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Nonlinear Waves and Dispersive Equations

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Abstract. Nonlinear dispersive equations are models for nonlinear waves in a wide range of physical contexts. Mathematically they display an interplay between linear dispersion and nonlinear interactions, which can result in a wide range of outcomes from finite time blow-up to solitons and scattering. They are linked to many areas of mathematics and physics, ranging from integrable systems and harmonic analysis to fluid dynamics, geometry, general relativity and probability.

Mathematics Subject Classification (2010): 35Qxx, 37Kxx, 74J40.

Introduction by the Organisers

The workshop *Nonlinear waves and dispersive equations*, organised by Herbert Koch (Bonn), Pierre Raphael (Nice), Danial Tataru (Berkeley), and Monica Visan (UCLA) was well attended, with around 50 participants with broad geographic representation from all continents. There was a strong preference in having talks by young mathematicians.

The field of nonlinear dispersive equations as a whole emerged in the late 80's and the early 90's, and has experienced a continuous growth during the last two decades. While a good number of problems were solved, many of the most difficult problems remain open. This attracts many strong mathematicians and is the engine for further growth. Topics of talks included:

(1) The notion of 'minimal blow-up solution', based on seminal work of Kenig-Merle, Tao-Visan, Dodson, has became an amazingly successful approach

to global well-posedness and scattering for critical dispersive equations. Supercritical problems have also been addressed, as well as ongoing progress toward various formulations of the soliton resolution conjecture and the classification of minimal elements.

(2) New insights in the asymptotics of blow-up for focusing problems have been presented, both in the mass and energy critical case and in the supercritical range.

(3) Geometric dispersive models such as wave maps, Schrödinger maps, and Yang-Mills flows continue to be the focus of active research. One goal here in recent years has been the Threshold Conjecture for energy critical problems.

(4) Over the last couple of years, space-time resonance methods (previously confined to the study of semilinear equations) have been extended to the quasilinear setting. In the regime of weak turbulence there are first promising rigorous results toward reducing the nonlinear Schrödinger equation to an equation describing only the resonant interaction.

(5) Wave equations with stochastic terms become an increasingly important area, where significant progress was achieved in the last years.

(6) New developments in harmonic analysis, dictated by the needs of dispersive PDE, were presented.

The main focus of the field nowadays seems to be toward understanding large data dynamics for a variety of models, blow up phenomena, generic flow properties, as well as the small data evolution for classes of strongly nonlinear dispersive equations which were out of reach until not very long ago.

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Workshop: Nonlinear Waves and Dispersive Equations

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Abstracts

Low Regularity Solutions of the Cubic Szegő equation

Patrick Gérard
(joint work with Herbert Koch)

The cubic Szegő equation on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is

$$i\partial_t u = \Pi(|u|^2 u) \tag{1}$$

where $\Pi : L^2(\mathbb{T}) \to L^2_+(\mathbb{T})$ denotes the orthogonal projector onto the closed subspace $L^2_+(\mathbb{T})$ of $L^2(\mathbb{T})$ defined by the cancellation of all negative Fourier modes,

$$\forall k < 0 \ , \ \hat{u}(k) = 0 \ .$$

Equation (1) was introduced by S. Grellier and the first author in [5], where a flow on $H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T})$, $s \geq \frac{1}{2}$, was defined, and where a Lax pair structure was discovered. In [8], this equation was identified as the time averaged effective system to the half wave equation on $\mathbb{T}$. In [6], more precise integrability properties were established, while in [7] an explicit formula for H^s solutions was derived. Finally, a general nonlinear Fourier transform was constructed in [9], where almost periodicity of solutions in $H^{\frac{1}{2}}_+$ and growth of higher Sobolev norms were proved. The interest of the cubic Szegő therefore lies in its quality of displaying both integrability and instability features.

Since Π is a pseudodifferential operator of order 0, it is natural to ask about solving Equation (1) for initial data with low regularity. For instance, the ordinary differential equation

$$i\partial_t u = |u|^2 u \tag{2}$$

is wellposed on $L^\infty(\mathbb{T})$, with the explicit formula

$$u(t, x) = \mathrm{e}^{-it|u(0,x)|^2} u(0, x) \ .$$

The results presented in this talk try to investigate how this property is modified by the action of the pseudodifferential operator Π. It is well known that Π is not bounded on $L^\infty(\mathbb{T})$. The space

$$BMO_+(\mathbb{T}) = \{\Pi(b), b \in L^\infty(\mathbb{T})\}$$

was identified by Fefferman [3] as the intersection of $L^2_+(\mathbb{T})$ with the John–Nirenberg space $BMO(\mathbb{T})$, see [10], [4]. It is also the dual of

$$L^1_+(\mathbb{T}) = \{h \in L^1(\mathbb{T}) : \forall k < 0 \ , \ \hat{h}(k) = 0\} \ .$$

For every $u \in BMO_+(\mathbb{T})$, we set

$$\|u\|_{BMO} = \inf\{\|b\|_{L^\infty}, b \in L^\infty(\mathbb{T}), \Pi(b) = u\} = \|u\|_{(L^1_+)'} \ .$$

Our main result is the following.

Theorem 1. *For every $u_0 \in BMO_+(\mathbb{T})$, there exists a unique function $u \in C^1(\mathbb{R}, L^2_+(\mathbb{T})) \cap C_{w*}(\mathbb{R}, BMO_+(\mathbb{T}))$, solution of the initial value problem*

$$i\partial_t u = \Pi(|u|^2 u)\ ,\ u(0) = u_0\ . \tag{3}$$

Furthermore, $\|u(t)\|_{BMO} = \|u_0\|_{BMO}$. Moreover, if u, v are two BMO solutions of (1) satisfying

$$\|u(0)\|_{BMO} + \|v(0)\|_{BMO} \leq M\ ,$$

there exists a constant K, depending only on M, such that, for every $t \in \mathbb{R}$,

$$\|u(t) - v(t)\|_{L^2} \leq K\, \|u(0) - v(0)\|_{L^2}^{\alpha(t)}\ ,\ \alpha(t) := \mathrm{e}^{-K|t|}\ . \tag{4}$$

The proof of Theorem 1 combines the Lax pair property of the cubic Szegő equation [5] involving Hankel operators, a theorem by Nehari [11], and the John–Nirenberg inequality [10], showing that the L^p norm of a BMO function grows at most linearly with p as p tends to infinity.

As a consequence of the stability estimate (4), we obtain a partial result of propagation of Sobolev regularity.

Corollary 2. *If $u_0 \in BMO_+(\mathbb{T}) \cap H^s(\mathbb{T})$, then $Z(t)(u_0) \in H^s(\mathbb{T})$ for every $s \geq \frac{1}{2}$. In the case $0 < s < \frac{1}{2}$, there exists $K > 0$, depending only on a bound of $\|u_0\|_{BMO}$, such that*

$$\forall t \in \mathbb{R}, Z(t)u_0 \in H^{s(t)}(\mathbb{T})\ ,\ s(t) := \frac{s\mathrm{e}^{-K|t|}}{1 - 2s + 2s\mathrm{e}^{-K|t|}}\ .$$

Remark 3. We do not know whether the above exponent $s(t)$ is optimal or not. If it is optimal, such a loss of regularity could be compared to the one established by Bahouri and Chemin in Theorem 1.3 of [1] for the bidimensional Euler flow with bounded vorticity.

In the beginning of this talk, we have seen that the ordinary differential equation (2) is well posed on $L^\infty(\mathbb{T})$. In contrast, using the John–Nirenberg definition given in [10], it is easy to prove that this equation is not wellposed on $BMO(\mathbb{T})$. Indeed, though $u_0(x) = \log|\sin x|$ belongs to $BMO(\mathbb{T})$, one can check that, for every $t \neq 0$, the function

$$u(t, x) = (\log|\sin x|)\mathrm{e}^{-it(\log|\sin x|)^2}$$

does not belong to $BMO(\mathbb{T})$. Somewhat symmetrically, the next result shows that the Szegő equation is illposed on L^∞. We denote by $C_+(\mathbb{T}) = C(\mathbb{T}) \cap L^2_+(\mathbb{T})$ the Banach space of continuous functions on the circle with nonnegative Fourier modes.

Theorem 4. *There exists a dense G_δ subset $\mathcal{G}$ of $C_+(\mathbb{T})$ such that, for every $u_0 \in \mathcal{G}$, the solution u of (3) satisfies*

$$\forall T > 0, u \notin L^\infty([0, T] \times \mathbb{T})\ .$$

The arguments for Theorem 4 are an adaptation of a method developed by Elgindi and Masmoudi in [2], which leads to ill–posedness for the incompressible Euler equation at the C^1 regularity. The crucial step is the following lemma.

Lemma 5. *Let $u_0 \in C_+(\mathbb{T})$. There exists a sequence (u^n) of smooth solutions to the* (1) *such that*

$$\|u^n(0) - u_0\|_{L^\infty} \to 0 \ ,$$

and a sequence of times $T_n > 0$ tending to 0 *such that*

$$\sup_{t\in[0,T_n]} \|u^n(t)\|_{L^\infty} \to \infty \ .$$

Acknowledgements. We are grateful to Daniel Tataru for suggesting an improvement leading to Corollary 2.

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Expanding global solutions of 3-D compressible Euler equations

Mahir Hadžić
(joint work with Juhi Jang)

We consider the three-dimensional free boundary compressible Euler system

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0 \qquad \text{in } \Omega(t)\,; \tag{1a}$$

$$\rho\,(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p = 0 \qquad \text{in } \Omega(t)\,; \tag{1b}$$

$$p = 0 \qquad \text{on } \partial\Omega(t)\,; \tag{1c}$$

$$\mathcal{V}(\partial\Omega(t)) = \mathbf{u}\cdot\mathbf{n}(t) \qquad \text{on } \partial\Omega(t)\,; \tag{1d}$$

$$(\rho(0,\cdot),\mathbf{u}(0,\cdot)) = (\rho_0,\mathbf{u}_0)\,,\ \Omega(0) = \Omega_0\,. \tag{1e}$$

The unknowns in the problem are the fluid density ρ, the pressure p, the velocity vector $\mathbf{u}$, and the moving domain $\Omega(t)$ on which the fluid pressure is supported. Moreover we assume the polytropic equation of state

$$p = \rho^\gamma, \quad \gamma > 1, \tag{2}$$

where γ is called an adiabatic index and it plays an important role in our work as it allows us to introduce a natural criticality scale for problem (1).

In [4] Sideris discovered a finite-parameter family of global-in-time compactly supported solutions to (1) referred to as *affine motions*. To describe them one first finds the unique global solution to the matrix ordinary differential equation

$$\ddot{A}(t) = \det A(t)^{1-\gamma} A(t)^{-\top}, \tag{3}$$

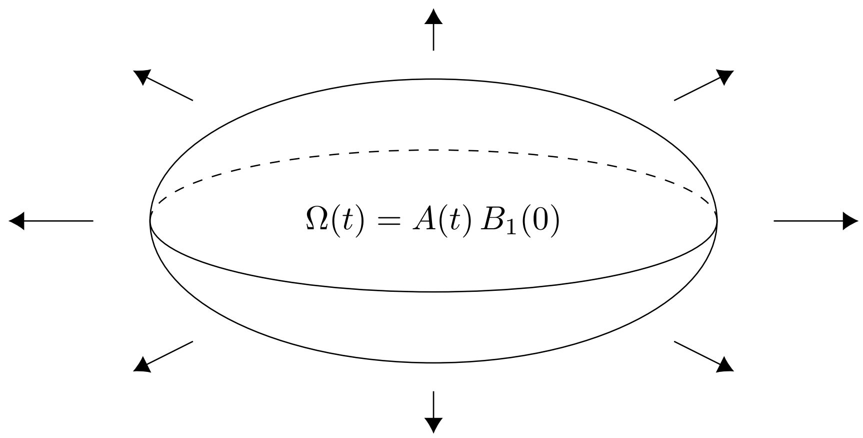
$$(A(0), \dot{A}(0)) = (A_0, A_1) \in \mathrm{GL}^+(3) \times \mathbb{M}^{3\times 3}, . \tag{4}$$

For any such a path $t \mapsto A(t) \in \mathrm{GL}^+(3)$ the fluid density and velocity solving (1) are given by

$$\rho_A(t,x) = \det A(t)^{-1} \left[\frac{(\gamma-1)}{2\gamma} (1 - |A^{-1}(t)x|^2) \right]^{\frac{1}{\gamma-1}}, \tag{5}$$

$$\mathbf{u}_A(t,x) = \dot{A}(t) A^{-1}(t) x, . \tag{6}$$

With a careful analysis of (3)–(4) it is possible to show that there exists an $A_\infty \in \mathrm{GL}^+(3)$ such that $\frac{A(t)}{t} \to_{t\to\infty} A_\infty$. In other words the support of the affine motions takes on the shape of an expanding ellipsoid:



Affine motions

Our main result is the construction of global-in-time solutions in the vicinity of affine motions:

Theorem 1 ([2], Main theorem-informal statement)**.** *Let $\gamma \in (1, \frac{5}{3}]$. Then the affine motions described above are nonlinearly stable. In other words, in a suitably rescaled set of coordinates small perturbations of affine motions lead to globally defined unique solutions of* (1)*. They remain close in a suitable high-order energy topology to the underlying moduli space of affine motions.*

Theorem 1 is the first global-in-time existence result to the physical vacuum free-boundary compressible Euler equations (1)–(2) without any symmetry assumptions on the initial data. Our approach relies of a re-interpretation of the affine motions as steady states of a suitably rescaled compressible Euler system. This quasi-conformal symmetry allows us to formulate the stability problem as a small data global existence question. The technical key to our strategy is the use of Lagrangian coordinates and the local existence theory developed in [1, 3].

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A non-linear adiabatic theorem for the Landau–Pekar equations

Rupert L. Frank

(joint work with Zhou Gang)

A polaron is a physical model for a particle accompanied by its polarization field. We treat a one-dimensional version of this model, where the particle is described by a complex-valued wave function $\psi \in L^2(\mathbb{R})$ and the (classical) field by a real-valued function $\phi \in L^2(\mathbb{R})$. The strength of the coupling between the particle and the field is described by a constant $\sqrt{\alpha} = \epsilon^{-1/4}$, which is assumed to be large.

Landau and Pekar [6] derived phenomenologically equations of motion, whose one-dimensional analogue reads

$$\begin{cases} i\epsilon\partial_t\psi &= (-\partial_x^2 + \phi)\psi\,, \\ -\partial_t^2\phi &= \phi + \frac{1}{2}|\psi|^2\,. \end{cases} \tag{1}$$

Note that the typical time scale of the electon is of order ϵ, whereas the time scale of the field is of order 1.

Our goal is to establish an adiabatic theorem, saying that if the initial condition $\psi|_{t=0}$ for the particle is the ground state of the Schrödinger operator $-\partial_x^2 + \phi|_{t=0}$ with potential given by the initial condition of the field, then up to times t of order one, $\psi(t)$ is (close to) the ground state of $-\partial_x^2 + \phi(t)$.

We work under the following

Assumption 1. *Let* $\phi_0, \dot{\phi}_0 \in \langle x\rangle^{-2}L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and assume that the operator* $-\partial_x^2 + \phi_0$ *in* $L^2(\mathbb{R})$ *has a single negative eigenvalue (denoted by* E_0*) and no zero energy resonance. Let* ψ_0 *a (not necessarily normalized) eigenfunction of* $-\partial_x^2 + \phi_0$ *corresponding to* E_0*.*

Lemma 2. *There are a (maximal) $T_* \in (0,\infty]$, time-dependent functions Q_*, V_* and a time-dependent number E_* such that for all $t \in (0, T_*)$*

$$-\partial_x^2 Q_* + V_* Q_* = E_* Q_* \,, \qquad -\partial_t^2 V_* = V_* + \frac{1}{2}|Q_*|^2 \,,$$

as well as $\|Q_*\|_{L^2} = \|\psi_0\|_{L^2}$ and, at $t=0$,

$$Q_*|_{t=0} = \psi_0 \,, \qquad V_*|_{t=0} = \phi_0 \,, \qquad \partial_t V_*|_{t=0} = \dot{\phi}_0 \,, \qquad E_*|_{t=0} = E_0 \,.$$

Let

$$T^* := \sup \big\{ T \in (0, T_*) : \ E_*(t) \text{ is the unique neg. ev. of } -\partial_x^2 + V_*(t) \text{ and there is no zero energy resonance } \forall t < T \big\}$$

The following is our main result.

Theorem 3. *For every $T < T^*$ there is an $\epsilon_T > 0$ such that for $0 < \epsilon \leq \epsilon_T$ the solution (ψ, ϕ) of (1) with initial conditions $\psi|_{t=0} = \psi_0$, $\phi|_{t=0} = \phi_0$, $\partial_t \phi_{t=0} = \dot{\phi}_0$ can be decomposed as*

$$\psi(t) = e^{i\epsilon^{-1}\int_0^t E(s)\, ds + i\gamma(t)} \left(Q(t) + R(t) \right) \,, \qquad \phi(t) = V(t) + W(t) \,,$$

where Q, V and E satisfy

$$-\partial_x^2 Q + VQ = EQ \,, \qquad -\partial_t^2 V = V + \frac{1}{2}|Q|^2 \,.$$

and where, for $t \in [0, T]$,

$$\|R\|_{L^2} \lesssim \epsilon \,, \qquad \|W\|_{L^1 \cap L^2} \lesssim \epsilon^2 \,, \qquad \left| \partial_t \|Q\|_{L^2}^2 \right| \lesssim \epsilon (1 + t/\epsilon)^{-3/2} \,, \qquad |\partial_t \gamma| \lesssim \epsilon \,.$$

Thus, the theorem says that up to the remainder terms $R(t)$ and $W(t)$, the function $\psi(t)$ is an eigenfunction of $-\partial_x^2 + \phi(t)$ for times of order 1. While this assertion is reminiscent of the (linear) adiabatic theorem in quantum mechanics, the mechanism of our proof is completely different. Namely, it is based on the *dispersive property* of the Schrödinger equation with a potential vanishing at infinity. This technique was used in a related context in [4] which, in turn, is inspired by works on asymptotic stability of ground states of the non-linear Schrödinger equation due to Soffer–Weinstein, Buslaev–Perelman and many others.

A key ingredient in our proof are adiabatic dispersive estimates for time-dependent Schrödinger operators. The non-resonance condition guarantees that the long time behavior of these dispersive estimates can be improved from $t^{-1/2}$ to $t^{-3/2}$, which is integrable at infinity.

Finally, we mention that it is an open problem to derive the Landau–Pekar equations from a microscopic model of a polaron (based on the Fröhlich Hamiltonian). For partial progress in this direction we refer to [3, 1, 5].

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On the Dirac–Klein–Gordon system

Sebastian Herr
(joint work with Ioan Bejenaru and Timothy Candy)

Consider the Dirac–Klein–Gordon system

$$\begin{aligned} -i\gamma^\mu\partial_\mu\psi + M\psi &= \phi\psi \\ \Box\phi + m^2\phi &= \psi^\dagger\gamma^0\psi \end{aligned} \tag{1}$$

for a spinor $\psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ and a scalar field $\phi : \mathbb{R}^{1+3} \to \mathbb{R}$. Here, $m, M \in \mathbb{R}$ are mass parameters, γ^μ are Dirac matrices, $\psi^\dagger$ denotes the conjugate transpose of ψ, and we use the summation convention. The Dirac–Klein–Gordon system arises as a model in classical field theory. Sufficiently nice solutions satisfy conservation of charge

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\psi(t,x)|^2 dx = 0,$$

and conservation of an energy which, however, ceases to be positive definite. In the mass-less case $m = M = 0$, solutions of the system are invariant under rescaling, which implies that

$$L^2(\mathbb{R}^3) \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$$

is the critical space for $(\psi, \phi, \partial_t\phi)$. In a joint work with Ioan Bejenaru [2] and a joint work with Timothy Candy [3] we pursued the goal of solving the initial value problem associated to (1) globally in time, for non-localized and rough initial data close to the critical regularity.

In [2] we covered the full sub-critical range in the non-resonant case, provided that the initial data are sufficiently small.

Theorem 1 (Bejenaru-H.)**.** *Suppose that $0 < m < 2M$ and $\epsilon > 0$. There exists $\delta > 0$, such that for initial data satisfying*

$$\|\psi(0)\|_{H^\epsilon} + \|\phi(0)\|_{H^{\frac{1}{2}+\epsilon}} + \|\partial_t\phi(0)\|_{H^{-\frac{1}{2}+\epsilon}} \leq \delta$$

the initial value problem associated to (1) *is globally well-posed and solutions scatter to free solutions.*

In [3] we consider small initial data in the critical Sobolev space with some additional angular regularity. In order to state the result, let $\langle\Omega\rangle^\sigma$ denote σ angular derivatives, defined via spherical Littlewood–Paley projections.

Theorem 2 (Candy-H.). *Suppose that either $0 < m \leq 2M$ and $\sigma > 0$, or $0 < 2M < m$ and $\sigma > 7/30$. There exists $\delta > 0$, such that for initial data satisfying*

$$\|\langle\Omega\rangle^\sigma \psi(0)\|_{L^2} + \|\langle\Omega\rangle^\sigma \phi(0)\|_{H^{\frac{1}{2}}} + \|\langle\Omega\rangle^\sigma \partial_t\phi(0)\|_{H^{-\frac{1}{2}}} \leq \delta$$

the initial value problem associated to (1) is globally well-posed and solutions scatter to free solutions.

Let us recall some selected previous results. In [4] special solutions have been constructed. In [1], smooth and localized initial data has been considered. Local well-posedness for initial data of sub-critical regularity has been established in [5]. Recently, small data global well-posedness and scattering for initial data in the critical Besov space with one additional angular derivative has been proved in the non-resonant case in [7].

For the two results discussed above there are two important non-linear structures of the system (1). The first one is the so-called null-structure, which, in the mass-less case $m = M = 0$, has been found previously in [5]. With the matrices

$$\Pi_\pm(\xi) = \frac{1}{2}\Big(I \pm \frac{1}{\langle\xi\rangle_M}(\xi_j\gamma^0\gamma^j + M\gamma^0)\Big)$$

one defines projections via $\widehat{\Pi_\pm\psi}(\xi) = \Pi_\pm(\xi)\widehat{\psi}(\xi)$. With $\psi_\pm = \Pi_\pm\psi$ we have $\psi = \psi_+ + \psi_-$ and (1) is equivalent to the first order system

$$\begin{aligned} -i\partial_t\psi_\pm \pm \langle\nabla\rangle_M\psi_\pm =& \Pi_\pm(\mathrm{Re}(\phi)\gamma^0\psi) \\ -i\partial_t\phi + \langle\nabla\rangle_M\phi =& \langle\nabla\rangle_m^{-1}(\psi^\dagger\gamma^0\psi). \end{aligned}$$

The key observation is

$$\big|\Pi_{\pm_1}(\xi)\gamma^0\Pi_{\pm_2}(\eta)\big| \lesssim \angle(\pm_1\xi, \pm_2\eta) + \frac{|\pm_1|\xi| \pm_2|\eta||}{\langle\xi\rangle\langle\eta\rangle},$$

which implies that parallel interactions are of lower order.

The second non-linear structure with an impact on the long-time behavior is the set of resonances. By duality, it is enough to provide estimates for the tri-linear expression

$$\int_{\mathbb{R}^{1+3}} \phi\psi_1^\dagger\gamma^0\psi_2 dxdt,$$

where ϕ and ψ_j are perturbations of $e^{-it\langle\nabla\rangle_m}\phi(0)$ and $e^{\mp_j it\langle\nabla\rangle_M}\psi_j(0)$, respectively. The temporal oscillations in the above integral depend on

$$M_{\pm_1,\pm_2}(\xi,\eta) = \big|\langle\xi-\eta\rangle_m \mp_1 \langle\xi\rangle_M \pm_2 \langle\eta\rangle_M\big|.$$

Based on the analysis of this function, we introduce the following terminology. The case $0 < m < 2M$ is called non-resonant, because there are the following two lower bounds

$$M_{\pm_1,\pm_2}(\xi,\eta) \gtrsim \min(\langle\xi\rangle,\langle\eta\rangle)\angle(\pm_1\xi,\pm_2\eta)^2,$$
$$M_{\pm_1,\pm_2}(\xi,\eta) \gtrsim \big(\min(\langle\xi\rangle,\langle\eta\rangle,\langle\xi-\eta\rangle)\big)^{-1}.$$

The case $0 < m = 2M$ is called weakly resonant, because the first bound still holds. In the remaining case $0 < 2M < m$ the function vanishes for some non-trivial interactions, hence it is called resonant.

Let us finally describe key ingredients of the proofs. First, in the non-resonant and sub-critical regime considered in Theorem 1, we use spectrally localized Strichartz norms and the $\dot{X}^{s,\frac{1}{2},\infty}$ Fourier restriction norm.

Second, in the (weakly) non-resonant and critical regime of Theorem 2 we use Fourier-localized V^2-norms and certain outer block norms.

Third, in the resonant and critical regime of Theorem 2, the key observation is that for very small modulation, the null-structure ceases to be effective, but resonant interactions are transversal. To exploit this fact, we develop robust versions of bi-linear Fourier (adjoint) restriction estimates, following Tao's approach [6]. In particular, for fixed $p > \frac{3}{2}$, we obtain the estimate

$$\|uv\|_{L^p(\mathbb{R}^{1+3})} \lesssim A\|u\|_{V^2_{\langle\nabla\rangle_{m_1}}}\|v\|_{V^2_{\langle\nabla\rangle_{m_2}}},$$

where $\|u\|_{V^2_{\langle\nabla\rangle_m}} = \|e^{-it\langle\nabla\rangle_m}u(t)\|_{V^2_t}$ for the quadratic variation norm $\|\cdot\|_{V^2}$, and provided that the spatial Fourier supports of the functions u and v are localized on the same dyadic scale and angularly or radially separated, with an explicit constant A depending on these scales. Compared with the available Strichartz estimates, it encodes stronger decay, and in conjunction with Strichartz estimates for solutions of the Klein–Gordon equation with additional angular regularity, this is enough to prove the required bounds for the tri-linear integral expression above.

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Energy-Critical Half-Wave Maps

Enno Lenzmann
(joint work with Armin Schikorra)

We consider the energy-critical half-wave maps equation which is given by

$$\partial_t u = u \wedge |\nabla| u \tag{1}$$

for a map $u : [0,T) \times \mathbb{R} \to \mathbb{S}^2$, where $\mathbb{S}^2$ denotes the standard unit two-sphere embedded in $\mathbb{R}^3$. Here $\wedge$ stands for the usual vector product in $\mathbb{R}^3$ and $|\nabla|$ denotes the fractional order derivative of order one defined by its multiplier $|\xi|$ in Fourier space. The evolution problem (1) is of Hamiltonian type with the conserved energy

$$E[u] = \frac{1}{2} \int_{\mathbb{R}} u \cdot |\nabla| u \, dx, \tag{2}$$

and hence the natural energy space is $\dot{H}^{\frac{1}{2}}(\mathbb{R}, \mathbb{S}^2) = \{u \in \dot{H}^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^3) : |u| = 1 \text{ a.e.}\}$. Note that the scaling $u(t,x) \mapsto u(\lambda t, \lambda x)$, which maps solutions of (1) into solutions, keeps the energy unchanged. With regard to physical motivation, we mention that the nonlinear equation (1) arises as a formal semi-classical and continuum limit from Haldane–Shastry quantum spin chains, which are exactly solvable systems with strong links to completely integrable systems (Calogero–Moser systems).

We present recent results obtained in [1]. That is, we give a complete classification of finite-energy traveling solitary waves $u(t,x) = Q_v(x - vt)$, where $v \in \mathbb{R}$ is a velocity parameter. Moreover, a complete spectral analysis of the linearized operator L around static solutions ($v = 0$) is carried out. In particular, we prove the nondegeneracy of L and determine all its resonances explicitly.

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On inelasticity of collisions for the energy critical wave equations in dimension five

Frank Merle
(joint work with Yvan Martel)

We prove that for parameters the collision of 2 solitons for the energy critical wave equation produce a channel of energy for large time (corresponding of radiative wave). This is done in dimension 5.

Hyperboloidal similarity coordinates and a globally stable blowup profile for supercritical wave maps

Roland Donninger
(joint work with Paweł Biernat and Birgit Schörkhuber)

Wave maps $U : \mathbb{R}^{1,3} \to \mathbb{S}^3$ are defined as critical points of the action principle

$$\int_{\mathbb{R}^{1,3}} \eta^{\mu\nu} \partial_\mu U^a \partial_\nu U^b g_{ab} \circ U.$$

By choosing standard spherical coordinates (t, r, θ, φ) on the Minkowski space $\mathbb{R}^{1,3}$ and hyperspherical coordinates (ψ, Θ, Φ) on the sphere $\mathbb{S}^3$, one may consider so-called co-rotational maps $U : \mathbb{R}^{1,3} \to \mathbb{S}^3$ which are of the form $U(t, r, \theta, \varphi) = (\psi(t, r), \theta, \varphi)$. Under this symmetry reduction the Euler-Lagrange equation associated to the wave maps action reduces to

$$\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\psi(t,r) + \frac{\sin(2\psi(t,r))}{r^2} = 0. \tag{1}$$

Eq. (1) is energy-supercritical and develops singularities in finite time. This is most easily evidenced by the explicit self-similar solution

$$\psi_T(t,r) = 2\arctan\left(\frac{r}{T-t}\right).$$

Here, $T > 0$ is a free parameter (the *blowup time*). The solution ψ_T is perfectly smooth for $t < T$ but develops a gradient blowup at the spacetime point $(t, r) = (T, 0)$. In [3, 4, 2, 1] it is proved that ψ_T is nonlinearly asymptotically stable in the backward lightcone of the blowup point $(t, r) = (T, 0)$ under small perturbations of the initial data. This leaves open two major questions which shall be addressed in the present work:

- How does the solution behave outside the backward lightcone?
- Is it possible to extend the solution beyond the blowup time?

Concerning the second question, we find a natural explicit extension of ψ_T given by

$$\psi_T^*(t,r) = 4\arctan\left(\frac{r}{T-t+\sqrt{(T-t)^2+r^2}}\right).$$

Interestingly, the boundary condition at the origin flips at blowup, i.e., $\psi_T^*(t, 0) = 0$ for $t < T$, whereas $\lim_{r\to 0+} \psi_T^*(t, r) = 2\pi$ if $t > T$. Asymptotically, as $t \to \infty$, the solution ψ_T^* settles down to the constant map 2π.

To answer the first question, we study the stability of ψ_T^* in a large portion of spacetime that reaches almost all the way up to the *Cauchy horizon* of the singularity, that is, the boundary of the future lightcone of the blowup point $(T, 0)$. The key ingredient for this stability analysis is the introduction of a novel coordinate system (s, η) which we call "hyperboloidal similarity coordinates". These coordinates are defined by the relation

$$t = T - e^{-s}h(\eta), \qquad r = e^{-s}\eta, \qquad h(\eta) := \sqrt{2+\eta^2} - 2$$

and depicted in Fig. 1. The coordinates (s, η) are "hyperboloidal" in the sense of [5, 6] but at the same time compatible with self-similarity, i.e., the ratio $\frac{r}{T-t} = -\frac{\eta}{h(\eta)}$ is independent of s. In particular, the blowup solution ψ_T^* is *static* in the new coordinates.

By employing semigroup methods, nonself-adjoint spectral theory, and ideas from infinite-dimensional dynamical systems, we develop a suitable perturbation theory that allows us to control the wave maps flow near the blowup solution ψ_T^* in the coordinates (s, η). This way, we obtain the stability of ψ_T^* everywhere outside the *future* lightcone of the singularity.

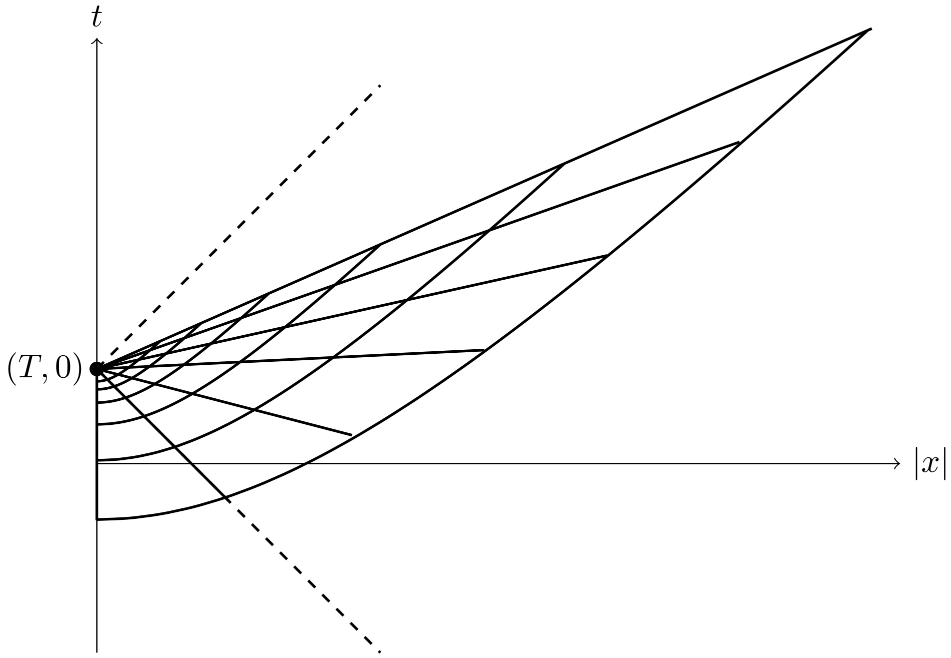

FIGURE 1. The hyperboloidal similarity coordinates (with $r = |x|$). The hyperboloids are the lines $s = \text{const}$ and the straight lines emerging radially from the blowup point $(T, 0)$ correspond to $\eta = \text{const}$. The dashed lines are the boundaries of the forward and backward lightcones of the singularity.

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The threshold theorem for the hyperbolic Yang–Mills equation

Sung-Jin Oh
(joint work with Daniel Tataru)

In this extended abstract, we report the proof of the author and D. Tataru of the threshold theorem for the $(4+1)$-dimensional Yang–Mills equation [8].

Statement of the main result. Let $\mathbf{G}$ be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$. Consider a connection D on a vector bundle η on $\mathbb{R}^{1+d}$ with structure group $\mathbf{G}$. We say that D is a *hyperbolic Yang–Mills connection* (or a *Yang–Mills wave*) if it is a critical point of the action[1]

$$\mathcal{S}[A] = \frac{1}{4}\int \langle F_{\mu\nu}, F^{\mu\nu}\rangle \, \mathrm{d}t\mathrm{d}x,$$

where $F_{\mu\nu}$ is the curvature form associated to D (which is a $\mathfrak{g}$-valued 2-form) and $\langle\cdot,\cdot\rangle$ is minus the Killing form on $\mathfrak{g}$ (which is positive-definite since $\mathbf{G}$ is compact semilinear). The corresponding Euler–Lagrange equation, called the *Yang–Mills equation*, is

$$D^{\mu} F[A]_{\nu\mu} = 0. \tag{YM}$$

More concretely, in a global trivialization of η, which exists since $\mathbb{R}^{1+d}$ is contractible, D takes the form $D = \mathrm{d} + A$, where d is the usual differential and A is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}^{1+d}$. The curvature form $F = F[A]$ is given by

$$F[A]_{\mu\nu} = (\mathrm{d}A + \tfrac{1}{2}[A\wedge A])_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu}, A_{\nu}].$$

The connection acts on F by the adjoint action, i.e., $D_{\mu}F_{\alpha\beta} = \partial_{\mu}F_{\alpha\beta} + [A_{\mu}, F_{\alpha\beta}]$. Thus (YM) may be interpreted as a 2nd-order PDE for A.

A fundamental tool for understanding long time dynamics of large data solutions is the *conserved energy* (which is constant in time for regular solutions)

$$\mathcal{E}_{\{t\}\times\mathbb{R}^d}[A] = \int_{\{t\}\times\mathbb{R}^d} \frac{1}{2}\sum_{\mu<\nu} \langle F[A]_{\mu\nu}, F[A]_{\mu\nu}\rangle \, \mathrm{d}x.$$

Its effectiveness at various scales is determined by its behavior with respect to the invariant scaling of (YM), namely $A \mapsto \lambda^{-1}A(\lambda^{-1}t, \lambda^{-1}x)$ for any $\lambda > 0$. We

[1] We identify the coordinate x^0 on $\mathbb{R}^{1+d}$ with time t. We adopt the standard conventions of using the Minkowski metric $\mathbf{m} = \mathrm{diag}(-1, +1, \cdots, +1)$ to raise and lower indices, and summing up repeated upper and lower indices. Greek alphabets stand for the space-time indices $0, 1, \dots, d$, and roman alphabets are for space indices $1, \dots, d$.

concentrate on the *energy critical* case $d = 4$, when the conserved energy does not change under this scaling. In the simplest form, our main theorem can be stated as follows.

Theorem 1 (Threshold theorem for the hyperbolic Yang–Mills equation [8])**.** *Let $\mathcal{E}_0$ be the ground state energy, i.e., the energy of the lowest-energy nontrivial harmonic Yang–Mills connection (i.e., critical point of the Dirichlet energy $\mathcal{D}[A] = \frac{1}{4}\int \langle F_{ij}, F^{ij}\rangle \,\mathrm{d}x$ on $\mathbb{R}^4$). The initial value problem for* (YM) *on $\mathbb{R}^{1+4}$ is "globally well-posed", and the solutions "scatter", for any initial data with energy strictly less than $\mathcal{E}_0$.*

The terms "globally well-posed" and "scatter" must be interpreted carefully, since the initial value problem for (YM) alone is *not* formally well-posed due to *gauge invariance*, i.e., if A is a solution to (YM) then so is $\tilde{A} = OAO^{-1} - \mathrm{d}OO^{-1}$ for any sufficiently regular $\mathbf{G}$-valued function O. In order to both properly state and prove this result, it is essential to fix the gauge choice in a favorable way. Addressing this issue turns out to be a significant part of our work, as we will discuss below.

The relevance of the harmonic Yang–Mills connections is that they are precisely the static solutions to (YM). The ground state is known to exist (thus $0 < \mathcal{E}_0 < \infty$) for any compact semi-simple Lie group $\mathbf{G}$. We note that Theorem 1 is sharp, as finite time blow-up solutions with energies exceeding but arbitrarily close to $\mathcal{E}_0$ were constructed [1, 9].

Some ideas of the proof. Our overall strategy of the proof of Theorem 1 is analogous to that in our previous work [5, 6, 7] on the closely-related Maxwell–Klein–Gordon system, which was in turn based on the strategy of Sterbenz–Tataru [10, 11] for wave maps. Due to space constraint, here we will content ourselves with discussion of one major aspect of our work which differs from [5, 6, 7], namely the use of caloric gauge instead of any other classical gauge choices.

We begin by motivating the need of a new gauge choice. For the proof of Theorem 1, some major requirements on the gauge choice are:

(1) (YM) is locally well-posed for regular data;
(2) the nonlinearity of (YM) exhibits nice cancellation (i.e., null structure);
(3) the gauge choice can be imposed on any data (in Theorem 1).

A classical gauge choice with nice structural properties (i.e., Properties (1) and (2) above) is the *Coulomb gauge*, defined by the condition $\partial^\ell A_\ell = 0$. Indeed, it was successfully used in the proof of global well-posedness and scattering for small energy [2]. However, great difficulties arise when one attempts to impose the Coulomb gauge for a large energy data (i.e., Property (3)). For instance, in order to solve (YM) in the Coulomb gauge we must invert the operator $\partial^\ell D_\ell$, but its spectral properties are unclear in the large energy case.

Instead, we use the so-called *caloric gauge*, which is constructed relying on regularity theory for another nonlinear PDE, namely the Yang–Mills heat flow.

We say that a connection A on $\mathbb{R}^4_x \times [0,\infty)_s$ is a *Yang–Mills heat flow*[2] if

$$F_{sj} = D^\ell F_{\ell j}. \qquad \text{(YMHF)}$$

A connection A on $\mathbb{R}^4$ is said to be *caloric* (or in the caloric gauge) if the Yang–Mills heat flow $A_{x,s}(s)$ with $A_x(s=0)=A$ exists and obeys $A_s(s)=0$ for all $s>0$ and $A_x(s=\infty)=0$.

An important property of a caloric connection A is that it obeys a *generalized Coulomb condition*:

$$\partial^j A_j = O(A,A), \qquad \text{(gCoulomb)}$$

where O is a bilinear operator with a 0th-order symbol. Essentially as a result, (YM) in the caloric gauge exhibits nice structural properties as the Coulomb gauge. In other words, the caloric gauge has Properties (1) and (2).

The caloric gauge can be imposed provided that the Yang–Mills heat flow $A_{x,s}(s)$ behaves regularly. We prove the following theorem, which ensures that this is the case, i.e., the caloric gauge has Property (3).

Theorem 2 (Threshold theorem for the Yang–Mills heat flow [8])**.** *For every subthreshold connection A on $\mathbb{R}^4$, there exists a global in parabolic-time Yang–Mills heat flow with $A_x(s=0)=A$. Furthermore, this solution has the property that the limit $A(s=\infty)$ exists in $\dot{H}^1$, and it is flat.*

Among the conclusions, we note that global regularity of the Yang–Mills heat flow for subthreshold data and weak convergence to a harmonic Yang–Mills connection as $s\to\infty$ essentially follow from the classical work of Struwe [12].

The idea of using the associated heat flow to define a high-quality gauge originates from Tao [13], in which the harmonic map heat flow on $\mathbb{R}^2$ was used to define a gauge for analyzing the wave map flow in $\mathbb{R}^{1+2}$. It was extended to the Yang–Mills setting at subcritical regularity by the author [3, 4].

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Invariance of the white noise for the cubic fourth order nonlinear Schrödinger equation on the circle

Yuzhao Wang

We consider the cubic fourth order nonlinear Schrödinger equation (4NLS) on the circle. In particular, we study the Cauchy problem for the renormalized 4NLS with the (spatial) Gaussian white noise as initial data. Due to the roughness of the white noise, the deterministic well-posedness theory is out of reach at this point. In order to overcome this difficulty, we introduce a random-resonant/nonlinear decomposition and prove (i) almost sure global well-posedness of the renormalized 4NLS with respect to the white noise as initial data and (ii) invariance of the white noise.

This is a joint work with Tadahiro Oh (University of Edinburgh) and Nikolay Tzvetkov (Université de Cergy-Pontoise).

1. Introduction

We consider the Cauchy problem for the cubic fourth order nonlinear Schrödinger equation (4NLS) on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$i\partial_t u = \partial_x^4 u + |u|^2 u, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{1}$$

where u is a complex-valued function. Our main goal is to study the well/ill-posedness issue of (1) in the low regularity setting from a probabilistic point of view. In particular, we study (1) with the (spatial) Gaussian white noise as initial data.

1.1. **Invariant measures for Hamiltonian PDEs.** Given a Hamiltonian dynamics on $\mathbb{R}^{2n}$: $\dot{p}_i = \frac{\partial H}{\partial q_i}$, $\dot{q}_i = -\frac{\partial H}{\partial p_i}$, $i = 1, \dots, n$, with Hamiltonian $H(\mathbf{p}, \mathbf{q}) = H(p_1, \cdots, p_n, q_1, \cdots, q_n)$, Liouville's theorem states that the Lebesgue measure on $\mathbb{R}^{2n}$ is invariant under the flow. Then, along with the conservation of the Hamiltonian H, it follows that the *Gibbs measure*: $d\mu = \exp(-H(\mathbf{p}, \mathbf{q}))\, d\mathbf{p}\, d\mathbf{q}$ is invariant. Moreover, if there exists another "nice" conserved quantity function $F(p, q)$ under the flow, then the measure μ_F defined by $d\mu_F = \exp(-F(\mathbf{p}, \mathbf{q}))\, d\mathbf{p}\, d\mathbf{q}$ is also invariant.

By drawing an analogy to the finite dimensional setting, one can study the transport property of the Gibbs measure of the form

$$\text{“}d\mu = Z^{-1}e^{-H(u)}du\text{”}$$

under the dynamics of a nonlinear Hamiltonian PDE such as NLS. In fact, Bourgain [1, 2] showed that the Gibbs measures are invariant under the dynamics of many nonlinear Hamiltonian PDEs on $\mathbb{T}^d$. Moreover, the L^2-norm of (smooth) solutions to some nonlinear Hamiltonian PDEs such as NLS and KdV is conserved. This suggests that the white noise, formally written as

$$d\mu_0 = Z^{-1}\exp\big(-\tfrac{1}{2}\int |u|^2 dx\big)du,$$

may be invariant under the dynamics. Note that a typical element ϕ under the white noise μ_0 is given by the following random Fourier series:

$$\phi^\omega(x) = \phi(x;\omega) = \sum_{n\in\mathbb{Z}^d} g_n(\omega)e^{2\pi i n\cdot x},$$

where $\{g_n\}_{n\in\mathbb{Z}^d}$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$. With this Fourier series representation, it is easy to see that the white noise μ_0 on $\mathbb{T}^d$ is supported on $H^{-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)\setminus H^{-\frac{d}{2}}(\mathbb{T}^d)$ for any $\varepsilon > 0$. The main difficulty in understanding the transport property of the white noise, even when $d = 1$, is due to its rough regularity. Nonetheless, it was shown that the white noise is indeed invariant under the flow of KdV on $\mathbb{T}$ [14, 3, 4, 5, 6]. Moreover, Oh-Quastel-Valkó [7] showed that the white noise is a weak limit of invariant measures for the cubic NLS on $\mathbb{T}$. While this result implies formal invariance of the white noise μ_0 for the (renormalized) cubic NLS on $\mathbb{T}$, it does *not* yield rigorous invariance due to the lack of well-defined flow in its support. Recalling that the L^2-norm is conserved for (1), we study the dynamics of (1) with the white noise μ_0 as initial data.

1.2. **Deterministic well-posedness.** In [9], Oh-Tzvetkov proved that (1) is globally well-posed in $L^2(\mathbb{T})$, while it is mildly ill-posed in negative Sobolev spaces in the sense that the solution map is not locally uniformly continuous. In a recent paper [11], we showed that (1) is in fact ill-posed in negative Sobolev spaces in a very strong sense (non-existence of weak solutions). In particular, one needs to renormalize the equation in order to study the dynamics with the white noise as initial data. In the following, we propose to study the following renormalized version of (1):[1]

$$i\partial_t u = \partial_x^4 u + \big(|u|^2 - 2\textstyle\int |u|^2 dx\big)u. \tag{2}$$

Note that, while (1) and (2) are equivalent in $L^2(\mathbb{T})$, they are not equivalent in negative Sobolev spaces. In the following, we *choose* to study (2).

In [11], we studied well-posedness of (2) in negative Sobolev spaces via the short-time Fourier restriction norm method and normal form method (with T. Oh).

[1]This renormalized equation (2) appears as an equivalent formulation of the Wick renormalization in Euclidean quantum field theory [8, 10]. For this reason, we will simply refer to (2) as the Wick ordered cubic 4NLS in the following.

Theorem 1. *The Wick ordered cubic 4NLS* (2) *is globally well-posed in* $H^s(\mathbb{T})$ $s > -\frac{1}{3}$.

2. Main result and a sketch of the proof

We now state our main result in [12] (with T. Oh and N. Tzvetkov).

Theorem 2. *The Wick ordered cubic 4NLS* (2) *is almost surely globally well-posed with respect to the white noise* μ_0. *Moreover, the white noise* μ_0 *is invariant under* (2).

The main difficulty in proving Theorem 2 is to construct almost sure local-in-time dynamics with respect to the white noise due to its low regularity $s < -\frac{1}{2}$, which goes beyond the regularity threshold in Theorem 1. Once we have almost sure local-in-time dynamics, we can apply Bourgain's invariant measure argument [1] to extend these local-in-time solutions globally in time and prove invariance of the white noise.

In the following, we briefly describe the idea for the almost sure local well-posedness argument. Given $\alpha \geq 0$, let us consider the random initial data ϕ_α^ω of the form

$$\phi_\alpha^\omega(x) = \phi(x;\omega) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle}^\alpha e^{2\pi i n x}.$$

A naive approach in establishing almost sure local well-posedness would be to write the solution u as $u = z_1 + v_1 := S(t)\phi_\alpha^\omega + v_1$ and try to solve the fixed point problem for v_1. Thanks to the gain of integrability[2] on the random linear solution z_1^ω, one can show that the residual part v_1 is smoother and lies in $L^2(\mathbb{T})$ for $\alpha > \frac{1}{6}$. In order to consider the white noise ($\alpha = 0$), we need to further elaborate this argument. By expanding the solution $u = z_1 + z_3 + \cdots + z_{2j-1} + v_j$ further in terms of the random linear solution z_1, we can actually prove almost sure local well-posedness of (2) with ϕ_α^ω as initial data for $\alpha > 0$ via a simple fixed point argument. In order to handle the white noise ($\alpha = 0$), however, we need to iterate this process indefinitely. By decomposing the dynamics into the *resonant* and *non-resonant* contributions, we observed that the bad behavior comes only from the resonant part. We therefore expanded the solution u in a power series only for the contributions coming from the resonant part while we hide the other part (i.e the non-resonant part) in the residual term: $u = \sum_{j=1}^\infty z_{\mathrm{res},2j-1} + v_\infty$, where $z_{\mathrm{res},2j-1}$ corresponds to the contribution of order $2j-1$ from the resonant part of the dynamics. One important observation is that $z_\infty^\omega = \sum_{j=1}^\infty z_{\mathrm{res},2j-1}$ satisfies the following resonant 4NLS:

$$i\partial_t z_\infty^\omega = \partial_x^4 z_\infty^\omega + \mathcal{R}(z_\infty^\omega)$$

with $z^\omega|_{t=0} = \phi^\omega$, where $\mathcal{R}(z^\omega)$ denotes the resonant part of the nonlinearity in (2). This motivated us to introduce the *random-resonant/nonlinear decomposition*:

[2]In the spirit of the classical work by Paley-Zygmund [13].

$u = z^\omega_\infty + v_\infty$ where v_∞ satisfies

$$i\partial_t v_\infty = \partial_x^4 v_\infty + \big[\mathcal{N}(v_\infty + z^\omega_\infty) - \mathcal{R}(z^\omega_\infty)\big]$$

with $v_\infty|_{t=0} = 0$. This decomposition together with a second iteration argument allowed us to establish a *probabilistic* a priori estimate as in [6] and thus construct almost sure local-in-time dynamics. We point out that z^ω does *not* belong to the Wiener homogeneous chaos of any finite order, i.e. it depends on arbitrarily higher powers of products of Gaussian random variables g_n and thus we needed to introduce a new stochastic estimate in handling z^ω_∞.

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Decay of nonlinear waves in one dimension and nonexistence of breathers

Claudio Muñoz
(joint work with Michal Kowalczyk and Yvan Martel)

Let us consider three different scalar field models coming from Physics, in 1+1 dimensions: the cubic nonlinear Klein-Gordon (NLKG)

$$\partial_t^2\phi - \partial_x^2\phi + \phi - \phi^3 = 0, \tag{1}$$

the ϕ^4 model (ϕ^4),

$$\partial_t^2\phi - \partial_x^2\phi - \phi + \phi^3 = 0, \tag{2}$$

and the Sine-Gordon (SG) equation:

$$\partial_t^2\phi - \partial_x^2\phi + \sin\phi = 0. \tag{3}$$

Each model above has $(\phi, \partial_t\phi) = (\phi, \partial_t\phi)(t, x) \in \mathbb{R}^2$, and $(t, x) \in \mathbb{R}^2$. Our question here is very simple: consider only "small solutions" in a certain sense in (1)-(3), can we show that there is always decay as in the linear case?

This question is far from being trivial, for many reasons. First, the linear decay in one dimension is the lowest possible ($1/\sqrt{t}$), and the nonlinearities are too weak to preserve standard scattering.

On the other hand, the answer to this question is depends on each equation above. For instance, (1) has the soliton solution $(Q, 0)$, where

$$Q(x) := \sqrt{2}\,\mathrm{sech}\, x.$$

On the other hand, (ϕ^4) has the kink solution $(H, 0)$, where

$$H(x) := \tanh\Big(\frac{x}{\sqrt{2}}\Big),$$

and finally, for any $\alpha, \beta > 0$ such that $\alpha^2 + \beta^2 = 1$, (SG) has the breather solution $(B, \partial_t B)$, where

$$B(t, x) := 4\arctan\Big(\frac{\beta}{\alpha}\frac{\sin(\alpha t)}{\cosh(\beta x)}\Big).$$

All these three solutions do not decay in time, and they are even, odd and even, respectively. Therefore, special attention must be put to these particular cases when considering the decay of solutions. In [2], we showed that both (1) and (3) satisfy the following simple decay result:

Assume that $(\phi, \partial_t\phi)(t = 0)$ *are small enough in the energy space* $(H^1 \times L^2)(\mathbb{R})$*, and they are odd. Then the corresponding global solution of* (1) *and* (3) *decays to zero locally on any spatial compact set* I*:*

$$\lim_{t\to\pm\infty} \|(\phi, \partial_t\phi)(t)\|_{(H^1\times L^2)(I)} = 0.$$

This result in particular implies that no small odd "breather" solution for (1) and (3) should persist in time. By breather, we mean a time-periodic, localized in space solution.

As for the (ϕ^4) model, things are more complicated. In [1], we showed the following *asymptotic stability* result:

Assume that $(\phi - H, \partial_t \phi)(t = 0)$ *are small enough in the energy space* $(H^1 \times L^2)(\mathbb{R})$*, and they are odd. Then the corresponding global solution of* (2) *converges to the kink locally on any spatial compact set* I*:*

$$\lim_{t \to \pm\infty} \|(\phi, \partial_t \phi)(t) - (H, 0)\|_{(H^1 \times L^2)(I)} = 0.$$

These two results are proved using new Virial identities in the spirit of previous works by Martel, Merle and Raphaël [3, 4]. Each virial identity must be adapted to the involved dynamics. Please see [1, 2] for more details on each particular proof and a full set of references to other works.

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Scattering for the 3D Gross-Pitaevskii equation

Zihua Guo

(joint work with Zaher Hani, Kenji Nakanishi)

Consider the Gross-Pitaevskii (GP) equation

$$i\psi_t + \Delta\psi = (|\psi|^2 - 1)\psi, \quad \psi : \mathbb{R}^{1+3} \to \mathbb{C} \tag{1}$$

with the boundary condition

$$\lim_{|x|\to\infty} \psi = 1. \tag{2}$$

Although the GP equation (1) is formally equivalent to the cubic Schrödinger equation, indeed, let $\phi = e^{-it}\psi$, then ϕ solves

$$i\phi_t + \Delta\phi = |\phi|^2\phi, \tag{3}$$

the non-vanishing boundary conditions bring remarkable effects on the space-time behaviour of the solutions. The nonzero boundary condition (2) or more generally $\lim_{|x|\to\infty} |\psi| = 1$, arises naturally in physical contexts such as Bose-Einstein condensates, superfluids and nonlinear optics, or in the hydrodynamic interpretation of NLS (see [2]).

The GP equation (1) has rich structures. Let $u = \psi - 1$ be the perturbation from the equilibrium. Then u satisfies

$$i\partial_t u + \Delta u - 2\Re u = u^2 + 2|u|^2 + |u|^2 u, \quad u|_{t=0} = u_0. \tag{4}$$

We have conservation of the energy: if u is a smooth solution to (4) then

$$E(u) := \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{(|u|^2 + 2\Re u)^2}{2} dx = E(u_0).$$

An unconditional global well-posedness for (4) in the energy space $\mathbb{E}$ was proved by Gérard [3], where

$$\mathbb{E} := \{f \in \dot{H}^1(\mathbb{R}^3) : 2\Re f + |f|^2 \in L^2(\mathbb{R}^3)\}$$

with the distance $d_{\mathbb{E}}(f,g)$ defined by

$$d_{\mathbb{E}}(f,g)^2 = \|\nabla(f-g)\|_{L^2}^2 + \frac{1}{2}\||f|^2 + 2\Re f - |g|^2 - 2\Re g\|_{L^2}^2.$$

Global well-posedness for (4) in a smaller space H^1 was previously proved in [1] where a-priori L^2-bound was derived by the Gronwall inequality (note we do not have L^2 conservation law). Both results were obtained by iterating the local well-posedness which was proved via Strichartz estimates and thus do not give good information on the asymptotic behavior.

For the asymptotic behavior, Gustafson-Nakanishi-Tsai proved scattering for suitable small solutions (see [7, 8, 9]). In these works they proved that under some decay and regularity conditions (weighted Sobolev space) on the initial data, the small solutions scatter to the solution of the linearized equation in dimension three and higher, and the weighted space rather than the energy space was essentially needed in 3D due to the quadratic nonlinearity.

In our recent work, we succeeded in proving scattering for small data in the natural energy space $\mathbb{E}$, with radial symmetry or with one-order additional angular regularity. This opens the possibility to study the large data problem at least in the radial case for which we believe smallness is not needed. There are two main ingredients. One is the generalized Strichartz estimates for the GP equation. This is inspired by our recent works on the Zakharov system ([5, 6, 4]). These generalized Strichartz estimates are very useful to deal with the 3D quadratic terms. The other is some nonlinear transform through which we can achieve some "null" structure. These structures are crucial to handle the difficulty caused by the weak control on the low-frequency component of the energy space data due to the lack of L^2 component.

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Degenerate dispersive equations and stability of compactons

Benjamin Harrop-Griffiths
(joint work with Pierre Germain, Jeremy L. Marzuola)

We recall that the (focusing) generalized Korteweg-de Vries equation and nonlinear Schödinger equation on $\mathbb{R}$ arise as the Hamiltonian flow associated to the Hamiltonian,

$$E[u] = \frac{1}{2}\int |\partial_x u|^2\,dx - \frac{1}{p+1}\int |u|^{p+1}\,dx,$$

with symplectic forms,

$$\omega_{\mathrm{gKdV}}(u,v) = \int u\partial_x^{-1}v\,dx, \qquad \omega_{\mathrm{NLS}}(u,v) = \mathrm{Im}\int u\bar{v}\,dx,$$

defined for real-valued, respectively complex-valued functions.

In this talk we introduce natural quasilinear variants of these equations given by the Hamiltonian flow associated to the Hamiltonian,

$$H[u] = \frac{1}{2}\int |u\partial_x u|^2\,dx - \frac{1}{p}\int |u|^p\,dx,$$

with the same symplectic forms. The corresponding equations are the quasilinear KdV equation,

$$u_t + (u(uu_x)_x + u^p)_x = 0, \tag{1}$$

and the quasilinear Schrödinger equation,

$$iu_t = \bar{u}(uu_x)_x + |u|^{p-1}u. \tag{2}$$

The equations (1) and (2) are amongst the simplest examples of quasilinear dispersive equations that exhibit *degenerate dispersion*: the dispersive effects may degenerate at a point in space (i.e. where $u = 0$). Degenerate dispersive phenomena appear in numerous physical situations, for example [1, 2, 3, 7, 8, 9, 11], yet the understanding of these models is still relatively primitive. We note that quasilinear KdV models similar to (1) originally appeared in [10] and the Hamiltonian model (1) later appeared in [6]. The Schrödinger model (2) appeared in [5] as a continuous approximation of the toy model for weak turbulence appearing in [4].

A natural starting point in the analysis of (1), (2) is to understand the traveling wave solutions. As we are primarily concerned with the case that solutions degenerate in space we restrict our attention to compactly supported traveling waves or *compactons*. For the KdV model (1) traveling wave solutions are given by the ansatz,

$$u(t,x) = \phi(x - ct).$$

By analyzing the corresponding ODE satisfied by ϕ we may find a 2-parameter family of compactons $\phi = \Phi_{B,c}$ where either $B > 0$, $c \in \mathbb{R}$ or $B = 0$, $c > 0$ that are even, compactly supported on an interval $I = (-x_{B,c}, x_{B,c})$ and decreasing on $(0, x_{B,c})$. For the Schrödinger model (2) traveling waves are given by the ansatz,

$$u(t,x) = \phi(x - vt)e^{-ict},$$

and we show that the compactons are given by a nonlinear Galilean shift of the KdV compactons,

$$\phi(x) = \Phi_{B,c}(x)e^{iv\theta_{B,c}(x)}, \qquad \theta'_{B,c} = -\frac{1}{2\Phi_{B,c}^2}.$$

Having discussed the existence and properties of traveling wave solutions we consider the problem of orbital stability. Here we seek solutions to the minimization problem,

$$\min H[u] \quad \text{subject to} \quad \|u\|_{L^2}^2 = M_0,$$

where the mass $M_0 > 0$ is a fixed constant. Traveling waves obtained through this minimization procedure are orbitally stable provided one can make sense of the flow around them. Using a concentration compactness argument we prove the following:

Theorem. For $2 < p < 8$, the above minimization problem admits a minimizer, which is (up to translation) one of the compactons $\Phi_{B,c}$. For $p = 4$, the minimizer is $\Phi_{0,c}$ with $c = \frac{M_0}{\sqrt{2}\pi}$.

In the case $p = 4$ we may find an explicit expression for the compactons,

$$\Phi_{B,c}(x) = \sqrt{c + \sqrt{4B + c^2}\cos(\sqrt{2}x)}, \qquad x \in (-x_{B,c}, x_{B,c}),$$

where $x_{B,c}$ is the smallest positive solution to $\cos(\sqrt{2}x) = -\frac{c}{\sqrt{4B+c^2}}$. In this case we present some numerical evidence that the $B = 0$ compacton is indeed stable under small perturbations.

We conclude our talk with a discussion of a linear equation arising from the linearization of the KdV model (1) about a fixed compacton ϕ in the case $p = 4$,

$$u_t = \partial_x \mathcal{L}_\phi u + f, \tag{3}$$

where $\mathcal{L}_\phi = -\phi(\partial_x^2 + 2)\phi$ is a degenerate Sturm-Liouville operator. A careful analysis of the operator $\mathcal{L}_\phi$ allows us to define associated energy spaces and obtain local well-posedness for the linear equation (3). This analysis will provide the basis for forthcoming work on the nonlinear problem.

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The wave equation on a model convex domain revisited

Fabrice Planchon
(joint work with Oana Ivanovici and Gilles Lebeau)

We improve on our recent work [5] by providing a different parametrix construction for the wave equation inside a 2D convex model domain Ω with Dirichlet boundary condition:

$$\begin{cases} (\partial_t^2 - \Delta_\Omega)u(t,x,y) = 0, & x \in \Omega \\ u|_{t=0} = u_0 \ \partial_t u|_{t=0} = u_1, & \\ u(0,y) = 0, \quad y \in \mathbb{R}. & \end{cases} \tag{1}$$

Here, $\Delta_\Omega = \partial_x^2 + (1+x)\partial_y^2$ and $\Omega = \{(x,y),\, x > 0\}$. At first order, this may be seen as the interior of the unit disk, with angle and distance to the boundary as coordinates.

Strichartz estimates are space-time estimates that uantify dispersive properties of the solutions to the linear wave equation: for given data in the natural energy space, the solution will have better decay for suitable time averages. On any riemanian manifold with empty boundary, the solution to (1) is such that, at least for a suitable $T < +\infty$,

$$h^\beta \|\chi(hD_t)u\|_{L^q([0,T],L^r)} \le C\Big(\|u(0,x)\|_{L^2} + \|hD_t u\|_{L^2}\Big), \tag{2}$$

where $\chi \in C_0^\infty$ is a smooth truncation in a neighborhood of 1. Let d be the spatial dimension of a general manifold Ω, one has $\beta = d\Big(\frac{1}{2} - \frac{1}{r}\Big) - \frac{1}{q}$, where (q, r) is a so-called admissible pair, e.g.

$$\frac{1}{q} \leq \frac{(d-1)}{2}(\frac{1}{2} - \frac{1}{r}), \quad q > 2. \tag{3}$$

On such boundary less Rimenanian manifold (Ω, g) this follows from proving sharp dispersion by non degenerate stationary phase on a Lax type parametrix (which may be constructed locally within a small ball, thanks to finite speed of propagation.)

On a manifold with boundary, picturing light rays becomes much more complicated, and one may no longer think that one is slightly bending flat trajectories. There may be gliding rays (along a convex boundary) or grazing rays (tangential to a convex obstacle) or combinations of both. Strichartz estimates outside a strictly convex obstacles were obtained in [6] and turned out to be similar to the free case (see [4] for the more complicated case of the dispersion). Strichartz estimates with losses were obtained later on general domains,[1], using short time parametrices constructions from [7], which in turn were inspired by works on low regularity metrics [8]. The main advantage of [1] is also its main weakness: by considering only time intervals that allow for no more than one reflection of a given wave packet, one may handle any boundary but one does not see the full effect of dispersion in the tangential directions.

In our recent work [5], a parametrix for the wave equation inside a model of stricly convex domain was constructed that provided optimal decay estimates. This yields by the usual argument Strichartz estimates with a range of pairs (q, r) such that

$$\frac{1}{q} \leq (\frac{(d-1)}{2} - \frac{1}{4})(\frac{1}{2} - \frac{1}{r}), \quad q > 2 \tag{4}$$

where, informally, the new $1/4$ factor is related to the $1/4$ loss in the dispersion estimate. On the other hand, earlier work [3] proved that Strichartz estimates on strictly convex domains can hold only if (q, r) are such that

$$\frac{1}{q} \leq (\frac{(d-1)}{2} - \frac{1}{12})(\frac{1}{2} - \frac{1}{r}), \quad q > 2\,, \quad r > 4\,. \tag{5}$$

Our result from [5] was later generalized to any strictly convex domain in [2]. By revisiting the parametrix construction from the general case, we can improve on both positive and negative results for the 2D model case.

Theorem 1. *Strichartz estimates* (2) *hold true on the domain* Ω *for pairs* (q, r) *such that*

$$\frac{1}{q} < (\frac{1}{2} - \gamma(2))(\frac{1}{2} - \frac{1}{r}), \quad \textit{with} \quad \gamma(2) = \frac{1}{9}\,. \tag{6}$$

In particular, for $r = +\infty$*, we have* $q > 5 + 1/7$*.*

The proof of Theorem 1 will rely on improving the bounds from [5] on the Green function in several directions as well as refining estimates on gallery modes from [3], all of which are of independent interest:

- the new parametrix construction may be done for initial data $\delta_{(x=a,y=0)}$ with $a > h^{2/3-\varepsilon}$, for any $\varepsilon > 0$, improving on the previous condition $a > h^{4/7}$;
- in relation to the invariance of the wave equation under the action of the operator $x + \Delta_y^{-1}\partial_x^2$, we may separate transverse waves and tangential ones, e.g. initial frequencies such that $|\xi| \leq \sqrt{a}$, where (ξ, η) is the Fourier variable corresponding to the spatial variable (x, y).
- Tangential waves require degenerate stationary phase; estimates in [5] may be refined to isolate precisely the space-time location of the worst case scenario of a swallowtail singularity. It turns out that such singularities only happen at an exceptional, discrete set of times;
- gallery modes satisfy the usual Strichartz estimates (as already proved in [3]) but with uniform constant with respect to the order of the mode: this allows to deal with the $a < h^{2/3-\varepsilon}$ region.

The same parametrix construction may also be used to improve on known counterexamples.

Theorem 2. *Strichartz estimates* (2) *may hold true on the domain* (Ω_2, g_F) *only if possible pairs* (q, r) *are such that*

$$\frac{1}{q} \leq (\frac{1}{2} - \frac{1}{10})(\frac{1}{2} - \frac{1}{r}) . \tag{7}$$

In particular, for $r = +\infty$*, we have* $q \geq 5$*.*

Counterexamples in [3] were constructed by carefully propagating a cusp starting in a suitable position around $a \sim h^{1/2}$. Here we start with such a suitably smoothed out cusp, around $a \sim h^{1/3}$ and let it propagate, estimating the resulting solution with the parametrix.

Both Theorems improve on earlier results. For counterexamples, in [3], the range of admissible pairs was only restricted with 1/10 replaced by 1/10. Regarding known results for strictly convex domains in 2D: for $d = 2$, [1] obtained $\gamma(2) = 1/6$ (but for any boundary), while [5] only provide the weaker $\gamma(2) = 1/4$ (in higher dimensions, Strichartz estimates derived from [5, 2] already improve all known results inside Strictly convex domains, and we expect our 2D result to yield further improvements, with a suitable numerology.)

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Probabilistic scattering for the 4D energy-critical defocusing nonlinear wave equation

Jonas Lührmann

(joint work with Benjamin Dodson and Dana Mendelson)

We consider the asymptotic behavior of solutions to the energy-critical defocusing nonlinear wave equation in four space dimensions for random initial data of super-critical regularity

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3 \text{ on } \mathbb{R}_t \times \mathbb{R}_x^4, \\ (u, \partial_t u)|_{t=0} = (f_0, f_1) \in H_x^s(\mathbb{R}^4) \times H_x^{s-1}(\mathbb{R}^4). \end{cases} \tag{1}$$

It is well known that for initial data at energy regularity $s = 1$, the solutions to (1) exist globally in time and scatter to free waves. However, the problem is ill-posed for initial data at super-critical regularity $s < 1$, see for example [6]. Our main result establishes almost sure global existence and scattering to free waves of strong solutions to (1) for randomized radially symmetric data for super-critical regularities $\frac{1}{2} < s < 1$. This work falls in the broader context of studying the super-critical initial data regime for nonlinear dispersive and hyperbolic equations from a probabilistic point of view, as initiated in the seminal works of Bourgain [2] and Burq-Tzvetkov [3, 4].

Previously, almost sure global existence for energy sub-critical defocusing nonlinear wave equations on $\mathbb{R}^3$ had been established by de Suzzoni [13, 14] and by the author and Mendelson [9, 10]. For energy-critical defocusing nonlinear wave equations almost sure global existence (without scattering) had been obtained by Pocovnicu [12] on $\mathbb{R}^d$, $d = 4, 5$, and by Oh-Pocovnicu [11] on $\mathbb{R}^3$.

Before stating the precise results, we introduce our randomization procedure which relies on a unit-scale decomposition of frequency space, see [15, 9, 1]. Let $\psi \in C_c^\infty(\mathbb{R}^4)$ be an even, non-negative bump function with $\text{supp}(\psi) \subseteq B(0,1)$ and such that

$$\sum_{k \in \mathbb{Z}^4} \psi(\xi - k) = 1 \quad \text{for all } \xi \in \mathbb{R}^4.$$

Let $s \in \mathbb{R}$ and let $f \in H^s_x(\mathbb{R}^4)$. For every $k \in \mathbb{Z}^4$, we define the function $P_k f : \mathbb{R}^4 \to \mathbb{C}$ by

$$(P_k f)(x) = \mathcal{F}^{-1}\Big(\psi(\xi - k)\hat{f}(\xi)\Big)(x) \quad \text{for } x \in \mathbb{R}^4. \tag{2}$$

By requiring the cut-off ψ to be even, we ensure that real-valued functions f satisfy the symmetry condition

$$\overline{P_k f} = P_{-k} f. \tag{3}$$

We let $\{(g_k, h_k)\}_{k\in\mathbb{Z}^4}$ be a sequence of zero-mean, complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the symmetry condition $g_{-k} = \overline{g_k}$ and $h_{-k} = \overline{h_k}$ for all $k \in \mathbb{Z}^4$. We assume that $\{g_0, \mathrm{Re}(g_k), \mathrm{Im}(g_k)\}_{k\in\mathcal{I}}$ are independent, zero-mean, real-valued random variables, where $\mathcal{I} \subset \mathbb{Z}^4$ is such that we have a disjoint union $\mathbb{Z}^4 = \mathcal{I} \cup (-\mathcal{I}) \cup \{0\}$, and similarly for the h_k.

Given a pair of real-valued functions $(f_0, f_1) \in H^s_x(\mathbb{R}^4) \times H^{s-1}_x(\mathbb{R}^4)$ for $s \in \mathbb{R}$, we define its randomization by

$$(f_0^\omega, f_1^\omega) := \bigg(\sum_{k\in\mathbb{Z}^4} g_k(\omega) P_k f_0, \sum_{k\in\mathbb{Z}^4} h_k(\omega) P_k f_1\bigg). \tag{4}$$

The key point of this randomization is that the free wave evolution of the resulting random data almost surely has much better space-time integrability properties. Crucially, such a randomization does not regularize at the level of Sobolev spaces. Moreover, the symmetry assumptions on the random variables as well as (3) ensure that the randomization of real-valued initial data is real-valued. In the following we denote the free wave evolution of the random initial data (f_0^ω, f_1^ω) by

$$S(t)(f_0^\omega, f_1^\omega) = \cos(t|\nabla|) f_0^\omega + \frac{\sin(t|\nabla|)}{|\nabla|} f_1^\omega.$$

Theorem 1 (Dodson-L-Mendelson [7]). *Let $\frac{1}{2} < s < 1$. For real-valued radially symmetric $(f_0, f_1) \in H^s_x(\mathbb{R}^4) \times H^{s-1}_x(\mathbb{R}^4)$, let (f_0^ω, f_1^ω) be the randomized initial data defined in* (4). *Then for almost every $\omega \in \Omega$, there exists a unique global solution*

$$(u, \partial_t u) \in \big(S(t)(f_0^\omega, f_1^\omega), \partial_t S(t)(f_0^\omega, f_1^\omega)\big) + C\big(\mathbb{R}; \dot{H}^1_x(\mathbb{R}^4) \times L^2_x(\mathbb{R}^4)\big)$$

to the energy-critical defocusing nonlinear wave equation (1) *with initial data $(u, \partial_t u)|_{t=0} = (f_0^\omega, f_1^\omega)$, which scatters to free waves as $t \to \pm\infty$ in the sense that there exist states $(v_0^\pm, v_1^\pm) \in \dot{H}^1_x(\mathbb{R}^4) \times L^2_x(\mathbb{R}^4)$ such that*

$$\lim_{t\to\pm\infty} \big\|\nabla_{t,x}\big(u(t) - S(t)(f_0^\omega + v_0^\pm, f_1^\omega + v_1^\pm)\big)\big\|_{L^2_x(\mathbb{R}^4)} = 0.$$

The starting point of our proof of Theorem 1 is the observation that in order to conclude almost sure global existence with scattering, it suffices to establish an a priori uniform-in-time energy bound for the nonlinear component

$$v(t) = u(t) - S(t)(f_0^\omega, f_1^\omega).$$

This observation can be proved by exploiting the Bahouri-Gérard a priori bounds on the scattering norms of solutions (at energy regularity) to the standard energy-critical defocusing nonlinear wave equation and by using a suitable perturbative argument. We then introduce an approximate Morawetz estimate for the forced cubic nonlinear wave equation satisfied by the nonlinear component $v(t)$ and combine it with Gronwall-type energy-growth estimates (due to Burq-Tzvetkov [5]). Given that the free wave evolution of the random data satisfies the following two global-in-time integrability properties

$$\big\| S(t)(f_0^\omega, f_1^\omega) \big\|_{L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^4)} < \infty \tag{5}$$

and

$$\big\| |x|^{\frac{1}{2}} S(t)(f_0^\omega, f_1^\omega) \big\|_{L_t^2 L_x^\infty(\mathbb{R}\times\mathbb{R}^4)} < \infty, \tag{6}$$

we can then infer the desired uniform-in-time energy bound for the nonlinear component $v(t)$ of the solution. While a proof that the free wave evolution of the random data $S(t)(f_0^\omega, f_1^\omega)$ almost surely satisfies (5) for a range of super-critical regularities is already contained in [9], a key novelty of this work is to establish almost surely the weighted space-time norm integrability property (6) for the free wave evolution of random initial data resulting from a radially symmetric pair $(f_0, f_1) \in H_x^s(\mathbb{R}^4) \times H_x^{s-1}(\mathbb{R}^4)$ for $\frac{1}{2} < s < 1$. To this end we derive a radial Sobolev-type estimate for the square-function associated with the unit-scale frequency projections (2) and combine it with a refinement of Strichartz estimates due to Klainerman-Tataru [8] as well as the fact that in the radial case a larger range of admissible Strichartz pairs is available.

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Global well-posedness for the massive Maxwell-Klein-Gordon equation with small critical Sobolev data

Cristian Gavrus

We prove global well-posedness and modified scattering for the massive Maxwell-Klein-Gordon equation in the Coulomb gauge on $\mathbb{R}^{1+d}$ ($d \geq 4$) for data with small critical Sobolev norm. This extends to the general case $m^2 > 0$ the results of Krieger-Sterbenz-Tataru ($d = 4, 5$) and Rodnianski-Tao ($d \geq 6$), who considered the case $m = 0$.

We proceed by generalizing the global parametrix construction for the covariant wave operator and the functional framework from the massless case to the Klein-Gordon setting. The equation exhibits a trilinear cancelation structure identified by Machedon-Sterbenz. To treat it one needs sharp L^2 null form bounds, which we prove by estimating renormalized solutions in null frames spaces similar to the ones considered by Bejenaru-Herr. To overcome logarithmic divergences we rely on an embedding property of $\Box^{-1}$ in conjunction with endpoint Strichartz estimates in Lorentz spaces.

We define the *covariant derivatives* and the *covariant Klein-Gordon operator* by

$$D_\alpha \phi = (\partial_\alpha + iA_\alpha)\phi, \quad \Box_m^A = D^\alpha D_\alpha + m^2$$

We will work under the Coulomb gauge condition

$$\mathrm{div}_x A = \partial^j A_j = 0 \tag{1}$$

Denoting $J_\alpha = -\Im(\phi \overline{D_\alpha \phi})$, we consider the equation

$$\begin{cases} \Box_m^A \phi = 0 \\ \Box A_i = \mathcal{P}_i J_x \\ \Delta A_0 = J_0, \ \Delta \partial_t A_0 = \partial^i J_i \end{cases} \tag{2}$$

where $\mathcal{P}$ denotes the Leray projection onto divergence-free vector fields

Theorem 1. *Let $d \geq 4$ and $\sigma = \frac{d}{2} - 1$. The MKG equation (2) is well-posed for small initial data on $\mathbb{R}^{1+d}$ with $m^2 > 0$, in the following sense: there exists a universal constant $\varepsilon > 0$ such that:*

Let $(\phi[0], A_x[0])$ *be a smooth initial data set satisfying the Coulomb condition and the smallness condition*

$$\|\phi[0]\|_{H^\sigma \times H^{\sigma-1}} + \|A_x[0]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} < \varepsilon. \tag{3}$$

Then there exists a unique global smooth solution (ϕ, A) *to the system* (2) *under the Coulomb gauge condition* (1) *on* $\mathbb{R}^{1+d}$ *with these data.*

For any $T > 0$*, the data-to-solution map* $(\phi[0], A_x[0]) \mapsto (\phi, \partial_t \phi, A_x, \partial_t A_x)$ *extends continuously to*

$$H^\sigma \times H^{\sigma-1} \times \dot{H}^\sigma \times \dot{H}^{\sigma-1}(\mathbb{R}^d) \cap \to C([-T, T]; H^\sigma \times H^{\sigma-1} \times \dot{H}^\sigma \times \dot{H}^{\sigma-1}(\mathbb{R}^d)).$$

The solution (ϕ, A) *exhibits modified scattering as* $t \to \pm\infty$*: there exist a solution* $(\phi^{\pm\infty}, A_j^{\pm\infty})$ *to the linear system*

$$\Box A_j^{\pm\infty} = 0, \qquad \Box_m^{A^{free}} \phi = 0, \qquad \text{such that}$$

$$\|(\phi - \phi^{\pm\infty})[t]\|_{H^\sigma \times H^{\sigma-1}} + \|(A_j - A_j^{\pm\infty})[t]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \to 0 \qquad \text{as } t \to \pm\infty.$$

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On the global regularity for a Wave-Klein-Gordon coupled system

Benoit Pausader

1. Statement of the main result

The following report presents the work [21] prepared in collaboration with A. Ionescu.

1.1. Global existence for a wave-Klein-Gordon system.

We consider the Wave-Klein-Gordon (W-KG) system in $3+1$ dimensions,

$$\begin{aligned} -\Box u &= A^{jk}\partial_j v \partial_k v + Dv^2, \\ (-\Box + 1)v &= uB^{jk}\partial_j \partial_k v, \end{aligned} \tag{1}$$

where u, v are real-valued functions, D is a real constant and A^{jk}, B^{jk} are real symmetric 3×3 matrices.

We prove that small data in an appropriate norm lead to global solutions which satisfy some modification to linear scattering. A simple variant of our main result reads as follows

Theorem 1. *Given $\varphi_0, \varphi_1, \psi_0, \psi_1 \in \mathcal{S}$, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, the solution with initial data*

$$u(t=0) = \varepsilon\varphi_0, \quad \partial_t u(t=0) = \varepsilon\varphi_1, \qquad v(t=0) = \varepsilon\psi_0, \quad \partial_t v(t=0) = \varepsilon\psi_1$$

remains globally regular. In addition, we have the following asymptotic description of the dynamics: there exists two functions u_∞ and v_∞ such that

$$\|\nabla_{x,t}(u(t) - u_\infty(t))\|_{L^2} + \|\nabla_{x,t}(v(t) - v_\infty(t))\|_{L^2} + \|v(t) - v_\infty(t)\|_{L^2} \to 0, \quad t \to \infty$$

and

$$\Box u_\infty = 0, \qquad \left(\partial_t^2 - \Theta\right) v_\infty = 0 \tag{2}$$

for some Fourier multiplier $\Theta(\nabla, t)$ depending on u and defined below.

We refer to [21] for a stronger and more precise version of the main theorem which in particular only requires the initial data to be in a fixed small ball in a given Sobolev-type space. We also note that a global existence result for compactly supported initial data was already obtained by a different method in [33].

The modified scattering operator in (2) is defined as follows

$$\Theta(\xi, t) = |\xi|^2 + 1 + B^{jk}\xi_j\xi_k \cdot u_{low}(t\frac{\xi}{\sqrt{1+|\xi|^2}}, t), \qquad u_{low} = P_{\leq(1+t)^{-2/3-}}u. \tag{3}$$

It corresponds to a modification of the free Klein-Gordon dispersion by the effect of the quasilinear perturbation $uB^{jk}\partial_j\partial_k v$ obtained by evaluating (a low frequency truncation of) u along the characteristices of the *free* Klein-Gordon equation and can also be obtained from (1) by assuming that u is a constant, in which case, one obtains the dispersion relation

$$-\tau^2 + |\xi|^2 + 1 + uB^{jk}\xi_k\xi_k = 0.$$

(The case when the quasilinear term involves time derivatives, $B = B^{\alpha\beta}$, is also treated in [21]; it follows from the dispersion relation above, but in this case, Θ is more naturally expressed as a product of 2 first order operators).

Since one has in general $u \gtrsim 1/t$ on a large portion of the interior of the light cone, the modification Θ is needed to obtain convergence of $v - v_\infty$. We note however, that when the quasilinear term is replaced by $\partial_m u B^{mjk}\partial_j\partial_k v$, one recovers scattering and this was already treated in [13, 26].

1.2. **Motivation.** The system (1) was derived by Wang [40] and LeFloch-Ma [34] as a model for the full Einstein-Klein-Gordon (E-KG) system

$$Ric(\mathbf{g})_{\alpha\beta} = \partial_\alpha\psi \cdot \partial_\beta\psi + (1/2)\psi^2 \cdot \mathbf{g}_{\alpha\beta}, \qquad \Box_{\mathbf{g}}\psi = \psi. \tag{4}$$

Intuitively, the deviation of the Lorentzian metric $\mathbf{g}$ from the Minkowski metric is replaced by a scalar function u, and the massive scalar field ψ is replaced by v. The system (1) retains the same linear structure as the Einstein-Klein-Gordon equations in harmonic gauge, but only keeps, schematically, quadratic interactions that involve the massive scalar field (the semilinear terms in the first equation and the quasilinear terms in the second equation coming from the reduced wave operator).

An interesting feature of (4), which also plays an important role in (1) comes from the fact that there are 2 modes of propagation of information: gravitational waves follow a wave equation at speed of light, while disturbances in the *massive* scalar field move at a speed strictly smaller than the speed of light according to a Klein-Gordon type evolution. A motivation for studying (4) is to develop robust methods to control the coupling of Einstein equations with matter fields which should also have different modes of propagation than gravitational waves.

Note that in contrast to the case of Einstein equations in vacuum, for which the stability of Minkowski space has been extensively studied [3, 5, 12, 32, 36, 37], there are comparatively few works devoted to (4) we mainly mention [34, 35, 41]. The presence of two *different* dispersion relations leads several classical methods for wave equations to break down (e.g. many interactions are no longer null, there are fewer symmetries and commuting vector fields).

2. Remarks about the proof

In order to prove Theorem 1, we use a combination of an energy method controlling the solution and some of its derivative along well-chosen vector fields (angular rotations and Lorentz boosts), and a dispersive analysis in order to extract the modification to scattering and obtain convergence of (the profile of) the solution in a well-chosen norm. This strategy borrows from earlier contributions [1, 2, 4, 6, 7, 8, 14, 15, 18, 24, 25, 27, 28, 29, 31, 38, 39] and was developed by the authors and coauthors to control many different quasilinear dispersive equations arising from plasma physics or water waves [9, 10, 11, 16, 17, 19, 20, 22, 23].

2.1. **Quadratic phase.** An important remark is that all nonlinear terms in (1) essentially only involve one quadratic phase. Indeed, controlling either of the equations in (1) by duality, one is led to consider integrals involving products of two Klein-Gordon-type unknowns and one wave-type unknown. The corresponding time oscillations involve only the phase

$$\Phi := \langle\xi_1\rangle \pm \langle\xi_2\rangle \pm |\xi_3|, \qquad \xi_1+\xi_2+\xi_3 = 0, \qquad |\Phi| \gtrsim \min\{1, |\xi_3|\}(1+|\xi_1|+|\xi_2|)^{-3}.$$

Since the phases are bounded from below, one only expects the case of very low frequencies for a wave unknown to play a significant role. It turns out however that this type of interaction does play a major role in the analysis. For example, in the case of

$$-\Box u = |\nabla_{x,t}v|^2 + v^2, \qquad (-\Box + 1)v = u\Delta v,$$

using positivity properties of the wave propagator in $3d$, one can construct solutions where u is nonnegative and $tu \gtrsim 1$ on large portions of the interior of the light cone, and this ultimately leads to the asymptotically nonlinear behavior of v in (3).

2.2. **Energy estimate.** A significant (and somewhat unexpected) difficulty in the study of (1) arises when considering energy estimates. While it is relatively easy to propagate high-order Sobolev norms, the special structure of (1) leads to difficulties when attempting to control vector fields. Indeed commuting with a vector field $\mathcal{V}$ leads to a system of the form

$$
\begin{aligned}
-\Box(\mathcal{V}u) &= v \cdot (\mathcal{V}v) + 2v \cdot \partial(\mathcal{V}v) + 2v \cdot [\mathcal{V}, \partial]v, \\
(-\Box + 1)(\mathcal{V}v) &= (\mathcal{V}u) \cdot \partial^2 v + \left\{u \cdot \partial^2(\mathcal{V}v) + u \cdot [\mathcal{V}, \partial^2]v\right\}
\end{aligned}
\tag{5}
$$

The first equation is easy to control and so are the terms inside the curly bracket in the second equation. However, the first term is problematic because the energy estimates only provide control on $\partial(\mathcal{V}u)$, but as we have seen, u can be significant at low frequency. This problem is solved by using the faster decay of the Klein-Gordon solution in both equations to recover *half* the derivative "loss" twice in two consecutive steps by controlling energies of the form

$$
\mathcal{E}_{\mathcal{V}} \simeq \||\nabla|^{-\frac{1}{2}} \nabla_{x,t} \mathcal{V}u(t)\|_{L^2}^2 + \|\nabla_{x,t} \mathcal{V}v(t)\|_{L^2}^2 + \|\mathcal{V}v(t)\|_{L^2}^2. \tag{6}
$$

Controlling this energy makes the first equation in (5) more difficult by multiplying the LHS by $|\nabla|^{-1/2}$, which, in the worse case corresponds to a growth of $t^{1/2}$, which is just compensated by the optimal decay of the Klein-Gordon unknown, while the second equation in (5) is now easier since the first term on the LHS is now only 1/2 derivative off. Once again, the worst case gives a growth of $t^{1/2}$ which is counteracted by the Klein-Gordon function decay. Note however that this requires obtaining optimal decay for the Klein-Gordon solution, which in turn is only possible once one identifies the correct asymptotic dynamics (2).

2.3. **Scope.** Using the analysis developed in [21], the result can be extended to prove stability of the Minkowski space for (4) for initial data with mild decay assumptions. However several additional hurdles have to be overcome: the wave solutions are no longer asymptotically free and require a correction to the asymptotic behavior similar to (3), as well as additional nonlinear forcing terms coming from non-null interactions of the metric. However, introducing a suitable Hodge-type decomposition, one can exhibit a special "2-step nilpotent" structure, already noticed in [36], which allows to contain the corresponding growth to $\|\nabla_{x,t} u(t)\|_{L^\infty} \simeq \ln(t)/t$. This in turns makes the energy estimates significantly more complicated. Another difficulty follows from the fact that the slow decay of the wave solutions does not allow such simple control of the energies as in (6) for wave-wave interactions. This is then compensated using appropriate bilinear estimates.

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On anisotropic type II blow-up for the energy supercritical semi-linear heat equation

Charles Collot
(joint work with Frank Merle, Pierre Raphaël)

1. Introduction

Solutions to the semi-linear heat equation

$$(NLH) \quad \begin{cases} \partial_t u(t,x) = \Delta u(t,x) + |u(t,x)|^{p-1} u(t,x), \\ u(0,x) = u_0(x) \end{cases}$$

where $p > 1$, $x \in \mathbb{R}^d$, $\Delta = \sum_{i=1}^d \partial_{x_i}^2$ and u is real-valued, may blow-up in finite time. In the usual functional spaces in which initial conditions u_0 are considered a regularising effect holds, and the solution belongs to $L^\infty(\mathbb{R}^d)$ immediately after the initial time $t = 0$. The solution is then said to blow up at a time $T > 0$ if $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \to +\infty$ as $t \uparrow T$. Similar blow-up phenomenon appear for other nonlinear evolution equations, and (NLH) belongs to the important class of that enjoying a scaling invariance. Namely, if $u(t,x)$ is a solution, then so is the translated and rescaled function

$$(t,x) \mapsto \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^2}, \frac{x-x_0}{\lambda}\right)$$

for $\lambda > 0$ and $x_0 \in \mathbb{R}^d$. A common belief is that the singularity formation can be described through suitable blow-up profiles that are renormalised using the above transformation. Solutions which reproduces exactly the same profile at smaller scales for (NLH) are called self-similar blow-up solutions and are of the form

$$u(t,x) = \frac{1}{(T-t)^{\frac{1}{p-1}}} \psi\left(\frac{x-x_0}{\sqrt{T-t}}\right)$$

These are associated to type I blow-up solutions that are solutions satisfying the growth condition $\limsup_{t\uparrow T}(T-t)^{1/(p-1)}\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} < +\infty$. This phenomenon is the only one in the so-called energy subcritical range $p < 1 + 4/(d-2)$ and is at the heart of the generic blow-up for (NLH). On this subject we refer to the book [12]. In the energy critical and supercritical settings, $p \geq 1 + 4/(d-2)$, another type of blow-up appears. This blow-up, at a point, is described by the concentration of a profile Q which is a stationary solution

$$\Delta Q + |Q|^{p-1} Q = 0, \tag{1}$$

and the solution u admits a main order description of the form

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x-x(t)}{\lambda(t)}\right) + \varepsilon(t,x)$$

where the scale of the leading order part shrinks to zero in finite time: $\lambda(t) \to 0$ as $t \uparrow T$, and where ε is some lower order remainder. The study of such blow-up was initiated in [5, 8, 14, 15] for example, and in [4, 10] for (NLH). This phenomenon is related to the type II blow-up, i.e. to solutions for which $\limsup_{t\uparrow T}(T-t)^{1/(p-1)}\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} = +\infty$ and we refer to [6] for precise statement of this fact in the radial case.

2. Anisotropic blow-up and main theorem

Despite substantial developments, the known results correspond to isotropic dynamics. Most of the results are proved for a radial blow-up at the origin, while the others involve several points blow-up [7], geometrical modifications, see [11] and [2] for type II blow-up for (NLH) in a domain, or blow-up on a sphere [13] etc. In various models however anisotropy is expected, for singularity formation of fluids at the boundary of a domain or for dispersive equations with anisotropic dispersion for example.

The main result revisits the radial analysis of [1, 4, 9, 10] to construct and describe precisely for the first time an anisotropic type II blow-up. The natural lift of a d-dimensional radial blow-up solution $u(t,|x|)$ to the $d+1$-dimension is the solution $U(t,x,y) = u(t,|x|)$ (where $(x,y) \in \mathbb{R}^d \times \mathbb{R}$). This solution blows up on the line $x = 0$ but is not well-localised in space. A suitable localisation is then given by

Theorem 1 ([3]). *For $d \geq 14$ and $p \geq 3$ there exists $\alpha = \alpha(d,p) > 0$ such that the following holds. For any $\ell \in \mathbb{N}^*$ with $\ell > \alpha/2$, there exists a finite codimensional set of initial data $u_0 \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$ with cylindrical symmetry, $u_0(x,y) = u_0(|x|,|y|)$, such that the corresponding solution blows up in finite time $0 < T < +\infty$ with the following asymptotics. It admits a decomposition*

$$u(t,x,y) = \frac{1}{\lambda(t,|y|)^{\frac{2}{p-1}}} Q\left(\frac{|x|}{\lambda(t,|y|)}\right) + \varepsilon(t,x,y)$$

where Q is the only radially symmetric solution to (1) with $Q(0)=1$, and:
1. Reconnection boundary*: there holds*

$$\lambda(t,|y|) \sim c(u_0)(T-t)^{\frac{\ell}{\alpha}} \left(1 + a(t) P_{2\ell}\left(\frac{|y|}{\sqrt{T-t}}\right)\right)^{\frac{1}{\alpha}}$$

where

$$P_m(z) = \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{k!(m-2k)!}(-1)^k z^{m-2k}$$

is the m-th one dimensional Hermite polynomial, and $a \in \mathcal{C}^1([0,T), \mathbb{R}_+^*)$ with:

$$a(t) = a^*(u_0)(1 + o_{t\uparrow T}(1)), \quad 0 < a^*(u_0) \ll 1.$$

2. Soliton profile and type II blow-up*:*

$$\lim_{t\to T} (T-t)^{\frac{\ell}{\alpha}\frac{2}{p-1}} \|\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R})} = 0, \quad \|u(t,\cdot)\|_{L^\infty} = \frac{c'(u_0)(1+o_{t\uparrow T}(1))}{(T-t)^{\frac{2}{p-1}\frac{\ell}{\alpha}}}.$$

3. Comments and open problems

A short explanation of the ideas of the proof of the above theorem can be found in the introduction of [3]. Some key features are the following. The y-dependence of the reconnection function $\lambda(t,y)$ changes with the d-dimensional blow-up speed ℓ. It shows that the radial type II blow up rates of the d-dimensional problem also exist for the $d+1$ dimensional problem. The method gives an iteration process for a further dimensional reduction procedure. The condition $d \geq 13$ and $p \geq 3$ is technical and can be improved to the optimal range $d \geq 11$ and $p > 1 + 4/(d - 4 - 2\sqrt{d-1})$.

Let us finish by some interesting open problems. The classification of type II blow-up profiles in the non-radial case is an important issue. The above result can be extended to other equations. Eventually, one should investigate the concentration in finite time of stationary profiles which are not radial.

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(In)stability of the Couette flow at high Reynolds numbers

Jacob Bedrossian
(joint work with Pierre Germain, Nader Masmoudi and Vlad Vicol)

The plane, periodic Couette flow has served as a canonical problem in the field of hydrodynamic stability since the late 19th century. Though much simpler than the more directly important problems of stability of isolated vortices in 2D or the stability of cylindrical pipe flow in 3D, the Couette flow has nonetheless raised many interesting questions. Indeed, obtaining a precise understanding the stability and instability of this flow in the nonlinear equations has remained elusive in both 2D and 3D. In this talk, I discussed some of the recent progress we have made in this direction in both 2D (joint with Nader Masmoudi, Vlad Vicol and Fei Wang) and 3D (joint with collaborators Pierre Germain and Nader Masmoudi). The dynamics of solutions have been determined in a variety of settings, and are governed by several important effects: namely inviscid damping and mixing-enhanced dissipation. The former is a kind of fluid mechanics analog of dispersion and the latter is an acceleration of the viscous (parabolic smoothing/dissipation) effects in the fluid due to mixing sending information to high frequencies.

The Couette flow is the equilibrium given by $u_E = (y,0)^t$ in 2D or $u_E = (y,0,0)^t$ in 3D. In 2D, we studied the problem on the cylinder $(x,y) \in \mathbb{T} \times \mathbb{R}$ and in 3D on the cylinder $(x,y,z) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}$. In both 2D and 3D, this set-up seems to be the simplest available equilibrium to study, and hence it is a very natural place to begin the nonlinear theory in earnest. All of the 2D works and the 3D works share one of several common themes. The main theme is that the background shear flow tends to stabilize the flow by suppressing the flow which depends on x. Hence, in 2D, the flow tends to converge to a shear flow: $u \to u_E(y) + (v(t,y),0)^t$ where v solves the 1D heat equation. We provide precise asymptotics both at infinite Reynolds number and in the high Reynolds number limits on how the fluid is attracted to the manifold of shear flows in the 2D Navier-Stokes equations. In 3D, the flow is attracted to a manifold of flows called 'streaks': $u \to u_E(y) + (v^1(t,y,z), v^2(t,y,z), v^3(t,y,z))^t$. Here $v^{2,3}$ will solve the standard 2D Navier-Stokes equations in $(y,z) \in \mathbb{R} \times \mathbb{T}$ and v^1 solves the effectively linear equations:

$$\partial_t v^1 + (v^2, v^3) \cdot \nabla v^1 = -v^2 + \nu \Delta v^1.$$

The transient growth induced by $-v^2$ is a major source of instability in the nonlinear problem, and is a primary culprit in the phenomenon known as *subcritical transition* in 3D fluid mechanics. The dynamics of the streaks is not entirely classified, but is much easier to classify and understand than the fully 3D flows. Our series of 3D works focus on understanding under which circumstances the manifold of streaks serves as a local attractor for the dynamics.

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Tidal energy for the Newtonian two-body motion

Sohrab Shahshahani

In this work we study the tidal energy for the motion of two gravitating incompressible fluid balls with free boundaries, obeying the Euler-Poisson equations. The orbital energy is defined as the mechanical energy of the center of mass of the two bodies. When the fluids are replaced by point masses, according to the classical analysis of Kepler and Newton, the conic curve describing the trajectories of the bodies is a hyperbola when the orbital energy is positive and an ellipse when the orbital energy is negative. If the point masses are initially very far, then the orbital energy, which is conserved in the case of point masses, is positive corresponding to hyperbolic motion. However, in the motion of fluid balls the orbital energy is no longer conserved, as part of the conserved energy is used in deforming the boundaries of the bodies. This energy is called the *tidal energy*. If the tidal energy becomes larger than the total energy during the evolution, the orbital energy must change its sign, signaling a qualitative change in the orbit of the bodies. We will show that under appropriate conditions on the initial configuration this change of sign occurs. Our analysis relies on a-priori estimates which we establish until the point of closest approach.

Threshold solutions for mass-subcritical NLS equations

Satoshi Masaki

1. Introduction

1.1. **Motivation.** We are interested in global behaviour of solutions to the nonlinear Schrödinger equations

$$i\partial_t u + \Delta u = l|u|^{p-1}u, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

Our aim is to study time global behavior of solutions to the equation, in particular, we want to give a classification of solutions according to their large time behavior. For this purpose, the focusing case $l = -1$ would be more interesting and difficult. As a first step, behavior of solutions around the special solution, such as the zero solution or the ground state, are studied. It is known that when $p > 1 + \frac{2}{d}$ small solutions (solutions around the zero solution) behaves like a free solution (**scattering**).

As a second step, we would like to find a threshold solution which lies on the boundary between small scattering solutions and solution with other behavior. Although there are many studies in this direction on the mass-critical case $p = 1+\frac{4}{d}$ or the mass-supercritical case $p > 1 + \frac{4}{d}$, the **mass-subcritical case** $p < 1 + \frac{4}{d}$ is less studied. Recalling the fact that the stability of the ground state changes at the mass-critical power, we can expect that the situation is slightly different.

To study the mass-subcritical case, we encounter several problems.

- Function space for small data scattering
  In the mass-subcritical case, constructing a solution is easy. Indeed, we can show global well-posedness in L^2 by the standard contraction map argument and conservation of mass. However, showing scattering is not easy. We want to start our study by the fact that a "small" solution scatters. Then, finding a right sense of "smallness" is a problem. For example, smallness in L^2 or H^1 do not yield scattering because we can construct small (in H^1) non-scattering solution from the scaling of ground state solution (if $l = -1$). The argument suggests that solving the equation in a scale critical space, or in a space embedded into a scale critical space, is necessary to start our study. So far, small data scattering is known in radial Sobolev space $\dot{H}^{s_c}_{\mathrm{rad}}$ ($s_c = \frac{d}{2} - \frac{2}{p-1} < 0$), weighted space $\mathcal{F}\dot{H}^{-s_c} := L^2(\mathbb{R}^d, |x|^{-2s_c}dx)$, and Fourier Lebesgue space $\mathcal{F}L^{(\frac{d(p-1)}{2})'}$, for instance.
- Non-polynomial nonlinearity
  Recall that when $p \leq 1 + \frac{2}{d}$ there is no L^2 solution that scatters. Hence, our range is $1 + \frac{2}{d} < p < 1 + \frac{4}{d}$. The range is
  $$\begin{cases} 3 < p < 5, & d = 1, \\ 2 < p < 3, & d = 2, \\ 5/3 < p < 7/3, & d = 3 \end{cases}$$
  and $p < 4/d \leq 2$ if $d \geq 4$. Hence, p is an integer only if $(d, p) = (1, 4), (3, 2)$. In particular, p is not an odd integer.
- No conservation law
  The well-known conserved quantity for NLS are mass, moment, and energy. But they are the quantities adopted to L^2-scaling, $\dot{H}^{1/2}$-scaling, and $\dot{H}^1$-scaling, respectively. Hence, it seems difficult to characterize behavior by a combination of these quantities.

1.2. **Formulation as a minimization problem.** Since there is no good conserved quantity, we regard the norm of a solution in a "good" function space (of space variable only) as a function of time, and then introduce a global quantity of the time function. The global quantity can be regarded as a kind of *conserved quantity*. Then, we shall find a threshold solution as a function which minimizes the global quantity *among all non-scattering solutions*. Here, we have at least two choices on the global quantity.

- The first example of the global quantity is

$$\inf_{t \in I_{\max}} \|u(t)\|_X. \tag{1}$$

  A minimizer with respect to the quantity gives us the best constant of the small data scattering.
- The second example is

$$\varlimsup_{t \to T_{\max}} \|u(t)\|_X, \tag{2}$$

where we assume that $u(t)$ does not scatter forward in time. In many cases, a minimizer to the quantity has a pre-compact orbit modulo symmetry. One may expect that it is a characterization of the ground state, which is true in some settings.

In the mass critical case $p = 1 + \frac{4}{d}$ with $X = L^2$, they coincide each other and equal to the norm of the ground state. This is merely a rephrase of results by Killip-Tao-Vişan, Killip-Vişan-Zhang, and Dodson. In the energy-critical case $p = 1 + \frac{4}{d-2}$ $(d \geq 4)$ with $X = \dot{H}^1$, these infimum values are *different* but we can evaluate the both. This follows from Kenig-Merle, Duyckaerts-Merle, Li-Zhang, Killip-Vişan, and Dodson. In the defocusing case, we can show that the second quantity is infinite, which implies boundedness and scattering are equivalent. See Kenig-Merle, Killi-Vişan, and series of works by Murphy. See [3] for the evaluation under the above settings.

The advantage of the above formulation is that we can obtain a threshold solution in a framework that no conserved quantities are available, which is often the case in the mass-subcritical case. However, one problem is that we do not know whether a given norm is reasonable, as an equivalent norm of X might give us a different minimizer.

2. Summary of results

1. Weighted space $\mathcal{F}H^1$, $\mathcal{F}\dot{H}^{-s_c}$

The problem (1) is considered in [4, 2]. However, we have to modify the above formulation because wighted space is not preserved by the NLS flow, as in the linear case. It is shown that there is a threshold solution which does not scatter and is not a standing wave solution. The problem (2) is considered in [1]. In this setting, the boundedness implies scattering even in the focusing case.

2. hat-Morrey space

The problems (1) and (2) are treated in a hat-Morrey type space. a set of functions which is a Fourier transform of a function in a generalization of a Morrey space in [3]. This kind of space arise in a context of refinement of Strichartz' estimate and is previously used in the analysis of mass-critical case. The space is wider than hat-Lebesgue space $\mathcal{F}L^{p'}$. By some reason, profile decomposition becomes harder in hat-Lebesgue spaces (see [3, 5]).

3. For generalized KdV equation.

The above formulation is not specific to NLS equation. We have similar results for generalized KdV equation (see [5, 6]). To work with a fractional power nonlinearity, we introduce in [5] an expansion of nonlinearity by means of the Fourier series expansion. The expansion technique is also applicable to analysis of large time behavior (cf. long range scattering).

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A critical scattering result for the defocusing cubic NLW in $\mathbb{R}^{1+3}$

Benjamin Dodson

In this talk we will several recent results concerning the cubic, nonlinear wave equation

$$u_{tt} - \Delta u \pm u^3 = 0, \qquad u(0,x) = u_0, \qquad u_t(0,x) = u_1. \tag{1}$$

In each case we will be interested in the radial equation, that is,

$$\frac{1}{r}(\partial_{tt} - \partial_{rr})(ru) \pm u^3 = 0. \tag{2}$$

In general, (1) is $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ - critical with $s_c = \frac{d-2}{2}$. This is because (1) is invariant under the scaling

$$u(t,x) \mapsto \lambda u(\lambda t, \lambda x), \tag{3}$$

and this scaling preserves the $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm.

Theorem 1 (Lindblad - Sogge)**.** (1) *is locally well - posed in dimensions $d \geq 3$ in $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$.*

In extending theorem 1 to a global result, dimension $d = 4$ has been of particular interest to researchers due to the fact that (1) has the conserved energy

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 + \frac{1}{2}\int u_t(t,x)^2 \pm \frac{1}{4}\int u(t,x)^4 dx. \tag{4}$$

In particular, in the defocusing (or +) case, (4) implies that the critical norm is globally bounded in dimension $d = 4$. This fact has been well - exploited to prove scattering.

Theorem 2. (1) *is globally well - posed and scattering in the defocusing case.*

Proof: This was proved by a number of authors, (Grillakis, Shatah - Struwe, Bahouri - Gerard).

In the focusing case in dimension $d = 4$ both type one and type two blowup has been shown to exist.

Type one blowup: Type one blowup occurs when

$$\sup_{t \in I} \|(u(t), u_t(t))\|_{\dot{H}^{s_c} \times \dot{H}^{s_c - 1}} = \infty, \tag{5}$$

where I is the maximal interval of existence for a solution to (1).

Type two blowup: Type two blowup occurs when

$$\sup_{t \in I} \|(u(t), u_t(t))\|_{\dot{H}^{s_c} \times \dot{H}^{s_c - 1}} < \infty, \tag{6}$$

but the solution fails to scatter. See the work of (Merle - Zaag) and (Donninger - Schorkhuber) for type one blowup results. See the work of (Duyckaerts - Kenig - Merle) for type two blowup results.

In this talk we will discuss some scattering results for (2) in dimensions three and five, where there is no critical conserved quantity.

Theorem 3 (D. - Lawrie, 2014)**.** *There is no type two blowup for* (2) *in dimensions* $d = 3, 5$*, either defocusing or focusing.*

Theorem 4 (D., 2016)**.** (2) *is globally well - posed and scattering in the defocusing case for initial data satisfying*

$$\|u_0\|_{\dot{H}^{1/2+\epsilon}} + \|u_1\|_{\dot{H}^{-1/2+\epsilon}} + \||x|^{2\epsilon} u_0\|_{\dot{H}^{1/2+\epsilon}} + \||x|^{2\epsilon} u_1\|_{\dot{H}^{-1/2+\epsilon}} < \infty, \tag{7}$$

for any $\epsilon > 0$.

Theorem 4 uses the I - method combined with the hyperbolic coordinates

$$u(t, x) \mapsto \frac{e^{\tau} \sinh s}{s} u(e^{\tau} \cosh s, e^{\tau} \sinh s). \tag{8}$$

Theorem 5 (D., 2016)**.** (2) *is globally well - posed and scattering in the defocusing case for initial data satisfying*

$$\|u_0\|_{B^2_{1,1}} + \|u_1\|_{B^1_{1,1}} < \infty. \tag{9}$$

for any $\epsilon > 0$.

The norm (9) is invariant under the scaling (3). Theorem 5 also uses the hyperbolic coordinates combined with splitting u into a finite energy part and a free solution.

The Radiation Field on Product Cones

Jeremy L. Marzuola
(joint work with Dean Baskin)

Suppose Y is a compact Riemannian manifold without boundary. The cone $X = C(Y)$ over Y is diffeomorphic[1] to $[0,\infty)_r \times Y$ and is equipped with the metric

$$g = dr^2 + r^2 h,$$

where h is a fixed Riemannian metric on Y.

The *forward radiation field* of u, denoted $\mathcal{R}_+[u]$ is the radiation pattern seen by a distant observer. More precisely,

$$\mathcal{R}_+[u](s,y) = \lim_{r\to\infty} r^{-\frac{n-1}{2}} u(s+r,r,y).$$

This talk announced the following Theorem:

Theorem 1. *Suppose u is the forward solution of $\Box u = f$ on $\mathbb{R}\times X$, where X is a product cone and $f \in C_c^\infty(\mathbb{R}\times X)$. The function u admits a joint asymptotic expansion of the following form:*

$$u(t,r,y) \sim (t+r)^{-(n-1)/2} \sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (t+r)^{-\ell}(t-r)^{-1/2-k-\sqrt{(n-2)^2/4+\mu_j^2}} a_{jk\ell}(y),$$

(once we have accounted for integer coincidence) where μ_j^2 are the eigenvalues of Δ_h on Y. In particular, the radiation field of u admits the following expansion as $s \to +\infty$:

$$\mathcal{R}_+[u](s,y) \sim \sum_{j=0}^{\infty} s^{-1/2-k-\sqrt{(n-2)^2/4+\mu_j^2}}$$

Remark 2. In the more general version of the theorem, you can in principle have logarithmic terms due to integer coincidences among the poles or owing to higher order poles from the resonances. We believe that in the case of a product cone you do not see these logarithmic terms for two reasons: a) the exact product structure means that the integer coincidences cannot occur, as there is no remainder term to deal with, and b) as seen in [1] all poles of the resolvent on the exact hyperbolic cone are simple poles.

It turns out that the radiation field on a product cone is related to an explicit calculation all of the resonances for a hyperbolic cone in terms of the eigenvalues of the cross-section. To fix notation, let X_h be a manifold of dimension $n+1$ diffeomorphic to $(\mathbb{R}_+)_r \times Y$, where Y is a compact n-manifold without boundary. Given a Riemannian metric h on Y, we equip X_h with the hyperbolic conic metric

[1] Our definition of a product cone has already resolved the conic singularity. We take X to be a smooth manifold with boundary and instead require that the metric be singular at $r = 0$.

$dr^2 + \sinh^2 r\, h$. Except in the special case of hyperbolic space, X has an isolated conic singularity at $r = 0$.

Given a hyperbolic cone X_h and its associated metric g, we define the resolvent

$$R(\lambda) = \left(-\Delta_g - \lambda^2 - \frac{n^2}{4}\right)^{-1},$$

which is a bounded operator $L^2(X_h, g) \to L^2(X_h, g)$ for $\operatorname{Im}\lambda > 0$. The resolvent $R(\lambda)$ admits a meromorphic continuation to the complex plane as an operator $L^2_c(X_h, g) \to L^2_{\mathrm{loc}}(X_h, g)$, i.e., from compactly supported functions to locally L^2 functions. The poles of this meromorphic continuation (aside from potentially finitely many eigenvalues lying in the upper half plane) are called *resonances*.

In [1], we establish the following theorem using explicit calculations.

Theorem 3. *Let $\{\mu_j^2\}_{j\in\mathbb{N}}$ be the eigenvalues of $-\Delta_h$. The resonances of $-\Delta_g$ are given by*

$$\lambda_{j,k} = -\imath\left(\frac{1}{2} + k + \sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}\right)$$

for $k \in \mathbb{N} = \{0, 1, 2, \dots\}$, and j so that

$$\sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2} \notin \frac{1}{2} + \mathbb{R}.$$

Here an eigenvalue μ_j^2 with multiplicity m for $-\Delta_h$ adds multiplicity m to $\lambda_{j,k}$.

If

$$\sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2} \in \frac{1}{2} + \mathbb{R},$$

then μ_j contributes no resonances to $-\Delta_g$.

With this theorem in hand, the proof of the main theorem follows closely the approach taken by my collaborator Dean Baskin with Andras Vasy and Jared Wunsch [3, 4]. The main idea is to use the Mellin transform to turn the foward problem into a Fredholm on the boundary (at infinity) of the spacetime.

The idea is to compactify to a domain as see in Figure 1. The compactification gives elliptic estimates near $C_\pm$ and asymptotically de Sitter estimates near C_0. A precise asymptotic expansion must be made at the boundary S_+ (for the forward problem), which requires careful propagation of singularities results near the North pole in C_+ and a good Carleman estimate for hyperbolic problems on de Sitter spaces. Once each component of the argument is established, the general framework gives us that the radiation field asymptotics are predicted precisely by the resonances computed in Theorem 3. Our theorem agrees well with direct computations of expected wave decay given by the explicit fundamental solution of Cheeger-Taylor in the works [5, 6].

Remark 4. The exponents in Theorem 1 are the resonances of the corresponding hyperbolic cone. This means that you might not see all of the exponents listed

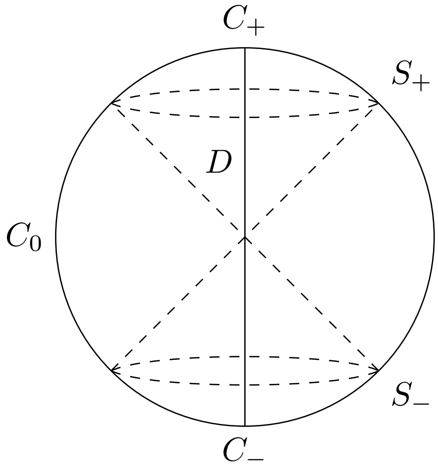

FIGURE 1. The compactified domain.

as they may not all be resonances. They are however still poles of our operator, but those that are supported exactly within S_+, much like the difference between the poles of the scattering matrix versus the poles of the resolvent in standard scattering theory.

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Multi-scale bilinear restriction estimates for general phases

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We give an overview of recent work which gives (adjoint) bilinear restriction estimates for general phases at different scales in the full mixed norm range [4]. As an application, we obtain a bilinear restriction estimate for wave/Klein-Gordon interactions, and a refined Strichartz inequality for the Klein-Gordon equation.

To describe the results in more detail, let $n > 1$ and $j = 1, 2$. Define phases $\Phi_j : \Lambda_j \to \mathbb{R}$ with $\Lambda_j \subset \mathbb{R}^n$. Given $f \in L^2(\mathbb{R}^n)$, we define the space-time function

$$e^{it\Phi_j(\nabla)} f = \int_{\mathbb{R}^n} e^{it\Phi_j(\xi)} \widehat{f}(\xi) e^{ix\cdot\xi} d\xi.$$

It is clear that this gives a solution to the linear PDE $-i\partial_t u + \Phi_j(\nabla)u = 0$. In particular, letting $\Phi_j(\xi) = \langle\xi\rangle_m = (m^2 + |\xi|^2)^{\frac{1}{2}}$ gives a solution to the Klein-Gordon equation, while $\Phi_j(\xi) = |\xi|^2$ gives a solution to the Schrödinger equation. Alternatively, $e^{it\Phi_j(\nabla)}f$ is essentially the extension operator (or adjoint restriction operator) for the surface

$$S_{\Phi_j} = \{(\Phi_j(\xi), \xi) \mid \xi \in \Lambda_j\} \subset \mathbb{R}^{1+n}.$$

We are interested in obtaining (adjoint) bilinear restriction estimates of the form

$$\big\| e^{it\Phi_1(\nabla)} f e^{it\Phi_2(\nabla)} g \big\|_{L^q_t L^r_x(\mathbb{R}^{1+n})} \leqslant \mathbf{C} \|f\|_{L^2_x} \|g\|_{L^2_x} \tag{1}$$

with a careful understanding of how the constant $\mathbf{C}$ depends on the phases Φ_j. These types of estimates have a number of applications, for instance to the linear restriction problem [12], to refinements of Strichartz inequalities [8, 5], and to well-posedness results for nonlinear PDE [3].

To gain some intuition into the estimate (1), we first consider the case of the wave equation $\Phi_j(\xi) = |\xi|$. If we take sets $\Lambda_j \subset \{|\xi| \approx 1\}$, then an application of Hölder together with the linear Strichartz estimates gives (1) in the range $\frac{1}{q} + \frac{n-1}{2r} \leqslant \frac{n-1}{2}$. However, this range can be improved significantly if we assume that the sets Λ_j have some angular separation. For instance, if $\Lambda_1 = \{|\xi| \approx 1, |\xi - e_1| \ll 1\}$ and $\Lambda_2 = \{|\xi| \approx 1, |\xi - e_2| \ll 1\}$ (here e_1 and e_2 are the first unit coordinate vectors), then we can improve the range given by the linear Strichartz estimates to $\frac{1}{q} + \frac{n+1}{2r} \leqslant \frac{n+1}{2}$. In particular, under an angular separation condition on the sets Λ_j, we essentially gain two dimensions over using linear estimates. The almost optimal range for the wave equation when $q = r$ was first obtained in the breakthrough work of Wolff [14]. The endpoint was obtained shortly thereafter, and extended to the non-unit scale case by Tao [10], the case $q \neq r$ is due to Tataru [13] and Lee-Vargas [6]. These results also included high-low interactions with an epsilon loss.

In the case of general phases, under suitable transversality and curvature assumptions on the surfaces S_j, it is known that, provided Φ_1 and Φ_2 are unit scale, the bilinear estimate (1) is also true in the full range $r = q > \frac{n+3}{n+1}$ [7, 1]. In fact, in recent joint work with Sebastian Herr, we have shown that the estimate (1) holds in the adapted function space $V^2_{\Phi_j}$ [3] in the full $q = r$ bilinear range. This bound was then applied to obtain global well-posedness for a resonant case of the Dirac-Klein-Gordon system. If the phases are not assumed to be unit scale, then only specific cases are known [11, 10, 9, 2]. In particular, the case of high-low Klein-Gordon interactions $\Phi_j = (m_j^2 + |\xi|^2)^{\frac{1}{2}}$ does not follow from the previous estimates. The main result we aim to present, is an estimate giving an explicit dependence of the constant $\mathbf{C}$ in (1) on general phases Φ_j.

Theorem 1 (Bilinear restriction for general phases [4])**.** *Let $n \geqslant 2$, $1 \leqslant q, r \leqslant 2$ and $\frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}$. Assume that Φ_j satisfy suitable curvature/transversality conditions and let $\mathcal{H}_j = \|\nabla\Phi_j\|_{L^\infty(\Lambda_j)}$, $\mathcal{V}_{max} = \sup_{\xi\in\Lambda_1, \eta\in\Lambda_2} |\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)|$.*

Then if $\mathcal{H}_2 \leqslant \mathcal{H}_1$ *and either* $\widehat{f}$ *or* $\widehat{g}$ *has support in a ball of radius* μ *we have*

$$\big\| e^{it\Phi_1(\nabla)} f e^{it\Phi_2(\nabla)} g \big\|_{L^q_t L^r_x(\mathbb{R}^{1+n})} \lesssim \mu^{n+1-\frac{n+1}{r}-\frac{2}{q}} \mathcal{V}_{max}^{\frac{1}{r}-1} \mathcal{H}_1^{1-\frac{1}{q}-\frac{1}{r}} \Big(\frac{\mathcal{H}_1}{\mathcal{H}_2}\Big)^{\frac{1}{q}-\frac{1}{2}} \|f\|_{L^2_x} \|g\|_{L^2_x}$$

The previous theorem also has a version in the adapted function space $U^2_{\Phi_j}$, which reflects the fact that the previous theorem holds for *vector valued* waves. It is also important to note that the dependence on the parameters μ, $\mathcal{H}_j$, and $\mathcal{V}_{max}$ is sharp, thus for general phases only the endpoint $\frac{1}{q} + \frac{n+1}{2r} = \frac{n+1}{2}$ remains open. The precise conditions required on the phases Φ_j can be found in [4], and are based on a normalised version of the conditions appearing in [3]. In particular, they hold for the Klein-Gordon equation.

Theorem 2 (Bilinear restriction for Klein-Gordon [4])**.** *Let* $1 \leqslant q, r \leqslant 2$, $\frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}$, $m_1, m_2 \geqslant 0$, *and* $\lambda \geqslant \mu > 0$. *Let* $0 < \alpha \leqslant 1$, *and suppose we have* $\xi_0, \eta_0 \in \mathbb{R}^n$ *such that* $\langle \xi_0 \rangle_{m_1} \approx \lambda$, $\langle \eta_0 \rangle_{m_2} \approx \mu$, *and*

$$\frac{\big| m_2|\xi_0| - m_1|\eta_0| \big|}{\lambda\mu} + \Big(\frac{|\xi_0||\eta_0| \mp \xi_0 \cdot \eta_0}{\lambda\mu}\Big)^{\frac{1}{2}} \approx \alpha.$$

Define $\beta = (\frac{m_1}{\alpha\lambda} + \frac{m_2}{\alpha\mu} + 1)^{-1}$. *If*

$$supp\ \widehat{f} \subset \big\{ \big||\xi| - |\xi_0|\big| \ll \beta\lambda,\ \big(|\xi||\xi_0| - \xi \cdot \xi_0\big)^{\frac{1}{2}} \ll \alpha\lambda \big\}$$

$$supp\ \widehat{g} \subset \big\{ \big||\xi| - |\eta_0|\big| \ll \beta\mu,\ \big(|\xi||\eta_0| - \xi \cdot \eta_0\big)^{\frac{1}{2}} \ll \alpha\mu \big\}$$

then we have the bilinear estimate

$$\big\| e^{it\langle -i\nabla \rangle_{m_1}} f e^{\pm it\langle -i\nabla \rangle_{m_2}} g \big\|_{L^q_t L^r_x} \lesssim \alpha^{n-1-\frac{n-1}{r}-\frac{2}{q}} \beta^{1-\frac{1}{r}} \mu^{n-\frac{n}{r}-\frac{1}{q}} \Big(\frac{\lambda}{\mu}\Big)^{\frac{1}{q}-\frac{1}{2}} \|f\|_{L^2_x} \|g\|_{L^2_x}$$

where the implied constant is independent of m_1, m_2.

Theorem 2 allows $m_1 = m_2 = 0$. In particular, we remove the ϵ high-low derivative loss appearing in the previous bounds for the wave equation due to [10] and Lee-Vargas [6]. As an application of Theorem 2, we obtain a refined Strichartz estimate for the Klein-Gordon equation, which is a Klein-Gordon counterpart to an estimate for the wave equation due to Ramos [8]. Similar results, but requiring additional regularity, have been obtained earlier for the Klein-Gordon equation by Killip-Stovall-Visan [5].

Given $\lambda \geqslant 1$, and $0 < \alpha < 1$, we define $\mathcal{A}_{\lambda,\alpha}$ to the collection of sets $A \subset \mathbb{R}^n$ of the form

$$A = A(\xi_0) = \big\{ \langle \xi \rangle \approx \lambda,\ \big||\xi| - |\xi_0|\big| \ll \tfrac{\alpha\lambda}{1+\alpha\lambda}\lambda,\ \big(|\xi||\xi_0| - \xi \cdot \xi_0\big)^{\frac{1}{2}} \ll \alpha\lambda \big\}$$

where the points $\xi_0 \in \mathbb{R}^n$ satisfy $\langle \xi_0 \rangle \approx \lambda$ and are chosen to ensure that the sets $A \in \mathcal{A}_{\lambda,\alpha}$ form a finitely overlapping cover of the annulus/ball $\{\langle \xi \rangle \approx \lambda\}$.

Theorem 3 (Refined Strichartz for Klein-Gordon [4])**.** *Let $n \geqslant 2$. There exists $0 < \theta < 1$ and $1 < r < 2$ such that*

$$\begin{aligned} &\big\| e^{it\langle \nabla \rangle} f \big\|_{L^{2\frac{n+1}{n-1}}_{t,x}(\mathbb{R}^{1+n})} \\ &\lesssim \Big(\sup_{\lambda \in 2^{\mathbb{N}}, \alpha \in 2^{-\mathbb{N}}} \sup_{A \in \mathcal{A}_{\lambda,\alpha}} \Big(\tfrac{\alpha\lambda}{1+\alpha\lambda} \Big)^{\frac{1}{n+1}} \lambda^{\frac{1}{2}} |A|^{\frac{1}{2}-\frac{1}{r}} \|\widehat{f}\|_{L^r_\xi(A)} \Big)^{\theta} \|f\|^{1-\theta}_{H^{\frac{1}{2}}}. \end{aligned}$$

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The generalized SQG patch equation: global stability of the half-plane stationary solution

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1. Introduction

In joint work [4] with Diego Córdoba and Javier Gómez-Serrano, we consider the generalized surface-quasigeostrophic equations (gSQG):

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \quad (x,t)\in\mathbb{R}^2\times\mathbb{R}_+, \\ u = -\nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\theta, \\ \theta_{|t=0} = \theta_0, \end{cases} \tag{1}$$

where $\alpha\in(0,2)$. The limiting case $\alpha=0$ corresponds to the 2D incompressible Euler equation, while the case $\alpha=2$ produces stationary solutions.

These are so-called active scalar equations, which have been originally introduced and studied in the setting of sufficiently smooth solutions θ. The equations (1) have also been analyzed extensively in the natural setting of the so-called α-*patches*, which are solutions for which θ is a step function

$$\theta(x,t) = \begin{cases} \theta_1, \text{ if } & x\in\Omega(t) \\ \theta_2, \text{ if } & x\in\Omega(t)^c, \end{cases}$$

where $\theta_1,\theta_2\in\mathbb{R}$ are constants, and $\Omega(t)$ is a regular domain that evolves in time under the influence of the induced velocity field. The evolution of a patch can be determined by looking just at the evolution of its boundary, called the interface. More precisely, the evolution equation for the interface of an α-patch, which we parametrize as $z: I\to\mathbb{R}^2$, $z(x)=(z_1(x),z_2(x))$, can be written as

$$\partial_t z(x,t) = -(\theta_2-\theta_1)C(\alpha)\int_I \frac{\partial_x z(x,t)-\partial_x z(x-y,t)}{|z(x,t)-z(x-y,t)|^\alpha}dy + c(x,t)z_x(x,t). \tag{2}$$

Here $I\subseteq\mathbb{R}$ is an interval (usually $I=[0,2\pi]$ in the case of bounded patches or $I=\mathbb{R}$ for unbounded patches), the presence of the function c has to do with the flexibility in parametrizing the curve, and $C(\alpha)\in(0,\infty)$ is a normalizing constant.

1.1. **Local regularity.** The local regularity theory for the equations (1)–(2) is generally well understood, starting with the work of Constantin–Majda–Tabak [3] and Held–Pierrehumbert–Garner–Swanson [6]. As expected, data with sufficient smoothness lead to local in time unique solutions that propagate the regularity of the initial data. See the papers of Rodrigo [10], Gancedo [5], and the references therein for such regularity results.

1.2. **Dynamical formation of singularities.** The problem of whether the SQG evolution can lead to finite time singularities is a challenging open problem both in the smooth case (1) and in the patch case (2). Numerical simulations appear to suggest (convincingly) the possibility of dynamical formation of singularities in certain scenarios, both in the smooth case and the patch case. However, we emphasize that no rigorous results are known. See, however, the recent work of

Kiselev–Ryzhik–Yao–Zlatos [9], where the authors introduced a new gSQG-patch model, with a fixed boundary, and proved the formation of finite time singularities in this model for certain patches that touch the boundary at all times. At this point it is unclear whether such a scenario can lead to singularities in the classical gSQG models considered here.

1.3. **Global regularity and rotating solutions.** The construction of nontrivial global solutions for the gSQG equations is also a challenging problem, both in the smooth and in the patch case. In fact, the only known non-stationary global solutions are very special rotating solutions. These solutions are periodic in time and evolve by rotating with constant angular velocity around their center of mass. See the recent papers of Castro–Córdoba–Gómez-Serrano [1] and [2] and the references therein for the construction of such solutions, both in the patch setting (where such solutions are known as V-states) and in the smooth setting.

2. The main theorem

Our goal in [4] is to initiate the study of *stable* global solutions of the equations (1) and (2). Such stable solutions cannot be periodic in time and their construction requires a different mechanism.

A natural way to look for families of global stable solutions is to perturb around certain explicit stationary solutions of the equation. In our case of the gSQG equations, one could start by perturbing around the trivial solution $\theta \equiv 0$ of the equation (1). However, there is no source of dispersion in this case and it is not clear to us how to control the solution beyond the natural time of existence $T_\varepsilon \approx \varepsilon^{-1}$ corresponding to data of size ε.

One could also start from the observation that all radial functions are stationary solutions of the gSQG equations, and look for global solutions that start as small perturbations of radial functions. Numerical simulations seem to suggest the existence of long-term (perhaps global) smooth solutions for the gSQG-patch equation (2), starting from certain small perturbations of a characteristic function of a ball of radius 1, but we have not been able to analyze this scenario rigorously so far.

In [4] we consider a simpler scenario, namely we perturb around the half-plane stationary solution corresponding to the straight interface $(z_1(x), z_2(x)) = (x, 0)$. For simplicity, we assume that $C(\alpha)(\theta_1 - \theta_2) = 1$, $c(x,t) = 0$ and $z_1(x,t) = x$. This choice yields the following equation for $z_2(x,t) \equiv h(x,t)$:

$$\partial_t h(x,t) = \int_{\mathbb{R}} \frac{h_x(x,t) - h_x(x-y,t)}{\big(|h(x,t) - h(x-y,t)|^2 + y^2\big)^{\alpha/2}} dy. \tag{3}$$

Notice that the integral in (3) is well defined for $\alpha \in (1,2)$.

At the linear level, the dynamics of solutions of (3) are determined by the equation

$$\partial_t \hat{h}(\xi,t) = i\Lambda(\xi)\hat{h}(\xi,t), \qquad \Lambda(\xi) := \gamma|\xi|^{\alpha-1}\xi, \tag{4}$$

where $\hat{h}(\xi, t)$ is the Fourier transform of $h(x, t)$ and $\gamma \in (0, \infty)$ is a constant. We notice that this linearized equation has *dispersive* character, due to the dispersion relation Λ. Thus one can hope to prove global regularity and decay for small and localized initial data. This is precisely our main theorem:

Theorem 1. *Assume $\alpha \in (1, 2)$, and let $N_0 := 20$ and $N_1 := 4$. Then there is a constant $\overline{\varepsilon} = \overline{\varepsilon}(\alpha)$ such that for all initial-data $h_0 : \mathbb{R} \to \mathbb{R}$ satisfying the smallness conditions*

$$\|h_0\|_{H^{N_0\alpha}} + \|x\partial_x h_0\|_{H^{N_1\alpha}} \leq \varepsilon_0 \leq \overline{\varepsilon} \tag{5}$$

there is a unique global solution $h \in C([0, \infty) : H^{N_0\alpha}(\mathbb{R}))$ of the evolution equation (3) with $h(0) = h_0$. Moreover, the solution h satisfies the slow growth energy bounds

$$\|h(t)\|_{H^{N_0\alpha}} + \|Sh(t)\|_{H^{N_1\alpha}} \lesssim \varepsilon_0(1+t)^{p_0}, \qquad t \in [0, \infty), \tag{6}$$

where $S := \alpha t\partial_t + x\partial_x$ is the scaling vector-field associated to the linear equation (4) and $p_0 := 10^{-7}(2-\alpha)$, and the sharp pointwise decay bounds

$$\big(2^{k/2} + 2^{N_2\alpha k}\big)\|P_k h(t)\|_{L^\infty} \lesssim \varepsilon_0(1+t)^{-1/2}, \qquad t \in [0, \infty),\ k \in \mathbb{Z}, \tag{7}$$

where P_k denote the standard Littlewood-Paley projections and $N_2 := 8$.

3. Main ideas of the proof

The equation (3) is a time reversible quasilinear equation. The classical mechanism to prove global regularity in such a situation has two main steps:

(1) Prove energy estimates to propagate control of high order Sobolev and weighted norms;

(2) Prove dispersion and decay of the solution over time.

More precisely, we prove energy bounds with slow growth of the form (6). To prove dispersion, we define a suitable norm, called the Z-norm, in such a way that $\|h(t)\|_Z$ is uniformly bounded as $t \to \infty$,

$$\|h(t)\|_Z \lesssim \varepsilon_0.$$

The precise choice of the Z-norm is important, since control of the Z-norm has to complement suitably the energy control proved in the first step.

The proof of Theorem 1 involves the construction of nonlinear profiles that lead to *modified scattering* and sharp pointwise decay (7). Our analysis in [4] has some similarities with the global analysis of water-wave models in 2D in [7] and [8].

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