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Mini-Workshop: Reflectionless Operators: The Deift and Simon Conjectures

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Abstract. Reflectionless operators in one dimension are particularly amenable to inverse scattering and are intimately related to integrable systems like KdV and Toda. Recent work has indicated a strong (but not equivalent) relationship between reflectionless operators and almost periodic potentials with absolutely continuous spectrum. This makes the realm of reflectionless operators a natural place to begin addressing Deift's conjecture on integrable flows with almost periodic initial conditions and Simon's conjecture on gems of spectral theory establishing correspondences between certain coefficient and spectral properties.

Mathematics Subject Classification (2010): 35Q53, 35B15, 47B36, 47A55.

Introduction by the Organisers

This mini-workshop was organized by David Damanik (Rice), Fritz Gesztesy (Baylor), and Peter Yuditskii (Linz). The program consisted of 15 lectures on a broad variety of problems related to the Deift and Simon conjectures, including perturbation theory, integrable systems, random matrix theory, character-automorphic Hardy spaces, and orthogonal polynomials. This workshop intended to provide a cutting-edge survey of new results for reflectionless operators, especially those results directed towards addressing Deift's conjecture regarding the almost periodicity of solutions to the KdV equation with almost-periodic initial data and Simon's conjecture regarding gems of spectral theory establishing a one-to-one correspondence between suitable classes of coefficient data and spectral data. This forum provided a great background for discussions of some of the extant open problems in the field.

Sixteen mathematicians took part in this mini-workshop, most of whom traveled from abroad to attend. The organizers and participants would like to extend their sincere gratitude towards the MFO for their hospitality and for providing a beautiful location to discuss and do mathematics.

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Mini-Workshop: Reflectionless Operators: The Deift and Simon Conjectures

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Abstracts

Chebyshev polynomials and Totik-Widom bounds

Jacob Stordal Christiansen
(joint work with Barry Simon, Peter Yuditskii, and Maxim Zinchenko)

Let $\mathsf{E} \subset \mathbb{R}$ (or $\mathbb{C}$) be a compact set of positive logarithmic capacity. The nth *Chebyshev polynomial* for E, denoted T_n, is the unique monic degree n polynomial of least deviation from zero on E. We write

$$t_n := \|T_n\|_{\mathsf{E}} = \inf\big\{\|P\|_{\mathsf{E}} : \deg(P) = n \text{ and } P \text{ is monic}\big\}$$

for the approximation error ($\|\cdot\|_{\mathsf{E}}$ denotes the sup-norm on E) and note that T_n is real whenever $\mathsf{E} \subset \mathbb{R}$.

The aim of this talk is to discuss the asymptotic behaviour of $T_n(z)$ and t_n as $n \to \infty$. In particular, we seek to understand precisely when the ratio $t_n/\operatorname{Cap}(\mathsf{E})^n$ is bounded. This scaling is natural since already Szegő proved that for $\mathsf{E} \subset \mathbb{C}$, one has that $t_n \geq \operatorname{Cap}(\mathsf{E})^n$ and

$$\lim_{n\to\infty} t_n^{1/n} = \operatorname{Cap}(\mathsf{E}).$$

In order to get more refined results for the asymptotics, we need further assumptions on E (as described below).

For the asymptotics of $T_n(z)$, it is natural to consider

$$\Phi_n(z) := \frac{T_n(z)B(z)^n}{\operatorname{Cap}(\mathsf{E})^n},$$

where B is a certain function (to be specified shortly). In fact, when E is a smooth Jordan curve, B is the Riemann map from the outer component onto $\mathbb{D}$ and $\Phi_n(z) \to 1$ (uniformly on the outer component). In the multiply connected setting, the function B becomes multivalued. It is determined by

$$|B(z)| = \exp\{-g(z)\},$$

where g is the Green's function for $\overline{\mathbb{C}} \setminus \mathsf{E}$ with pole at ∞, and the requirement that

$$B(z) = \operatorname{Cap}(\mathsf{E})/z + \mathcal{O}(1/z^2) \text{ near } \infty.$$

Going around a closed curve will change B by a phase factor and hence B is *character automorphic* with character, say χ_{E}. Note that $\Phi_n(z)$ cannot have a limit — for its character ($= \chi_{\mathsf{E}}^n$) is n dependent. Widom realized that the natural candidate for the asymptotics is the function, F_n, which among all character automorphic functions G on $\overline{\mathbb{C}} \setminus \mathsf{E}$ with character χ_{E}^n and with $G(\infty) = 1$ minimizes

$$\|G\|_\infty := \sup\{|G(z)| : z \in \overline{\mathbb{C}} \setminus \mathsf{E}\}.$$

In his landmark paper [4], he proved that for E a finite union of smooth Jordan curves, one has that $\Phi_n(z) \sim F_n(z)$ uniformly on compact subsets of the universal cover of $\overline{\mathbb{C}} \setminus \mathsf{E}$.

For more complicated sets, it is not clear whether the so-called *Widom minimizer*, F_n, exists and is unique. The main result to be presented here shows that for a large class of infinite gap sets $\mathsf{E} \subset \mathbb{R}$, this is indeed so and the expected asymptotic formula holds true. We shall always assume that E is regular (for potential theory).

Theorem 1 ([2])**.** *Suppose that* $\mathsf{E} \subset \mathbb{R}$ *obeys the PW and DCT conditions. Then*

$$\lim_{n\to\infty}\left[\frac{T_n(z)B(z)^n}{\mathrm{Cap}(\mathsf{E})^n} - F_n(z)\right] = 0$$

uniformly on compact subsets of the universal cover of $\overline{\mathbb{C}} \setminus \mathsf{E}$*. Moreover,* F_n *is almost periodic (as a function of n) and*

$$\lim_{n\to\infty}\frac{t_n}{\mathrm{Cap}(\mathsf{E})^n\,\|F_n\|_\infty} = 2.$$

Widom [4] conjectured this asymptotic behaviour of $T_n(z)$ for finite gap sets. After beeing open for more than 45 years, the issue was finally settled in [1]. The above result goes way beyond and [2] even has a simpler and more direct proof.

Let us briefly comment on the *Parreau–Widom* (PW) condition and the *direct Cauchy theorem* (DCT). Set $\Omega := \overline{\mathbb{C}} \setminus \mathsf{E}$ and let $\pi_1(\Omega)$ denote the fundamental group of Ω. By definition, E is a PW set if

$$PW(\mathsf{E}) := \sum\nolimits_j g(c_j) < \infty,$$

where $\{c_j\}$ are the critical points of g (i.e., points where Δg vanishes). Widom proved that E is PW if and only if $H^\infty(\Omega,\chi)$ contains a non-zero function for every character $\chi \in \pi_1(\Omega)^*$. This fact leads to the existence of Widom minimizers, F_χ, and a separate argument shows the uniqueness. The function $Q_\chi := F_\chi/\|F_\chi\|_\infty$ solves the dual maximization problem, that is,

$$Q_\chi(\infty) = \sup\bigl\{g(\infty) :\ g \in H^\infty(\Omega,\chi),\ \|g\|_\infty = 1\bigr\}.$$

A result of Hasumi and Hayashi states that E obeys DCT if and only if the function $\chi \mapsto Q_\chi(\infty)$ is *continuous* on $\pi_1(\Omega)^*$. Interestingly, these *dual Widom maximizers* turn out to be Blaschke products with at most one zero per gap of E.

As will be outlined, the key to the proof is to study the sets

$$\mathsf{E}_n := T_n^{-1}([-t_n,t_n]).$$

Clearly, $\mathsf{E}_n \supset \mathsf{E}$ and the inclusion is strict precisely when T_n has a zero in one of the gaps of E. By the alternation theorem, T_n has at most one such zero per gap. The proof also relies on the fact that

$$t_n \le D\cdot \mathrm{Cap}(\mathsf{E})^n \ \text{ for some constant } D.$$

Such a bound was established by Totik [3] for finite gap sets and this also follows implicitly from the results of Widom [4]. One of the central issues of [1] was to prove that every PW set obeys a Totik–Widom bound:

Theorem 2 ([1]). *For any PW set* $\mathsf{E} \subset \mathbb{R}$*, one has that*

$$t_n \le 2\exp\{PW(\mathsf{E})\} \cdot \operatorname{Cap}(\mathsf{E})^n.$$

It would be interesting to know if a bound of this type also applies to other infinite gap sets, for instance the middle 3rd Cantor set. The following question will be raised: *Does there exist a set* $\mathsf{E} \subset \mathbb{R}$ *which is not PW, but for which a Totik–Widom bound holds true? Moreover, can a set of Lebesgue measure zero obey a Totik–Widom bound?*

There are several examples in the literature of Cantor-type sets for which $t_n/\operatorname{Cap}(\mathsf{E})^n$ is unbounded and even examples of very thin sets for which the ratio grows subexponentially. But is PW the borderline?

We conclude by presenting a weak converse of the above theorem. A set $\mathsf{E} \subset \mathbb{C}$ is said to have a *canonical generator* if the orbit $\{\chi_{\mathsf{E}}^n\}_{n\in\mathbb{Z}}$ is dense in $\pi_1(\Omega)^*$. This property is generic (in various senses) but does *not* apply to, e.g., the middle 3rd Cantor set.

Theorem 3 ([2]). *Suppose that* E *has a canonical generator. If*

$$t_n \le D \cdot \operatorname{Cap}(\mathsf{E})^n \quad \textit{for some constant } D,$$

then E *is a PW set.*

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Universality for numerical calculations with random data.

Percy Deift

(joint work with Govind Menon, Sheehan Olver, and Thomas Trogdon)

In [5], the authors considered the computation of the eigenvalues of a random $n \times n$ matrix M using various standard algorithms. Let Σ_n denote the set of real $n \times n$ symmetric matrices. Associated with each algorithm $\mathcal{A}$ there is, in the discrete case, a map $\phi = \phi_{\mathcal{A}} : \Sigma_n \to \Sigma_n$ with the properties

- (isospectral) $spec(\phi_{\mathcal{A}}(H)) = spec(H)$
- (convergence) the iterates $X_{k+1} = \phi_{\mathcal{A}}(X_k), k \ge 0, X_0 = M$, converge to a diagonal matrix $X_\infty, X_k \to X_\infty$ as $k \to \infty$.

and in the continuous case, there is a flow $t \to X(t) \in \Sigma_n$ with the properties

- (isospectral) $spec(X(t) = spec(X(0)$

- (convergence) the flow $X(t), t \geq 0, X(0) = M$, converges to a diagonal matrix $X_\infty, X(t) \to X_\infty$ as $t \to \infty$.

In both cases, necessarily the diagonal entries of X_∞ are the eigenvalues of the given matrix M. The QR algorithm is a prime example of such a discrete algorithm, while the Toda algorithm is an example of the continuous case.

In the discrete case, the authors in [5] recorded the number of steps (i.e. the stopping time)

$$T(M) = T_{\epsilon,n,\mathcal{A},\mathcal{E}}(M)$$

for the algorithm $\mathcal{A}$ applied to a matrix $M \in \Sigma_n$ chosen from an ensemble $\mathcal{E}$, to compute the eigenvalues of M to an accuracy ϵ. They then plotted the histogram for the normalized stopping time

$$\tau(M) = \tau_{\epsilon,n\mathcal{A},\mathcal{E}}(M) = (T(M) - \langle T \rangle)/\sigma$$

where $\langle T \rangle$ and $\sigma^2 = \langle (T - \langle T \rangle)^2 \rangle$ denote the sample average and sample variance of the $T(M)'s$ taken over a large number (approx.15,000) of matrices M chosen from $\mathcal{E}$. What they found was that, for ϵ and n in a suitable scaling range (ϵ small, n large), the histogram for τ was *universal*, independent of the ensemble $\mathcal{E}$. The histogram does, however, depend on the algorithm. Similar phenomena were observed in the continuous case.

Subsequently in [2] the authors raised the question of whether the universality results in [5] were limited to eigenvalue algorithms for real symmetric matrices, or whether they were present more generally in numerical computations. And indeed, the authors in [2] found similar universality results for a wide variety of numerical algorithms, including

(a) other eigenvalue algorithms such as QR with shifts, the Jacobi eigenvalue algorithm, and also algorithms applied to complex Hermitian ensembles

(b) the conjugate gradient and GMRES iterative algorithms to solve linear linear $n \times n$ systems $Hx = b$ with H and b random

(c) an iterative algorithm to solve the Dirichlet problem $\Delta u = 0$ in a random star-shaped region $\Omega \subset \mathbf{R}^2$ with random boundary data f on $\partial\Omega$

(d) a genetic algorithm to compute the equilibrium measure for orthogonal polynomials on the line.

In [2] the authors also discussed similar universality results obtained by Bakhtin and Correll [1] in a series of experiments with live participants recording

(e) decision making times for a specified task.

All of the above results are numerical and experimental. In more recent work [3][4] the authors have proved universality rigorously for a number of algorithms: They showed in particular that the limiting histograms for these algorithms can be expressed in terms of the distribution of the inverse of the difference of the two top (or bottom, depending on the algorithm) eigenvalues of the matrices. The proofs of these universality results rely on the very latest results in random matrix theory.

Many problems remain open. In particular proving universality rigorously for more general algorithms, including those listed above (a) ...(e).

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Asymptotics for the recurrence coefficients of polynomials orthogonal with respect to a logarithmic weight.

Percy Deift

(joint work with Thomas Conway)

Let $d\mu(x)$ be a measure on the line with finite moments

$$\int_{\mathbf{R}} |x|^m d\mu(x) < \infty, \ m \geq 0,$$

and let $p_n(x) = \gamma_n x^n + \dots, \gamma_n > 0$, $n \geq 0$, be the associated orthonormal polynomials

$$\int_{\mathbf{R}} p_n(x) p_m(x) d\mu(x) = \delta_{n,m}, \ n, m \geq 0.$$

The polynomials automatically satisfy a three term recurrence relation

$$b_n p_{n+1}(x) + (a_n - x) p_n(x) + b_{n-1} p_{n-1}(x) = 0, \ n \geq 0$$

with recurrence coefficients $b_n > 0$, $a_n \in \mathbf{R}$ and $b_{-1} \equiv 0$. Given $d\mu(x)$, it is of basic interest to determine the asymptotic behavior of the $b_n's$ and $a_n's$ as $n \to \infty$.

In [1] the authors considered logarithmic weights $d\mu(x) = \log(2k/(1-x))$ on $[-1, 1]$, where $k > 1$. Such weights arise in various problems in physics and in mathematics. The main result in [1] is the following: As $n \to \infty$

$$(1) \ \ a_n = (2C)/(n \log n)^2 + O(1/n^2 \log n^3)$$
$$(2) \ \ b_n = 1/2 + 1/(16n^2) + C/(n \log n)^2 + O(1/n^2 \log n^3)$$

where $C = -3/32$.

This result, and more, was conjectured by A.Magnus [2], up to the precise value for the constant C.

The authors in [1] use Riemann-Hilbert/steepest-descent methods to prove (1)(2), but not in the standard way, as there is no known parametrix for the Riemann-Hilbert problem in a neighborhood of the logarithmic singularity at $x = 1$. The authors overcame this difficulty by using the operator theory that underlies Riemann-Hilbert problems, together with a new formula for the difference of the solutions of two Riemann-Hilbert problems on the same contour.

Many open problems remain. Firstly, to prove the analog of (1)(2) for the case $k = 1$ (note that for $k = 1$, we no longer have $\log(2/(1-x)) \geq c > 0$ for $x \in [-1, 1]$). Secondly, it is of great interest to compute the asymptotic behavior of the polynomials p_n themselves as $n \to \infty$, particularly in a neighborhood of the logarithmic singularity at $x = 1$.

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The Toda shock and rarefaction waves

Iryna Egorova

(joint work with Johanna Michor, Gerald Teschl)

We are concerned with the long-time asymptotics of the Cauchy problem for the Toda equation

$$\begin{aligned} \dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2), \\ \dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \end{aligned} \qquad (n,t) \in \mathbb{Z} \times \mathbb{R}_+, \tag{1}$$

with steplike initial data

$$\begin{aligned} &a(n,0) \to a > 0, \quad b(n,0) \to b \in \mathbb{R}, \quad \text{as } n \to -\infty, \\ &a(n,0) \to \frac{1}{2}, \qquad b(n,0) \to 0, \qquad \text{as } n \to +\infty. \end{aligned} \tag{2}$$

The initial value problem (1)–(2) is uniquely solvable for initial data which approach their limiting constants with a polynomial rate. Moreover, for each $t \neq 0$ the solution tends as $n \to \pm\infty$ to the same constants, and with the same rate as the initial data (cf. [7]). We study the asymptotic behaviour of the solution in the regime when $n \to \infty$, $t \to +\infty$, but the ratio $\xi = \frac{n}{t}$ slowly varying. Qualitatively (up to a phase shift), the long-time asymptotics are determined by the mutual location of the intervals $[b-2a, b+2a]$ and $[-1,1]$, and by the discrete spectrum $\lambda_1, ..., \lambda_N$ of the underlying Jacobi operator

$$H(t)y(n) := a(n-1,t)y(n-1) + b(n,t)y(n) + a(n,t)y(n+1), \quad n \in \mathbb{Z}. \tag{3}$$

Our goal is to rigorously justify these asymptotics by means of the nonlinear steepest descent (NSD) method developed by Deift and Zhou [6]. In fact, our investigation requires an extension of the original NSD analysis based on a suitably chosen g-function as first introduced in Deift, Venakides, and Zhou [5]. Here we restrict our considerations to the case of disjoint background spectra. For $b+2a < -1$ we deal with the Toda shock problem, the case $1 < b-2a$ is known as the Toda rarefaction problem.

The long-time asymptotics of the rarefaction problem were studied rigorously by Deift, Kamvissis, Kriecherbauer and Zhou [4] in the transitional region where $\xi := \frac{n}{t} \approx 0$ as $t \to +\infty$. To this end the authors applied the NSD approach for a **vector** Riemann–Hilbert problem. Using a similar approach, in [8] we show that there are four principal sectors with the following asymptotic behavior:

- In the region $n > t$, the solution $\{a(n,t), b(n,t)\}$ is asymptotically close to the constant right background solution $\{\frac{1}{2}, 0\}$ plus a sum of solitons corresponding to the eigenvalues $\lambda_j < -1$.
- In the region $0 < n < t$, as $t \to \infty$ we have

$$a(n,t) = \frac{n}{2t} + O\Big(\frac{1}{t}\Big), \quad b(n,t) = 1 + \frac{\frac{1}{2} - n}{t} + O\Big(\frac{1}{t}\Big). \tag{4}$$

- In the region $-2at < n < 0$, as $t \to \infty$ we have

$$a(n,t) = -\frac{n+1}{2t} + O\Big(\frac{1}{t}\Big), \quad b(n,t) = b - 2a - \frac{n + \frac{3}{2}}{t} + O\Big(\frac{1}{t}\Big). \tag{5}$$

- In the region $n < -2at$, the solution of (1)–(2) is asymptotically close to the left background solution $\{a, b\}$ plus a sum of solitons corresponding to the eigenvalues $\lambda_j > b + 2a$.

Note that the main terms of the asymptotics (4) and (5) are solutions of the Toda lattice equation. In turn, the error terms $O(t^{-1})$ are uniformly bounded with respect to n for $\varepsilon t \le n \le (1-\varepsilon)t$ in (4), and for $(-2a+\varepsilon)t \le n \le -\varepsilon t$ in (5), where $\varepsilon > 0$ is an arbitrary small value. In the two middle regions we also derive a precise formula for these error terms.

The first investigation of shock waves in the Toda lattice was done by Venakides, Deift, and Oba [10] employing the Lax–Levermore method. As their main result they showed (for $a = \frac{1}{2}$) that in a sector $|\frac{n}{t}| < \xi'_{cr}$ the solution can be asymptotically described by a period two solution, while in a sector $|\frac{n}{t}| > \xi_{cr}$ the particles are close to the unperturbed lattice. For the remaining region $\xi'_{cr} < |\frac{n}{t}| < \xi_{cr}$ the solution was conjectured to be asymptotically close to a modulated single-phase quasi-periodic solution but this case was not solved there. Investigation of the Toda shock problem by the NSD analysis was singled out by Deift [3] as an important open problem in the theory of nonlinear systems.

We perform this analysis for arbitrary $a > 0$, assuming that the discrete spectrum of the Jacobi operator (3) consists of the single point $\lambda_0 \in (2a+b, -1)$. A short qualitative description of the asymptotics for the Toda shock waves derived in [9] is the following.

There are five principal regions on the half plane (n,t) divided by rays $n/t = \tilde{\xi}$, with $\tilde{\xi} = \xi_{cr,1}$, $\xi'_{cr,1}$, $\xi_{cr,0}$, ξ'_{cr}, ξ_{cr} where $\xi_{cr,1} < \xi'_{cr,1} < \xi_{cr,0} < \xi'_{cr} < \xi_{cr}$. In the domain $\xi > \xi_{cr}$, the solution is asymptotically close to the constant right background solution $\{\frac{1}{2}, 0\}$, and in the domain $\xi < \xi_{cr,1}$ it is close to the left background $\{a, b\}$. In the domain $\xi'_{cr} < \xi < \xi_{cr}$, there appears a monotonous smooth function $\gamma(\xi) \in \mathbb{R}$ such that $\gamma(\xi'_{cr}) = b + 2a$, $\gamma(\xi_{cr}) = b - 2a$. When the parameter ξ starts to decay from the point ξ_{cr}, the point $\gamma(\xi)$ "opens" a band $[b - 2a, \gamma(\xi)]$ (the Whitham zone, cf. [2]). This interval and $[-1, 1]$ can be treated as the bands of a (slowly modulated) two band solution of the Toda lattice, which turns out to give the leading asymptotical term of our solution with respect to large t. This two band solution is defined uniquely by its initial divisor. We compute this divisor precisely via the values of the right transmission coefficient on the interval $[b - 2a, \gamma(\xi)]$. Thus, in a vicinity of any ray $\frac{n}{t} = \xi$ the solution of (1)–(2) is asymptotically finite-gap. This asymptotical term also can be treated as a function of n, t, and $\frac{n}{t}$ in the whole domain $t(\xi'_{cr} + \varepsilon) < n < t(\xi_{cr} - \varepsilon)$.

Next, in the domains $\xi_{cr,0} < \xi < \xi'_{cr}$ and $\xi'_{cr,1} < \xi < \xi_{cr,0}$, the asymptotic of the solution of (1)–(2) is described by two finite-gap solutions. They are connected with one and the same intervals $[b - 2a, b + 2a]$ and $[-1, 1]$ and the initial divisors (or shifts of the phase) do not depend on the slow variable ξ, but differ due to the presence of the point of the discrete spectrum λ_0 which generates a soliton. In particular, for $a = \frac{1}{2}$ our asymptotics in these domains agree with the asymptotics obtained in [10].

The situation in the domain $\xi_{cr,1} < \xi < \xi'_{cr,1}$ is similar to the Whitham zone described above. There appears a monotone smooth function $\gamma_1(\xi) \in \mathbb{R}$ such that $\gamma_1(\xi_{cr,1}) = 1$, $\gamma_1(\xi'_{cr,1}) = -1$. The finite-gap asymptotic here is again local along the ray, and is defined by the intervals $[b - 2a, b + 2a]$ and $[\gamma_1(\xi), 1]$.

An interesting open problem is to understand asymptotics in transitional regions. In particular, for the Toda shock problem in vicinities of the rays $\frac{n}{t} = \xi_{cr}$ and $\frac{n}{t} = \xi_{cr,1}$ one can expect the appearance of asymptotic solitons (see [1]). Another interesting problem is to describe long-time asymptotics of a steplike solution for (1)–(2) in the case of intersecting background spectra.

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Abelian coverings, discrete Schrödinger operators, and KdV-equation

Benjamin Eichinger
(joint work with Tom VandenBoom and Peter Yuditskii)

Let $E \subset \mathbb{R}$ be compact and $\mathbb{D}/\Gamma$ be a uniformization of $\overline{\mathbb{C}} \setminus E$. That is, there exists a local homeomorphism $\lambda : \mathbb{D} \to \overline{\mathbb{C}} \setminus E$ and Γ is the Fuchsian group of its deck transformations. If E has positive logarithmic capacity, then Γ is of convergent type, and hence the Blaschke product

$$b(z;\Gamma) = z \prod_{\gamma\in\Gamma} \frac{|\gamma(0)|}{\gamma(0)} \frac{\gamma(0) - z}{1 - \overline{\gamma(0)}z}$$

converges. Γ (or equivalently E) is called of Widom type, if in addition $b'(z;\Gamma)$ is of bounded characteristic in $\mathbb{D}$. Let Γ^* be the group of characters of Γ, i.e., the set of homomorphism from Γ to $\mathbb{R}/\mathbb{Z}$. A function f is called character automorphic with character $\alpha \in \Gamma^*$, if

$$f \circ \gamma = e^{2\pi i\alpha(\gamma)} f, \qquad \text{for all } \gamma \in \Gamma.$$

Note that $b(z;\Gamma)$ is character automorphic with some character $\mu \in \Gamma^*$. By $H^2(\alpha;\Gamma)$ we denote the Hardy space of character automorphic functions with character α on $\mathbb{D}$. That is, a function $f \in H^2(\alpha;\Gamma)$ is a multi-valued function on the Riemann surface $\mathbb{D}/\Gamma$. Widom [5] showed that if Γ is of Widom type, then $H^2(\alpha;\Gamma)$ is non-empty for all $\alpha \in \Gamma^*$.

Let $J(E)$ denote the isospectral torus of reflectionless Jacobi matrices whose spectrum is the set E. If E is of Widom type such that the so-called Direct Cauchy Theorem holds, then Sodin and Yuditskii [3] showed that $J(E)$ is homeomorphic to Γ^*. The homeomorphism is called generalized Abel map and associates to each Jacobi matrix $J \in J(E)$ a Hardy space $H^2(\alpha;\Gamma)$. As a corollary of this construction it follows that all elements in $J(E)$ are almost periodic. Hence, this model is based on the fact that there exist sufficiently many admissible analytic functions on the Riemann surface $\mathbb{D}/\Gamma$.

Assume now that E consists of three points, say $E = \{0, 1, \infty\}$. This is the smallest number of punctures in the Riemann sphere $\overline{\mathbb{C}}$ such that the universal covering is the unit disk $\mathbb{D}$. Lyons and McKean [2] proved that already in this case the commutator subgroup Γ' of Γ is of convergent type. In other words, the boundary of the Riemann surface $\mathbb{D}/\Gamma'$ has positive capacity. Moreover, note that if E has positive analytic capacity, then the Ahlfors function yields a single valued function on $\mathbb{D}/\Gamma'$. Hence, we conclude: By passing from the Riemann surface $\mathbb{D}/\Gamma$ to $\mathbb{D}/\Gamma'$ the "quality of the boundary", in the sense of number of admissible analytic functions, increases essentially. $\mathbb{D}/\Gamma'$ is called universal Abelian covering for $\mathbb{D}/\Gamma$.

Therefore, in this talk we shall discuss the Riemann surface $\mathbb{D}/\Gamma'$ in detail. We will describe it by means of the covering map λ and by means of $b(z;\Gamma)$, which becomes a single valued function on $\mathbb{D}/\Gamma'$. Moreover, we will discuss function theory on the Riemann surface $\mathbb{D}/\Gamma_0$, where $\Gamma_0 = \ker \mu$. In particular, we give a criterion by means of reproducing kernels of the Hardy space $H^2(\alpha;\Gamma_0)$ such that the Jacobi matrix $J(\alpha)$ is in fact a discrete Schrödinger operator.

Vinnikov and Yuditskii [4] gave an interpretation of the Toda flow an Jacobi matrices by means of the fact that multiplication by functions on $\mathbb{D}/\Gamma'$ trivially commute. We conjecture that the same would also be possible for the KdV hierarchy. To be more precise.

Conjecture. *Let*

$$E = [0, \infty) \setminus \bigcup_{j=1}^{\infty} (a_j, b_j)$$

such that E is of Widom type and the DCT holds in $\mathbb{C} \setminus E$. Assume in addition that for some $k \in \mathbb{N}$ we have

$$\sum_{j=1}^{\infty} b_j^{k+2} - a_j^{k+2} < \infty \tag{1}$$

Then there exists a function θ_k, on $\mathbb{C}_+/\Gamma'$ corresponding to the uniformization $(\mathbb{C}_+/\Gamma, \lambda)$ such that the kth element of the KdV hierarchy is generated by the functions θ_k and λ.

Let u_k^V denote the solution of the kth element of the KdV hierarchy with initial potential V. Deift asked the following question:

Question. *Does almost periodicity of the initial data V imply almost periodicity of the solution u_1^V in t-direction.*

As a consequence, we would get an affirmative answer to this question for initial data V, which are almost periodic and satisfy $\sigma(L_V) = \sigma_{ac}(L_V) = E$, where

$$L_V = -\partial_x^2 + V(x)$$

is the corresponding Schrödinger operator. This improves a result of Binder, Damanik, Goldstein and Lukic [1] in two directions. First, we weaken the assumption on the spectral set E and secondly under the regularity condition (1) we obtain the corresponding result for all elements of the KdV hierarchy, where the aforementioned authors only discuss the case $k = 1$.

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Limit-Periodic Schrödinger Operators with Zero-Measure Spectrum

Jake Fillman
(joint work with David Damanik, Milivoje Lukic)

We consider continuum Schrödinger operators acting in $L^2(\mathbb{R})$ via

$$L_V u = -u'' + Vu,$$

where the potential $V : \mathbb{R} \to \mathbb{R}$ is limit-periodic, that is, V is a uniform limit of continuous periodic functions. We denote the class of limit-periodic potentials by LP and equip it with the topology inherited from the L^∞ metric. For an archetypal example, consider

$$V(x) = \sum_{j=1}^{\infty} 2^{-j} \cos\left(\frac{2\pi x}{j!}\right).$$

These operators are interesting for spectral theory as they provide a large tractable class of aperiodic almost-periodic Schrödinger operators which may exhibit rich and subtle spectral properties. These operators and their discrete analogs may exhibit absolutely continuous spectrum [3, 8, 13, 15, 16], singularly continuous spectrum [2, 5, 6], or even pure point spectrum [7, 10, 17]. As the rate of approximation grows worse, the character of the associated quantum dynamics transitions from ballistic motion (free propagation of wave packets) [1, 11, 14] to localization [7, 11, 17].

Moreover, limit-periodic potentials are uniformly almost-periodic and hence also provide an interesting class of initial data for the Cauchy problem for the KdV equation:

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \quad u(x, 0) \equiv V_0(x).$$

Recent work of Binder–Damanik–Goldstein–Lukic achieved success in solving Deift's conjecture (cf. [12]) for reflectionless almost-periodic initial data as long as $\sigma(L_{V_0})$ is sufficiently thick [4]. It is then natural to ask how badly the hypotheses of the general results may fail, and how often such failures may occur within the class of limit-periodic potentials. Broadly speaking, within the class of limit-periodic potentials, the answers to these questions are "quite badly" and "rather often." To wit:

Theorem 1 (Damanik–F.–Lukic, (2015) [5])**.** *There is a dense G_δ subset $Z \subseteq \mathrm{LP}$ with the property that $\sigma(L_{\lambda V})$ is a perfect set of zero Lebesgue measure for every $V \in Z$ and every $\lambda > 0$. There is a dense set $H \subseteq \mathrm{LP}$ with the property that $\sigma(L_{\lambda V})$ is a perfect set of zero Hausdorff dimension for all $V \in H$ and every $\lambda > 0$.*

From the KdV point of view, the examples in this theorem are quite bad, as it shows that the (topologically) generic behavior of limit-periodic potentials lies well outside the tractable regime.

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On the Spectrum of Multi-Frequency Quasiperiodic Schrödinger Operators with Large Coupling

Michael Goldstein
(joint work with Wilhelm Schlag, Mircea Voda)

In this talk I will give a review of recent joint work M.Goldstein,W.Schlag,M.Voda on the spectrum of multi-frequency quasi-periodic operators at large coupling constant.

In the last 40 years after the groundbreaking paper [9] the theory of quasiperiodic Schrödinger operators has been developed extensively, see the monograph [5] for an overview and [14] for a survey of the more recent results. For shifts on a one-dimensional torus $\mathbb{T}$ most of the results have been established non-perturbatively, i.e., either in the regime of almost reducibility or in the regime of positive Lyapunov exponent, and Avila's global theory, see [3], gives a qualitative spectral picture, covering both regimes, for generic potentials. One of the main results of the one-dimensional theory is the fact that the spectrum is a Cantor set. For the case of the almost Mathieu operator (corresponding to a cosine potential), this result has been proved for any non-zero coupling and any irrational shift, see [19] and [1, 2]. For general analytic potentials in the regime of positive Lyapunov exponent with generic shift the Cantor structure of the spectrum has been obtained in [12].

On the other hand, shifts on a multidimensional torus $\mathbb{T}^d$ turned out to be harder to analyze and the theory is less developed, even in the perturbative setting. In particular, not much is known about the geometry of the spectrum for multidimensional shifts. In their pioneering paper [7], Chulaevsky and Sinai conjectured that in contrast to the shift on the one-dimensional torus, for the two-dimensional shift the spectrum can be an interval for generic large smooth potentials. In this paper we prove this conjecture for large analytic potentials.

Heuristically, gaps in the spectrum of the one-frequency operators are created by horizontal "forbidden zones" appearing at the points of intersection of the graphs of shifted finite scale eigenvalues parametrized by phase, see [20, 12]. In contrast to this, the heuristic principle underlying [7] is that for multiple frequencies, the intersection curves of the graphs of shifted finite scale eigenvalues may not be too flat, thus preventing the appearance of the horizontal "forbidden zones" and stopping the formation of gaps. It is clear that some genericity assumption on the potential function is needed for this to be true, since potentials like $V(x, y) = v(x)$ lead to flat intersection curves and have Cantor spectrum. Furthermore, the largeness of the potential is also needed. Indeed, it is known that for small potentials with atypical frequency vector the spectrum has gaps, see [4].

Implementing such an argument, appears to be very challenging for a number of reasons. First, the analytical techniques available in finite volume are less favorable (mainly the large deviation theorems and everything that depends on them) as compared to the case of one frequency. In particular, it is difficult to implement an approach based on finite scale localization as in [12]. This is due to the fact that it is hard to handle long chains of resonances and to control the intersections of the resonant curves with the level sets of the eigenvalues. Second, it is inevitable that the intersection curves of the graphs of shifted finite scale eigenvalues flatten near the absolute extrema and handling this situation seems to be a delicate matter.

In [13] we addressed some of the issues regarding the analytical techniques, including establishing finite scale localization. We will use most of the basic tools from [13]. However, for the purpose of this paper one would need a refined version of finite scale localization, beyond what is achieved in that paper. We analyze the spectrum of the operator $H_N(x)$, $x \in \mathbb{T}^d$, on a finite interval $[1, N]$ subject to Dirichlet boundary conditions. To keep this spectrum under control requires resolving the following problem. Given E let $\mathcal{R}_N(E)$ be the set of all phases x such that E is in the spectrum of the operator $H_N(x)$. One has to identify phases $x \in \mathcal{R}_N(E)$ for which $x + n\omega$ is not too close to $\mathcal{R}_N(E)$ as n runs in the interval $N \ll n < N^A$, $A \gg 1$. This issue, commonly referred to as *double resonances*, is well-known. Similar strategies, leading to the formation of intervals in the spectrum, have been implemented for the skew-shift in [15] and for continuous two-dimensional Schrödinger operators in [16]. The main new device that we develop in this work, consists of an elimination of double resonances for *all* shifts $x + h$, and not just the "arithmetic ones" $x + n\omega$. Of course the shift h cannot be too small. Although this problem looks less accessible, it turns out to provide more control on the resonant set $\mathcal{R}_N(E)$ of the previous scale. The level sets $V(x) = E$ of the potential in question must satisfy the requirements of this more general elimination in order to launch the multi-scale analysis.

Furthermore, in order to show that the spectrum is actually an interval, we develop a Cartan type estimate that controls the intersections of the level sets of an analytic function near a non-degenerate extremum with their shifts.

The core of our approach is *non-perturbative* and works in the regime of positive Lyapunov exponent. More precisely, we develop two non-perturbative inductive schemes, one leading to the formation of intervals in the bulk of the spectrum and the other leading to intervals at the edges of the spectrum. We will only use the largeness of the potential to check that the initial inductive conditions are satisfied.

We introduce some notation and definitions that we need to state our main result. We work with operators

$$[H_\lambda(x)\psi](n) = -\psi(n+1) - \psi(n-1) + \lambda V(x + n\omega)\psi(n), \tag{1}$$

with $\lambda > 0$ being a real parameter, and with the potential V a real analytic function on the torus $\mathbb{T}^d$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $d \geq 2$. We assume that the frequency vector $\omega \in \mathbb{T}^d$ obeys the standard Diophantine condition. We introduce also the class of "generic trigonometric polynomials of a given degree". The formal definition is pretty lengthy. Obviously it has to be a Morse function. On top of that the level sets shifts should be transversal to themselves unless the shift is too small. We denote the st of such trigonometric polynomials by $\mathfrak{G}$.

Theorem 1. *There exists* $\lambda_0 = \lambda_0(V, a, b, d)$ *such that the following statements hold for* $\lambda \geq \lambda_0$.

(a) Assume that V *attains its global minimum at exactly one non-degenerate critical point* $\underline{x}$. *Then there exists* $\underline{E} \in \mathbb{R}$, $|\lambda^{-1}\underline{E} - V(\underline{x})| < \lambda^{-1/4}$, *such that*

$$[\underline{E}, \underline{E} + \lambda \exp(-(\log \lambda)^{1/2})] \subset \mathcal{S}_\lambda \quad \text{and} \quad (-\infty, \underline{E}) \cap \mathcal{S}_\lambda = \emptyset.$$

An analogous statement holds relative to the global maximum of V *(using the notation* $\overline{x}, \overline{E}$*).*

(b) Assume that $V \in \mathfrak{G}$ *and let* $\underline{E}, \overline{E}$ *be as in (a). Then* $\mathcal{S}_\lambda = [\underline{E}, \overline{E}]$.

Remark 1. (a) The constant $\lambda_0(V, a, b, d)$ can be expressed explicitly, see the proof of Theorem 1.

(b) The genericity of the assumptions on V will be addressed in [21]. More precisely, the following result will be established. Consider real trigonometric polynomials of the form

$$V(x) = \sum_{m \in \mathbb{Z}^d : |m| \leq n} c_m e^{2\pi i m \cdot x}, \quad x \in \mathbb{R}^d$$

of a given cumulative degree $n \geq 1$, $|m| := \sum_{1 \leq j \leq d} |m_j|$. Then for almost all vectors $(c_m)_{|m| \leq n}$ one has $V \in \mathfrak{G}$.

(c) For the completeness of our paper we include a particular example of potential $V \in \mathfrak{G}$ that can be obtained by the methods from [21]. Namely, we show that

$$V(x, y) = \cos(2\pi x) + s \cos(2\pi y)$$

belongs to $\mathfrak{G}$ for all $s \in \mathbb{R} \setminus \{-1, 0, 1\}$. We note that as s approaches $\{-1, 0, 1\}$ our explicit value for λ_0 diverges to ∞ and the geometry of the spectrum cannot be decided by continuity. Of course, for $s = 0$ the spectrum is a Cantor set. However, for $s = \pm 1$, part (a) of Theorem 1 still applies and guarantees the existence of intervals at the edges of the spectrum.

As mentioned above, the derivation of Theorem 1 is based on two non-perturbative statements in the regime of positive Lyapunov exponent.

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Exponential estimates on the size of spectral gaps for quasi-periodic Schrödinger operators

Martin Leguil

(joint work with Jiangong You, Zhiyan Zhao, and Qi Zhou)

In the following, we consider one-dimensional discrete Schrödinger operators on $\ell^2(\mathbb{Z})$:

$$(H_{V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n, \quad \forall\, n \in \mathbb{Z},$$

for some phase $\theta \in \mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$, some analytic potential $V\colon \mathbb{T}^d \to \mathbb{R}$, and where the (multi-)frequency $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{T}^d$ is chosen in such a way that $(1, \alpha_1, \ldots, \alpha_d)$ is rationally independent. In this case, the spectrum of $H_{V,\alpha,\theta}$ is a compact subset of $\mathbb{R}$, independent of θ, denoted by $\Sigma_{V,\alpha}$. By the Gap-Labelling Theorem, it is of the form $\Sigma_{V,\alpha} = [\underline{E}, \overline{E}] \setminus \cup_{k\in\mathbb{Z}^d\setminus\{0\}} G_k(V)$ for some spectral gaps $G_k(V) = (E_k^-, E_k^+)$ labelled by integer vectors. For all $k \neq 0$, the restriction of the integrated density of states (IDS) $N_{V,\alpha}\colon \mathbb{R} \to [0,1]$ of $H_{V,\alpha,\theta}$ to the associate gap satisfies $N_{V,\alpha}|_{G_k(V)} = \langle k, \alpha\rangle \mod \mathbb{Z}$.

One particularly important example is given by almost Mathieu operators (AMO) $H_{\lambda,\alpha,\theta}$, in the case where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and the potential has the form $V = 2\lambda \cos 2\pi(\cdot)$ for some coupling constant $\lambda \in \mathbb{R}$. For simplicity, we denote by $G_k(\lambda) = (E_k^-, E_k^+)$ the gap with label k.

Estimates on spectral gaps. In what follows, given $d \geq 1$, $\gamma > 0$, $\tau \geq d-1$, we let

$$\mathrm{DC}_d(\gamma,\tau) := \left\{\varphi \in \mathbb{R}^d : \inf_{j\in\mathbb{Z}} |\langle n,\varphi\rangle - j| > \frac{\gamma}{|n|^\tau}, \quad \forall\, n \in \mathbb{Z}^d\backslash\{0\}\right\}$$

and we set $\mathrm{DC}_d := \bigcup_{\gamma>0,\,\tau>d-1} \mathrm{DC}_d(\gamma,\tau)$. Our first result is about exponential asymptotics on the size of spectral gaps for non-critical AMO with a Diophantine frequency.

Theorem 1 (L.-You-Zhao-Zhou [17]). *For $\alpha \in \mathrm{DC}_1$, and for any $0 < \xi_0 < 1$, there exist $C_0 = C_0(\lambda,\alpha,\xi_0) > 0$, $C_1 = C_1(\lambda,\alpha) > 0$, and a numerical constant $\xi_1 > 1$ such that*

$$\begin{array}{rcccll} C_1\lambda^{\xi_1|k|} & \leq & |G_k(\lambda)| & \leq & C_0\lambda^{\xi_0|k|}, & \text{if } 0<\lambda<1,\\ C_1\lambda^{-\xi_1|k|} & \leq & |G_k(\lambda)| & \leq & C_0\lambda^{-\xi_0|k|}, & \text{if } 1<\lambda<\infty, \end{array}$$

for all $k \in \mathbb{Z}\backslash\{0\}$, where $|G_k(\lambda)|$ denotes the length of $G_k(\lambda)$.

The study of lower bounds dates back to the long-standing conjecture of the "Ten Martini Problem" [20] (finally solved by Avila-Jitomirskaya [4], after partial results due to Puig [19] etc.) and the so-called "Dry Ten Martini Problem" (see [6] for recent progress on this question), which is a further elaboration asking whether for any $\lambda \neq 0$ and irrational α, all possible spectral gaps of $H_{\lambda,\alpha,\theta}$ predicted by the Gap-Labelling Theorem are non-collapsed.

Regarding the question of upper bounds, the first result is due to Moser-Pöschel. In [18], given an analytic potential $V\colon \mathbb{T}^d \to \mathbb{R}$, $d \geq 2$, and $\varpi \in \mathrm{DC}_d$, they consider the continuous quasi-periodic Schrödinger operator on $L^2(\mathbb{R})$:

$$(\mathcal{L}_{V,\varpi}y)(t) = -y''(t) + V(\varpi t)y(t).$$

Thanks to KAM techniques, Moser-Pöschel proved that if V is small enough, then $|G_k(V)|$ is exponentially small with respect to $|k|$ provided that $|k|$ is sufficiently large and $\langle k,\varpi\rangle$ is not too close to the other $\langle m,\varpi\rangle$. Later on, Amor [16] proved that in the same setting, the spectral gaps have sub-exponential decay for all $k \in \mathbb{Z}^d\backslash\{0\}$. Damanik-Goldstein [9] gave a stronger result: $|G_k(V)| \leq \varepsilon e^{-\frac{r_0}{2}|k|}$ if $V\colon \mathbb{T}^d \to \mathbb{R}$ has a bounded analytic extension to the strip $\{z \in \mathbb{C}/\mathbb{Z} : |\Im z| < r_0\}$ and $\varepsilon := \sup_{|\Im z|<r_0} |V(z)|$ is sufficiently small. We obtain:

Theorem 2 (L.-You-Zhao-Zhou [17]). *Let $\alpha \in \mathrm{DC}_d$ and let $V\colon \mathbb{T}^d \to \mathbb{R}$ be an analytic potential with a bounded analytic extension to the strip $\{z \in \mathbb{C}/\mathbb{Z} :$*

$|\Im z| < r_0\}$. *For any* $r \in (0, r_0)$, *there exists* $\varepsilon_0 = \varepsilon_0(V, \alpha, r_0, r) > 0$ *such that if* $\sup_{|\Im z|<r_0} |V(z)| < \varepsilon_0$, *then*

$$|G_k(V)| \le \varepsilon_0^{\frac{2}{3}} e^{-r|k|}, \quad \forall\, k \in \mathbb{Z}^d \backslash \{0\}.$$

Homogeneous spectrum. The exponential upper bounds on the size of spectral gaps in Theorem 2 can be used to prove homogeneity of the spectrum. Recall that a closed set $\mathcal{S} \subset \mathbb{R}$ is called homogeneous in the sense of Carleson if there exists $\mu > 0$ such that

$$|\mathcal{S} \cap (E - \epsilon, E + \epsilon)| > \mu\epsilon, \quad \forall\, E \in \mathcal{S},\ \forall\, 0 < \epsilon \le \mathrm{diam}\mathcal{S}.$$

Homogeneity of the spectrum plays an essential role in the inverse spectral theory of almost periodic potentials (for instance in the fundamental work of Sodin-Yuditskii [21, 22]).

Let us recall some recent results on the homogeneity of the spectrum. Building on the localization estimates developed in [9], Damanik-Goldstein-Lukic [11] proved that the spectrum of continuous Schrödinger operators $\mathcal{L}_{V,\varpi}$ with Diophantine ϖ and sufficiently small analytic potential V is homogeneous. For the discrete operator $H_{V,\alpha}$ in the positive Lyapunov exponent regime, Damanik-Goldstein-Schlag-Voda [12] proved that the spectrum is homogeneous for any $\alpha \in \mathrm{SDC}$, i.e., such that for some $\gamma, \tau > 0$, $\inf_{j\in\mathbb{Z}} |n\alpha - j| \ge \frac{\gamma}{|n|(\log|n|)^\tau}$, for all $n \in \mathbb{Z}\backslash\{0\}$. We show the following result:

Theorem 3 (L.-You-Zhao-Zhou [17])**.** *Let* $\alpha \in \mathrm{SDC}$. *For a (measure-theoretically) typical analytic potential* $V\colon \mathbb{T} \to \mathbb{R}$, *the spectrum* $\Sigma_{V,\alpha}$ *is homogeneous.*

Given $E \in \mathbb{R}$, we set $S_E^V(\cdot) := \begin{pmatrix} E - V(\cdot) & -1 \\ 1 & 0 \end{pmatrix}$ and we let (α, S_E^V) be the associate *Schrödinger cocycle*. The energy $E \in \Sigma_{V,\alpha}$ is called *supercritical* (resp. *subcritical*) if the Lyapunov exponent is positive, i.e., $L(\alpha, S_E^V) > 0$ (resp. $L(\alpha, S_E^V(\cdot + i\epsilon)) = 0$ for $|\epsilon| < \delta$). By Avila's global theory of one-frequency quasi-periodic Schrödinger operators [3], for a (measure-theoretically) typical analytic potential $V\colon \mathbb{T} \to \mathbb{R}$, any $E \in \Sigma_{V,\alpha}$ is either subcritical or supercritical. In particular, he shows that typically, there exists a finite number of intervals $(I_i)_i$ such that the set of all subcritical energies in the spectrum is $\Sigma_{V,\alpha}^{\mathrm{sub}} = \cup_i(\Sigma_{V,\alpha} \cap I_i)$. Since the supercritical regime was already handled in [12], we focus on energies E in the subcritical part of the spectrum. If $(p_n/q_n)_n$ denotes the sequence of best approximants of α, we let $\beta(\alpha) := \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}$. We obtain the following description of the subcritical spectrum and of the spectral gaps one of whose edge points is in $\Sigma_{V,\alpha}^{\mathrm{sub}}$:

Theorem 4 (L.-You-Zhao-Zhou [17])**.** *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ *satisfy* $\beta(\alpha) = 0$. *For typical analytic potentials* $V\colon \mathbb{T} \to \mathbb{R}$, *the following assertions hold.*

(1) *There exist constants* $C, \vartheta > 0$ *depending on* V, α, *such that*

$$|G_k(V)| \le Ce^{-\vartheta|k|}, \quad \forall\, k \in \mathbb{Z}\backslash\{0\} \text{ with } \overline{G_k(V)} \cap \Sigma_{V,\alpha}^{\mathrm{sub}} \ne \emptyset.$$

(2) *For any* $\eta > 0$*, there exists* $D = D(V, \alpha, \eta) > 0$ *such that*

$$\text{dist}(G_k(V), G_{k'}(V)) \geq De^{-\eta|k'-k|},$$

if $k \neq k' \in \mathbb{Z}$ *satisfy* $\overline{G_k(V)} \cap I_i \neq \emptyset$ *and* $\overline{G_{k'}(V)} \cap I_i \neq \emptyset$ *for some* i*.*

(3) *There exists* $\mu_0 \in (0,1)$ *such that*

$$|\Sigma_{V,\alpha} \cap (E - \epsilon, E + \epsilon)| > \mu_0 \epsilon, \quad \forall\, E \in \Sigma_{V,\alpha}^{\text{sub}},\ \forall\, 0 < \epsilon \leq \text{diam}\Sigma_{V,\alpha}.$$

Deift's conjecture. Deift's conjecture (Problem 1 of [13, 14]) asks whether for almost periodic initial data, the solutions to the KdV equation are almost periodic in the time variable. Tsugawa [24] proved local existence and uniqueness of solutions to the KdV equation when the frequency is Diophantine and the Fourier coefficients of the potential decay at a sufficiently fast polynomial rate. Damanik-Goldstein [10] then proved global existence and uniqueness for a Diophantine frequency and small quasi-periodic analytic initial datum. Recently, Binder-Damanik-Goldstein-Lukic [7] showed that in the same setting, the solution is in fact almost periodic in time, thus proving Deift's conjecture in this case. In our work, we consider the discrete version of Deift's conjecture, namely that for almost periodic initial data, the Toda flow is almost periodic in the time variable. Recall the Toda lattice equation

$$\left\{ \begin{array}{rcl} a_n'(t) & = & a_n(t)\left(b_{n+1}(t) - b_n(t)\right), \\ b_n'(t) & = & 2(a_n^2(t) - a_{n-1}^2(t)), \end{array} \right. \qquad n \in \mathbb{Z}. \tag{1}$$

In view of Theorem 12.6 in [23], given an initial condition $(a(0), b(0)) \in \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$, there is a unique solution $(a, b) \in \mathcal{C}^\infty(\mathbb{R}, \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z}))$ to (1). We can identify $(a(t), b(t))$ with a doubly infinite Jacobi matrix $J(t)$:

$$(J(t)u)_n := a_{n-1}(t)\, u_{n-1} + b_n(t)\, u_n + a_n(t)\, u_{n+1}. \tag{2}$$

As a consequence of homogeneity (Theorem 4) and purely absolutely continuous spectrum of subcritical Schrödinger operators [2], we prove a discrete version of Deift's conjecture for almost periodic initial data, building on an previous result of Vinnikov-Yuditskii [25]. We show the following generalization of the result of Binder-Damanik-Goldstein-Lukic [7] to Avila's subcritical regime (see also the recent paper [8] for related advance on this problem).

Theorem 5 (L.-You-Zhao-Zhou [17])**.** *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ *with* $\beta(\alpha) = 0$*. Let* $V : \mathbb{T} \to \mathbb{R}$ *be a subcritical analytic potential, i.e., such that* (α, S_E^V) *is subcritical for all* $E \in \Sigma_{V,\alpha}$*. We consider the Toda flow (1) with initial condition* $(a_n, b_n)(0) = (1, V(\theta + n\alpha))$*,* $n \in \mathbb{Z}$*. Then*

(1) *For any* $\theta \in \mathbb{T}$*, (1) admits a unique solution* $(a(t), b(t))$ *defined for all* $t \in \mathbb{R}$*.*

(2) *For every* t*, the Jacobi matrix* $J(t)$ *given by (2) is almost periodic and has constant spectrum* $\Sigma_{V,\alpha}$*.*

(3) *The solution* $(a(t), b(t))$ *is almost periodic in* t *in the following sense: there is a continuous map* $\mathcal{M}\colon \mathbb{T}^\mathbb{Z} \to \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$*, a point* $\varphi \in \mathbb{T}^\mathbb{Z}$ *and a direction* $\varpi \in \mathbb{R}^\mathbb{Z}$*, such that* $(a(t), b(t)) = \mathcal{M}(\varphi + \varpi t)$*.*

In particular, the above conclusion holds for $V = 2\lambda \cos 2\pi(\cdot)$ *with* $0 < \lambda < 1$.

Some ideas of the proofs. Our approach is from the perspective of dynamical systems, and is based on quantitative (strong) almost reducibility. To obtain bounds on the size of spectral gaps, we analyze the behavior of Schrödinger cocycles close to the boundary of some spectral gap. At the edge points, the cocycles are reducible to constant parabolic cocycles. The key points in our proof are the exponential decay of the off-diagonal coefficient of the parabolic matrix, and the subexponential growth of the conjugacy (in restriction to $\mathbb{T}$) with respect to the label k. We first consider the case of small analytic potentials, and we distinguish between two cases in the proof. If the frequency is Diophantine, we develop a new KAM scheme to show almost reducibility with nice estimates (this result works for multifrequencies, and for both continuous and discrete cocycles). Moreover, in order to get a sharp decay on the size of spectral gaps (Theorem 2), we prove almost reducibility of the cocycle in a fixed band, arbitrarily close to the initial band. On the other hand, for a one-dimensional frequency α satisfying $\beta(\alpha) = 0$, we use the almost localization argument (via Aubry duality) given by Avila [1] (initially developed by Avila-Jitomirskaya [5]); one key ingredient in the proof is the Corona Theorem. The generalization to the global subcritical regime is based on Avila's global theory of analytic $\mathrm{SL}(2,\mathbb{R})-$cocycles [3], especially his proof of the Almost Reducibility Conjecture [2, 3].

Homogeneity of the spectrum in the subcritical regime is derived from the upper bounds on the size of spectral gaps, together with Hölder continuity of the IDS. Thanks to Avila's global theory of one-frequency Schrödinger operators [3], one can then prove Theorem 3 by combining our results in the subcritical case with previous work of Damanik-Goldstein-Schlag-Voda [12] in the supercritical regime.

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KdV Equation with Quasi-Periodic Initial Data

Milivoje Lukic
(joint work with Ilia Binder, David Damanik, Michael Goldstein)

The KdV equation [1]

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0 \tag{1}$$

was introduced in the 19th century as a model for the propagation of shallow water waves in one dimension; in the 1960s it was found to have infinitely many conserved quantities [2] and a Lax pair representation [3], making KdV the first of many integrable partial differential equations to be discovered.

Integrability of the KdV equation was heavily used for the study of the Cauchy problem with rapidly decaying and for periodic initial data

$$u(0) = V. \tag{2}$$

In particular, for initial data $V \in H^n(\mathbb{T})$ for nonnegative integer n, the solution $u(t)$ is an $H^n(\mathbb{T})$-almost periodic function of t, which motivated the conjecture of

Deift [4] whether, for *almost periodic* initial data V, the solution of the Cauchy problem is almost periodic as a function of t. The analysis of the KdV Cauchy problem with almost periodic initial data presents significant new obstacles, and even short time existence of solutions is not known in general.

Our work has focused on quasiperiodic initial data V of the form

$$V(x) = \sum_{n \in \mathbb{Z}^\nu} c(n) e^{2\pi i n\omega x} \tag{3}$$

where $\omega \in \mathbb{R}^\nu$. In particular, we define for $\varepsilon > 0$ and $\kappa \in (0,1]$ the space $\mathcal{P}(\omega, \varepsilon, \kappa)$ of functions of the form (3) such that $|c(n)| \le \varepsilon \exp(-\kappa|n|)$ for $n \in \mathbb{Z}^\nu$. Our work proves Deift's conjecture for the case of small quasiperiodic analytic initial data with Diophantine frequency.

Theorem 1. [5] *Let $\omega \in \mathbb{R}^\nu$ obey the Diophantine condition*

$$|n\omega| \ge a_0 |n|^{-b_0}, \quad n \in \mathbb{Z}^\nu \setminus \{0\}$$

for some $0 < a_0 < 1$, $\nu < b_0 < \infty$. There exists $\varepsilon_0(a_0, b_0, \kappa) > 0$ such that, if $\varepsilon < \varepsilon_0$ and $V \in \mathcal{P}(\omega, \varepsilon, \kappa)$, then there exists a global solution u of (1), (2) with the following properties:

1. *for every $t \in \mathbb{R}$, $u(\cdot, t)$ is quasiperiodic in x and $u(\cdot, t) \in \mathcal{P}(\omega, \sqrt{4\varepsilon}, \kappa/4)$*
2. *u is $\mathcal{P}(\omega, \sqrt{4\varepsilon}, \kappa/4)$-almost periodic in t, i.e., there is a compact (finite or infinite dimensional) torus $\mathbb{T}^d$, a continuous map*
$$\mathcal{M} : \mathbb{T}^d \to \mathcal{P}(\omega, \sqrt{4\varepsilon}, \kappa/4),$$
a base point $\alpha \in \mathbb{T}^d$, and a direction vector $\zeta \in \mathbb{R}^d$ such that $u(t) = \mathcal{M}(\alpha + \zeta t)$
3. *the solution is unique, in the following sense: if $\tilde{u}$ is a solution of (1), (2) on $\mathbb{R} \times [-T, T]$ for some $T > 0$, and*
$$\tilde{u}, \partial_{xxx}\tilde{u} \in L^\infty(\mathbb{R} \times [-T, T]), \tag{4}$$
then $\tilde{u} = u$.

In the conclusion (2) above, $\mathcal{P}(\omega, \varepsilon, \kappa)$ can be taken as a metric space with the metric induced by the $L^\infty(\mathbb{R})$-norm, or the $W^{k,\infty}(\mathbb{R})$-norm for any $k \in \mathbb{N}$; all such metrics are mutually equivalent on $\mathcal{P}(\omega, \varepsilon, \kappa)$. The theorem therefore implies that besides u, derivatives of u are also almost periodic in t, and so is each Fourier coefficient $c(n, t)$ of $u(x, t)$.

In fact, Theorem 1 is a corollary of our more general result, which proves existence, uniqueness, and almost periodicity in t whenever V is almost periodic and the spectrum of the associated Schrödinger operator $-\partial_x^2 + V$ is absolutely continuous and not too thin, in a sense quantified by Craig-type conditions.

The spectrum $S = \sigma(H_V)$ is closed and bounded from below but not from above, so it can be written in the form

$$S = [\underline{E}, \infty) \setminus \bigcup_{j \in J} (E_j^-, E_j^+),$$

where $\underline{E} = \inf S$ and (E_j^-, E_j^+) are the maximal open intervals in $\mathbb{R} \setminus S$, called gaps. We denote

$$\gamma_j = E_j^+ - E_j^-$$

for $j \in J$ and

$$\eta_{j,l} = \operatorname{dist}((E_j^-, E_j^+), (E_l^-, E_l^+)), \qquad \eta_{j,0} = \operatorname{dist}((E_j^-, E_j^+), \underline{E})$$

for $j, l \in J$ (notationally, we assume here that our abstract index set J does not contain 0 as an element). We also denote

$$C_j = (\eta_{j,0} + \gamma_j)^{1/2} \prod_{\substack{l \in J \\ l \neq j}} \left(1 + \frac{\gamma_l}{\eta_{j,l}}\right)^{1/2}.$$

We assume that S satisfies a set of moment conditions and Craig-type conditions:

$$\sum_{j \in J} (1 + \eta_{j,0}^2)\gamma_j < \infty \tag{5}$$

$$\sum_{j \in J} \gamma_j^{1/2} < \infty, \quad \sup_{j \in J} \gamma_j^{1/2} \frac{1 + \eta_{j,0}}{\eta_{j,0}} C_j < \infty \tag{6}$$

$$\sup_{j \in J} \sum_{\substack{l \in J \\ l \neq j}} \left(\frac{\gamma_j^{1/2} \gamma_l^{1/2}}{\eta_{j,l}} \right)^a (1 + \eta_{j,0})(C_j + 1) < \infty \text{ for } a \in \left\{\tfrac{1}{2}, 1\right\}. \tag{7}$$

Theorem 2. [5] *Let the initial data $V : \mathbb{R} \to \mathbb{R}$ be uniformly almost periodic. Denote $S = \sigma(H_V)$ and assume that $S = \sigma_{\mathrm{ac}}(H_V)$ and that S obeys the Craig-type conditions* (5), (6), (7). *Then there exists a global solution u of* (1), (2) *with the following properties:*

1. *for every $t \in \mathbb{R}$, the function $u(\cdot, t)$ is uniformly almost periodic with frequency module equal to the frequency module of V;*
2. *u is almost periodic in t, in the following sense: there is a continuous map*

$$\mathcal{M} : \mathbb{T}^J \to W^{4,\infty}(\mathbb{R}),$$

a base point $\alpha \in \mathbb{T}^J$, and a direction vector $\zeta \in \mathbb{R}^J$ such that $u(\cdot, t) = \mathcal{M}(\alpha + \zeta t)$;

3. *the solution is unique, in the following sense: if $\tilde{u}$ is a solution of* (1), (2) *on $\mathbb{R} \times [-T, T]$ for some $T > 0$, which obeys* (4), *then $\tilde{u} = u$.*

The existence and almost periodicity of a solution of the KdV equation under Craig-type conditions were previously studied by Egorova [6] (with the analog for the nonlinear Schrödinger equation studied by Boutet de Monvel–Egorova [7]). Another paper of Egorova [8] used a different approach to construct almost periodic solutions for limit periodic initial data with superexponential periodic approximants.

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Long-time asymptotics for KdV with steplike initial data

Gerald Teschl
(joint work with Kyrylo Andreiev, Iryna Egorova, Till Luc Lange)

In this work we were concerned with the Cauchy problem for the Korteweg–de Vries (KdV) equation

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1}$$

with steplike initial data $q(x,0) = q_0(x)$ satisfying

$$\begin{cases} q_0(x) \to 0, & \text{as } x \to +\infty, \\ q_0(x) \to c^2, & \text{as } x \to -\infty. \end{cases} \tag{2}$$

This case is known as rarefaction problem. The corresponding long-time asymptotics of $q(x,t)$ as $t \to \infty$ are well understood on a physical level of rigor ([18, 14, 16]) and can be split into three main regions:

- In the region $x < -6c^2 t$ the solution is asymptotically close to the background c^2.
- In the region $-6c^2 t < x < 0$ the solution can asymptotically be described by $-\frac{x}{6t}$.
- In the region $0 < x$ the solution is asymptotically given by a sum of solitons.

For the corresponding shock problem we refer to [1, 6, 11, 12, 13, 15, 17].

Our aim was to rigorously justify these results. In addition, we were also able to compute the second terms in the asymptotic expansion, which was, to the best of our knowledge, not obtained before. Our approach is based on the nonlinear steepest descent method for oscillatory Riemann–Hilbert (RH) problems developed by Deift and Zhou [5] based on earlier work by Its and Manakov (see [10] for an easy introduction in the case $c = 0$). In turn, this approach rests on the inverse scattering transform for steplike initial data originally developed by Buslaev and Fomin [2] with later contributions by Cohen and Kappeler [3]. For recent developments and explicit conditions on the initial data q_0 ensuring unique solvability of the Cauchy problem we refer to [9, 8, 7].

As is known, the solution of the initial value problem (1), (2) can be computed by the inverse scattering transform from the right scattering data of the initial profile. Here the right scattering data are given by the reflection coefficient $R(k)$, $k \in \mathbb{R}$, a finite number of eigenvalues $-\kappa_1^2, \dots, -\kappa_N^2$, and positive norming constants $\gamma_1, \dots, \gamma_N$. The difference with the decaying case $c = 0$ consists of the fact, that the modulus of the reflection coefficient is equal to 1 on the interval $[-c, c]$. This implies that the deformation of the initial Riemann–Hilbert problem requires a new phase function, the so-called g function, as first outlined in [4]. At the point $k = 0$ the reflection coefficient takes the values ± 1 (cf. [3]). The case $R(0) = -1$ is known as the nonresonant case (which is generic), whereas the case $R(0) = 1$ is called the resonant case. Note, that the right transmission coefficient $T(k)$ can be reconstructed uniquely from these data (cf. [2]).

Our main results is the following:

Let the initial data $q_0(x) \in \mathcal{C}^8(\mathbb{R})$ of the Cauchy problem (1)–(2) satisfy

$$\int_0^{+\infty} \mathrm{e}^{\kappa x}(|q_0(x)| + |q_0(-x) - c^2|)dx < \infty, \tag{3}$$

for some small $\kappa > 0$. Let $q(x, t)$ be the solution of this problem. Then for arbitrary small $\epsilon_j > 0$, $j = 1, 2, 3$, and for $\xi = \frac{x}{12t}$, the following asymptotics are valid as $t \to \infty$ uniformly with respect to ξ:

A. In the domain $(-6c^2 + \epsilon_1)t < x < -\epsilon_1 t$:

$$q(x, t) = -\frac{x + Q(\xi)}{6t}(1 + O(t^{-1/3})), \quad \text{as } t \to +\infty, \tag{4}$$

where

$$Q(\xi) = \frac{2}{\pi}\int_{-\sqrt{-2\xi}}^{\sqrt{-2\xi}} \left(\frac{d}{ds}\log R(s) - 4\mathrm{i}\sum_{j=1}^N \frac{\kappa_j}{s^2 + \kappa_j^2}\right)\frac{ds}{\sqrt{s^2 + 2\xi}} \mp \frac{1}{2\sqrt{-2\xi}}, \tag{5}$$

with $\pm$ corresponding to the resonant/nonresonant case, respectively.

B. In the domain $x < (-6c^2 - \epsilon_2)t$ in the nonresonant case:

$$q(x, t) = c^2 + \sqrt{\frac{4\nu\tau}{3t}}\sin(16t\tau^3 - \nu\log(192t\tau^3) + \delta)(1 + o(1)), \tag{6}$$

where $\tau = \tau(\xi) = \sqrt{\frac{c^2}{2} - \xi}$, $\nu = \nu(\xi) = -\frac{1}{2\pi} \log\left(1 - |R(\tau)|^2\right)$ and

$$\begin{aligned}\delta(\xi) = &- \frac{3\pi}{4} + \arg(R(\tau) - 2T(\tau) + \Gamma(\mathrm{i}\nu)) \\ &- \frac{1}{\pi} \int_{\mathbb{R}\setminus[-\tau,\tau]} \log \frac{1 - |R(s)|^2}{1 - |R(\tau)|^2} \frac{s\, ds}{s^2 - c^2 - (\frac{c^2}{2} + \xi)^{1/2}(c^2 - s^2)^{1/2}}.\end{aligned}$$

Here Γ is the Gamma function.
C. In the domain $x > \epsilon_3 t$:

$$q(x,t) = -\sum_{j=1}^{N} \frac{2\kappa_j^2}{\cosh^2\left(\kappa_j x - 4\kappa_j^3 t - \frac{1}{2}\log\frac{\gamma_j}{2\kappa_j} - \sum_{i=j+1}^{N} \log\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j}\right)} + O(\mathrm{e}^{-\epsilon_3 t/2}).$$

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Reflectionless discrete Schrödinger operators are spectrally atypical

Tom VandenBoom

A discrete Schrödinger operator (DSO) is a self-adjoint linear operator H_V on $\ell^2(\mathbb{Z})$ which acts entrywise via

$$(H_V u)_n = u_{n+1} + u_{n-1} + V(n)u_n, \quad u \in \ell^2(\mathbb{Z}), \tag{1}$$

where V is a bounded potential funtion $V : \mathbb{Z} \to \mathbb{R}$. The Schrödinger operator and DSO tend to share many spectral characteristics, and as such the question of identifying the spectral characteristics of a DSO with a fixed almost-periodic potential V is thoroughly studied and reasonably well-understood.

On the other hand, one can likewise ask which bounded self-adjoint operators on $\ell^2(\mathbb{Z})$ demonstrate particular spectral characteristics. In this context, the Jacobi operator

$$(Ju)_n = a_n u_{n-1} + b_n u_n + a_{n+1}u_{n+1}, \quad u \in \ell^2(\mathbb{Z}) \tag{2}$$

is natural to consider. When a whole-line Jacobi operator is reflectionless – that is, when the diagonal entries of its resolvent matrix tend to purely imaginary values almost everywhere on the spectrum – one can reconstruct the sequences a and b from spectral data. Examples of reflectionless Jacobi operators abound, and are subtly but intimately related to the presence of absolutely continuous spectrum [3, 5, 7, 12].

Fix a positive-measure compact $E \subset \mathbb{R}$, and define the isospectral torus for E as

$$\mathcal{J}(E) := \{J : \sigma(J) \subset E \text{ and } J \text{ reflectionless on } E\}.$$

When E has uniformly positive Lebesgue density, $\mathcal{J}(E)$ is homeomorphic to a torus with dimension the number of spectral gaps in E [9]. For such compacts E, certain potential-theoretic properties are directly related to spectral properties of elements of $\mathcal{J}(E)$. For example, the logarithmic capacity of a finite-gap compact E can be determined as the limit of the geometric means of the off-diagonal sequences of Jacobi operators in the isospectral torus [8, 9]:

$$\operatorname{cap}(E) = \lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n}, \quad J(a, b) \in \mathcal{J}(E). \tag{3}$$

Note that, by (3), if $E = \sigma(H_V)$ for some DSO H_V, then $\operatorname{cap}(E) = 1$. Consequently, the following question is natural: for a compact E with $\operatorname{cap}(E) = 1$, does there exist a DSO $H_V \in \mathcal{J}(E)$? The titular result provides a negative answer:

Theorem 1. *For a full-measure dense G_δ of finite-gap compacts E having* $\mathrm{cap}(E) = 1$*, $\mathcal{J}(E)$ contains no DSO.*

This result is related to a result of Hur [4] regarding the sparsity of DSO m-functions among those for Jacobi operators; however, our result makes stronger claims about this atypicality.

In fact, this theorem arises as a straightforward corollary of a dynamical statement. The action $\mathcal{S}$ of conjugation by the left-shift $S : \delta_n \mapsto \delta_{n+1}$ preserves both the spectrum and the reflectionless condition, and thus $(\mathcal{J}(E), \mathcal{S})$ is a discrete-time dynamical system. This action is minimal if the $\mathcal{S}$ orbit of every point $J \in \mathcal{J}(E)$ is dense. We prove that

Theorem 2. *Fix a positive-measure compact $E \subset \mathbb{R}$, and suppose there exists a DSO $H_V \in \mathcal{J}(E)$. Then either V is a constant potential $V = C$, or the dynamical system $(\mathcal{J}(E), \mathcal{S})$ is not minimal.*

At first glance, this seems paradoxical, because it is not hard to find examples of reflectionless, finite-gap DSOs; in particular, any DSO having p-periodic potential function V has at most $p - 1$ spectral gaps. Another result suggests that further examples may not exist!

Theorem 3. *Suppose E has* $\mathrm{cap}(E) = 1$ *and* 0, 1, *or* 2 *spectral gaps. If there exists a DSO $H_V \in \mathcal{J}(E)$, then V (and the dynamical system $(\mathcal{J}(E), \mathcal{S})$) is periodic.*

That this result holds for 0-gap spectrum is an old and well-known result [1, 8]; similarly, this result has probably been observed for 1-gap spectra by way of trace formulas. That the theorem holds for 2-gap spectra is quite surprising, but follows via a certain novel application of the Toda flow, which we now describe.

Consider a bounded linear operator A on $\ell^2(\mathbb{Z})$. Denote by $A^\pm$ the restrictions of A to $\ell^2(\mathbb{Z}_\pm) \hookrightarrow \ell^2(\mathbb{Z})$, where the inclusion map is given by assigning zeros to the left- or right-half line. Fix a polynomial P of degree $n + 1 \geq 1$. The n^{th} Toda flow (for P) is the integral curve $J(t)$ of Jacobi operators satisfying the Lax pair

$$\partial_t J = [P(J)^+ - P(J)^-, J]. \tag{4}$$

There exist unique solutions to (4) for any bounded Jacobi initial condition J_0 [10, Theorem 12.6]. When there exists a monic polynomial P so that

$$[P(J)^+ - P(J)^-, J] = 0. \tag{5}$$

we say J is stationary for P. This definition can even be extended to bounded functions [11]. Stationary solutions to the Toda hierarchy are closely related to reflectionlessness; in a sense, isospectral tori are the level sets of commutators like in (5) [2, 10, 11]

For particular choices of polynomial P, the Toda flow induces a system of differential equations on the parametrizing sequences $a, b \in \ell^\infty$. The critical facts about the Toda flow that we employ are summarized in

Proposition 1. *For any non-constant polynomial P:*

(1) [6, Corollary 1.3] *Suppose* $J(t)$ *is the unique solution to* (4) *with* $J(0) = J_0 \in \mathcal{J}(E)$*, where* $E = \sigma(J_0)$*. Then* $J(t) \in \mathcal{J}(E)$ *for all* $t \in \mathbb{R}$*.*
(2) [10, Theorem 12.8] *The stationary solutions* $\partial_t J = 0$ *of* (4) *are finite-gap reflectionless Jacobi operators.*
(3) [10, Corollary 12.10]

$$\min\{\deg(P) : J \text{ stationary for } P\} = \#\{\text{spectral gaps in } \sigma(J)\} + 1,$$

where we interpret $\min\{\emptyset\} = \infty$*.*

We prove our theorems by leveraging these powerful results against the relative simplicity of the induced differential system under the assumption $a_n = 1$ for all n.

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Solutions to the KdV hierarchy via finite-gap approximation

Tom VandenBoom
(joint work with Benjamin Eichinger and Peter Yuditskii)

Consider the one-dimensional Schrödinger operator

$$L_q = -\Delta + q$$

where $\Delta = d^2/dx^2$ is the Laplacian and $q : \mathbb{R} \to \mathbb{R}$ is a potential function. For a potential function q of sufficient regularity (namely, $q \in C^{m+3}(\mathbb{R}, \mathbb{R})$), one can iteratively construct a sequence of differential polynomials depending on $L = L_q$ by

$$f_0(q) = 1 \tag{1}$$

$$\partial_x f_m(q) = -\frac{1}{4}\partial_x^3 f_{m-1} + q\partial_x f_{m-1} + \frac{1}{2} f_{m-1}\partial_x q \tag{2}$$

The k^{th} KdV hierarchy, whose initial member was proposed by Korteweg and de Vries in the late 19th century [4], is defined by

$$KdV_k(q) := -2\partial_x f_{k+1}(q).$$

We study the Cauchy problem for the KdV hierarchy; that is, we study solutions $q \in C^{2k+1,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ to the partial differential equation

$$\partial_{t_k} q = KdV_k(q), \tag{3}$$

$$q(\cdot, 0) = q_0 \tag{4}$$

for initial conditions q_0 satisfying certain regularity properties.

Consider a closed set $E \subset \mathbb{R}$ which is bounded from below such that the domain $\Omega = \mathbb{C} \setminus E$ is of Widom type. By translation, it is no assumption to let $\inf E = 0$. We thus can write E as the right half-line with an at-most countable set of maximal gaps removed; that is, E can be written as

$$E = [0, \infty) \setminus \bigcup_{j=1}^{\infty} (a_j, b_j). \tag{5}$$

Denote by $\mathcal{Q}(E)$ the set of potentials q for which L_q is reflectionless on its spectrum E. Joint with B. Eichinger and P. Yuditskii, we proved the following result:

Theorem 1. *Suppose $E \subset \mathbb{R}$ is closed and bounded below of the form* (5) *such that*

$$\sum_{j=1}^{\infty} b_j^{k+2} - a_j^{k+2} < \infty, \tag{6}$$

and suppose $q_0 \in \mathcal{Q}(E)$. Then there exists a classical solution $q = q(x, t_k) \in C^{2(k+1),1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ to the Cauchy problem (3), (4).

Furthermore, this solution is almost periodic in both the x *and* t_k *coordinates, in the sense that there exists a continuous map* $\mathcal{M} : \mathbb{T}^\infty \to \mathcal{Q}(E)$ *and vectors* $\alpha_0 \in \mathbb{T}^\infty$ *and* $\delta_x, \delta_{t_k} \in \mathbb{R}^\infty$ *so that*

$$q(x, t_k) = \mathcal{M}(\alpha_0 + x\delta_x + t_k\delta_{t_k}).$$

Results of this kind are contributed to a conjecture of Deift [2] and have been explored to a certain extent in a variety of previous results [1, 3].

The condition (6) is optimal for the existence of classical solutions to the Cauchy problem (3), (4). As an example, consider the case $k = 0$: if one only assumes finite total gap length, one can only conclude continuity of the associated potentials, which cannot in general be classical solutions to the associated KdV equation $\partial_t q = KdV_0(q) = -\partial_x q$. Explicit examples which are not differentiable can be constructed.

Our methods of proving Theorem 1 can be viewed to a certain extent as a refinement of those methods developed in [1] for KdV_1; specifically, in their paper they prove an analogous result (with the additional conclusion of uniqueness) to Theorem 1 via the following approximate scheme:

(1) Find the KdV flow on the Dirichlet data of an initial condition $q \in \mathcal{Q}(E)$
(2) Assume Craig-type conditions to achieve existence and uniqueness for the associated flowon Dirichlet data.
(3) Use previous results for finite-gap spectra to conclude almost-periodicity for finite-gap approximants.
(4) Pass to the infinite-gap limit under Craig-type conditions and uniform convergence.

Our methods replace items (1) and (2) above by the character-automorphism techniques of Sodin and Yuditskii [5]. Specifically, we find the flow associated to the KdV hierarchy on the characters of the finite-gap approximants, and pass to the limit using the condition (6). This allows for some simplification of the assumptions, but at the moment only allows for the weaker conclusion of existence. We believe that by using the theory of Abelian coverings, one can find the precise conditions for existence and uniqueness in an approach similar to that developed for the Toda flow by Vinnikov and Yuditskii [7].

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Killip-Simon problem and Jacobi flow on GMP matrices

Peter Yuditskii

One of the first and therefore most important theorems in perturbation theory claims that for an arbitrary self-adjoint operator A there exists a perturbation B of Hilbert-Schmidt class with arbitrary small operator norm, which destroys completely the absolutely continuous (a.c.) spectrum of the initial operator A (von Neumann). However, if A is the discrete free 1-D Schrödinger operator and B is an arbitrary Jacobi matrix (of Hilbert-Schmidt class) the a.c. spectrum remains perfectly the same (Deift-Killip [3]), that is, the interval $[-2, 2]$. Moreover, Killip and Simon [5] described explicitly the spectral properties for such $A + B$. Jointly with Damanik [2] they generalized this result to the case of perturbations of periodic Jacobi matrices in the non-degenerated case. Recall that the spectrum of a periodic Jacobi matrix is a system of intervals of a very specific nature. Christiansen, Simon and Zinchenko [1] posed in a review dedicated to F. Gesztesy the following question: "is there an extension of the Damanik-Killip-Simon theorem to the general finite system of intervals case?" In [7] this problem was solved completely. Our method deals with the Jacobi flow on GMP matrices. GMP matrices are probably a new object in the spectral theory. They form a certain Generalization of matrices related to the strong Moment Problem [4], the latter ones are a very close relative of Jacobi and CMV matrices. The Jacobi flow on them is also a probably new member of the rich family of integrable systems. Finally, related to Jacobi matrices of Killip-Simon class, analytic vector bundles and their curvature play a certain role in our construction and, at least on the level of ideology, this role is quite essential.

In this talk we concentrate on the functional model for periodic GMP matrices and prove the so-called "magic formula" for them as an evident consequence of this model.

For a finite gap set $E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \cup_{j=1}^{g} (\mathbf{a}_j, \mathbf{b}_j)$, let $\mathbb{D}/\Gamma \simeq \overline{\mathbb{C}} \setminus E$ be a uniformization of the given domain with the covering map function $\mathfrak{z} : \mathbb{D} \to \overline{\mathbb{C}} \setminus E$ and the Fuchsian group Γ, $\mathfrak{z} \circ \gamma = \mathfrak{z}, \forall \gamma \in \Gamma$. Let Γ^* be the corresponding group of unitary characters. For $\alpha \in \Gamma^*$ we define the Hardy space of character automorphic functions as

$$H^2(\alpha) = \{f \in H^2 \colon \ f \circ \gamma = e^{2\pi i \alpha(\gamma)} f, \ \gamma \in \Gamma\},$$

where H^2 denotes the standard Hardy class in $\mathbb{D}$.

We define two special functions: the so-called *Green function* $\mathfrak{b} = \mathfrak{b}_\infty$ of the group Γ, which is the Blaschke product with zeros at $\mathfrak{z}^{-1}(\infty) = \{\gamma(\zeta_0)\}_{\gamma \in \Gamma}$, and the *reproducing kernel* $k^\alpha = k^\alpha_\infty$ of the space $H^2(\alpha)$, i.e.,

$$\langle f, k^\alpha \rangle = f(\zeta_0) \quad \forall f \in H^2(\alpha).$$

Using these special functions, one can give a parametric description of the class of *reflectionless Jacobi matrices* $J(E)$ with the spectrum E (in a much more general form the following theorem was proved in [6]).

Theorem 1. *The system of functions*

$$\mathfrak{e}^\alpha_n(\zeta) = \mathfrak{b}^n(\zeta)\frac{k^{\alpha-n\mu}(\zeta)}{\sqrt{k^{\alpha-n\mu}(0)}}$$

forms an orthonormal basis in $H^2(\alpha)$ *for* $n \in \mathbb{N}$. *The multiplication operator by* $\mathfrak{z}$ *with respect to the basis* $\{\mathfrak{e}^\alpha_n(\zeta)\}_{n\in\mathbb{Z}}$ *is the Jacobi matrix* $J(\alpha)$ *with the coefficients* $\{a(n;\alpha), b(n;\alpha)\}_{n\in\mathbb{Z}}$*:*

$$\mathfrak{z}\mathfrak{e}^\alpha_n = a(n;\alpha)\mathfrak{e}^\alpha_{n-1} + b(n;\alpha)\mathfrak{e}^\alpha_n + a(n+1;\alpha)\mathfrak{e}^\alpha_{n+1}.$$

Moreover,

$$J(E) = \{J(\alpha) : \ \alpha \in \Gamma^*\}.$$

Recall that the set E is the spectrum of a periodic Jacobi matrix if and only if there exists a *polynomial* $T_n(z)$ such that $E = T_n^{-1}[-2, 2]$. In this case the isospectral set $J(E)$ can be described as a collection of Jacobi matrices J, which satisfies the following ("magic") formula

$$T_n(J) = S^n + S^{-n},$$

where S is the standard shift operator.

One of our main ideas in solving the Killip-Simon problem deals with the fact that for an arbitrary finite gap set E there exists an essentially unique *rational* function $\Delta(z)$ of the form

$$\Delta(z) = \lambda_0 z + \mathbf{c}_0 + \sum_{j=1}^{g} \frac{\lambda_j}{\mathbf{c}_j - z}, \ \lambda_j > 0, \ \mathbf{c}_j \in (\mathbf{a}_j, \mathbf{b}_j),$$

such that $E = \Delta^{-1}[-2, 2]$. Its Zhukovskii transform $\Psi(z)$, $\Delta(z) = \frac{1}{\Psi(z)} + \Psi(z)$, is a single valued function in the domain $\bar{\mathbb{C}} \setminus E$. Moreover, this is a product of the complex Green functions $\Psi(\mathfrak{z}(\zeta)) = \mathfrak{b}(\zeta)\prod_{j=1}^g \mathfrak{b}_{\mathbf{c}_j}(\zeta)$.

We substitute the orthonormal system $\{\mathfrak{e}^\alpha_n(\zeta)\}$ by the system

$$\text{(1)} \qquad \mathfrak{f}^\alpha_n = \mathfrak{f}^\alpha_n(\zeta; \mathbf{c}_1, \dots, \mathbf{c}_g) = \Psi^m \mathfrak{f}^\alpha_j, \quad n = (g+1)m + j, \ j \in [0, \dots, g]$$

where (with a suitable constants $\phi_j \in \mathbb{R}/\mathbb{Z}$)

$$\mathfrak{f}^\alpha_0 = \frac{e^{2\pi\phi_1 i} k^\alpha_{\zeta_1}}{\sqrt{k^\alpha_{\zeta_1}(\zeta_1)}}, \ \mathfrak{f}^\alpha_1 = \frac{e^{2\pi\phi_2 i}\mathfrak{b}_{\mathbf{c}_1} k^{\alpha-\mu_{\mathbf{c}_1}}_{\zeta_2}}{\sqrt{k^{\alpha-\mu_{\mathbf{c}_1}}_{\zeta_2}(\zeta_2)}}, \dots, \ \mathfrak{f}^\alpha_g = \frac{\prod_{j=1}^g \mathfrak{b}_{\mathbf{c}_j} k^{\alpha+\mu}}{\sqrt{k^{\alpha+\mu}(0)}}.$$

Theorem 2. *In the above notations the multiplication operator by $\mathfrak{z}$ with respect to the basis $\{\mathfrak{f}_n^\alpha\}_{n\in\mathbb{Z}}$ is a periodic GMP matrix $A(\alpha)$. Moreover, the isospectral set $A(E)$ of periodic GMP matrices has the form*

$$A(E) = \{A(\alpha) : \ \alpha \in \Gamma^*\},$$

and can be described as the collection of GMP matrices A, which satisfies the following ("magic") formula

$$\Delta(A) = S^{g+1} + S^{-(g+1)}.$$

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Lieb–Thirring inequalities for finite and infinite gap Jacobi matrices

Maxim Zinchenko

(joint work with Jacob Christiansen)

In 1976, Lieb and Thirring in their work on stability of matter [9, 10] obtained an upper bound on the moments of discrete eigenvalues of a Schrödinger operator. For one-dimensional Schrödinger operators the bound takes the form

$$\sum_{\lambda\in\sigma_d(A)} |\lambda|^{p-1/2} \le L_p \int_{\mathbb{R}} |V(x)|^p dx, \quad p \ge 1, \tag{1}$$

where L_p is a constant independent of V. The bound is false for $p < 1$. In the original work the inequality was derived for $p > 1$ and the endpoint result for $p = 1$ was proven only 20 years later by Weidl [11]. Lieb–Thirring inequalities have found applications in the studies of quantum mechanics, differential equations, and dynamical systems, see e.g., [7] for a history of the subject.

An analogous Lieb–Thirring inequality for discrete Schrödinger operators and more generally for Jacobi operators on $\ell^2(\mathbb{Z})$, represented by the tridiagonal Jacobi matrices

$$J = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_0 & b_1 & a_1 & & & \\ & & a_1 & b_2 & a_2 & & \\ & & & a_2 & b_3 & a_3 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2}$$

was obtained in [8] for perturbations of the discrete Laplacian and in [4] for perturbations of periodic Jacobi matrices. In either of these settings denoting the unperturbed Jacobi matrix by J_0, its spectrum by E, and letting the Jacobi matrix $J = J_0 + \delta J$ be a compact self-adjoint perturbation of J_0 leads to the following Lieb–Thirring bound for all $p \geq 1$,

$$\sum_{\lambda\in\sigma_d(J)} \operatorname{dist}\big(\lambda, \mathsf{E}\big)^{p-\frac{1}{2}} \leq L_{p,\,\mathsf{E}} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p, \tag{3}$$

where the constant $L_{p,\,\mathsf{E}}$ is independent of both J_0 and δJ and may depend only on p and E. In [5] the case $p = 1$ of the above Lieb–Thirring bound was proven for perturbations of Jacobi matrices from arbitrary finite gap isospectral tori $\mathcal{T}_{\mathsf{E}}$, hence extending the bound to perturbations of certain quasi-periodic Jacobi matrices.

In the joint work with J. S. Christiansen [2], we investigate possible extensions of the Lieb–Thirring bound to more general classes of Jacobi matrices, in particular, to perturbations of certain almost periodic Jacobi matrices with infinite gap spectrum. In [2] we obtain the following abstract version of the Lieb–Thirring bound. Suppose J_0 and $J = J_0 + \delta J$ are two-sided Jacobi matrices and δJ is a compact self-adjoint perturbation. Let E be the spectrum of J_0 and denote the gaps of E by (α_k, β_k), $k \geq 0$, so that $\mathsf{E} = \big[\beta_0, \alpha_0\big] \setminus \bigcup_{j\geq1}\big(\alpha_j, \beta_j\big)$. In addition, let $d\rho_n$ be the spectral measures of (J_0, δ_n), $n \in \mathbb{Z}$, and suppose that

$$\sup_{n\in\mathbb{Z}} \int_{\mathsf{E}} \frac{d\rho_n(t)}{|t-x|} \leq \frac{C_k}{\operatorname{dist}(x, \mathsf{E})^{1/2}}, \quad x \in (\alpha_k, \beta_k), \quad k \geq 0, \tag{4}$$

for some summable sequence $\{C_k\}_{k\geq0}$. Then $\sigma_{ess}(J) = \mathsf{E}$ and the discrete eigenvalues of J satisfy the Lieb–Thirring bound (3) with $p > 1$ and the constant $L_{p,\mathsf{E}}$ that depends only on p and the sequence $\{C_k\}_{k\geq0}$. We then show that the abstract result applies (i.e., assumption (4) is satisfied with the constants $\{C_k\}_{k\geq0}$ depending only on E) in particular situations of almost periodic J_0 from isospectral tori $\mathcal{T}_{\mathsf{E}}$ for sets E of three types: (i) arbitrary finite gap sets; (ii) Cantor sets $\mathsf{E} = \bigcap_{k=0}^{\infty} \mathsf{E}_k$, where $\mathsf{E}_0 = [\beta_0, \alpha_0]$ and E_k is obtained from E_{k-1} by removing the middle ϵ_k portion from each of the 2^{k-1} bands in E_{k-1}, with parameters $\{\epsilon_k\}_{k\geq1} \subset (0,1)$ satisfying $\limsup_{k\to\infty}(\epsilon_k)^{1/k} < 1/4$; (iii) infinite band sets $\mathsf{E} = \bigcap_{k=0}^{\infty} \mathsf{E}_k$, where $\mathsf{E}_0 = [\beta_0, \alpha_0]$ and E_k is obtained from E_{k-1} by removing the middle ϵ_k portion from the first of the k bands in E_{k-1}, with parameters $\{\epsilon_k\}_{k\geq1} \subset (0,1)$ satisfying

$\sum_{k=1}^{\infty}\sqrt{\epsilon_k}\log(1/\epsilon_k)<\infty$. The endpoint case $p=1$ of the Lieb–Thirring inequality (3) for perturbations of Jacobi matrices from infinite gap isospectral tori is an open problem. Another open problem is to find a characterization of infinite gap sets for which there is a Lieb–Thirring bound.

In recent years, Lieb–Thirring-type bounds have been also obtained for Schrödinger operators with complex potentials. In the setting of tridiagonal Jacobi matrices with complex coefficients

$$J=\begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_0 & b_1 & c_1 & & & \\ & & a_1 & b_2 & c_2 & & \\ & & & a_2 & b_3 & c_3 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{5}$$

a Lieb–Thirring-type bound was derived in [1, 6] for compact non-self-adjoint perturbations $J=J_0+\delta J$ of the discrete Laplacian J_0. The bound asserts that for every $0<\epsilon<1$,

$$\sum_{z\in\sigma_d(J)}\frac{\operatorname{dist}\big(z,[-2,2]\big)^{p+\epsilon}}{|z^2-4|^{\frac{1}{2}}}\leq L_{p,\epsilon}\sum_{n=-\infty}^{\infty}|\delta a_n|^p+|\delta b_n|^p+|\delta c_n|^p,\quad p\geq 1, \tag{6}$$

where the eigenvalues are repeated according to their algebraic multiplicity and the constant $L_{p,\epsilon}$ is independent of δJ. It is currently not known whether or not the inequality continues to hold for $\epsilon=0$.

In the joint work with J. S. Christiansen [3], we improve the non-self-adjoint Lieb–Thirring bound (6) and extend it to perturbations of periodic Jacobi matrices and more generally to perturbations of quasi-periodic Jacobi matrices from finite gap isospectral tori $\mathcal{T}_{\mathsf{E}}$. Give an arbitrary finite gap set $\mathsf{E}\subset\mathbb{R}$, let $J_0\in\mathcal{T}_{\mathsf{E}}$ and suppose that $J=J_0+\delta J$ is a compact non-self-adjoint perturbation of J_0. In [3] we show that for every $0<\epsilon<1$,

$$\sum_{z\in\sigma_d(J)}\frac{\operatorname{dist}\big(z,\mathsf{E}\big)^{p+\epsilon}(1+|z|)^{\frac{1}{2}-2\epsilon}}{\operatorname{dist}(z,\partial\mathsf{E})^{\frac{1}{2}}}\leq L_{\epsilon,p,\mathsf{E}}\sum_{n=-\infty}^{\infty}|\delta a_n|^p+|\delta b_n|^p+|\delta c_n|^p, \tag{7}$$

where the eigenvalues are repeated according to their algebraic multiplicity and the constant $L_{\epsilon,p,\mathsf{E}}$ is independent of both J_0 and δJ and may depend only on ϵ, p, and E. We note that (7) is new even when J_0 is the discrete Laplacian since, unlike (6), it is nearly optimal not only for small but also for large perturbations. As with (6), it is an open problem whether (7) remains true for $\epsilon=0$. Another open problem is to determine whether it is possible to replace $\operatorname{dist}(z,\partial\mathsf{E})^{\frac{1}{2}}$ in the denominator on the left hand-side by $\operatorname{dist}(z,\mathsf{E})^{\frac{1}{2}}$.

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