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## Automorphic Forms and Arithmetic

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ABSTRACT. The workshop brought together leading experts and young researchers at the interface of automorphic forms and analytic number theory to disseminate, discuss and develop important recent methods and results. A particular focus was on higher rank groups, as well as their arithmetic applications. This includes, for instance, the study of various aspects of  $L$ -functions, the fine distribution properties of their Fourier coefficients and Hecke eigenvalues, the mass distribution of automorphic forms on general symmetric spaces, and applications of results of algebraic geometry to automorphic forms.

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### Introduction by the Organisers

Automorphic forms are a very interdisciplinary topic in modern mathematics at the interface of number theory, analysis, representation theory and algebraic geometry. Among these different view points, the workshop focused in particular on the analytic theory of automorphic forms and their associated  $L$ -functions, and their interactions with number theory. Fifty-two leading experts and young researchers came together to exchange ideas, present newly developed methods and start or continue their collaboration on projects related to the subject of the workshop. The programme included 25 talks, all of which presented interesting new results. We highlight a few major topics:

The most important global analytic objects attached to an automorphic form or representation  $f$  are its  $L$ -functions, and they play naturally a major role in

the investigation of automorphic forms as well as in their own interest as a generalization of the Riemann zeta-function. R. Holowinsky presented a new look on Munshi's automorphic circle method, and Munshi presented a new application of it. D. Koukoulopoulos discussed the use of pretentious methods to prove automorphic prime number theorems. M. Radziwiłł explained his work (with collaborators) concerning the challenging arithmetic and analytic problems of evaluation of higher-moments of certain families of  $L$ -functions, while O. Balkanova discussed joint work with D. Frolenkov where new analytic techniques are used to evaluate moments of cusp forms in the fixed weight aspect. K. Soundararajan talked about an exciting new method (developed with M. Radziwiłł) that produces effective lower bounds for  $L$ -values whenever it is known from an analytic method that they are non-zero. M. Milinovich discussed new results concerning simple zeros of  $L$ -functions.

In arithmetic situations, special values of  $L$ -functions often encode periods of automorphic forms. One of the most general versions of this principle is the Gross-Prasad conjecture. An important (proven) special case of this concerns triple product  $L$ -functions, whose central value encodes information on the mass distribution of an automorphic form. As an analogue of the famous Quantum Unique Ergodicity Conjecture of Rudnick and Sarnak, one can consider the mass distribution of a holomorphic cusp of large weight or level (or both) and deduce in arithmetic situations a weak-star equidistribution result from a subconvexity bound of the corresponding triple product  $L$ -functions. P. Nelson presented an ingenious new method that is capable of reducing the problem to a subconvexity bound of degree 3 symmetric square  $L$ -functions, which in turn is accessible in a strong quantitative way (with power saving) by Munshi's circle method.

Another way of measuring the mass distribution of an automorphic form is given by the sup-norm, and several talks (by P. Maga, S. Marshall, D. Milićević and A. Saha) focussed on various aspects of this problem. This features a very beautiful combination of arithmetic and analysis and fit therefore nicely into the theme of this conference. The first step in most approaches to the sup-norm problem is an application of the trace formula or at least the spectral expansion of an automorphic kernel, often called a pre-trace formula. Trace formulae belong to the most powerful tools in the theory of automorphic forms. Being a vast generalization of the Poisson summation formula to a non-commutative setting, they translate spectral information of symmetric spaces into algebro-geometric information of the underlying group. Meanwhile the Arthur-Selberg trace formula has been developed to a point where equidistribution questions can be attacked by analytic means. This is an exciting new area with considerable potential, and was the subject of talks by J. Buttcane, J. Matz and M. Young.

A number of talks were devoted to interactions between automorphic methods and problems and various aspects of number theory. Z. Rudnick and W. Sawin presented problems concerning function fields, insisting on interesting phenomena. There were also talks on ergodic techniques (D. Kelmer and M. Lee), transfer operators (A. Pohl), Arakelov theory and invariants (A.-M. von Pippich).

A particularly remarkable application of modular forms to a problem of Fourier approximation on the real line was presented by M. Viazovska.

As fruitful as the talks presented at this workshop were informal discussion after lunch and after dinner that initiated several new projects. This included (on the very last day of the workshop), and following the talk of M. Risager on modular symbols and their distribution, the proof of a conjecture of Mazur and Rubin related to this topic. Wednesday evening was devoted to a lively and interesting problem session.

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## Abstracts

### Moments of $L$ -functions and Liouville-Green method

OLGA BALKANOVA

(joint work with Dmitry Frolenkov)

The Liouville-Green method (also called Liouville-Steklov method and WKB approximation) is a classical technique for finding approximate solutions to linear differential equations. The method was founded by Liouville and Green in 1837, and was further developed by physicists Wentzel, Kramers and Brillouin in 1920s. Nowadays it is a powerful tool to investigate a variety of problems in quantum physics and applied mathematics.

Our work concerns application of the Liouville-Green method in analytic number theory and demonstrates strong interaction between moments of  $L$ -functions and asymptotic analysis of special functions. More precisely, we address the following problems.

#### 1. MOMENTS OF CUSP FORM $L$ -FUNCTIONS

Motivated by the problem of non-existence of Landau-Siegel zeros for Dirichlet  $L$ -functions of real primitive characters, Iwaniec and Sarnak [6] studied non-vanishing of central  $L$ -values associated to primitive cusp forms  $H_{2k}(N)$  of large weight  $2k$  or large level  $N$ . In particular, a non-vanishing result in the weight aspect was proved in [6] by taking an extra smooth average over  $k \leq K$ .

In the last two decades, several attempts have been made with the purpose of removing the extra sum over weight. Fomenko [4] and Lau-Tsang [8] established that at least  $1/\log k$  of central  $L$ -values do not vanish as  $k$  tends to infinity. Proving an upper bound for the mollified second moment, Luo [10] showed that there is a strictly positive proportion of non-vanishing.

With the goal of obtaining a quantitative result, we proved in [1] an asymptotic formula for the mollified second moment and showed that the harmonic percentage of primitive forms of level one and weight  $4k \rightarrow \infty$ ,  $k \in \mathbf{N}$  for which the associated  $L$ -function at the central point is no less than  $(\log k)^{-2}$  is at least 20%.

Our proof is based on the Kuznetsov convolution formula for the twisted harmonic second moment of Hecke  $L$ -functions

$$(1) \quad \sum_{f \in H_{2k}(1)}^h \lambda_f(l) L_f^2(1/2) = (1 + (-1)^k) \left( \frac{\tau(l)}{\sqrt{l}} \left[ 2 \frac{\Gamma'}{\Gamma}(k) - \log l - 2 \log(2\pi) + 2\gamma \right] \right. \\ \left. + \frac{1}{2\sqrt{l}} \sum_{n=1}^{l-1} \tau(n) \tau(l-n) \phi_k \left( \frac{n}{l} \right) + \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n) \tau(n+l) \Phi_k \left( \frac{l}{n+l} \right) \right),$$

where  $\gamma$  is the Euler constant,  $\tau(n) = \sum_{d|n} 1$  and  $\phi_k(x)$ ,  $\Phi_k(x)$  are certain special functions defined in terms of the Gauss hypergeometric function. The Liouville-Green method serves to approximate special functions  $\phi_k(x)$ ,  $\Phi_k(x)$ , and as a

consequence, allows us to prove an asymptotic formula for the corresponding moment.

## 2. MOMENTS OF SYMMETRIC SQUARE $L$ -FUNCTIONS

Starting from the work of Iwaniec and Michel [5], moments of symmetric square  $L$ -functions received a lot of attention. Nethertheless, not much is known both in the level and weight aspects.

Our work concerns mainly the weight aspect and examines similarities and differences between the twisted first moment of symmetric square  $L$ -functions and the twisted second moment of Hecke  $L$ -functions. To this end, we prove in [2] the following exact formula for the harmonic average of symmetric square  $L$ -functions

$$(2) \quad \sum_{f \in H_{2k}(1)}^h \lambda_f(l^2) L(\text{sym}^2 f, 1/2) = \frac{1}{2\sqrt{l}} \left( -2 \log l - 3 \log 2\pi + \frac{\pi}{2} + 3\gamma + \frac{\Gamma'}{\Gamma}(k-1/4) + \frac{\Gamma'}{\Gamma}(k+1/4) \right) + \frac{\sqrt{2\pi}(-1)^k \Gamma(k-1/4)}{2\sqrt{l} \Gamma(k+1/4)} \mathcal{L}_{-4l^2}(1/2) + \frac{1}{\sqrt{l}} \sum_{1 \leq n < 2l} \mathcal{L}_{n^2-4l^2}(1/2) \psi_k \left( \frac{n^2}{4l^2} \right) + \frac{1}{l\sqrt{2}} \sum_{n > 2l} \mathcal{L}_{n^2-4l^2}(1/2) \sqrt{n} \Psi_k \left( \frac{4l^2}{n^2} \right),$$

where for  $\rho_q(n) := \#\{x \pmod{2q} : x^2 \equiv n \pmod{4q}\}$ , the generalized Dirichlet  $L$ -function is given by

$$(3) \quad \mathcal{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{\rho_q(n)}{q^s}, \quad \Re s > 1,$$

and  $\psi_k(x)$ ,  $\Psi_k(x)$  are certain special functions defined in terms of the Gauss hypergeometric function.

Comparing with exact formula (1), we note that the first sum over  $n$  in (2) is up to  $2l$  instead of  $l$ , and the generalized Dirichlet  $L$ -function  $\mathcal{L}_{n^2-4l^2}(1/2)$  in (2) corresponds to the product of two divisor functions in (1). Similarly to the case of Hecke  $L$ -functions, we apply the Liouville-Green method to study the functions  $\psi_k(x)$ ,  $\Psi_k(x)$ . Taking  $l = 1$ , exact formula (2) allows isolating an extra term of size square root of the main term. See also [9] for a similar result. For  $l > 1$ , we obtain an improvement upon the result of Ng [11].

## 3. MOMENTS OF GENERALIZED DIRICHLET $L$ -FUNCTIONS

Exact formula (2) can be applied to analyse the second moment of symmetric square  $L$ -functions. In this setting, the off-off-diagonal main term is encoded by the sums of  $\mathcal{L}_{n^2-4l^2}(1/2)$  with certain weight functions.

Furthermore, sums of special values of generalized Dirichlet  $L$ -functions are related to the Prime Geodesic Theorem as follows (see [7, 12] for details)

$$(4) \quad \Psi_{\Gamma}(x) = 2 \sum_{n \leq X} \sqrt{n^2 - 4} \mathcal{L}_{n^2-4}(1), \quad X = x^{1/2} + x^{-1/2}.$$



Currently, the best known result

$$\Psi_{\Gamma}(x) = x + O(x^{2/3+\theta/6+\epsilon})$$

is due to Soundararajan and Young [12], where  $\theta$  is a subconvexity exponent for Dirichlet  $L$ -functions. A possible approach to remove the dependence on  $\theta$  in the error term is to study moments of  $\mathcal{L}_{n^2-4}(1/2)$  in short or long intervals.

Motivated by these problems, we proved in [3] a convolution formula for

$$\sum_{n=1}^{\infty} \omega(n) \mathcal{L}_{n^2-4l^2}(s)$$

in terms of moments of symmetric square  $L$ -functions. As a consequence, using the Liouville-Green method, we derived the following analogue of the binary additive divisor problem

$$(5) \quad \sum_{2 < n < X} \mathcal{L}_{n^2-4}(1/2 + ir) = XP_1(\log X) + O(X^{2/3+2\theta/3+\epsilon}), \quad |r| < X^{\epsilon}.$$

Moreover, any improvement of the error term in (5) will result in an improvement of the error term in the Prime Geodesic Theorem.

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## Higher weight on $GL(3)$

JACK BUTTCANE

Up to isomorphism, the representations of  $K = SO(3, \mathbb{R})$  are given by the Wigner- $\mathcal{D}$  matrices  $\mathcal{D} : K \rightarrow SO(2d + 1, \mathbb{C})$ ,  $d \geq 0$ . For the particular groups  $G = SL(3, \mathbb{R})$  and  $\Gamma = SL(3, \mathbb{Z})$ , we define the spaces  $\mathcal{A}^d$  to be the set of functions  $f : \Gamma \backslash G \rightarrow \mathbb{C}^{2d+1}$ ,  $f(gk) = f(g)\mathcal{D}(k)$  which are smooth, bounded and have bounded derivatives. The details of the Langlands spectral decomposition are worked out in [2], and we may consider the space  $\mathcal{A}_{\text{cusp}}^d$  of cusp forms, whose integrals over the unipotent subgroups are zero.

For a cusp form  $\varphi \in \mathcal{A}_{\text{cusp}}^d$  which is an eigenfunction of the Casimir operators  $\Delta_i$ , we parameterize its eigenvalues as

$$\Delta_1 \varphi = \left(1 - \frac{\mu_1^2 + \mu_2^2 + \mu_3^2}{2}\right) \varphi, \quad \Delta_2 \varphi = \mu_1 \mu_2 \mu_3 \varphi,$$

where  $\mu \in \mathbb{C}^3$  satisfies  $\mu_1 + \mu_2 + \mu_3 = 0$ . The details of the structure of the representations and  $(\mathfrak{g}, K)$ -modules of these cusp forms and their minimal-weight Whittaker functions have been worked out in [11, 7, 10, 9, 8] and are collected in [3].

The cusp forms may be collected into families based on the type of representation they come from and the minimal-weight form contained therein. These families are the spherical principal series forms at  $d = 0$ , the non-spherical principal series forms at  $d = 1$ , and the generalized principal series forms at each  $d \geq 2$ . By computing a generalization of Stade's formula, a type of Kontorovich-Lebedev inversion can be proven on each of the spaces of spectral parameters

$$\begin{aligned} \mathcal{F}^0 &= \mathcal{F}^1 = \{\mu \in i\mathbb{R}^3 \mid \mu_1 + \mu_2 + \mu_3 = 0\}, \\ \mathcal{F}^d &= \left\{ \left( \frac{d-1}{2} + it, -\frac{d-1}{2} + it, -2it \right) \mid t \in \mathbb{R} \right\}, d \geq 2. \end{aligned}$$

Then a Kuznetsov-type trace formula may be built for each type of minimal-weight form by considering Poincaré series defined over an inverse Whittaker transform. The formulas generally express an arithmetically weighted spectral sum

$$\sum_{\varphi \in \mathcal{S}^{d*}} \frac{\overline{\lambda_\varphi(m)} \lambda_\varphi(n)}{L(\text{Ad}^2 \varphi, 1)} F(\mu_\varphi) + \text{Eisenstein series}$$

as sums of  $GL(3)$  Kloosterman sums

$$\begin{aligned} &\delta_{\substack{|m_1|=|n_1| \\ |m_2|=|n_2|}} \int_{\mathcal{F}^d} F(\mu) \text{spec}^d(\mu) d\mu + \text{intermediate Weyl-element terms} \\ &+ \sum_{\varepsilon \in \{\pm 1\}} \sum_{c \in \mathbb{N}^2} \frac{S_{w_l}(\psi_m, \psi_n, c)}{c_1 c_2} H_{w_l}^d(F; \left( \frac{\varepsilon_2 m_1 n_2 c_2}{c_1^2}, \frac{\varepsilon_1 m_2 n_1 c_1}{c_2^2} \right)). \end{aligned}$$

The spectral measure may be computed explicitly and  $H_{w_l}^d$  is given by a kernel integral transform of  $F$  where the kernel function can be expressed as a Mellin-Barnes integral. This is all done in the papers [4, 5]. Also in those papers,  $F(\mu)$  is taken to approximate a sharp cut-off, giving Weyl laws for each family.

A long-term goal is to apply this analysis to study exponential sums on  $GL(3)$ , and a simple application (using a theorem of Wallach [12]) gives the first arithmetic Kuznetsov formula [6] in the case  $m_1 n_2, m_2 n_1 > 0$ :

$$\sum_{c \in \mathbb{N}^2} \frac{S_{w_l}(\psi_m, \psi_n, c)}{c_1 c_2} f\left(\frac{2\sqrt{m_1 n_2 c_2}}{c_1}, \frac{2\sqrt{m_2 n_1 c_1}}{c_2}\right) = \text{weight 0 spectral sum} \\ + \text{weight 1 spectral sum,}$$

where the analytic weight function in the spectral sums is given by a Whittaker transform of the test function  $f$ . This in turn can be used to prove good cancellation in smooth sums

$$\sum_{c \in \mathbb{N}^2} \frac{S_{w_l}(\psi_m, \psi_n, c)}{c_1 c_2} f\left(\frac{X_1 m_1 n_2 c_2}{c_1^2}, \frac{X_2 m_2 n_1 c_1}{c_2^2}\right) \ll (X_1 X_2)^{\frac{5}{14} + \epsilon}.$$

This improves over [1] by isolating a particular choice of sign and dropping two terms  $X_i^{\frac{1}{2} + \epsilon}$  coming from a partial inversion formula used in that paper.

The last part of the talk was on the possibility of generalizing these constructions to  $GL(n)$  and other groups.

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## Archimedean Newforms for $\mathrm{GL}_n$

PETER HUMPHRIES

There is a well-known theory of decomposing spaces of automorphic forms into subspaces spanned by newforms and oldforms, and associated to a newform is its conductor. This theory can be reinterpreted as a local statement, and generalised to  $\mathrm{GL}_n$ , as distinguishing certain vectors in a generic irreducible admissible representation  $\pi$  of  $\mathrm{GL}_n(F)$ , where  $F$  is a nonarchimedean local field, and associating to  $\pi$  a conductor (or rather, a conductor exponent). Such a local theory was previously not well understood for archimedean fields  $F$ ; my goal is to introduce and prove this theory in this hitherto unexplored setting.

### 1. CLASSICAL THEORY

**1.1. The Newform.** Let  $\mathcal{S}_k(\Gamma_1(q))$  denote the space of holomorphic Hecke cusp forms of weight  $k$ , level  $q$ , and arbitrary nebentypus. This space has the decomposition

$$\bigoplus_{q_1 q_2 = q} \bigoplus_{d|q_2} \iota_d \mathcal{S}_k^*(\Gamma_1(q_1)),$$

where  $\iota_d f(z) = f(dz)$  and  $\mathcal{S}_k^*(\Gamma_1(q_1))$  denotes the subspace of  $\mathcal{S}_k(\Gamma_1(q_1))$  spanned by newforms. A similar decomposition holds for Hecke–Maaß cusp forms and for Eisenstein series. Given a newform  $f \in \mathcal{S}_k^*(\Gamma_1(q_1))$  with  $q_1 q_2 = q$  and  $q_1 < q$ , the space of oldforms in  $\mathcal{S}_k(\Gamma_1(q))$  associated to  $f$  is

$$\bigoplus_{d|q_2} \mathbb{C} \iota_d f,$$

which has dimension  $\tau(q_2) = \tau(\frac{q}{q_1})$ . The Mellin transform of a newform  $f \in \mathcal{S}_k^*(\Gamma_1(q))$  on the imaginary axis gives the completed  $L$ -function:

$$\Lambda(s, f) = \int_0^\infty f(iy) y^s \frac{dy}{y}.$$

This can be interpreted as the newform being a test vector for the (global)  $\mathrm{GL}_2 \times \mathrm{GL}_1$  Rankin–Selberg integral. This holds also for weight one and even weight zero Maaß newforms. For odd weight zero Maaß newforms, however, this integral vanishes; the correct test vector is not  $f$  but rather  $R_0 f$ , where  $R_0$  is the weight zero raising operator.

**1.2. The Conductor.** The conductor of a newform  $f \in \mathcal{S}_k^*(\Gamma_1(q))$  is  $q$ . The conductor of an Eisenstein series newform is multiplicative: given an Eisenstein series newform  $E_{\chi_1, \chi_2}(z, s)$  of conductor  $q$ ,  $\chi_1, \chi_2$  must be primitive characters of conductors  $q_1, q_2$  such that  $q_1 q_2 = q$ . Moreover, the conductor of a newform is inductive: to a Hecke Größencharakter  $\psi$  of a quadratic extension  $E$  of  $\mathbb{Q}$  with conductor  $\mathfrak{q}$  (as an ideal of  $\mathcal{O}_E$ ), one can associate via automorphic induction a (not necessarily cuspidal) newform  $f_\psi$  of conductor  $N_{E/\mathbb{Q}}(\mathfrak{q}) \Delta_{E/\mathbb{Q}}$ , where  $\Delta_{E/\mathbb{Q}}$  is the discriminant of  $E/\mathbb{Q}$ . Finally, the conductor of a newform appears in the functional equation in the form  $q^{-s/2}$ .

2. NONARCHIMEDEAN THEORY

The classical theory of newforms and conductors can be studied purely locally, and results analogous to those in the classical setting can be proven. Let  $F$  be a nonarchimedean field with ring of integers  $\mathcal{O}$  and maximal ideal  $\mathfrak{p}$ . The local analogue of  $\Gamma_1(q)$  for  $\mathrm{GL}_n$  is the congruence subgroup  $K_1(\mathfrak{p}^m)$  of  $\mathrm{GL}_n(\mathcal{O})$  defined by

$$K_1(\mathfrak{p}^m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_n(\mathcal{O}) : a \in \mathrm{GL}_{n-1}(\mathcal{O}), b \in \mathrm{Mat}_{(n-1) \times 1}(\mathcal{O}), \right. \\ \left. c \in \mathrm{Mat}_{1 \times (n-1)}(\mathfrak{p}^m), d \in 1 + \mathfrak{p}^m \right\}.$$

**2.1. The Newform.** Jacquet–Piatetski-Shapiro–Shalika [JP-SS81] proved that for a generic irreducible admissible representation  $\pi$  of  $\mathrm{GL}_n(F)$ , there exists a minimal nonnegative integer  $m = c(\pi)$ , such that  $\pi$  has a nontrivial  $K_1(\mathfrak{p}^m)$ -fixed vector. Moreover, the space of  $K_1(\mathfrak{p}^{c(\pi)})$ -fixed vectors is one-dimensional, so there exists a unique such vector up to scalar multiplication, the (local) newform of  $\pi$ . Reeder [Ree91] showed that the dimension of the space of  $K_1(\mathfrak{p}^m)$ -fixed vectors of  $\pi$  with  $m \geq c(\pi)$ , the space of (local) oldforms of level  $\mathfrak{p}^m$ , is  $\binom{m-c(\pi)+n-1}{n-1}$ . The newform of  $\pi$  in the Whittaker model is a test vector for the local  $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$  Rankin–Selberg integral for any unramified representation  $\pi'$  of  $\mathrm{GL}_{n-1}(F)$ , in that this integral is equal to the local  $L$ -function  $L(s, \pi \times \pi')$ .

**2.2. The Conductor Exponent.** The conductor exponent of  $\pi$  is  $c(\pi)$ . This is additive: if  $\pi = \boxplus_{j=1}^r \pi_j$  with each  $\pi_j$  an essentially square-integrable representation of  $\mathrm{GL}_{n_j}(F)$  such that  $n_1 + \dots + n_r = n$ , then  $c(\pi) = \sum_{j=1}^r c(\pi_j)$ . It is also inductive: if  $E/F$  is an extension of degree  $n_{E/F}$ , and

$$\pi = \mathrm{Ind}_{\mathrm{GL}_n(E)}^{\mathrm{GL}_{n n_{E/F}}(F)} \Pi$$

for some generic irreducible admissible representation  $\Pi$  of  $\mathrm{GL}_n(E)$  with conductor exponent  $c(\Pi)$ , then

$$c(\pi) = c \left( \mathrm{Ind}_{\mathrm{GL}_n(E)}^{\mathrm{GL}_{n n_{E/F}}(F)} \Pi \right) = f_{E/F} c(\Pi) + d_{E/F} n,$$

where  $d_{E/F}$  denotes the valuation of the discriminant of  $E/F$  and  $f_{E/F}$  denotes the residual degree of  $E/F$ . Finally, the conductor exponent appears in the epsilon factor, defined via the local functional equation, namely

$$\varepsilon(s, \pi, \psi) = \varepsilon(1/2, \pi, \psi) q^{-c(\pi)(s-1/2)},$$

where  $q = \#\mathcal{O}/\mathfrak{p}$ .

### 3. ARCHIMEDEAN THEORY

Let  $F \in \{\mathbb{R}, \mathbb{C}\}$  be an archimedean local field. I prove that there is an analogous theory of the newform and the conductor exponent of generic irreducible admissible representations  $\pi$  of  $\mathrm{GL}_n(F)$ . In this setting, there is no immediate analogue of a congruence subgroup; instead, the conductor exponent  $c(\pi)$  of  $\pi$  is a measure of the size of the minimal  $K$ -type of  $\pi$  whose restriction to  $K_{n-1}$  contains the trivial representation, and the newform is the unique vector up to scalar lying in this  $K$ -type that is  $K_{n-1}$ -invariant.

As in the nonarchimedean setting, there is a theory of oldforms (of essentially identical dimensions to Reeder's calculations [Ree91]); these correspond to the non-minimal  $K$ -types of  $\pi$  that contain the trivial representation of  $K_{n-1}$ . However, so far I cannot yet prove in full generality and can only conjecture and prove in certain cases that the newform in the Whittaker model is a test vector for the local  $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$  Rankin–Selberg integral.

I show that the conductor exponent is again additive and inductive, just as in the nonarchimedean setting. Finally, the epsilon factor  $\varepsilon(s, \pi, \psi)$  is equal to  $i^{c(\pi)}$ ; note that unlike the nonarchimedean setting, this only determines  $c(\pi)$  modulo 4.

**3.1. The Classical Picture.** We return to the classical setting to interpret these results. The infinite component  $\pi_\infty$  of the automorphic representation containing the adèlic lift of a holomorphic newform of weight  $k$  has conductor exponent  $c(\pi_\infty) = k$ ; similarly,  $c(\pi_\infty) = 0$  for an even weight zero Maaß newform and  $c(\pi_\infty) = 1$  for a weight one Maaß newform. Finally,  $c(\pi_\infty)$  is equal to 2 for an odd weight zero Maaß newform  $f$ ; this reflects the fact that  $R_0 f$ , the test vector for the (global)  $\mathrm{GL}_2 \times \mathrm{GL}_1$  Rankin–Selberg integral, is of weight two.

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### Subconvexity without the $\delta$ -method

ROMAN HOLOWINSKY

(joint work with Paul Nelson)

The subconvexity problem for automorphic  $L$ -functions of degree  $> 2$  has seen many recent advances, but a uniform approach to the problem remains elusive. For instance, consider the problem of bounding  $L(\pi \otimes \chi, \frac{1}{2})$ , where

- $\pi$  is a fixed cusp form on  $\mathrm{PGL}_3(\mathbb{Z})$ , not necessarily self-dual, and
- $\chi$  traverses a sequence of Dirichlet characters  $\chi$  of (say) prime conductor  $M$  tending off to  $\infty$ .

Munshi has recently established the first subconvex bound in this setting by showing that the estimate

$$|L(\pi \otimes \chi, \frac{1}{2})| \ll M^{3/4-\delta}$$

holds for any fixed  $\delta < 1/1612$  provided that  $\pi$  satisfies the Ramanujan–Selberg conjecture. Furthermore, in a recent preprint, Munshi removes the Ramanujan–Selberg assumption and extends the estimate to  $\delta < 1/308$ .

The main tool in his proof is the introduction of a novel “GL<sub>2</sub>  $\delta$ -symbol method,” whereby one detects an equality of integers  $m = n$  by averaging several instances of the Petersson trace formula, roughly like so:

$$\begin{aligned} \delta(m, n) &= \frac{1}{Q^*} \sum_{q \sim Q} \sum_{\psi(q)} \sum_{f \in S_k(q, \psi)} w_f \overline{\lambda_f(m)} \lambda_f(n) \\ &\quad - 2\pi i^{-k} \frac{1}{Q^*} \sum_{q \sim Q} \sum_{\psi(q)} \sum_{c \equiv 0(q)} \frac{S_\psi(m, n, c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \end{aligned}$$

This approach differs fundamentally from traditional moment method techniques, rather than embedding the  $L$ -function in a family, one views it as the “diagonal” contribution to the “auxiliary first moment” induced by the Petersson trace formula above. More specifically, Munshi begins by writing roughly  $L$  copies of the main object of interest in the following way:

$$(1) \quad \sum_{n \sim M^{3/2}} A(n)\chi(n) = \frac{1}{L^*} \sum_{\ell \sim L} \overline{\chi}(\ell) \sum_{n \sim M^{3/2}} A(n) \sum_{r \sim M^{3/2}L} \chi(r)\delta(r, n\ell).$$

The sum is over primes  $\ell$  of size  $L$  which are coprime with  $M$ , and  $L^*$  is the appropriate normalizing factor. The coefficients  $A(n)$  are those that would appear in the Dirichlet series expansion of  $L(\pi, s)$ . Using several copies of Petersson’s trace formula, Munshi then writes

$$\delta(r, n\ell) = \mathcal{F} - \mathcal{O}$$

with the sum of Fourier coefficients given roughly by

$$(2) \quad \mathcal{F} \approx \frac{1}{P^2} \sum_{p \sim P} \sum_{\psi(p)} \sum_{f \in S_k(pM, \psi)} \omega_f \overline{\lambda_f(r)} \lambda_f(n\ell)$$

and the sum of Kloosterman sums given roughly by

$$(3) \quad \mathcal{O} \approx \frac{1}{P^2} \sum_{p \sim P} \sum_{\psi(p)} \sum_{c \ll \sqrt{ML}/P} \frac{1}{cpM} S_\psi(r, n\ell; cpM) J_{k-1}\left(\frac{4\pi\sqrt{n\ell r}}{cpM}\right).$$

Munshi inserts this pair of formulas into (1) and bounds the resulting contributions separately through a sequence of dual summation formulas, spectral formulas, and local evaluations of various character sums, ultimately resulting in an application of Cauchy-Schwarz which breaks the original problem structure and allows Munshi to achieve his stated results.

In this talk, we demonstrate how one may avoid the use of the GL<sub>2</sub>  $\delta$ -method introduced by Munshi in order to obtain a more direct proof, with a quantitative

strengthening of the subconvexity exponent, by appealing instead to an application of Poisson summation. The main formula that one instead appeals to is the following: for any integer  $u$  and parameter  $q^\epsilon < R < q^{1-\epsilon}$  (say),

$$(4) \quad \epsilon(\bar{\chi})^2 \chi(u) = \frac{\sqrt{q}}{R} \sum_{r \ll R}^{\text{smooth}} \bar{\chi}(r) K_q(ur) - \sum_{0 \neq h \ll q/R}^{\text{smooth}} \frac{\sum_{a(q)} \bar{\chi}(a) K_q(ua) e_q(-ah)}{\sqrt{q}}.$$

Here  $K_q(a) := q^{-1/2} S(a, 1, q)$  is the Kloosterman sum and  $\epsilon(\bar{\chi})$  denotes the Gauss sum normalized so that  $|\epsilon(\bar{\chi})| = 1$ . The left hand side of (4) is obtained by degenerating the  $h \neq 0$  terms on the right hand side to  $h = 0$ . We interpret this identity as a formula for  $\chi(u)$ . It may be surprising that it is useful to write  $\chi(u)$  in such a manner, however, this formula rests hidden inside Munshi's argument as the key crucial identity.

We demonstrate how this key formula (4) is used to achieve subconvexity for  $L(\pi \otimes \chi, \frac{1}{2})$  and how one would discover similar formulas, in other applications, by first appealing to a sketch of proof that utilizes the  $GL_2$   $\delta$ -method as a guide and then proceeding to remove it. This is joint work with Paul Nelson.

### Shrinking targets for homogenous flows

DUBI KELMER

For a dynamical system on a probability space, the shrinking target problem studies the question of how fast can a sequence of targets shrink so that a typical orbit will keep hitting the targets infinitely often. A natural bound for this rate comes from the easy half of Borel-Cantelli, stating that for any sequence of sets,  $\{B_m\}_{m \in \mathbb{N}}$ , if  $\sum_{m=1}^{\infty} \mu(B_m) < \infty$  then for almost all starting points, the orbit will eventually miss the targets. For chaotic dynamical systems, it could be expected that this bound is sharp, and much work has gone into proving this in various examples of fast mixing dynamical systems (under some regularity restrictions on the shrinking sets).

In my talk I will describe joint work with Shucheng Yu, introducing a new method for attacking the problem by establishing effective mean ergodic theorem for flows on homogenous spaces. This method works for any monotone family of shrinking targets in a locally symmetric space and applies also for unipotent flows with arbitrarily slow polynomial mixing rate. In particular we prove a logarithm law for the first hitting time of a generic orbit to the shrinking targets.

To describe our result in more detail, let  $G$  denote a connected semisimple Lie group with finite center and no compact factors, let  $\Gamma \leq G$  be an irreducible lattice, and let  $\mu$  denote the  $G$ -invariant probability measure on  $\mathcal{X} = \Gamma \backslash G$ , coming from the Haar measure of  $G$ . We fix once and for all a maximal compact subgroup  $K \leq G$  say that a subset  $B \subseteq \mathcal{X}$  is spherical if it is invariant under the right action of  $K$  (note that spherical sets can be identified with subsets of the locally symmetric space  $\Gamma \backslash G/K$ ). We say that a family  $\{B_t\}_{t>0}$  is a monotone family of shrinking targets if  $B_t \subseteq B_s$  when  $t \geq s$  and  $\mu(B_t) \rightarrow 0$ , and we say it is a family of spherical shrinking targets if all sets are spherical.



One-parameter flows on  $\mathcal{X} = \Gamma \backslash G$  are given by the right action of one-parameter subgroups,  $\{h_t = \exp(tX_0) : t \in \mathbb{R}\} \leq G$ , and the corresponding discrete time flow is then given by the action of the discrete subgroup  $H = \{h_m\}_{m \in \mathbb{Z}}$ . We will always assume that the subgroup is unbounded in which case the action of any unbounded subgroup is ergodic and mixing. Our main result is the following.

**Theorem.** *Assume that either  $G$  has property (T), or that  $G$  is of real rank one, Let  $\{B_t\}_{t>0}$  denote a monotone family of spherical shrinking targets in  $\mathcal{X} = \Gamma \backslash G$ . Let  $\{h_m\}_{m \in \mathbb{Z}}$  denote an unbounded discrete time flow on  $\mathcal{X}$ , and let  $\tau_{B_n}(x) = \min\{m \in \mathbb{N} : xh_m \in B_n\}$ . Then for a.e.  $x \in \mathcal{X}$*

$$\lim_{t \rightarrow \infty} \frac{\log(\tau_{B_t}(x))}{-\log(\mu(B_t))} = 1.$$

In many cases we also prove a dynamical Borel-Canteli result, showing that the condition that  $\sum_{m=1}^{\infty} \mu(B_m) = \infty$  implies that for almost all starting points, the orbit will eventually hit the targets.

### Pretentious methods for $L$ -functions

DIMITRIS KOUKOULOPOULOS

(joint work with K. Soundararajan)

Let  $\{f(p)\}_{p \text{ prime}}$  be a sequence that we wish to estimate on average. A typical strategy is to extend  $f(p)$  to a multiplicative function  $f : \mathbb{N} \rightarrow \mathbb{C}$  and study first the partial sums of  $f(n)$  with  $n$  running over all integers. Often, this is accomplished by studying the corresponding Dirichlet series which, by multiplicativity, has an Euler product:

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left( 1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots \right).$$

For many important functions  $f$ , the Dirichlet series  $L(s, f)$  satisfies the following axioms:

- (i) The factors of the Euler product can be written as  $\prod_{j=1}^D (1 - \alpha_j(p)/p^s)^{-1}$ . The integer  $D$  is called the degree of  $L$ .
- (ii) We have  $|\alpha_j(p)| \leq 1$  for all  $j$  and all  $p$ . We then say that  $L(s, f)$  satisfies Ramanujan’s conjecture. In particular, this means that  $L(s, f)$  converges absolutely for  $\Re(s) > 1$ .
- (iii) Even though *a priori*  $L(s, f)$  is defined only for  $\Re(s) > 1$ , it admits a meromorphic continuation to  $\mathbb{C}$ , with its only poles located at  $s = 0$  and at  $s = 1$ .
- (iv)  $L(s, f)$  satisfies a certain functional equation that relates its value at  $s$  with its value at  $1 - s$ .

As Riemann understood, in this fortuitous circumstance the asymptotic behaviour of the sum  $\sum_{p \leq x} f(p)$  is controlled by the location of the zeroes of  $L(s, f)$  in the so-called *critical strip*  $\{s \in \mathbb{C} : 0 \leq \Re(s) \leq 1\}$ . The Generalized Riemann Hypothesis

is the conjecture that, under Axioms 1-4, all such zeroes lie on the central line  $\Re(s) = 1/2$ , that is a line of symmetry of  $L(s, f)$  and is called the *critical line*.

A careful examination of the classical proofs of the Prime Number Theorem reveals that we don't actually use the full strength of the above axioms, but rather two simple consequences of them:

- (a)  $|\Lambda_f| \leq D\Lambda$ , where  $\Lambda_f(n)$  are the coefficients of the Dirichlet series  $-(L'/L)(s, f)$ ;
- (b) the partial sums of  $f$  satisfy the asymptotic formula

$$(1) \quad \sum_{n \leq x} f(n) = xP_f(\log x) + O(x^{1-\delta}) \quad (x \geq q(f)^L),$$

where  $\delta > 0$  and  $L \geq 1$  are two constants,  $q(f)$  denotes the so-called *conductor* of  $L(s, f)$  that appears in its functional equation, and  $P_f$  is a polynomial whose degree equals  $k - 1$ , where  $k$  is the order of the pole of  $L(s, f)$  at  $s = 1$ .

A natural question then becomes what kind of general Prime Number Theorems we can prove for the extended family of multiplicative defined by Axioms (a) and (b). Building on the work in [4, 5] that dealt with the case  $D = 1$ , we prove a general theorem that characterizes functions satisfying a variation of the above axioms:

**Theorem 1.** *Let  $f$  be a multiplicative function such that  $|\Lambda_f| \leq D\Lambda$  and*

$$(2) \quad \sum_{n \leq x} f(n) \ll \frac{x}{(\log x)^A} \quad (x \geq 2).$$

*If  $A > D$ , then there exist real numbers  $\gamma_1, \dots, \gamma_k$ ,  $0 \leq k \leq D$ , such that*

$$(3) \quad \sum_{p \leq x} (f(p) + p^{i\gamma_1} + \dots + p^{i\gamma_k}) = o_{x \rightarrow \infty} \left( \frac{x}{\log x} \right).$$

*If, in addition, we know that  $L(1 + it, f) \neq 0$  for all  $t$ , then  $k = 0$  and*

$$\sum_{p \leq x} f(p) = o_{x \rightarrow \infty} \left( \frac{x}{\log x} \right).$$

The above theorem is proven by combining the theory of *pretentious multiplicative functions* with sieve methods. It is best possible, in the sense that if  $A < D$ , then we can find examples of functions  $f$  that satisfy its hypotheses, but for which (3) fails. The error term in right hand side of (3) admits a precise quantitative form and improves as  $A \rightarrow \infty$ .

A suitable adaptation of the methods leading to (1) under the stronger hypotheses imposed by Axioms (a) and (b) allows us to give a new proof of the classical zero-free regions for automorphic  $L$ -functions. As applications, we reprove general Prime Number Theorems with an error term that is as strong as the classical de la Vallée-Poussin type arguments permit.

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## Analytic continuation of the heat kernel and applications

JÜRIG KRAMER

## 1. MOTIVATION

Let  $\mathbb{H} := \{z = x + iy \in \mathbb{C} \mid y > 0\}$  denote the upper half-plane and let  $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$  be a Fuchsian subgroup of the first kind acting by fractional linear transformations on  $\mathbb{H}$ ; to simplify our exposition, we assume that  $\Gamma$  is cocompact and torsionfree. We let  $M$  denote the associated quotient space  $\Gamma \backslash \mathbb{H}$ , which has the structure of a compact Riemann surface.

The upper half-plane  $\mathbb{H}$  as well as the compact Riemann surface  $M$  are equipped with the hyperbolic metric having constant negative curvature equal to  $-1$ . The Laplacian with respect to this metric acting on smooth functions on  $\mathbb{H}$  is given as

$$\Delta_{\mathrm{hyp}} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right);$$

the corresponding Laplacian induced on  $M$  is denoted in the same way.

Introducing the function

$$(1) \quad K_{\mathbb{H}}(t; \rho) := \frac{\sqrt{2} e^{-t/4}}{(4\pi t)^{3/2}} \int_{\rho}^{\infty} \frac{u e^{-u^2/(4t)}}{(\cosh(u) - \cosh(\rho))^{1/2}} du \quad (t \in \mathbb{R}_{>0}; \rho \in \mathbb{R}_{\geq 0}),$$

the heat kernel on  $\mathbb{H}$  associated to  $\Delta_{\mathrm{hyp}}$  is given as

$$K_{\mathbb{H}}(t; z, z') = K_{\mathbb{H}}(t; \rho_{z, z'}) \quad (t \in \mathbb{R}_{>0}; z, z' \in \mathbb{H}),$$

where  $\rho_{z, z'} := \mathrm{dist}_{\mathrm{hyp}}(z, z')$  is the hyperbolic distance between  $z$  and  $z'$ . Now, the heat kernel on  $M$  associated to  $\Delta_{\mathrm{hyp}}$  is obtained by averaging over the group  $\Gamma$ , i.e., we have

$$K_M(t; z, z') = \sum_{\gamma \in \Gamma} K_{\mathbb{H}}(t; z, \gamma z') \quad (t \in \mathbb{R}_{>0}; z, z' \in M).$$

By means of the eigenvalues  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  of  $\Delta_{\text{hyp}}$  and the corresponding eigenfunctions  $\varphi_j$  ( $j = 0, 1, 2, \dots$ ), we obtain the spectral expansion

$$(2) \quad K_M(t; z, z') = \sum_{j=0}^{\infty} \varphi_j(z) \overline{\varphi_j}(z') e^{-\lambda_j t}.$$

From the expansion (2), we immediately see that the heat kernel on  $M$  can be analytically continued to the right half-plane  $\mathcal{H} := \{\zeta = t + is \in \mathbb{C} \mid t > 0\}$  by means of the formula

$$K_M(\zeta; z, z') = \sum_{j=0}^{\infty} \varphi_j(z) \overline{\varphi_j}(z') e^{-\lambda_j \zeta}.$$

The problem which we address in this talk is the question whether the analytic continuation  $K_M(\zeta; z, z')$  just obtained can be further continued beyond the right half-plane  $\mathcal{H}$ , in particular to the imaginary axis of the complex  $\zeta$ -plane.

## 2. MAIN RESULTS

In the sequel, we will use an alternative formula for the heat kernel on  $\mathbb{H}$  than the one given by (1).

**Lemma 1** (Gruet’s formula). *With the above notation, the function (1) can alternatively be represented as*

$$(3) \quad K_{\mathbb{H}}(t; \rho) = c(t) \int_0^{\infty} \frac{e^{-u^2/(4t)} \sinh(u) \sin(\pi u/(2t))}{(\cosh(u) + \cosh(\rho))^{3/2}} du,$$

with an elementary function  $c(t)$ . For a proof, we refer to [1].

By means of Gruet’s formula in combination with the Stieltjes integral representation of the heat kernel

$$K_M(\zeta; z, z') = \int_0^{\infty} K_{\mathbb{H}}(\zeta; \rho) dN(\rho; z, z')$$

with the counting function

$$N(\rho; z, z') := \#\{\gamma \in \Gamma \mid \rho_{z, \gamma z'} = \text{dist}_{\text{hyp}}(z, \gamma z') \leq \rho\},$$

we obtain an alternative way to analytically continue  $K_M(t; z, z')$  to the right half-plane  $\mathcal{H}$ .

In order to achieve a continuation of  $K_M(\zeta; z, z')$  beyond the right half-plane  $\mathcal{H}$ , we will shift the path of integration in (3) suitably into the complex  $w$ -plane ( $w = u + iv$ ), however in doing so, we have to take care of the multivaluedness and the emerging singularities of the complexified integrand in (3).

Letting  $\mathcal{D}$  denote the domain in the complex  $w$ -plane with vertical cuts from  $\pm \rho_{z, \gamma z'} + k\pi i$  to  $\pm \rho_{z, \gamma z'} + (k + 2)\pi i$  ( $k = \pm 1, \pm 5, \dots$ ), we observe that the function

$$f(w) := \sum_{\gamma \in \Gamma} \frac{1}{(\cosh(w) + \cosh(\rho_{z, \gamma z'}))^{3/2}}$$

becomes a univalued holomorphic function in the domain  $\mathcal{D}$ . Then, an application of Cauchy's theorem shows

**Lemma 2.** *Choosing a path  $\eta$  in  $\mathcal{D}$  starting at the origin with positive slope across the first quadrant and bypassing the cuts which are in the way, we have for  $\zeta \in \mathcal{H}$  that*

$$(4) \quad K_M(\zeta; z, z') = c(\zeta) \int_{\eta} f(w) e^{-w^2/(4\zeta)} \sinh(w) \sin(\pi w/(2\zeta)) dw + E(\zeta; \eta, \delta),$$

where

$$(5) \quad E(\zeta; \eta, \delta) = c(\zeta) \sum_{\gamma \in \Gamma} \sum_{\substack{k=1 \\ k \equiv 1 \pmod{4}}}^{\sim \rho_{z, \gamma z'}} \int_{\eta(\rho_{z, \gamma z'}, k; \delta)} \frac{e^{-w^2/(4\zeta)} \sinh(w) \sin(\pi w/(2\zeta))}{(\cosh(w) + \cosh(\rho_{z, \gamma z'}))^{3/2}} dw;$$

here  $\eta(\rho_{z, \gamma z'}, k; \delta)$  denotes a rectangular path around the cut from  $\rho_{z, \gamma z'} + k\pi i$  to  $\rho_{z, \gamma z'} + (k + 2)\pi i$  at distance  $\delta > 0$ , being small enough.

By construction, the first summand on the right-hand side of (4) can be analytically continued from  $\mathcal{H}$  across the imaginary axis (depending on the minimal positive slope of  $\eta$ ). With regard to the second summand on the right-hand side of (4), we have the following

**Theorem.** *By choosing  $\delta > 0$  sufficiently small, the quantity  $E(\zeta; \eta, \delta)$  is finite for  $\zeta = is$  provided that  $s > 0$  is small enough. Consequently, the heat kernel  $K_M(\zeta; z, z')$  can be continuously continued from  $\mathcal{H}$  to a suitable portion of the positive imaginary axis.*

For a proof, one shows first that the quantity

$$\sum_{\gamma \in \Gamma} \sum_{\substack{k=5 \\ k \equiv 1 \pmod{4}}}^{\sim \rho_{z, \gamma z'}} \int_{\eta(\rho_{z, \gamma z'}, k; \delta)} \frac{e^{-w^2/(4is)} \sinh(w) \sin(\pi w/(2is))}{(\cosh(w) + \cosh(\rho_{z, \gamma z'}))^{3/2}} dw$$

converges absolutely provided that  $s > 0$  is small enough. Rewriting the remaining contribution for  $\zeta \in \mathcal{H}$  as a Stieltjes integral yields

$$\int_0^{\infty} \int_{\eta(\rho, 1; \delta)} \frac{e^{-w^2/(4\zeta)} \sinh(w) \sin(\pi w/(2\zeta))}{(\cosh(w) + \cosh(\rho))^{3/2}} dw dN(\rho; z, z').$$

Next we integrate by parts and use the asymptotic expansion of  $N(\rho; z, z')$  given by a lead term  $\Sigma(\rho; z, z')$  and a remainder term  $R(\rho; z, z')$ . The contribution belonging to the lead term is then easily seen to have an analytic continuation from  $\mathcal{H}$  across the imaginary axis. As far as the contribution belonging to the remainder term is concerned, we use the fine structure of  $R(\rho; z, z')$  established in the article [2] to construct the desired analytic continuation (up to a rest term which can be handled separately by absolute convergence).

**Remark.** By treating more than just the first summand of the sum over  $k$  in (5) in the above manner, the range of the continuation  $K_M(\zeta; z, z')$  to the imaginary

axis can be arbitrarily extended. Another approach to achieve this effect is to use iteratively the semigroup law for the heat kernel.

**Applications.** As far as applications of our continuation of the heat kernel to the imaginary axis are concerned, it remains to control the growth of  $K_M(is; z, z')$  in  $s$ . Boundedness of our continuation for  $s \rightarrow \infty$  would yield the best possible bound for the sup-norm of the  $\varphi_j$ 's. However, this seems difficult to achieve. A more promising next step is to show that  $K_M(is; z, z') = O(s^\alpha)$  for some  $\alpha > 0$  as  $s \rightarrow \infty$ ; for example,  $\alpha = 1/2$  leads to the standard sup-norm bound for Maass forms.

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### Effective equidistribution of rational points on expanding horospheres

MIN LEE

(joint work with Jens Marklof)

For  $d \geq 2$ , let  $G = SL_d(\mathbb{R})$  and  $\Gamma = SL_d(\mathbb{Z})$ . Here  $G$  acts by right multiplication on the quotient space  $\Gamma \backslash G$ , which carries a unique  $G$ -invariant probability measure  $\mu$ . Set  $\Phi_t = \begin{pmatrix} e^t 1_{d-1} & \mathbf{0} \\ \mathbf{0} & e^{-(d-1)t} \end{pmatrix}$  for  $t > 0$ . Then the expanding horospherical subgroup  $H_+$  of  $G$  with respect to  $\{\Phi_t : t > 0\}$  can be explicitly written as

$$(1) \quad H_+ = \left\{ n_+(\mathbf{x}) = \begin{pmatrix} 1_{d-1} & \mathbf{0} \\ \mathbf{x} & 1 \end{pmatrix} : \mathbf{x} \in \mathbb{R}^{d-1} \right\}.$$

It is well-known that the translates of patches of expanding horospheres under  $\Phi_t$  becomes uniformly distributed in  $\Gamma \backslash G$  with respect to  $\mu$  as  $t \rightarrow \infty$ . We have the following equidistribution theorem :

**Theorem 1** (cf.[5]). *Let  $f : \Gamma \backslash G \times \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  be bounded continuous and  $\lambda$  a Borel probability measure on  $\mathbb{R}^{d-1}$  which is absolutely continuous with respect to the Lebesgue measure. Then*

$$(2) \quad \lim_{t \rightarrow \infty} \int_{\mathbb{R}^{d-1}} f(\Gamma n_+(\mathbf{x}) \Phi_t, \mathbf{x}) d\lambda(\mathbf{x}) = \int_{\Gamma \backslash G \times \mathbb{R}^{d-1}} f(g, \mathbf{x}) d\mu(g) d\lambda(\mathbf{x}).$$

For a positive integer  $q$ , let

$$(3) \quad \mathcal{R}_q = \{\mathbf{r} \in \mathbb{Z}^{d-1} \cap (0, q]^{d-1}, \gcd(q, \mathbf{r}) = 1\}.$$

In this talk, we study the distribution of rational points with denominator  $q$ . For  $\mathbf{r} \in \mathcal{R}_q$ , by [4, (3.52)],  $\Gamma n_+(\mathbf{r}/q) D(q) \in \Gamma \backslash \Gamma H$ , where  $D(q) = \Phi_{(d-1) \log q}$  and the

subgroup

$$(4) \quad H = \left\{ \begin{pmatrix} A & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix} : A \in SL_{d-1}(\mathbb{R}), \mathbf{x} \in \mathbb{R}^{d-1} \right\} \cong SL_{d-1}(\mathbb{R}) \ltimes \mathbb{R}^{d-1} =: ASL_{d-1}(\mathbb{R}).$$

Marklof [4] proved the equidistribution of rational points

$$(5) \quad \{\Gamma n_+(\mathbf{r}/q)D(q) : \mathbf{r} \in \mathcal{R}_q, 1 \leq q < Q\}$$

as  $Q \rightarrow \infty$ . This result has important applications to the asymptotic distribution of Frobenius numbers and the diameters of random circulant graphs. By extending this result, Einsiedler, Mozes, Shah and Shapira [2] proved the following remarkable equidistribution theorem.

**Theorem 2** ([2]). *Let  $f : \Gamma \backslash \Gamma H \times (\mathbb{R}/\mathbb{Z})^{d-1} \rightarrow \mathbb{R}$  be bounded continuous. Then*

$$(6) \quad \lim_{q \rightarrow \infty} \frac{1}{\#\mathcal{R}_q} \sum_{\mathbf{r} \in \mathcal{R}_q} f(\Gamma n_+(\mathbf{r}/q)D(q), \mathbf{r}/q) = \int_{\Gamma \backslash \Gamma H \times (\mathbb{R}/\mathbb{Z})^{d-1}} f(g, \mathbf{x}) d\mu_0(g) d\mathbf{x}.$$

Here  $\mu_0$  is the unique  $H$ -invariant probability measure on  $\Gamma \backslash \Gamma H$ .

The proof requires deep ergodic-theoretic tools, including Ratner’s measure classification theorem.

In the present talk, we provide a different proof of Theorem 2 in the case of  $d = 3$ , which uses

- harmonic analysis on  $ASL_2(\mathbb{Z}) \backslash ASL_2(\mathbb{R})$  ([6])
- Weil bounds on Kloosterman sums
- distribution of Hecke points ([1]).

Unlike the ergodic-theoretic approach pursued in [2], this provides an explicit estimate on the rate of convergence.

Let  $C_b^k(\Gamma \backslash \Gamma H \times (\mathbb{R}/\mathbb{Z})^2)$  be the space of  $k$  times continuously differentiable functions with all derivatives bounded. The following is our main result:

**Theorem 3** ([3]). *Let  $d = 3$ ,  $\epsilon > 0$  and  $k > 4$ . Then there is a constant  $c_{\epsilon,k} < \infty$  depending on  $\epsilon$  and  $k$  such that, for all  $q \in \mathbb{Z}_{\geq 1}$  and  $f \in C_b^k(\Gamma \backslash \Gamma H \times (\mathbb{R}/\mathbb{Z})^2)$ ,*

$$(7) \quad \left| \frac{1}{\sqrt{\#\mathcal{R}_q}} \sum_{\mathbf{r} \in \mathcal{R}_q} f(\Gamma n_+(\mathbf{r}/q)D(q), \mathbf{r}/q) - \int_{\Gamma \backslash \Gamma H \times (\mathbb{R}/\mathbb{Z})^2} f(g, \mathbf{x}) d\mu_0(g) d\mathbf{x} \right| \leq c_{\epsilon,k} \|f\|_{C_b^k} q^{-\frac{1}{2} + \epsilon} (q^\theta + q^{\frac{5}{2(k+1)}}).$$

Here  $\theta$  is the constant towards the Ramanujan conjecture, which asserts  $\theta = 0$  and  $\|f\|_{C_b^k}$  is the Sobolev norm.

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On the global sup-norm of  $GL(3)$  cusp forms

PÉTER MAGA

(joint work with Valentin Blomer, Gergely Harcos)

In the past few years, analytic number theory on higher-rank groups came into focus. In particular, the sup-norm problem (i.e. give as strong estimates on the sup-norm of eigenfunctions of the Hecke algebra as possible) has also been investigated. It turned out (in the work of Brumley-Templier) that automorphic forms on  $GL(n)$ , for  $n \geq 6$ , show high peaks near the cusp, that is, Sarnak’s general bound (referring to compact spaces or compact subsets of noncompact ones) does not hold. As a complement to Brumley-Templier (which gave lower bound on the sup-norm), we try to estimate the eigenfunctions from above.

Introduce the notation  $G = \mathrm{PGL}_3(\mathbb{R})$ ,  $\Gamma = \mathrm{PGL}_3(\mathbb{Z})$ ,  $K = \mathrm{PO}_3(\mathbb{R})$ , and set  $X = \Gamma \backslash G/K$ . Assume that  $\phi$  is a Hecke-Maaß cusp form on  $X$ , and denote its Laplace eigenvalue by  $\lambda_\phi$ . Assume further that  $\phi$  is arithmetically normalized, i.e. it has leading Fourier coefficient 1 with respect to Jacquet’s Whittaker function. We prove then that

$$(1) \quad \sup_{z \in X} |\phi(z)| \ll_\varepsilon \lambda_\phi^{39/40+\varepsilon}$$

holds for all  $\varepsilon > 0$ . To establish this bound, we consider two approaches.

The first approach is the Fourier-Whittaker expansion of  $\phi$ . Assuming that  $z \in X$  has diagonal Iwasawa factor  $\mathrm{diag}(y_1 y_2, y_1, 1)$  with  $y_1, y_2 \geq \sqrt{3}/2$ , after a careful analysis of Jacquet’s Whittaker function, we arrive at

$$\phi(z) \ll_\varepsilon \min(y_1, y_2) \left( \frac{\lambda_\phi^{1+\varepsilon}}{y_1 y_2} + \frac{\lambda_\phi^{3/2+\varepsilon}}{(y_1 y_2)^2} \right).$$

This bound is particularly strong when  $y_1 y_2$  is large, reflecting the exponential decay of the Whittaker function close to the cusp.



The second approach is via the pre-trace formula, leading us to

$$\phi(z) \ll \lambda_\phi^{3/4} + \lambda_\phi^{5/8} y_1 y_2.$$

This bound is strong when  $y_1 y_2$  is small, which must not be surprising in the light of the fact that the pre-trace formula approach is usually used for compact domains.

Balancing between these two estimates, we arrive at (1).

### Amplification and bounds for periods

SIMON MARSHALL

Let  $M$  be a compact Riemannian manifold of dimension  $n$ , and  $\psi$  a function on  $M$  satisfying  $(\Delta + \lambda^2)\psi = 0$  and  $\|\psi\|_2 = 1$ . A classical theorem of Avacumović [1] and Levitan [12] states that

$$(1) \quad \|\psi\|_\infty \ll \lambda^{(n-1)/2},$$

that is, the pointwise norm of  $\psi$  is bounded in terms of its Laplace eigenvalue. This bound is sharp on the round sphere  $S^n$  or a surface of revolution, but is far from the truth on flat tori. It is an interesting problem in semiclassical analysis to find conditions on  $M$  under which (1) can be strengthened, and such conditions often take the form of a non-recurrence assumption for the geodesic flow on  $M$ . One result of this kind is due to Bérard [2], who proves that if  $M$  has negative sectional curvature (or has no conjugate points if  $n = 2$ ) then we have

$$(2) \quad \|\psi\|_\infty \ll \frac{\lambda^{(n-1)/2}}{\sqrt{\log \lambda}}.$$

The problem of strengthening (1) for negatively curved  $M$  is an interesting one, because for generic  $M$  we expect that  $\|\psi\|_\infty \ll_\epsilon \lambda^\epsilon$ , whereas the strongest upper bound that is known in general is (2).

In [11], Iwaniec and Sarnak introduced a different condition on  $M$  and  $\psi$  which allows them to deduce quite a strong bound for  $\|\psi\|_\infty$ . They assume that  $M$  is a congruence hyperbolic manifold, in particular the quotient of  $\mathbb{H}^2$  by the group of units in an order in a quaternion division algebra over  $\mathbb{Q}$ , and that  $\psi$  is an eigenfunction of the Hecke operators on  $M$ . They then prove that  $\|\psi\|_\infty \ll_\epsilon \lambda^{5/12+\epsilon}$ . Moreover, one expects that the assumption on  $\psi$  is not necessary because the spectral multiplicities of negatively curved manifolds are always observed to be bounded. This bound is the strongest that is known for the supremum norm of eigenfunctions on a negatively curved surface.

We are interested in extending the methods of Iwaniec and Sarnak to higher dimensional manifolds, which requires considering eigenfunctions on general locally symmetric spaces. We shall only consider spaces of noncompact type, although the method of proof would apply equally well to spaces of compact type. We make this restriction partly for convenience, and partly because the multiplicities

of the Laplace spectrum on such manifolds are expected to be bounded as in the hyperbolic case. Although these manifolds have zero sectional curvature in certain directions, their eigenfunctions are expected to exhibit essentially the same chaotic behaviour that is observed on negatively curved manifolds.

We recall that locally symmetric spaces of noncompact type are constructed by taking a semisimple real Lie group  $G$ , a maximal compact subgroup  $K \subset G$ , and a lattice  $\Gamma \subset G$ , and defining  $Y = \Gamma \backslash G/K$ . We do not assume that  $Y$  is compact. We let  $n$  and  $r$  be the dimension and rank of  $Y$ . We consider functions  $\psi \in L^2(Y)$  that are eigenfunctions of the full ring of invariant differential operators, which is isomorphic to a finitely generated polynomial ring in  $r$  variables. This ring contains  $\Delta$ , and we continue to define  $\lambda$  by  $(\Delta + \lambda^2)\psi = 0$ .

If  $\Omega \subset Y$  is compact, Sarnak proves in [17] that  $\psi$  satisfies

$$(3) \quad \|\psi|_{\Omega}\|_{\infty} \ll \lambda^{(n-r)/2}.$$

The analogous problem to the one solved by Iwaniec and Sarnak for  $\mathbb{H}^2$  is to improve the exponent in this bound, under the assumptions that  $\Gamma$  is congruence arithmetic, and that  $\psi$  is an eigenfunction of the ring of Hecke operators. (Note that when  $r \geq 2$ ,  $\Gamma$  is automatically arithmetic by a theorem of Margulis.) This is often referred to as the problem of giving a subconvex, or sub-local, bound for the sup norm of a Maass form in the eigenvalue aspect. Besides the original work of Sarnak and Iwaniec, the pairs  $\Gamma \subset G$  for which it has previously been solved are  $SL_2(\mathcal{O}_F) \subset SL_2(F_{\infty})$  for any number field  $F$  by Blomer, Harcos, Maga, and Milićević [3, 4],  $Sp_4(\mathbb{Z}) \subset Sp_4(\mathbb{R})$  by Bomer and Pohl [9],  $SL_3(\mathbb{Z}) \subset SL_3(\mathbb{R})$  by Holowinsky, Ricotta, and Royer [10], and  $SL_n(\mathbb{Z}) \subset SL_n(\mathbb{R})$  for any  $n$  by Blomer and Mágá [5, 6]. There are also results bounding eigenfunctions on the round spheres  $S^2$  and  $S^3$  equipped with Hecke algebras [7, 8].

We note that much work has been done on variants of the sup-norm problem. One may consider Maass forms of varying level and eigenvalue, or bound the  $L^2$  norm of the restriction of  $\psi$  to a submanifold of positive dimension [13, 14].

**0.1. Statement of results.** We first state our result in a simple case.

**Theorem 1.** *Let  $F$  be a totally real number field, and let  $v_0$  be a real place of  $F$ . Let  $G/F$  be connected and semisimple. We make the following assumptions on  $G$ .*

- $G_v$  is compact for all real  $v \neq v_0$
- $G_{v_0}$  is  $\mathbb{R}$ -almost simple, quasi-split, and not isogenous to  $SU(n, n-1)$  for any  $n$ .

*Let  $Y$  be a congruence manifold associated to  $G$ , and let  $\Omega \subset Y$  be compact. Let  $\psi$  be a Hecke-Maass form on  $Y$  satisfying  $\|\psi\|_2 = 1$  and  $(\Delta + \lambda^2)\psi = 0$ . We then have  $\|\psi|_{\Omega}\|_{\infty} \ll \lambda^{(n-r)/2-\delta}$  for some  $\delta > 0$ .*

We deduce Theorem 1 from the following more general result. To state it, it will be convenient to make two definitions. The first is a condition on a real semisimple group  $G$ :

(WS):  $G$  is quasi-split, and not isogenous to a product of odd special unitary groups.

The second condition will be applied to the spectral parameters of our Maass form, to simplify the application of a theorem of Blomer-Pohl [9] and Matz-Templier [15] in the proof.

**Definition 1.** Let  $\mathfrak{g}$  be a real Lie algebra with Cartan decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  and Cartan subalgebra  $\mathfrak{a} \subset \mathfrak{p}$ . Let  $\mathfrak{g}_i$  be the  $\mathbb{R}$ -simple factors of  $\mathfrak{g}$ . We say that  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  is  $(A, \sigma)$ -balanced if its projections  $\lambda_i$  to  $\mathfrak{g}_{i, \mathbb{C}}^*$  satisfy  $\|\lambda_i\| \leq C\|\lambda_j\|^\sigma$ .

We may now state the general form of our main theorem.

**Theorem 2.** Let  $F$  be a number field, and let  $v_0$  be a real place of  $F$ . Let  $G/F$  be connected and semisimple, and assume that  $G_{v_0}$  satisfies (WS).

Let  $Y$  be a congruence manifold associated to  $G$ , and let  $\psi$  be a Hecke-Maass form on  $Y$  satisfying  $\|\psi\|_2 = 1$ . Let  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  be the spectral parameter of  $\psi$ . Let  $\lambda_0$  be the component of  $\lambda$  at  $v_0$ , and assume that  $\lambda_0$  is  $(A, \sigma)$ -balanced in  $\text{Lie}(G_{v_0})$ . Let  $\Omega_Y \subset Y$  be compact. Then there exists  $\delta = \delta(G, \sigma) > 0$  and  $C = C(\Omega_Y, A, \sigma)$  such that

$$(4) \quad \|\psi|_{\Omega_Y}\|_{\infty} \leq CD(\lambda)(1 + \|\lambda_0\|)^{-\delta}.$$

As Theorem 2 is rather general, we now give some examples of what one may prove by specializing it in various ways. First, Theorem 2 solves the sup norm problem for split groups over any number field  $F$ , subject to the balance condition on the spectral parameter.

**Corollary 1.** Let  $G/F$  be split. Let  $\psi$ ,  $\lambda$ , and  $\Omega_Y$  be as in Theorem 2. Assume that  $\lambda$  is  $(A, \sigma)$ -balanced in  $\text{Lie}(G_{\infty})$ . Then there exists  $\delta = \delta(G, \sigma) > 0$  and  $C = C(\Omega_Y, A, \sigma)$  such that

$$\|\psi|_{\Omega_Y}\|_{\infty} \leq C(1 + \|\lambda\|)^{(n-r)/2-\delta}.$$

*Proof.* If  $F$  has a real place, the corollary follows directly from Theorem 2. If  $F$  has only complex places, the  $\mathbb{Q}$ -group  $\text{Res}_{F/\mathbb{Q}}G$  satisfies (WS) at infinity so we may apply Theorem 2 to it. □

As a second example, we may apply Theorem 1 to groups with  $G(F_{v_0}) = SL(2, \mathbb{C})$  so that the associated symmetric spaces are congruence arithmetic hyperbolic 3-manifolds.

**Corollary 2.** Let  $Y$  be a compact congruence arithmetic hyperbolic 3-manifold. If the invariant trace field  $F$  of  $Y$  has a subfield of index 2, then any Hecke-Laplace eigenfunction  $\psi$  on  $Y$  that satisfies  $(\Delta + \lambda^2)\psi = 0$  and  $\|\psi\|_2 = 1$  also satisfies  $\|\psi\|_{\infty} \ll (1 + \lambda)^{1-\delta}$  for some  $\delta > 0$  depending only on  $Y$ .

We note that the condition on  $Y$  in Corollary 2 also arises in work of Milićević [16] on the sup norms of Maass forms.

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## Effective Plancherel equidistribution

JASMIN MATZ

(joint work with Tobias Finis)

Let  $G$  be a split semisimple algebraic group defined over  $\mathbb{Z}$ . We are interested in the distribution of the local components of cuspidal automorphic representations of  $G$  in the unitary dual of the local groups. To describe this problem more precisely, we need to introduce some notation: Fix suitable maximal compact subgroups  $\mathbf{K}_\infty \subseteq G(\mathbb{R})$ ,  $\mathbf{K}_p \subseteq G(\mathbb{Q}_p)$ , and set  $\mathbf{K} := \mathbf{K}_\infty \cdot \mathbf{K}_{\text{fin}} := \mathbf{K}_\infty \cdot \prod_{p < \infty} \mathbf{K}_p$ . Let  $\Omega \subseteq \widehat{G(\mathbb{R})}^{\text{ur,temp}}$  be a “nice” bounded subset of the tempered part of the spherical unitary dual of  $G(\mathbb{R})$ . We can identify  $\widehat{G(\mathbb{R})}^{\text{ur,temp}}$  with the real vector space  $\mathbb{R}^r$  with  $r$  the rank of  $G$  so that we can consider the sets  $t\Omega \subseteq \widehat{G(\mathbb{R})}^{\text{ur,temp}}$  for  $t \geq 1$ . Let  $\mathcal{F}_\Omega(t)$  denote the multiset (that is, appearing with the same multiplicity

as in  $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$  of all cuspidal automorphic representations  $\pi$  of  $G(\mathbb{A})$  which are spherical at the archimedean place and unramified at all  $p$  and such that  $\pi_\infty \in t\Omega$ . Studying the number of elements in  $\mathcal{F}_\Omega(t)$  as  $t \rightarrow \infty$  yields the distribution of the cuspidal automorphic spectrum in  $\widehat{G(\mathbb{R})}^{\text{ur}}$ . If  $\Omega$  is the unit ball, this is Weyl's law which is known to hold in many cases [14, 11, 12, 8].

For each prime  $p$  every  $\pi \in \mathcal{F}(t)$  determines a point  $\pi_p \in \widehat{G(\mathbb{Q}_p)}^{\text{ur}}$  in the unramified unitary dual of  $G(\mathbb{Q}_p)$ . An obvious question is how those  $\pi_p$  distribute in  $\widehat{G(\mathbb{Q}_p)}^{\text{ur}}$  as  $\pi \in \mathcal{F}_\Omega(t)$ ,  $t \rightarrow \infty$ . For small rank this question was studied in [13, 6, 1]. To make this more precise, we consider the Hecke algebra: Let  $\mathcal{H}_p$  denote the Hecke algebra of  $G(\mathbb{Q}_p)$  with respect to  $\mathbf{K}_p$ , that is, the convolution algebra of all smooth, compactly supported functions  $\tau_p : G(\mathbb{Q}_p) \rightarrow \mathbb{C}$  which are left- and right-invariant under  $\mathbf{K}_p$ . The characteristic function  $\chi_{\mathbf{K}_p}$  of  $\mathbf{K}_p$ , normalized by the measure of  $\mathbf{K}_p$ , is the unit element of  $\mathcal{H}_p$ . Let  $\mathcal{H}$  denote the convolution algebra generated by all functions  $\tau : G(\mathbb{A}_f) \rightarrow \mathbb{C}$  of the form  $\tau = \prod_{p < \infty} \tau_p$  with  $\tau_p \in \mathcal{H}_p$  for each  $p$  and  $\tau_p = \chi_{\mathbf{K}_p}$  for all but finitely many  $p$ .

For our main result, we need to make an assumption on the intertwining operators on our group, namely that the winding numbers of their normalizing factors do not grow too fast. More precisely, we assume that  $G$  satisfies property (TWN+) (tempered winding numbers), which was introduced in [4] in the context of the study of limit multiplicities. It is a natural condition on  $G$ , and known to hold for all classical groups and  $G_2$  [4].

We then have:

**Theorem.** Suppose that  $G$  satisfies property (TWN+). Then

- For any  $\epsilon > 0$  we have as  $t \rightarrow \infty$

$$(1) \quad \#\mathcal{F}_\Omega(t) = \text{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})) \int_{t\Omega} \beta(\lambda) d\lambda + O_\epsilon \left( t^{d-1/2+\epsilon} \right)$$

where  $d = \dim_{\mathbb{R}} G(\mathbb{R})/\mathbf{K}_\infty$ . Note that  $\int_{t\Omega} \beta(\lambda) d\lambda$  is asymptotic to a constant multiple of  $\text{vol}(\Omega)t^d$  as  $t \rightarrow \infty$ .

- For any  $\tau \in \mathcal{H}$ , and any  $\epsilon > 0$  we have

$$(2) \quad \sum_{\pi \in \mathcal{F}_\Omega(t)} \text{tr} \pi^{\mathbf{K}_\infty}(\tau) = \text{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})) \sum_{z \in Z(\mathbb{Q})} \tau(z) \int_{t\Omega} \beta(\lambda) d\lambda + O_\epsilon \left( \|\tau\|_{L^1(G(\mathbb{A}_f))} t^{d-1/2+\epsilon} \right)$$

where  $Z$  denotes the center of  $G$ . The implied constant is independent of  $\tau$  and  $t$ .

**Remark.** • (1) is essentially the Weyl law with a bound on the error term. Previously, for  $\text{PGL}(n)$  a stronger error term was established in [7], which is of order  $O(t^{d-1}(\log t)^{\max\{3,n\}})$ .

- For  $\text{PGL}(n)$  over  $\mathbb{Q}$  or an imaginary quadratic number field an estimate as in (2) was established in [9, 10] but with a worse error estimate.

- The error terms in (1),(2) can probably be improved.
- (2) can be used to prove that on average the family  $\mathcal{F}_\Omega(t)$  satisfies the Sato-Tate law.

The main tool in the proof of the theorem is the Arthur-Selberg trace formula for  $G$  and a good choice of test function. We rely on methods and ideas from [3, 7, 10, 2, 5]. The test function, or rather family of test functions, is constructed as in [3] via the Paley-Wiener theorem. More precisely, it depends on a spectral parameter  $\mu \in \widehat{G(\mathbb{R})}^{\text{ur,temp}}$  and the Hecke operator  $\tau \in \mathcal{H}$ . It is of the form  $F^{\mu,\tau} = f_\infty^\mu \tau$ .

The trace formula gives an identity

$$\int_{t\Omega} J_{\text{geom}}(F^{\mu,\tau}) d\mu = \int_{t\Omega} J_{\text{spec}}(F^{\mu,\tau}) d\mu.$$

The spectral side can basically be handled as for  $\text{GL}(n)$  in [7, 9]. The most work lies in the treatment of the geometric side. The main terms on the right hand sides of (1) and (2) come from the contribution of the center  $Z$  to the geometric side. Let  $J_{\text{geom}-Z}$  denote the geometric side of the trace formula with the central contribution removed. We then need to show that  $\int_{t\Omega} J_{\text{geom}-Z}(F^{\mu,\tau}) d\mu$  only contributes to the error term on the right hand side of (2). This is done by using results by Arthur and Finis-Lapid to reduce the problem to finding an upper bound for

$$(3) \quad \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})_{\leq T}} \sum_{\gamma \in G(\mathbb{Q}) \backslash Z(\mathbb{Q})} |F^{\mu,\tau}(g^{-1}\gamma g)| dg$$

where  $G(\mathbb{Q}) \backslash G(\mathbb{A})_{\leq T}$  indicates that we truncate the quotient  $G(\mathbb{Q}) \backslash G(\mathbb{A})$  as in Arthur's work. Further, we replace  $|F^{\mu,\tau}|$ , or more specifically  $|f_\infty^\mu|$  by a simpler function. For that we use an upper bound on the zonal spherical function on  $G(\mathbb{R})$ , which appears in the construction of the function  $f_\infty^\mu$ , as obtained in [2, 10]. Consequently, one obtains the upper bound

$$|f_\infty^\mu(g)| \ll_{\text{supp} f_\infty^\mu} (1 + \|\mu\|)^{d-r} \min\{1, (1 + \|\mu\|)^{-1/2} \|X(g)\|^{-1/2}\}$$

for any  $g \in \text{supp} f_\infty^\mu$  and  $\mu \in \widehat{G(\mathbb{R})}^{\text{ur,temp}}$ . Here  $X(g) \in \text{Lie } T_0(\mathbb{R})$  is such that  $g \in \mathbf{K}_\infty e^{X(g)} \mathbf{K}_\infty$  where  $T_0(\mathbb{R})$  is a suitable maximal split torus in  $G(\mathbb{R})$  such that the Cartan decomposition  $G(\mathbb{R}) = \mathbf{K}_\infty T(\mathbb{R}) \mathbf{K}_\infty$  holds. Note that the support of  $f_\infty^\mu$  is independent of  $\mu$  so that the implied constant is independent of  $\mu$  as well. We can therefore replace  $f_\infty^\mu$  by a suitable function depending only on the "radial part" of  $g$  for which upper bounds for (3) are easier to establish.

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## The sup-norm problem for $\mathrm{GL}(2)$ over number fields

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(joint work with Valentin Blomer, Gergely Harcos, Péter Maga)

Eigenfunctions of the Laplacian are basic building blocks of harmonic analysis on Riemannian manifolds. The sup-norm problem asks for nontrivial bounds on the pointwise values of an  $L^2$ -normalized eigenfunction in terms of its Laplacian eigenvalue  $\lambda$ , geometric properties of the underlying manifold  $X$ , or other increasing parameters. This question is closely connected to the multiplicity of eigenvalues, and it is motivated by the correspondence principle of quantum mechanics, where the high energy limit  $\lambda \rightarrow \infty$  provides a connection between classical and quantum mechanics. The sup-norm of an eigenform with large eigenvalue gives some information on the distribution of its mass on  $X$ , which sheds light on the question to what extent these eigenstates can localize (“scarring”). Exciting progress in arithmetic cases means that the sup-norm problem now occupies a prominent position at the interface of automorphic forms, analytic number theory, and analysis.

**1. Statement of results.** In the presented paper, we prove for the first time nontrivial bounds for the sup-norm of a spherical Hecke–Maaß cuspidal newform  $\phi$  on  $\mathrm{GL}(2)$  over a general number field  $F$  of squarefree level  $\mathfrak{n}$  and trivial central character, with a power saving over the local geometric bound simultaneously in the eigenvalue and the level aspect.

If the number field  $F$  admits  $r_1$  real embeddings and  $r_2$  conjugate pairs of complex embeddings, then the connected components of the underlying manifold

$X$  are quotients of  $(\mathcal{H}^2)^{r_1} \times (\mathcal{H}^3)^{r_2}$  by various level  $\mathfrak{n}$  congruence subgroups, where  $\mathcal{H}^2$  (resp.  $\mathcal{H}^3$ ) is the upper half-plane (resp. half-space). Assuming that  $\|\phi\|_2 = 1$  holds with respect to the probability measure coming from invariant measures on  $\mathcal{H}^2$  and  $\mathcal{H}^3$ , the local geometric bound reads

$$\|\phi\|_\infty \ll_\epsilon |\lambda|_\infty^{1/4+\epsilon} (\mathcal{N}\mathfrak{n})^{1/2+\epsilon},$$

where  $|\lambda|_\infty$  is the product of suitably normalized Laplacian eigenvalues at all archimedean places, and  $\mathcal{N}\mathfrak{n}$  is the norm of  $\mathfrak{n}$ . The first of our two main results is the following improvement.

**Theorem 1.** *For an  $L^2$ -normalized Hecke–Maaß cuspidal newform  $\phi$  on  $\mathrm{GL}_2(F)$  of square-free level  $\mathfrak{n}$ , trivial central character, and spherical at infinity,*

$$\|\phi\|_\infty \ll_\epsilon |\lambda|_\infty^{5/24+\epsilon} (\mathcal{N}\mathfrak{n})^{1/3+\epsilon} + |\lambda|_{\mathbb{R}}^{1/8+\epsilon} |\lambda|_{\mathbb{C}}^{1/4+\epsilon} (\mathcal{N}\mathfrak{n})^{1/4+\epsilon}.$$

This bound is particularly strong when  $F$  is totally real (when it recovers the best known hybrid result over the rationals) and features a Weyl exponent in the level-aspect.

For a general number field  $F$  with a maximal totally real subfield  $K \neq F$ , we establish as our second principal result the following hybrid bound, which saves uniformly in all aspects:

**Theorem 2.** *For  $[F : K] \geq 2$ , we have under the same assumptions as in Theorem 1*

$$\|\phi\|_\infty \ll_\epsilon (|\lambda|_\infty^{1/2} \mathcal{N}\mathfrak{n})^{\frac{1}{2} - \frac{1}{8[F:K]-4} + \epsilon}.$$

**2. Methods.** For the proof, in place of the amplified pre-trace formula, we apply a *pre-trace inequality* for a suitable positive operator on  $L^2(X)$ , which substantially streamlines the argument compared to the exact spectral average. This leads to a counting problem for  $2 \times 2$  matrices  $\gamma$  over  $F$  which lie suitably close to a certain maximal compact subgroup of  $\mathrm{GL}_2(F_\infty)$ .

We rely heavily on the Atkin–Lehner operators to show that the maximum in  $\|\phi\|_\infty$  is achieved at a point  $g = \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \begin{pmatrix} \theta & \\ & 1 \end{pmatrix}$  (with  $x \in F_\infty$ ,  $y \in F_\infty^\times$ , and  $\theta$  one of finitely many finite ideles) with  $|y|_\infty \gg 1/\sqrt{\mathcal{N}\mathfrak{n}}$ . We also develop the following uniform Fourier bound valid over an arbitrary number field, which is strong for large  $|y|_\infty$ :

$$|\phi(g)| \ll_\epsilon (|\lambda|_\infty^{1/12} + |\lambda|_\infty^{1/4} |y|_\infty^{-1/2})^{1+\epsilon} (\mathcal{N}\mathfrak{n})^\epsilon.$$

Away from the cuspidal regions, we use *geometry of numbers* for an efficient treatment of the counting problem over  $F$ , and we build a number of approaches that are strong in different ranges of parameters.

**3. New features over number fields.** As is well-known, the passage from  $\mathbb{Q}$  to a general number field introduces two abelian groups (the class group and the group of units), which cause considerable technical difficulties for arguments of analytic number theory. In this direction, we provide a general adelic counting scheme for efficient counting in possibly highly unbalanced boxes.



However, in our case, the difficulties go much deeper than dealing with the class group and the unit group. As soon as  $F$  has a complex place, the counting problem features conditions involving real and imaginary parts at each complex place separately. If  $F$  is not a CM-field, there is no global complex conjugation, and the global counting techniques that work over number fields like  $\mathbb{Q}$  or  $\mathbb{Q}(i)$  break down in general. In fact, the maximal compact subgroups of  $\mathrm{GL}_2(F_\infty)$  cannot be defined over  $F$  unless  $F$  is a totally real field or a CM-field. The Diophantine approximation route previously used over  $\mathbb{Q}$  and  $\mathbb{Q}(i)$  also fails since the Dirichlet approximation of required strength is not available over other number fields.

We therefore introduce a number of new devices to leverage the specific interplay between the maximal compact subgroups of  $\mathrm{GL}_2(F_\infty)$  and the arithmetic of  $F$ . One of these is the following *realness rigidity* statement:

**Lemma 3.** *If  $K$  is the maximal totally real subfield of a number field  $F$  of degree  $n$ ,  $F = K(\xi)$ , and, for all  $v \mid \infty$ ,  $|\xi_v| \leq A$  and  $|\mathrm{Im} \xi_v| \leq A\sqrt{\delta_v}$ , and if  $\mathfrak{l}$  is an ideal such that  $\mathfrak{l} \cdot (\xi)$  is an integral ideal, then*

$$((2A)^n (\mathcal{N}\mathfrak{l})^2)^{2([F:K]-1)} |\delta|_{\mathbb{C}} \geq 1.$$

Combined with a careful *choice of the amplifier*, in the hardest situation in which a correspondence  $\gamma$  is very close to the maximal compact subgroup and  $\det \gamma$  is a perfect square, this allows us to show that the rescaled trace  $\xi = \mathrm{tr}(\gamma) / \sqrt{\det(\gamma)}$  must in fact be an integer; a very strong conclusion.

On the other hand, by passing to a specific congruence subgroup of  $\Gamma_0(\mathfrak{n})$  (that behaves roughly like  $\Gamma_0(\mathfrak{n}) \cap \Gamma_1(\mathfrak{q})$ ) and thus artificially *extending the spectrum*, we can improve the performance of the pre-trace formula on the geometric side and achieve further non-archimedean localization (that is, congruence conditions) on the entries of  $\gamma$  and thus on  $\xi$ , finally allowing us to eliminate the possibility of non-parabolic correspondences in the most stubborn range of tiny distances.

In conclusion, the success of our method rests on three flexible tools introduced specifically to address the novel features of the sup-norm problem over number fields: passage to a suitably chosen congruence subgroup, a carefully designed amplifier equipped with arithmetic features, and realness rigidity for number fields, all of which appear to be of interest in other situations.

## Subconvexity and simple zeros of modular form $L$ -functions

MICAH B. MILINOVICH

(joint work with Andrew R. Booker, Nathan Ng)

Let  $f \in S_k(\Gamma_0(N), \xi)^{\mathrm{new}}$  be a primitive holomorphic cusp form of weight  $k$ , level  $N$ , and nebentypus character  $\xi$ . Writing the Fourier expansion at the cusp  $\infty$  as

$$f(z) = \sum_{n \geq 1} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z}, \quad \mathrm{Im}(z) > 0,$$

we denote the  $L$ -function associated to  $f$  as

$$L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}, \quad \operatorname{Re}(s) > 1,$$

so that  $\lambda_f(n) = 1$ ,  $|\lambda_f(n)| \leq d(n)$  (the divisor function), and the critical line of  $L(s, f)$  is  $\operatorname{Re}(s) = 1/2$ . We are interested in the following conjecture.

**Conjecture:** *The nonreal zeros of  $L(s, f)$  are all simple.*

There are examples of  $L(s, f)$  that have a multiple zero at the central point  $s = 1/2$ , and conjecturally there many such examples (for instance corresponding to elliptic curves over  $\mathbb{Q}$  of rank  $\geq 2$  via the Birch and Swinnerton-Dyer conjecture).

Recently, Booker [3] proved that infinitely many of the nontrivial zeros of  $L(s, f)$  are simple. Let

$$N_f^s(T) = \#\{0 < |\rho| \leq T : L(\rho, f) = 0, \rho \text{ simple}\}.$$

Then, assuming the generalized Riemann hypothesis for  $L(s, f)$ , Ng and I [6] proved the quantitative result that

$$N_f^s(T) \gg_\varepsilon T(\log T)^{-\varepsilon},$$

for any  $\varepsilon > 0$ . The three of us are now working together to try to prove unconditional quantitative estimates for  $N_f^s(T)$ .

Our starting point is a paper of Conrey and Ghosh [2] who proved a result of the following form for  $L$ -functions associated to  $f$  on the full modular group.

**Theorem 1:** *Let  $N = 1$  and suppose that: (1)  $L(s, f)$  has at least one simple nontrivial zero and (2) there is a  $\delta > 0$  such that  $|L(\frac{1}{2} + it, f)| \ll_\varepsilon |t|^{\frac{1}{2} - \delta + \varepsilon}$  for  $|t| \geq 3$  and all  $\varepsilon > 0$ . Then  $N_f^s(T) = \Omega(T^{\delta - \varepsilon})$  for all  $\varepsilon > 0$ .*

A few years earlier Good [5] established the following subconvexity estimate.

**Theorem 2:** *If  $N = 1$ , then  $|L(\frac{1}{2} + it, f)| \ll |t|^{\frac{1}{3}} (\log |t|)^{\frac{5}{6}}$  for  $|t| \geq 3$ .*

Combining Theorems 1 and 2 with Booker's result [3], it follows that  $N_f^s(T) = \Omega(T^{\frac{1}{6} - \varepsilon})$  for all  $\varepsilon > 0$  for any  $f$  with level 1. Generalizing this result to  $f$  with arbitrary level is a challenging problem. Booker handled the analogue of condition (1) of Conrey and Ghosh's criterion. Concerning the analogue of condition (2), there are  $t$ -aspect subconvexity results which prove the existence of some  $\delta > 0$  but only recently has the analogue of Good's result been established. In [4], modifying a method of Jutila, Booker, Ng, and I proved the following theorem.

**Theorem 3:** *For  $f \in S_k(\Gamma_0(N), \xi)^{new}$  we have  $|L(\frac{1}{2} + it, f)| \ll |t|^{\frac{1}{3}} \log |t|$  for  $|t| \geq 3$  where the implied constant is at most polynomial in the level  $N$ .*

A key innovation in our proof is a general form of Voronoi summation that applies to all fractions, even when the level is not squarefree. Using the resolution of the Sato-Tate conjecture, this inequality can be *slightly* improved to

$$|L(\frac{1}{2} + it, f)| \ll_{N,k} |t|^{\frac{1}{3}} (\log |t|)^{\frac{8}{9} + \frac{8}{27\pi}}, \quad \text{for } |t| \geq 3,$$

but with an implied constant that may no longer be polynomial in  $N$ .

Even with the analogues of conditions (1) and (2) in Theorem 1 now established, the proof does not seem to generalize to arbitrary level. It turns out that we must consider twists of  $L(s, f)$  by Dirichlet characters. The situation is in some ways analogous to trying to generalize Hecke's converse theorem to higher levels. Weil proved a version of the converse theorem for arbitrary level with additional assumptions for twists of  $L(s, f)$  by primitive characters  $\chi$  of conductor coprime to the level.

Using Theorem 3, we can prove that for any  $f \in S_k(\Gamma_0(N), \xi)^{\text{new}}$  there is a primitive Dirichlet character  $\chi$  such that the twisted  $L$ -function  $L(s, f \times \chi)$  satisfies  $N_{f \times \chi}^s(T) = \Omega(T^{\frac{1}{6} - \varepsilon})$  for any  $\varepsilon > 0$ . Under stronger assumptions, we can say a bit more. If we assume that  $L(s, f)$  has a wide enough zero-free of the Littlewood-type, i.e. no zeros in a region of the form

$$\sigma < 1 - \frac{c \log \log t}{\log t}, \quad s = \sigma + it,$$

for a sufficiently large constant  $c > 0$  when  $t$  is large, then we can show that  $N_f^s(T) = \Omega((\log T)^\alpha)$  for some constant  $\alpha > 0$  depending on  $c$ . This condition actually holds for some modular form  $L$ -functions as Coleman [1] has proved a zero-free region of Vinogradov-Korobov-type for Größencharakter  $L$ -functions.

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### Subconvexity problem for $L$ -functions

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In [2], [3], [4], [5] and [6] I have proposed a new approach to prove subconvex bounds for  $L$ -functions. In this short note I will use this technique to establish the Burgess bound -

$$L\left(\frac{1}{2}, \chi\right) \ll M^{3/16+\varepsilon}.$$

for  $\chi$  a primitive Dirichlet character modulo  $M$  (which we assume to be prime for simplicity). To get a subconvex bound for  $L(1/2, \chi)^2$  one is led via the approximate

functional equation to consider sums of the form

$$S(N) = \sum_{n \sim N} d(n)\chi(n)$$

where  $d(n)$  is the divisor function and  $N \ll M^{1+\varepsilon}$ . We want to establish a bound of the form  $|S(N)|/\sqrt{N} \ll M^{1/2-\theta}$  for some  $\theta > 0$ . The result follows if we are able to take  $\theta = 1/8$ . Our first step is to rewrite the sum  $S(N)$  as

$$L^{-1} \sum_{\ell \in \mathcal{L}} \sum_{n \sim NL} \sum_{r \sim N} d(n)\chi(r)\delta(n, r\ell)$$

where  $\mathcal{L}$  is a set of  $L$  primes of size  $L^{1+\varepsilon}$ , and  $\delta(.,.)$  is the Kronecker delta symbol. Our job is to save  $N^{3/2}LM^{\theta-1/2}$  for some  $\theta > 0$ . To this end we use the harmonics from the space  $S_k(pM, \psi)$  of cusp forms of weight  $k$ , level  $pM$  and nebentypus  $\psi$ , to detect the equation  $n = r\ell$ . Here  $p$  is a fixed prime and  $\psi$  is a non-primitive odd character modulo  $pM$  of conductor  $p$ . From the Petersson trace formula we have

$$\begin{aligned} \delta(n, r) &= \frac{1}{p-1} \sum_{\psi \bmod p} (1 - \psi(-1)) \sum_{f \in H_k(pM, \psi)} \omega_f^{-1} \lambda_f(n) \overline{\lambda_f(r)} \\ &- \frac{2\pi i}{p-1} \sum_{c=1}^{\infty} \frac{1}{cpM} \sum_{\psi \bmod p} (1 - \psi(-1)) S_{\psi}(r, n; cpM) J_{k-1} \left( \frac{4\pi\sqrt{nr}}{cpM} \right). \end{aligned}$$

When we substitute this formula in the above expression for  $S(N)$ , we get two terms - the off-diagonal contribution involving Kloosterman sums and the dual contribution involving the Fourier coefficients of cusp forms. The off-diagonal contribution is negligibly small if we pick  $p \gg NLM^\varepsilon/M$ , as it involves the  $J$ -Bessel function

$$J_{k-1} \left( \frac{4\pi\sqrt{nr\ell}}{cpM} \right),$$

with  $c$  a positive integer. Note that we are taking  $k$  to be large, like  $1/\varepsilon$ , and one has the bound  $J_{k-1}(x) \ll x^{k-1}$ . The dual term is given by

$$\sum_{\psi \bmod p} \sum_{\ell \in \mathcal{L}} \sum_{f \in H_k(pM, \psi)} \omega_f^{-1} \sum_{n \sim NL} d(n)\lambda_f(n) \sum_{r \sim N} \overline{\lambda_f(r\ell)}\chi(r),$$

where our job is to save  $N^{3/2}LM^{\theta-1/2}$  for some  $\theta > 0$ . Next we apply summation formulas on the sum over  $n$  and  $r$ . These can be derived, for example, from the functional equations of  $L(s, f)^2$  and  $L(s, f \otimes \chi)$  respectively. With this the above sum reduces to

$$\sum_{\psi \bmod p} \sum_{\ell \in \mathcal{L}} \sum_{f \in H_k(pM, \psi)} \omega_f^{-1} g_{\psi} \sum_{n \sim p^2M^2/NL} d(n)\overline{\lambda_f(np\ell)} \sum_{r \sim pM^2/N} \lambda_f(r)\chi(\bar{r}),$$

and we make a saving of size  $(NL/pM)(N/\sqrt{pM})$ . Here  $g_{\psi}$  stands for the Gauss sum associated with  $\psi$ . It now remains to save  $p^{3/2}M^{3/2+\theta}/\sqrt{N}$ . We now apply

the Petersson formula. The diagonal is easily seen to be small due to size of the variables. The off-diagonal is roughly of the form

$$\sum_{\psi \bmod p} g_\psi \sum_{\ell \in \mathcal{L}} \sum_{n \sim p^2 M^2 / NL} d(n) \sum_{r \sim p M^2 / N} \overline{\chi(r)} \sum_{c \ll p M / N} S_\psi(n p \ell, r; c p M).$$

In the off-diagonal the Petersson formula saves  $\sqrt{pM} / \sqrt{\text{size of } c} = \sqrt{N}$ , and then applying the Poisson summation formula on the sum over  $r$  we save  $M / \sqrt{N}$ . The sum over  $\psi$  saves  $\sqrt{p}$  more, and with this we arrive at the expression

$$\sum_{\ell \in \mathcal{L}} \sum_{n \sim p^2 M^2 / NL} d(n) \sum_{r \sim p} \sum_{c \ll p M / N} e\left(\frac{\bar{c} n \ell}{r M}\right) \chi(\bar{c}) \mathfrak{C}$$

where

$$\mathfrak{C} = \sum_{a \bmod M} \chi(\bar{a} + r) e\left(\frac{a \bar{c} n \ell}{M}\right).$$

Our job now is to save  $p M^{1/2+\theta} / \sqrt{N}$  in the above sum (beyond square root cancellation in the character sum  $\mathfrak{C}$ ). Applying Cauchy inequality we seek to save  $p^2 M^{1+2\theta} / N$  in

$$\sum_{n \sim p^2 M^2 / NL} \left| \sum_{\ell \in \mathcal{L}} \sum_{r \sim p} \sum_{c \ll p M / N} e\left(\frac{\bar{c} n \ell}{r M}\right) \chi(\bar{c}) \mathfrak{C} \right|^2.$$

We will now open the absolute value and apply the Poisson summation formula on the sum over  $n$ . The diagonal contribution is seen to be satisfactory as long as we have enough terms inside the absolute value, namely  $p^2 L M / N > p^2 M^{1+2\theta} / N$ , i.e.  $L > M^{2\theta}$ . On the other hand the off-diagonal is satisfactory as long as  $p^2 M^2 / N L M^{1/2} > p^2 M^{1+2\theta} / N$ , i.e.  $L < M^{1/2-2\theta}$ . At this stage we encounter a complete character sum of the form

$$\sum_{x \in \mathbb{F}_M - \{\text{few points}\}} \chi\left(\frac{P(x)}{Q(x)}\right)$$

where  $P$  and  $Q$  are quadratic polynomials. Exactly the same sum appeared in Burgess' method, and like him we appeal to Weil's results (Riemann hypothesis for curves over finite fields) to conclude square-root cancellation in the sum. Now we observe that the optimal choice for  $L$  is given by  $L = M^{1/4}$  and  $\theta$  is taken to be  $1/8$ . This establishes the bound

$$\frac{1}{\sqrt{N}} \left| \sum_{n \sim N} d(n) \chi(n) \right| \ll M^{1/2-1/8+\varepsilon}$$

for all  $N$ . From this we are able to conclude the Burgess bound. A careful reader will observe that the above sketch works even when the divisor function is replaced by Fourier coefficients of  $GL(2)$  forms.

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## Subconvex equidistribution of cusp forms

PAUL D. NELSON

## 1. SOME CONTEXT

Let  $M := \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$  denote the modular surface, and let  $\varphi : M \rightarrow \mathbb{C}$  traverse a sequence of Hecke–Maass cusp forms, thus

$$\varphi \in L^2(M), \quad \Delta\varphi = -\lambda\varphi, \quad T_n\varphi = \sqrt{n}\lambda_\varphi(n)\varphi$$

with eigenvalue  $\lambda \rightarrow \infty$ . We may then define a sequence of probability measures  $\mu_\varphi$  on  $M$  assigning to  $\Psi \in C_c(M)$  the values

$$\mu_\varphi(\Psi) := \frac{\langle \varphi, \Psi\varphi \rangle}{\langle \varphi, \varphi \rangle}.$$

The *AQUE theorem* of Lindenstrauss (2006) and Soundararajan (2010), answering a special case of the QUE conjecture of Rudnick–Sarnak (1994), asserts that for fixed  $\Psi$ , one has

$$\mu_\varphi(\Psi) \rightarrow \mu(\Psi) := \frac{\langle 1, \Psi \rangle}{\langle 1, 1 \rangle} \text{ as } \lambda \rightarrow \infty.$$

We consider the rate at which this convergence occurs. For a function  $\varphi \mapsto \varepsilon(\varphi)$ , we say that “ $\mu_\varphi \rightarrow \mu$  at rate  $\varepsilon(\varphi)$ ” if for each  $\Psi \in C_c^\infty(M)$  there exists  $C_\Psi \geq 0$  so that for all  $\varphi$  as above, one has

$$|\mu_\varphi(\Psi) - \mu(\Psi)| \leq C_\Psi \varepsilon(\varphi).$$

The proof of the AQUE theorem uses ergodic theory and gives “no rate.” On the other hand, it is expected (the “optimal AQUE conjecture”) that

$$\mu_\varphi \rightarrow \mu \text{ at rate } \lambda^{-1/4+\eta} \text{ for any fixed } \eta > 0;$$

Luo–Sarnak (1994) showed that this holds “on average,” Watson (2002) showed that it follows from GRH, and Luo–Sarnak (2004) and Peng Zhao (2010) showed that the exponent  $1/4$  is best possible. Unfortunately, even the weaker assertion (the “strong AQUE conjecture”) that

$$(1) \quad \mu_\varphi \rightarrow \mu \text{ at rate } \lambda^{-\delta} \text{ for some } \delta > 0$$

remains an open problem.

Sarnak (2001) and Liu–Ye (2002) showed that (1) holds in the very special case of *dihedral*  $\varphi$  on congruence covers of the modular surface. Holowinsky–Soundararajan (2010) proved an effective analogue of the AQUE theorem in the  $k \rightarrow \infty$  aspect for holomorphic forms, with rate  $(\log k)^{-\delta}$ . The tools used to prove all of these results are known to fall short of (1).

## 2. THE LEVEL ASPECT

The aim of the talk was to present a level aspect variant of (1). The problem of doing so had until recently seemed no more accessible than (1).

To that end, we consider a sequence of primes  $p \rightarrow \infty$ . We define the congruence covers  $M_p := \Gamma_0(p) \backslash \mathbb{H}$  of  $M$ , and consider a corresponding sequence  $\varphi$  of weight  $k = 2$  (say) newforms on  $\Gamma_0(p)$ . To each such  $\varphi$  we attach a probability measure  $\mu_\varphi$  on  $M$ , by the same formula as before. I had shown in 2011, using the Holowinsky–Soundararajan method, that  $\mu_\varphi \rightarrow \mu$  at rate  $(\log p)^{-\delta}$  for some  $\delta > 0$ .

## 3. MAIN NEW RESULT

We show in the above setting that  $\mu_\varphi \rightarrow \mu$  at rate  $p^{-\delta}$  for some  $\delta > 0$ , i.e., for each  $\Psi \in C_c^\infty(M)$  there exists  $C_\Psi \geq 0$  so that for all  $p$  and  $\varphi$  on  $\Gamma_0(p)$  as above,

$$(2) \quad \left| \frac{\langle \varphi, \Psi \varphi \rangle}{\langle \varphi, \varphi \rangle} - \frac{\langle 1, \Psi \rangle}{\langle 1, 1 \rangle} \right| \leq \frac{C_\Psi}{p^\delta}.$$

The proof combines a result of Munshi [1] from 2015 with my result [2] from earlier this year. We plan to present it in a joint paper.

## 4. DIVISION OF PROOF

By the spectral theory for  $L^2(M)$ , the proof of (2) divides into two cases: the Eisenstein case, in which  $\Psi$  is a unitary Eisenstein series, and the cuspidal case, in which  $\Psi$  is a Hecke–Maass cusp form. The triple product formula further reduces the problem in either case to a subconvex bound for the triple product  $L$ -function  $L(\varphi \times \bar{\varphi} \times \Psi, \frac{1}{2})$ , of degree 8. This  $L$ -function factors as a product of  $L$ -functions of degrees  $8 = 3 + 3 + 1 + 1$  in the Eisenstein case and of degrees  $8 = 6 + 2$  in the cuspidal case. Munshi’s result [1] from 2015 addresses the Eisenstein case by establishing a subconvex bound for the degree 3 factors. His technique does not seem to apply to the degree 6 factor arising in the cuspidal case.

My result [2] from earlier this year establishes the cuspidal case. The proof has the surprising feature of using the Eisenstein case as an input.

## 5. PROOF THAT THE EISENSTEIN CASE IMPLIES THE CUSPIDAL CASE

We first reduce to the problem of establishing the bound

$$(3) \quad \langle \varphi \theta, h \rangle \ll p^{-\delta}$$

where  $\theta(z) = y^{1/4} \sum e(n^2 z)$  is the Jacobi theta function and  $h$  is a certain half-integral weight theta lift of the fixed cusp form  $\Psi$  to  $\Gamma_0(p)$ . The important feature

of  $h$  is that  $\|h\| \ll 1$ . This reduction is achieved by combining a period formula of Qiu with certain local calculations; the proof laid out in full amounts to substituting into the definition

$$\mu_\varphi(\Psi) = \frac{\langle \varphi, \Psi \varphi \rangle}{\langle \varphi, \varphi \rangle}$$

the Shimizu-type identity

$$\frac{\varphi(z_1) \overline{\varphi(z_2)}}{\langle \varphi, \varphi \rangle} = \int_w \varphi(w) \Theta(z_1, z_2, w),$$

where  $\Theta(z_1, z_2, w)$  is a theta kernel attached to  $\left( \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}, \det \right)$ .

We then appeal to the amplification method of Duke–Friedlander–Iwaniec, implemented in the style of Michel–Venkatesh. This reduces the proof of (3) to that of the asymptotic formula

$$(4) \quad \|\varphi\theta\|^2 = \|\varphi\|^2 \|\theta\|^2 + O(p^{-\delta}),$$

together with its mild Hecke-twisted variants. For this we rewrite  $\|\varphi\theta\|^2$  as the inner product  $\langle |\varphi|^2, |\theta|^2 \rangle$ , which we then spectrally expand. The contribution of the constant function yields the required main term, while that of the unitary Eisenstein series may be adequately estimated using Munshi’s result. The cuspidal contribution is seen to vanish identically, thanks to the observation [3] that  $|\theta|^2$  is orthogonal to cusp forms.

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### Dynamical characterization of Maass forms

ANKE POHL

The interdependence of the geometric and the spectral data of Riemannian manifolds is of great interest in various areas, including dynamical systems, spectral theory, harmonic analysis, representation theory, number theory, and mathematical physics, in particular, quantum chaos. Over the last few years, this relation has been studied using an ever increasing number of methods which focus on the dynamics of the manifolds rather than on their (static) geometry. Among these dynamical methods are transfer operator techniques.

We discussed the development of transfer operator techniques for Riemannian surfaces (rather orbifolds)  $\Gamma \backslash \mathbb{H}$ , where  $\mathbb{H}$  denotes the hyperbolic plane and  $\Gamma$  is a geometrically finite, non-elementary Fuchsian group with at least one cusp.

The discretization for the geodesic flow on  $\Gamma \backslash \mathbb{H}$  provided in [11, 8] gives rise to a discrete dynamical system  $(D_\Gamma, F_\Gamma)$ , where  $D_\Gamma$  is a *finite* disjoint union of (the



cuspidal-free and funnel-free part of) intervals in  $\mathbb{R}$ , and  $F_\Gamma$  is piecewise given by fractional linear transformations by certain elements in  $\Gamma$ .

The associated transfer operator  $\mathcal{L}_{\Gamma,s}$  with parameter  $s \in \mathbb{C}$  is given by

$$\mathcal{L}_{\Gamma,s}f(x) := \sum_{y \in F_\Gamma^{-1}(x)} |F'_\Gamma(y)|^{-s} f(F_\Gamma(y)),$$

*a priori* acting on functions  $f \in \text{Fct}(D_\Gamma; \mathbb{C})$ . The structure of  $F_\Gamma$  yields that  $\mathcal{L}_{\Gamma,s}$  is a finite sum of slash-actions  $|_s g$  (multiplied with characteristic functions), where  $g$  runs through a finite subset of  $\Gamma$ . From this it follows immediately that  $\mathcal{L}_{\Gamma,s}$  also acts on functions defined on certain domains larger than  $D_\Gamma$ . Of major interest to us are eigenfunctions with eigenvalue 1 of  $\mathcal{L}_{\Gamma,s}$ .

For the modular group  $\text{PSL}_2(\mathbb{Z})$  we have  $D := D_{\text{PSL}_2(\mathbb{Z})} = (0, \infty) \setminus \mathbb{Q}$ . The self-map  $F_{\text{PSL}_2(\mathbb{Z})}: D \rightarrow D$  decomposes into the two branches

$$(0, 1) \setminus \mathbb{Q} \rightarrow (0, \infty) \setminus \mathbb{Q}, \quad x \mapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} .x$$

and

$$(1, \infty) \setminus \mathbb{Q} \rightarrow (0, \infty) \setminus \mathbb{Q}, \quad x \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} .x.$$

The associated transfer operator  $\mathcal{L}_s := \mathcal{L}_{\text{PSL}_2(\mathbb{Z}),s}$  reads

$$\mathcal{L}_s f(x) = f(x + 1) + (x + 1)^{-2s} f\left(\frac{x}{x + 1}\right), \quad x > 0,$$

or, equivalently,

$$\mathcal{L}_s = |_s \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + |_s \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

acting on  $\text{Fct}(\mathbb{R}_{>0}; \mathbb{C})$ . Eigenfunctions  $f$  with eigenvalue 1 of  $\mathcal{L}_s$  satisfy the *Lewis equation*

$$f(x) = f(x + 1) + (x + 1)^{-2s} f\left(\frac{x}{x + 1}\right), \quad x > 0.$$

As shown in [6, 2], the space of real-analytic functions  $f$  to the Lewis equation for which

$$(1) \quad x \mapsto \begin{cases} f(x) & \text{if } x > 0 \\ -|x|^{-2s} f\left(-\frac{1}{x}\right) & \text{if } x < 0 \end{cases}$$

extends  $C^\infty$  to 0 ('period functions') is isomorphic to the space of Maass cusp forms for  $\text{PSL}_2(\mathbb{Z})$  with spectral parameter  $s$  (see also [4]).

This kind of relation generalizes to other Fuchsian groups.

**Theorem 1** ([7, 10, 9, 8]). *Suppose that  $\Gamma$  is cofinite and  $s \in \mathbb{C}$ ,  $\text{Res} \in (0, 1)$ . Then the space of Maass cusp forms for  $\Gamma$  with spectral parameter  $s$  is isomorphic to the space of sufficiently regular eigenfunctions with eigenvalue 1 of the transfer operator  $\mathcal{L}_{\Gamma,s}$ .*

The regularity required in Theorem 1 is similar to the one for the case  $\mathrm{PSL}_2(\mathbb{Z})$ . The isomorphism from Maass cusp forms to  $\mathcal{L}_s$ -eigenfunctions is given by an integral transform, making it reasonable to consider the  $\mathcal{L}_s$ -eigenfunctions as *period functions*. The proof of Theorem 1 takes advantage of the characterization of Maass cusp forms in parabolic 1-cohomology by Bruggeman–Lewis–Zagier [3].

Theorem 1 naturally leads to several conjectures. It is reasonable to expect that also other Laplace eigenfunctions can be characterized as  $\mathcal{L}_{\Gamma,s}$ -eigenfunctions. Moreover, the construction of the transfer operators applies to non-cofinite  $\Gamma$ . In view of the results on representing Selberg zeta functions as Fredholm determinants of transfer operators [12, 1], we should expect that residues at (scattering) resonances are determined by  $\mathcal{L}_{\Gamma,s}$ -eigenfunctions.

Furthermore, finite-dimensional representations  $\chi: \Gamma \rightarrow \mathrm{GL}(V)$  can be accommodated by a transfer operator as a weight. In regard of the transfer operator approaches to  $\chi$ -twisted Selberg zeta functions for  $\chi$  having non-expanding cusp monodromy (e. g., if  $\chi$  is unitary) an analogue of Theorem 1 for  $(\Gamma, \chi)$ -automorphic functions or cusp forms should be expected [13, 1, 5].

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## An analytic class number type formula for $\mathrm{PSL}_2(\mathbb{Z})$

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(joint work with Gerard Freixas i Montplet)

### 1. THE SELBERG ZETA FUNCTION

Let  $\mathbb{H}$  denote the upper half plane and let  $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$  be a Fuchsian group of the first kind. The quotient space  $\Gamma \backslash \mathbb{H}$  admits a canonical structure of a Riemann surface. The points with non-trivial automorphisms are called elliptic fixed points. By adding a finite number of cusps, the Riemann surface  $\Gamma \backslash \mathbb{H}$  can be completed into a compact Riemann surface, which we denote by  $X$ . The hyperbolic metric on  $\mathbb{H}$  is given by

$$ds_{\mathrm{hyp}}^2 = \frac{dx^2 + dy^2}{y^2},$$

where  $x+iy$  is the usual parametrization of  $\mathbb{H}$ . As a metric on  $X$ , it has singularities at the cusps and the elliptic fixed points.

Let now  $H(\Gamma)$  denote a complete set of representatives of inconjugate, primitive, hyperbolic elements in  $\Gamma$ . For  $\gamma \in H(\Gamma)$ , we denote by  $\ell_{\mathrm{hyp}}(\gamma)$  the hyperbolic length of the closed geodesic determined  $\gamma$  on  $\Gamma \backslash \mathbb{H}$ . The Selberg zeta function  $Z(s, \Gamma)$  associated to  $\Gamma$  was introduced by Atle Selberg. For  $s \in \mathbb{C}$  with  $\mathrm{Re}(s) > 1$ , it is defined by the absolutely and locally uniformly convergent Euler product

$$Z(s, \Gamma) = \prod_{\gamma \in H(\Gamma)} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell_{\mathrm{hyp}}(\gamma)}).$$

The Selberg zeta function is known to have a meromorphic continuation to the whole complex  $s$ -plane, and its poles and zeros can be described in terms of the spectral theory of the hyperbolic Laplacian on  $\Gamma \backslash \mathbb{H}$ . In particular, the Selberg zeta function has a simple zero at  $s = 1$  and  $Z'(1, \Gamma)$  is a positive real number.

### 2. THE SPECIAL VALUE $Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$

In this section, we give an explicit formula for  $Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$  using Arakelov theory. More precisely, we apply an arithmetic Riemann–Roch theorem, namely Theorem 10.1 of [1], in the case of the coarse moduli scheme  $\mathbb{P}_{\mathbb{Z}}^1 \rightarrow \mathrm{Spec}(\mathbb{Z})$  of the Deligne–Mumford stack  $\mathcal{M}_1 \rightarrow \mathrm{Spec}(\mathbb{Z})$  of generalized elliptic curves. We interpret  $\mathbb{P}_{\mathbb{Z}}^1(\mathbb{C})$  as the Riemann surface  $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H} \cup \{\infty\}$ . The cusp at infinity and the elliptic fixed points  $i$  and  $\rho = e^{2i\pi/3}$  define integral sections  $\mathrm{Spec}(\mathbb{Z}) \rightarrow \mathbb{P}_{\mathbb{Z}}^1$ , which we denote by  $\sigma_{\infty}$ ,  $\sigma_i$ , and  $\sigma_{\rho}$ , having multiplicities  $m_{\infty} = \infty$ ,  $m_i = 2$ , and  $m_{\rho} = 3$ . In the notation of [1], we then have  $D = \sigma_{\infty} + (1/2)\sigma_i + (2/3)\sigma_{\rho}$ .

Let now  $\chi_i$  resp.  $\chi_{\rho}$  be the quadratic characters of  $\mathbb{Q}(i)$  and  $\mathbb{Q}(\rho)$ , respectively, and let  $L(s, \chi_i)$  and  $L(s, \chi_{\rho})$  denote the corresponding Dirichlet  $L$ -functions. Then, we have the following theorem.

**Theorem 1.** *The special value  $Z'(1, \text{PSL}_2(\mathbb{Z}))$  is given by*

$$\begin{aligned} \log Z'(1, \text{PSL}_2(\mathbb{Z})) &= \frac{1}{4} \frac{L'(0, \chi_i)}{L(0, \chi_i)} + \frac{13}{27} \frac{L'(0, \chi_\rho)}{L(0, \chi_\rho)} + \frac{73}{72} \frac{\zeta'(0)}{\zeta(0)} - \frac{37}{36} \frac{\zeta'(-1)}{\zeta(-1)} \\ &\quad - \frac{5}{36} \gamma + \frac{5}{12} \log 3 - \frac{167}{216} \log 2 - \frac{5}{6}, \end{aligned}$$

where  $\zeta(s)$  denotes the Riemann zeta function.

*Sketch of proof.* We only give a sketch of the proof; we use the notation of [1] and, for details, we refer the reader to [1]. To prove the statement, we employ Theorem 10.1 of [1], see also Theorem 2 of [2]. To compute the arithmetic degree  $\widehat{\text{deg}} \det H^\bullet(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1})_Q$  of the determinant of cohomology of the trivial sheaf, endowed with the Quillen metric, we first observe that  $H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}) = \mathbb{Z}$  and  $H^1(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}) = 0$ . Therefore, we get

$$12 \widehat{\text{deg}} \det H^\bullet(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1})_Q = -12 \log \|1\|_{L^2} + 6 \log (C(\text{PSL}_2(\mathbb{Z})) \cdot Z'(\text{PSL}_2(\mathbb{Z}), 1)),$$

where  $C(\text{PSL}_2(\mathbb{Z}))$  is a real positive constant, which can be explicitly expressed in terms of the multiplicities 2, 3, and  $\infty$ , the number  $c = 1$  of cusps, the number  $n = 3$  of cusps and elliptic fixed points, the genus  $g = 0$  of  $X$ , special values of the Riemann zeta function  $\zeta(s)$ , and the Euler–Mascheroni constant  $\gamma$ , see formula (1.2) of [1]. Theorem 10.1 of [1] thus implies the following equality of real numbers

$$\begin{aligned} \log Z'(\text{PSL}_2(\mathbb{Z}), 1) &= \frac{1}{6} (\omega_{\mathbb{P}_{\mathbb{Z}}^1/\mathbb{Z}}(D)_{\text{hyp}}, \omega_{\mathbb{P}_{\mathbb{Z}}^1/\mathbb{Z}}(D)_{\text{hyp}}) \\ (1) \quad &\quad - \frac{1}{6} \sum_{\substack{j, k \in \{i, \rho, \infty\} \\ j \neq k}} \left(1 - \frac{1}{m_j}\right) \left(1 - \frac{1}{m_k}\right) (\sigma_j, \sigma_k)_{\text{fin}} \\ &\quad + 2 \log \|1\|_{L^2} - \log C(\text{PSL}_2(\mathbb{Z})) - \frac{1}{6} \widehat{\text{deg}} \psi_W. \end{aligned}$$

It therefore remains to explicitly compute the contributions on the right-hand side of (1).

From the definition of the arithmetic self-intersection number of  $\omega_{\mathbb{P}_{\mathbb{Z}}^1/\mathbb{Z}}(D)_{\text{hyp}}$ , endowed with the hyperbolic metric, we derive using the relation  $\|\cdot\|_{\text{hyp}} = \frac{8}{(4\pi)^6} \|\cdot\|_{\text{Pet}}$  the equality

$$\begin{aligned} (\omega_{\mathbb{P}_{\mathbb{Z}}^1/\mathbb{Z}}(D)_{\text{hyp}}, \omega_{\mathbb{P}_{\mathbb{Z}}^1/\mathbb{Z}}(D)_{\text{hyp}}) &= \frac{1}{36} (\mathcal{M}_{12}(\Gamma(1))_{\text{Pet}}, \mathcal{M}_{12}(\Gamma(1))_{\text{Pet}}) \\ &\quad + \frac{1}{3} \log(2\pi) + \frac{1}{6} \log 2. \end{aligned}$$

Hence, employing the identity

$$(\mathcal{M}_{12}(\Gamma(1))_{\text{Pet}}, \mathcal{M}_{12}(\Gamma(1))_{\text{Pet}}) = -12 \left( \frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \right),$$

proven by Bost and Kühn, we conclude that

$$(2) \quad (\omega_{\mathbb{P}^1/\mathbb{Z}}(D)_{\text{hyp}}, \omega_{\mathbb{P}^1/\mathbb{Z}}(D)_{\text{hyp}}) = -\frac{1}{3} \left( \frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \right) + \frac{1}{3} \log(2\pi) + \frac{1}{6} \log 2.$$

In the next step, we prove that one has the following finite intersection numbers

$$(3) \quad \begin{aligned} (\sigma_\infty, \sigma_i)_{\text{fin}} &= 0, \\ (\sigma_\infty, \sigma_\rho)_{\text{fin}} &= 0, \\ (\sigma_i, \sigma_\rho)_{\text{fin}} &= \log(1728) = 6 \log 2 + 3 \log 3. \end{aligned}$$

Furthermore, the square-norm of 1 for the  $L^2$  metric is given by the volume

$$\|1\|_{L^2}^2 = \frac{1}{2\pi} \int_{\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}} \frac{dx \wedge dy}{y^2} = \frac{1}{2\pi} \frac{\pi}{3} = \frac{1}{6},$$

hence, we obtain

$$(4) \quad 2 \log \|1\|_{L^2} = -\log 2 - \log 3.$$

It remains to compute the arithmetic degree  $\widehat{\text{deg}} \psi_W$  of the  $\psi$ -bundle, endowed with the Wolpert metric. To this end, let  $E_i$  resp.  $E_\rho$  be the elliptic curves, defined over  $\mathbb{Q}$ , having complex multiplication by  $\mathbb{Q}(i)$  and  $\mathbb{Q}(\rho)$ , respectively. We denote by  $h_F(E_i)$  and  $h_F(E_\rho)$  their stable Faltings height. Then, one can prove that

$$\widehat{\text{deg}} \psi_W = 3h_F(E_i) + \frac{16}{3}h_F(E_\rho) - \frac{43}{18}(\sigma_i, \sigma_\rho)_{\text{fin}} + \frac{25}{6} \log(4\pi).$$

Consequently, by the Chowla–Selberg formula and (3), we get

$$\widehat{\text{deg}} \psi_W = -\frac{3}{2} \frac{L'(0, \chi_i)}{L(0, \chi_i)} - \frac{8}{3} \frac{L'(0, \chi_\rho)}{L(0, \chi_\rho)} + \frac{25}{6} \frac{\zeta'(0)}{\zeta(0)} - \frac{17}{2} \log 3 - \frac{15}{2} \log 2.$$

Inserting the explicit formula for  $C(\Gamma)$  together with (2), (3), and (4) into (1), finally yields the claimed formula for  $Z'(1, \text{PSL}_2(\mathbb{Z}))$ .  $\square$

Since there is a formal resemblance between the equality in Theorem 10.1 of [1] to the analytic class number formula of Dedekind zeta functions, we call the explicit expression for  $\log Z'(1, \text{PSL}_2(\mathbb{Z}))$  the analytic class number formula for  $\text{PSL}_2(\mathbb{Z})$ .

We finally remark that it would be interesting to have a direct “analytic number theoretic” evaluation of  $Z'(\text{PSL}_2(\mathbb{Z}), 1)$ , and differently see how the special values of Dirichlet  $L$ -functions above arise. The advantage of the Arakelov theoretic strategy is that the result has a geometric interpretation.

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## High Moments of Dirichlet $L$ -functions

MAKSYM RADZIWIŁŁ

(joint work with Vorropan Chandee, Xiannan Li, Kaisa Matömaki)

It is a simple consequence of the large sieve that,

$$(1) \quad \sum_{q \leq Q} \sum_{\chi \pmod{q}} |L(\tfrac{1}{2}, \chi)|^{2k} \ll Q^2 (\log Q)^{k^2}$$

for  $k = 1, 2, 3, 4$  and where the sum over  $\chi$  is over primitive characters  $\pmod{q}$ . Throughout the sum over  $\chi \pmod{q}$  will always refer to a sum over primitive characters. The upper bound (1) is tight up to a constant factor. In a recent paper [2] Conrey-Iwaniec-Soundararajan developed a technique known as the asymptotic large sieve with the aim of refining (1) to an asymptotic. For any fixed smooth function  $\psi(t)$  they obtained an asymptotic estimation of

$$(2) \quad \sum_{q \leq Q} \sum_{\chi \pmod{q}} \int_{\mathbb{R}} |L(\tfrac{1}{2} + it, \chi)|^6 \psi(t) dt$$

as  $Q \rightarrow \infty$ . The appearance of the smoothing over  $t$  is an unfortunate defect of their method. The asymptotic large sieve can only handle the so-called “central ranges” in the above moment problem. The smoothing over  $t$  is then introduced to eliminate the non-central ranges.

In a subsequent paper [1], Chandee-Li obtained an asymptotic for the 8th moment, assuming the Generalized Riemann Hypothesis. Precisely given a smooth function  $\psi(t)$ , they estimated asymptotically under GRH,

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \int_{\mathbb{R}} |L(\tfrac{1}{2} + it, \chi)|^8 \cdot \psi(t) dt.$$

as  $Q \rightarrow \infty$ . Currently one fundamental difference between the results for the sixth moment and the eight moment is that in the first case one obtains a power-saving where-as in the second case one gets by with a small logarithmic saving.

In this talk I discussed two further refinements of the above results. First of all in recent joint work we have obtained an asymptotic estimate for

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} |L(\tfrac{1}{2}, \chi)|^6$$

with a power saving in  $Q$  as  $Q \rightarrow \infty$ , thus eliminating the smoothing over  $t$  in (2). This builds on the method of Conrey, Iwaniec and Soundararajan but also imports automorphic methods to deal with the non-central ranges.

Secondly, we have also proved unconditionally the result of Chandee-Li on the 8th moment. The new techniques used there rely on sieve-theoretic ideas coming from the work of Matömaki-Radziwiłł [3] on multiplicative functions in short intervals. However we still save only a small power of the logarithm in the asymptotic estimate of the 8th moment. During the talk I presented a sketch of the proof of both results.

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## Arithmetic statistics of modular symbols

MORTEN S. RISAGER

(joint work with Yiannis N. Petridis)

Mazur, Rubin, and Stein have formulated a series of conjectures about statistical properties of modular symbols. We report on our recent work in this direction: Let  $f = \sum_{n=1}^{\infty} a_n n^{1/2} e(nz)$  be a holomorphic cusp form of weight 2 for  $\Gamma_0(q)$  with  $q$  squarefree. We consider the statistical properties of the modular symbols map

$$\mathbb{Q}/\mathbb{Z} \rightarrow i\mathbb{R}$$

$$r \mapsto \langle r \rangle = 2\pi i \int_{i\infty}^r \alpha,$$

where  $\alpha(z) = (f(z)dz + \overline{f(z)}d\bar{z})/2$ .

Let  $0 \leq x \leq 1$ . Our first result concerns the asymptotic of

$$G_c(x) = \frac{1}{c} \sum_{0 \leq \frac{a}{c} \leq x} \langle \frac{a}{c} \rangle.$$

Mazur, Rubin and Stein conjectures, based on heuristics and numerics, that  $G_c(x)$  converges to

$$g(x) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{\Re(a_n n^{1/2} (e(nx) - 1))}{n^2}$$

as  $c \rightarrow \infty$ . We prove that this conjecture holds on average:

**Theorem 1.**  $\frac{1}{M} \sum_{c \leq M} G_c(x) \rightarrow g(x)$  as  $M \rightarrow \infty$ .

Define now the usual mean and variance by

$$\mathbb{E}(f, c) = \frac{1}{\phi(c)} \sum_{\substack{a \bmod c \\ (a,c)=1}} \langle a/c \rangle, \quad \text{Var}(f, c) = \frac{1}{\phi(c)} \sum_{\substack{a \bmod c \\ (a,c)=1}} (\langle a/c \rangle - \mathbb{E}(f, c))^2.$$

We prove that the average asymptotic variance has an asymptotic expansion:

**Theorem 2.** *Let  $d|q$ . There exists  $\delta > 0$  (depending on the spectral gap) such that*

$$\frac{1}{\sum_{\substack{c \leq M \\ (c,q)=d}} \phi(c)} \sum_{\substack{c \leq M \\ (c,q)=d}} \phi(c) (\text{Var}(f, c) - C_f \log c) = D_{f,d} + M^{-\delta}, \text{ as } M \rightarrow \infty.$$

Here

$$C_f = B_q L(\text{sym}^2 f, 1), \quad D_{f,d} = A_{d,q} L(\text{sym}^2 f, 1) + B_q L'(\text{sym}^2 f, 1),$$

and

$$A_{d,q} = \frac{6 \left( -2^{-1} \log(q/d) - \sum_{p|q} \frac{\log p}{p+1} + \frac{12}{\pi^2} \zeta'(2) + \log(2\pi) \right)}{\pi^2 \prod_{p|q} (1 + p^{-1})},$$

$$B_q = -\frac{6}{\pi^2 \prod_{p|q} (1 + p^{-1})}.$$

This proves a conjecture of Mazur and Rubin on average: They conjecture the above expansion to hold for individual variances as  $c \rightarrow \infty$  through  $(c, q) = d$  (without specifying  $D_{f,d}$ ).

Finally we conclude by finding the asymptotic distribution of the modular symbols:

**Theorem 3.** *Let  $I \subseteq \mathbb{R}/\mathbb{Z}$  be an interval of positive length. be any interval of positive length, and consider for  $d|q$  the set  $Q_d = \{a/c \in \mathbb{Q}, (a, c) = 1, (c, q) = d\}$ . Then the values of the map*

$$\begin{array}{ccc} Q_d \cap I & \rightarrow & \mathbb{R} \\ \frac{a}{c} & \mapsto & \frac{\langle r \rangle}{(C_f \log c)^{1/2}} \end{array}$$

*ordered according to  $c$  have asymptotically a standard Gaussian distribution.*

The special case of  $I = \mathbb{R}/\mathbb{Z}$  was conjectured by Mazur and Rubin.

All of these results are proved through a careful analysis of the analytic properties of the following type of Eisenstein series defined originally by Goldfeld: For  $\Re(s) > 1$  it is defined by

$$E^{(k)}(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \left( 2\pi i \int_{i\infty}^{\gamma(i\infty)} \alpha \right)^k \Im(\gamma z)^s.$$

The relevance of this series to the problem we are studying is that its  $m$ th Fourier coefficients are explicitly related to the generating function

$$\sum_{\substack{c=1 \\ q|c}}^{\infty} \frac{\sum_{a \in (\mathbb{Z}/c)^*} \langle a/c \rangle^k e(ma/c)}{c^{2s}}.$$



We prove enough properties of this generating series to conclude the above theorems. In fact they are specializations of more general results for general co-finite Fuchsian group with cusps.

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**Angular distribution of Gaussian primes**

ZEÉV RUDNICK

In the lecture, we reported on joint work with Ezra Waxman [1] on the small-scale distribution of angles of Gaussian primes.

Fermat showed that every prime  $p = 1 \pmod{4}$  is a sum of two squares:  $p = a^2 + b^2$ , in which case  $a + ib$  is a prime in the ring  $\mathbb{Z}[i]$  of Gaussian integers. The representation is unique if we require  $a > b > 0$ , and we associate an angle  $\theta_p \in (0, \pi/4)$  so that  $a + ib = \sqrt{p} \exp(i\theta_p)$ . In 1919 Hecke [2] showed that these angles are uniformly distributed as  $p$  varies: If we denote by  $\mathcal{N}_{K,X}(\theta)$  the number of such angles  $\theta_p$ ,  $p \leq X$  in an arc of length  $(\pi/4)/K$  around  $\theta$ , and by  $N$  the number of primes  $p \leq X$  with  $p = 1 \pmod{4}$ , then Hecke proved that

$$(1) \quad \mathcal{N}_{K,X}(\theta) \sim \frac{N}{K}, \quad \text{as } X \rightarrow \infty.$$

In the 1950's Kubilius [3, 4] studied uniform distribution in short arcs, that is when  $K = K(X) \rightarrow \infty$  as  $X \rightarrow \infty$  in (1). Assuming the Generalized Riemann Hypothesis (GRH), the asymptotic count (1) holds as long as  $K \ll X^{1/2}$  but can fail for shorter arcs, e.g. there is a “forbidden region”: There are no primes  $p < X$  satisfying  $0 < \theta_p \ll \frac{1}{\sqrt{X}}$ .

To understand what happens for *typical* short arcs, we study the variance of  $\mathcal{N}_{K,X}(\theta)$ :

$$\text{Var}(\mathcal{N}_{K,X}) := \frac{1}{\pi/2} \int_0^{\pi/4} |\mathcal{N}_{K,X}(\theta) - \frac{N}{K}|^2 d\theta$$

We give an upper bound on this variance assuming GRH to obtain a new result on what is a very classical subject: Almost all short arcs of length slightly bigger than  $1/N$  contain a prime angle. This demonstrates the fundamental importance of the number variance.

Motivated both by a random matrix model, and by a function field analogue of this problem, we present a conjecture for the asymptotic behaviour of the number variance:

$$(2) \quad \frac{\text{Var}(\mathcal{N}_{K,X})}{N/K} \sim \min\left(1, 2 \frac{\log K}{\log N}\right), \quad X \rightarrow \infty$$

See Figure 1 for a numerical test of the conjecture. The result displays agreement with Poisson statistics ( $N$  random points) for short arcs ( $K \gg X^{1/2}$ ), but a surprising deviation from it for longer arcs ( $K \ll X^{1/2}$ ), see figure 1.

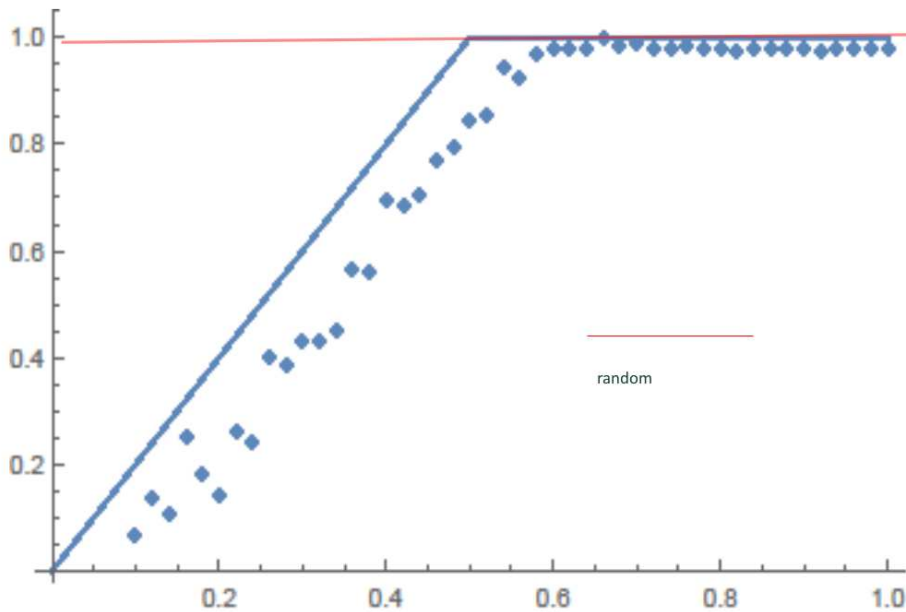


FIGURE 1. A plot of the ratio  $\text{Var}(\mathcal{N}_{K,x})/(N/K)$  as a function of  $\beta = \log K / \log N$ , for  $X \approx 10^8$ . The conjecture (2) is  $\min(1, 2\beta)$ . The variance for random points (Poisson) is the horizontal line  $\beta = 1$ . While we see a qualitative agreement with the conjecture, the fit for small  $\beta$  is not good.

Our function field model of the problem deals with representing prime polynomials as  $P(T) = A(T)^2 + TB(T)^2$ , possible if  $P(0)$  is a square in  $\mathbb{F}_q$ . The role of the Gaussian integers  $\mathbb{Z}[i]$  is played by the polynomial ring  $\mathbb{F}_q[\sqrt{-T}]$ . We assign a direction  $u(P) := (A(T) + \sqrt{-T}B(T))/(A(T) - \sqrt{-T}B(T)) \in \mathbb{F}_q[[\sqrt{-T}]]$  which is a (formal) power series in  $\sqrt{-T}$ , and plays the role of  $e^{2i\theta_p} = (a + ib)/(a - ib)$ . We then define a notion of “arcs”, or sectors, and divide the set  $\mathbb{S}^1$  of possible sectors into  $K$  equal sectors. That allows us to define a counting function  $\mathcal{N}_{K,X}$  counting the number of prime polynomials  $P$  with of degree  $n =: \log_q X$  which fall into a given sector. Its expected value is (by definition)  $E(\mathcal{N}_{K,X}) = N/K$ , where  $N$  is the number of prime polynomials of degree  $n = \log_q X$  with  $P(0)$  is a square in  $\mathbb{F}_q$ :  $N \sim \frac{1}{2}q^n/n = \frac{1}{2}X/\log_q X$ . We prove a precise asymptotic for the variance of  $\mathcal{N}_{K,X}$  as we average over all sectors, in the large finite field limit  $q \rightarrow \infty$  while

holding  $\log_q X = n$  fixed:

$$\frac{\text{Var}(\mathcal{N}_{K,X})}{N/K} \sim \begin{cases} 2 \frac{\log_q K}{\log_q N} - \frac{2}{\log_q N}, & \log_q K \leq \frac{1}{2} \log_q N + 1 \\ 1 + \frac{\eta(\log_q N) - 1}{\log_q N}, & \frac{1}{2} \log_q N + 1 \leq \log_q K \leq \log_q N. \end{cases}$$

This motivates our conjecture (2) for the Gaussian primes.

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### The sup-norm problem beyond newforms: Automorphic forms on $\text{GL}(2)$ of minimal type

ABHISHEK SAHA

(joint work with Yueke Hu, Paul D. Nelson)

Let  $\pi$  be a cuspidal automorphic representation of  $\text{GL}_2(\mathbb{A}_{\mathbb{Q}})$ . Many problems in the analytic number theory of  $\pi$  depend upon the choice of a specific  $L^2$ -normalized automorphic form  $\phi$  in the space of  $\pi$ . For example, the sup norm,  $L^p$ -norm and quantum unique ergodicity (QUE) problems have this feature, while the sub-convexity problem does not. In such problems, it is customary to work with factorizable vectors  $\phi = \otimes \phi_v$  for which

$$\phi_{\infty} = \text{lowest nonnegative weight vector in } \pi_{\infty}, \quad \phi_p = \text{newvector in } \pi_p.$$

But other reasonable choices are often possible, useful, and more natural.

In this talk, I described a particular choice for the local components  $\phi_p$  which turn out to have several remarkable properties. Briefly, assuming that  $\pi_p$  is supercuspidal and that its conductor is a fourth power, we consider  $\phi_p$  which are analogues of the lowest weight vectors in holomorphic discrete series representations of  $\text{PGL}_2(\mathbb{R})$ . For lack of better terminology, we refer to these vectors as *minimal vectors* or *vectors of minimal type*.

For automorphic forms  $\phi$  as above, we prove a sup-norm bound that is sharper than what is known in the newform case. In particular, if  $\pi_{\infty}$  is a holomorphic discrete series of lowest weight  $k$ , we obtain the optimal bound

$$C^{1/8-\epsilon} k^{1/4-\epsilon} \ll_{\epsilon} |\phi|_{\infty} \ll_{\epsilon} C^{1/8+\epsilon} k^{1/4+\epsilon}.$$

We prove also that these forms give analytic test vectors for the QUE period, thereby demonstrating the equivalence between the strong QUE and the subconvexity problems for this class of vectors. This finding contrasts the known failure of this equivalence for newforms of powerful level.

It is interesting to see what an automorphic form  $\phi$  of minimal type looks like classically. We can associate to  $\phi$  a function  $f$  on  $\mathbb{H}$  defined by  $f(z) = j(g_\infty, i)^k \phi(g_\infty)$  where  $g_\infty \in \mathrm{SL}_2(\mathbb{R})$  is any matrix such that  $g_\infty i = z$ . Then there exists an integer  $D$  and a character  $\chi_\pi$  on the “toric” congruence group

$$\Gamma_{T,D}(N) := \{(a \ b \ c \ d) \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \pmod{N}, c \equiv -bD \pmod{N}\}$$

such that

$$f|_k \gamma = \chi_\pi(\gamma) f, \quad \gamma \in \Gamma_{T,D}(N).$$

The character  $\chi_\pi$  turns out to be trivial on the principal congruence subgroup of level  $N^2$  which is contained in  $\Gamma_{T,D}(N)$ . Thus,  $f$  is a (very special) member of the space of (holomorphic or Maass) Hecke eigencuspforms of weight  $k \in 2\mathbb{Z}$  with respect to the principal congruence subgroup of level  $N^2$ .

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### On the Ramanujan conjecture for automorphic forms over function fields

WILL SAWIN

(joint work with Nicolas Templier)

Automorphic forms, in the adelic description, may be defined in a uniform way over arbitrary global fields. They have been studied much more heavily over number fields than over function fields of curves over finite field. Problems about automorphic forms in the function field setting are interesting in their own right, and may introduce techniques which can be applied over number fields, with the most striking example being Ngo Bau Cho’s use of Hitchin systems from geometric Langlands to prove the fundamental lemma [4].

Many of the most important problems about automorphic forms were solved over function fields in the  $GL_n$  case by Laurent Lafforgue. He proved the Langlands correspondence in that case, deriving as a corollary the Ramanujan conjecture and the Riemann hypothesis [2]. Vincent Lafforgue generalized the automorphic-to-Galois direction of the Langlands correspondence to general groups [3], which implies the Ramanujan conjecture and the Riemann hypothesis for forms satisfying a certain mild condition on their Langlands parameter, but it is not yet clear whether genericity, or any other representation-theoretic condition, implies this condition on the Langlands parameter side.

Some of the most important questions whose solution does not follow from these results are equidistribution questions in families of automorphic forms - The equidistribution of the local factors or of the L-functions of the set of automorphic representations with specified local conditions. Over number fields, the distribution of local factors is understood in great generality (e.g. [5] [6] [7]), but only partial results are known about the L-function. Over function fields, these can be studied in the level aspect or the  $q$  aspect (i.e. the size of the underlying finite field). In the level aspect, the situation is expected to be very similar to the number fields. In the  $q$  aspect, it may be possible to calculate the full distribution of the L-function, including all its moments and all statistics of the zeroes against arbitrary test functions, as was done by Katz for the family of all  $GL_1$ -forms with fixed squarefree conductor in [1].

Surprisingly, in the  $q$  aspect, the equidistribution of the local factors is more difficult. In fact, it is harder than the Ramanujan conjecture, and thus most likely cannot be established by a direct argument with the trace formula like that used to prove the equidistribution results over number fields, as we do not know how to use the trace formula alone to establish the Ramanujan conjecture. This can be proved by a reduction argument that deduces bounds for the Hecke eigenvalue from bounds for the average of a Hecke eigenvalue over a family, with error term going to 0 as  $q$  goes to  $\infty$ , and certain facts about cyclic base change that can be verified by local character computations.

Because the equidistribution of the local factors remains simpler than the equidistribution of the full L-function, any method to solve these equidistribution questions should also prove the Ramanujan conjecture. But it does not seem possible to prove any equidistribution results using the methods of Lafforgue that prove Ramanujan, as these are well-adapted to handling a single automorphic form at a time.

Motivated by this, Nicolas Templier and I are working on a new proof of the Ramanujan conjecture (under some local conditions). The method involves bounding the trace of a Hecke operator over the whole family, than using the previously mentioned reduction to deduce bounds for the individual Hecke eigenvalues. We bound the trace geometrically, along the lines of geometric Langlands, and this opens the possibility of geometrically studying the main term and thus attacking the equidistribution problems. Additionally, this argument should work for general  $G$ , and thus give Ramanujan for more groups than  $GL_n$ , again with local conditions.

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## Value distribution of $L$ -functions

KANNAN SOUNDARARAJAN

(joint work with Maksym Radziwiłł)

A fundamental result of Selberg [4] states that if  $t$  is chosen uniformly from  $[T, 2T]$  then  $\log |\zeta(\frac{1}{2} + it)|$  has an approximately Gaussian distribution with mean 0 and variance  $\sim \frac{1}{2} \log \log T$ . More recently, Keating and Snaith [2] have conjectured that an analogous result holds for central values of  $L$ -functions in families. To give three representative examples: (1) as  $\chi$  ranges over primitive characters  $(\bmod q)$ , one expects  $\log |L(\frac{1}{2}, \chi)|$  to be Gaussian with mean  $\sim 0$  and variance  $\sim \frac{1}{2} \log \log q$ ; (2) as  $d$  ranges over fundamental discriminants with  $|d| \leq X$ , one expects  $\log L(\frac{1}{2}, \chi_d)$  to be Gaussian with mean  $\frac{1}{2} \log \log X$  and variance  $\sim \log \log X$ ; (3) if  $f$  denotes a newform, and  $d$  runs over fundamental discriminants for which  $f \times \chi_d$  has root number 1, then we expect  $\log L(\frac{1}{2}, f \times \chi_d)$  to be Gaussian with mean  $-\frac{1}{2} \log \log X$  and variance  $\sim \log \log X$ .

These analogues for families of  $L$ -functions remain wide open, and one measure of their depth is that these conjectures imply that almost all of the central  $L$ -values in these families are non-zero. Recently Radziwiłł and I [3] described a method which (roughly speaking) shows that in any family where one can compute the first moment of central  $L$ -values (with a little extra room), one can also establish an upper bound for the frequency of large values that matches the Keating-Snaith conjecture. For example, we showed that given an elliptic curve  $E$ , the proportion of discriminants  $|d| \leq X$  such that the quadratic twist of  $E$  by  $d$  has root number 1 and satisfying

$$\frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2} \log \log |d|}{\sqrt{\log \log |d|}} \geq V$$

is at most

$$\frac{1}{\sqrt{2\pi}} \int_V^\infty e^{-t^2/2} dt + o(1).$$

In this talk, I described recent progress on the complementary problem of obtaining lower bounds for such frequencies. Such results are connected to methods for proving non-vanishing in families of  $L$ -functions, and our work bootstraps analytic methods for attacking the non-vanishing problem in order to extract further information on the size of the non-zero  $L$ -values that are produced. Here are two

sample results. In [5], I showed that for 7/8 of the fundamental discriminants  $|d| \leq X$  one has  $L(\frac{1}{2}, \chi_d) \neq 0$ . Refining this, we now establish that for any interval  $(\alpha, \beta)$  of  $\mathbb{R}$

$$\#\left\{|d| \leq X : \frac{\log |L(\frac{1}{2}, \chi_d)| - \frac{1}{2} \log \log X}{\sqrt{\log \log X}} \in (\alpha, \beta)\right\}$$

is

$$\geq \left(\frac{7}{8} \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt + o(1)\right) \#\{|d| \leq X\}.$$

The next result concerns the family of even quadratic twists of an elliptic curve. In general it is not known that a positive proportion of  $L$ -values for such twists are non-zero. Assuming the Generalized Riemann Hypothesis, Heath-Brown [1] showed that at least  $\frac{1}{4}$  of such quadratic twists do have non-zero central value. Refining Heath-Brown's result, we establish (again on GRH) that (with  $\mathcal{E}$  denoting the set of fundamental discriminants for which  $E_d$  has root number 1)

$$\#\left\{|d| \leq X : d \in \mathcal{E}, \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2} \log \log X}{\sqrt{\log \log X}} \in (\alpha, \beta)\right\}$$

is

$$\geq \left(\frac{1}{4} \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt + o(1)\right) \#\{|d| \leq X, d \in \mathcal{E}\}.$$

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### Fourier interpolation on the real line

MARYNA VIAZOVSKA

(joint work with Danylo Radchenko)

In this talk we present an explicit interpolation formula for Schwartz functions on the real line. The formula expresses the value of a function at any given point in terms of the values of the function and its Fourier transform on the set  $\{0, \pm\sqrt{1}, \pm\sqrt{2}, \pm\sqrt{3}, \dots\}$ . The functions in the interpolating basis are constructed in a closed form as an integral transform of weakly holomorphic modular forms for the theta subgroup of the modular group.

The main result presented in this talk is the following

**Theorem.** *There exists a collection of Schwartz functions  $c_0, a_n: \mathbb{R} \rightarrow \mathbb{R}$  with the property that for any Schwartz function  $f: \mathbb{R} \rightarrow \mathbb{R}$  and any  $x \in \mathbb{R}$  we have*

$$f(x) = c_0(x)f'(0) + \sum_{n=0}^{\infty} a_n(x)f(\sqrt{n}) + \sum_{n=0}^{\infty} \hat{a}_n(x)\hat{f}(\sqrt{n}),$$

where the right-hand side converges absolutely.

## Eisenstein series and the Bruggeman-Kuznetsov formula for newforms

MATTHEW YOUNG

### 1. INTRODUCTION

In a recent paper with I. Petrow [PY], the author developed a Petersson formula for newforms. As an application, we showed a cubic moment bound for twisted automorphic  $L$ -functions associated to holomorphic newforms of general square-free level (generalizing work of Conrey-Iwaniec [CI]), which in turn gave a new Weyl-type subconvexity bound for these  $L$ -functions. For some arithmetical applications, such as hybrid equidistribution problems, one wishes to generalize these results to hold for Maass forms. The first difficulty in doing so is in proving the Bruggeman-Kuznetsov formula for newforms. The only difficulty is in setting up the sieving, and showing that the same sieving procedure works equally well in both the cuspidal spectrum and the Eisenstein spectrum.

Let  $N$  be a positive integer, and consider the space of automorphic forms of level  $N$ , weight  $k \in \mathbb{Z}$ , and nebentypus  $\psi$  modulo  $N$ . There are at least two natural choices of how to decompose the space spanned by the Eisenstein series. One is to use Eisenstein series  $E_{\mathfrak{a}}(z, s, \psi)$  attached to cusps and the other is to use Eisenstein series  $E_{\chi_1, \chi_2}(z, s)$  attached to pairs of Dirichlet characters. In Theorems 1 and 2 below, we show that change-of-basis formulas relating these two bases. As a consequence, we may derive formulas for the inner product relations between Eisenstein series attached to characters; see Lemma 3 below. These formula turn out to be identical in shape to related formulas holding for cusp forms. This is the key property in sieving in the Bruggeman-Kuznetsov formula, and lets us derive a newform Bruggeman-Kuznetsov formula.

### 2. DEFINITIONS

**2.1. Eisenstein series attached to cusps.** Let  $\mathfrak{a}$  be a cusp for  $\Gamma$ , and let  $\sigma_{\mathfrak{a}}$  be a scaling matrix for  $\mathfrak{a}$ , which means  $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$ , and  $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_{\infty} = \{\pm\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} : b \in \mathbb{Z}\}$ . Let  $\tau_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}$ , so that  $\pm\tau_{\mathfrak{a}}$  generate  $\Gamma_{\mathfrak{a}}$ , the stabilizer of  $\mathfrak{a}$  in  $\Gamma$ . We say that  $\mathfrak{a}$  is *singular* for  $\psi$  if  $\psi(\tau_{\mathfrak{a}}) = 1$ . The Eisenstein series of nebentypus  $\psi$  and weight  $k$  attached to the cusp  $\mathfrak{a}$  is defined by

$$(1) \quad E_{\mathfrak{a}}(z, s, \psi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\psi}(\gamma) j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k} (\operatorname{Im} \sigma_{\mathfrak{a}}^{-1}\gamma z)^s,$$

initially for  $\operatorname{Re}(s) > 1$ .



Every singular cusp may be expressed in the form  $u/f$  where  $f|N$  and  $(u, N) = 1$ . Two such cusps  $u_1/f_1$  and  $u_2/f_2$  are  $\Gamma_0(N)$ -equivalent if and only if  $f_1 = f_2$  and  $u_1 \equiv u_2 \pmod{(f, N/f)}$ . Moreover, the cusp  $u/f$  is singular for  $\psi$  if and only if  $\psi$  is periodic modulo  $\frac{N}{(f, N/f)}$ .

**2.2. Eisenstein series attached to characters.** Let  $\chi_1, \chi_2$  be Dirichlet characters modulo  $q_1, q_2$ , respectively, with  $\chi_1(-1)\chi_2(-1) = (-1)^k$ . Define

$$(2) \quad E_{\chi_1, \chi_2}(z, s) = \frac{1}{2} \sum_{(c,d)=1} \frac{(q_2 y)^s \chi_1(c) \chi_2(d)}{|c q_2 z + d|^{2s}} \left( \frac{|c q_2 z + d|}{c q_2 z + d} \right)^k.$$

3. CHANGE OF BASIS FORMULAS

**Theorem 1.** *We have*

$$(3) \quad E_{\frac{u}{f}}(z, s, \psi) = \frac{(f, N/f)^s}{N^s} \frac{1}{\varphi((f, N/f))} \sum_{q_1 | \frac{N}{f}} \sum_{\substack{q_2 | f \\ \chi_1 \pmod{q_1} \\ \chi_2 \pmod{q_2} \\ \chi_1 \overline{\chi_2} \sim \psi}}^* \overline{\chi_1}(-u) \frac{L(2s, \chi_1 \chi_2)}{L(2s, \chi_1 \chi_2 \chi_{0, N})} \\ \sum_{\substack{a|f \\ (a, q_2)=1}} \sum_{\substack{b| \frac{N}{f} \\ (b, q_1)=1}} \frac{\mu(a)\mu(b)\chi_1(b)\chi_2(a)}{(ab)^s} E_{\chi_1, \chi_2} \left( \frac{bf}{aq_2} z, s \right),$$

where the sum is over primitive characters  $\chi_i$  modulo  $q_i$ , and  $\chi_1 \overline{\chi_2} \sim \psi$  means that both sides are induced by the same primitive character.

The inversion formula for (3) is given by the following:

**Theorem 2.** *Let  $\chi_i, i = 1, 2$ , be primitive characters modulo  $q_i$  with  $q_1 q_2 | N$ , and write  $N = q_1 q_2 L$ . Suppose  $B|L$ , and write  $L = AB$ . Then*

$$(4) \quad E_{\chi_1, \chi_2}(Bz, s) = \sum_{d|A} \sum_{\substack{e|B \\ (d,e)=1}} \frac{\chi_1(d)\chi_2(e)}{(de)^s} \\ \times \left( \frac{N}{(q_2 \frac{Bd}{e}, q_1 \frac{Ae}{d})} \right)^s \sum_{u \pmod{(q_2 \frac{Bd}{e}, q_1 \frac{Ae}{d})}}^* \chi_1(-u) E_{\frac{u}{q_2 \frac{Bd}{e}}}(z, s, \psi).$$

Here the sum is over  $u$  is over a set of representatives for  $(\mathbb{Z}/(q_2 \frac{Bd}{e}, q_1 \frac{Ae}{d})\mathbb{Z})^*$ , chosen coprime to  $N$ , and  $\psi$  is modulo  $N$ , induced by  $\chi_1 \overline{\chi_2}$ .

4. ORTHOGONALITY PROPERTIES

Let  $\mathcal{E}_{t, \psi}(N)$  be the finite-dimensional vector space defined by

$$\mathcal{E}_{t, \psi}(N) = \text{span}\{E_{\mathfrak{a}}(z, 1/2 + it, \psi) : \mathfrak{a} \text{ is singular for } \psi\},$$

and define a formal inner product  $\langle \cdot, \cdot \rangle_{\text{Eis}}$  on this space by

$$(5) \quad \frac{1}{4\pi} \langle E_{\mathfrak{a}}(\cdot, 1/2 + it, \psi), E_{\mathfrak{b}}(\cdot, 1/2 + it, \psi) \rangle_{\text{Eis}} = \delta_{\mathfrak{a}\mathfrak{b}},$$

extended bilinearly.

Let  $M = q_1q_2$ ,  $N = ML$ , and let  $\chi_i$  be primitive modulo  $q_i$ . Define the inner product

$$(6) \quad I_{\chi_1, \chi_2}(B_1, B_2; N) := \frac{1}{4\pi} \langle E_{\chi_1, \chi_2}(B_1z, 1/2 + it), E_{\chi_1, \chi_2}(B_2z, 1/2 + it) \rangle_N,$$

where  $B_1, B_2 | L$ , and the inner product is on Eisenstein series of level  $N$ . The corresponding formula for cusp forms was worked out by various authors with various degrees of generality [AU] [ILS] [BM] [Hum] [S-PY].

**Lemma 3.** *Let notation be as above. Then*

$$(7) \quad \frac{I_{\chi_1, \chi_2}(B_1, B_2; N)}{I_{\chi_1, \chi_2}(1, 1; N)} = A_{\chi_1, \chi_2} \left( \frac{B_2}{(B_1, B_2)} \right) \overline{A_{\chi_1, \chi_2} \left( \frac{B_1}{(B_1, B_2)} \right)},$$

where  $A_{\chi_1, \chi_2}(n)$  is the multiplicative function defined for  $B \geq 1$  by

$$(8) \quad A_{\chi_1, \chi_2}(p^B) = \frac{\lambda_{\chi_1, \chi_2}(p^B) - \chi_1 \overline{\chi_2}(p) p^{-1} \lambda_{\chi_1, \chi_2}(p^{B-2})}{p^{B/2}(1 + \chi_0(p) p^{-1})}.$$

Here  $\lambda_{\chi_1, \chi_2}(n) = \sum_{ab=n} \chi_1(a) \overline{\chi_2}(b) (a/b)^{it}$ , and where for  $B = 1$  we define  $\lambda_{\chi_1, \chi_2}(p^{-1}) = 0$ . Moreover,  $\chi_0$  is the principal character modulo  $q_1q_2$ .

The form of (8) is in perfect accord with the cuspidal case of [Hum, Lemma 3.13].

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