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Mini-Workshop: Interplay between Number Theory and Analysis for Dirichlet Series

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Abstract. In recent years a number of challenging research problems have crystallized in the analytic theory of Dirichlet series and its interaction with function theory in polydiscs. Their solutions appear to require unconventional combinations of expertise from harmonic, functional, and complex analysis, and especially from analytic number theory. This MFO workshop provided an ideal arena for the exchange of ideas needed to nurture further progress and to solve important problems.

Mathematics Subject Classification (2010): 11xx, 30xx.

Introduction by the Organisers

The workshop 'Interplay between number theory and analysis for Dirichlet series', organised by Frédéric Bayart (Clermont Université), Kaisa Matomäki (University of Turku), Eero Saksman (University of Helsinki) and Kristian Seip (NTNU, Trondheim) was held October 29th – November 4th, 2017. This meeting was well attended with around 25 participants coming from a number of different countries, including participants form North America. The group formed a nice blend of researchers with somewhat different mathematical backgrounds which resulted in a fruitful interaction.

About 17 talks, of varying lengths, were delivered during the five days. The talks were given by both leading experts in the field as well as by rising young researchers. Given lectures dealt with e.g. connections between number theory and random matrix theory, operators acting on Dirichlet series, distribution of Beurling primes, growth of L^p-norms of Dirichlet polynomials, growth and density of values of the Riemann zeta on boundary of the critical strip, Rado's criterion for kth powers, Sarnak's and Elliot's conjectures, and Hardy type spaces of general Dirichlet series. In addition, a problem session was held on Wednesday evening. Two of the speakers gave wider expositions, each comprising of two talks: Hugh Montgomery's beautiful review of the theory of mean values of Dirichlet polynomials was much appreciated by the participants. In turn, Adam Harper exposed his impressive solution of Helson's conjecture, which has interesting consequences both to the analytic theory of Dirichlet series and to the growth of random or Dirichlet character sums. It also provided a surprising connection to the theory of multiplicative chaos.

The meeting stimulated many new collaborative research projects. Besides the high level scientific program, most of the participants took part in the the classical social activity, i.e the Wednesday afternoon hike to St. Roman for coffee and Black Forest cake. The weather was nice during the hike and almost all participants managed to return from the long walk before it got really dark. Overall, the atmosphere was very relaxed and stimulated discussions and free scientific gatherings of the participants – during the last day of the conference a considerable audience gathered into the seminar room to listen to improvised extra lectures given around midnight! *Acknowledgement:* The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Mini-Workshop: Interplay between Number Theory and Analysis for Dirichlet Series

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Abstracts

Sums of greatest common divisors and metric number theory

Christoph Aistleitner

In summer 2012 I started a collaboration with Kristian Seip from NTNU Trondheim on questions concerning so-called “GCD sums”, that is, sums involving greatest common divisors of the form

$$\sum_{1 \le k,l \le N} c_k c_\ell \frac{(\gcd(n_k,n_l))^{2\alpha}}{(n_k n_l)^\alpha}.$$

Here $n_1, \dots, n_N$ are distinct positive integers, and (c_k) are real coefficients which are normalized to $\sum c_k^2 \le 1$. Furthermore, α is a real parameter which in the interesting cases is from the range $[1/2, 1]$. The role of such sums in the theory of Diophantine approximation was recognized since the early 20th century. For example, the problem of finding upper bounds for such GCD sums in the case of $\alpha = 1$ was stated as a prize problem by the Royal Dutch society (following a proposal of Erdős), and was solved by Gál [5] in 1949. The problem in the case $\alpha = 1/2$ is discussed in detail in Harman's book on *Metric number theory* [6], together with many applications, but the full problem was settled only very recently as a result of the collaboration with Kristian Seip mentioned at the beginning (see [1], [4] and [7]). The research on this problem took a surprising turn when a connection to the question concerning large values of the Riemann zeta function in the critical strip was established; this led to a series of papers, which culminated in the breakthrough obtain by Bondarenko and Seip [4] concerning large values of the zeta function on the critical line.

However, after the question asking for the maximal size of such GCD sums was settled, these estimates were also used very successfully in the field which they were originally intended for. More precisely, the new strong bounds for GCD sums led to several remarkable results in analysis and metric number theory, concerning for example the convergence of series of dilated functions [1], the discrepancy of parametric sequences [2], and the metric theory of pair correlations [3]. In my talk at the MFO I explained how GCD sums arise naturally in such questions as the “variance” of a sum of functions, by using simple methods from Fourier analysis, and illustrated as a particular case how to obtain the results from [3] which link the asymptotic distribution of the pair correlations of a parametric sequence $(n_k\alpha)_{k\ge1}$ mod 1 with the so-called additive energy of the sequence $(n_k)_{k\ge1}$.

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A remark on Sarnak's conjecture

Régis de la Bretèche
(joint work with Gérald Tenenbaum)

We investigate Sarnak's conjecture on the Möbius function in the special case when the test function is the indicator of the set of integers for which a real additive function assumes a given value.

According to a general pseudo-randomness principle related to a famous conjecture of Chowla and recently considered by Sarnak, the Möbius function μ does not correlate with any function ξ of low complexity. In other words,

$$\sum_{n\leq x}\mu(n)\xi(n) = o\Big(\sum_{n\leq x}|\xi(n)|\Big) \qquad (x\to\infty).$$

It is known since Halász that

$$Q(x;f) := \sup_{m\in\mathbb{R}} \sum_{\substack{n\leq x\\ f(n)=m}} 1 \ll \frac{x}{\sqrt{1+E(x)}}$$

where we have put

$$E(x) := \sum_{\substack{p\leq x\\ f(p)\neq 0}} \frac{1}{p}\cdot$$

Here and in the sequel, the letter p denotes a prime number.

As a first investigation of the above described problem, we would like to show that

$$Q(x;f,\mu) := \sup_{m\in\mathbb{R}} \Big|\sum_{\substack{n\leq x\\ f(n)=m}} \mu(n)\Big|$$

is generically smaller than $Q(x;f)$. Of course we have to avoid the case when $f(p)$ is constant, for then $\mu(n)$ does not oscillate on the set of squarefree integers n with $f(n)=m$. Therefore we seek an estimate which coincides with those of $Q(x;f)$ when $f(p)$ is close to a constant and which has smaller order of magnitude otherwise. When $f(p)$ is restricted to assume the values 0 or 1 only, we thus expect a significant improvement when

$$F(x) := \sum_{p\leq x} \frac{1-f(p)}{p}$$

is large. Indeed, in this simple case we obtain the following estimate.
Let f denote a real additive arithmetic function such that $f(p) \in \{0,1\}$ for all p. Then, with the above notation and $c = (2\pi-4)/(3\pi-2) \approx 0.30751$, we have

$$Q(x;f,\mu) \ll \frac{x\{1+F(x)\}e^{-cF(x)}}{\sqrt{1+E(x)}}.$$

Moreover, we obtain an asymptotic relation for

$$N_m(x;f,\mu) := \sum_{\substack{n\leq x\\ f(n)=m}} \mu(n)$$

providing $F(x)$ is not too large.
Let $\kappa \in]0,1[$ and let f denote a strongly additive function such that $f(p) \in \{0,1\}$ for all primes p. Assume furthermore that

$$\sum_{p\leq y}\{1-f(p)\}\log p \ll \frac{y}{(\log_2 y)}.$$

Uniformly in the range $\kappa E(x) \leq m \leq E(x)/\kappa$, we have

$$N_m(x;f,\mu) = (-1)^m N_m(x;f)\left\{\lambda_f e^{-2F(x)} + O\left(\frac{1}{(\log_2 x)^b}\right)\right\},$$

with

$$\lambda_f := \prod_{f(p)=0} \frac{1-1/p}{1+1/p} e^{2/p}, \quad b := \frac{\kappa}{2(4-\kappa)}.$$

The key ingredient of all our proofs comes from the new and powerful results of Tenenbaum on effective mean value of multiplicative functions [2].

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Non linear chains in the study of functional equations

Jean-François Burnol

The Dedekind zeta function $\zeta_{\mathbb{Q}(i)}(s)$ verifies the functional equation

$$\pi^{-s}\zeta_{\mathbb{Q}(i)}(s) = \frac{\Gamma(1-s)}{\Gamma(s)}\pi^{-(1-s)}\zeta_{\mathbb{Q}(i)}(1-s)\ . \tag{1}$$

We study functional equations of the type

$$f(s) = \pm\frac{\Gamma(\frac{\nu}{2}+1-s)}{\Gamma(\frac{\nu}{2}+s)}f(1-s)\ . \tag{2}$$

The Riemann zeta function $\zeta(s)$ can be put into this mold by considering $\pi^{-w/2}\zeta(w)$ (whose functional equation involves $\Gamma(\frac{w}{2})$), after the change of variable $w-\frac{1}{2} = 2(s-\frac{1}{2})$. This will become (2) with $\nu = -\frac{1}{2}$. and similarly for Dirichlet L-series for even characters. Those for odd characters are related to (2) with $\nu = +\frac{1}{2}$.
For technical reasons, some of the discussion that follows requires $\nu > -1$, but this can also be lifted to some extent after a while. At first stages we do insist that ν is real because the ratio of Gamma factors in (2) is then unitary on the $\frac{1}{2}$-line.

Hence it is associated with some unitary operator H_ν on $L^2(0,+\infty;\mathrm{d}x)$, from the formula

$$\widehat{H_\nu(\phi)}(s) = \frac{\Gamma(\frac{\nu}{2}+1-s)}{\Gamma(\frac{\nu}{2}+s)}\widehat{\phi}(1-s) \tag{3}$$

where $\widehat{\phi}(s) = \int_0^\infty \phi(x)x^{-s}\mathrm{d}x$. This $\phi \to \widehat{\phi}$ (Mellin transform) identifies unitarily $L^2(0,+\infty;\mathrm{d}x)$ with $L^2(\Re(s)=\frac{1}{2};\frac{|ds|}{2\pi})$.

The operator H_ν is the operator with kernel $J_\nu(2\sqrt{xy}) = \sum_{n=0}^\infty (-1)^n \frac{x^{n+\frac{\nu}{2}}y^{n+\frac{\nu}{2}}}{n!\Gamma(n+\nu+1)}$.

The study of functional equation (2) ultimately leads to functions verifying the Riemann Hypothesis when we try to elucidate those $f(s)$ which additionally belong to the (conformally invariant) Hardy space $\mathbb{H}^2_{\Re(s)>\frac{1}{2}}$ of the right half-plane or more generally to some $a^{-s+\frac{1}{2}}\mathbb{H}^2_{\Re(s)>\frac{1}{2}}$. This fact was known to de Branges as an incarnation ("Sonine spaces") of his general theory of Hilbert spaces of entire functions. I elucidated some structure problem in [1] via the study of integral equations associated to operators H_ν.

I was led myself to such spaces starting from the study of functions such as $\zeta(s)/(s-\rho)$. I found Hilbert spaces of entire functions, then learned there was a general de Branges theory, and even that the spaces already had a name, but their structure was basically unknown. Except for $\nu = 0$ and to a lesser extent for $\nu \in \mathbb{N}$ for which the spaces can be explained in terms of Bessel functions.

My theory started in [1]. I then found the link with Fredholm determinants and Dirac type equations in [2]. That paper was dedicated to the cosine and sine transforms ($\nu = \pm\frac{1}{2}$), and I latter gave a detailed exposé of my approach in the case $\nu = 0$, applicable with very light modifications to general $\nu > -1$, in some chapters of [3].

These techniques were recently applied and extended by Suzuki ([5], [6]) to a more direct study of the Riemann ξ-function (and Selberg class.)

My talk focused on results obtained after [3] and which revolve around a connection with Painlevé transcendants. Aspects of this were published in [4], which is connected actually not with the operators H_ν but with toy-operators H_ν^N where the Gamma factors have been replaced with rational functions with N poles along an arithmetic progression. Then the theory leads to PVI transcendants, whereas the H_ν-theory is a confluence limit related to PIII and PV transcendants. Although I obtained that part of the theory earlier I have not yet published it.

To state a typical result, I need to make a few definitions. Let $a > 0$, P_a orthogonal projection to $L^2(0, a)$ and $\phi^\pm_{\nu,a}$ the unique solution to the integral equation on $(0, a)$, $(1 \pm P_a H_\nu P_a)f = J_\nu(2\sqrt{ax})$. The $\phi^\pm_{\nu,a}$ are analytic with a branch cut on the negative real axis. The functions $\frac{\sqrt{a}}{2}\Gamma(s + \frac{\nu}{2})(a^{-s} \pm \int_a^\infty \phi^\pm_{\nu,a}(x)x^{-s}\mathrm{d}x)$ are the entire functions I mentioned earlier which solve the structure problem of the Sonine spaces ([1]). They are involved in differential system ([2]) whose coefficients are expressible with the Fredholm determinants of $1 \pm P_a H_\nu P_a$, in particular via the function

$$\mu_\nu(a) = a\frac{d}{da}\log\frac{\det(1 + P_a H_\nu P_a)}{\det(1 - P_a H_\nu P_a)} \tag{4}$$

Let $\tau_\nu(a) = (2a)^{\nu^2} e^{a^2} \det\Big(1 - (P_a H_\nu P_a)^2\Big)$, then the Toda chain relation holds:

$$\left(a\frac{d}{da}\right)^2 \log\tau_\nu = \frac{\tau_{\nu-1}\tau_{\nu+1}}{\tau_\nu^2}\ . \tag{5}$$

The quantities μ_ν and $\mu_{\nu+1}$ are related one with the other by a non-linear differential first order system, connecting them to PIII and PV, and such systems exist in relevant Physics literature. But the more involved analogous system where there is also the parameter N, and which connects the $\mu_{\nu,N}$ to PVI ([4]) seems to possibly not have been known in Painlevé litterature. My theory is autonomous and uses nothing from Painlevé litterature.

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A proof of Helson's conjecture

Adam J. Harper

Let $(f(p))_{p \text{ prime}}$ be a sequence of independent Steinhaus random variables, i.e. independent random variables distributed uniformly on the unit circle $\{|z| = 1\}$. We define a *Steinhaus random multiplicative function* f, by setting $f(n) := \prod_{p^a || n} f(p)^a$ for all natural numbers n (where $p^a || n$ means that p^a is the highest power of the prime p that divides n, so $n = \prod_{p^a || n} p^a$). Thus f is a random function taking values in the complex unit circle, that is totally multiplicative.
I gave two lectures discussing the following result, taken from my preprint [1].

Theorem 1. *If $f(n)$ is a Steinhaus random multiplicative function, then uniformly for all large x and $0 \le q \le 1$ we have*

$$\mathbb{E}|\sum_{n \le x} f(n)|^{2q} \asymp \left(\frac{x}{1 + (1-q)\sqrt{\log\log x}}\right)^q .$$

I also discussed an analogous result for sums of Dirichlet characters $\chi(n)$ (for which the random multiplicative function $f(n)$ is often proposed as a good model).

Theorem 2. *Let r be a large prime. Then uniformly for any $1 \le x \le r$ and any $0 \le q \le 1$, we have*

$$\frac{1}{r-1} \sum_{\chi \bmod r} |\sum_{n \le x} \chi(n)|^{2q} \ll \left(\frac{x}{1 + (1-q)\sqrt{\min\{\log\log \mathcal{L}, \log\log\log r\}}}\right)^q ,$$

where $\mathcal{L} := 10 \min\{x, \frac{r}{x}\}$.

In the special case where $q = 1/2$, Theorem 1 implies that $\mathbb{E}|\sum_{n \le x} f(n)| \asymp \left(\frac{x}{\sqrt{\log\log x}}\right)^{1/2}$, and in particular that $\mathbb{E}|\sum_{n \le x} f(n)| = o(\sqrt{x})$ as $x \to \infty$. This proved a conjecture of Helson [2], as referenced in the title of my talks. Part of the interest of these results is that, in view of the Bohr correspondence and Theorem 1, we have

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\sum_{n \le x} n^{-it}|^{2q} dt = \mathbb{E}|\sum_{n \le x} f(n)|^{2q} = o(x^q) \quad \forall 0 < q < 1.$$

On the other hand, Riemann–Stieltjes integration implies that

$$\int_0^1 |\sum_{n\leq x} \frac{1}{n^{1/2+it}}|^{2q} dt = \int_0^1 |\frac{x^{1/2-it}}{1/2-it} + O(1)|^{2q} dt \asymp x^q,$$

so we see that there cannot exist any universal constants C_{2q} such that

$$\int_0^1 |P(1/2+it)|^{2q} dt \leq C_{2q} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^{2q} dt$$

for all Dirichlet polynomials $P(s)$. This gave a negative answer to the so-called *embedding problem* for Dirichlet polynomials (see Question 2 of [3], or Problem 2.1 of [4]) for all exponents $0 < 2q < 2$.

The proof of Theorem 1 breaks into two (somewhat uneven) parts. Firstly, we need to establish a connection between $\mathbb{E}|\sum_{n\leq x} f(n)|^{2q}$ and an average involving the corresponding Euler product

$$F(s) := \prod_{p\leq x} (1 - \frac{f(p)}{p^s})^{-1} = \sum_{\substack{n=1,\\ p|n\Rightarrow p\leq x}}^{\infty} \frac{f(n)}{n^s}.$$

This will be useful for several reasons, notably because in the Euler product the different factors $(1 - \frac{f(p)}{p^s})^{-1}$ are independent, whereas in the sum $\sum_{n\leq x} f(n)$ the contributions from the underlying independent $f(p)$ are entangled with one another in a highly non-trivial way.

The standard way to achieve such a connection would be using Perron's formula:

$$\sum_{n\leq x} f(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} ds, \quad c > 0, \quad x \notin \mathbb{N}.$$

However, if we choose $c = 1/2$ and apply the triangle inequality (the natural way to proceed since, to first order, we expect $\sum_{n\leq x} f(n)$ to typically be around $\sqrt{x}$ in size), we get

$$|\sum_{n\leq x} f(n)| \leq \frac{x^{1/2}}{2\pi} \int_{-\infty}^{\infty} |F(1/2+it)| \frac{1}{|1/2+it|} dt.$$

Here the integral on the right will actually diverge, but we could correct this by introducing a smoother weight on the left, which would replace the factor $1/|1/2+it|$ by something like $1/|1/2+it|^2$, whose integral converges. However, in any case, to deduce (the upper bound in) Theorem 1 from this we would need $|F(1/2+it)|$ to typically have size $o(1)$, which is not the case. So we need a different approach.

One can actually show, very roughly speaking, that

$$\mathbb{E}|\sum_{n\leq x} f(n)|^{2q} \approx x^q \mathbb{E}\left(\frac{1}{\log x} \int_{-1/2}^{1/2} |F(1/2+it)|^2 dt \right)^q.$$

This is done by splitting up $\sum_{n \leq x} f(n)$ into double sums like

$$\sum_{\sqrt{x} < p \leq x \text{ prime}} f(p) \sum_{m \leq x/p} f(m),$$

and then conditioning on the inner random sums (which are independent of the outer random variables $(f(p))_{\sqrt{x}<p\leq x}$) and applying Khintchine's inequality to relate the conditional $2q$-th moment to a mean square. Having done this, we can apply Parseval's identity to relate an integral square average of $\sum_{n\leq x} f(n)$ to an integral square average of $F(1/2+it)$. The underlying point here is that, if one tries to apply Perron's formula and the triangle inequality, one loses something because any cancellation from the oscillations of x^{it} in the Perron integral is lost. If one can first move everything to the level of mean squares, there are no oscillations to be lost and one can hope to obtain a sharp connection with the Euler product $F(s)$.

The larger second part of the proof is to analyse $x^q \mathbb{E}\left(\frac{1}{\log x}\int_{-1/2}^{1/2} |F(1/2+it)|^2 dt\right)^q$. To set the scene we note that, by Hölder's inequality,

$$x^q \mathbb{E}\left(\frac{1}{\log x}\int_{-1/2}^{1/2} |F(1/2+it)|^2 dt\right)^q \leq x^q \left(\frac{1}{\log x}\int_{-1/2}^{1/2} \mathbb{E}|F(1/2+it)|^2 dt\right)^q$$

for all $0 \leq q \leq 1$. A standard calculation shows $\mathbb{E}|F(1/2+it)|^2 \asymp \log x$, and inserting this yields the trivial upper bound $\mathbb{E}|\sum_{n\leq x} f(n)|^{2q} \ll x^q$. The major contribution to this expected size of $|F(1/2+it)|^2$ comes from the fairly rare event that $|\log|F(1/2+it)| - \log\log x| \leq \sqrt{\log\log x}$, but if integrating over $[-1/2, 1/2]$ roughly corresponded to taking $\log x$ independent samples of $|F(1/2+it)|$ (because $F(s)$ varies with s on a scale of $1/\log x$), one might indeed typically find a few such values of $\log|F(1/2+it)|$ with $|t| \leq 1/2$. So the essence of Theorem 1 is that, when looking at $\mathbb{E}\left(\frac{1}{\log x}\int_{-1/2}^{1/2} |F(1/2+it)|^2 dt\right)^q$ with q a little smaller than 1, integrating over $[-1/2, 1/2]$ *does not correspond* to taking $\log x$ independent samples of $|F(1/2+it)|$, so the above application of Hölder's inequality is wasteful. It turns out that $\frac{1}{\log x}\int_{-1/2}^{1/2} |F(1/2+it)|^2 dt$ is fairly close to (the total mass of a truncation of) a probabilistic object called *critical multiplicative chaos*, and one can analyse it using an iterative argument drawing on ideas from that field. We refer to the preprint [1] for further details.

We finish with a couple of remarks about Theorem 2. There is a well known duality between the character sum $\sum_{n\leq x} \chi(n)$ and the character sum $\sum_{n\leq r/x} \chi(n)$, arising from Poisson summation (alternatively known, in this context, as the approximate functional equation or the Pólya Fourier expansion). Indeed, roughly speaking we have $|\sum_{n\leq x} \chi(n)| \approx \frac{x}{\sqrt{r}} |\sum_{n\leq r/x} \chi(n)|$. In view of this and Theorem 1, it is natural to think that the sharp upper bound in Theorem 2 would be $\left(\frac{x}{1+(1-q)\sqrt{\log\log \mathcal{L}}}\right)^q$, i.e. that the minimum in the denominator ought not to be there. It is work in progress of the author to prove this. To prove Theorem

2 as stated, one shows that the behaviour of a randomly chosen character $\chi(n)$ on small primes is roughly the same as the behaviour of a random multiplicative function. Using this and a suitable adaptation of the conditioning argument from the random case (so one averages out all the large primes), one can then deduce Theorem 2 from Theorem 1.

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Problems concerning Beurling's generalized primes

Titus W. Hilberdink

We discuss various problems of generalized prime systems. First we give an overview of the results connecting the counting functions $\pi(x)$ and $N(x)$ of generalized primes and integers respectively; in particular where one of the functions behaves like its counterpart in the actual system of primes/integers. For example, Beurling showed that

$$N(x) = cx + O\Big(\frac{x}{(\log x)^\gamma}\Big) \implies \pi(x) \sim \frac{x}{\log x}$$

whenever $\gamma > \frac{3}{2}$ but fails for $\gamma = \frac{3}{2}$, showing this is sharp. Another case was proven by Landau: if $\alpha < 1$, then

$$N(x) = cx + O(x^\alpha) \implies \pi(x) = \mathrm{li}(x) + O(xe^{-a\sqrt{\log x}})$$

for some $a > 0$. This was proven to be sharp in [1].

In the reverse direction,

$$\pi(x) = \mathrm{li}(x) + O(x^\alpha) \implies N(x) = cx + O(xe^{-a\sqrt{\log x \log\log x}})$$

(as shown in [4]), but here it is not known whether this is best possible. This is an open problem.

Another open problem concerns the generalized Möbius function, $\mu_{\mathcal{P}}(n)$. Assuming $N(x) = x^{1+o(1)}$, how small can we make the function

$$M(x) = \sum_{n \le x, n \in \mathcal{N}} \mu_{\mathcal{P}}(n)?$$

For $\mathbb{N}$, on RH, $M(x) = O(x^{\frac{1}{2}+\varepsilon})$ for all $\varepsilon > 0$ (but no $\varepsilon < 0$ due to the Riemann zeros). Is it possible to make it $O(x^{\alpha})$ for some $\alpha < \frac{1}{2}$ in some other system?

Further, we discussed Mellin transforms $\hat{N}(s) = \int_{1-}^{\infty} x^{-s} dN(x)$ for which $N(x) - x$ is periodic with period 1 in order to investigate 'flows' of such functions to Riemann's $\zeta(s)$ and the possibility of proving the Riemann Hypothesis with such an approach. In [3] it was shown that, excepting the trivial case where $N(x) = x$, the supremum of the real parts of the zeros of any such function is at least $\frac{1}{2}$. We asked whether it is possible to adapt this method to have better (but non-periodic) approximations to $[x]$ for which the suprema of the real parts of the zeros would increase (to $\frac{1}{2}$).

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Random Matrix Theory and the Maximum of the Riemann Zeta Function

Christopher Hughes

It is well known that the number of zeros of the Riemann zeta function up to height T is

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$

Note that this implies the zeros get more closely spaced together as one goes up the critical line. Rescaling them by via $\tilde{\gamma} = \gamma \times \frac{1}{2\pi} \log \frac{\gamma}{2\pi}$ means the mean spacing of $\tilde{\gamma}$ is unity.

In the 1970's Hugh Montgomery [1] first investigated correlations between these scalings, famously proposing the conjecture that roughly says for test functions f,

$$\frac{1}{N(T)} \sum_{0<\tilde{\gamma},\tilde{\gamma}'\leq T} f(\tilde{\gamma} - \tilde{\gamma}') \sim \int_{-\infty}^{\infty} f(x) \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right) \mathrm{d}x$$

for large T where δ denotes the Dirac delta function.

In a conversation between Hugh Montgomery and Freeman Dyson, it was realised this correlation was the same as one would get for random matrices. Not historically accurate, but for the purposes of this talk we'll take a random matrix to mean a unitary matrix chosen with Haar measure. Haar measure is the unique left (and right) invariant measure on the group, that is for any fixed unitary V, if U is drawn with Haar measure, $VU \overset{\text{law}}{=} U$. Since the matrix is unitary, all its eigenvalues lie on the unit circle, and for a Haar distributed random unitary matrix these eigenvalues are correlated—large and small gaps are avoided.

Summary 1. *The scaled zeros of the Riemann zeta function are believed to be statistically distributed like the scaled eigenangles of Haar distributed random unitary matrices.*

Keating and Snaith [2] modelled the Riemann zeta function (not just its zeros) with RMT, using the characteristic polynomial of the random unitary matrix

$$\begin{aligned} Z_{U_N}(\theta) &:= \det(I_N - U_N e^{-\mathrm{i}\theta}) \\ &= \prod_{n=1}^{N} (1 - e^{\mathrm{i}(\theta_n - \theta)}) \end{aligned}$$

where U_N is an $N \times N$ unitary matrix chosen with Haar measure. The matrix size N is connected to the height up the critical line T via

$$N = \log \frac{T}{2\pi}$$

and they proved that this was a good model in the sense that its value distribution was normal, in analogy with the normal distribution of the Riemann zeta function on the critical line.

Summary 2. *The value distribution of the Riemann zeta function can be modelled by characteristic polynomials of random unitary matrices.*

More importantly, they calculated the moments of the characteristic polynomial

Theorem (Keating-Snaith)**.**

$$\mathbb{E}\left[|Z_{U_N}(0)|^{2k}\right] \sim \frac{G^2(k+1)}{G(2k+1)} N^{k^2}$$

where G is the Barnes' G-function

This theorem led to their celebrated conjecture for the moments of the Riemann zeta function

Conjecture (Keating-Snaith)**.**

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^{2k} \, \mathrm{d}t \sim a(k) \frac{G^2(k+1)}{G(2k+1)} \left(\log \frac{T}{2\pi}\right)^{k^2}$$

where

$$a(k) = \prod_{\substack{p \\ \text{prime}}} \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-m}$$

The point that was made in the talk is that $a(k)$ term, which does not appear in any of the random matrix calculations, was well-known and natural to number theorists, but, up until that point, number theory had been unable to provide even a conjecture for the Barnes' G-function term, which arose in the random matrix calculation.

Summary 3. *The Keating-Snaith conjecture used random matrix theory to predict the moments of the Riemann zeta function.*

One way to understand the interplay between the random matrix contribution and the arithmetic contribution to the moments comes from the following theorem which I proved with Steve Gonek and Jon Keating [3] over a decade ago.

Theorem (Gonek, Hughes, Keating)**.** *A simplified form of our theorem is:*

$$\zeta(\tfrac{1}{2}+\mathrm{i}t) = P_X(t) Z_X(t) + \textit{errors}$$

where

$$P_X(t) = \prod_{p \le X} \left(1 - \frac{1}{p^{\frac{1}{2}+\mathrm{i}t}}\right)^{-1}$$

and

$$Z_X(t) = \exp\left(\sum_{\gamma_n} \mathrm{Ci}(|t-\gamma_n| \log X)\right)$$

We were able to prove that if $X = O(\log T)$ then

$$\frac{1}{T}\int_T^{2T} |P_X(t)|^{2k} \; \mathrm{d}t \sim a(k) (e^\gamma \log X)^{k^2}$$

and we made a random matrix calculation that led to a conjecture (proven in the cases $k=1$ and $k=2$) that if $X, T \to \infty$ such that $\frac{\log T}{\log X} \to \infty$, then we have

$$\frac{1}{T}\int_0^{T} |Z_X(t)|^{2k} \; \mathrm{d}t \sim \frac{G^2(k+1)}{G(2k+1)} \left(\frac{\log T}{e^\gamma \log X}\right)^{k^2}$$

and combining these, plus an assumption about independence (also proven in the case $k=1$ and $k=2$), we recover the Keating-Snaith conjecture.

Summary 4. *We can write $\zeta = P_X Z_X$ where P_X is a term made of primes, and Z_X is a term made from zeros of zeta, which can be modeled by RMT.*

This result also plays a key role in a conjecture on the maximum of the Riemann zeta function which was made in a joint paper with David Farmer and Steve Gonek [4]

Conjecture (Farmer, Gonek, Hughes)**.**

$$(*) \qquad \max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log T \log\log T}\right)$$

This conjecture is one of the tantalising links with the topic of the Oberwolfach workshop, as it is of the same shape as appears in the bounds for the Sidon constants [5]. The known bounds for zeta are (under RH)

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = O\left(\exp\left(C\frac{\log T}{\log\log T}\right)\right)$$

where the value of C has been steadily decreasing from Littlewood's existence result in the 1902's to Chandee and Soundararajan's [6] value of $\frac{1}{2}\log 2$. In the other direction, very recent work of Bondarenko and Seip [7] yield

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| > \exp\left((1+o(1))\sqrt{\frac{\log T\log\log\log T}{\log\log T}}\right)$$

To obtain the conjecture of the true growth rate of zeta, note that simply taking the largest value of a characteristic polynomial doesn't work. Instead, split the interval $[0,T]$ up into $M = \frac{T\log T}{N}$ blocks, each containing approximately N zeros.

Model each block with the characteristic polynomial of an $N \times N$ random unitary matrix, and find the smallest $K = K(M,N)$ such that choosing M independent characteristic polynomials of size N, almost certainly none of them will be bigger than K.

Note that

$$\mathbb{P}\left\{\max_{1\le j\le M}\max_{\theta}|Z_{U_N^{(j)}}(\theta)| \le K\right\} = \mathbb{P}\left\{\max_{\theta}|Z_{U_N}(\theta)| \le K\right\}^M$$

and we were able to show that for $0<\beta<2$, if $M = \exp(N^\beta)$, and if

$$K = \exp\left(\sqrt{\left(1-\tfrac{1}{2}\beta+\varepsilon\right)\log M\log N}\right)$$

then

$$\mathbb{P}\left\{\max_{1\le j\le M}\max_{\theta}|Z_{U_N^{(j)}}(\theta)| \le K\right\} \to 1$$

as $N\to\infty$ for all $\varepsilon>0$, but for no $\varepsilon<0$.

Recall that $\zeta(\frac{1}{2}+\mathrm{i}t) \approx P_X(t)Z_X(t)$ and that, in the previous section, we showed that $Z_X(t)$ can be modelled by characteristic polynomials of size $N = \frac{\log T}{e^\gamma \log X}$. Therefore the previous theorem suggests the conjecture that if $X = \log T$, then

$$\max_{t\in[0,T]} |Z_X(t)| = \exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log T\log\log T}\right)$$

Since the Prime Number Theorem implies that if $X=\log T$ then for any $t\in[0,T]$,

$$P_X(t) = O\left(\exp\left(C\frac{\sqrt{\log T}}{\log\log T}\right)\right)$$

we see how one is led to the max values conjecture, $(*)$.

An alternative argument, leading to the same conclusion, can be given in terms of Steinhaus random variables. First note that

$$P_X(t) = \exp\left(\sum_{p\le X} \frac{1}{p^{1/2+\mathrm{i}t}}\right) \times O(\log X)$$

Treat $p^{-\mathrm{i}t}$ as independent random variables, U_p, distributed uniformly on the unit circle. It follows, from a slight generalisation of the Central Limit Theorem, that the distribution of

$$\Re \sum_{p\le X} \frac{U_p}{\sqrt{p}}$$

tends to Gaussian with mean 0 and variance $\frac{1}{2}\log\log X$ as $X \to \infty$.

We let $X = \exp(\sqrt{\log T})$ and model the maximum of $P_X(t)$ by finding the maximum of the Gaussian random variable sampled $T(\log T)^{1/2}$ times. This suggests, for that X,

$$\max_{t\in[0,T]} |P_X(t)| = O\left(\exp\left((\frac{1}{\sqrt{2}} + \varepsilon)\sqrt{\log T \log\log T}\right)\right)$$

for all $\varepsilon > 0$ and no $\varepsilon < 0$. For such a large X, random matrix theory suggests that $\max_{t\in[0,T]} |Z_X(t)| = O\left(\exp\left(\sqrt{\log T}\right)\right)$ and so doesn't contribute to the main term in the exponent, and thus this gives another justification of the large values conjecture, $(*)$.

Summary 5. *We presented two arguments, one coming from considering the zeros of zeta (through random matrix theory) and one coming from considering primes, that suggest*

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2} + \mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log T \log\log T}\right)$$

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Rado's criterion for k'th powers

Sofia Lindqvist
(joint work with Sam Chow, Sean Prendiville)

We say that an equation is *partition regular* if any finite colouring of $\mathbb{N}$ has a (nontrivial) monochromatic solution. One of the earliest examples of such a result is Schur's Theorem.

Theorem (Schur [1]). *Let $k \in \mathbb{N}$ and let $N > k!e$. Then any k-colouring of $\{1,\dots,N\}$ has x,y,z the same colour such that $x+y=z$.*

For linear equations the property of being partition regular was fully classified by Rado.

Theorem (Rado [2]). *The equation*

$$a_1x_1 + \cdots + a_sx_s = 0$$

is partition regular if and only if there is some nonempty subset $I \subset \{1,\cdots,s\}$, such that $\sum_{i\in I} a_i = 0$.

Here the coefficients are assumed to be nonzero integers. If there does exist a subset I such that $\sum_{i\in I} a_i = 0$ we say that the $\{a_i\}$ satisfy Rado's criterion.

In the case of nonlinear equations much less is known. It is an open problem to prove that the pythagorean equation $x^2+y^2=z^2$ is partition regular [3]. We are not able to deal with this equation, but by adding in two more variables we can prove the weaker statement that $x_1^2+x_2^2+x_3^2+x_4^2=x_5^2$ is partition regular. In fact we prove the following more general result.

Theorem 1 (Chow, L., Prendiville). *Let $k \in \mathbb{N}$ and assume that $s \geq k^2+1$. Then the equation*

$$a_1x_1^k + a_2x_2^k + \cdots + a_sx_s^k = 0$$

is partition regular if and only if the coefficients $\{a_i\}$ satisfy Rado's criterion.

For comparison, if the coefficients satisfy $\sum_{i=1}^{s} a_i = 0$ instead of just some strict subset summing to zero, one can prove that any dense subset of $\mathbb{N}$ has infinitely many (nontrivial) solutions. This was done by Browning–Prendiville [4] in the case of squares and generalised to k'th powers by Chow [5]. Their proofs use Green's transference principle. Very broadly speaking, transference allows one to transfer results about solutions to linear equations in dense sets to solutions in a sparse set satisfying various “pseudorandomness” conditions. The idea is then to transfer a result on the linear equation $\sum_i a_i x_i = 0$ to the sparse set of k-th powers.

In the case of Theorem 1 one can't blindly follow the same approach, as this relies on translation invariance of the equation, something one doesn't have when $\sum_i a_i \neq 0$. In order to overcome this obstacle we introduce the following notion.

Definition. *A set $A \subset \mathbb{N}$ is M-homogeneous if for any $q \in \mathbb{N}$ one has*

$$A \cap \{q, 2q, \cdots, Mq\} \neq \emptyset.$$

The idea is then to only restrict the translation invariant part of our equation to a particular set, while allowing the other variables to come from a homogeneous set, in the following sense.

Proposition. *Let $A \subset \{1, \cdots, N\}$ satisfy $|A| \geq \delta N$ and let B be M-homogeneous. Then there are $\gg_{\delta,M} N^{5/2}$ tuples $(x_1, x_2, y_1, y_2, y_3) \in A^2 \times B^3$ such that*

$$x_1 - x_2 = y_1^2 + y_2^2 + y_3^2.$$

Using the transference principle one can prove the corresponding result with A being a subset of the squares. Finally, given an r-colouring of $\{1, \ldots, N\}$ one can consider the largest colour class. If this colour class is homogeneous we are done by this result, and if not, by definition there is some arithmetic progression $q, 2q, \cdots, Mq$ which is $(r-1)$-coloured. This allows one to finish the proof by an induction on the number of colours.

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Mean values of Dirichlet polynomials

Hugh L. Montgomery

A *Dirichlet polynomial* is a sum of the form

$$D(s) = \sum_{n=1}^{N} a_n n^{-s}.$$

For such a sum we have an approximate Parseval identity,

$$(1) \qquad \int_0^T |D(it)|^2 \, dt = \big(T + O(N)\big) \sum_{n=1}^{N} |a_n|^2.$$

We can prove such an estimate in two different ways (both due to Selberg). In the first method, we use the Selberg functions S^+ and S^- which provide one-sided approximations that are sharp in the L^1 norm. The second method involves a generalized form of the Hilbert inequality. A weighted form of this allows us to establish a slight refinement of the above, namely that

$$\int_0^T |D(it)|^2\,dt = T\sum_{n=1}^N |a_n|^2 \;+\; O\Big(\sum_{n=1}^N n|a_n|^2\Big).$$

Expositions of these items are found in [4] and in Chapter 7 of [5].
For purposes of proving zero density estimates for the Riemann zeta function, we need to know how often a Dirichlet polynomial can be large. From (1) we see that if $|a_n| \le 1$ for all n, then

$$\text{meas}\{t \in [0,T] \,:\, |D(it)| \ge V\} \ll \frac{(T+N)N}{V^2}. \tag{2}$$

By applying this argument with D replaced by

$$D(s)^2 = \sum_{n=1}^{N^2} b_n n^{-s}$$

we find that

$$\text{meas}\{t \in [0,T] \,:\, |D(it)| \ge V\} \ll \frac{\big(T+N^2\big)N^{2+\varepsilon}}{V^4}. \tag{3}$$

If $N^2 \le T$ then this is a sharper bound than (2), while (2) is sharper if $N > T$. The critical configuration arises when $T^{1/2} < N < T$. If we could take fractional exponents, we would take κ so that $N^\kappa = T$, and then show that

$$\int_0^T |D(it)|^{2\kappa}\,dt \ll T^{2+\varepsilon}$$

when $|a_n| \le 1$ for all n. We conjecture that the above is true. It is not hard to show that the Density Hypothesis follows from the above. Also, it is not hard to show that the Majorant Conjecture of Hardy and Littlewood [2] implies the above. Unfortunately, it is now known that the Majorant Conjecture is false (see Bachelis [1]), but the known counter-examples do not seem to be very related to our situation.

Although the Hardy–Littlewood Majorant Conjecture is false, there are nevertheless valid majorant principles. We note in particular that if $|a_n| \le A_n$ for all n, then

$$\int_{T-C}^{T+C} \Big|\sum_{n=1}^N a_n n^{-it}\Big|^2\,dt \le 3\int_{-C}^{C} \Big|\sum_{n=1}^N A_n n^{-it}\Big|^2\,dt$$

for any real T and $C > 0$. Earlier forms of this inequality are found in the writings of Wiener and of Halász, but it seems that Wirsing (unpublished) was the first to obtain the sharp constant 3, which Logan [3] showed is best possible.

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Large values of Dirichlet polynomials

Hugh L. Montgomery

Fifty years ago, Gabor Halász discovered that more can be said about large values of Dirichlet polynomials than merely the bounds that follow from mean value theorems. Essesntially, he noted that if $0 \le t_1 < t_2 < \cdots < t_R \le T$ and $t_{r+1} - t_r \ge 1$ for all r, then it is rewarding to consider the bilinear form inequality

$$\sum_{r=1}^{R} \Big| \sum_{n=1}^{N} a_n n^{-it_r} \Big|^2 \le \Delta \sum_{n=1}^{N} |a_n|^2.$$

By duality, the assertion that the above holds for all choices of the a_n is equivalent to the assertion that

$$\sum_{n=1}^{N} \Big| \sum_{r=1}^{R} y_r n^{-it_r} \Big|^2 \le \Delta \sum_{r=1}^{R} |y_r|^2$$

for all y_r. By expanding the modulus-squared on the left hand side, and taking the sum over n inside, we see that the left hand side is

$$= \sum_{r=1}^{R} \sum_{s=1}^{R} y_r \overline{y_s} \sum_{n=1}^{N} n^{i(t_s - t_r)}.$$

When $r = s$ then inner sum is N, but when $r \ne s$ the sum is smaller, and we have tools for estimating it (e.g., van der Corput's method, Vinogradov's method of exponential sums). Moreover, if the Lindelöf Hypothesis is true, then this inner sum is $\ll N^{1/2} T^{\varepsilon} + N/|t_r - t_s|$ when $r \ne s$. Thus we find that a Dirichlet polynomial is large much less frequently than had been realized before.

The classical method for deriving zero density estimates for the zeta function involves 'Littlewood's Lemma', which is a bound for rectangles analogous to Jensen's inequality for discs. However, Littlewood's Lemma relates zeros to a mean value, not to occasional large values. In order for Halász's results to have an impact on zero densities, a new zero-detection method is needed. This was supplied by Turán. Halász and Turán [1] showed (among other things) that if the Lindelöf Hypothesis is true, then $N(\sigma, T) < T^{\varepsilon}$ for any fixed $\sigma > 3/4$, and $T > T_0(\varepsilon)$. Here

$N(\sigma, T)$ denotes the number of zeros $\rho = \beta + i\gamma$ of the zeta function such that $\beta \geq \sigma$ and $0 < \gamma \leq T$. This is an amazing result, but the fact that it is restricted to $\sigma > 3/4$ rather than to $\sigma > 1/2$ reflects what seems to be an imperfection in Halász's method. We focus on the problem of removing this blemish.

Since Halász's initial result there have been two significant advances: Huxley's Trick, which involves cutting the interval of length T into intervals of some smaller length T_1 and then multiplying the estimate obtained by T/T_1. Secondly, Bourgain has constructed an example from which we learn that it is better to work from the hypothesis that $|a_n| \leq 1$ for all n than it is to work with $\sum_{n=1}^{N}$ alone. Otherwise, there has not been much progress, despite the efforts of Bourgain and others. A q-analogue of Huxley's Trick has been devised by Iwaniec and his collaborators — they call it the 'δ method'.

We propose that to make significant progress, one must make greater use of the fact that the frequencies $\log n$ are logarithms of rational integers. For example, perhaps by use of the Mellin transform of the Gamma function we could transform our question concerning a Dirichlet polynomial into one concerning a trigonometric polynomial.

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Optimal comparison of the p-norms of Dirichlet polynomials

Antonio Pérez
(joint work with Andreas Defant)

In his early works, Bohr [5, 6] studied analiticity and convergence features of Dirichlet series in terms of abscissas. Among other results, he established that the maximal width T of the strip on which a Dirichlet series converges uniformly but not absolutely satisfies $T \leq 1/2$, and later Bohnenblust and Hille [4] proved that actually $T = 1/2$. More recently, Queffélec [16] purposed a different approach to this phenomenon in terms of the so-called Sidon constants: for each $x \geq 1$, let $\mathcal{S}(x)$ be the best (i.e. smallest) constant satisfying

$$\sum_{n \leq x} |a_n| \leq \mathcal{S}(x) \sup_{t \in \mathbb{R}} \Big| \sum_{n \leq x} \frac{a_n}{n^{it}} \Big| \quad \text{for every } (a_n)_{n \leq x} \text{ in } \mathbb{C}.$$

The problem of finding asymptotic estimations of $\mathcal{S}(x)$ as $x \to +\infty$ is closely related to the search for the value of T. Indeed, an elementary argument ([15]) shows that T can be described as the infimum of all $\sigma > 0$ such that $\mathcal{S}(x) = O(x^\sigma)$.

This problem has a long story (see [7, 8, 9, 13]) and the last breakthrough was reached by Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip in [9], namely

$$\mathcal{S}(x) = \sqrt{x}\,\exp\left[\left(-\frac{1}{\sqrt{2}} + o(1)\right)\,\sqrt{\log x\,\log\log x}\right]. \tag{1}$$

Our aim is to carry out an analogous study but comparing the convergence of Dirichlet series with respect to the following norms originally considered by Besicovitch [3] in the framework of almost periodic functions: for each Dirichlet polynomial $D(s) = \sum_{n\leq x} a_n n^{-s}$

$$\|D\|_{\mathcal{H}_p} = \lim_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T}|D(it)|^p dt\right)^{1/p}, \ 1 \leq p < \infty \tag{2}$$

The limit in (2) in fact exists (this follows from Birkhoff's ergodic theorem, see e.g. [1] or [17] for details). The completion of the linear space of all Dirichlet polynomials together with the p-norm leads to the Hardy space $\mathcal{H}_p$. These Banach spaces were first considered and investigated in [1, 12], and now form one of the fundamental objects of the modern analysis of Dirichlet series (see also [17]). Let us define define the main constant that we aim to estimate asymptotically: for each $1 \leq p < q < \infty$ and $x \geq 1$, let $\mho(q,p,x)$ be the best (i.e. smallest) constant satisfying

$$\Big\|\sum_{n\leq x}\frac{a_n}{n^s}\Big\|_{\mathcal{H}_q} \leq \mho(q,p,x)\cdot\Big\|\sum_{n\leq x}\frac{a_n}{n^s}\Big\|_{\mathcal{H}_p} \quad \text{for every } (a_n)_{n\leq x} \text{ in } \mathbb{C}.$$

Our main result [11] states that

$$\mho(q,p,x) = \exp\left[\frac{\log x}{\log\log x}\left(\log\sqrt{\frac{q}{p}} + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right]. \tag{3}$$

Despite notorious differences, the proofs of (1) and (3) posses some common points. A fundamental tool in both cases is the following striking idea glimpsed by Bohr: Using that every $n \in \mathbb{N}$ has a unique factorization into primes $n = \mathfrak{p}^\alpha = \mathfrak{p}_1^{\alpha_1}\mathfrak{p}_2^{\alpha_2}\ldots$ for some $\alpha = (\alpha_1, \alpha_2, \ldots)$, we can identify formal Dirichlet series with formal power series in infinitely many variables in the form

$$\sum_{n\in\mathbb{N}} a_n n^{-s} \ \longleftrightarrow \ \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \quad \text{where } a_n = c_\alpha \text{ provided } n = \mathfrak{p}^\alpha \ . \tag{4}$$

Remarkably, for a Dirichlet polynomials $D(s)$ the corresponding power series is an ordinary analytic polynomial $Q(z)$ on the compact group $\mathbb{T}^{\mathbb{N}}$ satisfying ([1, 17])

$$\|D\|_{\mathcal{H}_p} = \|Q\|_{L_p(\mathbb{T}^{\mathbb{N}})}.$$

This fact allows to translate our problem in terms of comparing the L_p and L_q norm of certain classes of trigonometric polynomials on $\mathbb{T}^{\mathbb{N}}$. Indeed, a main ingredients for the proof of (3) is the next inequality due to Bayart [2]: Given $1 \leq p \leq q < \infty$

and $m \in \mathbb{N}$, for every m-homogeneous (analytic) polynomial $Q(z)$ it holds that

$$\|Q\|_{L_q(\mathbb{T}^{\mathbb{N}})} \leq \left(\sqrt{\frac{q}{p}}\right)^m \|Q\|_{L_p(\mathbb{T}^{\mathbb{N}})}. \tag{5}$$

To prove the upper estimation of $\mho(q,p,x)$, we first extend Bayart's inequality (5) to degree-m polynomials, and combine it with the decomposition method of Konyagin and Queffélec [13] and some deep number theoretical results. On the other hand, the lower estimation of $\mho(q,p,x)$ follows an argument based on the central limit theorem which was used in [14] to give optimal bounds for the constants in the Khintchine-Steinhaus inequality, namely we construct an explicit family of Dirichlet polynomials $D_x(s) = \sum_{n\leq x} a_n n^{-s}$ such that the growth of $\|D_x\|_{\mathcal{H}_q}/\|D_x\|_{\mathcal{H}_p}$ as $x \to +\infty$ provides with the desired estimation.

Problems:

(i) Estimate asymptotically $\mho(q,p,x)$ for values $0 < p < q < +\infty$.

(ii) For each $1 \leq p \leq q < +\infty$ and $m \in \mathbb{N}$, search for the optimal constant $C_{q,p,m}$ in (5), that is, satisfying $\|Q\|_{L_q(\mathbb{T}^{\mathbb{N}})} \leq C_{q,p,m} \|Q\|_{L_p(\mathbb{T}^{\mathbb{N}})}$ for every m-homogeneous polynomial Q. In this sense, it is proved in [10] that $\sup_m \sqrt[m]{C_{q,p,m}} = \sqrt{q/p}$. On the other hand, simple estimations for particular values of p and q suggest that likely $C_{q,p,m} \leq m^{\frac{1}{2q}-\frac{1}{2p}} (\sqrt{q/p})^m$.

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Approximation numbers of composition operators in dimension two or more

Hervé Queffélec

Let φ be a non degenerate analytic self-map of a domain $\Omega \subset \mathbb{C}^d$ and H a Hilbert space of analytic functions on Ω, as well as C_φ, $C_\varphi(f) = f \circ \varphi$, the associated composition operator. Motivated by a joint result with K. Seip ([1]) on singular numbers $a_n(C_\varphi)$ of composition operators C_φ on the space $\mathcal{H}^2$ of *Dirichlet series,* for which $\Omega = \{\Re s > 1/2\} =: \mathbb{C}_{1/2}$, $\varphi \in \mathcal{G}$, the Gordon-Hedenmalm class, and behind which the Bohr lift is lurking, we study the situation on the *polydisk* $\mathbb{D}^d$ in dimension d, with $2 \leq d < \infty$.

In that case ([2]), we always have the lower bound

$$a_n(C_\varphi) \geq c e^{-Cn^{1/d}}.$$

The upper bound $a_n(C_\varphi) \leq C e^{-cn^{1/d}}$ holds when $\|\varphi\|_\infty < 1$, and if $d = 1$ fails to hold as soon as $\|\varphi\|_\infty = 1$ (as a consequence of a "spectral radius formula").
Here, our main result ([5]) is a counterexample for $d = 2$: we may have $\|\varphi\|_\infty = 1$ and yet $a_n(C_\varphi) \leq C e^{-cn^{1/2}}$, which is a significant difference.

This counterexample unexpectedly involves weighted composition operators $T = M_w\, C_\varphi$, where $w \in H^\infty(\mathbb{D})$, $M_w(f) = wf : H^2(\mathbb{D} \to H^2(\mathbb{D})$, the singular numbers of which we recently studied with G. Lechner, D. Li and L. Rodríguez-Piazza ([4]). The Monge-Ampère, or Bedford-Taylor, pluricapacity is also underlying, as well as related results of Nivoche and Zaharyuta.

Finally ([3]), for $d = \infty$, we show that compact, and even p-Schatten composition operators still exist on $\mathbb{D}^\infty \cap \ell^1$, even if their singular numbers cannot decay very fast:

$$\sum_{n \geq 1} \frac{1}{(\log 1/a_n)^p} = \infty \text{ for all } \ p < \infty.$$

This corresponds to one joint work with F. Bayart, D. Li, L. Rodríguez-Piazza ([2]) and two joint works ([3], [5]) with D. Li, L. Rodríguez-Piazza.

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Carlson's Theorem for Different Measures

Meredith Sargent

We use an observation of Bohr connecting Dirichlet series in the right half plane $\mathbb{C}_+$ to power series on the polydisk to interpret Carlson's theorem about integrals in the mean as a special case of the ergodic theorem by considering any vertical line in the half plane as an ergodic flow on the polytorus. Of particular interest is the imaginary axis because Carlson's theorem for Lebesgue measure does not hold there. In this talk, we construct measures for which Carlson's theorem does hold on the imaginary axis for functions in the Dirichlet series analog of the disk algebra $\mathcal{A}(\mathbb{C}_+)$.

Hardy type spaces of general Dirichlet series

Ingo Schoolmann
(joint work with Andreas Defant)

A general Dirichlet series is a formal series $\sum a_n e^{-\lambda_n s}$, where s is a complex variable, (a_n) is a complex sequence of coefficients, and (λ_n) a frequency, i.e., a strictly increasing non-negative real sequence which tends to $+\infty$. Choosing $\lambda_n = \log n$ leads to ordinary Dirichlet series $\sum a_n n^{-s}$. In analogue to the ordinary case we define the spaces $\mathcal{H}_\infty(\lambda)$ as the space of all λ-Dirichlet series which converge on the right half plane $[Re > 0]$ and define a bounded function there. Endowed with the sup norm on $[Re > 0]$ this space becomes a normed space. In general $\mathcal{H}_\infty(\lambda)$ is not complete, but there is a manageable sufficient condition on λ for completeness (and more) introduced by Bohr (see [3]), called Bohr's condtion (BC):

$$\exists C > 0 : \sum_{n=1}^{\infty} \frac{1}{\lambda_{n+1} - \lambda_n} e^{-C\lambda_n} < \infty.$$

It would be interesting to know if the space $\mathcal{H}_\infty(\log\log(n)))$ is complete. This seems unknown.

Due to an ingenious idea of H. Bohr the recent $\mathcal{H}_p$-theory of ordinary Dirichlet series is intimately linked with Fourier analysis on the infinite dimensional polytorus $\mathbb{T}^\infty$. Inspired by ideas of Bohr and Helson we indicate that in the more general situation of general Dirichlet series a natural substitute of the polytorus $\mathbb{T}^\infty$ is given by the Bohr compactification $\overline{\mathbb{R}}$ of $\mathbb{R}$, which is the group of all group homomorphism $\omega\colon (\mathbb{R},+) \to \mathbb{T}$. If the group $(\mathbb{R},+)$ carries the discrete topolgy, then $\overline{\mathbb{R}}$ together with the compact open topology forms a compact abelian group and therefore has a Haar measure. Since the Fourier transform of $f \in L_1(\overline{\mathbb{R}})$ is a function on $\mathbb{R}$ (which is the dual group of $\overline{\mathbb{R}}$), we define for $1 \leq p \leq \infty$ the space $H_p^\lambda(\overline{\mathbb{R}})$ as the (closed) subspace of all $f \in L_p(\overline{\mathbb{R}})$ such that $\hat{f}(x) \neq 0$ implies $x \in (\lambda_n)_n$ for all $x \in \mathbb{R}$. To each $f \in H_p^\lambda(\overline{\mathbb{R}})$ we formally assign the λ-Dirichlet series $\sum \hat{f}(\lambda_n)e^{-\lambda_n s}$. In this way the so called Bohr map

$$\mathcal{B}\colon H_p^\lambda(\overline{\mathbb{R}}) \hookrightarrow \mathcal{D}(\lambda),\ f \mapsto D$$

produces new spaces of λ-Dirichlet series, which in the case $p = \infty$ actually turn out to be exactly the spaces $\mathcal{H}_\infty(\lambda)$ provided λ satisfies (BC).

Theorem 1. *If (BC) holds for λ, then the Bohr map*

$$\mathcal{B}\colon H_\infty^\lambda(\overline{\mathbb{R}}) \to \mathcal{H}_\infty(\lambda),\ f \mapsto D(s) := \sum_{n=1}^{\infty} \widehat{f}(\lambda_n)e^{-\lambda_n s}.$$

is an isometry onto that preserves the Fourier- and Dirichlet coefficients.

The ordinary case $H_\infty(\mathbb{T}^\infty) = \mathcal{H}_\infty$, an important theorem from [4], follows easily.

Moreover, we define $\mathcal{H}_p(\lambda) := \mathrm{Im}\ \mathcal{B}(H_p^\lambda(\overline{\mathbb{R}}))$. These spaces also have an internal description in terms of λ-Dirichlet polynomials. With

$$\|D\|_p := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\sum_{n=1}^{N} a_n e^{-\lambda_n it}\right|^p dt\right)^{\frac{1}{p}}$$

the space of all λ-Dirichlet polynomials $\mathrm{poly}(\lambda)$ becomes a normed space, and it can be shown that $\mathcal{H}_p(\lambda)$ is exactly the completion of $\mathrm{poly}(\lambda)$. Since $\mathcal{H}_p(\log(n)) = \mathcal{H}_p$ holds isometrically, this new definition of $\mathcal{H}_p(\log(n))$-Dirichlet series coincides with the well-established definition of $\mathcal{H}_p$ for ordinary Dirichlet series due to Bayart in [2]. We prove that if $D \in \mathcal{H}_p(\lambda)$, $1 < p \leq \infty$, then almost all 'vertical limits' of D converges on $[Re > 0]$. More precisely:

Theorem 2. *Let $1 < p \leq \infty$ and $D(s) = \sum a_n e^{-\lambda_n s} \in \mathcal{H}_p(\lambda)$. Assume that (BC) holds for λ. Then for almost all $\omega \in \overline{\mathbb{R}}$ the Dirichlet series $D_\omega(s) = \sum a_n \omega(\lambda_n) e^{-\lambda_n s}$ converges on $[Re > 0]$ and coincides on $[Re = u]$ with the convolution $P_u * f_\omega(t)$ almost everywhere on $\mathbb{R}$.*

This extends Helson's result from [5] for the case $p = 2$ to the range $1 < p \leq 2$. For ordinary Dirichlet series the theorem is again due to Bayart [2], and even true for $p = 1$. In this case the proof relies on the hypercontractivity of the Poisson kernel on $\mathbb{T}^N$. It seems like an interesting problem to establish such concepts for general Dirichlet series.

Let us finally mention an application of this circle of ideas – the 'brother Riesz type theorem' for the Bohr compactification $\overline{\mathbb{R}}$ fixing a frequency λ. In the ordinary case the Hardy space $H_1(\mathbb{T}^\infty)$ is isometrically isomorphic to the space $M_+(\mathbb{T}^\infty)$ of all bounded, regular and analytic Borel measures on $\mathbb{T}^\infty$. This result, which again follows easily, is due to Helson and Lowdenslager [6], was recently reproved by [1], and extends the classical brother Riesz theorem to the infinite dimensional torus.

Theorem 3. *Let* (BC) *hold for* λ. *Then the Bohr map*

$$\mathcal{B}\colon H_1^\lambda(\overline{\mathbb{R}}) \to M_\lambda(\overline{\mathbb{R}}),\ f \mapsto f\ dm$$

is an isometry onto, where $M_\lambda(\overline{\mathbb{R}}) := \{\mu \in M(\overline{\mathbb{R}}) \mid \hat{\mu}(x) \neq 0 \Rightarrow x \in (\lambda_n)_n\}$ *and* $M(\overline{\mathbb{R}})$ *is the space of all bounded and regular Borel measures on* $\overline{\mathbb{R}}$.

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On correlations of multiplicative functions

Joni Teräväinen
(joint work with Terence Tao)

An arithmetic function g is called multiplicative, if it satisfies the functional equation $g(mn) = g(m)g(n)$ whenever m and n are coprime. Many of the most interesting arithmetic functions in number theory have the multiplicativity property, including the Möbius function $\mu(n)$. The mean values $\frac{1}{x}\sum_{n\le x} g(n)$ of multiplicative functions have been studied widely, and are connected for instance to sieve methods and to probabilistic number theory.

We consider a far-reaching generalization of these mean values, namely the correlation averages

$$\frac{1}{x}\sum_{n\le x} g_1(n+h_1)\cdots g_k(n+h_k),$$

where $g_1, \dots, g_k$ are 1-bounded multiplicative functions and $h_1, \dots, h_k \ge 0$ are distinct shifts. A well-known conjecture of Elliott [1] states that the correlation average above should converge to 0 as $x \to \infty$, whenever one of the functions g_j does not "pretend to be" a twisted Dirichlet character $n \mapsto \chi(n)n^{it}$ in the sense of the pretentious distance for multiplicative functions (see [6] for a precise statement). Thus Elliott's conjecture can be interpreted as saying that shifts of bounded multiplicative functions are asymptotically independent, except in the case of "pretentious" functions.

Since 2015, a number of approximations to Elliott's conjecture have been proved, but the full conjecture remains out of reach. Importantly, Matomäki, Radziwiłł and Tao [3] showed that Elliott's conjecture holds on average over the shifts h_i, and Tao [4] settled the two-point case $k = 2$ of Elliott's conjecture when ordinary averages are replaced with logarithmic ones. These breakthroughs have lead to numerous new advances and applications, and in particular Tao used in [5] his two-point result to solve the long-standing Erdős discrepancy problem in combinatorics.

In my talk, I discussed a joint work [5] with T. Tao, where we extended the work of [4] to the higher order case $k > 2$, under an additional assumption. More precisely, if the functions g_j are as above, we showed that the logarithmic average

$$\frac{1}{\log x}\sum_{n\le x}\frac{g_1(n+h_1)\cdots g_k(n+h_k)}{n}$$

converges to 0 as $x \to \infty$, provided that the product $g_1 \cdots g_k$ does not "weakly pretend" to be any Dirichlet character $\chi(n)$. This in particular enabled us to settle the odd order cases of the logarithmically averaged Chowla conjecture, stating that

$$\frac{1}{\log x}\sum_{n\le x}\frac{\mu(n+h_1)\cdots \mu(n+h_k)}{n}$$

converges to 0 as $x \to \infty$ for odd values of k. Recently in [7], we gave another proof of these odd order logarithmic Chowla conjectures, using combinatorial ideas instead of the ergodic theory machinery in [6]. The even order cases of the logarithmically averaged Chowla conjecture remain open, except for the $k = 2$ case that Tao [4] settled. Although our proofs do not directly use Dirichlet polynomials, the earlier mentioned results [3], [4] draw crucial input from the result of Matomäki and Radziwiłł [2] on multiplicative functions in short intervals, whose proof is based on a careful analysis of Dirichlet polynomials.

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Open problems session

Problem 1 (Håkan Hedenmalm)

Let $f \in \mathcal{S}(\mathbb{R}^n)$ be a Schwarz function. Let

$$r_1 < r_2 < r_3 < \dots$$
$$\rho_1 < \rho_2 < \rho_3 < \dots$$

be two sequences of radii tending to infinity, and suppose that $f(x) = 0$ for $|x| = r_j$, and that the Fourier transform $\hat{f}$ satisfies $\hat{f}(\xi) = 0$ for $|\xi| = \rho_j$. Does this imply $f \equiv 0$?

Problem 2 (Erdös and Ingham, communicated by Michel Balazard)

Let

$$D(s) = 1 + 2^{-s} + 3^{-s} + 5^{-s}.$$

Question 1. Is it true that $D(s) \neq 0$ for all $\mathrm{Re}(s) = 1$?

Question 2. Is it true that if $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is non-decreasing and

$$f(x) + f\left(\frac{x}{2}\right) + f\left(\frac{x}{3}\right) + f\left(\frac{x}{5}\right) = \left(\frac{61}{30} + o(1)\right) x$$

as $x \to \infty$, then necessarily $f(x) = (1 + o(1))x$?
As Erdős and Ingham [1] showed , these two questions are equivalent.

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Problem 3 (Titus Hilberdink)

Let $f : \mathbb{N} \to \mathbb{R}$ be a nonngeative multiplicative function. Define

$$F(x) = \sum_{n \le x} f(n), \quad F_2(x) = \sum_{\substack{n \le x \\ n \text{ odd}}} f(n).$$

Suppose that $F(\lambda x) = (1 + o(1))F(x)$ for any fixed $\lambda > 0$ as $x \to \infty$. Is it true that then

$$\lim_{x\to\infty} \frac{F_2(x)}{F(x)}$$

exists? This limit should equal to

$$\frac{1}{\sum_{k=0}^{\infty} \frac{f(2^k)}{2^k}}.$$

We remark that if f is completely multiplicative, then one easily sees that

$$\frac{F_2(x)}{F(x)} = \frac{F(x) - f(2)F\left(\frac{x}{2}\right)}{F(x)} = 1 - \frac{f(2)}{2} + o(1),$$

which agrees with the above prediction for the value of the limit.

Problem 4 (Frédéric Bayart)

Let

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^\infty(\mathbb{C}_0)$$

be a Dirichlet series with partial sums

$$S_N(f)(s) = \sum_{n=1}^{N} a_n n^{-s}.$$

Let

$$\sigma_N(f) = \inf_{\sigma>0} \{\|S_N(f)\|_{\mathcal{H}^\infty(\mathbb{C}_\sigma)} \leq \|f\|_{\mathcal{H}^\infty(\mathbb{C}_\sigma)}\},$$

and further let

$$\sigma_N = \sup_{f\in\mathcal{H}^\infty(\mathbb{C}_0)} \sigma_N(f).$$

Is it the case that $\sigma_N \to 0$ as $N \to \infty$?

Problem 5 (Jean-François Burnol)

Consider the Hilbert space

$$H = L^2(\mathbb{R}, d\mu) \quad \text{with} \quad d\mu = \left|\zeta\left(\frac{1}{2} + ix\right)\right|^2 dx.$$

Does there exists $\tau > 0$ and $f \not\equiv 0$ an entire function of exponential type τ belonging to H such that f is orthogonal to all functions in H of strictly smaller exponential type?

Note that for example the function

$$f(s) = \left(e^{is} - e^{-1}\right)(s - i)^{-1}$$

is of finite exponential type (of type 1) and belongs to H.

Problem 6 (Michel Balazard)

Construct a Dirichlet series f such that f has exactly one zero in some half-plane (and converges there).

Observe that the function $f(s) = \frac{1}{\zeta(s)}$ would work if the Riemann hypothesis held, or even under the weaker assumption that there are no zeros of $\zeta(s)$ with $\mathrm{Re}(s) \geq 1 - \varepsilon$ for some $\varepsilon > 0$.

Reporter: Joni Teräväinen