $K_0(X) \simeq R(T)/I_{sc}^W \cap R(T)$ and a finite set of generators of $I_{sc}^W \cap R(T)$ is not known in general. Note that by definition we have inclusions of abelian groups $I^W \subseteq I_{sc}^W \cap R(T) \subseteq I_{sc}^W$ which all coincide if taken with $\mathbb{Q}$-coefficients. However, there are examples of semisimple groups (see [12, Ex.3.1] and [1]) where both quotients $I_{sc}^W \cap R(T)/I^W$ and $I_{sc}^W/I_{sc}^W \cap R(T)$ are non-trivial.

As one of the applications we describe cohomological invariants of degree 3 of G for these semi-simple groups. According to Garibaldi-Merkurjev-Serre [3, p.106], a degree d *cohomological invariant* is a natural transformation of functors $a\colon H^1(\,\cdot\,, G) \to H^d(\,\cdot\,, \mathbb{Q}/\mathbb{Z}(d-1))$ on the category of field extensions over F, where the functor $H^1(\,\cdot\,, G)$ classifies G-torsors, $H^d(\,\cdot\,, \mathbb{Q}/\mathbb{Z}(d-1))$ is the Galois cohomology. Following Merkurjev [11], an invariant is called *decomposable* if it is given by a cup-product of invariants of smaller degrees; the factor group of (normalized) invariants modulo decomposable is called the group of *indecomposable* invariants. For $d = 3$ the latter has been computed for all simple split groups in [11] and [2]; for some semi-simple groups of type A in [10] and [1]; for adjoint semisimple groups in [9].

Another key subgroup of *semi-decomposable* invariants introduced in [12] consists of invariants given by a cup-product of invariants up to some field extensions. For $d = 3$ it coincides with the group of decomposable invariants for all simple groups [12]. It was also shown that these groups are different for $G = SO_4$ [12, Ex.3.1] and for some semisimple groups of type A (see [1]).

We compute the groups of decomposable, indecomposable and semi-decomposable invariants of degree 3 for new examples of semisimple groups (e.g. G^{sc}/μ_2, products of adjoint groups), hence, extending the results of [11], [2], [1], [12], [10]. In particular, we essentially extend the examples [12, Ex.3.1] and [1]; we show that

• The factor group of semi-decomposable invariants of G modulo decomposable is nontrivial if and only if G is of classical type A, B, C, D. Moreover, we determine all the factor groups (and indecomposable groups) for an arbitrary product of simply-connected simple groups of the same Dynkin type modulo the central subgroups μ_2.

• If G is of type A, then the factor group of semi-decomposable invariants modulo decomposable (and the group of indecomposable invariants) can have an arbitrary order and contains any homocyclic p-group.

• If G is of type B or C, then it is always a product of cyclic groups of order 2.

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Chow rings of generic flag varieties

Nikita Karpenko

This is an exposition of results of [1], were further references are given.

Let k be a field, G a split semisimple affine algebraic group over k, and $B \subset G$ a Borel subgroup. For an imbedding $G \hookrightarrow \mathrm{GL}(n)$, the generic fiber E of the quotient map $\mathrm{GL}(n) \to \mathrm{GL}(n)/G$ is a G-torsor over the function field $F := k(\mathrm{GL}(n)/G)$ called a *generic G-torsor*. The F-variety $X := E/B$ is a *generic flag variety*.

We consider the Chow ring $\mathrm{CH}(X)$, the Grothendieck ring $K(X)$ endowed with the topological filtration, and the associated graded ring $GK(X)$.

Conjecture. The canonical epimorphism

$$\mathrm{CH}(X) \to GK(X)$$

is an isomorphism.

Remark. The ring $\mathrm{CH}(X)$ does not depend (canonically) on the choice of the imbedding $G \hookrightarrow \mathrm{GL}(n)$.

Remark. The topological filtration on $K(X)$ is known to coincide with the γ-filtration, which is computable. Therefore Conjecture provides a way to compute $\mathrm{CH}(X)$.

Remark. The Borel subgroup B can be replaced by any *special* parabolic subgroup $P \subset G$, where the adjective *special* means for P that any P-torsor over any field extension of k is trivial. For any given G and P, the new statement obtained this way is equivalent to the old statement for G (and B).

Remark. To prove Conjecture for any field k (and a given G over k) it suffices to prove it for $k = \mathbb{Q}$ (and the corresponding split semisimple group over $\mathbb{Q}$). In particular, is suffices to prove it in characteristic 0.

Remark. If Conjecture holds for groups G_1 and G_2, then it also holds for the product $G_1 \times G_2$.

Remark. Conjecture is known to hold for the following split *simple* groups:

- all groups of type A_n;
- all groups of type C_n;
- the special orthogonal groups (of type B_n as well as of type D_n);
- G_2, F_4, and simply-connected E_6.

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Fusion systems on Lie algebras

Oksana Yakimova

(joint work with László Héthelyi and Magdolna Szőke)

Fusion systems (also called Frobenius categories) on finite groups have enjoyed a lot of interest and attention. Books have been written on the subject [1, 3]. We define a similar notion for Lie algebras. The original fusion system is defined on a Sylow p-subgroup. The natural counterpart on the Lie theory side is a maximal unipotent subgroup.

Let the ground field be $\mathbb{C}$ and let $\mathfrak{u} = \operatorname{Lie} U$ be the Lie algebra of a unipotent affine algebraic group.

Definition. A fusion system $\mathcal{F} = \mathcal{F}(\mathfrak{u})$ on $\mathfrak{u}$ is a category, whose objects are the Lie subalgebras of $\mathfrak{u}$ and the morphisms are certain injective Lie algebra homomorphisms such that the following axioms are satisfied:

(*i*) for $g \in U$ and $\mathfrak{q} \subset \mathfrak{u}$, $\mathrm{Ad}(g)_{|\mathfrak{q}} \in \mathrm{Hom}_{\mathcal{F}}(\mathfrak{q}, \mathfrak{r})$ if $Ad(g)\mathfrak{q} \subset \mathfrak{r}$;

(*ii*) for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\mathfrak{q}, \mathfrak{r})$, also $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\mathfrak{q}, \varphi(\mathfrak{q}))$;

(*iii*) if $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\mathfrak{q}, \mathfrak{r})$ is an isomorphism, then $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\mathfrak{r}, \mathfrak{q})$;

(*iv*) $\mathrm{Aut}_{\mathcal{F}}(\mathfrak{q})$ is closed in $\mathrm{GL}(\mathfrak{q})$ for each $\mathfrak{q} \subset \mathfrak{u}$.

Examples. (1) Let $U \subset G$ be a maximal unipotent subgroup of an affine algebraic group G. Then $\mathcal{F} = \mathcal{F}_G(\mathfrak{u})$ is defined by

$$\mathrm{Hom}_{\mathcal{F}}(\mathfrak{q}, \mathfrak{r}) = \{\mathrm{Ad}(g)_{|\mathfrak{q}} \mid g \in G, \mathrm{Ad}(g)\mathfrak{q} \subset \mathfrak{r}\}.$$

(2) Taking $G = U$, the obtain $\mathcal{F}_U(\mathfrak{u})$, the smallest fusion system on $\mathfrak{u}$.
(3) In the *universal* fusion system $\Omega(\mathfrak{u})$ any injective morphism between subalgebras $\mathfrak{q}$ and $\mathfrak{r}$ is allowed.
(4) Take an Abelian $\mathfrak{u}$. Then any morphism in $\Omega = \Omega(\mathfrak{u})$ comes from $\mathrm{GL}(\mathfrak{u})$ and $\mathrm{Aut}_{\Omega}(\mathfrak{u}) = \mathrm{GL}(\mathfrak{u})$.

A fusion system has to be *saturated* in order to be of any interest. One of the requirements is that $\mathrm{Inn}(\mathfrak{u})$ is a maximal unipotent subgroup of $\mathrm{Aut}_{\mathcal{F}}(\mathfrak{u})$. In particular, the universal system on an Abelian $\mathfrak{u}$ with $\dim \mathfrak{u} > 1$ is not saturated. Using the fact that all maximal unipotent subgroups are conjugate, one shows that $\mathcal{F}_G(\mathfrak{u})$ is always saturated.

Many group theoretic concepts and results were translated to the language of fusion systems. One of them is Alperin's fusion theorem dealing originally with Sylow subgroups, their intersections and conjugations. In [4], it is shown that a similar statement holds for each saturated fusion system $\mathcal{F}(\mathfrak{u})$. Our theorem states that each morphism in $\mathcal{F}(\mathfrak{u})$ can be decomposed into a product of morphisms of *essential* subalgebras and possibly one morphism of $\mathfrak{u}$ itself. A subalgebra $\mathfrak{s} \subset \mathfrak{u}$ has to satisfy many conditions in order to be essential, the most important of them is that the group $\mathrm{Aut}_{\mathcal{F}}(\mathfrak{s})/\mathrm{Inn}(\mathfrak{s})$ is reductive and its semisimple part is locally isomorphic to SL_2 if non-trivial.

Suppose that $\mathcal{F} = \mathcal{F}_G(\mathfrak{u})$ and G is reductive. By the Borel-Tits theorem [2] or by a result of Weisfeiler [5], the essential subalgebras in $\mathcal{F}$ are exactly the nilpotent radicals $\mathfrak{u}_i$ of the minimal parabolic subalgebras $\mathfrak{p}_i$ of $\mathfrak{g} = \mathrm{Lie}\,G$. Combining the above observations, one obtains the following statement.

Proposition (An analogue of the Alperin theorem for reductive Lie algebras, [4]). Suppose that $\mathrm{Ad}(g)\mathfrak{q} = \mathfrak{r}$ for two subalgebras $\mathfrak{q}, \mathfrak{r} \subset \mathfrak{u}$ and $g \in G$. Then there is $\tilde{g} = s_t \dots s_1 b$ such that $\mathrm{Ad}(\tilde{g})\mathfrak{q} = \mathfrak{r}$, the element b normalises U, $s_i \in P_i$ with P_i being a minimal parabolic subgroup, and

$$\mathrm{Ad}(b)\mathfrak{q} \subset \mathfrak{u}_1,\ \mathrm{Ad}(s_i \dots s_1 b)\mathfrak{q} \subset \mathfrak{u}_i \ \text{ for any } \ i \leq t.$$

The result appears to be new.

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The Shareshian-Wachs Conjecture

Patrick Brosnan

(joint work with Timothy Chow)

The following extended abstract describes joint work with Timothy Chow (Center for Communications Research, Princeton) proving the Shareshian-Wachs conjecture [3]. It is very similar to the abstract for my talk in the Oberwolfach conference "Algebraic Cobordism and Projective Homogeneous Varieties" held January 31 to Febrary 6, 2016. However, I have corrected a few typos in that document (some of which are fairly serious), and, to save space, I have deleted the sketch of the proof which appeared in the previous abstract. I remark that there is another, completely independent and very interesting, proof of the conjecture given by M. Guay-Paquet [7].

0.1. **The Stanley-Stembridge Conjecture.** Suppose $G = (V, E)$ is a finite graph (with vertex set V and edge set E). A *coloring* of G is a map $\kappa : V \to \mathbb{Z}_+$ such that $\kappa(v) \neq \kappa(w)$ if v and w are adjacent. Write $C(G)$ for the set of all colorings. Let Λ denote the $\mathbb{C}$-algebra of all symmetric functions in infinitely many variables $x_1, x_2, \ldots$. For a coloring $\kappa \in C(G)$, we set $x_\kappa := \prod_{v \in V} x_{\kappa(v)}$. R. Stanley defined the chromatic symmetric function

$$X_G(x) := \sum_{\kappa \in C(G)} x_\kappa. \tag{1}$$

Suppose $n \in \mathbb{Z}_+$. A *Hessenberg function* for n is a non-decreasing sequence $m_1, \ldots, m_n$ of positive integers such that, for all i, $i \le m_i \le n$. Given a Hessenberg sequence $\mathbf{m}$, let $G(\mathbf{m}) = (V, E)$ denote the graph with vertex set $V = \{1, \ldots, n\}$ and with i and j adjacent for $i < j$ if and only if $j \le m_i$. In this language, we can formulate the following long-standing conjecture of Stanley and J. Stembridge [9, 10] .

Conjecture 1 (Stanley-Stembridge)**.** Suppose $G = G(\mathbf{m})$ for a Hessenberg function $\mathbf{m}$. Then $X_G(x)$ is a non-negative sum of elementary symmetric functions.

Remark 1. In fact, Stanley and Stembridge conjecture something which seems more general. But, in [6], Guay-Paquet proved that the general conjecture reduces to Conjecture 1.

0.2. **The Shareshian-Wachs polynomial.** Now suppose $G = (V, E)$ is a graph with $V \subset \mathbb{Z}_+$. For a coloring κ of G define

$$\operatorname{asc}(\kappa) := \#\{\{v, w\} \in E : v < w, \kappa(v) < \kappa(w)\}.$$

In [8], J. Shareshian and M. Wachs prove the following remarkable theorem.

Theorem 1 (Shareshian-Wachs)**.** Suppose $G = G(\mathbf{m})$. Then

$$X_G(x, t) := \sum_{\kappa \in C(G)} t^{\operatorname{asc}\kappa} x_\kappa$$

is a polynomial in $\Lambda[t]$.

Examples 0.3. Write e_k for the k-th elementary symmetric function.

(1) If $\mathbf{m} = (1, 2, \ldots, n)$, then $X_{G(\mathbf{m})}(x, t) = e_1^n$.
(2) If $\mathbf{m} = (2, 3, 3)$, then $X_{G(\mathbf{m})}(x, t) = e_3 + t(e_3 + e_2 e_1) + t^2 e_3$.
(3) If $\mathbf{m} = (n, n, \ldots, n)$, then $X_{G(\mathbf{m})}(x, t) = \sum_{w \in S_n} t^{\ell(w)} e_n$.

0.4. **Hessenberg varieties.** Suppose $\mathbf{m} = (m_1, \ldots, m_n)$ is a Hessenberg function. Let s be an $n \times n$-matrix in $\mathfrak{g} := \mathfrak{gl}_n$. Let X denote the variety of complete flags F in n-dimensional space. Set

$$\mathscr{H}(\mathbf{m}, s) := \{F \in X : \forall i, sF_i \subset F_{m_i}\}.$$

This is called the *Hessenberg variety* of type $\mathbf{m}$. These varieties were introduced by DeMari, Procesi and Shayman in [4]. In fact, [4] studies a generalization of the varieties defined above inside the variety $\mathscr{B}$ of Borel subgroups of an arbitrary reductive group. In my work with Chow, only the type A case appears.

Write $\mathfrak{g}^{\mathrm{rs}}$ for the Zariski open subset of $\mathfrak{g}$ consisting of regular semi-simple matrices. Then [4] shows that, for $y \in \mathfrak{g}^{\mathrm{rs}}$, $\mathscr{H}(\mathbf{m}, y)$ is smooth. Moreover, the centralizer $Z(y)$ of y, which is a maximal torus in $G = \mathbf{GL}_n$ acts on $\mathscr{H}(\mathbf{m}, y)$. And the fixed point set $\mathscr{H}(\mathbf{m}, y)^{Z(y)}$ coincides with $X^{Z(y)}$. Note that the set

$X^{Z(y)}$ is a torsor for the Weyl group $W \cong S_n$ of $Z(y)$. By Białynicki-Birula, it follows that the cohomology $H^*(\mathscr{H}(\mathbf{m}, y))$ is freely generated by one element for each fixed point [2]. Thus, $H^*(\mathscr{H}(\mathbf{m}, y))$ is (non-canonically) freely generated by one element for every element of W.

0.5. **Tymoczko's Dot Action.** Pick $y \in \mathfrak{g}^{\mathrm{rs}}$, and set $T = Z(y)$. In [11], J. Tymoczko defines an action (called the *dot* action) of the Weyl group W of T on the equivariant cohomology $H_T^*(\mathscr{H}(\mathbf{m}, y))$. Moreover, Tymoczko shows that this action descends to a W action (also called the *dot* action) on $H^*(\mathscr{H}(\mathbf{m}, y))$.

0.6. **Frobenius Character.** For each positive integer n write $p_n := \sum x_i^n$ for the power-sum symmetric function, and for each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ write $p_\lambda := \prod p_{\lambda_i}$. For $w \in S_n$, write $\lambda(w)$ for the partition corresponding to the cycle decomposition of w. Then for a representation V of S_n, set

$$\mathrm{ch}V := \frac{1}{n!} \sum_{w \in S_n} \chi_V(w) p_{\lambda(w)}.$$

It is well-known (see [5]), that ch is then an isomorphism from the space of $\mathbb{C}$-valued class functions of S_n to the space Λ_n of symmetric functions of degree n. We write ω for the involution on Λ that takes $\mathrm{ch}V$ to $\mathrm{ch}V \otimes \mathrm{sgn}$.

1. Shareshian-Wachs

The main result of my joint paper [3] is the following theorem which was conjectured by Shareshian and Wachs in [8].

Theorem 2. Suppose $\mathbf{m} = (m_1, \dots, m_n)$ is a Hessenberg function, and $y \in \mathfrak{g}^{\mathrm{rs}}$. Then we have

$$\omega X_{G(\mathbf{m})}(t) = \sum_{k \in \mathbb{Z}} t^k \mathrm{ch} H^k(\mathscr{H}(\mathbf{m}, y)).$$

As mentioned above, Guay-Paquet has given an independent proof in [7]. While the proof in [3] is geometric, the main ideas behind [7] are combinatorial. The crucial tool in Guay-Paquet's proof is a theorem of Aguiar, Bergeron and Sottile on the universality of a certain Hopf algebra of quasi-symmetric functions [1]. It is, in fact, very tempting to try to combine the ideas from both proofs to search for a proof of Conjecture 1.

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Tamagawa numbers of linear algebraic groups over function fields

Zev Rosengarten

Remark 1. All of the author's results that are referred to in this abstract will appear in upcoming manuscripts.

Weil observed that for any connected linear algebraic group G over a global field k, there is a canonical Haar measure, called the Tamagawa measure, on the group $G(\mathbf{A})_1$ of "norm-one" adelic points of G. The volume $\tau(G)$ of the space $G(\mathbf{A})_1/G(k)$ with respect to the Tamagawa measure is called the Tamagawa number of G. It is a highly nontrivial theorem that this volume is always finite (due to Borel in the number field case, and to B. Conrad over function fields). This number contains very interesting arithmetic information. For example, given a quadratic form q over the rational numbers, the equation $\tau(SO_q) = 2$ is equivalent to the Siegel Mass Formula.

In 1981, Sansuc obtained a beautifully simple formula for Tamagawa numbers of reductive groups (modulo some then as-yet unknown deep results on the arithmetic of simply connected groups which have since been proven). Before we can state it, we need to introduce some notation. For an algebraic group G, let $\mathrm{Pic}(G)$ denote the Picard group of G, i.e., the set of line bundles on G up to equivalence; and let

$$Ш(G) := \ker\left(\mathrm{H}^1(k, G) \to \prod_v \mathrm{H}^1(k_v, G)\right)$$

denote the Tate-Shafarevich set of G. This latter set is finite for any linear algebraic group, as was shown by Borel and Serre in the number field case, and by B. Conrad over function fields. (It is also true, and not nearly as deep, that $\mathrm{Pic}(G)$ is finite for any connected reductive G over any field whatsoever.) Then Sansuc's result is the following:

Theorem 1. (Sansuc, [San], Th. 10.1) Let G be a connected reductive group over the global field k. Then

$$\tau(G) = \frac{\#\mathrm{Pic(G)}}{\#\text{Ш}(G)} \tag{1}$$

One may easily deduce that the same formula holds for Tamagawa numbers of all connected linear algebraic groups over *number fields*, for the following reason. If G is such a group, then we have the exact sequence

$$1 \longrightarrow U \longrightarrow G \longrightarrow R \longrightarrow 1 \tag{2}$$

where U is the k-unipotent radical of G and R is the pseudo-reductive quotient. (That is, the k-unipotent radical of R is trivial.) Since the number field k is *perfect*, U is split unipotent and R is reductive, not just pseudo-reductive. (The unipotent radical over $\overline{k}$ descends all the way down to k, by Galois descent.) The behavior of Tamagawa numbers in exact sequences combined with the very simple nature of split unipotent groups together imply that Sansuc's formula for G is equivalent to the same formula for the reductive group R, hence the formula holds for all connected linear algebraic groups over number fields. The problem of computing Tamagawa numbers of linear algebraic groups over number fields is therefore completely solved.

The situation for function fields is much more difficult, for two reasons both arising from the imperfection of function fields: (i) The group U in sequence (2) is not necessarily split, and indeed may be quite complicated. (Indeed, even computing Tamagawa numbers of unipotent groups over function fields is a highly nontrivial problem. Oesterlé [Oes] has a paper on the subject.); and (ii) The group R is not necessarily reductive, only pseudo-reductive. Over any imperfect field (such as function fields), there exist many examples of pseudo-reductive groups that are not reductive.

In fact, Sansuc's formula (1) is not correct in general for linear algebraic groups over function fields. This is in some sense unsurprising. Whereas the quantities $\tau(G), \text{Ш}(G)$ depend quite crucially on the group structure on G, the quantity $\mathrm{Pic}(G)$ depends only on the structure of G as a scheme. We should therefore seek a substitute that remembers this group-theoretic structure. To this end, we introduce the group

$$\mathrm{Ext}^1(G, \mathbf{G}_m) := \{\mathcal{L} \in \mathrm{Pic}(G) \mid m^*\mathcal{L} \simeq p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}\}$$

of primitive line bundles on G, where $m, p_i : G \times G \to G$ are the multiplication and projection maps, respectively. The reason for the notation is that for any extension

$$1 \longrightarrow \mathbf{G}_m \longrightarrow E \longrightarrow G \longrightarrow 1$$

of algebraic groups, E is in particular a $\mathbf{G}_m$-torsor over G. One easily checks that the resulting torsor is primitive. We therefore get a homomorphism $\mathrm{Ext}^1(G, \mathbf{G}_m) \to \{\mathcal{L} \in \mathrm{Pic}(G) \mid m^*\mathcal{L} \simeq p_1 * \mathcal{L} \otimes p_2^*\mathcal{L}\}$ (where the group on the left is now the Yoneda group of algebraic group extensions), and one can show that this map is an isomorphism for any smooth connected k-group scheme G.

We now propose the following modification of Sansuc's formula:

Conjecture 1. (R.) For any connected linear algebraic group G over the global field k, one has

$$\tau(G) = \frac{\#\mathrm{Ext}^1(G, \mathbf{G}_m)}{\#\text{Ш}(G)}$$

It turns out that for any perfect field k, one has the equivalence $\mathrm{Ext}^1(G, \mathbf{G}_m) = \mathrm{Pic}(G)$ for any connected linear algebraic k-group G, and both groups are finite. In particular, Conjecture 1 recovers Sansuc's formula when k is a number field. But it is truly different in the function field setting. Before trying to prove Conjecture 1, it is good to know as a sanity check that the group $\mathrm{Ext}^1(G, \mathbf{G}_m)$ appearing in Conjecture 1 is finite.

Theorem 2. (R.) If G is a connected linear algebraic group over the global function field k, then $\mathrm{Ext}^1(G, \mathbf{G}_m)$ is finite.

Remark 2. The above finiteness statement fails over every local function field and over every imperfect separably closed field, even for forms of $\mathbf{G}_a$, so it is a truly arithmetic result.

Our progress toward proving Conjecture 1 thus far consists of the following two results.

Theorem 3. (R.) Conjecture 1 holds for connected commutative linear algebraic groups.

Theorem 4. (R.) If $\mathrm{char}(k) > 3$, then Conjecture 1 holds for all pseudo-reductive k-groups.

Remark 3. One may remove the assumption on $\mathrm{char}(k)$ in Theorem 4 if one assumes that G is a *standard* pseudo-reductive group, cf. [CGP], Definition 1.4.4. The difficulty in small characteristics arises from the existence of non-standard pseudo-reductive groups. Basic exotic pseudo-reductive groups that are not pseudo-split seem to be the most difficult case; all other cases seem quite doable (but the author has yet to treat them).

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R-matrices and cohomological Hall algebras

Eric Vasserot
(joint work with Olivier Schiffmann)

Maulik and Okounkov have given a new way to construct solutions of the Yang-Baxter equation with spectral parameters using symplectic geometry and the quiver varieties associated to an arbitrary quiver Q. To these new R-matrices one associates a quantum group Y_Q (a Yangian) using the well-known $RT_1T_2 = T_2T_1R$ formalism. Maulik and Okounkov have proved that Y_Q is a deformation of the enveloping algebra of the current algebra of a Lie algebra $\mathfrak{g}_Q$. If the quiver Q is of finite type, then Y_Q is the Yangian associated with the Lie algebra of type Q, i.e., $\mathfrak{g}_Q$ is the Lie algebra associated with the Cartan matrix of Q. If Q is the Jordan quiver then Y_Q is the Yangian analogue of the Elliptic Hall Algebra, which was introduced previously by Schiffmann and Vasserot in [3]. In this case $\mathfrak{g}_Q$ is an Heisenberg Lie algebra. For an arbitrary quiver Q, there is no algebraic presentation of $\mathfrak{g}_Q$ or Y_Q. In particular their graded dimensions are not known.

The goal of this talk is to present another conjectural description of Y_Q. It has the advantage to allowing to compute the graded dimension of $\mathfrak{g}_Q$. Our conjecture would imply a conjecture of Okounkov which claims that the dimension of the root subspaces of $\mathfrak{g}_Q$ should be the value at 1 of the corresponding Kac polynomial.

By construction, the Yangian Y_Q acts on the equivariant cohomology groups of the quiver variety associated with Q. The COHA we consider is given by an associative multiplication on the equivariant cohomology groups of an analogue of the Lusztig nilpotent variety associated with the quiver. The nilpotence condition we use differs from Lusztig's one when the quiver has some edge-loops. This nilpotent variety is Lagrangian, according to a previous work of Bozec. The multiplication resemble the multiplication law of usual Hall algebras. One replaces extensions in an Abelian category by convolution relative to an induction diagram in the moduli space of representations of the double quiver associated with Q. Our main results are the following :

- The Deligne mixed Hodge structure on the COHA is pure.
- We can compute the number of points of the nilpotent variety over finite fields, giving a formula for the graded dimension of the COHA, see [1].
- We can compute a nice system of generators of the COHA, see [4].
- There is an inclusion COHA$\subseteq Y_Q$, see [5].

Finally, as mentionned above we conjecture that COHA$= Y_Q$ up to some central elements.

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Cocharacter-closure and the rational Hilbert-Mumford Theorem

Gerhard Röhrle
(joint work with M. Bate, S. Herpel, and B. Martin)

For a field k, let G be a reductive k-group and V an affine k-variety on which G acts. Using the notion of cocharacter-closed $G(k)$-orbits in V, we prove a rational version of the celebrated Hilbert-Mumford Theorem from geometric invariant theory in [1, Thm. 1.3]. There we initiate a study of applications stemming from this rationality tool. An example is discussed to illustrate the concept of cocharacter-closure and to highlight how it differs from the usual Zariski-closure.

The concept of a closed orbit is fundamental in geometric invariant theory over algebraically closed fields. A first problem is to devise a suitable analogue of this idea for $G(k)$-orbits. One can define the notion of a k-orbit over G [6, 10.2, Def. 4], or study the Zariski closure of a $G(k)$-orbit, but such constructions do not appear to be helpful here, e.g. see the discussion in [3, Rem. 3.9]. Instead we adopt an approach involving cocharacters, as follows. Let $Y_k(G)$ denote the set of k-defined cocharacters of G. In [3, Def. 3.8], we made the following definition.

Definition 1. Let $v \in V$ (not necessarily a k-point). The orbit $G(k) \cdot v$ is *cocharacter-closed over* k provided for all $\lambda \in Y_k(G)$, if $v' := \lim_{a\to 0} \lambda(a) \cdot v$ exists, then $v' \in G(k) \cdot v$.

In [1], we extend this definition to cover arbitrary subsets of V, and introduce the cocharacter-closure of a subset of V:

Definition 2. (a) Given a subset X of V, we say that X is *cocharacter-closed (over k)* if for every $v \in X$ and every $\lambda \in Y_k(G)$ such that $v' := \lim_{a\to 0} \lambda(a) \cdot v$ exists, $v' \in X$. Note that this definition coincides with the one above if $X = G(k) \cdot v$ for some $v \in V$.
(b) Given a subset X of V, we define the *cocharacter-closure of X (over k)*, denoted $\overline{X}^c$, to be the smallest subset of V such that $X \subseteq \overline{X}^c$ and $\overline{X}^c$ is cocharacter-closed over k. (This makes sense because the intersection of cocharacter-closed subsets is clearly cocharacter-closed.)

It follows from the Hilbert-Mumford Theorem that $G \cdot v$ is cocharacter-closed over $\overline{k}$ if and only if $G \cdot v$ is closed. It is obvious that $\overline{G \cdot v}^c$ is contained in $\overline{G \cdot v}$. Note, however, that this containment can be proper: e.g., see [1, Ex. 11.1].

Here is the rational version of the Hilbert-Mumford Theorem:

Theorem 1 ([1, Thm. 1.3])**.** Let $v \in V$. Then there is a unique cocharacter-closed $G(k)$-orbit $\mathcal{O}$ inside $\overline{G(k) \cdot v}^{c}$. Moreover, there exists $\lambda \in Y_k(G)$ such that $\lim_{a \to 0} \lambda(a) \cdot v$ exists and lies in $\mathcal{O}$.

By a standard fact, the closure of a geometric G-orbit is again a union of G-orbits, [5, I 1.8 Prop.]. Thanks to [1, Lem. 3.3(i)], the rational counterpart holds for the cocharacter-closure of a $G(k)$-orbit in V. Therefore, we can mimic the usual "degeneration" partial order on the G-orbits in V in this rational setting:

Definition 3. Given $v, v' \in V$, we write $G(k) \cdot v' \prec G(k) \cdot v$ if $v' \in \overline{G(k) \cdot v}^{c}$.

Then it is clear that $\prec$ is reflexive and transitive, so $\prec$ gives a preorder on the set of $G(k)$-orbits in V. In general, the behavior of the $G(k)$-orbits can be quite pathological; e.g., see Example 1, see also [1, §7]. Our next result holds under some mild hypothesis on the centralizer G_v of v in G.

Theorem 2. Let $v \in V$. Suppose that G_v is k-defined. Then the following hold:

(i) If $G \cdot v$ is Zariski-closed, then $G(k) \cdot v$ is cocharacter-closed over k.
(ii) Let k'/k be an algebraic field extension and suppose that $G(k') \cdot v$ is cocharacter-closed over k'. Then $G(k) \cdot v$ is cocharacter-closed over k. Moreover, the converse holds provided $v \in V(k)$ and k'/k is separable.
(iii) Let S be a k-defined torus of G_v and set $L = C_G(S)$. Then $G(k) \cdot v$ is cocharacter-closed over k if and only if $L(k) \cdot v$ is cocharacter-closed over k.
(iv) Let $w \in V$ and suppose that both $G(k) \cdot w \prec G(k) \cdot v$ and $G(k) \cdot v \prec G(k) \cdot w$. Then $G(k) \cdot v = G(k) \cdot w$.

Note that the rationality condition on the centralizer G_v in Theorem 2 is satisfied in many instances, e.g., if $v \in V(k)$ and k is perfect (cf. [1, Prop. 7.4]).

The notion of a cocharacter-closed $G(k)$-orbit has already proved very useful in the context of Serre's notion of G-complete reducibility over k. In [3, Thm. 5.9] we gave a geometric characterisation of the latter using the former. In [1, Thm. 9.3] this is strengthened further by removing the connectedness assumption on G from [3, Thm. 5.9].

Definition 4. A subgroup H of G is said to be *G-completely reducible (G-cr)* if whenever H is contained in an R-parabolic subgroup P of G, there exists an R-Levi subgroup of P containing H. Similarly, a subgroup H of G is said to be *G-completely reducible over k* if whenever H is contained in a k-defined R-parabolic subgroup P of G, there exists a k-defined R-Levi subgroup of P containing H.

In [2, Cor. 3.7], we show that G-complete reducibility has a geometric interpretation in terms of the action of G on G^n, the n-fold Cartesian product of G with itself, by simultaneous conjugation.

Definition 5. Let H be a subgroup of G and let $G \hookrightarrow \mathrm{GL}_m$ be an embedding of algebraic groups. Then $\mathbf{h} \in H^n$ is called a *generic tuple of H for the embedding $G \hookrightarrow \mathrm{GL}_m$* if $\mathbf{h}$ generates the associative subalgebra of Mat_m spanned by H. We call $\mathbf{h} \in H^n$ a *generic tuple of H* if it is a generic tuple of H for some embedding $G \hookrightarrow \mathrm{GL}_m$.

Note that generic tuples exist for any embedding $G \hookrightarrow \mathrm{GL}_m$ provided n is sufficiently large. We now give the characterization of G-complete reducibility over k in terms of geometric invariant theory mentioned above.

Theorem 3 ([1, Thm. 9.3])**.** Let H be a subgroup of G and let $\mathbf{h} \in H^n$ be a generic tuple of H. Then H is G-completely reducible over k if and only if $G(k)\cdot\mathbf{h}$ is cocharacter-closed over k.

The following elementary example illustrates some of the complexities that can arise, even over a field of characteristic 0. For more examples, see [1, §10, §11].

Example 1 ([1, Ex. 4.3]). Let $k = \mathbb{R}$ and consider the group $G = \mathbb{G}_m$ acting on $V = \mathbb{A}^1$ by $a \cdot z := a^2 z$. The group $G(k) = \mathbb{G}_m(k)$ is just the multiplicative group of the field $\mathbb{R}$, and there are three orbits of $G(k)$ on k-points of V: $G(k)\cdot(-1) = \{x \in \mathbb{R} \mid x < 0\}$, $G(k)\cdot 0 = \{0\}$ and $G(k)\cdot 1 = \{x \in \mathbb{R} \mid x > 0\}$. We have $\overline{G(k)\cdot(-1)}^c = G(k)\cdot(-1) \cup \{0\}$ and $\overline{G(k)\cdot 1}^c = G(k)\cdot 1 \cup \{0\}$. On the other hand, since the non-zero $G(k)$-orbits $G(k)\cdot 1$ and $G(k)\cdot(-1)$ are both infinite subsets of V, their Zariski closures are the whole of $\mathbb{A}^1$. We also have $G\cdot 1 = G\cdot(-1) = \{z \in \mathbb{A}^1 \mid z \neq 0\}$. This gives an example of how the cocharacter-closure isn't the same as the closure (or the closure intersected with the set of k-points) and how different parts of the same G-orbit may be inaccessible from each other when viewed as $G(k)$-orbits.

We close with an application of the notion of cocharacter-closedness over k to characterize k-anisotropy, see [1, Thm. 1.6]. Recall that G is k-anisotropic provided $Y_k(G) = \{0\}$. Part (i) of Theorem 4 gives a characterization of k-anisotropic reductive groups over an arbitrary field k in terms of cocharacter-closed orbits. In the special case when k is perfect, we recover in part (ii) a result of Kempf [7, Thm. 4.2]. Characterizing anisotropy over perfect fields in terms of closed orbits was a question of Borel, [4, Rem. 8.8 (d)].

Theorem 4. (i) G is k-anisotropic if and only if for every k-defined affine G-variety W and every $w \in W(k)$, the orbit $G(k)\cdot w$ is cocharacter-closed over k.

(ii) Suppose k is perfect. Then G is k-anisotropic if and only if for every k-defined affine G-variety W and every $w \in W(k)$, the orbit $G \cdot w$ is closed in W.

Part (ii) of Theorem 4 follows from part (i), Theorem 2(ii) and the Hilbert-Mumford Theorem. Note that Theorem 4(ii) fails for non-perfect fields; see [1, Rem. 5.9(ii)].

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Involutions on the affine Grassmannian and moduli spaces of principal bundles

Anthony Henderson

Let G be a simply connected semisimple algebraic group over $\mathbb{C}$, with chosen Borel subgroup B and maximal torus T. Let Λ^+ denote the resulting set of dominant coweights, which is a complete set of representatives for the G-conjugacy classes of homomorphisms $\mathbb{G}_m \to G$.

Let $\mathsf{Gr} = G(\!(t)\!)/G[\![t]\!]$ be the affine Grassmannian of G, an integral ind-scheme. It is the disjoint union of $G[\![t]\!]$-orbits Gr^λ indexed by $\lambda \in \Lambda^+$. We consider the open subset $\mathsf{Gr}_0 = G[t^{-1}]G[\![t]\!]/G[\![t]\!]$ of Gr and its G-stable strata $\mathsf{Gr}_0^\lambda = \mathsf{Gr}^\lambda \cap \mathsf{Gr}_0$. (Some of what follows can be generalized to the intersections of Gr^λ with transverse slices to smaller orbits Gr^μ; here we consider just the $\mu = 0$ case.)

We make the obvious identification of Gr_0 with $G[t^{-1}]_1 = \ker(G[t^{-1}] \to G)$. The involution $\iota : \mathsf{Gr}_0 \to \mathsf{Gr}_0 : \gamma(t) \mapsto \gamma(-t)^{-1}$ arose in the study [1] of relationships between the affine Grassmannian and the nilpotent cone. There it was observed that $\iota(\mathsf{Gr}_0^\lambda) = \mathsf{Gr}_0^{-w_0\lambda}$ where w_0 is the longest element of the Weyl group. Let $\Lambda_1^+ = \{\lambda \in \Lambda^+ \,|\, \lambda = -w_0\lambda\}$. For $\lambda \in \Lambda_1^+$ we have an induced involution ι of Gr_0^λ, and the aim of the present work is to describe the fixed-point subvariety $(\mathsf{Gr}_0^\lambda)^\iota$.

We use a moduli-space interpretation of Gr_0^λ due to Braverman and Finkelberg [2]. Let X_G be the scheme denoted $\mathrm{Bun}_G(\mathbb{P}^2, \ell_\infty)$ in [3]; this is the moduli space of pairs $(\mathcal{F}, \Phi)$ where $\mathcal{F}$ is a principal G-bundle on $\mathbb{P}^2$ and Φ is a trivialization of $\mathcal{F}$ on the line at infinity ℓ_∞. On X_G we have an action of $G \times \mathrm{SL}_2$, where G acts by changing the trivialization and SL_2 acts on the base $\mathbb{P}^2$ preserving ℓ_∞.

Consider the fixed-point set $X_G^{\mathbb{G}_m}$ where $\mathbb{G}_m$ means the diagonal torus in SL_2. If the isomorphism class of a pair $(\mathcal{F}, \Phi)$ is stable under $\mathbb{G}_m$ then, by considering the fibre of $\mathcal{F}$ at the $\mathbb{G}_m$-fixed point of $\mathbb{A}^2$, we obtain a homomorphism $\mathbb{G}_m \to G$ which is well defined up to G-conjugacy. Hence there is a disconnected union decomposition $X_G^{\mathbb{G}_m} = \coprod_{\lambda \in \Lambda^+} X_G^{\mathbb{G}_m, \lambda}$. Part of the Braverman–Finkelberg result [2, Theorem 5.2] is that for any $\lambda \in \Lambda^+$ there is a G-equivariant isomorphism

$$\Psi_\lambda : \mathsf{Gr}_0^\lambda \xrightarrow{\sim} X_G^{\mathbb{G}_m,\lambda}.$$

Our first result is that, under these isomorphisms Ψ_λ, the involution ι corresponds to the action of the non-identity component of $N = N_{\mathrm{SL}_2}(\mathbb{G}_m)$. Now, by the same argument as for $\mathbb{G}_m$, we have a disconnected union decomposition $X_G^N = \coprod_{\xi\in\Xi} X_G^{N,\xi}$ where Ξ is the set of G-conjugacy classes of homomorphisms $N \to G$. As a consequence, for any $\lambda \in \Lambda_1^+$ one has a disconnected union decomposition $(\mathsf{Gr}_0^\lambda)^\iota = \coprod_{\xi\in\Xi(\lambda)} (\mathsf{Gr}_0^\lambda)^{\iota,\xi}$ where $\Xi(\lambda)$ denotes the set of G-conjugacy classes of homomorphisms $N \to G$ whose restriction to $\mathbb{G}_m$ is G-conjugate to λ. Here $(\mathsf{Gr}_0^\lambda)^{\iota,\xi} = \Psi_\lambda^{-1}(X_G^{N,\xi})$. (With hindsight, this disconnected union decomposition of $(\mathsf{Gr}_0^\lambda)^\iota$ can be defined purely in terms of Gr, but the connection with X_G gives a more intrinsic explanation for it.)

To motivate the study of the varieties $(\mathsf{Gr}_0^\lambda)^{\iota,\xi}$, or the isomorphic varieties $X_G^{N,\xi}$, note that the connected components of the SL_2-fixed-point set $X_G^{\mathrm{SL}_2}$ are isomorphic to the nilpotent orbits of G, by a result of Kronheimer [5, Theorem 1]. Thus, for at least two reductive subgroups Γ of SL_2, namely $\Gamma = \mathbb{G}_m$ and $\Gamma = \mathrm{SL}_2$ itself, the connected components of the fixed-point set X_G^Γ are important objects in geometric representation theory; the results of [2] suggest that the same can be said when Γ is a finite cyclic subgroup of SL_2. Thus it is natural to consider the other reductive subgroups of SL_2, of which N should be the easiest to treat.

For general G, we do not know for which $\xi \in \Xi(\lambda)$ the variety $(\mathsf{Gr}_0^\lambda)^{\iota,\xi}$ is nonempty, or whether it can ever be disconnected. However, these questions can be answered when $G = \mathrm{SL}_n$ using the theory of Nakajima quiver varieties [6, 7], which indeed sprang from the Atiyah–Drinfeld–Hitchin–Manin description of X_{SL_n}, or rather of the closely related moduli space of torsion-free sheaves on $\mathbb{P}^2$ (see [8, Theorem 2.1]). In this theory, each variety $X_{\mathrm{SL}_n}^{N,\xi}$ is identified with an open subset of a Nakajima quiver variety of type D (this being the type of N in the McKay correspondence). This implies an explicit combinatorial criterion for nonemptiness of $(\mathsf{Gr}_0^\lambda)^{\iota,\xi}$, and shows that this variety is connected whenever it is nonempty.

By combining the above results with [4, Theorem 4.4], one obtains an appealing description of $(\mathsf{Gr}_0^\lambda)^\iota$ in the $G = \mathrm{SL}_2$ case. In this case, $\Lambda_1^+ = \Lambda^+ = \mathbb{N}\alpha^\vee$. It is trivial to see that $(\mathsf{Gr}_0^{m\alpha^\vee})^\iota$ is empty if $m \geq 2$ is even.

For any $m \geq 1$, let $\mathcal{O}_{(2m)}$ denote the regular nilpotent orbit of $\mathfrak{sl}_{2m}$, and $\mathcal{S}_{(m,m)}$ the Slodowy transverse slice to the two-row nilpotent orbit $\mathcal{O}_{(2m)}$. Then there is an isomorphism $\mathsf{Gr}_0^{m\alpha^\vee} \cong \mathcal{O}_{(2m)} \cap \mathcal{S}_{(m,m)}$ under which the involution ι corresponds to the restriction of a Lie algebra involution of $\mathfrak{sl}_{2m}$: namely, negative transpose with respect to a nondegenerate bilinear form on $\mathbb{C}^{2n}$ which is $(-1)^m$-symmetric. We conclude that, when $m \geq 2$ is even, $(\mathsf{Gr}_0^{m\alpha^\vee})^\iota \cong \mathcal{O}_{(2m)}^{\mathfrak{sp}_{2m}} \cap \mathcal{S}_{(m,m)}^{\mathfrak{sp}_{2m}}$.

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R-matrices and convolution algebras arising from Grassmannians

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(joint work with Vassily Gorbounov, Christian Korff)

We describe how to construct interesting convolution algebras from equivariant cohomology of Grassmannians and give a geometric construction of certain R-matrices arising in the 5-vertex models. Details appear in [GKS17].

1. Integrable systems

In the following let N be a fixed natural number. We fix as the ground field the complex numbers $\mathbb{C}$ and let $\mathbb{C}[t]$ be the polynomial ring in a variable t. For any finite dimensional vector space W denote $W[t] = W \otimes \mathbb{C}[t]$, the $\mathbb{C}[t]$-module obtained by scalar extension, and $W[t_1, t_2, \ldots, t_N] = W \otimes \mathbb{C}[t_1, t_2, \ldots t_N]$.

Let $V = \mathbb{C}^2$ with a fixed basis v_0, v_1. It induces a standard (or tensor) basis of $V \otimes V$. Then $V[t] = V \otimes \mathbb{C}[t]$ has $\mathbb{C}[t]$-basis v_0, v_1. We identify $V[t]^{\otimes N} = V \otimes [t_1, t_2, \ldots, t_N]$ as vector space (in the obvious way identifying t_i with t from the ith tensor factor). A *Lax matrix* is a 2×2-matrix

$$L = L(x,t) = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{pmatrix} \quad \in \quad \mathrm{M}(2\times 2, \mathrm{M}(2\times 2, \mathbb{C}[x,t])) \tag{1}$$

with entries in $\mathrm{M}(2\times 2, \mathbb{C}[x,t])$. It defines a $\mathbb{C}[x,t]$-linear endomorphism of $V \otimes V$. The *monodromy matrix* $M = M(x, t_1, t_2, \ldots, t_N)$ is the endomorphism

$$L_{0,N}(x,t_N)\ldots L_{0,2}(x,t_2)L_{0,1}(x,t_1) = \begin{pmatrix} A(x,t_1,t_2,\ldots,t_N) & B(x,t_1,t_2,\ldots,t_N) \\ C(x,t_1,t_2,\ldots,t_N) & D(x,t_1,t_2,\ldots,t_N) \end{pmatrix}$$

of $V[x]\otimes V^{\otimes N}[t_1,\dots,t_N]$. A pair $(R(x,y),L(x,t))$ of Lax matrices *satisfies the Yang-Baxter equation* if $L_{2,3}(y,t)L_{1,3}(x,t)_{1,2}R(x,y)=R_{1,2}(x,y)L_{1,3}(x,t)R_{2,3}(y,t)$ as endomorphisms of $V[t]\otimes V[x]\otimes V[y]$ or equivalently

$$\begin{aligned} & M_2(x_2,t_1,\dots,t_N)M_1(x_1,t_1,\dots,t_N)R_{1,2}(x_1,x_2) \\ = \; & R_{1,2}(x_1,x_2)M_1(x,t_1,\dots,t_N)M_2(x_1,t_1,\dots,t_N). \end{aligned} \tag{2}$$

Important examples appearing in the physics literature are the so-called 5-vertex models which are certain (not well understood) degeneration of the (well-known) 6-vertex model and given by the following pairs $(R(x,y),L(x,y))$ with $z=x-y$.

$$\left(\begin{pmatrix}1&0&0&0\\0&z&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&z&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\right)\qquad\left(\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&-z&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}z&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}\right)$$

The *Yang-Baxter algebra* is the $\mathbb{C}[t_1,\dots t_N]$-subalgebra of endomorphisms of $V^{\otimes N}$ generated by the coefficients $A_i(t_1,\dots t_N)$, $B_i(t_1,\dots,t_N)$, $C_i(t_1,\dots,t_N)$, and $D_i(t_1,\dots,t_N)$ of the x^i of the entries of M. It is easy (although maybe not very useful) to write down an explicit presentation of the algebras ,let us call them Y_N respectively Y'_N, for the above special choices of $(R(x,y),L(x,t))$.

Lemma 1.1. The action of Y_N and of Y'_N on $V[t]^{\otimes N}$ commutes with the action of the symmetric group S_N permuting the factors.

2. Geometry of Grassmannians

Let $X_k=\mathrm{Gr}(k,N)$ be the Grassmannian of k-dimensional subspaces in $\mathbb{C}^N$ with its standard torus T (=diagonal matrices)-action. If we fix the standard basis $e_1,\dots,e_N$ of $\mathbb{C}^N$ then the T-fixed points are precisely the coordinate spaces $p_I=\langle e_i\mid i\in I\rangle$, where I is a k-element subset of $\mathbb{I}=\{1,2,\dots,N\}$. For each regular integral cocharacter (i.e. $\chi_{\mathbf{a}}:\mathbb{C}^*\to T$ sending $t\in\mathbb{C}^*$ to the diagonal matrix $(t_1^{a_1},\dots,t^{a_N})$ with $a_i\in\mathbb{Z}$ paarwise distinct) we have the attracting cells $C_I=\{x\in X_n\mid \lim_{t\to 0}\chi_{\mathbf{a}}(t).x=p_I\}$. They do in fact not depend on the tuple $\mathbf{a}$, but only on the Weyl chamber containing it. Let us identify Weyl chambers with Weyl group elements by mapping the antidominant chamber to the identity element. Then we obtain geometric interpretations of integrable systems bases:

Proposition 1. Each Weyl chamber, hence each Weyl group element w, defines a basis S_I^w, $I\subset\mathbb{I}$, $|I|=k$, of the T-equivariant cohomology ring $H_T(X_k)$.

(1) Sending a basis vector $v_{i_1}\otimes\ldots\otimes v_{i_N}$ of $V[t]^{\otimes N}=V\otimes[t_1,t_2,\dots,t_N]$ to S_I^e where $I=\{j\mid i_j=1\}$ defines an isomorphism of $\mathbb{C}[t_1,t_2,\dots,t_N]$-modules

$$\Phi:\quad V[t]^{\otimes N}\quad\cong\quad\bigoplus_{k=0}^{N}H_T(X_k)$$

after localisation at all $t_i-t_j, i\neq j$ or at all symmetric polynomials.

(2) Hereby the normalized Bethe basis vectors for the subalgebra generated by the A_i's are mapped to the geometric fixed point basis vectors.

The first R-matrix from the 5-vertex model has a beautiful interpretation (the second can be obtained by some renormalizing):

Theorem 2 (Wall-crossing)**.** The base change from S_I^w to $S_I^{ws_i}$ (that is the wall crossing in the ith wall) is given via Φ by $R_{a,b}(t_b, t_a)$ acting on the ath and bth tensor factor of $V[t]^{\otimes N}$, where $a = w(i)$ and $b = w(i+1)$.

Theorem 3. The Yang Baxter-algebras Y_N and of Y_N' can be realized via certain correspondences $X_k \longleftarrow X_{k,k+1} \longrightarrow X_{k+1}$ involving the two-steps partial flag varieties $X_{k,k+1}$.

Remark 4. The construction can be seen as an analogue of the Maulik-Okounkov construction [MO12]. Instead of working with cotangent bundles and their symplectic structure we work instead with the base X_k itself. The Schubert varieties $\overline{C_I}$ are then classical analogues of their stable manifolds. Our assignment $p_I \mapsto [\overline{C_I}]$ from the set of T-fixed points to bases of $H_T(X_k)$ form a *stabilization map.*

Remark 5. The subalgebra of endomorphisms generated by Y_N and Y_N' is isomorphic to the image of the obvious $U(\mathfrak{gl}_2[t])$-action on $V[t]^{\otimes N}$. Taking the limit $N \to \infty$ allows to mimic the construction of [BLM90] for $\mathfrak{gl}_n$ now for current algebras. Our construction works more general for $U(\mathfrak{gl}_n[t])$ by using partial flag varieties up to n steps; although the formulas are not anymore totally explicit.

Remark 6. Geometric constructions of the current Lie algebras seem to be not available, in construst to Yangian (a certain quantum deformation of the loop algebra). The R-matrix corresponding to the Yangian corresponds to the "generic" 6-vertex model, and via for instance [MO12] to the geometry of the cotangent bundle of partial flag varieties and Nakajima quiver varieties. Working with the base instead specialises the deformation parameter and corresponds to a non-trivial interesting degeneration on the side of R-matrices. This is the passage to the 5-vertex model. As fas as we understand, these degenerations cannot be deduced easily from the "generic" model. Our stable manifolds do not arise from the construction in [MO12] by pushing their stable manifolds down to the base space of the cotangent bundle.

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Support of the spherical module of rational Cherednik algebras

Daniel Juteau

(joint work with Stephen Griffeth)

Given a complex reflection group W acting on a space V, the associated rational Cherednik algebra can be defined by a faithful representation on $\mathbb{C}[V]$: it is generated by polynomials acting by multiplication, the group algebra of W, and Dunkl operators, which are a deformation of differential operators depending on some parameters c (as many as there are conjugacy classes of reflections in W). One can define a category $\mathcal{O}$ in a similar way as for semisimple Lie algebras [2]. The determination of the support of simple modules (seen as coherent sheaves on V) is a central problem in the theory; as a subproblem, in contains the question of determining when the simple modules (which are in bijection with characters of W) are finite dimensional (which happens exactly when the support is $\{0\}$). The particular case of the spherical module, the unique simple quotient of the polynomial representation, has attracted particular interest: Varagnolo and Vasserot obtained a criterion for Weyl group in the case of equal parameters [3]; this criterion has been generalized Etingof to all Coxeter groups and arbitrary parameters [1]. I talked about a further generalization to all complex reflections groups (and arbitrary parameters), modulo some hypotheses about their Hecke algebras (they should be projective and symmetric over their parabolic subalgebras).

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Combinatorics of canonical bases and cluster duality

Bea Schumann

(joint work with Volker Genz and Gleb Koshevoy)

Let G be a simple, simply connected, simply laced algebraic group over $\mathbb{C}$ and $U \subset G$ its unipotent radical. We investigate the interplay of several classical and new parametrizations of canonical vector space bases of the ring of regular functions on the base affine space G/U. Since $\mathbb{C}[G/U]$ splits into the multiplicity-free direct sum of all finite dimensional irreducible G-representation, the definition of a canonical basis $\mathbb{B}$ should include the compatibility of $\mathbb{B}$ with respect to this decomposition.

Building on a (corrected version of a) conjecture of Fock-Goncharov, Gross-Hacking-Keel-Kontsevich approached the question of constructing a canonical basis of $\mathbb{C}[G/U]$ by means of cluster duality. To fix notation, let B, B^{-} be two opposite Borel subgroups of G, w_0 the longest Weyl group element, N the length of w_0 and n the rank of G.

There exists an open embedding of the $\mathcal{A}$-cluster variety $G^{e,w_0} := B \cap B^- w_0 B^-$ into G/U. The feature of being an $\mathcal{A}$-cluster variety means that, up to codimension 2, the space G^{e,w_0} is the union of open copies of $(\mathbb{C}^*)^{N+n}$ which are glued via subtraction-free birational transformations, called $\mathcal{A}$-cluster mutations. This construction always comes with a dual picture: The Fock-Goncharov dual $\mathcal{X}$-cluster variety. This is again the union of open copies of $(\mathbb{C}^*)^{N+n}$, running over the same index set as used in the construction of the $\mathcal{A}$-cluster variety, glued by (an appropriate notion of) dual birational transformations, called $\mathcal{X}$-cluster mutations. The tori in the two dual toric systems are related by a regular map. Since all maps involved in the construction are subtraction-free, we may apply the machinery of tropicalisation. In the following we mean by the tropical points $[T]_{trop}$ of a torus T its cocharacter lattice and by the tropicalisation $[f]_{trop}$ of a positive rational function f on T the usual $(\min, +)$-tropicalisation.

In [3], Gross-Hacking-Keel-Kontsevich construct a canonical basis $\mathcal{B}_{\text{can}}$ of $\mathbb{C}[G/U]$ and a regular function W on the associated $\mathcal{X}$-cluster variety (called potential) such that for any copy T of $(\mathbb{C}^*)^{N+n}$ in the $\mathcal{X}$-cluster variety, $\mathcal{B}_{\text{can}}$ is natural parametrized by the polyhedral cone $\mathcal{C}_T = \{x \in [T]_{trop} \mid [W\big|_T]_{trop}(x) \geq 0.\}$.

There is a well-known classical canonical basis $\mathbb{B}$ of $\mathbb{C}[G/U]$ constructed by Kashiwara and Lusztig, independently. The relation of $\mathcal{B}_{\text{can}}$ and $\mathbb{B}$ is an open question. Both Lusztig's and Kashiwara's construction yield parametrizations of $\mathbb{B}$, one for each reduced word $\mathbf{i}$ of w_0, in terms of polyhedral cones in $\mathbb{N}^{N+n}$ called the string cone $\mathcal{S}_\mathbf{i}$ and the cone of Lusztig's parametrization $\mathcal{L}_\mathbf{i}$, respectively. We relate the various parametrizations in the following theorem.

Theorem 1 ([2]). For every reduced word $\mathbf{i}$ there exists an open torus $T_\mathbf{i}$ in the toric atlas of the $\mathcal{X}$-cluster variety associated to G^{e,w_0} and explicit lattice isomorphisms t_1 and t_2 yielding bijections:

$$\mathcal{S}_\mathbf{i} \xrightarrow{t_1} \mathcal{C}_{T_\mathbf{i}} \xleftarrow{t_2} \mathcal{L}_\mathbf{i}.$$

Investigating the type A situation further hints at a much deeper relation between the different combinatorial studies of canonical bases of $\mathbb{C}[G/U]$. Let from now on $G = \mathrm{SL}_{n+1}$. By specializing an appropriate set of frozen coordinates to 1 we arise at the cluster variety $L^{e,w_0} = U \cap B^{e,w_0}$ with partial compactification given by the unipotent radical U. We denote the corresponding potential by $\overline{W}$ and the projections of the tori $T_\mathbf{i}$ along the specialized frozen coordinates by $\overline{T}_\mathbf{i}$.

Let f_a denote the crystal operator on Lusztig's parametrization corresponding to the simple root α_a and let

$$R_a = \{v \in \mathbb{N}^N \mid (\lambda, f_a v - v) \in \mathcal{L}_\mathbf{i} \text{ for a } \lambda \in \mathbb{N}^n\}.$$

There is an intimate relation between potentials and crystal operations given as follows.

Theorem 2 ([1]). The function $\overline{W}\big|_{\overline{T}_{\mathbf{i}}}$ is the pullback of $r = \sum_{a=1}^{n} r_a$ via an explicit isomorphism of tori, where

$$r_a = \sum_{v \in R_a} \sum_{j=1}^{N} x_i^{v_i} \in \mathbb{C}[x_1^{\pm 1}, \dots, x_N^{\pm 1}].$$

As shown in [2], we have

$$W\big|_{T_{\mathbf{i}}} = \overline{W}\big|_{\overline{T}_{\mathbf{i}}} + X$$

and the summand X of $W\big|_{T_{\mathbf{i}}}$ is easy to compute explicitly. We obtain as a corollary of Theorem 2 and the explicit description of the set R_a obtained in [1] that $W\big|_{T_{\mathbf{i}}}$ is Laurent polynomial without constant term, all coefficients in $\{0,1\}$ and all exponents in $\{0,-1\}$. Furthermore we believe that Theorem 2 holds true in the simply laced situation.

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Non-Levi branching rules via Littelmann paths: a new approach

Jacinta Torres

(joint work with Bea Schumann)

Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let $\mathrm{R} \subset \mathfrak{h}^*$ be the corresponding root system and $\mathrm{X} \subset \mathfrak{h}^*$ the integral weight lattice. We fix a choice of simple roots $\Delta = \{\alpha_i\}_{i \in I} \subset \mathrm{R}$. This defines a set of dominant integral weights X^+ and a dominant Weyl chamber $C^+ \subset \mathfrak{h}^*_{\mathbb{R}}$. For $\lambda \in \mathrm{X}^+$, we will denote the associated simple module of $\mathfrak{g}$ by $\mathrm{L}(\lambda)$.

1. The Littelmann path model

A *Littelmann path model* for $\mathrm{L}(\lambda)$ is a set of paths

$$\mathcal{P}(\lambda) \subset \{\pi : \pi : [0,1] \to \mathfrak{h}^*_{\mathbb{R}}\}$$

with the following properties:

- The startpoint of a path $\pi \in \mathcal{P}(\lambda)$ is always the origin $\pi(0) = 0$.
- The formal character $\mathrm{chL}(\lambda)$ can be written as $\mathrm{chL}(\lambda) = \sum_{\pi \in \mathcal{P}(\lambda)} e^{\pi(1)}$.

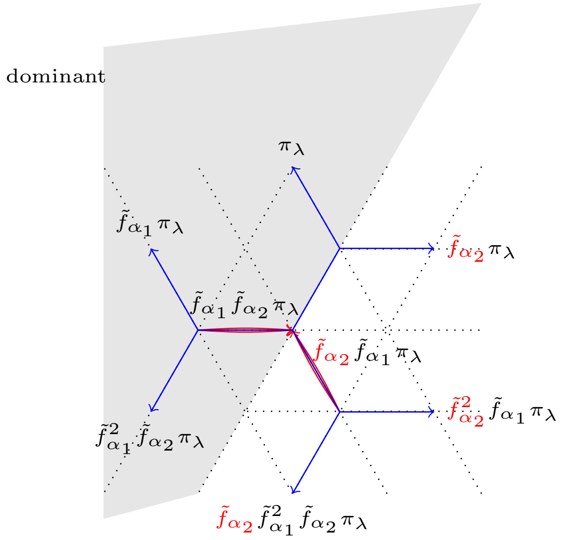

Figure 1. A Littelmann path model for the adjoint representation of $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. The paths contained in the shaded region are those $\mathfrak{g}^J$ dominant for $J = \{2\} \subset \{1, 2\} = I$

- There exists a unique path $\pi_\lambda \in \mathcal{P}(\lambda)$ that is *dominant* (i.e. $\pi_\lambda([0,1]) \subset C^+$) and endpoint $\pi_\lambda(1) = \lambda$. From this path π_λ one obtains the rest of the paths in $\mathcal{P}(\lambda)$ by applying root operators $\tilde f_\alpha$, one for each simple root $\alpha \in \Delta$, which are defined combinatorially on paths (see [1]). These operators endow the set $\mathcal{P}(\lambda)$ with the structure of a crystal isomorphic to $\mathrm{B}(\lambda)$.

Moreover, if we choose *any* dominant piecewise linear path π'_λ with endpoint $\pi'_\lambda(1) = \lambda$, the set of paths obtained by successively applying root operators to it is also a crystal isomorphic to $\mathrm{B}(\lambda)$. The following two theorems are a sample of the remarkable properties of $\mathcal{P}(\lambda)$. They were proven by Peter Littelmann in [1].

Generalised Littlewood Richardson rule. Let $\lambda, \mu \in \mathrm{X}^+$ be two dominant integral weights. Then the following decomposition holds:

$$\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu) = \bigoplus_{\eta = \pi * \nu} \mathrm{L}(\eta(1)),$$

where $\pi * \nu$ denotes the concatenation the paths π and ν.

Levi branching rule. Let $J \subset I$ be a subset of simple roots and $\mathfrak{g}^J \subset \mathfrak{g}$ the corresponding Levi subalgebra, and $\lambda \in \mathrm{X}^+$ a dominant integral weight. Then

$$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{g}^J} \mathrm{L}(\lambda) = \bigoplus_{\eta \in \mathcal{P}(\lambda)^{J,+}} \mathrm{L}(\eta(1))$$

where $\mathcal{P}(\lambda)^{J,+}$ are the paths in a Littelmann path model $\mathcal{P}(\lambda)$ which are dominant for $\mathfrak{g}^J$.

2. A new approach

Now consider $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$, and let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the automorphism induced by folding the Dynking diagram along the middle vertex. The fixed point set $\mathfrak{g}^\sigma$ is a Lie algebra isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$. If we take $\mathfrak{h}$ to be the Cartan subalgebra of diagonal matrices, then the fixed point set $\mathfrak{h}^\sigma$ is a Cartan subalgebra for $\mathfrak{g}^\sigma$. By restricting the simple roots $\alpha_i^\sigma := \alpha_i|_{\mathfrak{h}^\sigma}$ for $i = 1, \cdots, n$, we get a set of (simple) roots for $(\mathfrak{g}^\sigma, \mathfrak{h}^\sigma)$ and therefore a choice of dominant chamber. Now, given a path $\pi : [0, 1] \to \mathfrak{h}^*_\mathbb{R}$, we define a new restricted path

$$\begin{aligned} \mathrm{res}(\pi) : [0,1] &\to (\mathfrak{h}^\sigma)^*_\mathbb{R} \\ t &\mapsto \pi(t)|_{\mathfrak{h}^\sigma}. \end{aligned}$$

Consider all of the restricted paths in a path model $\mathcal{P}(\lambda)$ that are dominant for $\mathfrak{g}^\sigma$:

$$\mathrm{domres}(\mathcal{P}(\lambda)) = \{\eta = \mathrm{res}(\pi) : \pi \in \mathcal{P}(\lambda), \eta \text{ is dominant}\}.$$

In the following, $\mathcal{P}(\lambda)$ is the Littelmann path model consisting of lattice paths (see [2]). The proof [3] of Theorem 3 consists in establishing a combinatorial bijection.

Theorem 3. The following decomposition holds:

$$\mathrm{L}(\lambda) = \bigoplus_{\pi \in \mathrm{domres}(\mathcal{P}(\lambda))} \mathrm{L}^\sigma(\pi(1)).$$

Theorem 3 was conjectured by Naito-Sagaki in [2]. It does not hold for all path models. The sense behind this mystery, as well as the possibility to generalise to other Dynkin diagram automorphisms, remain open.

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Spectra of quantum integrable systems, Langlands duality and category $\mathcal{O}$ for quantum affine algebras

David Hernandez

R-matrices give power tools to study the spectra of quantum integrable systems. A better understanding of transfer-matrices obtained from R-matrices led us to the proof of several conjectures. Our approach is based on the study of a category $\mathcal{O}$ *of representations of a Borel subalgebra of a quantum affine algebra.*

This is based on several joint works with M. Jimbo, E. Frenkel and B. Leclerc. Supported by ERC with Grant Agreement number 647353 Qaffine.

Main motivations (Quantum integrable systems and Langlands duality)
The partition function Z of a (quantum) integrable system is crucial to understand its physical properties. For important examples, it can be expressed as $Z = \mathrm{Tr}(T^M)$ where T is the transfer matrix and M is the size of system. Therefore, one needs to find the spectrum of T : the spectrum of the system.

The ODE/IM correspondence (Ordinary Differential Equations/Integrable models) was discovered at the end of the 90's (Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov) and gives a surprising relation between functions associated to Schrödinger differential operators of the form $-\partial^2 + x^{2M} + \ell x^{-2}$ (with $M > 0$ integer and $\ell \in \mathbb{C}$) and the spectrum of quantum systems called "quantum KdV". The functions for the Schrödinger systems are the spectral determinants defined as coefficients of the expansion of certain solutions with remarkable asymptotical properties (subdominant solutions) towards a natural basis of solutions.

Feigin-Frenkel [FF] have proposed a large generalization and an interpretation of this correspondence in terms of Langlands duality. The Schrödinger operators are generalized to affine opers (without monodromy), associated to Langlands dual of the affine Lie algebra attached to the quantum KdV system. This conjecture is largely open, but it is a fruitful source of researches. In particular, a remarkable system of relations (the $Q\tilde{Q}$-system) was observed [MRV] to be satisfied by spectral determinants of certain solutions of affine opers. These $Q\tilde{Q}$-systems are particularly important as they imply the famous Bethe relations. What is the explanation for such a system and does it hold on the Integrable Model side ?

1. Category $\mathcal{O}$

Let us consider these questions in terms of representation theory. Let $\hat{\mathfrak{g}}$ be an untwisted affine Kac-Moody algebra and $q \in \mathbb{C}^*$ which is not a root of 1. Let $n = rk(\mathfrak{g})$ and for $i \in I = \{1, \cdots, n\}$, $q_i = q^{r_i}$ with $B_{i,j} = r_i C_{i,j}$ where C (resp. B) is the (resp. symmetrized) Cartan matrix of $\mathfrak{g}$. Consider the corresponding quantum affine algebra $U_q(\hat{\mathfrak{g}})$. This is a quantum group of Drinfeld-Jimbo.

Let $U_q(\hat{\mathfrak{b}}) \subset U_q(\hat{\mathfrak{g}})$ be a Borel subalgebra (in the sense of Chevalley). It is a Hopf subalgebra of $U_q(\hat{\mathfrak{g}})$. A simple finite-dimensional representation of $U_q(\hat{\mathfrak{g}})$ is still simple when restricted to $U_q(\hat{\mathfrak{b}})$. $U_q(\hat{\mathfrak{b}})$ has itself a triangular decomposition deduced from the Drinfeld realization of $U_q(\hat{\mathfrak{g}})$. This lead [HJ] to the definition of a category $\mathcal{O}$ of $U_q(\hat{\mathfrak{b}})$-modules whose weight spaces (for the analog of the finite type Cartan algebra) are finite-dimensional and whose weights satisfy the same axiomatic properties as for the usual category $\mathcal{O}$ of $\mathfrak{g}$.

Theorem 1 [Hernandez-Jimbo, 2012] *The simple objects in the category $\mathcal{O}$ are parametrized by n-tuples $(f_i(z))_{i\in I}$ of $f_i(z) \in \mathbb{C}(z)$ regular at the origin.*

For example, for $i \in I$, $a \in \mathbb{C}^*$ we have the prefundamental representation $L_{i,a}$ associated to $\mathbf{\Psi}_{i,a} = (1, \cdots, 1, 1 - za, 1, \cdots, 1)$ with $1 - za$ in position i. It was constructed in [HJ] as (the dual of) a limit of finite-dimensional representations. For $\mathfrak{g} = sl_2$ it was constructed explicitly by Bazhanov-Lukyanov-Zamolodchikov.

The Grothendieck ring $K_0(\mathcal{O})$ has a very rich structure. In [HL] we used this category $\mathcal{O}$ to obtain new monoidal categorifications of cluster algebras. For $\mathfrak{g} = sl_2$ and V simple of dimension 2, we get a Fomin-Zelevinsky mutation relation

$$[V \otimes L_{1,aq}] = [\omega][L_{1,aq^{-1}}] + [-\omega][L_{1,aq^3}]. \tag{1}$$

where $a \in \mathbb{C}^*$ and $[\pm\omega]$ are invertible representations of dimension 1. This is a categorified realization of the Baxter's TQ-relation. Our cluster algebra framework lead to a natural generalization (the QQ^*-system established in [HL]) :

$$[L_{i,a}^*][L_{i,a}] = \prod_{j, C_{i,j}<0} [L_{j,aq^{-B_{j,i}}}] + [-\alpha_i] \prod_{j, C_{i,j}<0} [L_{j,aq^{B_{i,j}}}] \text{ for } i \in I,\, a \in \mathbb{C}^*,$$

with $[-\alpha_i]$ of dimension 1 and $L_{i,a}^*$ simple corresponds to $\mathbf{\Psi}_{i,a}^{-1} \prod_{j, C_{i,j}<0} \mathbf{\Psi}_{j,aq^{-B_{j,i}}}$.

2. Transfer-matrices

A very important property of $U_q(\hat{\mathfrak{g}})$ is the existence of the universal R-matrix

$$\mathcal{R}(z) \in (U_q(\hat{\mathfrak{g}}) \hat{\otimes} U_q(\hat{\mathfrak{g}}))[[z]],$$

solution of the Yang-Baxter equation in a (slight) completion of the tensor square. Given V in the category $\mathcal{F}$ of finite-dimensional representations of $U_q(\mathfrak{g})$, we have

$$t_V(z) = \mathrm{Tr}_V(\pi_{V(z)} \otimes \mathrm{Id})(\mathcal{R}) \in U_q(\hat{\mathfrak{g}})[[z]],$$

the transfer-matrix where $V(z)$ is a twist of V for a natural grading of $U_q(\hat{\mathfrak{g}})$ and Tr_V is the (graded) trace on V. As a consequence of the Yang-Baxter equation the coefficients of transfer-matrices generate a commutative subalgebra of $U_q(\hat{\mathfrak{g}})$. As the first factor of $\mathcal{R}(z)$ lies in $U_q(\hat{\mathfrak{b}})$, $t_V(z)$ can also be defined for V in $\mathcal{O}$.

The transfer-matrix construction gives rise to various families of quantum integrable systems with an action of $K_0(\mathcal{F})$ (and of $K_0(\mathcal{O})$) on a space W. For XXZ-type models W is a tensor product of simple objects in $\mathcal{F}$ and for quantum KdV models W is the Fock space of a quantum Heisenberg algebra.

A representation V in $\mathcal{F}$ has a q-character $\chi_q(V) \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ [FR]. For $\mathfrak{g} = \hat{sl}_2$ and V simple of dimension 2, $\chi_q(V) = Y_{1,a} + Y_{1,aq^2}^{-1}$ where $a \in \mathbb{C}^*$.

Theorem 2 [Frenkel-Hernandez 2015, conjectured by Frenkel-Reshetikhin 1998] *Let V, W as for a XXZ-type model above. The eigenvalues λ_k of $t_V(z)$ on W are obtained from $\chi_q(V)$ by replacing each variable $Y_{i,a}$ by a quotient*

$$\frac{f_i(azq^{-1}) Q_{i,k}(zaq_i^{-1})}{f_i(azq_i) Q_{i,k}(zaq)}$$

where the functions $f_i(z)$ do not depend on λ_k and $Q_{i,k}$ is a polynomial.

Our proof [FH1] is based on the study of the category $\mathcal{O}$. We establish relations in $K_0(\mathcal{O})$ generalizing the relation (1) and we prove the transfer-matrix associated to $L_{i,a}$ are polynomial on W up to a scalar. For $\mathfrak{g} = sl_2$ and V of dimension 2, we recover the Baxter formula $\lambda_k = A(z)\frac{Q_k(zq^2)}{Q_k(z)} + D(z)\frac{Q_k(zq^{-2})}{Q_k(z)}$ with A, D universal.

The functions $f_i(z)$ can be computed. What about the polynomials $Q_{i,k}$?

3. $Q\tilde{Q}$-systems and consequences.

The $Q\tilde{Q}$-system of [MRV] for two families of functions $(Q_i(z))_{i\in I}$, $(\tilde{Q}_i(z))_{i\in I}$ is

$$\begin{aligned} &Q_i(zq_i^{-1})\tilde{Q}_i(zq_i) - Q_i(zq_i)\tilde{Q}_i(zq_i^{-1}) \\ &= \prod_{j | C_{i,j<0}} Q_j(zq^{C_{i,j}+1})Q_j(zq^{C_{i,j}+3})\cdots Q_j(zq^{-C_{i,j}-1}). \end{aligned}$$

Theorem 3 [Frenkel-Hernandez, 2016] *There is a natural family of simple objects $\tilde{L}_{i,a}$ $(i \in I, a \in \mathbb{C}^*)$ in the category $\mathcal{O}$ such that $Q_i(z) = [L_{i,z}]$, $\tilde{Q}_i(z) = [\tilde{L}_{i,z}]$ satisfy the $Q\tilde{Q}$-system in $K_0(\mathcal{O})$.*

This gives an explanation [FH2] for the results of [MRV]. We also derive informations on the the root of the Baxter's polynomials, the Bethe Ansatz equations conjectured by various authors (see [FR, H]) : for w a (generic) root of $Q_{i,k}$,

$$v_i^{-1} \prod_{j\in I} \frac{Q_{j,k}(wq^{B_{i,j}})}{Q_{j,k}(wq^{-B_{i,j}})} = -1, \tag{2}$$

where the v_i are the parameters of the twisted trace. The genericity condition was dropped by Feigin-Jimbo-Miwa-Mukhin by using the QQ^*-systems as in [HL].

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On the tensor semigroup of affine Kac-Moody Lie algebras

Nicolas Ressayre

In this note, we are interested in the tensor product decomposition of simple representations of an untwisted affine Lie algebra. Let $\dot{\mathfrak{g}} \supset \dot{\mathfrak{b}} \supset \dot{\mathfrak{h}}$ be a complex simple Lie algabra with fixed Borel and Cartan subalgebras. Consider the associated affine Kac-Moody Lie algebra

$$\mathfrak{g} := \dot{\mathfrak{g}} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is the central element and d the derivation. Let $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the standard Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}^* = \dot{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda \oplus \mathbb{C}\delta$ its dual. See [4, Chap. 13] for more precise definitions. Set P_+ be the set $\lambda = \dot\lambda + \ell\Lambda + b\delta$ in $\mathfrak{h}^*$ such that $\dot\lambda$ is a dominant weight of $\dot{\mathfrak{h}}$, ℓ and b are integers such that $\langle \dot\lambda, \dot\theta^\vee \rangle \leq \ell$. Here $\dot\theta^\vee$ is the dual root of the heighest root of $\dot{\mathfrak{g}}$. For $\lambda \in P_+$, we denote by $L(\lambda)$ the integrable representation of $\mathfrak{g}$ of heighest weight λ. Given two elements $\lambda_1 = \dot\lambda_1 + \ell_1\Lambda + b_1\delta$ and $\lambda_2 = \dot\lambda_2 + \ell_2\Lambda + b_2\delta$ in P_+, the tensor product $L(\lambda_1) \otimes L(\lambda_2)$ decomposes as

$$L(\lambda_1) \otimes L(\lambda_2) = \oplus_{\mu \in P_+} m^{\mu}_{\lambda_1\,\lambda_2} L(\mu),$$

for some multiplicities $m^{\mu}_{\lambda_1\,\lambda_2} \in \mathbb{N}$.

The first aim of this talk is to describe the set of nonzero multiplicities. One can easily check that one may assume that

(1) $\ell = \ell_1 + \ell_2$;
(2) $b_1 = b_2 = 0$;
(3) $\mu - \lambda_1 - \lambda_2$ belongs to the root lattice;
(4) $\ell_1 > 0$ and $\ell_2 > 0$.

Set $\mathcal{A} = \{(\lambda_1 = \dot\lambda_1 + \ell_1\Lambda, \lambda_2 = \dot\lambda_2 + \ell_2\Lambda, \bar\mu = \dot\mu + (\ell_1 + \ell_2)\Lambda) \in (P_+)^3 \,:\, \ell_1 > 0 \text{ and } \ell_2 > 0\}$. Then $S(\mathrm{lg}) = \{(\lambda_1, \lambda_2, \bar\mu, b) \in \mathcal{A} \times \mathbb{Z} \,:\, m^{\bar\mu + b\delta}_{\lambda_1\,\lambda_2} \neq 0\}$ is a semigroup and

$$S_{\mathbb{Q}}(\mathfrak{g}) := \{(\lambda_1, \lambda_2, \bar\mu, b) \in \mathcal{A} \otimes \mathbb{Q} \times \mathbb{Q} \,:\, \exists N > 0 \quad (N\lambda_1, N\lambda_2, N\mu) \in S(\mathfrak{g})\}$$

is a convex cone, that we are going to describe.

Fix $(\lambda_1, \lambda_2, \bar\mu) \in \mathcal{A}$ and consider the formal series in z

$$F(z) = \sum_{b \in \mathbb{Z}} m^{\mu}_{\lambda_1\,\lambda_2} z^b.$$

Using the action of a Virazoro algebra, one can show that $F(z)$ has the following form

$$F(z) \in m_0(\lambda_1, \lambda_2, \bar\mu) z^{b_0(\lambda_1, \lambda_2, \bar\mu)} (1 + \mathbb{N}z^{-1} + \mathbb{N}^* z^{-2} + \mathbb{N}^* z^{-3} + \dots),$$

where $m_0(\lambda_1, \lambda_2, \bar\mu)$ is a positive integer and $b_0(\lambda_1, \lambda_2, \bar\mu) \in \mathbb{Z}$. Denote by m_0 and b_0 these integers for short. Obviously $m_0 = m^{\bar\mu + b_0\delta}_{\lambda_1\,\lambda_2}$. In particular, by setting

$$B(\lambda_1, \lambda_2, \mu) = \sup_{N \in \mathbb{Z}_{>0}} \frac{b_0(N\lambda_1, N\lambda_2, N\bar\mu)}{N},$$

one gets $S_{\mathbb{Q}}(\mathfrak{g}) = \{(\lambda_1, \lambda_2, \bar{\mu}, b) \in \mathcal{A} \otimes \mathbb{Q} \times \mathbb{Q} \,:\, b \leq B(\lambda_1, \lambda_2, \bar{\mu})\}$. Our aim is to determine $S_{\mathbb{Q}}(\mathfrak{g})$ that is to compute the function $B : \mathcal{A} \longrightarrow \mathbb{R}$. We need to introduce some notation.

Let $\mathfrak{h}^* \supset \{\alpha_0, \dots, \alpha_l\} =: \Delta$ be the set of simple roots of $\mathfrak{g}$. Fix $(\varpi_{\alpha_0^\vee}, \dots, \varpi_{\alpha_l^\vee}) \subset \mathfrak{h}_{\mathbb{Q}}$ be elements dual to the simple roots. Let W be the Weyl group of $\mathfrak{g}$. Let G be the minimal Kac-Moody group and B its standard Borel subgroup. To any simple root α_i, is associated a maximal standard parabolic subgroup P_i, its standard Levi subgroup L_i, its Weyl group $W_{L_i} \subset W$ and the set W^{P_i} of minimal length representative of elements of W/W_{L_i}. We also consider the partial flag ind-variety $X_i = G/P_i$ containing the Schubert varieties $X_w = \overline{BwP_i/P_i}$, for $w \in W^{P_i}$. Let $\{\epsilon_w\}_{w \in W^{P_i}} \subset \mathrm{H}^*(X_i, \mathbb{Z})$ be the Schubert basis dual to the basis of the singular homology of X_i given by the fundamental classes of X_w. As defined by Belkale-Kumar [1, Section 6] in the finite dimensional case, Brown-Kumar defined in [2, Section 7] a deformed product $\odot_0$ in $\mathrm{H}^*(X_i, \mathbb{Z})$, which is commutative and associative.

Theorem 1. Let $(\lambda_1, \lambda_2, \mu) \in P^3_{+,\mathbb{Q}}$ such that $\lambda_1(c) > 0$ and $\lambda_2(c) > 0$ that is λ_1 and λ_2 are not W-stable.

Then, $(\lambda_1, \lambda_2, \mu) \in S_{\mathbb{Q}}(\mathfrak{g})$ if and only if

$$\mu(c) = \lambda_1(c) + \lambda_2(c), \text{ and} \tag{1}$$

$$\langle \mu, v\varpi_{\alpha_i^\vee} \rangle \leq \langle \lambda_1, u_1\varpi_{\alpha_i^\vee} \rangle + \langle \lambda_2, u_2\varpi_{\alpha_i^\vee} \rangle \tag{2}$$

for any $i \in \{0, \dots, l\}$ and any $(u_1, u_2, v) \in (W^{P_i})^3$ such that ϵ_v occurs with coefficient 1 in the deformed product $\epsilon_{u_1} \odot_0 \epsilon_{u_2}$.

Note that the statement makes for any symmetrizable Kac-Moody group. The finite dimensional case was proved in [1] by Belkale-Kumar. The case of $\tilde{A}_1$ is due to Brown-Kumar [2] that have conjectured that it is true for any symmetrizable Kac-Moody group.

Let us now explain how the proof works. Consider the projective ind-variety $X = (G/B^-)^2 \times G/B$. Given $(\lambda_1, \lambda_2, \bar{\mu}) \in \mathcal{A}$ and $b \in \mathbb{Z}$, there is a G-linearized line bundle $\mathcal{L}$ in X such that

$$\mathrm{H}^0(X, \mathcal{L}) \simeq \mathrm{Hom}(L(\lambda_1)^\vee \otimes L(\lambda_2)^\vee \otimes L(\mu), \mathbb{C}),$$

by the Borel-Weil theorem (see [4]). Here $\mu = \bar{\mu} + b\delta$. Then $(\lambda_1, \lambda_2, \bar{\mu}, b) \in S_{\mathbb{Q}}(\mathfrak{g})$ if and only if $\mathrm{H}^0(X, \mathcal{L})^G \neq \{0\}$. Similarly, $(\lambda_1, \lambda_2, \bar{\mu}, b) \in S(\mathfrak{g})$ if and only if

$$\exists N > 0 \qquad \mathrm{H}^0(X, \mathcal{L}^{\otimes N})^G \neq \{0\}.$$

If G has **finite dimension** this condition is equivalent to the existence semi-stable points for the action of G on X for the line bundle $\mathcal{L}$. In [1] (see also [3]), several tools of Geometric Invariant Theory are used to prove the result: Hilbert-Mumford theorem, Kempf-Rousseau's optimal one parameter subgroup, Hesselink stratification... We observe that these results have no known equivalent in our setting (action of a Kac-Moody group on a projective ind-variety). We use a completely different strategy that we explain now.

Fix a Schubert data $\underline{sc} = (u_1, u_2, v, i)$ as in Theorem 1. An explicit computation allows to prove that inequality (2) is equivalent to

$$b \leq \varphi_{\underline{sc}}(\lambda_1, \lambda_2, \bar{\mu}), \tag{3}$$

for some well defined linear function $\varphi_{\underline{sc}}$ on $\mathcal{A}$. Set

$$\varphi = \inf_{\underline{sc}} \varphi_{\underline{sc}},$$

where the infimum is over any $\underline{sc} = (u_1, u_2, v, i)$ as in Theorem 1. It reamains to prove that $\varphi = B$ as functions on $\mathcal{A}$. The fact that the ineaqualities (2) are fulfilled by the points of $S(\mathfrak{g})$ is proved is [2]; this implies that $\varphi \geq B$.

Step 1. Show that φ is locally piecewise linear.

We use, in this step, inequalities implied by the assumption " ϵ_v occurs in the expansion of $\epsilon_{u_1}.\epsilon_{u_2}$ in the Schubert basis".

Step 1 reduces the proof of the inequality $\varphi \leq B$ to the set of points $(\lambda_1, \lambda_2, \bar{\mu}) \in \mathcal{A}$ such that $\varphi(\lambda_1, \lambda_2, \bar{\mu}) = \varphi_{\underline{sc}}(\lambda_1, \lambda_2, \bar{\mu})$, for some fixed $\underline{sc} = (u_1, u_2, v, i)$ as in the theorem.

Step 2. Reduction for mulitplicities on the boundary.

Set $C = L_i u_1^{-1} B^-/B^- \times L_i u_2^{-1} B^-/B^- \times L_i v^{-1} B/B$; it is a closed subvariety of X isomorphic to the product of three copies of the complete flag manifold of the finite dimensional reductive group L_i.

Theorem 2. Let α_i be a simple root of $\mathfrak{g}$. Let $(u_1, u_2, v) \in W^{P_i}$ such that ϵ_v occurs with coefficient 1 in the deformed product $\epsilon_{u_1} \odot_0 \epsilon_{u_2}$.

Fix $(\lambda_1, \lambda_2, \bar{\mu}) \in \mathcal{A}$ such that $b = \varphi_{\underline{sc}}(\lambda_1, \lambda_2, \bar{\mu}) \in \mathbb{Z}$. Let $\mathcal{L}$ be the G-linearized line bundle on X associated to $(\lambda_1, \lambda_2, \bar{\mu} + b\delta)$. Then the restriction map induces an isomorphism

$$\mathrm{H}^0(X, \mathcal{L})^G \simeq \mathrm{H}^0(C, \mathcal{L}_{|C})^{L_i}.$$

Step 3. Induction.

Let $(\lambda_1, \lambda_2, \bar{\mu}, b) \in \mathcal{A}$ satisfying all the ineqaulities of Theorem 1 and such that $b = \varphi_{\underline{sc}}(\lambda_1, \lambda_2, \bar{\mu})$.

It is sufficient to prove that there exists $N > 0$ such that $b_0(N\lambda_1, N\lambda_2, N\bar{\mu}) = Nb$. By Theorem 2, this is equivalent to

$$\exists N > 0 \qquad \mathrm{H}^0(C, \mathcal{L}_{|C}^{\otimes N})^{L_i} \neq \{0\}. \tag{4}$$

By the Belkale-Kumar theorem in the finite dimensional case, it remains to prove that the line bundle $\mathcal{L}_{|C}$ satisfies some explicit inequalities. Working on the condition ϵ_v occurs with coefficient 1 in $\epsilon_{u_1} \odot_0 \epsilon_{u_2}$, we prove that inequalities (2) of Theorem 1 imply the inequalities for $\mathcal{L}_{|C}$.

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Macdonald's formula for Kac-Moody groups over local fields

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(joint work with Nicole Bardy-Panse and Guy Rousseau)

The Macdonald's formula that is discussed here is the one giving the image of the Satake isomorphism between the spherical Hecke algebra and the ring of symmetric functions in the context of Kac-Moody groups. If the group is semisimple, the formula was proven by Macdonald. In the case of an affine Kac-Moody group, it is a result of Braverman, Kazhdan and Patnaik.

Let $\mathcal{F}$ be a local field, denote its ring of integers by $\mathcal{O}$, fix a uniformizer t and suppose that the residue field is of cardinality q. Let $G = G(\mathcal{F})$ be a (split) minimal Kac-Moody group over $\mathcal{F}$. Let $T \subset G$ be a maximal torus, denote the $\mathbb{Z}$-lattice of coweights by $Y = Hom(\mathcal{F}^*, T)$, the $\mathbb{Z}$-lattice of coroots by $Q^\vee$, and the real roots of (G,T) by Φ. Let W be the Weyl group of (G,T).

To these data, Guy Rousseau in [6] associate a masure $\mathcal{I} = \mathcal{I}(G,\mathcal{F})$, which is a generalization of the Bruhat-Tits building. The masure $\mathcal{I}$ is covered by apartments, all isomorphic to the standard one $\mathbb{A} = Y \otimes_{\mathbb{Z}} \mathbb{R}$. The group G acts on $\mathcal{I}$ such that $K := Stab_G(0) = G(\mathcal{O})$. The preorder associated to the positive Tits cone in $\mathbb{A}$ extends to a G-invariant preorder $\geq$ on $\mathcal{I}$. Then $G^+ = \{g \in G \mid g\cdot 0 \geq 0\}$ is a subsemigroup of G and we show in [4] that

$$G^+ = \bigsqcup_{\lambda \in Y^{++}} Kt^\lambda K,$$

where Y^{++} is the set of dominant coweights. Note that $G^+ = G$ if, and only if, G is semisimple. Let c_λ be the characteristic function of the double coset $Kt^\lambda K$. The spherical Hecke algebra is the set

$${}^s\mathcal{H} = \{\varphi = \sum_{\lambda \in Y^{++}} a_\lambda c_\lambda \mid supp(\varphi) \subset \cup_{i=1}^n \mu_i - Q_+^\vee,\ \mu_i \in Y^{++}\}$$

where the algebra structure is given by the convolution product. Consider now the Looijenga's coweights algebra $\mathbb{Z}[q^{\pm}1][[Y]]$, with the same kind of support condition as above. We show in [4] using an extended tree inside the masure $\mathcal{I}$ the following

Theorem. The spherical Hecke algebra ${}^s\mathcal{H}$ is isomorphic, via the Satake isomorphism S, to the commutative algebra $\mathbb{Z}[q^{\pm}1][[Y]]^W$ of Weyl invariant elements in $\mathbb{Z}[q^{\pm}1][[Y]]$.

Now, to compute the image $S(c_\lambda)$ of c_λ by the Satake isomorphism, we use essentially an idea of Casselmann [3]: write $S(c_\lambda)$ as a sum (indexed by the Weyl group W) of more simple elements $J_w(c_\lambda)$ and then compute these $J_w(c_\lambda)$. Originally this decomposition as a sum was obtained using intertwinning operators; this same idea was still used in [2] for the affine case. Here we get this decomposition using our interpretation of $S(c_\lambda)$ with paths in the masure, and we are able to prove a recursive formula for the $J_w(c_\lambda)$, still only using the masure.

Independently, we introduce some algebraic symmetrizers deduced from the Bernstein-Lusztig-Hecke algebra ${}^{BL}\mathcal{H}$, which was defined in [1] as an abstract version of the affine Iwahori-Hecke algebra of the Kac-Moody group. This involves the definition and study of a representation of ${}^{BL}\mathcal{H}$ (due to Cherednik, see also Macdonald [5]) leading to a general Kac-Moody version of Cherednik's identity. In this representation, one of the symmetrizers satisfies the same recursive formula as for the $J_w(c_\lambda)$. Comparing the first terms on both sides, we are able to prove the Macdonald's formula. To state it, set

$$\Delta = \prod_{\alpha \in \Phi_+} \frac{1 - q^{-1} e^{-\alpha^\vee}}{1 - e^{-\alpha^\vee}},$$

$W(q^{-1}) = \sum_{w \in W} q^{-\ell(w)}$ and $W_\lambda(q^{-1}) = \sum_{w \in W_\lambda} q^{-\ell(w)}$, where $W_\lambda = Stab_W(\lambda)$. Of course, we have to pay attention since we are dealing with infinite sums or products and there are some completions to be considered, in order that these expressions make sense.

Macdonald's formula. Let $\lambda \in Y^{++}$. Then

$$S(c_\lambda) = q^{\rho(\lambda)} \left(\frac{W(q^{-1})}{\sum_{w \in W} {}^w\Delta} \right) \left(\frac{\sum_{w \in W} {}^w\Delta . e^{w\lambda}}{W_\lambda(q^{-1})} \right),$$

where ρ is a root taking value 1 on each simple coroot.

Actually, our result is still valid in the more general framework of an abstract masure as defined in [6]. In particular, we can deal with the case of an almost split Kac-Moody group over a local field and we may have unequal parameters in the definition of the Hecke algebras.

The right hand side of the equality is the Hall-Littlewood polynomials $P_\lambda(t)$ defined by Viswanath [7] in this context, where its t corresponds to our q^{-1}. In particular, $P_\lambda(0)$ is the character of the irreducible representation $V(\lambda)$ of highest weight λ of the Langlands dual Kac-Moody group.

Finally, the factor $\mathfrak{m} = \frac{W(q^{-1})}{\sum_{w \in W} {}^w\Delta}$ is equal to 1 in the finite dimensional setting. Whereas, in the affine setting, $\mathfrak{m}$ has an expression as an infinite product involving the minimal positive coroot and the exponents of the underlying semisimple group. In the general Kac-Moody case, no such formula is known. However, it would be very interesting to get one to study Eisenstein series on Kac-Moody groups.

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Representation ring of Levi subgroups versus cohomology ring of flag varieties

Shrawan Kumar

Let us begin by recalling the classical result that the cup product structure constants for the singular cohomology with integral coefficients H^* of the Grassmannian of r-planes coincide with the Littlewood-Richardson tensor product structure constants for GL_r. Specifically, the result asserts that there is a $\mathbb{Z}$-algebra homomorphism $\phi : \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r) \to H^*(\mathrm{Gr}(r,n))$, where $\mathrm{Gr}(r,n)$ denotes the Grassmannian of r-planes in $\mathbb{C}^n$, $\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r)$ denotes the polynomial representation ring of GL_r and ϕ takes the irreducible polynomial representation $V(\lambda)$ of GL_r corresponding to the partition $\lambda : \lambda_1 \geq \cdots \geq \lambda_r \geq 0$ to the Schubert class $\epsilon_{v_A(\lambda)}$ corresponding to the same partition λ if $\lambda_1 \leq n-r$, where $v_A(\lambda)$ is a certain Weyl group element associated to λ. If $\lambda_1 > n-r$, then $\phi(V(\lambda)) = 0$.

This work seeks to achieve one possible generalization of this classical result for GL_r *and the Grassmannian* $\mathrm{Gr}(r,n)$ *to the Levi subgroups of any reductive group* G *and the corresponding flag varieties.*

Let G be a connected reductive group over $\mathbb{C}$ with a Borel subgroup B and maximal torus $T \subset B$. Let P be a standard parabolic subgroup with the Levi subgroup L containing T. Let W (resp. W_L) be the Weyl group of G (resp. L). Let $V(\lambda)$ be an irreducible almost faithful representation of G with highest weight λ (i.e., the corresponding map $\rho_\lambda : G \to \mathrm{Aut}(V(\lambda))$ has finite kernel). Then, Springer defined an adjoint-equivariant regular map with Zariski dense image $\theta_\lambda : G \to \mathfrak{g}$ (depending upon λ). Then, θ_λ takes the maximal torus T to its Lie algebra $\mathfrak{t}$. This induces a $\mathbb{C}$-algebra homomorphism $(\theta_{\lambda|T})^* : \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[T]$ on the corresponding affine coordinate rings. Since θ_λ is equivariant under the adjoint actions, $(\theta_{\lambda|T})^*$ takes $\mathbb{C}[\mathfrak{t}]^{W_L} = S(\mathfrak{t}^*)^{W_L}$ to $\mathbb{C}[T]^{W_L}$. Moreover, $(\theta_{\lambda|T})^*$ is injective. Let $\mathrm{Rep}^{\mathbb{C}}(L)$ be the complexified representation ring of the Levi subgroup L. As it is well known,

$$\mathrm{Rep}^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L}$$

induced from the restriction of the character to T. We call the image of $\mathbb{C}[\mathfrak{t}]^{W_L}$ under $(\theta_{\lambda|T})^*$, the *λ-polynomial subring* $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L)$ of $\mathrm{Rep}^{\mathbb{C}}(L)$.

For $G = \mathrm{GL}_n$ and $V(\lambda)$ the defining representation $\mathbb{C}^n$, the ring $\mathrm{Rep}_{\lambda-\mathrm{poly}}(G) := \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(G) \cap \mathrm{Rep}(G)$ coincides with the standard notion of polynomial representation ring of GL_n.

The Borel homomorphism $\beta : S(\mathfrak{t}^*) \to H^*(G/B, \mathbb{C})$ (which is surjective) from the symmetric algebra of $\mathfrak{t}^*$ restricticted to the W_L-invariants gives a surjective $\mathbb{C}$-algebra homomorphism $\beta^P : S(\mathfrak{t}^*)^{W_L} \to H^*(G/P, \mathbb{C})$. Thus, we get a surjective $\mathbb{C}$-algebra homomorphism $\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) \to H^*(G/P, \mathbb{C})$, which is our main result.

Specializing the above result to the case when $G = \mathrm{GL}_n$, λ is the first fundamental weight (so that $V(\lambda)$ is the standard defining representation $\mathbb{C}^n$) and $P = P_r$ (for any $1 \le r \le n-1$) is the maximal parabolic subgroup so that the flag variety G/P_r is the Grassmannian $\mathrm{Gr}(r, n)$, we recover the above classical result.

We determine the λ-polynomial representation ring $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(G)$, for λ the first fundamental weight ω_1 (i.e., $V(\lambda)$ is the defining representation) of the classical groups: $\mathrm{SO}_n, \mathrm{Sp}_{2n}$. In this case, the Springer morphism coincides with the classical Cayley transform. Recall that the defining representations of the classical groups have minimum Dynkin index. We believe that for the exceptional groups as well, the irreducible representation $V(\lambda)$ with minimum Dynkin index might be most 'appropriate' to consider the Springer morphism. Recall that for the exceptional groups: G_2, F_4, E_6, E_7, E_8, the representation $V(\lambda)$ has minimum Dynkin index for $\lambda = \omega_1, \omega_4, \omega_1$ (and ω_6), ω_7, ω_8 respectively.

We partially determine the homomorphism $\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) \to H^*(G/P, \mathbb{C})$ (with respect to the defining representation: $\lambda = \omega_1$) for all the maximal parabolic subgroups P in the classical groups $\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$ and SO_{2n}.

We determine the homomorphism $\xi^B_{\omega_1} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(T) \to H^*(G/B, \mathbb{C})$ for the Borel subgroups B in the classical groups $\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$ and SO_{2n}. (In this case, T is of course the Levi subgroup of B.)

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Non commutative algebra and invariant theory

Claudio Procesi
(joint work with E. Aljadeff, A. Giambruno, and A. Regev)

Abstract. Polynomial identities are at the crossroad between Non commutative Algebra and Representation Theory. I will point out some of the main aspects of the theory, also as advertisement of a forthcoming book: *Rings with polynomial identities and finite dimensional representations of algebras E. Aljadeff, A. Giambruno, C. Procesi, A. Regev*

A basic tool in algebra, which is also the basis of most algorithms and computer programs is *Symbolic calculus.*

In my talk I discuss only non commutative associative algebras for which we need to use *non commutative polynomials.*

Polynomial identities

Definition A non zero non commutative polynomial $f(x_1, \dots, x_m) \in F\langle X\rangle$ is a *polynomial identity* for an algebra R if it vanishes identically when computed in R.

(1) Which algebras satisfy polynomial identities?
(2) In other words what are the implications of the existence of some polynomial identities on an algebra?
(3) Which are the polynomial identities of a given algebra? For instance a matrix algebra.
(4) What about the resulting symbolic calculus?

The theory was prompted by Kurosh problem

The Burnside problem posed by William Burnside in 1902: Is every finitely generated torsion group G finite?

Kurosh is every finitely generated algebraic algebra A, over a field F, finite dimensional?

Both solved in the negative In 1964, Golod and Shafarevich.

In the bounded Kurosh problem one assumes that every element $x \in A$ satisfies some polynomial $x^n + a_1x^{n-1} + \ldots + a_1x = 0,\ a_i \in F$ for *fixed n*.

The bounded Burnside problem has sometimes a positive sometimes a negative answer while the bounded Kurosh problem has always a positive answer: An algebraic algebra of bounded degree satisfies a polynomial identity (Jacobson).

A finitely generated algebraic algebra A which satisfies a polynomial identity is finite dimensional. (Levitzki and Kaplansky).

The theory of polynomial identities mixes methods of commutative algebra with methods of finite dimensional algebras. The blending agent is representation theory and invariant theory.

The algebra $M_n(A)$ of $n \times n$ matrices over a commutative ring A is the basic example. The identity of minimal degree is given by the

Theorem [Amitsur–Levitzki] The algebra of $n \times n$ matrices over any commutative ring A satisfies the standard polynomial St_{2n}

$$St_{2n} := \sum_{\sigma \in S_{2n}} \epsilon_\sigma x_{\sigma(1)} \dots x_{\sigma(2n)}$$

The free algebra $F\langle x_1, \dots, x_m \rangle$ modulo the ideal J_n of polynomial identities of $n \times n$ matrices should be thought of as the algebra of polynomial functions of matrix variables.

- $F\langle \xi_1, \dots, \xi_m \rangle$ is an integral domain (Amitsur).
- $F\langle \xi_1, \dots, \xi_m \rangle$ has a quotient division algebra of fractions $D(m,n)$ of dimension n^2 over its center $Z(m,n)$ (Amitsur).
- $Z(m,n)$ is the field of rational functions on the space $M_n(F)^m$ of m–tuples of matrices, invariant under conjugation (Procesi).

This suggests to study the ring of polynomial functions $F : M_n(F)^m \to M_n(F)$ which are equivariant under conjugation.

$$f(gX_1g^{-1}, \dots, gX_ng^{-1}) = gf(X_1, \dots, X_n)g^{-1}, \ \forall g \in GL(n,F).$$

We can thus consider

(1) The ring T of invariants generated by all the coefficients of the characteristic polynomials of elements f of $F\langle \xi_1, \dots, \xi_m \rangle$, in fact generated just when f is a *primitive monomial*.
(2) The ring $T\langle \xi_1, \dots, \xi_m \rangle$ generated by $F\langle \xi_1, \dots, \xi_m \rangle$ and T.
(3) We have natural inclusions

$$F\langle \xi_1, \dots, \xi_m \rangle \subset T\langle \xi_1, \dots, \xi_m \rangle \subset D(m,n).$$

A MAIN THEOREM is that $T\langle \xi_1, \dots, \xi_m \rangle$ equals the algebra of polynomial functions $F : M_n(F)^m \to M_n(F)$ which are equivariant under conjugation.

This is a finitely generated module over the ring $T(m,n) := T$ of polynomial functions $F : M_n(F)^m \to F$ which are invariant under conjugation.

The spectrum of $T(m,n)$ The ring of invariants T parametrizes equivalence classes of n–dimensional semi–simple representations of the free algebra in m variables.

Except for the trivial case $n = 1$ or $n = m = 2$, its smooth part parametrizes irreducible n–dimensional representations. We call this smooth variety $X(m,n)$.

The spectrum of a PI (polynomial identity) algebra

1. spectrum

For PI algebras one has several theorems which resemble the theorems of commutative algebra, the main difference is that the spectrum is divided into natural strata, let us see it in a special case.

Nullstellensatz Let $R = F[a_1, \dots, a_m]$ be a a finitely generated algebra over F algebraically closed and satisfying a PI of degree $2d$ then.

Theorem[Nullstellensatz Procesi–Razmyslov] If M is a maximal ideal of R then $R/M \sim M_k(F),\ k \leq d$.

$\cap_{M \text{ maximal ideal}} M$ is a maximal nilpotent ideal.

The spectrum of generic matrices Denote the spectrum of m generic $n \times n$ matrices by $Y(m, n)$ then

$$Y(m, n-1) \subset Y(m, n), \quad Y(m, n) \setminus Y(m, n-1) = X(m, n)$$

recall $X(m, n)$ is a smooth variety parametrizing irreducible n–dimensional representations of the free algebra in m variables. As a consequence for the spectrum we have:

$$Y(m, n) = \cup_{i=1}^{n} X(m, i).$$

Azumaya algebras An Azumaya algebra R over its center Z of fixed rank n^2 is a *non split form of matrices* that is an algebra which, under a faithfully flat (even étale) extension of its center $Z \subset B$, becomes matrices

$$R \otimes_Z B = M_n(B).$$

It should be though of geometrically as a *a principal $PGL(n, F)$ bundle* over *$Spec(Z)$*.

Theorem[Artin] Assume that R is an algebra which satisfies *all polynomial identities of $n \times n$ matrices* for some n.

Assume further that there is no quotient R/I which satisfies a polynomial identity of $n-1 \times n-1$ matrices which is not an identity of $n \times n$ matrices. Then R is a rank n^2 Aumaya algebra over its center Z.

Finally a relatively free algebra that is a free algebra with finitely many variables modulo a non–zero T–ideal, has a nilpotent radical and modulo this it is an algebra of generic matrices. The entire algebra has a finite canonical filtration such that the factors are finitely generated modules over rings of invariants of matrices.

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