

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 44/2017

DOI: 10.4171/OWR/2017/44

**Mini-Workshop: Lattice Polytopes: Methods, Advances,
Applications**

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17 September – 23 September 2017

ABSTRACT. Lattice polytopes arise naturally in many different branches of pure and applied mathematics such as number theory, commutative algebra, combinatorics, toric geometry, optimization, and mirror symmetry. The mini-workshop on “Lattice polytopes: methods, advances, applications” focused on two current hot topics: the classification of lattice polytopes with few lattice points and unimodality questions for Ehrhart polynomials. The workshop consisted of morning talks on recent breakthroughs and new methods, and afternoon discussion groups where participants from a variety of different backgrounds explored further applications, identified open questions and future research directions, discussed specific examples and conjectures, and collaboratively tackled open research problems.

Mathematics Subject Classification (2010): 52B20.

Introduction by the Organisers

The Mini-Workshop on Lattice Polytopes: Methods, Advances, Applications organized by Takayuki Hibi (Osaka), Akihiro Higashitani (Kyoto), Katharina Jochemko (Stockholm) and Benjamin Nill (Magdeburg) was held September 17th–23th, 2017. The focus of this Mini-Workshop was on learning about the methods that led to recent advances in lattice polytope theory with a view towards further applications. It was aimed at experts as well as young researchers in order to learn from and with each other, to discuss future applications and research trends, and to discover common goals and build strong research ties. Recently, a couple of breakthroughs were achieved – quite a few of these by members of the Mini-Workshop.

Among these were a proof of a conjecture going back to Hensley from 1983 on volume bounds, an extension of White's classification to lattice tetrahedra with few lattice points, and a proof of the unimodality conjecture for h^* -polynomials of s -lecture hall polytopes. New methods have been introduced or successfully adapted from other areas such as lattice segmental fibrations, interlacing polynomials and valuations. New results had in turn applications in other fields like optimization, mirror symmetry and statistics.

During the week of the Mini-Workshop there were two contributed one-hour talks every morning followed by extensive discussions, setting the theme for the afternoon working groups concerning the above topics. Monday morning began with Gennadiy Averkov surveying the known results about maximal hollow lattice polytopes and Hensley's conjecture and Monica Blanco providing her tools for the recent classification of 3-dimensional lattice polytopes. This was followed on Tuesday morning with an introductory talk by Matthias Beck on various results about unimodality question of h^* -vectors, and the survey talk of interlacing polynomials by Katharina Jochemko including a beautiful application to the unimodality question. On Wednesday morning, Raman Sanyal gave a precise explanation about combinatorial mixed valuations and Fu Liu talked about her detailed investigation of Ehrhart positivity. On Thursday morning, Mateusz Michałek supplied several old and new open problems on normality (IDP) and very ampleness of lattice polytopes, and Johannes Hofscheier gave a talk about spanning polytopes which sheds new light on a direction towards unimodality of h^* -vectors of IDP polytopes. On Friday morning, we organized a seminar talk by Stefano Urbinati jointly with the parallel Mini-Workshop group Positivity in Higher-dimensional Geometry: Higher-codimensional Cycles and Newton-Okounkov Bodies on the connection between lattice polytopes and higher-dimensional algebraic geometry. Finally, Ivan Soprunov discussed positivity and strict monotonicity of mixed volumes. There were also two more informal 30-minutes talks on Monday and Tuesday evening. Gabriele Balletti gave a talk about his partial results concerning Hensley's conjecture, and Liam Solus described an interesting example of a family of lattice polytopes to which one may apply the methods of interlacing polynomials.

In the afternoons, after a short discussion and brainstorming round, we split into several working groups which then reported again on their findings in the large group before dinner. The groups worked on several conjectures which arose from current hot topics in the theory of lattice polytopes, several of them were introduced and posed in the morning talks. For instance, we tried to solve Oda's conjecture in dimension 3, Hensley's or Duong's conjecture in dimension 3, open problems about 2-level polytopes, alcoved polytopes, the existence of very ample (or IDP) polytopes with non-unimodal (or non-log concave) h^* -vectors, and volume bounds on Minkowski sums.

These working groups were very dynamic, stimulating and full of excitement. The small group of researchers allowed for an open and accepting atmosphere. Also the diverse range of backgrounds of the participants in combinatorics, commutative

algebra, algebraic geometry and optimization stimulated an intensive exchange of ideas.

As a direct outcome of the working groups and collaborations during the workshop we would like to explicitly mention:

- a proof of Oda's conjecture in dimension 3 for centrally symmetric polytopes;
- a proof of Duong's conjecture for the case of aligned interior lattice points;
- to our knowledge the first explicit example of a very ample (but not-normal) polytope having non-log concave h^* -vectors, which yields the possibility to construct further counterexamples for the unimodality question.

In what follows we present, in addition to summaries of the talks, brief accounts on the outcome of brainstorming sessions and working groups. We are confident that the research initiated and developed at this Mini-Workshop will continue to be pursued by the participants. As a direct continuation of the Mini-Workshop and in order to foster further interactions, Gennadiy Averkov has already set up an online forum for the participants as a place for quick questions and informal discussions.

The organizers and participants sincerely thank the institute for providing excellent working conditions and the unique Oberwolfach spirit. We are also grateful for funding from the NSF grant supporting young US-based participants, which allowed an extra participant to attend. Let us finish by an example of how effective the methods of MFO (from shuffling napkins to changing the lecture rooms) indeed are in increasing the interactions even between different workshops. After one of the organizers forgot the power cable of his computer in one of the lecture rooms, he went in to get it and noticed on the board a question of the Mini-Workshop on Positivity in Higher-dimensional Geometry about 2-Fano n -folds. Talking in the break to a participant of this workshop, they decided to combine the expertise from both communities and to start a collaborative research project on this problem.

Akihiro Higashitani
Katharina Jochemko
Benjamin Nill

Acknowledgement: The workshop organizers would like to thank the MFO for hosting this workshop and the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Unfortunately, the fourth organizer, Takayuki Hibi could not take part at the workshop.

Mini-Workshop: Lattice Polytopes: Methods, Advances, Applications

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Abstracts

Classification of hollow lattice polyhedra

GENNADIY AVERKOV

My talk was about the interplay between the following two properties of convex sets: having no interior lattice points, on the one hand, and being a polytope with all vertices lying in a lattice, on the other hand. Both properties attracted attention of experts from different areas of mathematics, including mixed-integer optimization and algebraic geometry. The underlying terminology differs from area to area. A closed d -dimensional closed convex set in \mathbb{R}^d without interior lattice points is called *lattice-free* in optimization and *hollow* in algebraic geometry. A polytope P in \mathbb{R}^d with all vertices belonging to the lattice \mathbb{Z}^d is called an *integral polytope* in optimization and a *lattice polytope* in algebraic geometry. I will stick to optimization terminology. The notion of *integral polytope* can be extended to the unbounded case. A polyhedron $P \subseteq \mathbb{R}^d$ is called *integral* if $P = \text{conv}(\mathbb{Z}^d \cap P)$.

Given a set $X \subseteq \mathbb{R}^d$ and a lattice-free set $L \subseteq \mathbb{R}^d$, we say that L is X -maximal if and only if, for every $x \in X \setminus L$, the interior of $\text{conv}(L \cup \{x\})$ contains points of \mathbb{Z}^d . \mathbb{R}^d -maximal lattice-free sets have been studied since the work of Lovász [Lov89]. In particular, Lovász gave a clear geometric characterization of them. Motivated by applications in integer optimization, Christian Wagner, Robert Weismantel and I [AWW11] asked if among all integral lattice-free polyhedra the notions of \mathbb{Z}^d -maximality and \mathbb{R}^d -maximality coincide. In this same paper we showed that, in each dimension d , there are only finitely \mathbb{Z}^d -maximal integral lattice-free polyhedra. The same result was also obtained by Benjamin Nill and Günter Ziegler [NZ11]. Thus, the two family of integral polyhedra involved in this question are essentially finite.

For $d \leq 2$, the question clearly has the positive answer, because $[0, 1]$ is the only \mathbb{Z}^1 -maximal integral lattice-free polyhedron and $[0, 1] \times \mathbb{R}$ and $\text{conv}(o, 2e_1, 2e_2)$ are the only \mathbb{Z}^2 -maximal lattice-free polyhedra and the \mathbb{R}^d -maximality of these polyhedra is easy to check. As for higher dimensions, for each $d \geq 4$, Benjamin Nill and Günter Ziegler [NZ11] constructed a d -dimensional integral polytope which is \mathbb{Z}^d -maximal but not \mathbb{R}^d -maximal.

The main result of my talk was the following result of Jan Krümpelman, Stefan Weltge and me [AKW17] that settles the remaining open case of dimension three.

Theorem 1. *Among all three-dimensional integral polyhedra, \mathbb{Z}^3 -maximality is equivalent to \mathbb{R}^3 -maximality.*

Another way to phrase the above theorem is this: if an integral lattice-free polyhedron is a subset of a larger lattice-free set, then it is also a subset of a larger integral lattice-free polyhedron.

As all integral \mathbb{R}^3 -maximal polyhedra were characterized by Christian Wagner, Robert Weismantel and me [AWW11], the above theorem yields a complete classification of all \mathbb{Z}^3 -maximal integral polyhedra.

Corollary 2. *Up to affine unimodular transformations, the family of all \mathbb{Z}^3 -maximal integral lattice polyhedra consists of the 12 bounded and 2 unbounded integral lattice-free polyhedra from [AWW11]. The 12 bounded ones are depicted in Fig. 1. The two unbounded ones are $[0, 1] \times \mathbb{R}^2$ and $\text{conv}(0, 2e_1, 2e_2) \times \mathbb{R}$.*

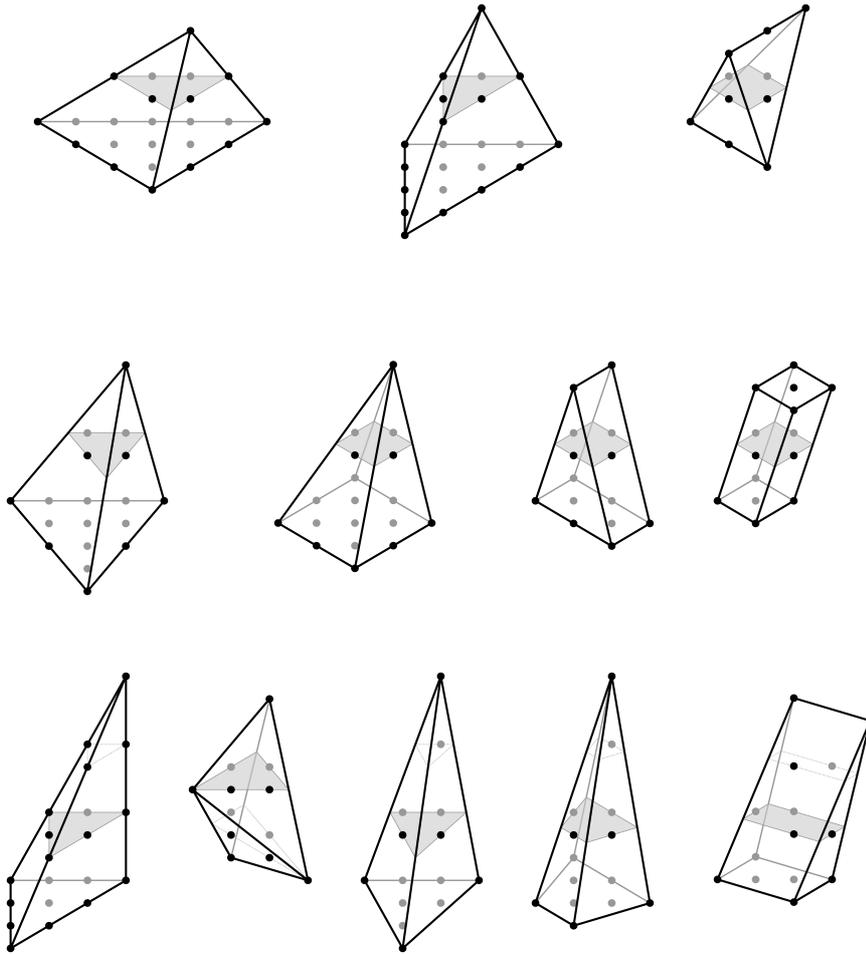


FIGURE 1. All \mathbb{Z}^3 -maximal integral lattice-free polytopes. The polytopes in the first two rows have lattice-width two, while the last row contains polytopes of lattice-width three.

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Enumerating lattice polytopes by number of lattice points

MONICA BLANCO

(joint work with Christian Haase, Jan Hofmann, Francisco Santos)

In this talk I will present the enumeration (up to unimodular transformation) of lattice d -polytopes by their *size* (total number of lattice points they contain). In dimension 1, $[0, n - 1]$ is the unique lattice segment of size n , for each $n \geq 2$. In dimension 2, Pick's formula gives an upper bound on the volume of lattice polygons of size n , for each $n \geq 3$, which in turns implies that there are only finitely many of them.

For each $d \geq 3$ and each $n \geq d + 1$, there exist infinitely many lattice d -polytopes of size n . In this talk I review the following two results:

- [BHHS17+] We prove that in every dimension $d \geq 3$ there exists a constant $w^\infty(d) \in \mathbb{N}$, depending solely on d , such that all but finitely many lattice d -polytopes of size n have width $\leq w^\infty(d)$, for each $n \geq d + 1$. We call $w^\infty(d)$ the *finiteness threshold width* in dimension d .

We show that $w^\infty(d)$ equals the maximum width of a lattice $(d - 1)$ -polytope Q such that there exist infinitely many lattice polytopes of bounded size projecting to Q . We also show that every Q with that property is hollow and is not a simplex. This allows us to prove that $d - 2 \leq w^\infty(d) \leq O(d^{3/2})$ for every $d \geq 3$ and, more particularly, that $w^\infty(4) = 2$ and $w^\infty(5) \geq 4$. That $w^\infty(3) = 1$ was already established by Blanco-Santos.

- [BS17] We develop an algorithm for the exhaustive enumeration of lattice 3-polytopes of width > 1 and size n , of which there are only finitely many for each $n \geq 4$.

For $n = 4$, White proved that no empty tetrahedron has width > 1 . In previous papers, Blanco-Santos enumerated all lattice 3-polytopes of width > 1 and sizes 5 and 6. For larger sizes we first prove that if P is a lattice 3-polytope of width > 1 and size $n \geq 7$ then one of the following happens:

- (1) All except three of the lattice points of P lie in a rational parallelepiped of width one with respect to every facet. (We call them *boxed* polytopes).
- (2) P projects in a very specific manner to one of a list of seven particular lattice polygons. (*Spiked* polytopes)

- (3) P has (at least) two vertices u and v such that both $\text{conv}(P \cap \mathbb{Z}^3 \setminus \{u\})$ and $\text{conv}(P \cap \mathbb{Z}^3 \setminus \{v\})$ still have width larger than one. (*Merged polytopes*).

Boxed polytopes have at most 11 lattice points; in particular they are finitely many, and we enumerate them completely with computer help. Spiked polytopes are infinitely many but admit a quite precise description (and enumeration) for each $n \geq 7$. Merged polytopes of size n are computed as a union (*merging*) of two polytopes of width > 1 and size $n - 1$, which are part of the input.

We have implemented the algorithm and run it until obtaining the following: there are 9, 76, 496, 2675, 11698, 45035 and 156464 lattice 3-polytopes of width larger than one and of sizes 5, 6, 7, 8, 9, 10 and 11, respectively.

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On the maximum dual volume of a canonical Fano polytope

GABRIELE BALLETTI

(joint work with Alexander M. Kasprzyk, Benjamin Nill)

Let $P \subset N_{\mathbb{R}}$ be a d -dimensional lattice polytope. We say that P is a *canonical Fano polytope* if it contains exactly one point in its interior. We can assume that this interior point is the origin of the lattice. As a consequence of results by Hensley [2] and Lagarias–Ziegler [5], there are finitely many canonical Fano polytopes (up to unimodular equivalence) in each dimension d .

Canonical Fano polytopes in dimensions $d \leq 3$ have been classified [3], and we find that $\text{vol}(P) \leq 12$. For $d \geq 4$ it is conjectured that the volume of a d -dimensional canonical Fano polytope is bounded by

$$(1) \quad \text{vol}(P) \leq \frac{1}{d!} 2(s_d - 1)^2,$$

where s_i denotes the i -th term of the *Sylvester sequence*:

$$s_1 := 2, \quad s_{i+1} := s_1 \cdots s_i + 1 \text{ for } i \in \mathbb{Z}_{\geq 1}.$$

Moreover, the case of equality in (1) is expected to be attained only by the canonical Fano simplex

$$R_{(d)} := S_{(d)} - \sum_{i=1}^d e_i, \quad \text{where } S_{(d)} := \text{conv}\{\mathbf{0}, 2(s_d - 1)e_d, s_{d-1}e_{d-1}, \dots, s_1e_1\}.$$

This conjecture is hinted at in [8, 5], explicitly stated in [6, Conjecture 1.7], and proved by Averkov–Krümpelmann–Nill [1] for the case when P is a canonical Fano simplex. The conjecture remains open for a general canonical Fano polytope. The currently best upper bound on the volume of a canonical Fano polytope that is not a simplex is established in [1, Theorem 2.7] (improving upon a result by Pikhurko [7]), however this is presumed to be far from sharp:

Instead of bounding $\text{vol}(P)$, it is also natural to consider the volume of the dual polytope P^* . In this case we are able to prove the following bound.

Theorem 1. *Let $P \subset N_{\mathbb{R}}$ be a d -dimensional canonical Fano polytope, where $d \geq 4$. Then*

$$\text{vol}(P^*) \leq \frac{1}{d!} 2(s_d - 1)^2,$$

with equality if and only if $P = R_{(d)}^$.*

In three dimensions, the expected bound $\text{vol}(P^*) \leq 12$ is proved in [3, Theorem 4.6]. In this case, however, equality is obtained by the duals of two distinct simplices:

$$(2) \quad P_{1,1,1,3} = \text{conv}\{e_1, e_2, e_3, -e_1 - e_2 - 3e_3\} \quad \text{and} \quad P_{1,1,4,6} = R_{(3)}^*.$$

The analogue of Theorem 1 is proved in [1, Theorem 2.5(b)] for d -dimensional canonical Fano simplices.

Probably one of the most studied class of canonical Fano polytopes are the *reflexive polytopes*, consisting of those $P \subset N_{\mathbb{R}}$ such that the dual P^* is also a canonical Fano polytope. Note that $R_{(d)}$ is a reflexive simplex [6]. An immediate consequence of Theorem 1 is a proof of the conjectured inequality (1) in the case of reflexive polytopes:

Corollary 2. *Let $P \subset N_{\mathbb{R}}$ be a d -dimensional reflexive polytope, where $d \geq 4$. Then*

$$\text{vol}(P) \leq \frac{1}{d!} 2(s_d - 1)^2,$$

with equality if and only if $P = R_{(d)}$.

Translated into toric geometry, this gives the following sharp upper bound on the anti-canonical degree $(-K_X)^d$ of a d -dimensional toric Fano variety X with at worst canonical singularities.

Corollary 3. *Let X be a d -dimensional toric Fano variety with at worst canonical singularities, where $d \geq 4$. Then*

$$(3) \quad (-K_X)^d \leq 2(s_d - 1)^2,$$

with equality if and only if X is isomorphic to the weighted projective space

$$\mathbb{P}(1, 1, 2(s_d - 1)/s_{d-1}, \dots, 2(s_d - 1)/s_1).$$

Our strategy to prove Theorem is as follows. We first reduce the problem to canonical Fano polytopes satisfying some minimality condition. Such polytopes have been previously studied [4, 3], and admit a decomposition into canonical Fano

simplices, for which the statement is already known [1]. We use this decomposition, together with the monotonicity of the normalised volume, to prove Theorem 1 in the majority of cases. Finally, the remaining cases are proved using a mixture of integration techniques and explicit classification.

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Unimodality Among Ehrhart h^* -polynomials

MATTHIAS BECK

In 1962, Eugène Ehrhart established the following fundamental theorem for a **lattice polytope**, i.e., the convex hull of finitely many integer points in \mathbb{R}^d .

Theorem 1 (Ehrhart [8]). *If $\mathcal{P} \subset \mathbb{R}^d$ is a lattice polytope and $n \in \mathbb{Z}_{>0}$ then*

$$\text{ehr}_{\mathcal{P}}(n) := \#(n\mathcal{P} \cap \mathbb{Z}^d)$$

*evaluates to a polynomial in n (the **Ehrhart polynomial** of \mathcal{P}). Equivalently, the accompanying generating function (the **Ehrhart series** of \mathcal{P}) evaluates to a rational function:*

$$\text{Ehr}_{\mathcal{P}}(x) := 1 + \sum_{n>0} \text{ehr}_{\mathcal{P}}(n) x^n = \frac{h_{\mathcal{P}}^*(x)}{(1-x)^{\dim(\mathcal{P})+1}}$$

for some polynomial $h_{\mathcal{P}}^(x)$ of degree at most $\dim(\mathcal{P})$, the **Ehrhart h^* -polynomial** of \mathcal{P} .¹*

We remark that the step from an Ehrhart polynomial to its rational generating function is a mere change of variables: the coefficients of $h_{\mathcal{P}}^*(x)$ express $\text{ehr}_{\mathcal{P}}(n)$ in the binomial-coefficient basis $\binom{n}{k}$, $\binom{n+1}{k}$, \dots , $\binom{n+k}{k}$, where $k = \dim(\mathcal{P})$. Stanley [12] proved the coefficients of $h_{\mathcal{P}}^*(x)$ are nonnegative integers, and several other linear constraints on the coefficients of $h_{\mathcal{P}}^*(x)$ are known; see [3] for an overview.

¹The h^* -polynomial is also known by the names of *Ehrhart h -vector* and *δ -vector/polynomial*.

It is natural to ask, for any combinatorial nonnegative sequence of integers, when the given sequence is **unimodal**, i.e., the sequence increases up to some point and then decreases. Unimodality is implied by the stronger condition of **log-concavity** ($c_j^2 \geq c_{j-1} c_{j+1}$), which in turn is implied by the even stronger condition of the associated polynomial having only real roots; see [5], which contains all of the results mentioned in this extended abstract, and [6].

In general, $h_{\mathcal{P}}^*(x)$ is not unimodal, not even when its coefficients are symmetric [9]. However, there are several families of polytopes whose h^* -polynomial do exhibit unimodality, and there are even more families for which unimodality is conjectured. A natural starting point is given by unit cubes $\mathcal{P} = [0, 1]^d$, for which

$$\text{Ehr}_{\mathcal{P}}(x) = \sum_{n \geq 0} (n + 1)^d x^n = \frac{h_{\mathcal{P}}^*(x)}{(1 - x)^{\dim(\mathcal{P})+1}}$$

and so, essentially by definition, $h_{\mathcal{P}}^*(x)$ is an **Eulerian polynomial**, which is well known to be real rooted and thus unimodal. An indication that unimodality questions are subtle is that the next natural family of polytopes, namely, lattice parallelepipeds were shown to have unimodal $h_{\mathcal{P}}^*(x)$ only a few years ago [11]. The recent paper [2] proves the more general statement that $h_{\mathcal{P}}^*(x)$ is real rooted when \mathcal{P} is a lattice zonotope, i.e., the Minkowski sum of line segments.

The following families of lattice polytopes are conjectured to have unimodal, possibly even real-rooted, h^* -polynomials:

- hypersimplices $\{\mathbf{x} \in [0, 1]^d : x_1 + \dots + x_d = k\}$;
- order polytopes $\{\mathbf{x} \in [0, 1]^P : x_j \leq x_k \text{ if } j \preceq k \text{ in } P\}$ for a given poset P ;
- alcoved polytopes (whose facet normals are either of the form \mathbf{e}_j or $\mathbf{e}_j - \mathbf{e}_k$);
- polytopes that admit unimodular triangulations (i.e., into lattice simplices of minimal volume $\frac{1}{d!}$);
- polytopes with the integer decomposition property (every lattice point in $k\mathcal{P}$ is the sum of k lattice points in \mathcal{P}).

With the exception of the first two (which are both alcoved polytopes), each family is contained in the one below it. Any of these families can also be studied with additional constraints. An example is given by reflexive polytopes that admit regular unimodular triangulations, which have been proved to come with unimodal h^* -polynomials [1]; see also [7, 15].

We finish with a potential *ansatz* to prove polynomiality which we find particularly intriguing. Fix a triangulation T of the lattice polytope \mathcal{P} . Then a theorem of Betke and McMullen [4] says that

$$h_{\mathcal{P}}^*(x) = \sum_{\Delta \in T} h_{\text{link}(\Delta)}(x) B_{\Delta}(x).$$

Here the **link** of a simplex $\Delta \in T$ is

$$\text{link}(\Delta) := \{\Omega \in T : \Omega \cap \Delta = \emptyset, \Omega \subseteq \Phi \text{ for some } \Phi \in T \text{ with } \Delta \subseteq \Phi\},$$

with accompanying h -vector (in the face-number sense) $h_{\text{link}(\Delta)}(x)$, and its **box polynomial** is

$$B_{\Delta}(x) := \sum_{\mathbf{m} \in \Pi(\Delta) \cap \mathbb{Z}^{d+1}} x^{\text{height}(\mathbf{m})}$$

where

$$\Pi(\Delta) := \left\{ \sum_{\mathbf{v} \text{ vertex of } \Delta} \lambda_{\mathbf{v}}(\mathbf{v}, 1) : 0 < \lambda_{\mathbf{v}} < 1 \right\}$$

and $\text{height}(\mathbf{m})$ denotes the last coordinate of \mathbf{m} . (Geometrically, $\Pi(\Delta)$ is the open fundamental parallelepiped of the cone over Δ .) For the empty simplex \emptyset of a triangulation, we set $B_{\emptyset}(x) = 1$.

The Betke–McMullen Theorem was greatly extended by Stanley in (and served as some motivation to) his work on local h -vectors of subdivisions [13]; see also [10]. It has a powerful consequence when \mathcal{P} has an interior lattice point; this consequence was fully realized only by Stapledon [15] who extended it to general lattice polytopes. Namely, if a lattice polytope \mathcal{P} has an interior lattice point, it admits a regular triangulation that is a cone (at this point) over a boundary triangulation. This has the charming effect that each $h_{\text{link}(\Delta)}(x)$ is palindromic (due to the Dehn–Sommerville equations). Since the box polynomials are palindromic and both kinds of polynomials have nonnegative coefficients, a little massaging yields (unique) polynomials $a(x)$ and $b(x)$ with nonnegative coefficients such that

$$h_{\mathcal{P}}^*(x) = a(x) + x b(x),$$

$a(x) = x^d a(\frac{1}{x})$, and $b(x) = x^{d-1} b(\frac{1}{x})$. The identities for $a(x)$ and $b(x)$ say that $a(x)$ and $b(x)$ are palindromic polynomials; the degree of $a(x)$ is necessarily d , while the degree of $b(x)$ is $d-1$ or smaller; in fact, $b(x)$ can be zero—this happens if and only if \mathcal{P} is the translate of a reflexive polytope.

Back to unimodality, it is well known that $h_{\text{link}(\Delta)}(x)$ is unimodal when, as above, T is the cone over a regular boundary triangulation. Thus it might be worth studying unimodality-like properties of box polynomials to deduce further results about unimodality of h^* -polynomials.

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h^* -polynomials of dilated polytopes

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Dilated lattice polytopes often play the role of a first testing ground for various conjectures in Ehrhart theory. An important such one is the unimodality conjecture, which states that every polytope having the integer decomposition property (IDP) has a unimodal h^* -vector. It was formulated by Schepers and Van Langenhoven [11] and can be traced back to a conjecture of Stanley [12] for standard graded Cohen-Macaulay domains. In [4] Bruns, Gubeladze and Trung proved that the r -th dilate of a lattice polytope has the IDP property whenever the dilation factor r is greater or equal to the dimension of the polytope minus one. In the light of the unimodality conjecture it is therefore natural to investigate the h^* -polynomials of dilated polytopes for unimodality properties. Brenti and Welker [3] and Diaconis and Fulman [6] proved that for every lattice polytope P there is a constant N such that the h^* -polynomial of rP has only negative, real roots, and therefore unimodal coefficient vector, whenever $r \geq N$. Beck and Stapledon [1] strengthened this result by proving that there is an absolute constant N that only depends on the dimension of the polytope. They furthermore conjectured the following.

Conjecture 1 ([1]). *For a d -dimensional lattice polytope P the h^* -polynomial of rP has only real zeros for all integers $r \geq d$.*

Towards a proof of this conjecture, Higashitani [8] showed that when $r \geq \deg h^*(P)$ then the coefficient vector is log-concave, a property that implies unimodality but is weaker than real-rootedness. In [9], the author of this abstract proved Conjecture 1 by showing the following.

Theorem 2. [9] Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers such that

$$\sum_{n \geq 0} a_n t^n = \frac{h(t)}{(1-t)^d}$$

for some integer d and some polynomial $h \neq 0$ of degree s with nonnegative coefficients. Let

$$\sum_{n \geq 0} a_{rn} t^n = \frac{U_r^d h(t)}{(1-t)^d}.$$

Then $U_r^d h(t)$ has only real roots for all $r \geq s$.

In the talk, the main ideas of the proof were presented. In particular, a short introduction to a rather elementary, yet very powerful tool was given: *interlacing polynomials*. For further reading on this topic a book draft by Fisk [7] and a survey article by Brändén [2] are recommended.

Definition 3. Let $f, g \in \mathbb{R}[t]$ be real-rooted polynomials with roots $\{s_1 \geq \dots \geq s_n\}$, respectively, $\{t_1 \geq \dots \geq t_m\}$. Then f interlaces g and we write $f \preceq g$, if

$$\dots \leq s_2 \leq t_2 \leq s_1 \leq t_1$$

From the intermediate value theorem it follows easily that if f interlaces g then $f + g$ is real-rooted. This rather harmless looking observation turns out to be the building block for a powerful machinery for proving real-rootedness of polynomials.

The richness of that definition comes to light when passing from two polynomials to sequences of polynomials.

Definition 4. A sequence $f_1, \dots, f_m \in \mathbb{R}[t]$ is called *interlacing* if $f_i \prec f_j$ whenever $i < j$.

For interlacing sequences the following can be deduced by the intermediate value theorem.

Theorem 5 ([7, 5]). Let f_1, \dots, f_m be an interlacing sequence of polynomials with positive leading coefficients. Then $c_1 f_1 + \dots + c_m f_m$ is real-rooted for all $c_1, \dots, c_m \geq 0$.

To prove that a sequence is interlacing can be very challenging. It is therefore desirable to investigate operations that allow to build new interlacing sequences from old ones. The following is an example for such an operation.

Proposition 6 ([7, 10]). Let f_1, \dots, f_m be a sequence of interlacing polynomials with only nonnegative coefficients. For $1 \leq l \leq m$ let

$$g_l = t f_1 + \dots + t f_{l-1} + f_l + \dots + f_m.$$

Then also g_1, \dots, g_m is an interlacing sequence.

In this case, the sequence g_1, \dots, g_m is obtained from f_1, \dots, f_m in a linear manner. In [2] Brändén gave a complete characterization of all linear operators

on interlacing sequences with nonnegative coefficients that preserve the interlacing property.

To prove Theorem 2 we studied in [9] refinement operators on formal power series. For every formal power series $f \in \mathbb{R}[[t]]$ and every integer $r \geq 1$ there are uniquely determined $f_0, \dots, f_{r-1} \in \mathbb{R}[[t]]$ such that

$$f(t) = f_0(t^r) + t f_1(t^r) + \dots + t^{r-1} f_{r-1}(t^r).$$

For the components f_i we write $f^{(r,i)}$.

Example 7. Let $f(t) = 2 + 3t + 5t^3 + 4t^4$ and $r = 2$. Then

$$f_0 = 2 + 4t^2 \quad , \quad f_1 = 3 + 5t$$

The following can be deduced from Proposition 6.

Proposition 8 ([7]). Let f be a polynomial with only nonnegative coefficients and such that $f^{(r,r-1)}, f^{(r,r-2)}, \dots, f^{(r,0)}$ is an interlacing sequence. Let

$$g(t) := (1 + t + \dots + t^{r-1})f(t).$$

Then also $g^{(r,r-1)}, g^{(r,r-2)}, \dots, g^{(r,0)}$ is an interlacing sequence.

For $0 \leq i < r$ we consider the polynomials

$$a_d^{(r,i)}(t) := \left((1 + t + \dots + t^{r-1})^d \right)^{(r,i)}.$$

From Proposition 8 and induction on d it follows that $a_d^{(r,r-1)}, \dots, a_d^{(r,0)}$ is an interlacing sequence, and therefore also $a_d^{(r,0)}, ta_d^{(r,r-1)}, ta_d^{(r,r-2)}, \dots, ta_d^{(r,0)}$ form an interlacing sequence. Starting with the observation that

$$\sum_{n \geq 0} a_{rn} t^n = \left(\sum_{n \geq 0} a_{rn} t^n \right)^{(r,0)}$$

it can be deduced that $U_r^d h(t)$ is a positive linear combination of the latter polynomials whenever $r \geq \deg h$, and therefore $U_r^d h(t)$ is real-rooted by Theorem 5. An even more careful analysis shows that in many cases all roots of $U_r^d h(t)$ are distinct. This yields a proof of the full conjecture of Beck and Stapledon given in [1]. We refer the reader to [9] for details.

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Simplices for Numeral Systems: or How I Learned to Stop Worrying and Stratify Ehrhart Unimodality among Weighted Projective Spaces

LIAM SOLUS

Where there is the Eulerian polynomials there are real-roots and *certainly* unimodal polynomials! This simple declaration of intuition has guided generalizations of well-studied polytopes to large and combinatorially intriguing families whose associated h^* -polynomials often (or always!) exhibit unimodal Ehrhart h^* -polynomials. For instance, the *lecture hall simplex* generalizes to the family of *s -lecture hall simplices* [7], all of whose h^* -polynomials are real-rooted and unimodal [8]; or the unit cube $[0, 1]^n$ whose is secretly the *order polytope* of the antichain on n elements [10]. Our efforts to better understand the phenomenon of Ehrhart unimodality is guided in part by the mysteriously challenging conjecture of Hibi and Ohsugi which states that any Gorenstein lattice polytope with the Integer Decomposition Property (IDP) will have a unimodal h^* -polynomial [5]. Over a decade of efforts aimed at testing, proving, and/or disproving this conjecture has resulted in a zoo of combinatorially exotic polytopes who provably satisfy the conjectural hypotheses (in part or in full) and/or its elusive conclusion. On one front, the ongoing saga to discover a Gorenstein and IDP lattice polytope with a non-unimodal h^* -polynomial (or to prove that no such beast exists) has left us mining the fields of lattice simplices whose associated toric varieties are weighted projective spaces. Special amongst these lattice simplices are those of the form

$$\Delta_{(1,q)} := \text{conv}(e_1, \dots, e_n, -q) \subset \mathbb{R}^n,$$

where e_1, \dots, e_n are the standard basis vectors in \mathbb{R}^n and $q := (q_1, \dots, q_n) \in \mathbb{R}^n$ is a sequence of weakly increasing positive integers. A decade before Hibi and Ohsugi presented their challenge, Hibi conjectured a more general statement: any Gorenstein lattice polytope is Ehrhart unimodal [3]. Thirteen years later, Mustata and Payne showed us counterexamples to this conjecture across even dimensional spaces [4], and three years after that Payne gave counterexamples of the form $\Delta_{(1,q)}$ in each dimension greater than five [6]. It is then natural, of course, to search this same space for counterexamples to the refined conjecture of Hibi and

Ohsugi. In the family of $\Delta_{(1,q)}$'s, the Gorenstein condition reduces to reflexivity, and Conrads [2] showed that $\Delta_{(1,q)}$ is reflexive if and only if

$$q_i \mid 1 + \sum_{j \neq i} q_j \quad \text{for all } j \in [n].$$

Extending this characterization to a statement refined for the new conjecture, Braun, Davis, and the author of this abstract provide a characterization of when $\Delta_{(1,q)}$ is reflexive and IDP in terms of divisibility conditions on the entries of the sequence q [1]. A natural goal then is to hope that we could be so lucky (or perhaps as clever) as Payne to select a $\Delta_{(1,q)}$ satisfying this condition that fails to have a unimodal h^* -polynomial. But what does our intuition say? How likely are we to avoid Ehrhart unimodality among the $\Delta_{(1,q)}$'s?

When asked the latter question, we must respond with our valiant cry of combinatorial intuition: Where there is the Eulerian polynomials, there are real-roots and unimodal h^* -polynomials! But how might we find our favorite polynomials among the $\Delta_{(1,q)}$'s? We begin by searching for a method to capture one of their fundamental properties: real-rootedness.

In this talk, we discussed the *simplices for numeral systems* [9], which are simplices of the form $\Delta_{(1,q)}$, and many of which have provably real-rooted (and thus unimodal) h^* -polynomials. The common approach to proving a polynomial is a real-rooted is to search for recursions that allow us to study the roots of the polynomials and how they interact. To relate simplices $\Delta_{(1,q)}$ across dimensions in such a way as to discover recursions relating their h^* -polynomials, we associate them to *place values of numeral systems*.

Simply put, a numeral system is a sequence of numbers $a = (a_n)_{n=0}^{\infty}$ that we use to encode numbers which satisfies $a_0 := 1$ and $a_n < a_{n+1}$ for all $n \geq 0$. For example, the sequence $a = (2^n)_{n=0}^{\infty}$ is the sequence of place values for the *binary numeral system*. To encode a nonnegative integer b uniquely with respect to a numeral system, we select the largest place value of the sequence that divides b , say a_n , and record as the n^{th} entry in a string of *digits* the number of times a_n divides into b . We then record, as the $(n-1)^{\text{st}}$ digit, the analogous quotient for the resulting remainder with divisor now a_{n-1} , and so on until we have reached a_0 . For instance, to write the number $b = 102$ in binary, we consider the expression

$$102 = 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0.$$

The representation of the number 102 in the binary numeral system $a = (2^n)_{n=0}^{\infty}$ is then 1100110. It turns out that if we pick $q = (1, 2, 4, \dots, 2^{n-1})$ to define an n -simplex $\Delta_{(1,q)}$, then its h^* -polynomial counts the number nonnegative integers less than 2^n by the number of 1's in their binary representations! From this fact, we recover quickly that this h^* -polynomial is real-rooted and therefore unimodal. In this talk, we observed that a similar construction yields $\Delta_{(1,q)}$'s, one in each dimension, for every numeral system of the form $a = (r^n)_{n=0}^{\infty}$ with real-rooted h^* -polynomials. In fact, we even saw that the unique symmetric polynomials $a(z)$ and $b(z)$ decomposing the h^* -polynomial as $a(z) + zb(z)$ are also real-rooted for this family [9]. What an abundance of real-rooted and unimodal h^* -polynomials

that pepper our fertile testing ground for the Hibi-Ohsugi conjecture! But where lie our old friends the Eulerian polynomials? In fact they are here! Their numeral system is the *factoradic numeral system* $a = ((n+1)!)_{n=0}^{\infty}$ and their corresponding $q \in \mathbb{R}^n$ in each dimension n is given by $q := (b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,n}) \in \mathbb{R}^n$, where $b_{n+1,k}$ is the coefficient of z^k in the generating polynomial

$$B(z) := \sum_{\pi \in S_{n+1}} z^{\max \text{Des}(\pi)},$$

in which $\max \text{Des}(\pi)$ denotes the maximum element of the descent set for the permutation $\pi \in S_{n+1}$ [9]. At last, our old friends the Eulerian polynomials emerge as h^* -polynomials of these simplices, suggesting a hidden wealth of Ehrhart unimodal polytopes lying just out of sight...

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Combinatorial mixed valuations

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(joint work with Katharina Jochemko)

Minkowski showed that for convex polytopes $P_1, \dots, P_d \subset \mathbf{R}^d$ the function

$$\text{Vol}(\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_d P_d)$$

agrees with a homogeneous polynomial of degree d for all $\lambda_1, \dots, \lambda_d \geq 0$; see, for example, Schneider [10]. The coefficients of that polynomial are called the *mixed volumes* $MV(P_1, \dots, P_d)$. Mixed volumes play an important role in many areas, most prominently in convex geometry and algebraic geometry. The mixed

volumes are continuous, symmetric, positively linear (with respect to Minkowski sums), and with the normalization $MV(K, K, \dots, K) = d! \text{vol}(K)$ they are unique. Said differently, the mixed volume arise from the polarization of $\text{Vol}(\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_d P_d)$.

A fundamental but non-trivial property is that mixed volumes are nonnegative and monotone with respect to inclusion, i.e.,

$$0 \leq MV(P_1, \dots, P_d) \leq MV(Q_1, \dots, Q_d),$$

for polytopes $P_1 \subseteq Q_1, \dots, P_d \subseteq Q_d$. If P_1, \dots, P_d are lattice polytopes, then the nonnegativity can be seen as a consequence of the Bernstein–Khovanskiĭ–Kushnirenko theorem; see [3].

An equally important invariant in the study of lattice polytopes is the *discrete volume* $E(P) := |P \cap \mathbf{Z}^d|$. Ehrhart [4] showed that for a lattice polytope P of dimension r , the function $E_P(n) := E(nP)$ agrees with a polynomial in $n \in \mathbf{Z}_{\geq 1}$ of degree r . Bernstein [1] and McMullen [7] independently showed that for lattice polytopes $P_1, \dots, P_k \subset \mathbf{R}^d$ the function $E(n_1 P_1 + \dots + n_k P_k)$ agrees with a polynomial of degree $r = \dim(P_1 + \dots + P_k)$. It is tempting to ask if the coefficients of this polynomial are suitable counterparts in a theory of *discrete mixed volumes*. This, however, is not true. The coefficients of $E_P(n)$ in the usual monomial basis are in general not nonnegative (as can be seen at the Reeve tetrahedra) and not monotone (as follows from Pick’s formula for the linear coefficient).

One possible reason that the naive approach to discrete mixed volumes fails is that in contrast to the volume, the discrete volume is not homogeneous. In this talk we elaborated on the notion of a *combinatorial mixed valuations* to overcome this problem. To that end, we adopt the more general perspective of translation-invariant valuations. For $\Lambda \in \{\mathbf{Z}^d, \mathbf{R}^d\}$, we denote by $\mathcal{P}(\Lambda)$ the family of polytopes with vertices in Λ . A Λ -valuation is a map $\varphi : \mathcal{P}(\Lambda) \rightarrow \mathbf{R}$ (or to any other abelian group) such that $\varphi(\emptyset) = 0$ and

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)$$

for any polytopes $P, Q \in \mathcal{P}(\Lambda)$ for which $P \cup Q$ and $P \cap Q$ are also in $\mathcal{P}(\Lambda)$ and such that

$$\varphi(P + t) = \varphi(P)$$

for all $P \in \mathcal{P}(\Lambda)$ and $t \in \Lambda$. McMullen [7] showed that for a fixed polytope P , the function $n \mapsto \varphi_P(n) := \varphi(nP)$ agrees with a polynomial of degree at most $\dim P$. Since for a fixed Q the map $P \mapsto \varphi(P + Q)$ is a Λ -valuation, it follows that $\varphi(n_1 P_1 + \dots + n_k P_k)$ agrees with a polynomial of degree at most $\dim(P_1 + \dots + P_k)$. Hence, it may be pondered if a *mixed theory* for general Λ -valuations can be established.

For Λ -valuation φ and $r \geq 0$ we define the r -th *combinatorial mixed valuation* of φ on polytopes P_1, \dots, P_r by

$$\text{CM}\varphi(P_1, \dots, P_r) := \sum_{I \subseteq [r]} (-1)^{r-|I|} \varphi(P_I),$$

where $P_I := \sum_{i \in I} P_i$ and $P_\emptyset := \{0\}$. The combinatorial mixed valuations can be algebraically characterized as the coefficients of $\varphi(n_1 P_1 + \cdots + n_k P_k)$. In particular $\text{CM}_r \varphi \equiv 0$ whenever $r > d$. If $\varphi = \text{Vol}$ is the volume, then Minkowski showed that

$$\text{CM}_d \text{Vol}(P_1, \dots, P_d) = \text{MV}(P_1, \dots, P_d).$$

If $\varphi = E$ is the discrete volume, then Bernstein [1] showed that again

$$\text{CM}_d E(P_1, \dots, P_d) = \text{MV}(P_1, \dots, P_d).$$

More recently, Bihan [2] that $\text{CM}_r E(P_1, \dots, P_r) \geq 0$ for all r and $P_1, \dots, P_r \in \mathcal{P}(\mathbf{Z}^d)$. Nonnegativity and, more generally, monotonicity does not hold for all (nonnegative) Λ -valuations. For a Λ -valuation φ , one usually defines

$$\varphi(\text{relint } P) := \sum_F (-1)^{\dim P - \dim F} \varphi(F),$$

where the sum is over all faces $F \subseteq P$. In [5], we introduced the notion of *weakly combinatorial monotone* valuations as those satisfying

$$\varphi(\text{relint } P) + \varphi(\text{relint } F) \geq 0$$

for any $P \in \mathcal{P}(\Lambda)$ and facet $F \subset P$.

Theorem [6]. If φ is a weakly combinatorial monotone Λ -valuation, then

$$0 \leq \text{CM}_r(P_1, \dots, P_r) \leq \text{CM}_r(Q_1, \dots, Q_r)$$

for any $r \geq 0$ and $P_i \subseteq Q_i$, $i = 1, \dots, r$.

This recovers the nonnegativity and monotonicity of mixed volumes and strengthens Bihan's result. In particular, this result gives an interesting consequence for the interpretation of $\text{CM}_r E(P_1, \dots, P_r)$ as a (motivic) arithmetic genus of non-compact complete intersections given in [9].

The second part of the talk focused on the proof of the theorem. For that, we gave an introduction to the algebra of polytopal simple functions as developed by Khovanskii–Pukhlikov [8] and McMullen's polytope algebra. The polytope algebra plays the role of an universal object with respect to translation-invariant valuations. Properties of valuations, such as monotonicity, can be modeled as cones (or submonoids) in the polytope algebra. Differences of the combinatorial mixed valuations associated to this universal valuation can be interpreted as nonnegative linear combinations of half-open products of simplices. For weakly combinatorial monotone Λ -valuations, this directly translates into the statement of the theorem above.

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Ehrhart positivity and McMullen's formula

FU LIU

Introduction: Ehrhart positivity. For a polytope $P \subset \mathbb{R}^d$, and for any non-negative integer t , let $i(P, t) := |tP \cap \mathbb{Z}^d|$ be the function that counts the number of lattice points in the t th dilation of P . In the 1960's Eugène Ehrhart [Ehr62] discovered that as long as P is an *integral polytope*, that is, a polytope whose vertices are lattice points, the function $i(P, t)$ is a polynomial in t of degree $\dim(P)$ (see [Sta97] for a proof). Thus, we call $i(P, t)$ the *Ehrhart polynomial* of P . Three coefficients of $i(P, t)$ are well-understood: the leading coefficient is equal to the normalized volume of P , the second coefficient is one half of the sum of the normalized volumes of facets, and the constant term is always 1. Although these three coefficients can be described in terms of volumes (considering 1 to be the volume of a point), and thus are positive, it is not true that all the remaining coefficients of $i(P, t)$ are positive. The first counterexample comes up in dimension 3, known as *Reeve's tetrahedra* [BR15, Example 3.22].

We say a polytope has *Ehrhart positivity* or is *Ehrhart positive* if it has positive Ehrhart coefficients. We are interested in the following question:

Question 1. *Which families of integral polytopes are Ehrhart positive?*

This question turns out to be very difficult. Even though multiple families of polytopes have been shown to be Ehrhart positive in the literature, the techniques involved are (almost) all different. In particular, Theorem 1.2 in [Liu09] states that for any rational polytope P , there exists a polytope P' with the same face lattice such that P' is Ehrhart positive. This result indicates that Ehrhart positivity is *not* a combinatorial property. Therefore, I am interested in other geometric methods to prove Ehrhart positivity. McMullen's formula is such a tool.

McMullen’s formula and α -positivity. In 1975 Danilov asked, in the context of toric varieties, if it is possible to assign values $\Psi(C)$ to all pointed cones C such that

$$(1) \quad |P \cap \mathbb{Z}^d| = \sum_{F: \text{a face of } P} \alpha(F, P) \text{ nvol}(F),$$

where $\alpha(F, P)$ is set to be Ψ of the “pointed feasible cone” of P at F . (The pointed feasible cone is a cone that captures local property of P at F .) McMullen [McM93] was the first to confirm the existence of (1) in a non-constructive way. Hence, we refer to the above formula as *McMullen’s formula*. Subsequently, different constructions were given by Pommersheim and Thomas [PT04], Berline and Vergne [BV07], and Schurmann-Ring [SR]. We are mostly interested in Berline-Vergne’s construction for its nice valuation and symmetry properties. We refer to their construction of Ψ and α as *BV-construction* and *BV- α -valuation*, respectively.

One notices that even the existence of McMullen’s formula has interesting consequences, providing a refinement of Ehrhart positivity. Note that pointed feasible cones do not change when we dilate a polytope. Thus, applying McMullen’s formula to tP and rearranging coefficients, we obtain a formula for the coefficient of t^k in $i(P, t)$:

$$[t^k]i(P, t) = \sum_{F: k\text{-dimensional face of } P} \alpha(P, F) \text{ nvol}(F).$$

Hence, Ehrhart positivity follows if all α -values from a certain construction are positive. This motivate us to say a polytope P has *α -positivity* or is *α -positive* if all α ’s associated to P are positive. In particular, we will say *BV- α -positivity* if we use Berline-Vergine’s construction. By above discussion, proving α -positivity is a natural approach of attacking conjectures on Ehrhart positivity.

Positivity of generalized permutohedra. In [DLHK09] De Loera, Haws, and Koeppel studied matroid base polytopes and conjectured them to be Ehrhart positive. Note that matroid base polytopes fit into a bigger family: generalized permutohedra considered by Postnikov. In joint work with Castillo [CL15, CL], we generalized conjecture of De Loera et al. to all integral generalized permutohedra, and made progress on proving our conjecture. Our discussion on McMullen’s formula implies that our conjecture can be reduced to proving all integral generalized permutohedra are BV- α -positive. However, it follows from the valuation property of BV-construction, one sees that the problem can be reduced further to the following conjecture:

Conjecture 2. *Every regular permutohedron Π_d is BV- α -positive.*

By studying this conjecture, we obtained partial results on our initial conjecture on the Ehrhart positivity of generalized permutohedra. First, we show that our conjecture is true for dimension up to 6. Next, instead of focusing on all the coefficients of Ehrhart polynomials, we study certain special coefficients, and are able to show that the third and fourth coefficients of the Ehrhart polynomial of any integral generalized permutohedra are positive. (Note that the first and second Ehrhart coefficients are always positive.) Finally, using the symmetry property

of the $BV\text{-}\alpha$ -valuation, we were able to show the linear coefficient of the Ehrhart polynomial of any integral generalized permutohedra dimension at most 100 are positive.

Some negative results. BV -construction (or probably any other construction for McMullen's formula) is not only a good geometric method for showing Ehrhart positivity, but can also be used in producing negative results. In [Bru13, Question 7.1], Bruns asked whether all smooth integral polytopes are Ehrhart positive. In joint work with Castillo-Nill-Paffenholz [CLNP], we show the answer is false by presenting counterexamples in dimensions 3 and higher. The main ideas used in our paper was chiseling cubes and searching for negative $BV\text{-}\alpha$ -values. Furthermore, we were able to construct a smooth normal fan for dimension 7 and higher, such that one of the $BV\text{-}\alpha$ -values associated to it is negative whereas any integral polytope with this fan as its normal fan is Ehrhart positive. This implies that $BV\text{-}\alpha$ -positivity is strictly stronger than Ehrhart-positivity.

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Normal and Very Ample Polytopes - old and new open problems

MATEUSZ MICHAŁEK

Normal and very ample polytopes play a central role in toric geometry. Let us consider a lattice polytope $P \subset M \otimes \mathbb{R} = \mathbb{R}^n$. Assume that the polytope P is spanning, e.g. for simplicity $0 \in P$ and the integral combinations of lattice points of P generate the whole M i.e. $\mathbb{Z}(P \cap M) = M$.

We say that a (spanning) polytope P is *normal* if and only if for every integer $k > 0$ for any $p \in kP \cap M$ there exist $p_1, \dots, p_k \in P \cap M$ such that $p = \sum_{i=1}^k p_i$. The definition of *very ample* is similar, but instead of the condition holding for every k we only require it for k large enough.

We note that given P (or in fact any set of points in \mathbb{Z}^n) we may associate to it a projective variety $X_P \subset \mathbb{P}^{|P \cap M| - 1}$ as the closure of the image of the map $(\mathbb{C}^*)^n \rightarrow \mathbb{P}^{|P \cap M| - 1}$ defined by the Laurent monomials corresponding to lattice points $P \cap M$. The variety X_P is normal if and only if P is very ample. Equivalently, if and only if X_P is the toric variety represented by the normal fan of P . For more information about these constructions we refer to [6, 7, 10, 16] and further characterizations of very ample polytopes to [1, Proposition 2.1].

There are many open problems related to normal and very ample polytopes. Below we present just a selection of 'our favorite'.

Conjecture 1.

- (1) [4, Remark 6.6] *Let B be a ball in \mathbb{R}^n . Is $\text{conv}(B \cap \mathbb{Z}^n)$ a normal polytope?*
- (2) *Does there exist a very ample, nonnormal simple polytope (or even a simplex)?*

The second point would be a step towards (finding a counterexample to) the famous Oda's conjecture:

Conjecture 2 (Oda). *Is every smooth polytope normal?*

Let us point out that very ample polytopes can be very far from normal, as shown e.g. in the following cases [11]:

- For any integer n there exists a lattice polytope P , such that for $k < n$, kP is not normal.
- There exists a very ample polytope P such that aP and bP are normal, but $(a + b)P$ is not.

The constructions are based on [1] and answer several questions from [8, 5].

Another series of very interesting questions concerns unimodality. One of the motivating questions was asked by Stanley, however not in the realm of toric geometry.

Conjecture 3. [15, Conjecture 4(a)] *Let $R = R_0 \oplus R_1 \oplus \dots$ be a graded (noetherian) Cohen-Macaulay (or perhaps Gorenstein) domain over a field $K = R_0$, which is generated by R_1 , and has Krull dimension d . Let $H(R, m) = \dim_K R_m$,*

be the Hilbert function of R , and write

$$\sum_{m \geq 0} H(R, m)x^m = (1-x)^{-d} \sum_{i=0}^s h_i x^i.$$

Then the sequence h_0, h_1, \dots, h_s is log-concave.

There are known counterexamples to Conjecture 3 in case of Cohen-Macaulay rings. Let us present them in detail as there seems to be confusion in the literature. In [3] the counterexamples are not presented, but referred to [12] and [13] (we have not found the latter preprint, however we believe that its published version is [14]).

Example 4. Let p be the homogenization in $R = \mathbb{C}[X, Y, Z, T, W]$ of the ideal $(X^{18} - X - 1 - T, Y - X^3, Z - XY) \subset \mathbb{C}[X, T, Z, T]$. Then R/p is a Cohen-Macaulay domain with h -vector $(1, 3, 5, 4, 4, 1)$.

The counterexamples indeed are Cohen-Macaulay and do not have a log-concave h -vector. However, all of them are unimodal. Meanwhile, some authors started referring to Conjecture 3 as Stanley's unimodality conjecture, e.g. [2, p. 696]. As far as we know however the following conjecture is open (we are not sure to who it should be attributed).

Conjecture 5. For a standard graded Cohen-Macaulay domain the h -vector is unimodal.

Further the counterexamples we referred to do not define normal projective varieties. This has following consequences. First, in toric world normal projective toric varieties correspond to very ample polytopes. Thus in case of nonnormal projective toric varieties the Hilbert polynomial differs from the Ehrhart polynomial - this is in fact one of the characterizations of very ample polytopes. Thus, in such a case the Hilbert polynomial does not provide information about h^* -vector. Second, it motivates the following questions that we find interesting:

Conjecture 6. Do normal, standard graded Cohen-Macaulay domains have log-concave h -vectors? What if we only require their Proj to be normal?

In toric setting we ask for the following:

Conjecture 7. Do very ample polytopes always have unimodal h^* -vectors?

For further conjectures with additional assumptions on normality or Gorenstein we refer to [9] and the report of L. Solus present in this volume.

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Spanning Lattice Polytopes

JOHANNES HOFSCHEIER

In this note we explore spanning lattice polytopes with a view toward Ehrhart theory and their relation to smooth, very ample and IDP polytopes.

Let $M \cong \mathbb{Z}^d$ be a lattice and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ the associated vector space. An M -spanning lattice polytope $P \subseteq M_{\mathbb{R}}$ (or just “spanning lattice polytope” if the ambient lattice is clear from the context) is a lattice polytope whose lattice points affinely span the ambient lattice M . These polytopes are sometimes also called *primitive* (see, for instance, [2]). From now on, let $M = \mathbb{Z}^d$.

Every 2-dimensional lattice polytope is spanning whereas the Reeve-simplex $\text{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + r\mathbf{e}_3) \subseteq \mathbb{R}^3$ is not spanning for $r \geq 2$ where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{Z}^3$ denotes the standard basis.

The spanning-assumption is a mild condition for lattice polytopes. Indeed, one can associate a spanning polytope \tilde{P} to every lattice polytope P : \tilde{P} is the same polytope as P , but considered as a lattice polytope with respect to the lattice spanned by the lattice points in P . Moreover, the class of spanning lattice polytopes seems to be manageable and it has many interesting properties.

The h^* -polynomial of a lattice polytope P is the numerator polynomial of the generating series of the *Ehrhart polynomial* which counts lattice points in dilations

of the polytope. The coefficient-vector $h^*(P)$ of the h^* -polynomial is called its h^* -vector. We refer the reader to [1] for a beautiful introduction to this subject and for further details.

A sequence a_0, \dots, a_n of real numbers has *no internal zeros* if there do not exist integers $0 \leq i < j < k \leq n$ satisfying $a_i a_k \neq 0$ and $a_j = 0$. The sequence is called *unimodal* if for some $0 \leq j \leq n$ we have $a_0 \leq a_1 \leq \dots \leq a_j \geq a_{j+1} \geq \dots \geq a_n$.

It is a challenging and fascinating problem to determine what geometric, arithmetic or combinatorial conditions on lattice polytopes lead to unimodality or no internal zeros for their h^* -vectors. The following theorem can be seen as a partial answer to this question.

Theorem 1 ([6]). *Let P be a spanning lattice polytope of degree s , i.e., the right-most nonzero entry of the h^* -vector $h^*(P) = (h_0^*, h_1^*, \dots, h_s^*)$ has index s . Then, for every $i, j \in \mathbb{N}$ with $i + j < s$, we have*

$$(1) \quad h_1^* + h_2^* + \dots + h_i^* \leq h_{j+1}^* + h_{j+2}^* + \dots + h_{j+i}^*.$$

In [9, Proposition 3.4], Stanley showed these inequalities under the stronger assumption that P is IDP which stands for “integer decomposition property” and means that for every integer $k > 0$ and any $\mathbf{x} \in kP \cap \mathbb{Z}^d$, there exist $\mathbf{x}_1, \dots, \mathbf{x}_k \in P \cap \mathbb{Z}^d$ such that $\mathbf{x} = \sum_{i=1}^k \mathbf{x}_i$. The inequalities of Theorem 1 imply many other known inequalities, e.g., the special case $i = 1$ implies that the h^* -vector of a spanning lattice polytope has no internal zeros (see [7]) and can be considered as a generalisation of Hibi’s Lower Bound Theorem:

Corollary 2 (Hibi’s Lower Bound Theorem [5]). *For a d -dimensional lattice polytope P with an interior lattice point, we have $h_1^*(P) \leq h_j^*(P)$ for $1 \leq j < d$.*

Proof. Since \tilde{P} is obtained by coarsening the ambient lattice, its h^* -vector satisfies $h_j^*(P) \geq h_j^*(\tilde{P})$. As P and \tilde{P} contain the same number of lattice points, it follows that

$$h_1^*(P) = h_1^*(\tilde{P}) \leq h_j^*(\tilde{P}) \leq h_j^*(P). \quad \square$$

The proof of Theorem 1 uses arguments from algebraic geometry applied to the *Ehrhart ring* of the polytope. In the spanning-case, in contrast to the IDP-case, the Ehrhart ring needs not to be generated in degree 1, and thus the geometry takes place in a *weighed* projective space where surprisingly not everything goes the same as for the “straight” projective space (see [3]). We refer the reader to the upcoming paper [6] for further details of the proof.

Let us complete this note with a comparison of the spanning-property to other conditions on lattice polytopes. A lattice polytope P is called *smooth* if its normal fan is smooth, i.e., every top-dimensional cone is generated by a lattice basis. The definition of a *very ample* lattice polytope is similar to the IDP one, but here the condition needs only to be true for k sufficiently large.

Figure 1 depicts the relations (known and conjectured ones) between the above mentioned notions. No arrow in Figure 1 can be reversed as can be seen by the following examples.

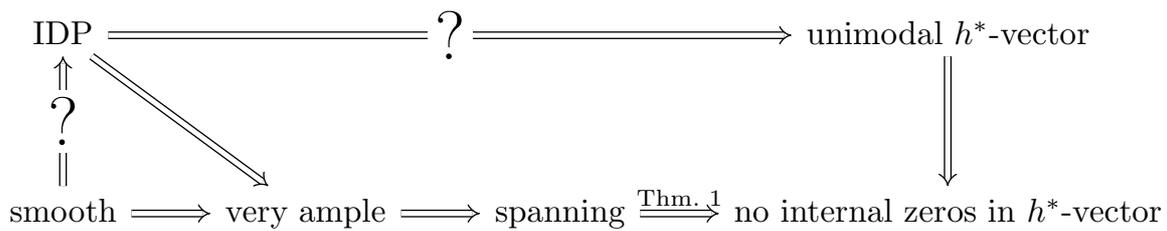


FIGURE 1. The conjectured implications are known as the *Unimodality Conjecture in Ehrhart Theory* and (a special case of) *Oda's Conjecture*. The rightmost horizontal implication in the second row follows by Theorem 1 and can be considered as a modest analogue of the unimodality conjecture.

Example 3. *The lattice polytope $P = \text{conv}(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_4, 5(\mathbf{e}_1 + \dots + \mathbf{e}_4) + 8\mathbf{e}_5) \subseteq \mathbb{R}^5$ is a spanning but not very ample. Indeed, as $2(\mathbf{e}_1 + \dots + \mathbf{e}_4) + 3\mathbf{e}_5$ is contained in P , it is spanning. Using the software *polymake* (see [4]), one can check that P is not very ample. Moreover, the h^* -vector of P equals $(1, 1, 2, 1, 2, 1)$, and thus is not unimodal.*

Example 4. *In [8], one can find several very ample but not IDP polytopes. These polytopes are also not smooth.*

We have the following question that we find interesting:

Question 5. *How can Figure 1 be extended to comprise real-rooted or log-concave h^* -vectors?*

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Positivity and strict monotonicity of the mixed volume

IVAN SOPRUNOV

(joint work with Frederic Bihan)

One of the fundamental results in the theory of Newton polytopes is Bernstein’s theorem which gives an upper bound for the number of isolated solutions to a system of Laurent polynomials. Recall that the *support* $\mathcal{A}(f)$ of a Laurent polynomial $f \in \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ is the set of lattice points $a \in \mathbb{Z}^d$ such that the corresponding Laurent monomial $t^a = t^{a_1} \dots t^{a_d}$ appears in f . The Newton polytope $P(f)$ is the convex hull of $\mathcal{A}(f)$, which is a lattice polytope in \mathbb{R}^d . Then Bernstein’s theorem says that the number of isolated solutions in $(\mathbb{C}^*)^d$ of a system $f_1 = \dots = f_d = 0$ is at most $d!V(P_1, \dots, P_d)$ solutions, where $P_i = P(f_i)$ for $i = 1, \dots, d$. The quantity $V(P_1, \dots, P_d)$ is the *mixed volume*, which is the polarization of the Euclidean volume form. In particular, it is symmetric, Minkowski additive, and $V(P, \dots, P)$ coincides with $\text{vol}(P)$, the Euclidean volume of P .

The mixed volume is non-negative and monotone with respect to the inclusion, that is

$$V(P_1, \dots, P_d) \leq V(Q_1, \dots, Q_d),$$

whenever $P_i \subseteq Q_i$ for $i = 1, \dots, d$. In the Einstein workshop on Lattice Polytopes in December of 2016, Frédéric Bihan asked the following question. Let P_1, \dots, P_d be polytopes in \mathbb{R}^d and Q be the convex hull of $P_1 \cup \dots \cup P_d$. Can we characterize when

$$V(P_1, \dots, P_d) = \text{vol}(Q)$$

geometrically? He was motivated by the problem of improving Bernstein’s estimate, in the case when the Newton polytopes are equal to one polytope Q , by applying an invertible linear transformation to the coefficients of the system. This operation preserves the solution set and the union of the supports of the system, but may change the individual supports. In other words, it produces an equivalent system with some Newton polytopes P_1, \dots, P_d satisfying $Q = \text{conv}(P_1 \cup \dots \cup P_n)$. Thus, if $V(P_1, \dots, P_d) < \text{vol}(Q)$ we get a better bound.

Our main result is a geometric criterion for the strict inequality

$$V(P_1, \dots, P_d) < V(Q_1, \dots, Q_d)$$

in terms of essential collections of “touched faces”, see [1]. For two polytopes $P \subseteq Q$ we say P *touches* a face F of Q if $P \cap F \neq \emptyset$. In particular, we show that if polytopes P_1, \dots, P_d are contained in a d -dimensional polytope Q then $V(P_1, \dots, P_d) < \text{vol}(Q)$ if and only if Q has a face F of dimension $k < d$ which is touched by at most k of the P_i . This criterion has a simple interpretation in

the theory of Newton polytopes. Let $f_1 = \cdots = f_d = 0$ be a Laurent polynomial system with the same Newton polytope Q . Each face F of Q defines a submatrix C_F of the coefficient matrix of the system. Then the system has strictly less than $d! \operatorname{vol}(Q)$ isolated solutions if $\operatorname{rank} C_F < \dim F + 1$.

The geometric criterion for strict monotonicity holds for arbitrary real polytopes. In the case of lattice polytopes one should be able to say more about the difference $V(Q_1, \dots, Q_d) - V(P_1, \dots, P_d)$, as the normalized mixed volume $d!V(P_1, \dots, P_d)$ is always a non-negative integer. For example, suppose lattice polytopes P_1, \dots, P_d lie in a lattice polytope Q and there is a *facet* $F \subset Q$ not touched by m of the P_i . Then one can show that $d! \operatorname{vol}(Q) - d!V(P_1, \dots, P_d) \geq m$. It would be interesting to see if this could be generalized to faces of smaller dimension.

Another possible application of our criterion is to the problem of classifying collections of lattice polytopes P_1, \dots, P_d with the fixed value of the mixed volume. A recent result by Esterov and Gusev [2] provides such a classification in the case of normalized mixed volume one. For full-dimensional collections their classification is particularly simple. One can show that there exist only finitely many collections of d full-dimensional lattice polytopes with fixed mixed volume, up to unimodular transformations and independent translations. Classifying them for small values of the mixed volume or in small dimensions seems feasible.

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Discussion session: “Hensley’s and Duong’s conjecture in dimension 3”

GENNADIY AVERKOV, GABRIELE BALLETTI, MÓNICA BLANCO, BENJAMIN NILL, IVAN SOPRUNOV

Conjecture 1 (Duong’s conjecture). *Let $\Delta \subseteq \mathbb{R}^3$ be a clean tetrahedron with exactly $k > 0$ interior lattice points. Then the normalized volume of Δ is at most $12k + 8$.*

Moreover, this volume is uniquely achieved by the following clean tetrahedron, for each $k \geq 1$:

$$T_k := \operatorname{conv} \{(-1, -1, 1), (1, -1, -2), (1, 0, 2k - 1), (-1, 2, 0)\}$$

Remember that a lattice polytope is clean if the only lattice points in the boundary are the vertices.

Conjecture 2 (Hensley’s conjecture for dimension 3). *Let $P \subseteq \mathbb{R}^3$ be a lattice 3-polytope with exactly $k > 0$ interior lattice points. Then the normalized volume of P is at most $36(k + 1)$.*

Moreover, this volume is uniquely achieved by the following polytope, for each $k \geq 2$:

$$S_k^3 := \text{conv}\{(0, 0, 0), (2, 0, 0), (0, 3, 0), (0, 0, 6(k+1))\}$$

For $k = 1$ another polytope, different than S_1^3 , also achieves this volume. In both T_k and S_k^3 , the k interior points are collinear, so we study whether Duong's conjecture is true or not by independently asking the following questions:

- (1) Is the conjecture true for the case where the interior lattice points are collinear?
- (2) If the interior lattice points are not collinear, is it true then that the volume is not maximized?

For question (1) we are confident that we can find a tight upper bound as follows:

Claim 3. *Let $\Delta \subset \mathbb{R}^3$ be a clean tetrahedron with $k > 0$ collinear interior points. Let $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the lattice projection that maps the segment of interior lattice points to the origin. Then if $k \geq 3$, $\pi(\Delta)$ is a reflexive polygon.*

With this, the number of projections (reflexive triangles or quadrilaterals) is finite, and with the cleanness condition, the enumeration of the possible tetrahedra that project to each of these polygons is relatively small and easy to derive, and a tight bound on the volume of them can be achieved.

To determine a volume maximizer within the family of *all* clean tetrahedra Δ it thus remains to study the case where the set X of all interior lattice points of Δ is not collinear (meaning that the set is of dimension at least 2). In this case, we want to refine the arguments of Pikhurko for deriving volume bounds of non-hollow lattice simplices. First of all, X contains a point lying 'rather centrally' in the interior of Δ . This allows Pikhurko to derive a bound on the volume. The main ingredients of Pikhurko's approach are: his 'jump' method for localizing the position of the most central points, and a lemma that allows to bound the volume of a simplex using Van der Corput's theorem (which bounds the barycentric coordinates of the interior lattice points). We recall that Van der Corput's theorem links the number of interior lattice points and the volume for origin-symmetric convex bodies. It turns out that if X is at least two-dimensional, we can apply a modification of Van der Corput's theorem that has both a strengthened assumption and a strengthened assertion.

It might be that using a modified Van der Corput's inequality we can also say something about the following

Conjecture 4 (Hensley's conjecture for 3-dimensional simplices). *Let $\Delta \subseteq \mathbb{R}^3$ be a lattice tetrahedron with exactly $k > 0$ interior lattice points. Then the normalized volume of Δ is at most $36(k+1)$.*

It would be nice to at least confirm the bound in the case where X is of dimension 2.

We notice that Claim 3 is also true for non-clean, non-simplices in dimension 3, which implies that this falls under the conditions of Hensley's conjecture for

collinear points. Then maybe both conjectures in dimension 3 can be attacked at the same time. So we forget about the cleanness condition and consider *all* lattice 3-polytopes with $k > 0$ interior lattice points. Then we look at the restrictions that the shape of these interior points imposes to the lattice points in the boundary of the polytope.

We consider the case when the interior lattice points in the polytope form a 2-dimensional polygon Q . We again have arguments to limit the possible positions of the boundary lattice points. More specifically:

Claim 5. *For most possibilities of Q all the boundary lattice points are at lattice distance ≤ 1 from the plane spanned by Q . That is, in particular the polytope has width 2.*

Now we need to address two different problems:

- Study the exceptional cases that do not allow for these bound on the distance. We believe that we can prove that they can only happen for small values of k , using volume bounds on canonical and hollow 3-polytopes.
- The next problem that arises is to try and bound the volume of the whole polytope, based on the maximum area of its intersection with the plane that contains Q . But for this we need to know a bound on the area of a rational polytope that has Q as its set of interior lattice points, knowing only the number k of lattice points of Q .

Discussion session: “Oda’s conjecture”

MATTHIAS BECK, CHRISTIAN HAASE, AKIHIRO HIGASHITANI, JOHANNES HOFSCHEIER, KATHARINA JOCHEMKO, LUKAS KATTHÄN, MATEUSZ MICHAŁEK

We have investigated Oda’s conjecture which states that smooth polytopes are normal. It is one of the most intriguing open problems in toric geometry. While many experts believe there may exist high dimensional counterexamples, still a proof for three dimensional polytopes is very much sought for. Let P be a smooth polytope. In dimension 3 it is sufficient to prove that every lattice point in $2P$ is a sum of two lattice points of P . Equivalently that for any lattice point z if the intersection of P and $-P + z$ is nonempty then it contains a lattice point, or also equivalently every half integer point in P is a mid point of a lattice segment with endpoints in P . Christian Haase recalled us the following result: for any facet F of P let F_{-1} be a polytope obtained as an intersection of P and a hyperplane parallel to F and of lattice distance 1 to F . Then the convex hull of F and F_{-1} is a normal lattice polytope. It means that a neighborhood of the boundary is covered by normal polytopes and (quite counterintuitively) the only ‘problematic’ points are ‘deep in interior’. We decided that it is worth to try to prove Oda’s conjecture for centrally symmetric polytopes. The main reason is that in that case for any lattice point $p \in P$ the (rational) polytope $\frac{1}{2}(P + p)$ contains 0. Hence, we can cover a large part of the polytope with a star neighbourhood of 0 that is

a sum of polytopes of the previous type. An article proving Oda's conjecture for three dimensional smooth centrally symmetric polytopes will be written.

Discussion session: "Alcoved polytopes"

MATTHIAS BECK, CHRISTIAN HAASE, JOHANNES HOFSCHEIER, KATHARINA JOCHEMKO, LUKAS KATTHÄN, FU LIU, RAMAN SANYAL, AKIYOSHI TSUCHIYA

Our plan was to consider the question whether the h^* -vector of alcoved polytopes is unimodal. Recall that a polytope is *alcoved* if is of the form

$$P = \{x \in \mathbb{R}^d \mid x_i - x_j \leq a_{ij}, l_i \leq x_i \leq u_i \text{ for } 1 \leq i, j \leq d\}$$

for some $a_{ij}, l_i, u_i \in \mathbb{Z}$. These polytopes admit a canonical triangulation which is balanced, i.e., its vertices can be colored such that no two vertices of the same color are in the same simplex.

Our approach to showing unimodality was via exhibiting a Lefschetz element. For this, we guessed that the sums over the color classes should yield a linear system of parameters. This is true for balanced simplicial complexes, and we hope that one can use a deformation argument to lift this to our setting. Further, we tried to find an explicit vector space basis for the Artinian reduction of the Ehrhart ring in terms of a half-open decomposition of the triangulation. This seems to be difficult.

Finally, we guessed that the sum over the vertices of a single simplex should be a Lefschetz element. We verified this in two small examples.

Discussion session: "Multivariate stable h^* -polynomials"

AKIHIRO HIGASHITANI, KATHARINA JOCHEMKO, MATEUSZ MICHALEK,
ALEXANDER KASPRZYK, RAMAN SANYAL, LIAM SOLUS

Stable polynomials are multivariate generalizations of real-rooted univariate polynomials. In the light of the unimodality conjecture a current hot topic is to find and investigate lattice polytopes with real-rooted h^* -polynomials. The aim of this discussion group was to discuss multivariate stable polynomials in the context of Ehrhart theory. We discussed two appearances of multivariate polynomials in Ehrhart theory. The first one was counting lattice points in Minkowski sums of lattice polytopes. The Bernstein-McMullen Theorem [2, 3] asserts that for lattice polytopes $P_1, \dots, P_k \in \mathbb{Z}^d$, $E_{P_1, \dots, P_k}(n_1, \dots, n_k) := |n_1 P_1 + \dots + n_k P_k \cap \mathbb{Z}^d|$ agrees with a multivariate polynomial for integers $n_1, \dots, n_k \geq 0$ of degrees $d_i := \deg_{n_i}(E_{P_1, \dots, P_k}) \leq \dim P_i$. Therefore, the multivariate generating series can be written in the form

$$\sum_{n_1, \dots, n_k \geq 0} E_{P_1, \dots, P_k}(n_1, \dots, n_k) t_1^{n_1} \dots t_k^{n_k} = \frac{h_{P_1, \dots, P_k}(t_1, \dots, t_k)}{\prod_{i=1}^k (1 - t_i)^{d_i+1}},$$

where $h_{P_1, \dots, P_k}(t_1, \dots, t_k)$ is a polynomial of degree at most d_i in t_i . Recently, Beck, Jochemko and McCullough [1] proved that h^* -polynomials of zonotopes are

real-rooted. Towards a multivariate analog we proved in the discussion round that $h_{P_1, \dots, P_k}(t_1, \dots, t_k)$ is stable whenever P_1, \dots, P_k generate an axes-parallel box. However, this is not true for general lattice parallelepipeds.

We also discussed multivariate generating functions of the form

$$\sum_{n \geq 0} \sum_{\mathbf{a} \in nP \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{a}} = \frac{h(\mathbf{x})}{\prod_{\mathbf{a} \in P \cap \mathbb{Z}^d} (1 - \mathbf{x}^{\mathbf{a}})}.$$

We observed that already in small dimensions calculating the numerator polynomial gets quickly involved. It might be worthwhile to study families of polytopes for which the numerator is stable or at least hyperbolic with respect to certain directions.

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Discussion session: “2-level Polytopes”

JOHANNES HOFSCHEIER, FU LIU, ALEXANDER KASPRZYK, RAMAN SANYAL
AND AKIYOSHI TSUCHIYA

A polytope $P \subseteq \mathbb{R}^d$ (not necessarily integral) is called *2-level* if for every facet F of P there is a translation of its supporting hyperplane which contains the vertices not in F . For instance, hypersimplices are 2-level. These polytopes have a particular nice combinatorial structure, e.g., up to affine linear isomorphisms (not necessarily unimodular) they are equivalent to the intersection of a unit hypercube $[0, 1]^n$ with an affine hyperplane which does not intersect any edge of the hypercube.

From the definition it straightforwardly follows that a polytope is centrally symmetric and 2-level if and only if its dual is centrally symmetric and 2-level. This motivates the following question:

Question 1. *What are the centrally symmetric 2-level reflexive polytopes?*

For instance $Q = \text{conv}(P_G \times \{1\}, -P_G \times \{-1\})$ is a centrally symmetric 2-level reflexive polytope where P_G denotes the stable set polytope of a perfect graph G . Indeed, Q is reflexive by [HT17, Theorem 1.1 (b)] and 2-level by a straightforward argument using the following theorem:

Theorem 2 ([GPT10, Corollary 4.9.]). *A polytope is 2-level with a simple vertex if and only if it is affinely isomorphic to the stable set polytope of a perfect graph.*

This result also motivates our final question:

Question 3. *Can the previous statement be generalised for bigger classes of 2-level polytopes? Is there a similar description for all 2-level polytopes?*

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Discussion session: “Very Ampleness, IDP, Log-concavity, and Unimodality”

GABRIELE BALLETTI, AKIHIRO HIGASHITANI, JOHANNES HOFSCHEIER,
MATEUSZ MICHAŁEK, BENJAMIN NILL, LIAM SOLUS

One of the fundamental projects of this mini-workshop was to study when a d -dimensional lattice polytope P has a unimodal Ehrhart h^* -polynomial, a polynomial denoted by $h^*(P; z) := h_0^* + h_1^*z + \cdots + h_d^*z^d$ in this summary. Some of the most challenging open conjectures on Ehrhart unimodality (i.e. the unimodality of $h^*(P; z)$) arise in the context of more general conjectures pertaining to graded semigroup algebras. The first such algebraic conjecture that we will consider was proposed by Stanley in his survey article on log-concave and unimodal sequences in algebra and combinatorics.

Conjecture 1. [11, Conjecture 4(a)] *Let $R = \bigoplus_{m=0}^{\infty} R_m$ be a graded (Noetherian) Cohen-Macaulay (or perhaps Gorenstein) domain over a field $R_0 := K$, which is standard (i.e. generated by R_1) and has Krull dimension d . Let $H(R, m) := \dim_K R_m$ be the Hilbert function of R and write*

$$\sum_{m \geq 0} H(R, m)z^m = (1 - z)^{-d} \sum_{i=0}^s h_i z^i.$$

Then the polynomial $\sum_{i=0}^s h_i z^i$ is log-concave.

This conjecture was disproven in its full generality by Niesi and Robbiano [6]. Using the computer algebra system CoCoA, they showed, for instance, that the Cohen-Macaulay domain

$$\mathbb{Q}[x, y, z]/\langle x^{18} - x - 1, y - x^3, z - xy \rangle$$

has $\sum_{i=0}^s h_i z^i = 1 + 3z + 5z^2 + 4z^3 + 4z^4 + z^5$, which is not log-concave. Less than a year later, in a new survey article on the same topic, Brenti updated Stanley’s original conjecture as follows:

Conjecture 2. [2, Conjecture 5.1] *Let R be a graded standard Gorenstein domain. Then the polynomial $\sum_{i=0}^s h_i z^i$ defined in Conjecture 1 is unimodal.*

Brenti goes on to note that the statement of Conjecture 2 is also open if we substitute the word “Gorenstein” with “Cohen-Macaulay” or if we substitute the word “unimodal” with the word “log-concave.” This story has a place in the context of this workshop since each of these conjectures yields, as a special case, a conjecture on the h^* -polynomials of lattice polytopes.

To uncover the special cases of each of these conjectures that are relevant to this mini-workshop, we recall that the Ehrhart ring of a d -dimensional lattice polytope $P \subset \mathbb{R}^n$ is defined as follows: Let x_1, \dots, x_n, x_{n+1} be indeterminants over some field K , and let $mP := \{m\alpha : \alpha \in P\}$ be the m^{th} dilate of P for $m \in \mathbb{Z}_{>0}$. Let $A(P)_m$ denote the vector space over K spanned by the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_{n+1}^m$ for all $(\alpha_1, \dots, \alpha_n) \in mP \cap \mathbb{Z}^n$. The graded algebra

$$A(P) := \bigoplus_{m=0}^{\infty} A(P)_m$$

is called the *Ehrhart ring* of P . The polynomial $\sum_{i=0}^s h_i z^i$ for the ring $A(P)$ is the h^* -polynomial of P , and by a theorem of Hochster [4] we know that $A(P)$ is a Cohen-Macaulay domain. Moreover, $A(P)$ is standard if and only if P is IDP (i.e. for every $m \in \mathbb{Z}_{>0}$ each lattice point in mP can be written as the sum of m lattice points in P). Additionally, $A(P)$ is Gorenstein if and only if $h^*(P; z)$ is a symmetric polynomial [10]. The collection of conjectures given by Stanley and Brenti that remain open in the context of the Ehrhart rings $A(P)$ can be stated in terms of the lattice polytopes P and their h^* -polynomials as follows:

Conjecture 3.

- (1) If P is IDP then $h^*(P; z)$ is unimodal.
- (2) If P is IDP the $h^*(P; z)$ is log-concave.
- (3) If P is Gorenstein and IDP then $h^*(P; z)$ is unimodal.
- (4) If P is Gorenstein and IDP then $h^*(P; z)$ is log-concave.

These conjectures have also appeared in the discrete geometry setting. Conjecture 3 (1) appeared in [9, Question 1.1] and Conjecture 3 (3) is often attributed to [8]. Evidence supporting these conjectures is ever growing in abundance, however it remains less than clear how strong the IDP hypothesis is in regards to Ehrhart unimodality. Just as unimodality is a weakening of log-concavity, IDP has a weakening of its own. A polytope P is called *very ample* if for any sufficiently large $m \in \mathbb{Z}_{>0}$ every lattice point in mP can be written as the sum of m lattice points in P . An IDP polytope P is very ample, but not every very ample polytope is IDP [3, 5]. To better estimate the strength of the IDP hypothesis in the above conjectures, we may consider the analogous family of questions:

Question 4.

- (1) If P is very ample, is $h^*(P; z)$ unimodal?
- (2) If P is very ample, is $h^*(P; z)$ log-concave?
- (3) If P is Gorenstein and very ample, is $h^*(P; z)$ unimodal?
- (4) If P is Gorenstein and very ample, is $h^*(P; z)$ log-concave?

Given Conjecture 3, it is natural to search for counterexamples to Questions 4 (1),(2),(3), and (4) arising from the family of very ample but non-normal lattice polytopes. For starters, using a construction due to Lasoń and Michałek [5], we can produce a counterexample to Question 4 (2). In particular, the 9-dimensional

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 101 \end{pmatrix}.$$

FIGURE 1. The columns of A are the vertices of a very ample non-normal polytope with a non-log-concave h^* -polynomial.

lattice polytope P with vertices the columns of the matrix A in Figure 1 has h^* -polynomial $h^*(P; z) = 1 + 8z + 8z^2 + 8z^3 + 104z^4$, and this polynomial is not log-concave. The polytope P is a *segmental fibration* [1, Definition 2.2] over the *edge polytope* P_{C_8} (defined in [7]) of the cycle graph C_8 on eight nodes. Lasoń and Michałek showed that taking similar segmental fibrations over edge polytopes for other even length cycles produces non-normal but very ample polytopes. Empirically, these fibrations can further be chosen to produce numerous counterexamples to Question 4 (2). However, answers to the remaining questions and open conjectures discussed in this summary are still needed. This line of investigation was motivated in part by the talk given by Mateusz Michałek during this Oberwolfach mini-workshop.

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Discussion session: “Mixed volume”

GENNADIY AVERKOV, BLANCO MONICA, FU LIU, BENJAMIN NILL, IVAN SOPRUNOV

We were working on the following problem. Let P_1, \dots, P_d be d -dimensional lattice polytopes in \mathbb{R}^d . Give a sharp upper bound on the normalized volume of the Minkowski sum $P_1 + \dots + P_d$ when the value of the normalized mixed volume $V(P_1, \dots, P_d)$ is a fixed integer m . We conjectured that the upper bound equals $(m + d - 1)^d$. Clearly, it is attained at $P_1 = m\Delta$, $P_2 = \dots = P_d = \Delta$, where Δ is the standard d -simplex. We confirmed the conjecture in the two-dimensional case.

Discussion session: “Fibonacci and Fibonacci-like Polynomials as h^* -polynomials of Lattice Polytopes”

CHRISTIAN HAASE, LIAM SOLUS

The n^{th} *Fibonacci (or Jacobsthal) polynomial* [3] is defined by the recursion

$$F_n(z) = F_{n-1}(z) + zF_{n-2}(z),$$

with the initial conditions $F_0 = F_1 = 1$. More generally, a *Fibonacci-like polynomial* is a polynomial $a_n(z)$ defined by a recursion of the form

$$a_n(z) = a_{n-1}(z) + za_{n-2}(z)$$

with some chosen initial conditions $a_0, a_1 \in \mathbb{Z}_{>0}$. For example, the n^{th} *Lucas polynomial* is a well-studied Fibonacci-like polynomial which is given by taking the initial conditions $a_0 = 2$ and $a_1 = 1$. The Fibonacci-like polynomials, similar to the Eulerian polynomials [5], are a class of real-rooted and unimodal generating polynomials that appear throughout enumerative combinatorics and combinatorial optimization. For instance, the Fibonacci and Lucas polynomials appear as *independence polynomials* for claw-free graphs [4] and as generating polynomials for *Markov equivalence classes of DAG models* [6]. They also admit well-studied multivariate generalizations [1]. However, unlike the Eulerian polynomials, the Fibonacci-like polynomials rarely appear as the h^* -polynomials of lattice polytopes (or more generally as the h -polynomials of Hilbert series of graded algebras). In fact, the only such occurrence recorded in the literature thus far is for n odd, the n^{th} Lucas polynomial arises as the h^* -polynomial of an $(n - 1)$ -dimensional r -stable (n, k) -hypersimplex [2].

A major goal of this mini-workshop is to study when and why a lattice polytope P is *Ehrhart unimodal*; i.e., when its h^* -polynomial $h^*(P; z)$ is unimodal. One increasingly popular method by which to prove Ehrhart unimodality is to show that $h^*(P; z)$ is real-rooted; that is, that all roots of $h^*(P; z)$ are real numbers. It is therefore natural to ask which families of real-rooted generating polynomials may arise as h^* -polynomials of lattice polytopes. Despite their suggested rarity in the field thus far, the following theorem verifies that all Fibonacci-like polynomials can arise as the h^* -polynomial of a lattice polytope.

Theorem 1. *Let $P \subset \mathbb{R}^n$ be a lattice polytope, $Q \prec P$ and $e \in \mathbb{Z}^n$ primitive such that $P \subset Q + \mathbb{R}_{\geq 0}e$. Then*

$$R := \text{conv}(P \times \{0\} \cup \{(0, 1), (-e, 1)\}) \subset \mathbb{R}^{n+1}$$

has h^* -polynomial $h^*(R; z) = h^*(P; z) + zh^*(Q; z)$.

Recursively applying Theorem 1 with the primitive e_n being the n^{th} standard basis vector in \mathbb{R}^n and initializing with Q the one-dimensional line segment

$$Q := \text{conv}((0, 0), (0, 2)),$$

and P the two-dimensional lattice polytope

$$P := \text{conv}((0, 0), (0, 2), (1, 1), (1, 0)),$$

will produce a lattice polytope $R_n \subset \mathbb{R}^n$ for which $h^*(R_n; z) = F_n(z)$ for all $n \geq 2$. Similarly, for the same choice of primitive in each dimension, and initial conditions

$$Q := \text{conv}((0, 0), (0, 3)) \subset P := \text{conv}((0, 0), (0, 3), (1, 1), (1, 0)),$$

produces a lattice polytope $R_n \subset \mathbb{R}^n$ with $h^*(R_n; z)$ the n^{th} Lucas polynomial. Analogous choices of P and Q in dimensions two and one, respectively, allow us to realize all Fibonacci-like polynomials as h^* -polynomials of lattice polytopes. Future work on this problem will aim at generalizing these methods to describe which real-rooted generating polynomials can be realized as h^* -polynomials.

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