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Copositivity and Complete Positivity

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Abstract. A real matrix A is called copositive if $x^T Ax \geq 0$ holds for all $x \in \mathbb{R}^n_+$. A matrix A is called completely positive if it can be factorized as $A = BB^T$, where B is an entrywise nonnegative matrix. The concept of copositivity can be traced back to Theodore Motzkin in 1952, and that of complete positivity to Marshal Hall Jr. in 1958. The two classes are related, and both have received considerable attention in the linear algebra community and in the last two decades also in the mathematical optimization community. These matrix classes have important applications in various fields, in which they arise naturally, including mathematical modeling, optimization, dynamical systems and statistics. More applications constantly arise.

The workshop brought together people working in various disciplines related to copositivity and complete positivity, in order to discuss these concepts from different viewpoints and to join forces to better understand these difficult but fascinating classes of matrices.

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Introduction by the Organisers

A real matrix A is called copositive if $x^T Ax \geq 0$ holds for all $x \in \mathbb{R}^n_+$. Obviously, every positive semidefinite matrix is copositive, and so is every entrywise symmetric nonnegative matrix. However, for $n \geq 5$ the cone of $n \times n$ copositive matrices is considerably larger and more complex than both the semidefinite and symmetric nonnegative matrix cones. Its dual cone is the cone of $n \times n$ completely positive matrices, that is, matrices that have a representation $A = \sum b_i b_i^T$ with $b_i \in \mathbb{R}^n_+$.

The cone of completely positive matrices is contained in the cone of doubly nonnegative matrices, i.e., matrices which are both positive semidefinite and entrywise nonnegative. For $n \leq 4$ these two cones are equal [35], but for $n \geq 5$, there are $n \times n$ doubly nonnegative matrices which are not completely positive. Both the copositive and the completely positive matrix cones are closed, convex, full dimensional and pointed [7]. They are, however, not polyhedral, but rather their boundaries have both polyhedral parts and "curved parts." For the copositive cone, a characterization in terms of its supporting hyperplanes is known, but a complete characterization of its extremal (generating) rays is an open question. For the completely positive cone, the converse is true: the extremal rays are known, but characterizing the supporting hyperplanes is an open problem. Checking membership in each of these cones is (theoretically and algorithmically/computationally) challenging. These open problems and several others are highlighted in [3].

Both matrix cones possess highly interesting properties, and have attracted interest in the linear algebra community for many years. For surveys on copositive matrices see [31, 30] and for one on completely positive matrices see [7].

Several necessary and sufficient conditions for copositivity are known. Schur complement properties have been proposed [11], special subclasses of copositive matrices have been characterized (e.g., matrices whose entries are ± 1 or 0, cf. [29]), spectral properties have been studied, and a partial characterization of extremal copositive matrices has been given. The extremal 5×5 copositive matrices have recently been fully described [28]. A nonnegative vector x is a zero of a copositive matrix A if $x^T Ax = 0$. These vectors play an important role in the study of extreme copositive matrices [27]. It is known that testing copositivity is a co-NP-complete problem [37]. Nonetheless, several algorithmic copositivity tests have been developed [16, 13, 46, 44, 19].

On the dual side, necessary and sufficient conditions for complete positivity have also been introduced. Often, these involve the zero-nonzero pattern of the matrix, described by a graph. For example, the necessary condition of being doubly nonnegative was shown to be also sufficient if and only if the graph of the matrix has no long odd cycle [5, 4, 33, 1]. A sufficient condition in terms of the so-called comparison matrix of the given matrix was introduced, and shown to be also necessary when the graph of the matrix is triangle free [23]. Finding a representation $A = \sum_{i=1}^{r} b_i b_i^T$ with $b_i \in \mathbb{R}^n_+$ is an open question even in cases where the matrix A is known to be completely positive. The minimal number r of rank-one summands needed in such a representation is called the cp-rank of A, see [7]. Upper bounds for the cp-rank of matrices are known, and it was conjectured [23] that cp-rank$(A) \leq \frac{n^2}{4}$ for any $n \times n$ completely positive matrix A. This bound has been shown to hold in some special cases [23, 8, 34, 43, 42, 20]. However the conjecture in general has been recently refuted, and an asymptotic bound has been given [15, 14]. Testing a rational matrix for complete positivity was shown to be NP-hard [22]. Although it is an open question to determine a representation $A = \sum_{i=1}^{r} b_i b_i^T$ with $b_i \in \mathbb{R}^n_+$ of a general completely positive matrix A, some algorithms have been suggested [21, 39, 2, 26, 45, 10]. The computational question of determining whether or not a given symmetric nonnegative matrix is completely positive is studied in [9].

From the above, it is clear that copositivity and complete positivity are highly interesting concepts from the point of view of linear algebra. They also have many important applications. One of the early motivations to study complete positivity was its relevance to block designs. Other applications include a maximin efficiency-robust test, a proposed mathematical model of energy demand, exchangeable probability distributions on finite sample spaces and a Markovian model of DNA evolution [7]. Some recent applications include hard and probabilistic clustering [47], scheduling of stochastic arrivals [32], and dynamical systems [6]. One of the most important applications is to mathematical optimization, which is briefly described below.

Many combinatorial and nonconvex quadratic optimization problems have been shown to possess an equivalent formulation as linear problems over the copositive or completely positive cone. This was first shown for the so-called standard quadratic programming problem. Letting $Q \in \mathbb{R}^{n\times n}$ be an arbitrary symmetric matrix and denoting by e the all-ones vector, it has been shown by Bomze et al. [12] that any nonconvex quadratic problem over the simplex

$$\min\{x^T Qx : e^T x = 1, x_i \geq 0 \text{ for all } i\}, \tag{1}$$

has an equivalent completely positive formulation (with $E = ee^T$)

$$\min\{\langle Q, X\rangle : \langle E, X\rangle = 1, X \text{ is completely positive}\}. \tag{2}$$

This equivalence is remarkable, since it transforms a nonconvex NP-hard optimization problem into a linear problem in matrix variables over a convex cone of matrices. The local minima of (1) disappear, and there is a one-to-one correspondence between the global minima of (1) and the local (= global) minima of (2). The difficulty of the problem is thus shifted entirely into the cone constraint, which makes understanding the cone crucial for tackling the problem.

Generalizing this result, Burer [17] showed that any quadratic problem with linear and binary constraints

$$\begin{array}{ll} \min & x^T Qx + 2c^T x \\ \text{st.} & a_i^T x = b_i \quad (i = 1, \dots, m) \\ & x \geq 0, \quad x_j \in \{0, 1\} \quad (j \in B), \end{array}$$

with Q not necessarily positive semidefinite and $B \subset \{1, \dots, n\}$, can equivalently be written in the form

$$\begin{array}{ll} \min & \langle Q, X\rangle + 2c^T x \\ \text{st.} & a_i^T x = b_i \quad (i = 1, \dots, m) \\ & \langle a_i a_i^T, X\rangle = b_i^2 \quad (i = 1, \dots, m) \\ & x_j = X_{jj} \quad (j \in B) \\ & \begin{pmatrix} 1 & x \\ x & X \end{pmatrix} \text{ is completely positive,} \end{array}$$

which is again a linear problem over the cone of completely positive matrices. Because of the binary constraints, this latter setting includes a broad class of combinatorial problems and shows that most of them can be formulated as linear problems over the completely positive cone.

Combinatorial problems for which copositive formulations have been studied include the clique number [12], the chromatic number [25], the quadratic assignment problem [40], and certain graph partitioning problems [41].

Note that the dual problem of a completely positive problem is a problem over the dual cone, i.e., the copositive cone, and vice versa. Under mild regularity conditions, strong duality holds between the problems, thus any progress in understanding either of the cones will help solving these difficult optimization problems.

We mention that copositivity also plays a role in modeling optimization under uncertainty [38], in complementarity problems [18], and matrix games [36]. For more details, we refer to the survey papers [24] and [30].

As outlined above, copositivity and complete positivity are highly interesting and relevant topics that have attracted increasing attention in the last two decades. These topics are of interest in various specialty areas, both in pure and applied mathematics. The workshop brought together researchers working in different areas where copositivity and complete positivity arise, including some who had not met in person before. The following abstracts give an idea about the different views onto this key concept and the fruitful discussions that followed.

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Workshop: Copositivity and Complete Positivity

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Abstracts

An Algorithm to Compute the CP-Factorization of a Completely Positive Matrix

Kurt Anstreicher
(joint work with Sam Burer and Peter Dickinson)

Let $\mathcal{S}_n$ denote the set of $n \times n$ real symmetric matrices. The cone of $n \times n$ completely positive (CP) matrices is $\mathcal{C}_n = \{X \in \mathcal{S}_n \mid \exists A \geq 0, X = AA^T\}$. The dual of $\mathcal{C}_n$ is the cone of $n \times n$ copositive (CoP) matrices, $\mathcal{C}_n^* = \{X \in \mathcal{S}_n \mid y^T Xy \geq 0\ \forall y \in \mathbb{R}_+^n\}$. Recent interest in CP and CoP matrices from the optimization community stems from fact that a large class of NP-Hard optimization problems can be written as linear optimization problems over these cones [4].

A fundamental problem for CP matrices is to determine if a given matrix C is CP, and if so compute a CP-factorization $C = AA^T$ where A is a nonnegative matrix. Virtually all literature on this problem is remarkably recent. Berman and Rothblum [3] show that the question of whether or not $C \in \mathcal{C}_n$ can be resolved by a finite algorithm using quantifier elimination, with an operation complexity of $n^{O(n^5)}$. Several subsequent papers [12, 13, 14, 8] consider more practical approaches, resulting in algorithms which are implementable but lack a complexity bound. Our goal is to create an implementable method to determine whether a given $C \in \mathcal{C}_n$, but which also has a complexity bound in terms of n and the geometry of $\mathcal{C}_n$. If $C \in \mathcal{C}_n$ then the algorithm should compute a CP-factorization $C = AA^T$, while if $C \notin \mathcal{C}_n$ the algorithm should produce a "certificate" $X \in \mathcal{C}_n^*$ with $\langle C, X\rangle < 0$.

Our approach is based on considering an optimization problem of the form

$$\text{CPTest}: \quad z^* = \min \ \langle C, X\rangle \quad \text{s.t.} \ \frac{1}{2} \leq \langle S, X\rangle \leq \frac{3}{2}, \quad X \in \mathcal{C}_n^*$$

where $S \in \text{Int}(\mathcal{C}_n)$. Then $C \in \mathcal{C}_n \iff z^* \geq 0$. We consider a cutting-plane algorithm for CPTest, using the "fat slice" $\frac{1}{2} \leq \langle S, X\rangle \leq \frac{3}{2}$ in place of a normalization such as $\langle S, X\rangle = 1$ so as to have a non-empty interior for the feasible region.

The cutting-plane algorithm requires a separation oracle for $X \in \mathcal{C}_n^*$. A particularly simple oracle can be based on a result of Hadeler [11] on almost copositive matrices. A matrix $X \in \mathcal{S}_n$ is called *almost copositive* if X is not copositive, but every principal submatrix of X is copositive. Hadeler proved that if X is almost copositive, then X is nonsingular and $X^{-1}e < 0$. Note that for such a matrix, $v = -X^{-1}e > 0$ and $v^T Xv = -e^T v < 0$, certifying that $X \notin \mathcal{C}_n^*$. It is also easy to see that if X is not copositive, then X has a principle submatrix that is almost copositive. Enumerating all principle submatrices of X then either obtains $v \geq 0$ with $v^T Xv < 0$, or proves that X is copositive. The resulting oracle for membership in $\mathcal{C}_n^*$ has a complexity of $O(2^n n^3)$ using standard linear algebra. If X has rational components and $X \notin \mathcal{C}_n^*$, the size of the certificate $v \geq 0$ with $v^T X v < 0$ is also polynomially bounded in the size of X [9]. A more complex oracle that has the possibility of running faster, particularly in the case where $X \in \mathcal{C}_n^*$, can be based on a finite branch-and-bound algorithm for nonconvex QP [5].

Let $\mathcal{F}$ denote feasible region of CPTest. We assume that $\mathcal{F}$ is contained in $B_R(0)$, a ball of radius R in Frobenius norm centered at the origin, and contains a ball of radius r. For $X \in S_n$, let svec(X) be the vector in $\mathbb{R}^N$, $N = n(n+1)/2$ obtained by "stacking" the elements in the upper triangle of X. The cutting-plane algorithm for CPTest is based on maintaining a sequence of ellipsoids E_k, $k \geq 0$ each of which must contain the solution $x^* = \text{svec}(X^*)$ of CPTest. The goal is to obtain an ϵ-optimal solution. We will specify the algorithm to run for a fixed number of iterations K, and give a complexity result for K in terms of r, R and ϵ.

There are several possibilities for obtaining E_{k+1} from E_k. For example, using the central-cut ellipsoid algorithm [9], an ϵ-optimal solution of CPTest is obtained in $K = \lceil 2N^2 \ln(4R^2\|C\|/(\epsilon r))\rceil$ iterations. For CPTest with $S = I + \frac{1}{4n}$ we can take $r = \frac{2}{5n}$ and $R = (3\sqrt{2})n$, and an ϵ-optimal solution is obtained in

$$K = \left\lceil \tfrac{n^2(n+1)^2}{2} \ln(45 n^{2.5}\|C\|/\epsilon)\right\rceil$$

iterations. If the algorithm returns $v_{\text{best}} = \langle C, X_{\text{best}}\rangle \geq \epsilon$, then we have a proof that $C \in \mathcal{C}_n$, while if $v_{\text{best}} < 0$ then we know that $C \notin \mathcal{C}_n$ (with certificate $X_{\text{best}} \in \mathcal{C}_n^*$). If the algorithm returns $0 \leq v_{\text{best}} < \epsilon$ then the status of C is indeterminate. In the case that $v_{\text{best}} \geq \varepsilon$ we can solve an auxiliary linear program to obtain an explicit CP-factorization of C from the cuts generated in the course of running the algorithm. This factorization has the form

$$C = \sum_{1\leq i\leq I} \lambda_i v_i v_i^T + \sum_{1\leq i\leq j\leq n} u_{ij}(e_i+e_j)(e_i+e_j)^T,$$

where v_i, $i = 1,\ldots,I$, $I \leq K$ are the set of cuts returned by the copositive separation oracle and the vectors (e_i+e_j) arise from a polyhedral outer approximation of $\mathcal{C}_n^*$ for which intersection with the "fat slice" is contained in $B_R(0)$. The variables λ_i, $1 \leq i \leq I$ and u_{ij}, $1 \leq i \leq j \leq n$ are all nonnegative and at most N of these variables are strictly positive.

As C approaches the boundary of $\mathcal{C}_n$, the solution value $z^* \to 0$, so we need $\varepsilon \to 0$ to demonstrate that $C \in \mathcal{C}_n$ and produce a CP-factorization. It is easy to relate the required ε to the interiority of C. Note that if $B_\delta(C) \in \mathcal{C}_n$, and X is in the fat slice $\frac{1}{2} \leq \langle S, X\rangle \leq \frac{3}{2}$, then

$$\begin{aligned}\langle C - \frac{\delta}{\|S\|}S, X\rangle &= \langle C, X\rangle - \frac{\delta}{\|S\|}\langle S, X\rangle \;\geq\; 0\\ \langle C, X\rangle &\geq \frac{\delta}{\|S\|}\langle S, X\rangle \;\geq\; \frac{\delta}{2\|S\|} \;\geq\; \frac{2\delta}{5\sqrt{n}},\end{aligned}$$

so if $B_\delta(C) \in \mathcal{C}_n$ it suffices to take $\varepsilon = \frac{2\delta}{5\sqrt{n}}$. Using the ellipsoid algorithm with rational arithmetic, if C is rational we can polynomially bound the size of all cuts and intermediate solutions produced by the algorithm. As a corollary, we obtain the result that if $C \in \text{Int}(\mathcal{C}_n)$ is rational then C has a rational CP factorization, as shown independently in [10]. Note that the size of this factorization will also increase polynomially in δ. This fact is closely related to the question of whether or not determining that $C \in \mathcal{C}_n$ is actually in NP [6].

Using the volumetric cutting-plane algorithm [15, 1, 2] in place of the ellipsoid algorithm reduces the number of oracle calls by a factor of $O(N)$, and also reduces the potential number of cut matrices $v_i v_i^T$ needed to obtain a CP-factorization from $O^*(N^2)$ to $O(N)$. Alternatively one could apply the analytic center cutting-plane method [7], in which case one loses polynomiality in ε but expects good performance in practice.

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Completely Positive Matrices – Real, Rational and Integral

Abraham Berman

A matrix A is completely positive (cp) if it can be decomposed as $A = BB^T$, where the entries of B are nonnegative.

Completely positive matrices have important applications as is evident in this workshop. The talk is a survey of results on cp matrices and two open problems.

A necessary condition for an $n \times n$ matrix A to be completely positive is that it is doubly nonnegative (dnn), i.e. positive semidefinite and nonnegative. This condition is also sufficient if $n < 5$, but for every $n > 4$ there exists a doubly nonnegative matrix that is not completely positive.

A graph G has the property that every doubly nonnegative matrix realization of G is completely positive if and only if G does not contain an odd cycle of length greater than 4.

A sufficient condition for a nonnegative matrix A to be completely positive is that its comparison matrix M ($m_{ii} = a_{ii}, m_{ij} = -a_{ij}$ if $i \neq j$) is positive semidefinite. This condition is, in general, not necessary, but it is also necessary if $G(A)$, the graph of A, is triangle free.

If $A = BB^T$, where the entries of B are rational and nonnegative, we say that A has a rational cp factorization.

Question 1. *Does every rational completely positive matrix have a rational cp factorization?*

Cases when this is true include:

- $G(A)$ is triangle free,
- $G(A)$ does not contain an odd cycle of length greater than 4,
- $M(A)$ is positive semidefinite,
- A is in the interior of the cone of completely positive matrices.

If $A = BB^T$, where the entries of B are integral and nonnegative, we say that A has an integral cp factorization.

If G is a graph that has a vertex of degree greater than 1, then there is an integral completely positive matrix realization of G that does not have an integral cp factorization.

Question 2. *Does every* 2×2 *integral completely positive matrix have an integral cp factorization?*

The structure of completely positive matrices according to their CP-rank and CP-plus-rank

Peter J.C. Dickinson

(joint work with Immanuel M. Bomze and Georg Still)

This talk is a presentation of a joint paper by the speaker, Peter J.C. Dickinson, together with co-authors, Immanuel M. Bomze and Georg Still [3].

An important property of completely positive matrices is their cp-ranks (see e.g. [1, Chapter 3]), and here we introduce the closely related cp-plus-rank. The definitions of these for a nonzero symmetric matrix $\mathsf{A} \in \mathcal{S}^n \setminus \{\mathsf{O}\}$ are respectively

$$\mathrm{cp}(\mathsf{A}) := \inf \left\{ p \in \mathbb{N} : \exists \mathsf{B} \in \mathbb{R}_{+}^{p\times n} \text{ s.t. } \mathsf{A} = \mathsf{B}\mathsf{B}^{\mathsf{T}} \right\},$$

$$\mathrm{cp}^{+}(\mathsf{A}) := \inf \left\{ p \in \mathbb{N} : \exists \mathsf{B} \in \mathbb{R}_{++}^{p\times n} \text{ s.t. } \mathsf{A} = \mathsf{B}\mathsf{B}^{\mathsf{T}} \right\},$$

where $\mathbb{R}_+^{p\times n}$ denotes the set of entrywise nonnegative $p \times n$ matrices and $\mathbb{R}_{++}^{p\times n}$ denotes the set of entrywise strictly positive $p \times n$ matrices.

Due to the subtle similarities and differences between the cp-plus-rank and the cp-rank, the analysis of the cp-plus-rank is highly useful in the investigation of the cone of completely postive matrices.

One result that is presented is on how cp-plus-ranks vary in a neighbourhood. For any matrix $\mathsf{A} \in \mathcal{S}^n$, there is an open neighbourhood of A in $\mathcal{S}^n$ such that the cp-ranks of matrices in this neighbourhood are greater than or equal to the cp-rank of A [5]. For the cp-plus-rank the opposite is true: for any matrix $\mathsf{A} \in \mathcal{S}^n$, not on the boundary of the completely positive cone, there is an open neighbourhood of A in $\mathcal{S}^n$ such that the cp-plus-ranks of matrices in this neighbourhood are less than or equal to the cp-plus-rank of A. As a result we have that, in the interior of the completely positive cone, the set of matrices whose cp-rank and cp-plus-rank both equal a fixed number is an open set.

Further analysis is provided by considering Perron-Frobenius vectors and semi-algebraic sets, showing that:

- If A is a complete positive matrix with an entrywise strictly positive eigenvector $\mathbf{x}$ then for all $\mu > 0$ the cp-plus-rank of $\mathsf{A} + \mu\mathbf{x}\mathbf{x}^\mathsf{T}$ is less than or equal to the cp-rank of A.
- Letting p_n be the maximum finite cp-rank of matrices in $\mathcal{S}^n$ (bounds on which are known [1, 2, 4]), we have that the maximum finite cp-plus-rank of matrices in $\mathcal{S}^n$ is between p_n and $p_n + 1$.
- Generically the cp-plus-rank of a matrix is equal to its cp-rank.

Two open questions connected to this research are:

(1) What is the maximum finite difference between the cp-plus-rank and the cp-rank of a matrix?
(2) What is the maximum finite cp-plus-rank of matrices in $\mathcal{S}^n$?

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Global Optimization Techniques for Copositivity Testing

Gabriele Eichfelder
(joint work with Immanuel M. Bomze, Carmo Bras and Joaquim Judice)

This talk intends to give an introduction to the basic techniques used in recent numerical algorithms [3, 7, 6, 1, 2] for testing a matrix on copositivity. This means that given a real symmetric matrix $Q \in \mathcal{S}_n$ these algorithms aim on deciding whether this matrix is copositive or not. Thereby we write

$$Q \in \mathcal{COP} \text{ if and only if } \forall\, x \in \mathbb{R}^n_+ :\ x^\top Qx \geq 0.$$

Within this talk we are only interested in tests which are implemented, apply to general symmetric matrices without any structural assumptions or dimensional restrictions and which are not recursive, i.e., do not rely on information taken from all principal submatrices.

The relation between copositivity testing and global optimization is obvious in view of the following quadratic optimization problem:

$$\text{(QP-Q)} \qquad \begin{array}{rl} \min & f(x) := \frac{1}{2}x^\top Qx \\ \text{s.t.} & e^\top x = 1 \\ & x \in \mathbb{R}^n_+ \end{array}$$

where $e \in \mathbb{R}^n$ denotes the all-one vector. If Q is not positive semidefinite then the optimization problem (QP-Q) has a nonconvex objective function and solving it numerically requires techniques from global optimization. As the feasible set is compact, (QP-Q) has always an optimal solution and the optimal value is nonnegative if and only if Q is copositive.

The algorithms [3, 7, 6, 1] are all based on a branch-and-bound scheme, where the branching corresponds to a partitioning of the standard simplex

$$\Delta^s := \{x \in \mathbb{R}^n_+ \mid e^\top x = 1\}.$$

Assume that one partitions the standard simplex Δ^s into two subsimplices B_1 and B_2 with $B_1 \cup B_2 = \Delta^s$ and with $B_i = \text{conv}(v_i^1, \ldots, v_i^n)$, $i = 1, 2$ with $v_i^j \in \Delta^s$ for all $i = 1, 2$, $j = 1, \ldots, n$. Then one uses that a matrix $Q \in \mathcal{S}_n$ is copositive if and only if

$$x^\top Qx \geq 0 \text{ for all } x \in B_1 \text{ and } x^\top Qx \geq 0 \text{ for all } x \in B_2,$$

i.e. if and only if $\min_{x \in B_1} x^\top Qx \geq 0$ and $\min_{x \in B_2} x^\top Qx \geq 0$.

For the bounding within the branch-and-bound scheme efficient sufficient criteria for copositivity are required which can easily be applied also to subsimplices. To give an example from [3]:

$$\forall\, j, k \in \{1, \ldots, n\} :\ (v_1^j)^\top Qv_1^k \geq 0 \ \Rightarrow\ \min_{x \in B_1} x^\top Qx \geq 0.$$

Thus here one tries to verify for the subsimplex B_1 that $\min_{x \in B_1} x^\top Qx \geq 0$ as this implies that the set B_1 does no longer need to be considered. This requires to evaluate a finite number of inequalities only. While evaluating this sufficient condition one might also find a vector $v \in B_1$ with $v^\top Qv < 0$. This implies that the matrix Q is not copositive and the algorithm can be stopped. Another criterium which we present in this talk was proposed in [1] and uses the representation of the objective function of (QP-Q) as a difference of two convex functions, called d.c. decomposition. Thereby one uses the spectral decomposition of the matrix Q,

$$Q = Q_+ - Q_- \quad \text{with } Q_+ \ \text{ and } \ Q_- \ \text{ positive semidefinite,}$$

which we define as follows: let $Q = U\Lambda U^\top$ with the matrix $U := [u^1, \ldots, u^n]$ consisting of orthonormal eigenvectors u^i of Q, $\Lambda := \mathrm{diag}[(\lambda_i)]_i$ the diagonal matrix with the real eigenvalues of Q,

$$\Lambda_+ := \mathrm{diag}[(\lambda_i)_+]_i \quad \text{with} \quad (\lambda_i)_+ := \max\{0, \lambda_i\}, \ i = 1, \ldots, n,$$

and

$$Q_+ := U\Lambda_+ U^\top \quad \text{and} \quad Q_- := Q_+ - Q \ .$$

Then

$$Q \in \mathcal{COP} \ \text{ if and only if } \ \inf\{x^\top Q_+ x \mid x^\top Q_- x = 1, \ x \in \mathbb{R}^n_+\} \geq 1.$$

Relaxations of the latter optimization problem which uses this d.c. decomposition lead to sufficient conditions for copositivity which can also be modified to be applicable to subsimplices.

Moreover, in this talk convergence results and test sets for the numerical evaluation of these algorithms are shortly discussed. Typical random test matrixes which are used are of the form $P + N$ for P positive semidefinite and N a nonnegative matrix. Then all these matrices are copositive. Typically, also for large random matrices as for $n = 200$ copositivity can be verified easily by the mentioned algorithms. Random matrices with unit diagonal and with off-diagonal entries in the interval $[-1, 1]$ are with a high probability non-copositive and the algorithms also detect that generally within a few iterations.

The smallest matrix which tends to cause numerical difficulties is the famous Horn-matrix H, see [5], which is copositive but not of the form $P + N$:

$$H = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix} .$$

Other hard test instances are a result of the following reformulation of the maximum clique problem: given a simple (i.e. loopless and undirected) graph a clique is a subset of the node set such that every pair of nodes in the clique is connected by an edge of the graph. A clique is said to be a maximum clique if it contains the most elements among all cliques, and its size $\omega(G)$ is called the (maximum) clique number. It holds that the matrices $Q = \lambda(E_n - A_G) - E_n$ with E_n the $n \times n$ all-ones matrix, $A_G = [a_{ij}]_{i,j}$ the adjacency matrix of the graph G, and $\lambda \geq 0$ are copositive if and only if $\lambda \geq \omega(G)$. To be more concrete [6]:

$$\lambda(E_n - A_G) - E_n \begin{cases} \in \mathrm{int}\mathcal{COP} & \text{if } \lambda > \omega(G) \\ \in \mathrm{bd}\mathcal{COP} & \text{if } \lambda = \omega(G) \\ \notin \mathcal{COP} & \text{if } \lambda < \omega(G). \end{cases}$$

where int$\mathcal{COP}$ denotes the interior of the cone of copositive matrices and bd$\mathcal{COP}$ its boundary. If $\lambda(E_n - A_G) - E_n \notin \mathcal{COP}$ for some $\lambda \in \mathbb{N}$, then we can conclude that $\omega(G) \geq \lambda + 1$ and by that one can generate lower bounds for the maximum clique number. In general, only lower bounds are calculated as verifying copositivity tends to be more difficult than testing for non-copositivity. For the concrete test instances see [4].

Finally, we also point out a relation between copositivity testing and the following mathematical program with linear complementarity constraints:

$$\text{(MPLCC)} \qquad \begin{array}{lrcl} \min & \frac{1}{2}\lambda & & \\ \text{s.t.} & w & = & Qx - \lambda e \\ & x & \geq & 0,\ w \geq 0 \\ & x^\top w & = & 0,\ e^\top x = 1 \\ & x \in \mathbb{R}^n,\ \lambda & \in & \mathbb{R},\ w \in \mathbb{R}^n. \end{array}$$

It holds that $Q \in \mathcal{COP}$ if and only if (MPLCC) has a (globally) minimal solution $(\bar{x}, \bar{\lambda}, \bar{w})$ with $\bar{\lambda} \geq 0$.

Instead of solving (MPLCC) directly one studies instead the following linear complementarity problem:

$$\text{(LCP)} \qquad \begin{array}{rcl} \multicolumn{3}{c}{\text{Find } x \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^n \text{ such that}} \\ w & = & Qx - \lambda e \\ x & \geq & 0,\ w \geq 0 \\ x^\top w & = & 0. \end{array}$$

Pairs $(\bar{x}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^n$ which satisfy the (LCP) with $\bar{x} \neq 0$ deliver feasible points for the (MPLCC).

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On Continuous Powers of Certain Positive Matrices

Shaun Fallat
(joint work with Charles Johnson and Alan Sokal)

A matrix is called totally positive TP (resp. nonnegative TN) if all of its minors are positive (resp. nonnegative). It is known that such matrices are closed under conventional multiplication, but not necessarily closed under entry-wise or Hadamard multiplication. On the other hand, a real matrix is called positive definite (resp. semidefinite) if it is symmetric and has positive (resp. nonnegative) principal minors. In this case, such matrices need not be closed under matrix multiplication, but are closed under Hadamard multiplication. One important observation to note is that any symmetric TN or TP matrix is both completely positive and co-positive.

In my talk, I began with a detailed survey of existing work on continuous Hadamard and conventional powers of both totally positive and entry-wise nonnegative positive definite matrices (along with their closures). Most of the existing work has been done on the case of doubly nonnegative matrices, whereas the analogous results in the totally positive case are rather new and still have a number of distinct differences.

In the doubly nonnegative case, the most celebrated result is due to Horn and Fitzgerald [2], where it is established that the critical exponent for this class of matrices under Hadamard (or entry-wise) powers is $n-2$, here n is the order of the matrix. If $A = (a_{ij})$ is a matrix and $t > 0$ is an integer, the *Hadamard power* (or *entrywise power*) $A^{\circ t}$ is defined to be the matrix with elements $(A^{\circ t})_{ij} = a_{ij}^t$. For each $n \geq 1$, we call $m(n)$ the *critical exponent* if $A^{\circ\alpha}$ is in one of matrix classes (DN, TP, etc.) for all $\alpha \geq m(n)$, and $m(n)$ is the smallest such real number with this property.

In [2], it is shown that the critical exponent for doubly nonnegative matrices is $n-2$. Let $n \geq 2$, and let A be a n-by-n doubly nonnegative matrix. Then, for all real $t \geq n-2$ ($t > 0$ if $n = 2$), the Hadamard power $A^{\circ t}$ is positive semidefinite (resp. positive-definite). Conversely, for every $\alpha < n-2$, there exists a rank two doubly nonnegtive matrix A (which must also be completely positive), such that $A^{\circ\alpha}$ is not doubly nonnegative.

One of our main objectives was to find a critical exponent for TP matrices. We established that for $n \leq 3$, the critical exponent for TP matrices is $n-2$. However, for $n \geq 4$, we proved that there is no critical exponent for TP matrices under Hadamard powers. Even if symmetry is imposed as a hypothesis, then we showed that the critical exponent such matrices with $n \leq 4$ is $n-2$, but that no critical exponent exists for $n \geq 5$ (for more detailed information, please consult [1]).

In this case of Hankel matrices, we can say a bit more which offers a glimpse into the connection between the doubly nonnegative case and the TP case. Recall that an m-by-n matrix $A = (a_{ij})_{1\leq i\leq m,\, 1\leq j\leq n}$ is said to be a *Hankel matrix* if a_{ij} depends only on $i+j$, i.e. $a_{ij} = a_{i'j'}$ whenever $i+j = i'+j'$. Hankel matrices are characterized combinatorially by the following simple but important fact:

For an m-by-n matrix A, the following are equivalent:

(a) A is Hankel.
(b) Every contiguous submatrix of A is Hankel.
(c) Every contiguous submatrix of A is symmetric.
(d) Every contiguous 2-by-2 submatrix of A is symmetric.

We proved that the critical exponent for Hankel TP matrices was again $n - 2$.
I closed my talk by discussing critical exponents under conventional matrix powers, where much less is known in general. However, when the critical exponent is well defined, it does seem to be $n - 2$ as in the Hadamard case. Unfortunately, the critical exponent need not always exist even when conventional powers are well-defined, which is not always the case, such as in the co-positive setting.

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A Copositive Approach to Adjustable Robust Optimization with Uncertain Recourse

Markus Gabl
(joint work with Immanuel M. Bomze)

Robust Optimization deals with optimization problems under uncertainty. Parts or all of the parameters that describe the instance are affected by a vector valued uncertainty parameter. The goal is to find the best solution amongst those who will be feasible in any case, i.e. for all possible realizations of the uncertainty vector (see [6]).

This approach, however, leads to very conservative strategies as the feasible set might be very small compared to the case without uncertainty. But if the application allows for a two-stage design, where for parts of the variables the decision can be delayed until the uncertainty is removed, we have a significantly less conservative way of modelling the decision problem at our disposal. This is the domain of adjustable robust optimization (see [7]). Here the second stage variables are defined as functions that model the adjustment of the second stage decision to the first stage decision and the uncertainty parameter. Thus we look for the best solution amongst those who in any case will allow for a feasible adjustment of the second stage variables. This increases the feasible space for the first stage decision variables compared to the setting where all variables are treated as first stage type as is the case in the robust framework. Of course, the computational cost rises, also for problems where the constraint-coefficients of the second stage variables are affected by uncertainty as well (uncertain recourse), or if the data depends quadratically on the uncertainty vector. In both cases bilinear terms emerge, such that standard reformulation strategies can not be applied. However, recently a new methodology in quadratic optimization emerged, that seems to be fit to tackle precisely this issue and to render a broader array of modeling choices viable. The idea is to reformulate non-convex quadratic problems as linear problems in lifted variables, where the feasible set is described as the convex hull of extreme points of the lifted feasible set (some pioneering work in this field has been achieved by [1, 2, 3, 4, 5]). In many interesting cases, these sets have a characterization based on convex cones. Thus, a conic-duality argument can be employed in order to characterize sets of quadratic forms which are nonnegative over a given domain. As a consequence we can for example generalize the well known S-Lemma (see [8]) in a way such that we can characterize, by means of a linear matrix inequality, non-negativity of a quadratic function over a domain described by an arbitrary number of quadratic functions. This allows us to handle uncertainty sets which are ellipsoids with holes. Introducing more structure to the uncertainty set allows for modeling more information into the set of possible outcomes of the uncertainty, thus reducing the conservativeness of the model.

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Factorizations for completely positive matrices based on projection approaches

Patrick Groetzner
(joint work with Mirjam Dür)

A matrix A is called completely positive, if there exists an entrywise nonnegative matrix B such that $A = BB^T$. The set $\mathcal{CP}_n$ of completely positive matrices can be described as $\mathcal{CP}_n := \operatorname{conv}\{xx^T \mid x \in \mathbb{R}^n_+\} = \{BB^T \mid B \in \mathbb{R}^{n\times r}_+\}$. Therefore, $\mathcal{CP}_n$ is a proper subset of the positive semidefinite cone. Moreover, it is a closed, pointed, convex and full dimensional matrix cone, see for example [1].

These matrices play a major role in combinatorial and quadratic optimization. As shown in [3], even non-convex quadratic problems can be reformulated as convex problems over the completely positive cone, where the whole complexity is moved into the cone constraint. Therefore it is not surprising that checking whether a given matrix is completely positive is NP-hard, as shown in [4]. So far it is still an open question whether checking $A \in \mathcal{CP}_n$ is also in NP.

For a given matrix $A \in \mathcal{CP}_n$ it is nontrivial to find a cp-factorization $A = BB^T$ with $B \in \mathbb{R}_+^{n\times r}$, since this factorization would provide a certificate for the matrix to be completely positive. But this factorization is not only important for the membership to the completely positive cone, it can also be used to recover the solution of the underlying quadratic or combinatorial problem.

Moreover, it is not known a priori how many columns the matrix B in a factorization $A = BB^T$ has. The minimum possible number of columns is called the cp-rank of A and is defined as

$$\mathrm{cp}(A) = \inf\{r \in \mathbb{N} \mid \exists B \in \mathbb{R}^{n\times r},\ B \geq 0,\ A = BB^T\}.$$

So far it is still an open question how the cp-rank of a given matrix can be computed. But, as shown in [2], we have

$$\mathrm{rank}(A) \leq \mathrm{cp}(A) \leq \mathrm{cp}_n^+, \text{ where } \mathrm{cp}_n^+ := \begin{cases} n & \text{for } n \in \{2,3,4\} \\ \frac{1}{2}n(n+1) - 4 & \text{for } n \geq 5. \end{cases}$$

In this talk I will propose a factorization algorithm which computes the nonnegative factorization BB^T of a given completely positive matrix A. This method is based on the following lemma, which can be used to transform one factorization to another.

Lemma 1. *Let $B, C \in \mathbb{R}^{n\times r}$. Then $BB^T = CC^T$ if and only if there exists an orthogonal matrix $Q \in \mathbb{R}^{r\times r}$ with $BQ = C$.*

The basic idea of the factorization algorithm is to start from an initial factorization $A = \tilde{B}\tilde{B}^T$ with $\tilde{B} \in \mathbb{R}^{n\times n}$ not necessarily nonnegative, and to extend $\tilde{B}$ to a matrix $B \in \mathbb{R}^{n\times r}$, where $r \geq \mathrm{cp}(A)$ and $A = BB^T$. Then Lemma 1 provides the means to transform this factorization $A = BB^T$ into a cp-factorization.

This problem can be formulated as a nonconvex feasibility problem and solved by a method which is based on alternating projections. In this talk I will show a local convergence result for the algorithm, which is based on results from [5] for alternating projection between semialgebraic sets.

For the algorithm it is necessary to solve a second order cone problem in every projection step, which is very costly. Therefore, I will provide a heuristic extension which improves the numerical performance of the algorithm. Extensive numerical tests show that the factorization method is very fast in most instances. In addition, I will show how to derive a certificate for a matrix to be in the interior of the completely positive cone.

As a second application, this method can be extended to find a general nonnegative matrix factorization for a given matrix $A \in \mathbb{R}^{m\times n}$, see for example [6].

For the symmetric nonnegative matrix factorization it is necessary to solve the following problem:

Given $A \in \mathbb{R}_+^{n\times n}$ symmetric and $k \ll n$, find a solution $B \in \mathbb{R}_+^{n\times k}$ of

$$\min_{B\geq 0} \|A - BB^T\|_2^2.$$

For this, it becomes necessary to add a low-rank constraint to our factorization algorithm. Here, in contrast to the cp-factorization, the number of columns in the factorization matrix B is smaller than the order of A. If we consider the general, non-symmetric case of nonnegative matrix factorization, we are looking for a solution to the following problem:

Given $A \in \mathbb{R}_+^{n\times m}$ and $k \ll \min\{n, m\}$, find solutions $B \in \mathbb{R}_+^{n\times k}$ and $C \in \mathbb{R}_+^{k\times m}$ of

$$\min_{B,C\geq 0} \|A - BC\|_2^2.$$

Here it is necessary to extend the algorithm to the non-symmetric case. Therefore I will show an adapted version of Lemma 1 such that the main ideas of the cp-factorization algorithm can be reused to generate a nonnegative matrix factorization. I will also present numerical results for the nonnegative matrix factorization, indicating that the presented algorithm can be extended to other nonnegative factorization problems.

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Extremal copositive matrices: approach via zero support sets

Roland Hildebrand

Since the early days of research on copositive matrices the zeros of a copositive matrix and their support sets have been an important tool for extracting information on the facial structure of the copositive cone $\mathbf{C}^n$, in particular, for describing the extreme rays of this cone. The extreme rays are of use when checking exactness of inner approximations of the copositive cone. Another application is the study of the facial structure of the dual cone, the completely positive cone. In this talk we give a review on existing and new results on extreme rays of the copositive cone and on the methods which have been developed to investigate these rays.

In particular, we present the concept of reduced copositive matrices, which is a weaker condition than extremality but easier to handle. A copositive matrix A is called *reduced* with respect to a non-zero copositive matrix B if $A - \varepsilon B$ is not copositive for all $\varepsilon > 0$. A *zero* of a copositive matrix $A \in \mathbf{C}^n$ is a non-zero vector $x \in \mathbb{R}_+$ such that $x^T Ax = 0$. The *support* $\operatorname{Supp} x$ of a zero x is the index set of its positive elements. An *exceptional* copositive matrix is a copositive matrix which cannot be represented as a sum of a positive semi-definite matrix and an element-wise nonnegative matrix.

The zeros and their support sets have been used in [1] to establish sufficient conditions for a copositive matrix to be reduced with respect to all element-wise nonnegative matrices, which in turn is necessary for the matrix to be extremal exceptional. These conditions have been weakened to necessary and sufficient conditions in [2]. In [4] a subclass of zeros has been introduced, the *minimal zeros*. A zero of a copositive matrix A is called minimal if its support is minimal with respect to inclusion among all supports of zeros of A. The necessary and sufficient conditions for reducedness with respect to element-wise nonnegative matrices have been reformulated in terms of the minimal zeros of the matrix and extended to the case of reducedness with respect to positive semi-definite matrices.

A number of necessary combinatorial conditions on the supports of minimal zeros of an exceptional extremal copositive matrix have been established in [4] which allowed a coarse classification of the extreme rays of the copositive cone. Examples are given for the dimensions $n = 5$ and $n = 6$. In [3] extremality of a copositive matrix A and reducedness with respect to an arbitrary copositive matrix B have been described in terms of a solution set of a linear system of equations constructed from the elements of A (and B) and from its minimal zeros.

In [6] the extremal exceptional copositive matrices have been classified which possess only minimal zeros with supports of cardinality two. They are closely liked to the matrices with elements from $\{-1, 0, +1\}$ which have been constructed by Hoffman and Pereira in [7]. Finally, some results are presented on extremal exceptional copositive matrices in $\mathbf{C}^n$ which possess only minimal zeros with supports of cardinality $n - 2$, which is the maximal possible cardinality for an exceptional extremal matrix. This class can be reduced to the subset of copositive matrices whose minimal zero support set is circulant, which have been considered in [5].

A number of directions for further research have been proposed.

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The Max-Cut Polytope, the Unit Modulus Lifting, and their set-completely-positive representations

Florian Jarre

(joint work with Felix Lieder, Ya-Feng Liu, and Cheng Lu)

A generalization of the "max-cut polytope" $\text{conv}\{xx^T \mid |x_k| = 1 \text{ for } 1 \le k \le n\}$ in the space of real symmetric $n \times n$-matrices with all-ones-diagonal is considered, namely a complex "unit modulus lifting"

$$\mathcal{UML}_{\mathbb{C}} := \text{conv}\{xx^* \mid |x_k| = 1 \text{ for } 1 \le k \le n\}$$

in the space of complex Hermitian $n \times n$-matrices with all-ones-diagonal. (Here, x^* denotes the transpose of the complex conjugate of x.) The problem of minimizing a linear objective function over $\mathcal{UML}_{\mathbb{C}}$ arises for example in digital communication, robust optimization and robust control. While it is possible to generalize the Goemans-Williamson approach [2] to such problems, see [4, 1], these problems are NP-hard in general, and thus, tight convex relaxations are of interest.

An explicit description of $\mathcal{UML}_{\mathbb{C}}$ by inequality constraints is not known. Set-completely positive representations, however are possible and presented here. The notion of cp-rank is generalized to set-completely-positive sets. Set-completely-positive representations of the max-cut polytope and of $\mathcal{UML}_{\mathbb{C}}$ are compared, and a set of matrices at the boundary of max-cut polytopes in dimension $n \times n$ is defined for which the generalized cp-rank is not monotone with n.

For $\mathcal{UML}_{\mathbb{C}}$ the generalized cp-rank is used to bound the number of variables in a nonconvex formulation for the membership problem for $\mathcal{UML}_{\mathbb{C}}$.

For $n = 3$, it is shown that $\mathcal{UML}_{\mathbb{C}}$ coincides with its semidefinite relaxation and for $n = 4$ matrices belonging to the semidefinite relaxation are defined that do not belong to $\mathcal{UML}_{\mathbb{C}}$. A candidate for a separating hyperplane (H, α) is discussed along with rotations of this hyperplane that also form valid inequalities for $\mathcal{UML}_{\mathbb{C}}$ if (H, α) is valid. While the verification of the validity of (H, α) may be tractable using global optimization packages (and can then be applied to 4×4 submatrices also in higher dimensions) compact representations of these rotations remain open. Also the question of whether rotations and permutations of these hyperplanes describe $\mathcal{UML}_{\mathbb{C}}$ for $n = 4$ remains open for now.

A preprint associated with this talk was posted on [3] after the workshop in Oberwolfach.

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Quadratic Optimization with Uncertainty in the Objective Function: Theory and Practice of a Robust Approach

Michael Kahr

(joint work with Immanuel M. Bomze and Markus Leitner)

Numerous practically relevant optimization problems confront decision makers with uncertain input data, which naturally arises, for instance, if the true data is revealed in future but decisions need to be made now. In order to tackle that uncertainty, two strategies asserted themselves during the last decades, i.e., *Stochastic Optimization* [3] and *Robust Optimization* [7]. The former is typically applied if the probability distribution of the data realizations is known, whereas the latter is usually used if only bounds on the data realizations can be estimated.

In this talk we focus on the framework of Robust Optimization, in which the uncertain data realizations are assumed to be bounded by *uncertainty sets*. Thereby, the objective is to identify optimal solutions that are robust against all data realizations therein.

We consider a robust variant of the *Standard Quadratic Problem* (StQP) which is, despite of its simplicity, quite versatile and has numerous applications in research areas from different domains in which data uncertainty may arise, e.g., Finance (Markowitz portfolio selection), Economics (evolutionary algorithms), Graph Theory (graph clustering), Machine Learning (image analysis) and Ecology (replicator dynamics). In particular we discuss the robust counterpart of the StQP in which uncertainty affects the objective function and its corresponding copositive reformulation [1]. It turns out that for the StQP, the usual box-, spectrahedron-, or ball-shaped uncertainty sets do not add complexity to the robust counterpart.

We also discuss the case of polyhedral uncertainty sets, in which case we propose a copositive relaxation which can be obtained in two ways (a) by applying Sion's theorem to the conic formulation or (b) by relaxing the robust counterpart (resulting in a QCQP) by a copositive formulation as in [2].

The findings are then applied to a robust variant of the *Dominant-Set Clustering Problem* (DSCP) introduced by Pavan [5] which aims to identify homogeneous clusters in a graph. Application areas include, e.g, video analysis, image segmentation, human action recognition, and community detection in social networks. Pavan showed that the DSCP has an equivalent StQP formulation, i.e., optimizing the quadratic form of the weighted adjacency matrix of the input graph, generalizing the famous Motzkin/Straus theorem [4]. For a comprehensive, recent review on the DSCP see [6].

We show that dominant sets are connected, and derive conditions under which dominant sets form cliques. Our computational experiments indicate that considering box- and spectrahedron-shaped uncertainty sets tend to underestimate the sizes of dominant sets, while the opposite is true for ball-shaped uncertainty sets.

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$(0,1)$ Matrices and the Analysis of Social Networks

Steve Kirkland

A two–mode network can be represented by a rectangular $(0,1)$ matrix, where rows represent agents and columns represent events, with a 1 in the (i,j) position if agent i participates in event j, and a 0 in that position otherwise. Sociologists analyse these matrices mathematically in order to understand the relative importance of, or relationships between, the various agents and/or events. Given an $m \times n$ $(0,1)$ matrix A representing a two–mode network, one approach to that analysis is to consider the related (completely positive) matrices AA^T and A^TA. Both AA^T and A^TA represent single–mode networks, and can be easier to work with mathematically than the original rectangular matrix A. Further, the rows and columns of the single–mode networks are of the same type (i.e. they all represent agents, or they all represent events), so comparisons made between those entities arise more naturally from AA^T and A^TA, avoiding the 'apples by oranges' properties of two–mode networks.

In [1], the authors pose the question as to whether knowledge of both AA^T and A^TA is sufficient to reconstruct A. In general the answer is in the negative, as there are examples where A is not specified uniquely by AA^T and A^TA; such examples exhibit data loss, since the pair of matrices AA^T, A^TA does not contain enough information to specify the $(0,1)$ matrix A.

In this work, we use tools from combinatorial matrix theory in order to construct pairs of distinct $m \times n$ $(0,1)$ matrices A, B such that $AA^T = BB^T$ and $A^TA = B^TB$, and to illuminate the relationship between such pairs of matrices. Our approach relies on the simple observation that if we have such a pair A, B, then both matrices have the same vector of row sums and the same vector of column sums; this is because those vectors are the diagonals of $AA^T = BB^T$ and $A^TA = B^TB$, respectively. Consequently, the matrix $E \equiv B - A$ is a $(0,1,-1)$ matrix with all row and column sums equal to 0.

This last observation informs the following strategy. Start with a given $m \times n$ matrix E that is $(0,1,-1)$ with all row and column sums equal to 0; now look for a $(0,1)$ matrix A such that i) $B \equiv A + E$ is also $(0,1)$, ii) $AA^T = BB^T$, and iii) $A^TA = B^TB$. The problem of finding such an A (when E is specified in advance) is equivalent to finding a $(0,1)$ solution to a linear system having $\binom{m}{2} + \binom{n}{2}$ equations, and unknowns indexed by the positions in E corresponding to zero entries. It turns out that this linear system (which is, in general, non–homogeneous) is always consistent, and finding such an A is equivalent to finding a $(-1,1)$ solution to the associated homogeneous linear system. Using this strategy, we can produce a large infinite family of pairs of $(0,1)$ matrices A, B satisfying i)–iii).

In order to do so, first recall that a tournament matrix T of order n is a $(0,1)$ matrix satisfying $T + T^T = J - I$, where J is the $n \times n$ all ones matrix. A tournament matrix of order n is called regular if its row sums are all equal to $\frac{n-1}{2}$ (note that necessarily n has to be odd in that case). The following result of McKay [3] gives an asymptotic expression for the number of regular tournament matrices of (odd) order n.

Proposition 1. *As $n \to \infty$ through odd values, then for any $\epsilon > 0$, the number t_n of regular tournament matrices of order n is given by*

$$t_n = \left(\frac{2^{n+1}}{\pi n}\right)^{\frac{n-1}{2}} \left(\frac{n}{e}\right)^{\frac{1}{2}} \left(1 + O\left(n^{-\frac{1}{2}+\epsilon}\right)\right).$$

We have the following result from [2], which uses regular tournament matrices in order to construct pairs of $(0,1)$ matrices satisfying i)–iii).

Theorem 1. *Suppose that $n \in \mathbb{N}$ and denote the all ones vector in $\mathbb{R}^n$ by* $\mathbf{1}$. *Consider the following $(n+1) \times 2n$ matrix:*

$$E = \left[\begin{array}{c|c} -I & I \\ \hline \mathbf{1}^T & -\mathbf{1}^T \end{array}\right].$$

There is an $(n+1) \times 2n$ $(0,1)$ matrix A such that i) $A+E$ is also $(0,1)$, ii) $AA^T = (A+E)(A+E)^T$, and iii) $A^TA = (A+E)^T(A+E)$ if and only if n is odd.

When n is odd, each $(0,1)$ A satisfying i)–iii) has the form

$$A = \left[\begin{array}{c|c} T+I & T^T \\ \hline 0^T & \mathbf{1}^T \end{array}\right], \tag{1}$$

where T is a regular tournament matrix of order n. Conversely, for any regular tournament matrix T of order n, the matrix A of (1) satisfies i)–iii).

In view of Proposition 1, we see that asymptotically, the number of such pairs of matrices in Theorem 1 is quite large. Observe also that the matrices of Theorem 1 are highly structured. It remains to be seen whether there are are two–mode networks that arise in empirical settings that give rise to $(0,1)$ pairs A, B for which $AA^T = BB^T$ and $A^TA = B^TB$.

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Approximating the cone of copositive kernels to estimate the stability number of infinite graphs

Olga Kuryatnikova

(joint work with Juan C. Vera)

It has been shown by Dobre, Dür, Frerick and Vallentin [3] that the stable set problem in certain infinite graphs, and particularly the kissing number problem, reduces to a minimization problem over the cone of copositive kernels. Optimizing over this cone is NP-hard, so we propose two converging inner hierarchies approximating the cone and implement their first two levels to compute upper bounds on the kissing number κ_n.

Let $V \subset \mathbb{R}^n$ be a compact set and $\mathcal{COP}(V)$ be the cone of opositive kernels over V. Our approximations of $\mathcal{COP}(V)$ extend existing inner hierarchies for copositive matrices, $\mathcal{C}_r^n$ by De Klerk and Pasechnik [2] and Q_r^n by Peña, Vera and Zuluaga [5]. To do the extension, we represent these hierarchies via tensors, similarly to Dong [4]. We denote the new sets by $\mathcal{C}_r^V$ and Q_r^V and show that $\mathcal{C}_r^V \subseteq Q_r^V \subseteq \mathcal{COP}(V)$ for every level of the hierarchies r.

Now let us consider the kissing number problem. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. The kissing number κ_n is the optimal value of an LP over $\mathcal{COP}(S^{n-1})$. Denote by γ_r and ν_r upper bounds on κ_n obtained by replacing $\mathcal{COP}(S^{n-1})$ in this problem with $\mathcal{C}_r^{S^{n-1}}$ and $Q_r^{S^{n-1}}$ respectively. Then

Theorem 1. $\gamma_r \downarrow \kappa_n$, $\nu_r \downarrow \kappa_n$.

Further we concentrate on ν_r as it provides stronger upper bounds. Using invairance of S^{n-1} under orthogonal group O_n, we characterize $Q_r^{S^{n-1}}$ uintroducing a notion of generalized Jacobi polynomials. For $r = 1$, our formulation is close to the best existing upper bound formulation by Bachoc and Vallentin [1], but the bounds we obtain are a little weaker.

The main goal of further research is to implement ν_r for $r = 2$. An open question is the connection between our hierarchies and other existing approximations for the kissing number problem.

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Rank Restrictions in the NIEP

Thomas Laffey

(joint work with Helena Šmigoc)

The *nonnegative inverse eigenvalue problem* (NIEP) asks: which lists of complex numbers can be the spectrum of some entry-wise nonnegative matrix. If a list of complex numbers σ is the spectrum of some entry-wise nonnegative matrix A, we say that σ is *realisable*, and that A *realises* σ. The NIEP is a difficult open problem, however, several partial results are known. For the source of literature on the problem we refer the reader to the following works and the citations that appear in them: [3, 13, 14, 12, 9, 8, 5].

Motivated by applications in ergodic theory, Boyle and Handelman [1] solved a related question: which lists of complex numbers can be the nonzero spectrum of a nonnegative matrix. In particular, they proved that if $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a list of complex numbers such that the power sums $s_k = \sum_{i=1}^{n} \lambda_i^k > 0$ for all positive integers k, and $\lambda_1 > |\lambda_i|$ for $i = 2, 3, \ldots, n$, then there exists a nonnegative integer N such that the list obtained by appending N zeros to σ is realizable by a nonnegative $(n+N) \times (n+N)$ matrix. It is easy to show that the least N required here is in general not bounded as a function of n. Their proof is not constructive and does not enable one to determine the size of the minimal N required for the realizability in the general case. A constructive approach to the Boyle and Handelman result that provides a bound on the minimal such N, was found by this author in [6].

Several other variants of the NIEP are considered in the literature. The one that has attracted most attention is the symmetric nonnegative inverse eigenvalue problem (SNIEP), where we demand that the realising nonnegative matrix is symmetric. The corresponding question about the nonzero spectrum of a symmetric matrix is open. Unlike in the general case, the number of zeros needed to be added to the nonzero spectrum of a symmetric nonnegative matrix in order to obtain a nonnegative symmetric realisation is bounded by a function of the number of nonzero elements in the list.

Theorem 1. [4] *Let $A \in M_n(\mathbb{R})$ be a symmetric nonnegative matrix of rank k. Then, there exists a symmetric nonnegative matrix $\tilde{A} \in M_{k(k+1)/2}(\mathbb{R})$ with the same nonzero spectrum as A.*

This result was used in the first proof that the symmetric nonnegative inverse eigenvalue problem is different from the real nonnegative inverse eigenvalue problem, the problem of determining which lists of real numbers are realisable. The bound provided in the theorem above is believed not to be tight. In fact, examples of lists where one zero added makes the list symmetrically realisable are known, for example $\sigma = (\frac{10}{3}, \frac{8}{3}, -2, -2, -2)$ is not symmetrically realizable but $\sigma \cup \{0\}$ is; however, there are no known examples where three or more zeros are required in symmetric realisability.

Here, we consider diagonal realisability. We show that if a list is the nonzero spectrum of a diagonalisable nonnegative matrix with k nonzero eigenvalues, then it can be realised by a nonnegative diagonalisable matrix of order $k + k^2$.

Our approach depends on the study of a principal sub-matrix A_{11} of the original matrix A that has the same rank as A.

Theorem 2. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n(\mathbb{R})$$

be a nonnegative matrix, where $A_{11} \in M_m(\mathbb{R})$ has rank equal to the rank of A and the rank of A_{21} is equal to r. Furthermore, we assume $n > m + mr$.

Then there exists a nonnegative matrix $\tilde{A} \in M_{m+mr}(\mathbb{R})$ whose nonzero spectrum is the same as the nonzero spectrum of A. Moreover, the Jordan canonical forms of A and $\tilde{A}$, denoted by $J(A)$ and $J(\tilde{A})$ respectively, satisfy: $J(A) = J(\tilde{A}) \oplus 0_{n-m-mr}$.

Theorem 3. *Let $A \in M_n(\mathbb{R})$ be a nonnegative matrix with l nonzero eigenvalues and rank k. Then there exists a nonnegative matrix $\tilde{A}$ of order $\tilde{n} = (2k-l) + (2k-l)^2$ whose nonzero spectrum is the same as the nonzero spectrum of A and whose Jordan canonical form $J(\tilde{A})$ satisifes: $J(A) = J(\tilde{A}) \oplus 0_{n-\tilde{n}}$.*

These results show, that in the Boyle-Handelman result, the component of the Jordan canonical form of a realizing matrix corresponding to the eigenvalue zero, may be required to have large rank.

The Nonnegative Inverse Elementary Divisors Problem

The nonnegative inverse elementary divisors problem (NIEDP) asks: for a given realisable spectrum $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, what are the possible Jordan forms of realising matrices. Of course, if σ has no repeated entries, this problem reduces to the NIEP for σ. It is conjectured that if σ is realisable, then it is realisable by a nonnegative nonderogatory matrix, but this appears to be still open.

Minc [11] proved that if σ is diagonalisably realisable by a positive matrix A, then for every Jordan form J with spectrum σ, σ is realisable by a positive matrix similar to J. This result is conjectured to hold without requiring the positivity of A but this also seems to be open at present.

We consider the classic example

$$\sigma(t) = (3 + t, 3 - t, -2, -2, -2).$$

We ask the question what is the minimal t for which $\sigma(t)$ is realisable by a nonnegative matrix with a given Jordan canonical form $J_i(t)$ associated with $\sigma(t)$, where $J_1(t)$ is a diagonal matrix, $J_2(t)$ is the Jordan canonical form with its minimal polynomial of degree 4, and $J_3(t)$ is nonderogatory. We will denote the minimal t in each case by t_i.

It is shown by Cronin and the author in [2] that $\sigma(t)$ is realisable by a diagonalisable nonnegative matrix only for $t \geq 1$, i.e. $t_1 = 1$. In this case, the condition for diagonalisable realisability and symmetric realisability coincide. Also, in the same paper it is shown that

$$\sigma = (3 + t, 3 - t, -2.09, -2, -2.1)$$

is realisable for $t \geq \frac{1}{10}\sqrt{120\sqrt{3166} - 3899} \approx 0.435$, while by the McDonald-Neumann inequality, given in [10], $t \geq 0.9$ is necessary for symmetric realisability.

On the other hand, $\sigma(t)$ is realisable for $t \geq \sqrt{16\sqrt{6} - 39} \approx 0.438$. This is shown in [7], where a nonderogatory matrix with spectrum $\sigma(t_3)$, $t_3 = \sqrt{16\sqrt{6} - 39}$, is provided.

Here we present the matrix

$$A(t) = \begin{pmatrix} 0 & 2 & \frac{1}{2} & 0 & 0 \\ 2 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{256}{t^2+7} - 32 & \frac{256}{t^2+7} - 32 & 0 & 1 & 0 \\ 0 & 0 & \frac{t^4+78t^2-15}{2(t^2+7)} & 0 & \frac{1}{2}\sqrt{2t^2+30} \\ 0 & 0 & \frac{2\sqrt{2}\left(3t^4+58t^2+3\right)}{(t^2+7)\sqrt{t^2+15}} & \frac{1}{2}\sqrt{2t^2+30} & 0 \end{pmatrix}.$$

$A(t)$ has eigenvalues $(3 + t, 3 - t, -2, -2, -2)$, it is nonnegative, and it has Jordan canonical form $J_2(t)$, for $\sqrt{16\sqrt{6} - 39} \leq t < 1$. This shows that $t_2 = t_3$.

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Maximal exponents of polyhedral cones

Raphael Loewy

The purpose of this talk is to generalize the notions of nonnegative (elementwise), primitive matrices and their exponents. Given an $n \times n$ nonnegative matrix A (denoted $A \geq 0$), A is said to be *primitive* if there exists a positive integer l such that A^l is positive (denoted $A^l > 0$). In that case, the smallest such l is called the *exponent* of A, denoted $\gamma(A)$.

A primitive matrix A can be recognized combinatorially, namely from the directed graph $D(A)$ associated with A. Its vertex set is $v = \{1, 2, \ldots, n\}$, and given two vertices i and j there is a directed edge from i to j if and only if $a_{ij} > 0$. The graph $D(A)$ is called *primitive* if it is strongly connected and the greatest common divisor of the lengths of its cycles is equal to 1. It is known that A is primitive if and only if $D(A)$ is a primitive graph.

There has been great interest in exponents of primitive matrices, in particular establishing upper bounds using combinatorial and linear algebraic parameters. The following result is due to Wielandt [Wi], although a proof appeared later. In order to state his result we define, for any positive integer n, the graph W_n as follows: Its vertex set is $\{1, 2, \ldots, n\}$, and the edge set is $\{(i, i+1) for\ i = 1, 2, \ldots, n-1, (n, 1), (n, 2)\}$.

Theorem 1. *Let A be an $n \times n$ nonnegative, primitive matrix. Then $\gamma(A) \leq (n-1)^2 + 1 = n^2 - 2n + 2$, and inequality holds if and only if $D(A)$ is isomorphic to W_n.*

An $n \times n$ nonnegative matrix can be considered as a linear operator mapping the nonnegative orthant $\mathbb{R}^n_+ = \{x = (x_i) \in \mathbb{R}^n : x_i \geq 0, i = 1, 2, \ldots, n\}$ into itself. A natural generalization is the following.

Suppose that K is a proper cone in $\mathbb{R}^n$, that is, a cone which is convex, pointed, closed, and with non-empty interior. Given an $n \times n$ matrix A, we say that it is *K-nonnegative* if $AK \subset K$; it is *K-positive* if $Ax \in intK$ for every nonzero x, $x \in K$; A is *K-primitive* if A is K-nonnegative and there exists a positive integer l such that A^l is K-positive, and in that case the smallest such l is called the *exponent* of A, denoted by $\gamma_K(A)$. Following a question by S. Kirkland in 1999 we are interested to establish upper bounds for exponents of K- primitive matrices. The results appear in [LT1, LT2, LPT].

Our main results obtain appropriate generalizations of Theorem 1. We restrict our attention to *polyhedral* cones, that is proper cones which have finitely many extreme rays. More precisely, we fix positive integers m,n such that $m \geq n$, and consider cones in $\mathbb{R}^n$ with m extreme rays.

Theorem 2. *Let K be a proper cone in $\mathbb{R}^n$ with m extreme rays, and let A be K-primitive. Then,*

$$\gamma_K(A) \leq (n-1)(m-1) + \tfrac{1}{2}(1 + (-1)^{(n-1)m}).$$

Note that the case $m = n$ in Theorem 2 reduces to the Wielandt bound.

Theorem 3. *The inequality in Theorem 2 is best possible in the sense that there exist K in $\mathbb{R}^n$ with m extreme rays and A which is K-primitive such that equality holds in this inequality.*

A key tool in the proof of Theorem 2 and Theorem 3 is the use of a graph introduced by Barker and Tam [BP1, BP2], which generalizes the graph $D(A)$ in the nonnegative case. Now suppose that K is a proper polyhedral cone in $\mathbb{R}^n$ and A is K-nonnegative. We define the graph $D_K(A)$ as follows: Its vertices correspond to the extreme rays of K. Given extreme rays $R_1 = \{\alpha x : \alpha \geq 0\}$ and $R_2 = \{\alpha y : \alpha \geq 0\}$, for $x, y \in K$, both nonzero, there is a directed edge from R_1 to R_2 if and only if y belongs to the face of K generated by Ax.

It should be pointed out that a significant problem in our work is when is a directed graph on m vertices realizable as $D_K(A)$ for suitable K and A.

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An exact copositive programming formulation for the Discrete Ordered Median Problem

Justo Puerto

The discrete ordered median problem (**DOMP**) represents a generalization of several well-known discrete location problems, such as p-median, p-center or $(k_1 + k_2)-$trimmed mean, among many others. The problem was introduced in [10] and later studied by [11], [2], and [13], [9] and [1] among many other papers. **DOMP** is an NP-hard problem as an extension of the p-median problem.

In 2001, Nickel [10] first presented a quadratic integer programming formulation for **DOMP**. However, no further attempt to deal directly with this formulation was ever considered. Furthermore, that approach was never exploited in trying to find alternative reformulations or bounds; instead several linearizations in different spaces of variables have been proposed to solve **DOMP** some of them being rather promising, [9] and [8].

Motivated by the recent advances in conic optimization and the new tools that this branch of mathematical programming has provided for developing bounds and approximation algorithms for NP-hard problems, as for instance max-cut, QAP, and other hard combinatorial problems [6], we present in this talk an exact alternative reformulation of **DOMP** as a continuous, linear conic problem. Our interest is mainly theoretical and tries to borrow tools from continuous optimization to be applied in some discrete problems in the field L.A. To the best of our knowledge reformulations of that kind have never been studied before for **DOMP** nor even in the wider field of L.A.

The goal of our presentation was to prove that the natural binary, quadratically constrained, quadratic formulation for **DOMP** admits a compact, exact reformulation as a continuous, linear problem over the cone of completely positive matrices.

The talk was organized as follows. The first part formally defined the ordered median problem and set its elements. The next part was devoted to describe a folk result that formulates the problem of sorting numbers as a feasibility binary linear program. This is used later as a building block to present the binary quadratic, quadratically constrained formulation of **DOMP**. The following section contained the main result: **DOMP** is equivalent to a continuous, linear conic problem. Obviously, there are no shortcuts and the problem remains NP-hard but it allows to shed some lights onto the combinatorics of this difficult discrete location problem. Moreover, it permits to borrow also the tools from continuous optimization to the area of L.A. Finally, we addressed some conclusions and pointers to future research.

1. Definition and formulation of the problem

1.1. **Problem definition.** Let $S = \{1, ..., n\}$ denote the set of n sites. Let $C = (c_{j\ell})_{j,\ell=1,...,n}$ be a given nonnegative $n \times n$ cost matrix, where $c_{j\ell}$ denotes the cost of satisfying demand point (client) j from a facility located at site ℓ. We also assume the so called, *free self-service* situation, namely $c_{jj} = 0$ for all $j = 1, \ldots, n$. Let $p < n$ be the desired number of facilities to be located at the candidate sites. A solution to the facility location problem is given by a set $\mathcal{X} \subseteq S$ of p sites.

We assume, that each new facility has unlimited capacity. Therefore, each client j will be allocated to a facility located at site ℓ of $\mathcal{X}$ with lowest cost, i.e.

$$c_j = c_j(\mathcal{X}) := \min_{\ell \in X} c_{j\ell}.$$

The costs for supplying clients, $c_1(\mathcal{X}), ..., c_n(\mathcal{X})$, are sorted in nondecreasing order. We define $\sigma_{\mathcal{X}}$ to be a permutation on $\{1, ..., n\}$ for which the inequalities

$$c_{\sigma_{\mathcal{X}}(1)}(\mathcal{X}) \leq \cdots \leq c_{\sigma_{\mathcal{X}}(n)}(\mathcal{X})$$

hold.

Now, for any nonnegative vector $\lambda \in \mathbb{R}^n_+$, the Discrete Ordered Median Problem (**DOMP**) consists of finding $\mathcal{X}^* \subset S$ with $|\mathcal{X}^*| = p$ such that:

$$\sum_{k=1}^{n} \lambda_k c_{\sigma_{\mathcal{X}^*}(k)}(\mathcal{X}^*) = \min_{\mathcal{X} \subset S, |\mathcal{X}|=p} \sum_{k=1}^{n} \lambda_k c_{\sigma_{\mathcal{X}}(k)}(\mathcal{X}).$$

2. Main result

In this talk we prove that **DOMP** admits a reformulation as a mixed-binary quadratic objective, quadratically constrained problem. This problem can be always relaxed using the corresponding matrix variables that replace the original quadratic terms.

Our main result is that this second reformulation, using matrix variables, is indeed exact and therefore, **DOMP** is a new hard combinatorial optimization problems that falls into the class of completely positive conic linear programs. See [3, 4, 5, 7, 12] and the references therein for a detailed literature review of this topic.

Theorem 1. ***DOMP*** *belongs to the class of continuous, convex, conic problems. Moreover, an explicit formulation of* ***DOMP*** *as a completely positive convex problem is given as CP-DOMP. Problem CP-DOMP is equivalent to the quadratic reformulation of* ***DOMP****: (i) they share the same objective value, (ii) if* Φ^* *is* *an optimal solution for Problem CP-DOMP then a suitable linear transformation of Φ^* is in the convex hull of optimal solutions of the quadratic reformulation of* ***DOMP***.

3. Concluding remarks

The result in this talk states, for the first time, the equivalence of a difficult NP-hard discrete location problem, namely **DOMP**, with a continuous, convex problem. This new approach can be used to start new avenues of research by applying tools in continuous optimization to approximate or numerically solve some hard discrete location problems.

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SPN graphs: when copositive = SPN

Naomi Shaked-Monderer

A real symmetric matrix A is *copositive* if $\mathbf{x}^T A\mathbf{x} \geq 0$ for every nonnegative vector $\mathbf{x}$. A matrix is *SPN* if it is a sum of a real positive semidefinite matrix and a nonnegative one. Every SPN matrix is copositive, but the converse does not hold for matrices of order greater than 4. We define a graph G to be an *SPN graph* if every copositive matrix with graph G is SPN, and consider the problem of characterizing such graphs.

We present sufficient conditions for a graph to be SPN (in terms of its possible blocks) and necessary conditions for a graph to be SPN (in terms of forbidden subgraphs), and make some conjectures regarding the remaining gap.

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Constrctructive Techniques in the Symmetric Nonnegative Inverse Eigenvalue Problem

Helena Šmigoc

(joint work with Richard Ellard)

A list of real numbers is *symmetrically realisable* if it is the spectrum of some (entrywise) nonnegative symmetric matrix. The Symmetric Nonnegative Inverse Eigenvalue Problem (SNIEP) is the problem of characterising all symmetrically realisable lists. If we do not insist that the realising matrix be symmetric, then the resulting problem is called the *Real Nonnegative Inverse Eigenvalue Problem* (RNIEP).

Several constructions have been developed that enable us to construct nonnegative matrices with given spectrum. We consider a construction presented in [7] that joins two smaller matrices to construct a bigger one. Let B be an $l \times l$ nonnegative symmetric matrix with Perron eigenvalue c, Perron eigenvector u, ($Bu = cu$ and $u^T u = 1$) and spectrum $(c, \nu_2, \nu_3, \ldots, \nu_l)$. Let

$$A := \begin{bmatrix} A_1 & a \\ a^T & c \end{bmatrix},$$

where A_1 is an $(k-1) \times (k-1)$ nonnegative symmetric matrix and $a \in \mathbb{R}^{k-1}$ is nonnegative, have eigenvalues $(\mu_1, \mu_2, \ldots, \mu_k)$. Then

$$C := \begin{bmatrix} A_1 & av^T \\ va^T & B \end{bmatrix}$$

is a nonnegative symmetric matrix with eigenvalues $(\mu_1, \mu_2, \ldots, \mu_k, \nu_2, \nu_3, \ldots, \nu_l)$. If A and B are positive semidefinite, then so is C. However, the question, if completely positive matrices A and B produce a completely positive matrix C, is open.

We denote by $\mathcal{H}_n$ the set of all symmetrically realisable lists, that can be obtained recursively, starting with lists of length 2 and repeatedly applying the construction given above. The realisable family obtained in this way has several interesting properties, [2]:

(1) Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{H}_n$ have the Perron eigenvalue λ_1, and let $\epsilon > 0$. Then

$$(\lambda_1 + \epsilon; \lambda_2 - \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n) \in \mathcal{H}_n \text{ and } (\lambda_1 + \epsilon; \lambda_2 + \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n) \in \mathcal{H}_n.$$

(2) Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then $(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n$ if and only if there exist $0 \leq \epsilon \leq \frac{1}{2}(\lambda_1 - \lambda_2)$ and a partition

$$\{3, 4, \ldots, n\} = \{p_1, p_2, \ldots, p_{l-1}\} \cup \{q_1, q_2, \ldots, q_{n-l-1}\},$$

such that

$$(\lambda_1 - \epsilon; \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{l-1}}) \in \mathcal{H}_l \text{ and} (\lambda_2 + \epsilon; \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{n-l-1}}) \in \mathcal{H}_{n-l}.$$

(3) If $(\lambda_1; \lambda_2, \ldots, \lambda_n, 0) \in \mathcal{H}_{n+1}$, then $(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n$.

These properties allow us to make connections between a number of sufficient conditions in the SNIEP developed over forty years, starting with the work of Fiedler [4] in 1974. In particular, they enable us to show that a condition due to Borobia, Moro and Soto [1] called *C-realisability* is sufficient for the existence of a symmetric realising matrix, and that the set of C-realisable lists is the same as $\mathcal{H}_n$, [2].

Soules' approach to the SNIEP focuses on constructing the eigenvectors of the realising matrix A. Starting from a positive vector $x \in \mathbb{R}^n$, Soules [6] showed how to construct a real orthogonal $n \times n$ matrix R with first column x such that for all $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, the matrix $R\Lambda R^T$—where $\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$—is nonnegative. This motivated Elsner, Nabben and Neumann [3] to make the following definition: Let $R \in \mathbb{R}^{n\times n}$ be an orthogonal matrix with columns $r_1, r_2, \ldots, r_n$. R is called a *Soules matrix* if r_1 is positive and for every diagonal matrix $\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, the matrix $R\Lambda R^T$ is nonnegative.

Let $\mathcal{S}_n$ denote the set of all lists of eigenvalues of symmetric nonnegative matrices of the form $R\Lambda R^T$, where R is a Soules matrix and Λ is a diagonal matrix. Then $\mathcal{S}_n = \mathcal{H}_n$, [2]. This result together with a result by Shaked-Monderer [5] that states, that nonnegative matrix generated by a Soules matrix is a completely positive matrix with cp-rank equal to the rank, connects the construction introduced earlier to cp-matrices.

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Positive polynomials on unbounded domains

Juan C. Vera
(joint work with Luis F. Zuluaga)

Certificates of non-negativity such as Putinar's Positivstellensatz have been used to obtain powerful numerical techniques to solve polynomial optimization (PO) problems. Putinar's certificate uses sum-of-squares (sos) polynomials to certify the non-negativity of a given polynomial over a domain defined by polynomial inequalities. This certificate assumes the Archimedean property of the associated quadratic module, which in particular implies compactness of the domain.

We present a new certificate of non-negativity for polynomials over the possibly unbounded set obtained from the intersection of a closed domain S and $h^{-1}(0) = \{x \in \mathbb{R}^n : h(x) = 0\}$, the zero set of a given polynomial $h(x)$. It is evident that if $p(x)$ is non-negative on the domain S, then $p(x) + h(x)q(x)$ is non-negative on the domain $S \cap h^{-1}(0)$ for any polynomial $q(x)$. In [7] it is shown that (modulo an appropriate closure) the converse is true when the domain S is compact, thereby establishing a certificate of non-negativity for polynomials on $S \cap h^{-1}(0)$ in terms of non-negative polynomials on S.

We show that under suitable conditions on $h(x)$ and S, the non-negativity of a polynomial over the set $S \cap h^{-1}(0)$ can be certified in terms of the non-negative polynomials on Seven if the set S is unbounded. Moreover, a characterization of the sets $S \cap h^{-1}(0)$ for which the certificate of non-negativity exists is provided in terms of an appropriate condition on S and h.

Unlike previous related results in [2, 5, 6], the proposed certificate of non-negativity is independent of the polynomial defining the objective of an associated PO problem. Instead, the certificate is written purely in terms of the set of non-negative polynomials over a set S and the ideal generated by $h(x)$. Also, unlike the recent related results in [4], and as a result of the use of the non-negative polynomials on a set S, the associated quadratic module and the Archimedean property are not used to characterize the cases in which the proposed certificate of non-negativity holds.

The certificate of non-negativity presented here readily allows the use of copositive polynomials to certify the non-negativity of a polynomial (as opposed to the more common use of sums-of-squares polynomials to certify non-negativity). In particular it encompasses the important results of Burer [1] and Peña et al., [8]. So a natural question is the role this certificate could play to generalize this type of result to other contexts.

We are interested in studying the consequences of this new type of certificate. For instance, the fact that copositive polynomials can be used in the proposed certificate of non-negativity, together with Polya's Positivstellensatz (see, e.g., [3]), means that convergent linear programming (LP) hierarchies can be constructed to approximate the solution of general PO problems. Also, the new certificate could be used to provide an interesting bridge between the results on certificates of non-negativity in algebraic geometry (such as Schmüdgen's and Putinar's Positivstellensatz) and Polya's Positivstellensatz.

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Mixed Integer Linear Programming Formulations of Standard Quadratic Programs

E. Alper Yildirim

Standard quadratic programs have numerous applications and play an important role in copositivity detection. We consider reformulating a standard quadratic program as a mixed integer linear programming problem. We discuss the advantages and drawbacks of such a reformulation.

1. Introduction

A standard quadratic program involves minimizing a (nonconvex) quadratic form (i.e., a homogeneous quadratic function) over the unit simplex. Formally, it can be stated in the following form:

$$\text{(StQP)} \quad \nu(Q) = \min\{x^T Q x : x \in \Delta_n\},$$

where $\Delta_n \subset \mathbb{R}^n$ denotes the unit simplex given by

$$\Delta_n = \{x \in \mathbb{R}^n : e^T x = 1, \quad x \in \mathbb{R}^n_+\},$$

and $Q \in \mathcal{S}^n$ and $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices, $x \in \mathbb{R}^n$, $e \in \mathbb{R}^n$ denotes the vector of all ones, and $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$.

Standard quadratic programs have important applications in portfolio optimization, quadratic resource allocation problem, maximum (weighted) stable set problem, and copositivity detection (see, e.g., [1]).

2. Copositivity and Optimality Conditions

In this talk, we focus on solving (StQP) to global optimality. For (StQP), the necessary and sufficient global optimality conditions can be stated as follows: Let $x^* \in \Delta_n$. Then, x^* is a global optimal solution of (StQP) (i.e., $\nu(Q) = (x^*)^T Q x^*$) if and only if the matrix $\left(Q - \left((x^*)^T Q x^*\right) ee^T\right)$ is copositive (see, e.g., [1]). Since checking copositivity is a co-NP-complete problem, we aim to exploit optimality conditions from a different perspective.

Using the Karush-Kuhn-Tucker optimality conditions, if $x \in \Delta_n$ is an optimal solution of (StQP), then there exist $s \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that

$$
\begin{aligned}
&(1) & Qx - \lambda e - s &= 0, \\
&(2) & e^T x &= 1, \\
&(3) & x &\in \mathbb{R}^n_+, \\
&(4) & s &\in \mathbb{R}^n_+, \\
&(5) & x_j s_j &= 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

By (1), (2), and (5), if $x \in \Delta_n$ is an optimal solution of (StQP), then $\nu(Q) = x^T Q x = \lambda$.

3. A Mixed Integer Linear Programming Reformulation

We can linearize the nonconvex complementarity constraints (5) by using binary variables and big-M constraints. Therefore, (StQP) can be reformulated as the following mixed integer linear programming problem:

$$
\begin{array}{llrcl}
\text{(MILP)} & \min & & \lambda & \\
& & Qx - \lambda e - s & = & 0, \\
& & e^T x & = & 1, \\
& & x_j & \leq & y_j, \quad j = 1, \ldots, n, \\
& & s_j & \leq & M_j(1 - y_j), \quad j = 1, \ldots, n, \\
& & x & \geq & 0, \\
& & s & \geq & 0, \\
& & y_j & \in & \{0, 1\}, \quad j = 1, \ldots, n.
\end{array}
$$

In this formulation, we need to specify the values of $M_j,\ j = 1, \ldots, n$. By the first constraint of (MILP), we have $Qx - \lambda e - s = 0$, which implies,

$$s_j = e_j^T Qx - \lambda, \quad j = 1, \ldots, n,$$

where $e_j \in \mathbb{R}^n$ denotes the jth unit vector, $j = 1, \ldots, n$. Since $x \in \Delta_n$, we have $e_j^T Qx = x^T Q e_j \leq \max_{i=1,\ldots,n} Q_{ij}$. Therefore, any lower bound on λ (equivalently, a lower bound on $\nu(Q)$) can be used to obtain an upper bound on $s_j,\ j = 1, \ldots, n$.

The first simple lower bound on $\nu(Q)$ is given by

$$\text{(LB1)} \quad \nu(Q) \geq \ell_0(Q) := \min_{1 \leq i \leq j \leq n} Q_{ij}.$$

A tighter lower bound on $\nu(Q)$ can be obtained by

$$\text{(LB2)} \quad \nu(Q) \geq \ell_{\text{cop}}(Q) := \max\{\lambda : Q - \lambda ee^T \in \mathcal{C}^n\},$$

where $\mathcal{C}^n$ is the cone of matrices that can be written as the sum of a componentwise nonnegative matrix and a positive semidefinite matrix.

Note that $\ell_0(Q)$ can be computed in $O(n^2)$ time whereas the computation of $\ell_{\text{cop}}(Q)$ requires solving a semidefinite program. However, $\ell_{\text{cop}}(Q)$ can be approximated by using an accelerated proximal gradient method.

Therefore, we can substitute $M_j := \max_{i=1,\ldots,n} Q_{ij} - \ell$ in the constraint $s_j \leq M_j(1 - y_j),\ j = 1, \ldots, n$ of (MILP), where $\ell \in \{\ell_0(Q), \ell_{\text{cop}}(Q)\}$.

4. Computational Experiments

We compare the performances of the MILP formulations by using the two alternative lower bounds $\ell_0(Q)$ and $\ell_{\text{cop}}(Q)$ in the computation of the upper bound $M_j,\ j = 1, \ldots, n$. We test our formulations on the maximum stable set problem: Let $G = (V, E)$ be a simple, undirected graph. A set $S \subseteq V$ is a *stable set* if each pair of vertices in S is mutually nonadjacent. The maximum stable set problem is concerned with finding a stable set with the largest cardinality, denoted by $\alpha(G)$. This problem can be formulated as an instance of (StQP) as follows [2]:

$$\frac{1}{\alpha(G)} = \min\left\{x^T(I + A_G)x : x \in \Delta_n\right\},$$

where $A_G \in \mathcal{S}^n$ is the vertex adjacency matrix of G.

We generated random graphs with a specified density $d \in (0, 1)$. Each instance is denoted by (n, d, s), where s is the seed of the random number generator. Our implementation uses the Matlab-Cplex interface (Matlab 2017b; Cplex 12.7.1). Our results are summarized in Tables 1 and 2:

Table 1. $n = 100; d = 0.5$

		$\ell_0(Q)$	$\ell_{\text{cop}}(Q)$		
Instance	$\alpha(G)$	MILP Time	MILP Time	Lower Bound	Total Time
$(100, 0.5, 1)$	10	4	5	202	207
$(100, 0.5, 2)$	9	5	5	188	193
$(100, 0.5, 3)$	9	5	5	186	191
$(100, 0.5, 4)$	9	5	6	239	245
$(100, 0.5, 5)$	9	6	5	118	123
$(100, 0.5, 6)$	9	7	7	106	113
$(100, 0.5, 7)$	9	5	4	156	160
$(100, 0.5, 8)$	9	4	5	178	183
$(100, 0.5, 9)$	9	6	4	165	169
$(100, 0.5, 10)$	9	5	5	99	104
Average	-	5	5	164	169

Table 2. $n = 100; d = 0.25$

		$\ell_0(Q)$	$\ell_{\text{cop}}(Q)$		
Instance	$\alpha(G)$	MILP Time	MILP Time	Lower Bound	Total Time
$(100, 0.25, 1)$	16	1040	426	186	**612**
$(100, 0.25, 2)$	16	572	292	170	**462**
$(100, 0.25, 3)$	17	467	367	117	**484**
$(100, 0.25, 4)$	16	608	373	191	**564**
$(100, 0.25, 5)$	17	812	595	199	**794**
$(100, 0.25, 6)$	18	1759	307	79	**386**
$(100, 0.25, 7)$	17	643	359	214	**573**
$(100, 0.25, 8)$	17	913	392	210	**602**
$(100, 0.25, 9)$	17	**354**	300	76	376
$(100, 0.25, 10)$	16	**225**	135	224	359
Average	-	739	354	167	**521**

Our preliminary computational results illustrate that, on certain instances, the MILP formulation using the improved lower bound $\ell_{\text{cop}}(Q)$ seems to outperform that with the simple lower bound $\ell_0(Q)$, even when we include the computation time for the improved lower bound. Therefore, the MILP formulation can be used to solve certain instances of (StQP) to global optimality. We intend to incorporate valid inequalities in the near future.

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CP rank of Graphs

Xiao-Dong Zhang
(joint work with Jiong-Sheng Li)

In this talk, we introduce some properties of CP rank of complete positive graphs and prove that the set of all factorization indices of a completely positive graph has no gaps. In other words, we give an affirmative answer to a question raised by N. Kogan and A. Berman in the case of completely positive graphs.

An $n \times n$ symmetric matrix A is *doubly non-negative,* denoted by $A \in DN_n$, if it is non-negative and positive semidefinite. An $n \times n$ symmetric matrix A is *completely positive,* denoted by $A \in CP_n$, if there exists an $n \times m$ non-negative matrix B such that $A = BB^T$, where B^T is the transpose of B. The smallest number m of columns in any such matrix B is called the factorization index of A and is denoted by $CPrankA$. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. An $n \times n$ doubly non-negative (resp. completely positive) matrix $A = (a_{ij})$ is a doubly non-negative (resp. completely positive) *realization* of G if $a_{ij} > 0$ if and only if v_i and v_j are adjacent for any $1 \leq i \neq j \leq n$. A graph G is *completely positive,* or CP for short, if every doubly non-negative realization of G is completely positive. The set of all factorization indices of CP realizations of G is denoted by $I(G)$ and the maximum number in $I(G)$ is denoted by $CPrankG$. Kogan and Berman proposed the question as follows:

Question 1. *For any graph G, if $a, b \in T(G)$, does $I(G)$ contain all integers between a and b?*

In order to study this question, we need the following notation. An $n \times n$ CP matrix A is called *critical* if $A - \varepsilon E_{ii}$ is not CP for any $\varepsilon > 0$ and for each $i = 1, \ldots, n$, where E_{ii} is the matrix of order n with (i, i) entry 1, and 0 elsewhere. Furthermore, a CP graph G is called CP^1 if there exists one critical CP realization A of G such that $CPrankG = CPrankA$. A CP graph G is called CP^2 if there does not exist a singular CP realization A of G such that $CPrankG = CPrankA$, and for any $h \in I(G) \setminus \{CPrankG\}$, there exists one critical CP realization A of G such that $CPrankA = h$. we prove the following results.

Theorem 1. *We have:*

(1) *K_2 and K_3 are CP^2.*
(2) *K_4 is CP^1.*
(3) *If G is a connected bipartite graph of order n, then*

$$I(G) = \begin{cases} \{n-1, n\}, & \text{if } G \text{is acyclic,} \\ \{|E(G)|\}, & \text{otherwise,} \end{cases}$$

where $|E(G)|$ is the number of edges of G. Furthermore, G is CP^2 if G is acyclic, and G is CP^1 if G has a cycle.

(4) *If $n \geq 4$, then $I(T_{n-2}) = \{n-2, \ldots, 2n-4\}$ and T_{n-2} is CP^1, where where $T_{n-2} = (V(T_{n-2}), E(T_{n-2}))$ is a graph of order n consisting of $n-2$ triangles with a common base.*

Theorem 2. *If G is a CP graph of order n, then: (1) $I(G)$ has no gaps; (2) G is either CP^1 or CP^2.*

Alternative SDP and SOCP Approximations for Polynomial Optimization

Luis F. Zuluaga
(joint work with Xiaolong Kuang, Bissan Ghaddar, and Joe Naoum-Sawaya)

Many real-world problems can be modeled as a polynomial optimization problem (POP); that is, an optimization problem in which both the objective and constraints are multivariate polynomials on the decision variables. Thus devising new approaches to globally solve POPs is an active area of research. In his seminal work, Lasserre [11] showed that *semidefinite programming* (SDP) relaxations based on *sum of square* (SOS) polynomials can provide global bounds for POPs. However, the SDP constraints are computationally expensive and thus even using low-orders of the hierarchy to approximate large-scale POPs becomes computationally intractable in practice. To improve the computational performance of the SDP based hierarchies to approximate the solution of POPs, prior work has focused on exploiting the problem's sparsity [10, 9] and symmetry [3, 6], improving the relaxation by generating and adding appropriate valid inequalities [8], using bounded SOS polynomials [13] and more recently by devising more computationally efficient hierarchies such as linear programming (LP) and second-order cone programming (SOCP) hierarchies [7, 14, 1, 4, 5].

Here, we consider alternative ways to use SOCP restrictions of the SOS condition introduced by [1]. In particular, we show that SOCP hierarchies can be effectively used to strengthen hierarchies of LP relaxations for general POPs. Such hierarchies of LP relaxations have received little attention in the POP literature (a few noteworthy exceptions are [2, 12, 15, 4, 5]). However, in this paper we show that this solution approach is substantially more effective in finding solutions of certain POPs for which the more common hierarchies of SDP relaxations are known to perform poorly (see, e.g., [7]). Furthermore, when the feasible set of the POP is compact, these SOCP hierarchies converge to the POP's optimal value. Note that for the well-known SDP based hierarchies introduced in [11], the *quadratic module* (QM) associated with the feasible set of the POP is required to be *Archimedean*, which implies the compactness of the POP's feasible set.

1. Hierarchies

In particular, we consider for the polynomial optimization problem

$$\min\{f(x) : g_i(x) \geq 0, i = 1, \ldots, m\}. \tag{1}$$

In particular we consider the Lasserre hierarchy (QM-SOS$_r$) below applied to problem (1) in which SOS polynomials are constructed to create the hierarchy. Also below, we consider the (QM-SDOS$_r$) hierarchy proposed in [1], in which the hierarchy (QM-SDOS$_r$) is weakened by considering SDSOS polynomials, instead of SOS polynomials.

(QM-SOS$_r$ (QM-SDOS$_r$))

$$\begin{aligned} \max_{\lambda, s_i(x)} \quad & \lambda \\ \text{st} \quad & f(x) - \lambda = s_0(x) + \sum_{i=1}^{m} s_i(x) g_i(x), \\ & s_0(x) \in SOS_{2r}(SDOS_{2r}), \\ & s_i(x) \in SOS_{2\lfloor r - \deg(g_i)/2 \rfloor}(SDOS_{2\lfloor r - \deg(g_i)/2 \rfloor}), \quad i = 1, \dots, m, \\ & \lambda \in \mathbb{R}. \end{aligned}$$

On the other hand, we consider below two hierarchies that can be derived from the non-negative coefficient polynomial hierarchies proposed for (1) in [14]. The Po-SOS$_r$ hierarchy in which the non-negative coefficient polynomials are replaced with SOS polynomials, and the Po-SDOS$_r$ in which the non-negative coefficient polynomials are replaced with SDSOS polynomials. Thus, these two hierarchies strengthen the one proposed in [14].

(Po-SOS$_r$ (Po-SDOS$_r$))

$$\begin{aligned} \max_{\lambda, p_{\alpha,\beta}(x)} \quad & \lambda \\ \text{st} \quad & (1 + \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} g_j(x))^r (f(x) - \lambda) = \sum_{(\alpha,\beta) \in \mathbb{N}^{n+m}} p_{\alpha,\beta}(x) x^\alpha g(x)^\beta, \\ & p_{\alpha,\beta}(x) \in SOS_2(p_{\alpha,\beta}(x) \in SDOS_2), \text{ for all } (\alpha, \beta) \in \mathbb{N}^{n+m}, \\ & \lambda \in \mathbb{R}, \end{aligned}$$

2. Numerical Results

We test a POPs from [8], which are highly non-convex and require a high level of Lasserre's hierarchy to converge to their global optimum.

Example 1. *Consider the following quadratic POP with 5 variables:*

$$\begin{aligned} \min_{x \in \mathbb{R}^5} \quad & 2x_1 - x_2 + x_3 - 2x_4 - x_5 \\ \text{st} \quad & (x_1 - 2)^2 - x_2^2 - (x_3 - 1)^2 - (x_5 - 1)^2 \geq 0, \\ & x_1 x_3 - x_4 x_5 + x_1^2 \geq 1, \\ & x_3 - x_2^2 - x_4^2 \geq 1, \\ & x_1 x_5 - x_2 x_3 \geq 2, \\ & x_1 + x_2 + x_3 + x_4 + x_5 \leq 14, \\ & x_i \geq 0, i = 1, \dots, 5. \end{aligned}$$

As shown in Table 3, the (QM-SOS$_r$) hierarchy converges to the global optimum when $\hat{d} = 8$ with a computational time of 49.82 seconds, while the hierarchy (Po-SOS$_r$) converges to global optimum when $\hat{d} = 6$ with only 8.21 seconds

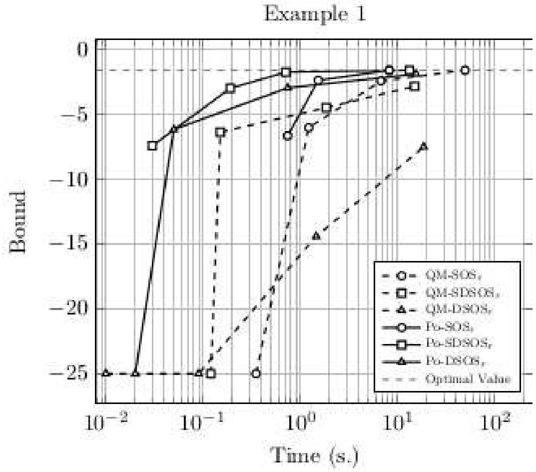

FIGURE 1. Bound and Time Comparison of Different Hierarchies for Example 1.

of computational time. Hierarchy (QM-SDOS$_r$) fails to converge to the global optimum up to $\hat{d} = 8$. However, the hierarchy (Po-SDOS$_r$) also converges to the global optimum when $\hat{d} = 8$ with 13.28 seconds of computational time.

Although the degree $\hat{d}$ provides an approximate measure of the size (variables and constraints) involved in the formulations of the hierarchies' problems, a better comparison of the hierarchies can be done by illustrating the trade-off between the solution time and the quality of the bound obtained from each hierarchy. In Figure 1, the different line plots show the bound and solution time associated with increasing orders of each of the hierarchies. Clearly, within one second, the (Po-SDOS$_r$) hierarchy gives the best bound; within ten seconds, the (Po-SOS$_r$) hierarchy gives the optimal value while there is still a gap between the problem's optimal value (illustrated by the dashed horizontal line) and the bounds obtained by other hierarchies. Clearly, the hierarchies proposed have better performance over the Lasserre-type hierarchies for this problem.

	QM-SOS$_r$		QM-SDOS$_r$		Po-SOS$_r$		Po-SDOS$_r$	
$\hat{d}$	Bound	T	Bound	T	Bound	T	Bound	T
2	-25.00	0.35	-25.00	0.12	-6.63	0.74	-7.40	0.03
4	-6.01	1.22	-6.35	0.15	-2.35	1.53	-2.96	0.19
6	-2.40	6.75	-4.46	1.85	*-1.57	8.21	-1.72	0.71
8	*-1.57	49.82	-2.81	15.00			*-1.57	13.28

$*$: Optimal value is obtained.

TABLE 3. Bound and Time Comparison of Different Hierarchies for Example 1.

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