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Reflection Positivity

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Abstract. The main theme of the workshop was reflection positivity and its occurences in various areas of mathematics and physics, such as Representation Theory, Quantum Field Theory, Noncommutative Geometry, Dynamical Systems, Analysis and Statistical Mechanics. Accordingly, the program was intrinsically interdisciplinary and included talks covering different aspects of reflection positivity.

Mathematics Subject Classification (2010): 17B10, 22E65, 22E70, 81T08.

Introduction by the Organisers

The workshop on *Reflection Positivity* was organized by Arthur Jaffe (Cambridge, MA), Karl-Hermann Neeb (Erlangen), Gestur Ólafsson (Baton Rouge) and Benjamin Schlein (Zürich) during the week November 27 to December 1, 2017. The meeting was attended by 53 participants from all over the world. It was organized around 24 lectures each of 50 minutes duration representing major recent advances or introducing to a specific aspect or application of reflection positivity.

The meeting was exciting and highly successful. The quality of the lectures was outstanding. The exceptional atmosphere of the Oberwolfach Institute provided the optimal environment for bringing people from different areas together and to create an atmosphere of scientific interaction and cross-fertilization. In particular, people from different subcommunities exchanged ideas and this lead to new collaborations that will probably stimulate progress in unexpected directions.

Reflection positivity (RP) emerged in the early 1970s in the work of Osterwalder and Schrader as one of their axioms for *constructive quantum field theory* ensuring the equivalence of their euclidean setup with Wightman fields. From the perspective of mathematics, a physical system corresponds to a unitary representation of the corresponding symmetry group. Here reflection positivity corresponds to transforming a unitary representation of one symmetry group to a unitary representation of the other symmetry group. In the classical QFT context, one passes from the euclidean motion group to the Poincaré group. For stochastic processes RP is weaker than the Markov property and specificies processess arising in lattice gauge theory, in analysis it is a crucial condition that leads to inequalities such as the Hardy–Littlewood–Sobolev inequality. RP also plays a central role in the mathematical study of phase transitions and symmetry breaking.

The recent years have seen a variety of interesting new developments around the theme of reflection positivity distributed over various subfields of mathematics and physics. The talks included in particular the following aspects of reflection positivity:

- RP in the representation theory of symmetric Lie groups
- Constructing quantum fields by modular localization
- Lattice systems, KMS conditions, and old and new connections with RP
- RP in renormalization theory
- Transfer matrices and dynamical systems
- RP in QFT models based on noncommutative geometry
- Interacting quantum fields and Hadamard states on curved space times
- Aspects of RP in Topological Quantum Field Theory
- Applications of RP in Analysis (Hardy–Littlewood–Sobolev inequalities)
- A geometric understanding and topological interpretation of RP

The discussions during the conference showed that the method of modular localization which recently developed into an effective tool to construct quantum fields with good control over their properties has rather close connections with representation theory and RP which are still far from sufficiently understood. Several projects in this direction were initiated during the meeting. A possible outcome may be a better understanding of the role of modular localization in euclidean field theories and which flows on causal manifolds correspond to modular automorphism groups of von Neumann algebras.

In particular the recent work of Barata, Jäkel and Mund shows that a systematic exploitation of symmetry groups leads to quite robust methods to construct interacting quantum fields on curved spacetimes and from there to fields on flat space by well-controlled deformation procedures.

There also seem to be connections between KMS conditions, RP and transfer operators arising in the context of dynamical systems related to locally symmetric spaces and their dynamical ζ-functions which have to be made more explicit and exploited.

The understanding of reflection positivity by Jaffe and Liu within the framework of mathematical picture languages (starting from planar algebras and their generalizations) had led to new insights and a geometric understanding of reflection positivity. Related problems in two and three dimensional topological field theories and their generalizations lead to fascinating open questions concerning putative higher dimensional topological field theories, as well as new lattice statistical mechanical models. One hopes these insights will be helpful toward construction of new relativistic quantum field theories.

More specific information is contained in the abstracts which follow in this volume.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Shamil Shakirov and Christian D. Jäkel in the "Simons Visiting Professors" program at the MFO.
The organizers thank the director Prof. Dr. Gerhard Huisken, and the MFO staff for offering us outstanding support in all phases of the planning and throughout the workshop.

Workshop: Reflection Positivity

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Abstracts

Reflection Positivity Then and Now

Arthur Jaffe

Dedicated to the memory of Robert Schrader

Konrad Osterwalder and Robert Schrader discovered *reflection positivity* (RP) in the summer of 1972 at Harvard University.[1] At the time they were both my postdoctoral fellows, and together we formed the core of a group at Harvard who studied quantum field theory from a mathematical point of view.

RP has since blossomed into an active research area; this conference in Oberwolfach marks the 45^{th} anniversary of its discovery. When Palle Jorgensen, Karl-Hermann Neeb, Gestur Ólafsson, and I first envisioned this meeting in 2014, we had hoped that both Robert and Konrad would be here. Unfortunately neither is: Robert died in November 2015, and Konrad had to cancel because of a conflict. So I have been called upon to make some remarks to set the stage.

These comments not only relate to the early discovery of RP, but they also illustrate that RP is still an active and interesting area of research. The original discovery of RP arose from an effort to relate two different mathematical subjects in a specific way: what property is needed to start from a classical probability theory of fields (i.e. a statistical mechanics of random fields), and end up with a quantum theory of fields? Of course one wanted to have a framework that would apply to the putative quantum fields that physicists believe describe the interactions of elementary particles.

The original work focused into finding a set of axioms for Euclidean-covariant Green's functions, that are equivalent to the Wightman axioms for the vacuum expectation values of relativistic quantum fields [42, 43]. The Euclidean Green's functions are objects in classical probability theory, while vacuum expectation values describe the quantum mechanics of fields.

One can regard the relation between the analytically continued expectations (in time) to probability theory, as generalizing the "Feynman-Kac formula." This formula gives a Wiener integral representation that provides solutions to the heat-diffusion equation, namely the analytic continuation of the Schrödinger equation. One side of the equation is purely classical; the other side is an analytic continuation (justified by the positivity of the energy) of quantum theory.

An immediate application of RP was to provide a framework for the first existence proof for examples of non-linear quantum field theories with scattering [12]. These are quantum, non-linear, scalar wave equations with energy density $\lambda\mathcal{P}(\varphi)_2$. Here $\lambda\mathcal{P}$ is a polynomial bounded from below, and the subscript 2 denotes two-dimensional space time. **Theorem:** *For $0 < \lambda$ sufficiently small, these models exist and satisfy all the Wightman axioms for a quantum field theory.* The models

[1] This is based on the opening talk on November 20, 2017 at the conference "Reflection Positivity," held at the Mathematical Research Institute, Oberwolfach, Germany.

actually satisfy as well the more-detailed Haag-Ruelle axioms for scattering, which include the existence of an lower and upper mass gap, leading to an isolated eigenvalue m in the mass spectrum. In addition they yield local algebras that satisfy the Haag-Kastler axioms. For these reasons the discovery of RP marked a turning point in the study of relativistic quantum physics.

Amazingly, the RP property not only arises in quantum physics. But RP connects to areas of research in fields ranging from operator algebras, mathematical analysis, probability, and representation theory on one hand, to statistical physics and the study of phase transitions on the other. RP has enabled the proofs of numerous interesting and deep results in far-flung areas of mathematics and physics. Similar positivity conditions appear in other subjects, so one might dream that new areas of relevance for reflection positivity will appear in the future, in other mathematical areas.

Since RP is now an enormous field, I apologize in advance for my sparse list of references; these are only meant to be personal and impressionistic, and they cover a small selection of papers from an enormous universe of possibilities. However, I am confident that the other speakers during this week will fill the gap by citing many other important papers.

1. The Background Story

The analytic continuation of the expectations of fields to imaginary time was proved in the general framework of Wightman in 1956 [53]. This result is a consequence of the assumption that the spectrum of the energy H is non-negative, and that the time-translation automorphism arises from the unitary group e^{itH} acting on the Hilbert space $\mathcal{H}$ of quantum theory. Furthermore one assumes that the Poincaré group acting on $\mathcal{H}$ has an invariant vector (a vacuum).

This analytic continuation is the non-perturbative parallel to "Wick rotation" of time introduced in the physics literature to make sense of the terms in the perturbation expansion of a Lorentz-invariant theory [51]. Jost studied properties of analytically continued Wightman functions to Schwinger functions, including their symmetry at imaginary time, as detailed in his 1957 book [30]. Schwinger emphasized the notion of Euclidean quantum field theory in his well-known 1958 paper [46]. According to Miller [37], Schwinger claimed to have discovered these results about Euclidean fields seven years before he published them.

In any case, the advent of Euclidean expectation values and Euclidean fields set the stage for the pioneering work of Kurt Symanzik, who proposed in 1964 that the Euclidean field could be formulated as the classical Markov field. Symanzik described these ideas in preliminary form in his widely-circulated Courant Institute Report [47] and in the paper [48]. He also presented a monumental course in his 1968 Varenna summer school lectures [49].

The Varenna lectures mark the end of Symanzik's research on Euclidian quantum field theory, which he explains in his introductory lecture. There he estimates the efficacy of the different approaches known at the time for constructing an example of an interacting field. Unfortunately Symanzik under-estimated the value of his own approach. In fact when it was understood some five years later, it turned out to play an important role.

Symanzik's approach cried out for building a mathematical foundation by which to understand it. His vision appeared to be a promising blueprint for finding a non-trivial field theory. Symanzik made a valiant attempt to implement his program for the φ^4 theory in two space-time dimensions, by working on this problem with S.R.S. Varadhan. But the gap between their results (described in the appendix to [49]), and the much more extensive results necessary to establish the existence of an interacting Euclidean field theory—even in two dimensions—remained impossible to bridge at the time.

Furthermore, even if one could solve that problem of constructing the probability distribution for a Euclidean classical field, there remained another fundamental question: how can one relate a classical Euclidean field to a relativistic quantum field? In other words, as there is no Hamiltonian for the Euclidean classical field, how can one justify analytically continuing back from Euclidean time to real time? Namely, how can one justify an inverse Wick rotation? In the rest of this work I will call this question the *reconstruction problem*; similarly I call a solution to this problem a *reconstruction theorem*.

How to prove a reconstruction theorem baffled mathematical physicists during my student days, and it became the center of much discussion. But there were no viable suggestions of how to resolve it. The same question also baffled particle physicists, including Schwinger[2] who regarded it as important. This latter statement is documented in Schwinger's response to a question asked by Pauli about understanding spin and statistics in the Euclidean framework; it followed Schwinger's talk at a 1958 CERN conference. "The question of to what extent you can go backwards, remains unanswered, i.e. if one begins with an arbitrary Euclidean theory and one asks: when do you get a sensible Lorentz theory? This I do not know. The development has been in one direction only; the possibility of future progress comes from the examination of the reverse direction, and that is completely open."

The first big step in the reconstruction from Euclidean to real-time was achieved in 1972 by Edward Nelson. He had been fascinated by Symanzik's Markov field approach, and formulated that framework in mathematical terms. Nelson discovered a reconstruction theorem that starts from a Markov field and yields a scalar bosonic quantum field [38]. He then showed that the free (Gaussian) Euclidean field satisfies the hypotheses of that construction [39].

[2] In those days, discussions after lectures at important conferences were recorded or transcribed; then they were published along with the lectures in the conference proceedings. In perspective, the discussions are often of greater interest than the talks. I was grateful to learn in [37] about the reference [45], where one can read this fascinating exchange. The book that I quote can be found online by following the hyperlink to CERN, and the quoted passage appears at the end of page 140.

2. The Discovery of Reflection Positivity

This was the situation in the summer of 1972 when I began to read the preprint of Nelson's reconstruction paper. This led to several discussions with Konrad Osterwalder on the topic. Not only did I wish to understand Nelson's construction, but I hoped that one could discover a more robust reconstruction method that also applied to fermions. In the Euclidean world fermions are Grassmann, rather than abelian, so they did not appear to fit into a Markov setting.

Furthermore, the Markov property that Nelson assumed was a strong one, and it had not been proved for the examples of two-dimensional interacting fields known at the time, although Nelson could verify it in the case of the free field. Konrad was looking for an interesting project to work on, and I thought a new way to do reconstruction could be very fruitful. I was deeply engrossed at the time in trying to understand aspects of the three-dimensional φ^4 theory.

Meanwhile Robert Schrader was on vacation, visiting his friend (later wife) in Germany. Since one could not park a car on the street for more than one night near the Cambridge address where Robert lived, I had suggested that he leave his car in Northampton, Massachusetts. This is a small college town where my wife worked, and I knew that there were no overnight-parking laws like in Cambridge. Just after Robert returned, I planned to drive to Northampton, and from near there to fly to Chicago for a conference. So I took Robert with me to pick up his car, and for nearly two hours we discussed the reconstruction project while I drove.

I explained to Robert what Konrad and I had been thinking about, and I encouraged Robert to think more about the question. But Robert immediately resisted. He too had seen Nelson's paper and thought that Nelson's construction was very natural, leaving little new to be discovered. As the trip progressed, I was getting more and more upset—and afterward I recalled worrying about paying too little attention to the road. For not only did I feel that the question was extremely interesting, but I really hoped that Robert and Konrad could make some progress on the problem while I was in Chicago. Finally I seemed to break through, only shortly before we arrived at Robert's car.

About one week later, when I returned to Cambridge, Massachusetts, I found Konrad and Robert enthusiastic, proud, and delighted. They thought that they knew the key to a new reconstruction method: they had discovered the reflection positivity property! A reflection-invariant functional ω on an algebra $\mathfrak{A} = \mathfrak{A}_- \otimes \mathfrak{A}_+$ is reflection positive on $\mathfrak{A}_+ \ni A$ with respect to the antilinear reflection homomorphism Θ mapping $\mathfrak{A}_+ \mapsto \mathfrak{A}_-$, if $0 \leqslant \omega(\Theta(A)A)$. And in their example, the RP form provided the inner product space a Hilbert space $\mathcal{H}$ for quantum theory. Their original celebrated publication [42] appeared a few months later. This is the statement for bosons; more generally one uses a twisted product $\Theta(A) \circ A$ that reduces to $\Theta(A)A$ for bosons, see [42, 18, 20].

The RP form actually provides the quantization of the classical system. If ω is invariant under a time-translation *-automorphism σ_t for $t \in \mathbb{R}$, and α_t acts on $\mathfrak{A}_+$ for $t \geqslant 0$, then the quantization of α_t yields a positive Hamiltonian H on

$\mathcal{H}$. The importance of RP in quantum physics stems from the fact that the inner product and the Hilbert space for almost every quantum mechanics or relativistic field theory arises from the quantization of some classical system that has the reflection-positivity property. Often one calls this property "Osterwalder-Schrader positivity."

It was discovered that the original Osterwalder-Schrader paper contained an error involving continuity for a tensor product of distributions. Although some persons claimed that the gap was serious, I was sure that the problem was minor; that is what turned out to be the case. They corrected their method in a second paper [43], where they also gave a stronger version of their reconstruction theorem–showing the equivalence of their assumptions to a modified version of Wightman's theory.

It is also meaningful that Edward Nelson explained in his 1973 Erice Lectures [40], how his Markov field fits into the Osterwalder-Schrader (OS) framework, even though at that point the correction to the OS method had not yet been found [41]. But by that time, Nelson had come to regard the OS reconstruction as the natural way to pass from Euclidean theory to quantum theory.

3. The Proliferation of Results Related to Reflection Positivity

Since the appearance of RP in quantum field theory, it has permeated many other fields. Here we only give an impressionistic view. Robert Schrader left his position as postdoctoral fellow at Harvard in 1973 to go to Germany. His successor as postdoctoral fellow was Jürg Fröhlich, who became one of the first persons to investigate RP in great detail. Right in the beginning he made an interesting study of the generating functionals and the reconstruction theorem via RP-functional integrals, both the zero-temperature and also finite-temperature fields [6]. I refer the reader to the notes of his presentation at this meeting for several of the many other perspectives that Jürg pursued [7].

It also turns out that RP has close ties with preceding work: including with Widder's investigation of the Laplace transform in the 1930's [52], as was discovered by Klein and Landau [32]. Furthermore, the reflection Θ in RP is related in various ways to the reflection J in the Tomita-Takesaki theory of operator algebras [50], as well as to other reflections in its predecessors. Some references on these latter connections can be found in the introduction to [19] and of course the talks of Fröhlich [7] and of Longo [36].

3.1. **Phase Transitions and Symmetry Breaking.** Physicists believed for many years in symmetry breaking and vacuum degeneracy for certain quantum field theories. The idea of the proof goes back to Peierls' analysis of phase transitions in the Ising model in 1936, but it was completed mathematically by Griffiths and by Dobrushin some thirty years later.

One expects that a $\lambda^2(\varphi^2 - 1/\lambda^2)^2$ field theory with a "W-shaped" potential will also have a phase transition for the real parameter λ sufficently small. In this case the field will behave much like an Ising system at low temperature: the average of the field will be approximately localized near the two minima of the potential at $\varphi = \pm\lambda$ that are separated by a large potential barrier of height λ^{-2} at $\varphi = 0$. This became a much-sought-after result, which was announced without proof in 1973 by Dobrushin and Minlos in a widely-cited paper [3].

Another early application of RP that one might not have anticipated, was the important use of the Schwarz inequality arising from RP to establish global estimates of local perturbations—by reflecting the perturbation multiple times. In particular this was an important component in the first mathematical proof of the existence of symmetry breaking and vacuum degeneracy in quantum field theory. One used multiple-reflection RP bounds in $\lambda\varphi^4$ theory to estimate the deviation between local fluctuations in that model from a Peierls' type estimate, yielding a degenerate ground state for λ sufficiently large [13].

3.2. **Statistical Physics.** RP played a major role in lattice statistical physics. There is a very large literature, with early work by Fröhlich, Simon, and Spencer [10] on continuous symmetry breaking, by Fröhlich, Israel, Lieb, and Simon [8] on establishing RP, and by Dyson, Lieb, and Simon [4] on establishing phase transitions in quantum spin systems. Lieb used these methods to analyze the ground state vortices in some models [34], see also [2]. One can consult the review of Biskup [1] for much other work.

3.3. **Relations to Mathematics.**

3.3.1. *Representation Theory.* The first work relating RP to representation theory for quantum fields arose from the desire to obtain representations of the Poincaré group from the analytic continuation of quantization of the representations of the Euclidean group. After the initial work of Osterwalder and Schrader [42, 43], this was investigated abstractly by Fröhlich, Osterwalder, and Seiler [9] and by Klein and Landau [32, 33].

This was eventually developed into an entire subfield of representation theory and stochastic analysis studied by Klein and developed extensively by Palle Jorgensen, Karl-Hermann Neeb, and Gestur Ólafsson, as well as their collaborators. See [31, 28, 29] and the citations in these papers.

3.3.2. *Relations to PDE.* For a free scalar field, the RP property is equivalent to a statement about monotonicity of the Green's functions with respect to a change from Dirichlet to Neumann boundary conditions. Let $C = (-\Delta + 1)^{-1}$ denote the Green's operator for the Laplacian on $\mathbb{R}^d$, and let C_D denote the Green operator obtained by imposing vanishing Dirichlet boundary conditions on the time-zero hyperplane. Let C_N denote the corresponding Neumann Green's function for vanishing normal derivatives on the time-zero hyperplane. Then RP is equivalent to the following statement of operator monotonicity on $L^2(\mathbb{R}^d)$,

$$C_D \leqslant C_N \ .$$

This can be seen by expressing C_D and C_N using the method of "image charges," familiar in physics [11]. One can generalize this to obtain a condition for RP on reflection-invariant spaces $\Sigma = \Sigma_- \cup \Sigma_0 \cup \Sigma_+$ with an involution Θ that exchanges $\Sigma_\pm$ and leaves Σ_0 fixed.

3.3.3. *Fourier Analysis and the Inequalities.* For the approach to classical Fourier analysis through RP, see the contribution of Frank to these proceedings [5]. It is interesting that it is possible to prove Fourier bounds in subfactor theory to prove uncertainty principles, and this is related to the picture analysis that I describe in §4 below [25, 26]. This area of research on the analytical inequalities for pictures in a non-commutative algebra is just emerging; it seems to offer potential exciting new insights.

3.3.4. *Relations to Tomita-Takesaki Theory and the KMS Property.* This is another enormous subject. The reflection Θ arises in several different ways as the Tomita-Takesaki reflection operator J. It is also related to the KMS property, discovered by Haag, Hugenholtz, and Winnink [14].

3.3.5. *Stochastic Quantization.* The study of classical stochastic PDE's has received a great deal of attention recently. Physicists have proposed that a classical equation with a white noise linear forcing term can be used to quantize a classical equation—the method of *stochastic quantization.* The quantum distribution arises as the limit of the distribution of solutions at infinite stochastic time. Unfortunately reflection positivity does not hold for the distribution of the stochastic field at any finite stochastic time [15].[3] Therefore it is difficult to see how to use this method to obtain a limit that satisfies RP–except when one can analyze the limit in closed form.

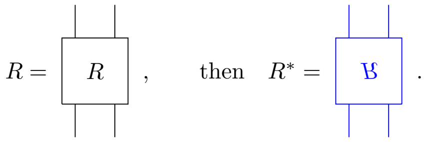
4. Some Elementary Remarks on the Mathematics of Pictures

When Zhengwei Liu came to Harvard as a postdoctoral fellow in the summer of 2015, we spent a good deal of time telling each other about each of our current areas of research. After a while we realized that we could combine our work, by defining a pictorial framework that we called "planar para algebras." This led us to a new way to think about the RP property. We implemented all this in the framework of statistical mechanics models of parafermions on a lattice—a setting we call PAPPA. In these examples, we found a geometric interpretation and proof of RP [20] for a very wide class of Hamiltonians. We then found that our models could be viewed as languages; we had first used this principle in joint work with Alex Wozniakowski on quantum information [22, 20], and later found that it provides very interesting insights into other fields of mathematics [21]. Hopefully they will also contribute in the future to physics.

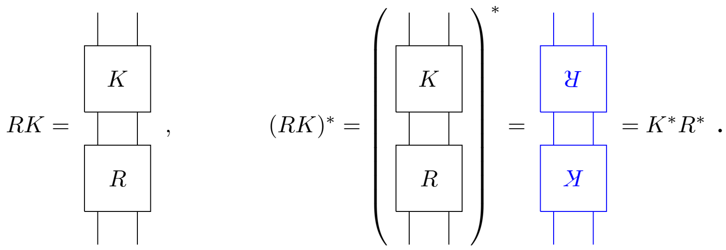
Parenthetically in the case of parafermions of degree $d = 2$, the fermionic case, elementary parafermions are called Majoranas. As quadratic functions of Majoranas represent classical and quantum lattice spins, the RP for parafermions leads to RP proofs in these cases [24, 18, 19].

[3]We have only proved this for a linear field. One sees in this case that RP does hold in the infinite-stochastic-time limit. If the Wightman functions for a non-linear theory are continuous (as expected) in the perturbation parameter, then RP will also not hold at finite stochastic time. In that case, however, one needs an independent method to establish RP for the limit.

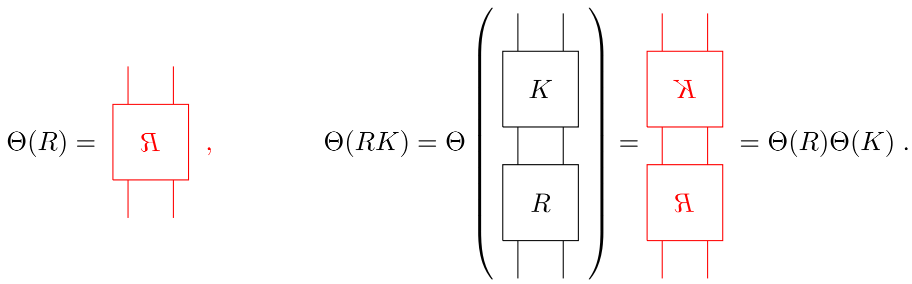
I believe that the simple fundamental idea embodied in our new pictorial approach to RP will prove fruitful in many other contexts. Let me explain some central ideas that lead to RP, without much elaboration. Planar algebra is based on the mathematics of a $*$-algebra of pictures. Multiplication is vertical composition of pictures, with algebraic right-to-left order represented by top-to-bottom composition of pictures. For a picture T, one represents the adjoint map $T \mapsto T^*$ in the algebra by vertical reflection of the picture. If



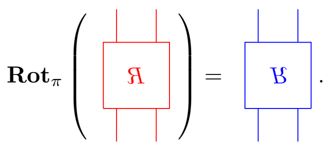
Composed with multiplication, the vertical reflection gives an anti-linear anti-homomorphism that reverses the order of multiplication of pictures,



It is natural to consider a second reflection Θ that is horizontal. This reflection defines an anti-linear homomorphism of pictures,

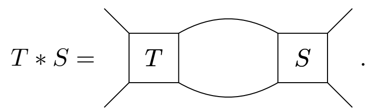


The basic idea of the geometric notion of RP arises from the fact that the two Θ-reflection of pictures is related to the *-reflection of pictures by a 180-degree rotation that we donote $\mathbf{Rot}_\pi$. Algebraically $\mathbf{Rot}_\pi(\Theta(R)) = R^*$, while in pictures,

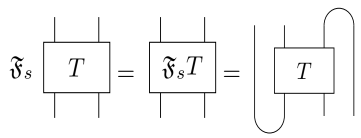


The horizontal reflection is the reflection for RP. The vertical reflection, on the other hand is associated with the fact that R^*R is positive, if the pictures have a positive expectation defining a non-negative form that can be used to define an inner product and a Hilbert space through the GNS representation.

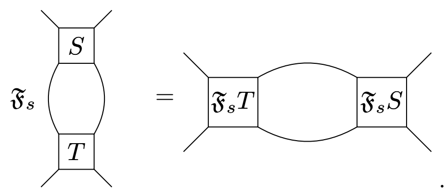
In order to understand the geometric proof of RP, one needs to analyze the horizontal multiplication of pictures. This horizontal multiplication defines the convolution product of operators,



As there are two types of multiplication, rotation of pictures by 90 degrees is a very important transformation. Denote this by $\mathfrak{F}_s$; in terms of the input and output strings, this permutes them cyclically by one:



We named $\mathfrak{F}_s$ the *string Fourier transform* (SFT) as it generalizes the normal Fourier series. The SFT sends a product of neutral 1-qudit transformations to the convolution of two SFT's,



For a transformation with two input and two output strings this is a rotation by 90 degrees.

One needs a state on the algebra of pictures, in order to apply the GNS construction and to recover the identity of pictures with elements of a Hilbert space. This arises from attaching the input strings to the output strings, and it defines a trace functional on the algebra of pictures.

4.1. **The Geometric Interpretation of Reflection Positivity.** One visualizes the new geometric proof of RP from looking at Figure 1. This illustrates how one relates the horizontal reflection $\Theta(R)$ of R with its vertical reflection R^*. The circle product has the pictorial meaning that $\Theta(R)$ and R appear at the same vertical level.

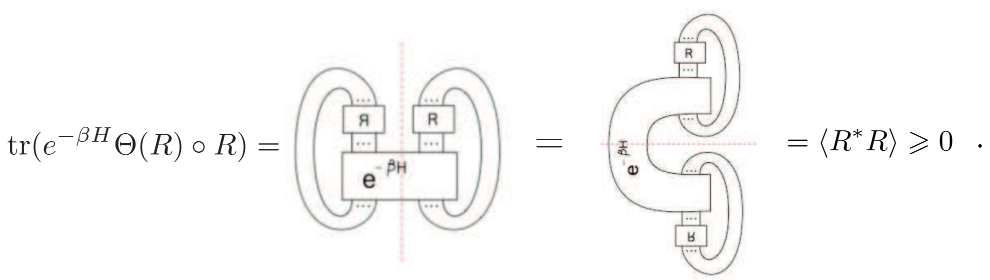


FIGURE 1. The Picture Proof of Reflection Positivity.

One then rotates part of the picture to obtain an expectation of the positive Hilbert-space expectation of the quantity R^*R. In the picture we see that the positive-temperature expectation of $\Theta(R) \circ R$ is given by another expectation of R^*R, and this expectation also involves the rotation of the Hamiltonian H.

There are many technical points, and in [20] you will find the complete proof as well as the explanation of all the details. One must begin by proving that our pictorial language is well-defined as a mathematical tool, and that it maps onto the objects of interest in mathematical physics. For this we implemented the parafermion algebra as an example of our model. We call the example PAPPA (for parafermion planar para algebra).

One must establish the para-isotopy invariance of our pictures. It is necessary to show that our expectation of pictures is a trace, and that it is positive. In this way one can use the GNS construction to obtain a Hilbert space representation of the pictures.

This led us to prove a variation of Jones' famous index theorem [27] for the quantization of the quantum dimension. Finally we had to understand how the one-string rotation $\mathfrak{F}_s$ of a picture (the SFT of the picture) is actually a generalization of the Fourier transform of the function the picture represents. One needs to pin down the relation between the SFT and reflection invariance of H leading to RP. It turns out that we can require that $\mathfrak{F}_s(H)$ is positive, as it occurs in the expectation $\langle \cdot \rangle$ in Figure 1. It is not necessary that the Hamiltonian itself be positive for RP to hold, but only that its SFT is positive, This can be expressed as H having a hermitian expansion in terms of a natural basis.

Having established the RP result for parafermions, one can ask how this relates to RP for classical or quantum spin systems. Many spin systems reduce to the case for Majoranas, namely parafermions of degree $d = 2$. An extensive analysis of the relation between RP for Majoranas and RP for classical and quantum spin systems can be found in [18].

Acknowledgement

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Direct construction of pointlike observables in the Ising model

Daniela Cadamuro

Relativistic quantum field theories are described by their set of local observables. These are linear bounded or unbounded operators associated with regions of Minkowski space. They form $*$-algebras that are expected to satisfy, e.g., the Haag-Kastler axioms, which are relevant to their interpretation as physical "measurements".

The problem of constructing models of quantum field theory, i.e., exhibiting algebras of local observables with such properties, is a notoriously hard task due to the complicated structure of local observables in the presence of interaction.

Quantum integrable models in $1+1$-dimensional Minkowski space are simplified models of interaction, rendering the mathematical structure of quantum field theory more accessible. In these models, the scattering of n particles is the product of two particle scattering processes, namely the S-matrix is said to be "factorizing", a property connected to integrability. Examples include the Ising model, the $O(N)$ nonlinear sigma models and the Sine-Gordon model.

We are interested in studying the content of local observables in these theories. This can be investigated in various mathematical frameworks: as Wightman fields [1], as algebras of bounded operators [2], or as closed operators affiliated with those algebras. For example, the task of constructing the Wightman n-point functions in integrable models from a given S-matrix has been widely studied, see, e.g., [3], but convergence of the associated series expansions has not been established so far, despite some progress [4].

An alternative approach considers fields localized in unbounded wedge-shaped regions as an intermediate step to the construction of sharply localized objects, which is handled indirectly [5, 6, 7, 8], thus avoiding explicit computation of point-like fields. The existence proof of local observables is reduced to an abstract condition on the underlying wedge algebras. While the generators of the wedge algebras are explicitly known, the passage to the von Neumann algebras includes the weak limit points of this set. These limit points include the elements of local algebras, but of these much less is known. Our task is to gain more information on the structure of these local observables.

For that, we characterize the local observables in terms of a family of coefficient functions $f^{[A]}_{m,n}$ in the following series expansion [9]:

$$A = \sum_{m,n=0}^{\infty} \frac{d^m\boldsymbol{\theta} d^n\boldsymbol{\eta}}{m!n!} f^{[A]}_{m,n}(\boldsymbol{\theta},\boldsymbol{\eta}) z^\dagger(\theta_1)\cdots z^\dagger(\theta_m) z(\eta_1)\cdots z(\eta_n), \tag{1}$$

where $z^\dagger, z$ are "interacting" creators and annihilators fulfilling a deformed version of the CCR relations which involves the scattering function.

Due to the form of this expansion, local observables are defined as quadratic forms in a suitable class. We denote $\mathcal{H}^{\omega,f}$ the dense space of finite particle number states Ψ fulfilling the condition $\|e^{\omega(H/\mu)}\Psi\| < \infty$, where H is the Hamiltonian, $\mu > 0$ is the mass, and $\omega : [0,\infty) \to [0,\infty)$ is a function with the properties of [10, Definition 2.1]. In (1), A is a quadratic form on $\mathcal{H}^{\omega,f} \times \mathcal{H}^{\omega,f}$ such that

$$\|Q_k A e^{-\omega(H/\mu)} Q_k\| + \|Q_k e^{-\omega(H/\mu)} A Q_k\| < \infty$$

for any $k \in \mathbb{N}_0$, where Q_k is the projector onto the space of k or fewer particles. We denote this class of quadratic forms by $\mathcal{Q}^\omega$.

In order to characterize the coefficients $f^{[A]}_{m,n}$ in terms of the localization of A in spacetime, we need a notion of locality which is adapted to quadratic forms in the class $\mathcal{Q}^\omega$: We say that $A \in \mathcal{Q}^\omega$ is ω*-local* in the double cone $\mathcal{O}_{x,y} := \mathcal{W}_x \cap \mathcal{W}'_y$ (where $\mathcal{W}_x$ denotes the right wedge with edge at x and $\mathcal{W}'_y$ the left wedge with edge at y, with x to the left of y) if and only if $[A, \varphi(f)] = [A, \varphi'(g)] = 0$ for all $f \in \mathcal{D}^\omega(\mathcal{W}'_y)$ and all $g \in \mathcal{D}^\omega(\mathcal{W}_x)$, as a relation in $\mathcal{Q}^\omega$. Here φ, φ' are the left and right wedge-local fields, respectively, $\mathcal{D}^\omega(\mathcal{W}_x)$ is the space of smooth functions compactly supported in $\mathcal{W}_x$ with the property that $\theta \mapsto e^{\omega(\cosh\theta)} f^\pm(\theta)$ is bounded and square integrable ($f^\pm$ is positive and negative frequency part of the Fourier transform, respectively.)

The notion of ω-locality is weaker than the usual notion of locality in the net of C^*-algebras $\mathcal{A}(\mathcal{O}_{x,y})$. It does not imply that A commutes with unitary operators

$e^{i\varphi(f)^-}$, or with an element $B \in \mathcal{A}(\mathcal{W}_x)$: if A is just a quadratic form, it would not be possible to write down these commutators in a meaningful way. We therefore clarify how ω-locality is related to the usual locality:

Proposition 1.

(i) Let A be a bounded operator; then A is ω-local in $\mathcal{O}_{x,y}$ for some $x, y \in \mathbb{R}^2$ if and only if $A \in \mathcal{A}(\mathcal{O}_{x,y})$.

(ii) Let A be a closed operator with core $\mathcal{H}^{\omega,f}$, and $\mathcal{H}^{\omega,f} \subset \operatorname{dom} A^*$. Suppose that

$$\forall g \in \mathcal{D}^\omega_{\mathbb{R}}(\mathbb{R}^2) \;:\; \exp(i\varphi(g)^-)\mathcal{H}^{\omega,f} \subset \operatorname{dom} A. \qquad (*)$$

Then A is ω-local in $\mathcal{O}_{x,y}$ if and only if it is affiliated with $\mathcal{A}(\mathcal{O}_{x,y})$.

(iii) In the case $S = -1$, statement (ii) is true even without the condition $(*)$.

This proposition gives criteria for affiliation of closed operators to local algebras, but in examples, closability of a quadratic form A is difficult to characterize in terms of the coefficients in the expansion (1). Moreover, not much is known about the domain of the closed operator. We therefore look for sufficient (but not necessary) conditions that allow to apply Proposition 1. We will understand (1) as an absolutely convergent sum on a certain domain, using summability conditions on the norms of the coefficients $f^{[A]}_{m,n}$. The following proposition provides a sufficient criterion for closability of A as an operator:

Proposition 2. *Let $A \in \mathcal{Q}^\omega$. Suppose that for each fixed n,*

$$\sum_{m=0}^{\infty} \frac{2^{m/2}}{\sqrt{m!}} \left(\| f^{[A]}_{m,n} \|^\omega_{m\times n} + \| f^{[A]}_{n,m} \|^\omega_{n\times m} \right) < \infty.$$

Then, A extends to a closed operator A^- with core $\mathcal{H}^{\omega,f}$, and $\mathcal{H}^{\omega,f} \subset \operatorname{dom}(A^-)^*$.

To apply Proposition 1 we therefore need to fulfill the condition in Proposition 2 and to show ω-locality of A. Hence, we formulate the ω-locality condition in terms of properties of the functions $f^{[A]}_{m,n}$. This is the content of [10, Theorem 5.4], which we summarize briefly: A is localized in the standard double cone $\mathcal{O}_r$ of radius r if and only if the coefficients $f^{[A]}_{m,n}$ are boundary values of meromorphic functions $(F_k)_{k=0}^\infty$ on $\mathbb{C}^k$ (with $k = m + n$) with a certain pole structure, which are S-symmetric, S-periodic, and fulfill certain bounds in the real and imaginary directions, depending on ω and r, and which fulfill the *recursion relations*

$$\operatorname{res}_{\zeta_n - \zeta_m = i\pi} F_k(\boldsymbol{\zeta}) = -\frac{1}{2\pi i} \Big(\prod_{j=m}^{n} S(\zeta_j - \zeta_m) \Big) \Big(1 - \prod_{p=1}^{k} S(\zeta_m - \zeta_p) \Big) F_{k-2}(\hat{\boldsymbol{\zeta}}).$$

The problem is now to find examples of functions $(F_k)_{k=0}^\infty$ fulfilling the above conditions of ω-locality and closability via Proposition 2. In the case $S = -1$ (Ising model) this is possible, and we aim at constructing a large enough set of observables so that they have the Reeh-Schlieder property.

To that end, let $k \geq 0$, let $g \in \mathcal{D}(\mathcal{O}_r)$ with some $r > 0$, and let P be a symmetric Laurent polynomial of $2k$ variables. We define the analytic functions

$$F_{2k}^{[2k,P,g]}(\boldsymbol{\zeta}) := \tilde{g}(p(\boldsymbol{\zeta}))P(e^{\boldsymbol{\zeta}}) \sum_{\sigma \in \mathfrak{S}_{2k}} \operatorname{sign} \sigma \prod_{j=1}^{k} \sinh \frac{\zeta_{\sigma(2j-1)} - \zeta_{\sigma(2j)}}{2}, \tag{2}$$

and $F_j^{[2k,P,g]} = 0$ for $j \neq 2k$. For these the properties above hold with respect to this r and for example with $\omega(p) := \ell \log(1 + p)$ for some $\ell > 0$.

Another example, involving the F_j for odd j, is the non-terminating sequence

$$F_{2j+1}^{[1,P,g]}(\boldsymbol{\zeta}) := \frac{1}{(2\pi i)^k} \tilde{g}(p(\boldsymbol{\zeta}))P_{2j+1}(e^{\boldsymbol{\zeta}}) \prod_{1 \leq \ell < r \leq 2j+1} \tanh \frac{\zeta_\ell - \zeta_r}{2}, \tag{3}$$

where $g \in \mathcal{D}^\omega(\mathcal{O}_r)$, and $P = (P_{2j+1})_{j=0}^\infty$ are symmetric Laurent polynomials in $2j+1$ variables such that $P_{2j+1}(p, -p, \mathbf{q}) = P_{2j-1}(\mathbf{q})$. We set $F_{2j}^{[1,P,g]} = 0$. Also for these F_j, the properties above hold with respect to r and $\omega(p) = p^\alpha$ with $\alpha \in (0, 1)$.

Hence in both examples the associated quadratic form A given by (1) is ω-local in the double cone $\mathcal{O}_r$. Additionally, the families of functions fulfill the condition of Proposition 2, which implies that A extends to a closed operator affiliated with the local algebras $\mathcal{A}(\mathcal{O}_r)$. This is in fact trivial for $(F_j^{[2k,P,g]})_{j=0}^\infty$ as the sequence terminates; but for $(F_j^{[1,P,g]})_{j=0}^\infty$, it involves careful norm estimates of a sequence of singular integral operators, as one is concerned precisely with the boundary values of the function at the poles of the hyperbolic tangent.

Further, by choosing different polynomials P we can generate a large set of observables which has the Reeh-Schlieder property.

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Quadratic Hamiltonians and their renormalization

Jan Dereziński

Quantum bosonic quadratic Hamiltonians, or *bosonic Bogoliubov Hamiltonians* are formally given by expressions of the form

$$\hat{H} \quad = \quad \sum h_{ij}\hat{a}_i^*\hat{a}_j + \frac{1}{2}\sum g_{ij}\hat{a}_i^*\hat{a}_j^* + \frac{1}{2}\sum \bar{g}_{ij}\hat{a}_i\hat{a}_j + c, \tag{1}$$

where $h = [h_{ij}]$ is a Hermitian matrix, $g = [g_{ij}]$ is a symmetric matrix, c is an arbitrary real number (possibly, infinite!) and $\hat{a}_i^*, \hat{a}_j$ are the usual bosonic creation/annihilation operators. They are often used in quantum field theory to describe free theories interacting with a given external classical field [7, 3].

Bogoliubov Hamiltonians that are bounded from below are especially useful. Their infimum $E := \inf \hat{H}$ is often interesting physically.

Bogoliubov Hamiltonians have a surprisingly rich mathematical theory [1, 2, 6, 8, 4]. In infinite dimension this theory sometimes involves interesting pathologies. For instance, $\hat{H}$ is often ill defined, but one can define its "infimum" E. In some situations, one needs to perform an infinite renormalization in order to define $\hat{H}$, or at least to compute E. This is typical for Bogoliubov Hamiltonians that are motivated by relativistic quantum field theory [3]. The most popular choice is probably $c = 0$, corresponding to the *normally (Wick) ordered Hamiltonian*. It will be denoted $\hat{H}^{\mathrm{n}}$. The choice $c = \frac{1}{2}\sum_i h_{ii}$, which we call the *Weyl Bogoliubov Hamiltonian* and denote $\hat{H}^{\mathrm{w}}$, has its advantages as well. In some situations, however, one needs to consider other quantizations, where the constant c may turn out to be infinite, and can be viewed as a renormalization counterterm. One particular possibility, which we call the *second order renormalized quantization* and denote $\hat{H}^{\mathrm{2ren}}$, plays an important role in Quantum Field Theory in $1 + 3$ dimensions. In the language of Feynman diagrams $\hat{H}^{\mathrm{2ren}}$ corresponds to discarding loops of order 2 or less.

We will use the following notation for the infimum of the three main Bogoliubov Hamiltonians that we discuss:

$$E^{\mathrm{w}} := \inf \hat{H}^{\mathrm{w}}, \quad E^{\mathrm{n}} := \inf \hat{H}^{\mathrm{n}}, \quad E^{\mathrm{2ren}} := \inf \hat{H}^{\mathrm{2ren}}. \tag{2}$$

Physicists often compute the vacuum energy without worrying whether the corresponding quantum Hamiltonian is well defined as a self-adjoint operator. Following this philosophy, we may consider E^{n} or E^{2ren} under conditions that are more general than the conditions for the existence of the corresponding Hamiltonians.

The Weyl Hamiltonian $\hat{H}^{\mathrm{w}}$ is the most natural. In fact, it is invariant wrt symplectic transformations. Unfortunately, it is often ill defined.

The normally ordered Hamiltonian $\hat{H}^{\mathrm{n}}$ is naturally defined given a Fock representation.

As the first example, consider the neutral massive scalar quantum field $\hat{\varphi}(\vec{x})$. Its conjugate field is denoted $\hat{\pi}(\vec{x})$ with the usual equal time commutation relations

$$\begin{aligned} [\hat{\varphi}(\vec{x}), \hat{\varphi}(\vec{y})\} = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] &= 0, \\ [\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] &= \mathrm{i}\delta(\vec{x}-\vec{y}). \end{aligned} \tag{3}$$

The free Hamiltonian is defined in the standard way:

$$\hat{H}_0^{\mathrm{n}} := \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\hat{\varphi}(\vec{x})\big)^2 + \frac{1}{2}m^2\hat{\varphi}^2(x)\Big):\mathrm{d}\vec{x}, \tag{4}$$

where the double dots denote the normal ordering.

Suppose that the mass squared is perturbed by a Schwartz function $\kappa(\vec{x})$. One can check that the normally ordered full Hamiltonian does not exist. However, the 2nd order renormalized Hamiltonian is well-defined. Formally, it can be written as

$$\hat{H}^{2\mathrm{ren}} := \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\hat{\varphi}(\vec{x})\big)^2 + \frac{1}{2}(m^2 + \kappa(\vec{x}))\hat{\varphi}^2(\vec{x})\Big):\mathrm{d}\vec{x} - E_2, \tag{5}$$

where the infinite counterterm E_2 is the contribution of loop diagram with 2 vertices.

(5) is well-defined, but physically somewhat artificial. To obtain a physically more satisfactory Hamiltonian, one needs to perform an additional finite subtraction, adding to (5) E_2^{ren}, the renormalized value of E_2. The renormalization can be performed with help of any method described in textbooks of QFT, e.g. by the Pauli-Villars method, by dispersion relations or by dimensional regularization. All these methods are equivalent and one obtains a renormalized Hamiltonian with only *local* counterterms, which formally can be written as

$$\hat{H}^{\mathrm{ren}} := \int :\Big(\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}\big(\vec{\partial}\hat{\varphi}(\vec{x})\big)^2 + \frac{1}{2}(m^2 + \kappa(\vec{x}))\hat{\varphi}^2(\vec{x})\Big):\mathrm{d}\vec{x} - C\int |\kappa(\vec{x})|^2\mathrm{d}\vec{x}, \tag{6}$$

where C is infinite. This example is discussed in detail in Chap. III Subsect. C14 of [3], see also [5].

The second example is more singular. Consider the charged massive scalar quantum field $\hat{\psi}(\vec{x})$, with $\hat{\psi}^*(\vec{x})$ denoting its Hermitian adjoint. The conjugate field will be denoted $\hat{\eta}(\vec{x})$, so that we have the commutation relations

$$[\hat{\psi}(\vec{x}), \hat{\psi}(\vec{y})] = [\hat{\psi}(\vec{x}), \hat{\eta}(\vec{y})] = [\hat{\eta}(\vec{x}), \hat{\eta}(\vec{y})] = 0, \tag{7}$$

$$[\hat{\psi}(\vec{x}), \hat{\eta}^*(\vec{y})] = [\hat{\psi}^*(\vec{x}), \hat{\eta}(\vec{y})] = \mathrm{i}\delta(\vec{x}-\vec{y}). \tag{8}$$

The free Hamiltonian is of course

$$\hat{H}_0^{\mathrm{n}} = \int :\Big(\hat{\eta}^*(\vec{x})\hat{\eta}(\vec{x}) + \vec{\partial}\hat{\psi}^*(\vec{x})\vec{\partial}\hat{\psi}(\vec{x}) + m^2\hat{\psi}^*(\vec{x})\hat{\psi}(\vec{x})\Big):\mathrm{d}\vec{x}.$$

Suppose now that we consider an external stationary electromagnetic potential, described by, say, Schwartz functions $(A^0, \vec{A})$. A candidate for the full Hamiltonian is

$$\hat{H}^{2\mathrm{ren}} = \int \mathrm{d}\vec{x}\Big(\hat{\eta}^*(\vec{x})\hat{\eta}(\vec{x}) + \mathrm{i}eA_0(\vec{x})\big(\hat{\psi}^*(\vec{x})\hat{\eta}(\vec{x}) - \hat{\eta}^*(\vec{x})\hat{\psi}(\vec{x})\big)$$
$$+(\partial_i - \mathrm{i}eA_i(\vec{x}))\hat{\psi}^*(\vec{x})(\partial_i + \mathrm{i}eA_i(\vec{x}))\hat{\psi}(\vec{x}) + m^2\hat{\psi}^*(\vec{x})\hat{\psi}(\vec{x})\Big)$$
$$-E_0 - E_2, \tag{9}$$

where E_0, E_2 are infinite counterterms that come from the expansion in e. ($E_1 = 0$ by the Furry Theorem). Again, physically one prefers to add to (9) E_2^{ren}, the renormalized value of E_2, so that all counterterms are local. One obtains the renormalized Hamiltonian formally written as

$$\hat{H}^{\mathrm{ren}} = \int \mathrm{d}\vec{x}\Big(\hat{\eta}^*(\vec{x})\hat{\eta}(\vec{x}) + \mathrm{i}eA_0(\vec{x})\big(\hat{\psi}^*(\vec{x})\hat{\eta}(\vec{x}) - \hat{\eta}^*(\vec{x})\hat{\psi}(\vec{x})\big)$$
$$+(\partial_i - \mathrm{i}eA_i(\vec{x}))\hat{\psi}^*(\vec{x})(\partial_i + \mathrm{i}eA_i(\vec{x}))\hat{\psi}(\vec{x}) + m^2\hat{\psi}^*(\vec{x})\hat{\psi}(\vec{x})\Big)$$
$$-E_0 - C\int \big(\partial_\mu A_\nu(\vec{x}) - \partial_\nu A_\mu(\vec{x})\big)\big(\partial^\mu A^\nu(\vec{x}) - \partial^\nu A^\mu(\vec{x})\big)\mathrm{d}\vec{x}, \tag{10}$$

where C is infinite. This example is worked out in detail in Chap. VI, Subsec. B17 of [3], see also [5].

Unfortunately, the classical dynamics is implementable only if the vector potential $\vec{A}$ vanishes everywhere. Therefore, both $\hat{H}^{2\mathrm{ren}}$ and $\hat{H}^{\mathrm{ren}}$ are well defined only in this case. However, the infimum of (10) is a well defined gauge-invariant number also for nonzero $\vec{A}$.

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Ward identities as a tool to study functional integrals.

Margherita Disertori
(joint work with T. Spencer, M. Zirnbauer)

In the context of field theory Ward identities are functional identites generated by internal symmmetries of the model. They generally appear as relations between Feynman diagrams, allowing to simplify the pertubative expansion, and in some cases even to close the Schwinger-Dyson equation. Such applications require nevertheless the model to be described by a small perturbation of a free (quadratic) action. A natural question is whether symmetry-generated identities may also help studying models where standard renormalization group techniques do not apply. In this context, in collaboration with M. Zirnbauer and T. Spencer [2], we considered the so-called $H^{2|2}$ supersymmetric nonlinear sigma model, introduced in [1] as a toymodel for quantum diffusion. It is also a key ingredient in the construction and study of certain stochastic processes with memory (cf. [4] [5] [6]). For this model we constructed a multiscale procedure whose key ingredient is a infinite family of Ward identities generated by supersymmetry. We hope a similar strategy may extend to other models with and without supersymmetry.

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Symmetric *R*-spaces, reflection positivity and the Berezin form

Jan Frahm
(joint work with Gestur Ólafsson, Bent Ørsted)

We give a brief introduction to the Berezin form on symmetric R-spaces and explain its relation to reflection positivity. The main new result is a complete answer to the question for which parameters the Berezin form is positive semidefinite, or in other words, for which parameters the reflection positivity condition holds.

1. Symmetric R-spaces

Roughly speaking, a symmetric R-space is a compact symmetric space K/L which is, at the same time, a homogeneous space for a larger non-compact semisimple group G. Typical examples are the Grassmannians $X = \mathrm{Gr}_p(\mathbb{R}^{p+q})$ consisting of all p-dimensional $\mathbb{R}$-linear subspaces of $\mathbb{R}^{p+q}$, where $p, q \geq 1$. The group $G = \mathrm{SL}(p+q, \mathbb{R})$ acts transitively on X by $g \cdot b = gb$. Fixing the base point $b_0 = \mathbb{R}e_1 + \cdots + \mathbb{R}e_p$ we can identify $X \simeq G/P$, where P is the stabilizer of b_0, a maximal parabolic subgroup of G. Since the maximal compact subgroup $K = \mathrm{SO}(p+q) \subseteq G$ already acts transitively on X, we further have $X \simeq K/L$ with $L = P \cap K = \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$, which expresses X as a compact symmetric space.

For simplicity, we focus on the example $X = \mathrm{Gr}_n(\mathbb{R}^{2n})$, i.e. $p = q = n$, for the rest of this note and refer the interested reader to [5] for the general statements.

2. The Berezin form

For $\lambda \in \mathbb{C}$ we define a representation π_λ of G on $\mathcal{E} = C^\infty(X)$ by

$$\pi_\lambda(g)f(b) = j(g^{-1}, b)^{-\frac{\lambda+n}{2}} f(g^{-1}b), \qquad g \in G, b \in X,$$

where $j(g, b) = \det(\mathrm{pr}_b \circ g^t g \circ i_b)$ with $i_b : b \hookrightarrow \mathbb{R}^{2n}$ the natural embedding and $\mathrm{pr}_b : \mathbb{R}^{2n} \twoheadrightarrow b$ the orthogonal projection with respect to the standard inner product on $\mathbb{R}^{2n}$. This normalization is chosen, so that π_λ extends to an irreducible unitary representation on $L^2(X)$, the *unitary principal series*, if and only if $\lambda \in i\mathbb{R}$. For $\lambda \in \mathbb{R}$ there is a $\pi_\lambda(G)$-invariant Hermitian form on $\mathcal{E}$ given by

$$\langle f_1, f_2 \rangle_\lambda = \int_X \int_X |\mathrm{Cos}(b_1, b_2^\perp)|^{\lambda-n} f_1(b_1) \overline{f_2(b_2)}\, db_1\, db_2,$$

where $b^\perp \in X$ is the orthogonal complement of $b \in X$ and the kernel function is $\mathrm{Cos}(b_1, b_2) = \mathrm{Vol}(\mathrm{pr}_{b_2}(E_{b_1}))$ for any convex subset $E_{b_1} \subseteq b_1$ of volume 1 containing the origin (see e.g. [7]). We remark that this form depends meromorphically on λ and has to be regularized at all simple poles of the integral kernel. The form $\langle \cdot, \cdot \rangle_\lambda$ (or its regularization) is positive definite if and only if $\lambda \in (-1, 1)$. The corresponding irreducible unitary representations of G are called *complementary series representations*.

Now consider the involutive automorphism τ of G given by

$$\tau(g) = I_{n,n} g^{-\top} I_{n,n},$$

where $I_{n,n} = \mathrm{diag}(I_n, -I_n)$. Its fixed point group is given by $H = G^\tau = \mathrm{SO}(n, n)$ and the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ into ± 1 eigenspaces of τ. The real form $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ of $\mathfrak{g}_\mathbb{C}$ is given by $\mathfrak{g}^c = \mathfrak{su}(n, n)$.

On the representation space $\mathcal{E}$ we also have an involution

$$\tau_* : \mathcal{E} \to \mathcal{E}, \quad \tau_* f(b) = f(I_{n,n} b^\perp)$$

which is compatible with τ in the sense that

$$\pi_\lambda(\tau(g)) = \tau_* \circ \pi_\lambda(g) \circ \tau_*.$$

Therefore, twisting the $\pi_\lambda(G)$-invariant Hermitian form $\langle\cdot,\cdot\rangle_\lambda$ by τ_* gives a $\pi_\lambda(H)$- and $\pi_\lambda(\mathfrak{g}^c)$-invariant Hermitian form

$$\langle f_1, f_2\rangle_{\tau,\lambda} := \langle f_1, \tau_* f_2\rangle_\lambda = \int_X \int_X |\mathrm{Cos}(b_1, I_{n,n} b_2)|^{\lambda-n} f_1(b_1)\overline{f_2(b_2)}\, db_1\, db_2,$$

which is called the *Berezin form* and was previously studied in [1, 2, 3].

3. Reflection positivity

Since the Berezin form is H- and $\mathfrak{g}^c$-invariant, we can restrict it to $\mathcal{E}_+ = C_c^\infty(\mathcal{O})$ for any open H-orbit $\mathcal{O} \subseteq X$ and obtain an H- and $\mathfrak{g}^c$-invariant Hermitian form on $\mathcal{E}_+$. We ask the following natural question:

Question. *For which open H-orbits $\mathcal{O}$ in X and for which parameters $\lambda \in \mathbb{R}$ is the Berezin form $\langle\cdot,\cdot\rangle_{\tau,\lambda}$ positive semidefinite on $\mathcal{E}_+ = C_c^\infty(\mathcal{O})$?*

The positivity of the Berezin form is nothing else than reflection positivity for the involution τ_* with respect to the subspace $\mathcal{E}_+$. In this case, we hope that the Lie algebra representation π_λ^c of $\mathfrak{g}^c$ on $\mathcal{E}_+$ yields a unitary representation of the 1-connected group G^c with Lie algebra $\mathfrak{g}^c$ on the Hilbert space completion of $\mathcal{E}_+$ with respect to the Berezin form $\langle\cdot,\cdot\rangle_{\tau,\lambda}$.

The open H-orbits in X are given by

$$\mathcal{O}_j = \{b \in X : \omega|_{b\times b} \text{ has signature } (n-j, j)\} \qquad (0 \le j \le n).$$

Every open H-orbit is a symmetric space, more precisely $\mathcal{O}_j \simeq \mathrm{SO}(n,n)/\mathrm{S}(\mathrm{O}(n-j,j)\times \mathrm{O}(j,n-j))$. In particular, every $\mathcal{O}_j$ has an H-invariant pseudo-Riemannian metric which is Riemannian if and only if $j=0$ or $j=n$.

Theorem (see [5]). *The restriction of the Berezin form $\langle\cdot,\cdot\rangle_{\tau,\lambda}$ to $\mathcal{E}_+ = C_c^\infty(\mathcal{O}_j)$ $(0 \le j \le n)$ is positive semidefinite if and only if $j \in \{0, n\}$ and $\lambda \in (-\infty, 1) \cup \{1, 2, \ldots, n\}$ or if $j \in \{1, \ldots, n-1\}$ and $\lambda = n$.*

The first case leads to (scalar type) unitary highest weight representations of the group $G^c = \widetilde{\mathrm{SU}}(n,n)$. This was first observed by Enright [4] and Schrader [8] in special cases and later generalized by Jørgensen–Ólafsson [6].

In the second case, the Berezin kernel is trivial, and the corresponding unitary representation π_λ^c of $G^c = \mathrm{SU}(n,n)$ is the trivial representation.

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Inversion positivity and the sharp Hardy–Littlewood–Sobolev inequality

Rupert L. Frank
(joint work with Elliott H. Lieb)

For $0<\lambda<N$ and functions f and g on $\mathbb{R}^N$ we abbreviate

$$I_\lambda[f,g] := \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\overline{f(x)}\; g(y)}{|x-y|^\lambda}\, dx\, dy\,.$$

According to the Hardy–Littlewood–Sobolev (HLS) inequality there is a constant $\mathcal{H}_{N,\lambda}$ such that for all $f,g\in L^p(\mathbb{R}^N)$ with $p=2N/(2N-\lambda)$,

$$\Big| I_\lambda[f,g] \Big| \le \mathcal{H}_{N,\lambda} \|f\|_p \|g\|_p\,. \tag{1}$$

In [7] Lieb computed the sharp (that is, smallest possible) constant $\mathcal{H}_{N,\lambda}$ in (1) and characterized the cases of equality. An alternative proof was given in [1]. The precise statement is

Theorem 1. *Let $0<\lambda<N$ and $p=2N/(2N-\lambda)$. Then (1) holds with*

$$\mathcal{H}_{N,\lambda} = \pi^{\lambda/2} \frac{\Gamma((N-\lambda)/2)}{\Gamma(N-\lambda/2)} \left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{1-\lambda/N}\,.$$

Equality holds if and only if

$$f(x) = \alpha\left(\beta+|x-\gamma|^2\right)^{-(2N-\lambda)/2} \quad \text{and} \quad g(x) = \alpha'\left(\beta+|x-\gamma|^2\right)^{-(2N-\lambda)/2},$$

for some $\alpha,\alpha'\in\mathbb{C}$, $\beta>0$ and $\gamma\in\mathbb{R}^N$.

Our goal here is to sketch our proof [2] of Theorem 1 *under the additional assumption $N-2\le\lambda<N$ if $N\ge 3$*. In contrast to the proofs in [7] and [1], which rely on the technique of Schwarz symmetrization, our proof in [2] relies on reflection positivity. We also refer to [3] for a variation of the ideas in [2] and the extension of our method to the so-called logarithmic HLS inequality. Yet another rearrangement-free proof of Theorem 1 was given in [5], this time in the whole range $0<\lambda<N$. The work [5] was motivated by our proof of the sharp HLS inequality on the Heisenberg group [4], which is a situation where one cannot expect rearrangement techniques to work. It is hoped that the methods from

[2, 3, 4, 5] will be relevant in other situations, where rearrangement techniques cannot be used effectively.

Inequality (1) is clearly invariant under translations and dilations. It is less obvious that it is invariant under the whole conformal group [7, 1]. This fact will play a crucial role in our proof.

Reflection positivity. Let $B = \{x \in \mathbb{R}^N : \ |x-a| < r\}$, $a \in \mathbb{R}^N$, $r > 0$, be a ball and denote by

$$\Theta_B(x) := \frac{r^2(x-a)}{|x-a|^2} + a$$

the inversion of a point x through the boundary of B. This map on $\mathbb{R}^N$ is lifted to an operator acting on functions f on $\mathbb{R}^N$ according to

$$(\Theta_B f)(x) := \left(\frac{r}{|x-a|}\right)^{2N-\lambda} f(\Theta_B(x))\,.$$

One easily finds that with $p = 2N/(2N-\lambda)$

$$I_\lambda[f] = I_\lambda[\Theta_B f] \qquad \text{and} \qquad \|f\|_p = \|\Theta_B f\|_p\,,$$

where we abbreviated $I_\lambda[f] := I_\lambda[f,f]$. Similarly, let $H = \{x \in \mathbb{R}^N : \ x \cdot e > t\}$, $e \in \mathbb{S}^{N-1}$, $t \in \mathbb{R}$, be a half-space and denote by

$$\Theta_H(x) := x + 2(t - x\cdot e)$$

the reflection of a point x on the boundary of H. The corresponding operator is defined by

$$(\Theta_H f)(x) := f(\Theta_H(x))$$

and it again satisfies

$$I_\lambda[f] = I_\lambda[\Theta_H f] \qquad \text{and} \qquad \|f\|_p = \|\Theta_H f\|_p\,.$$

Our first ingredient in the proof of Theorem 1 is the following

Theorem 2 (**Reflection and inversion positivity**). *Let* $0 < \lambda < N$ *if* $N = 1, 2$, $N-2 \leq \lambda < N$ *if* $N \geq 3$, *and let* $B \subset \mathbb{R}^N$ *be either a ball or a half-space. If* $f \in L^{2N/(2N-\lambda)}(\mathbb{R}^N)$ *and*

$$f^i(x) := \begin{cases} f(x) & \text{if } x \in B\,, \\ \Theta_B f(x) & \text{if } x \in \mathbb{R}^N \setminus B\,, \end{cases} \qquad f^o(x) := \begin{cases} \Theta_B f(x) & \text{if } x \in B\,, \\ f(x) & \text{if } x \in \mathbb{R}^N \setminus B\,, \end{cases}$$

then

$$\frac{1}{2}\left(I_\lambda[f^i] + I_\lambda[f^o]\right) \geq I_\lambda[f]\,.$$

If $\lambda > N-2$ *then the inequality is strict unless* $f = \Theta_B f$.

For half-spaces and $\lambda = N-2$ this theorem is well known. The half-space case with $N-2 < \lambda < N$ was apparently first proved by Lopes and Mariş [8]. The case of balls seems to be new for all λ. Our original proof [3] was simplified by E. Carlen, to whom we are grateful, using the conformal invariance [2].

The Li–Zhu lemma. Our second ingredient in the proof of Theorem 1 is a geometric characterization of the optimizing functions $\alpha\left(\beta+|x-\gamma|^2\right)^{-(2N-\lambda)/2}$, extending a result of Li and Zhu [6].

Theorem 3 (**Characterization of inversion invariant measures**). *Let μ be a finite, non-negative measure on $\mathbb{R}^N$. Assume that*

(A) for any $a\in\mathbb{R}^N$ there is an open ball B centered at a such that

$$\mu(\Theta_B^{-1}(A))=\mu(A)\qquad \text{for any Borel set } A\subset\mathbb{R}^N\,.$$

and for any $e\in\mathbb{S}^{N-1}$ there is an open half-space H with interior unit normal e such that

$$\mu(\Theta_H^{-1}(A))=\mu(A)\qquad \text{for any Borel set } A\subset\mathbb{R}^N\,.$$

Then μ is absolutely continuous with respect to Lebesgue measure and

$$d\mu(x)=\alpha\left(\beta+|x-y|^2\right)^{-N}dx$$

for some $\alpha\geq 0$, $\beta>0$ and $y\in\mathbb{R}^N$.

For absolutely continuous measures $d\mu=v\,dx$ assumption (A) is equivalent to the fact that for any $a\in\mathbb{R}^N$ there is an $r_a>0$ such that for any x

$$v(x)=\left(\frac{r_a}{|x-a|}\right)^{2N}v\left(\frac{r_a^2(x-a)}{|x-a|^2}+a\right),$$

and similarly for reflections.

Finally, let us deduce Theorem 1 from Theorems 2 and 3. We make use of the fact that there is an optimizer f for the inequality in Theorem 1. Given $a\in\mathbb{R}^N$, we apply Theorem 2 to the ball B centered at a with $\int_B|f|^p\,dx=(1/2)\int_{\mathbb{R}^N}|f|^p\,dx$ (or, given $e\in\mathbb{S}^{N-1}$ to the half-space H with interior unit normal e such that $\int_H|f|^p\,dx=(1/2)\int_{\mathbb{R}^N}|f|^p\,dx$). We infer that both f^i and f^o are optimizers. Moreover, if $\lambda>N-2$, then the strictness statement in Theorem 2 implies that $f=\Theta_B f$. The same conclusion holds for $\lambda=N-2$, but in this case we need an additional argument based on unique continuation. Therefore, we obtain in any case that assumption (A) in Theorem 3 is satisfied for the measure $d\mu=|f|^p\,dx$. We conclude from that theorem that f has the claimed form.

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RP, PCT, KMS, Etc.

Jürg Fröhlich
(joint work with L. Birke, K. Osterwalder, E. Seiler, and others)

Dedicated to the memory of *R. Haag, R. Jost* and *R. Schrader*

In my lecture I gave a brief survey of results on general quantum statistical mechanics, all related to *"Osterwalder-Schrader Positivity"*, or *"Reflection Positivity" (RP)*, and to the *"KMS condition"* for so-called "temperature-ordered Green functions", and I discussed some consequences of these results for relativistic quantum field theory and representation theory. My lecture was originally prepared in 2004 but was presented at the Oberwolfach meeting, in November 2017, for the first time. Earlier results, which my research and the lecture presented in Oberwolfach have been based on, are alluded to below, and some relevant papers are listed in the bibliography. Time did not permit to discuss some of the rather spectacular applications of *Reflection Positivity* to concrete models of statistical mechanics, in particular to the theory of phase transitions and spontaneous symmetry breaking in certain classes of classical and quantum lattice systems and lattice gauge theories; but see [21] and references given there.

A. Remarks on Early History of Reflection Positivity and the KMS condition: The first seed of Reflection Positivity can be found in the book [3]. Subsequently, it was formulated explicitly and used to reconstruct (real-time) vacuum expectation values of local quantum fields from Euclidean (imaginary-time) Green functions by *K. Osterwalder* and *R. Schrader* in their celebrated work [10]. Inspired by their proposal, related results appeared in [11].

The first shadow of the KMS condition and an example of a "modular conjugation" – namely the anti-unitary PCT symmetry operation of local relativistic quantum field theory – appeared in the context of a proof of the PCT theorem in the framework of the Wightman axioms in Jost's famous paper [1]. The KMS condition was formulated explicitly in [2]. An understanding of this condition as a general characterization of thermal equilibrium states in quantum statistical mechanics, as well as very interesting mathematical consequences thereof appeared in seminal work of Haag, Hugenholtz and Winnink [4]. An analysis of thermal Green functions and, in particular, of their analyticity properties in the time variables was presented in [5]. Results on analyticity properties of thermal Green functions of concrete systems of quantum statistical mechanics (dilute quantum gases described in terms of “reduced density matrices”) were sketched in [8]. The findings in [4] apparently played a role in the genesis of the deep discoveries, due to Tomita, that gave rise to the modular theory of von Neumann algebras, details of which are described in [7]; (see also references given there).

B. More recent developments related to RP and the KMS condition:
Cousins of the Osterwalder-Schrader reconstruction theorem for “temperature-ordered” (imaginary-time) Green functions of thermal equilibrium states of simple quantum field models at positive temperatures appeared in [12, 13]. An attempt to formulate and prove a general reconstruction theorem of real-time thermal Green functions from imaginary-time, temperature-ordered Green functions was undertaken in a course on equilibrium statistical mechanics I taught at Princeton in 1977. Although various technical details were not straightened out in my course, yet, it did lead to the rather important result on selfadjoint extensions of locally densely defined symmetric semigroups published in [15]. A crucial idea used in my proof of this result was generously contributed by Edward Nelson. A related result was subsequently proven in [16]. It has been pointed out to me by Arthur Jaffe at the Oberwolfach meeting that an earlier (seemingly slightly weaker) result on selfadjoint extensions of symmetric semigroups was proven by A. E. Nussbaum in [6], almost ten years earlier! A first general reconstruction theorem (in the context of stochastic processes) appeared in [17]. A very general theorem on the reconstruction of real-time thermal Green functions from temperature-ordered Green functions appeared in [20], (the genesis of which was independent of [17]). The reason these results are of interest is that, in many examples of quantum many-body systems, the temperature-ordered Green functions are more directly accessible to construction than the real-time Green functions.

In [18], Reflection Positivity, the KMS condition and the semigroup theorem alluded to above were first applied to problems in the representation theory of Lie groups: “virtual representations of symmetric spaces and their analytic continuation”. The notion of “virtual representations” was coined in this paper. This line of research caught the attention of a number of mathematicians, including P. E. T. Jorgensen, K.-H. Neeb, G. Olafsson, and their followers, working in the field of group representation theory. New results have been described at the Oberwolfach meeting by Jan Frahm whose notes I refer the reader to.

A very general understanding of a construction of local quantum theories with infinitely many degrees of freedom from generalized “imaginary-time Green functionals” emerged from the line of work begun in [18]. It has been described in detail in [18, 20]. This work has given rise to – among other results – very simple proofs of the celebrated PCT theorem and of the connection between spin and statistics, (which were sketched in my lecture). The PCT theorem says that the product of space reflection (P), charge conjugation (C) and time reversal (T) is a symmetry of every local relativistic quantum field theory on space-times of *even* dimension; this implies that to every species of particles there corresponds a species of anti-particles with the same mass and the same quantum numbers, but of *opposite* charge. The spin-statistics theorem says that local quantum fields of half-integer spin located at space-like separated points in space-time anti-commute (Fermi-Dirac statistics), while local quantum fields of integer spin located at space-like separated points commute (Bose-Einstein statistics), implying what one calls "Einstein causality". The proofs of these results described in [20] are somewhat inspired by results of Bisognano and Wichmann [14] concerning the KMS condition satisfied by the vacuum state of a local relativistic quantum field theory with respect to one-parameter subgroups of Lorentz boosts; (results that, in turn, continue the thought processes initiated in [1, 4]).

A variant of Reflection Positivity – connected to Markov traces on Hecke algebras and to "planar algebras" and related objects – has first appeared and been used in subfactor theory and then in the general theory of tensor categories. This line of work was initiated by Vaughan Jones [19] and is still going on. It has important implications in the theory of knots and links, (low-dimensional) local quantum field theory and quantum information theory. In his lecture at Oberwolfach, Arthur Jaffe has sketched some of the recent research in this direction that he and his collaborators have undertaken. I refer the reader to Jaffe's notes.[1]

C. Applications of Reflection Positivity to Statistical Mechanics:
Some of the most spectacular uses of Reflection Positivity and of some of its consequences have occurred in the theory of *phase transitions* and of *spontaneous breaking of (continuous) symmetries* in classical and quantum systems – a theory that belongs to equilibrium statistical mechanics; see [21]. The first such use appeared in a proof of existence of a phase transition accompanied by the spontaneous breaking of the $(\varphi \mapsto -\varphi)$ - symmetry in the $\lambda\varphi^4$ - theory in two space-time dimensions, due to Glimm, Jaffe and Spencer. They used a more powerful version of estimates in [9], which they derived directly from Reflection Positivity, to carry out a so-called *Peierls argument* patterned on a method first discovered by Sir Rudolf Peierls for the example of the two-dimensional Ising model, which proves the existence of a phase transition. In work of Simon, Spencer and myself, a new method to exhibit phase transitions in lattice models of magnets and in $\lambda|\vec{\varphi}|^4$ - theories with continuous symmetries in three or more (space-time) dimensions was discovered. It draws inspiration from the so-called Källen-Lehmann representation of vacuum expectation values of two fields in local relativistic quantum field theory. A variant of this representation, called "infrared bounds", can be established using Reflection Positivity. Infrared bounds have since been very widely used. Many interesting results can be found in [21].

Files of my lecture at Oberwolfach and of my Vienna lectures [21], IV., are available on request.

[1]In the past, I have contributed various results to the theory of braided tensor categories and their applications in the theory of superselection sectors in quantum field theory in collaborations led by Thomas Kerler, Ingo Runkel and Christoph Schweigert.

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Periodic striped ground states in Ising models with competing interactions

Alessandro Giuliani

(joint work with J. Lebowitz, E. Lieb, R. Seiringer)

In this talk, I will review some selected results obtained in the last few years on the existence of periodic minimizers in two- and three-dimensional spin systems with competing interactions. The model that we consider is an Ising model in dimension d (the most interesting cases being $d = 2$ and $d = 3$), with short range ferromagnetic and long range, power-law decaying, anti-ferromagnetic interactions. The Hamiltonian describing the energy of the system is

$$H = -J \sum_{\langle x,y\rangle} (\sigma_x\sigma_y - 1) + \sum_{\{x,y\}:\ x\neq y} \frac{(\sigma_x\sigma_y - 1)}{|x-y|^p}, \tag{1}$$

where $J > 0$ is the ratio between the strengths of the ferromagnetic and of the anti-ferromagnetic interaction, and $p > d$ is the decay exponent of the long-range interaction. The first sum ranges over pairs of nearest-neighbor sites in the discrete torus $\mathbb{T}_L^d := \mathbb{Z}^d/L\mathbb{Z}^d$, while the second over pairs of distinct sites in $\mathbb{T}_L^d$. The spins σ_x, $x \in \mathbb{T}_L^d$, take values in $\{\pm 1\}$, and the constant -1 appearing in the two terms is chosen in such a way that the energy of the homogeneous configuration $\sigma_x \equiv +1$, is equal to zero. A physically relevant case is $d = 2$ and $p = 3$, in which case (1) models the low-temperature equilibrium properties of thin magnetic films, embedded in the three-dimensional space, with the easy-axis of magnetization coinciding with the axis orthogonal to the film; in this case, the long range term models the dipolar interaction among the localized magnetic moments, while the short-range term models a ferromagnetic exchange interaction.

The goal is to characterize the structure of the ground states of the system, for any (even, sufficiently large) $L \in \mathbb{N}$. Ideally, one would also like to characterize the low-temperature infinite volume Gibbs states, but this is beyond our current abilities. Note that the short-range interaction favors a homogeneous state, that is $\sigma_x \equiv +1$ or $\sigma_x \equiv -1$, while the long-range term favors an anti-ferromagnetic 'Nèel' state, that is $\sigma_x = (-1)^{x_1+\cdots+x_d}$ or $\sigma_x = (-1)^{x_1+\cdots+x_d+1}$. The fact that the long-range contribution to the energy is minimized by the Nèel state is not obvious, and was proved in [2] by Reflection Positivity (RP) methods.

In the presence of both terms, the competition between the short-range ferromagnetic and the long-range anti-ferromagnetic interaction induces the system to form domains of minus spins in a background of plus spins, or vice versa. This happens in an intermediate range of values of J: in fact, if J is sufficiently small, the ground state is the same as for $J = 0$, that is, it is the Nèel state [2]; if J is sufficiently large and $p > d + 1$[1], the ground state is the same as for $J = +\infty$, that is, it is the homogeneous state. For intermediate values of J the ground state is characterized by non-trivial structures, whose typical length scale diverges as $J \to J_c(p)$ from the left; here $J_c(p)$ is the critical value of J, beyond which the ground state is homogeneous. It coincides with the value of J at which the surface tension of an infinite straight domain wall, separating a half space of minuses from a half space of pluses, vanishes [5]. It is expected that, for values of J close to $J_c(p)$ and slightly smaller than it, all the ground states are quasi-one-dimensional (i.e., they are translationally invariant in $d - 1$ directions), and periodic, provided the box size L is an integer multiple of an 'optimal period' $2h^*$, which can be explicitly computed. We shall refer to these expected ground states as the 'optimal periodic striped states': they consist of 'stripes' (in $d = 2$, or 'slabs', in $d = 3$) of spins all of the same sign, arranged in an alternating way (that is, neighbouring stripes have opposite magnetization), and all of the same width h^*.

The conjecture that optimal periodic striped states are ground states of (1) has been first proved in [3, 4], via a generalization of the standard RP technique, which we named 'block reflection positivity', because the reflections are performed across the bonds that separate a block of plus spins from a block of minus spins. The same proof shows that in any dimension, optimal periodic striped states are the states of minimal energy, among all the possible quasi-one-dimensional states.

More recently, in a work in collaboration with R. Seiringer, we succeeded in proving this conjecture [6], for all dimensions $d \geq 1$ and sufficiently large decay exponents, namely $p > 2d$. The result has been recently extended to the continuum setting and $p > d + 2$ [1]. The proof is based on the following main steps:

(1) We re-express the energy of the spin configuration as the energy of an equivalent droplet configuration. Here the droplets are the connected regions of minus spins, in a background of plus spins. The energy, if expressed in terms of droplets, consists of (i) a sum of droplet self-energies, which include the ferromagnetic contribution to the surface tension, plus the long range interaction of the minus spins in each droplet δ with a 'sea' of plus spins in the complement of the droplet $\delta^c = \mathbb{T}_L^d \setminus \delta$, and (ii) a droplet-droplet pair interaction, which is repulsive. Remarkably, the long

[1] for $p \leq d + 1$, the homogeneous state is *not* the ground state, for any finite value of J. In these cases, which include the case $d = 2$, $p = 3$ mentioned above, it would be interesting to characterize the ground states for J sufficiently large; unfortunately, we do not have rigorous results to report on this case yet, with the only exception of the one-dimensional case $d = 1$.

range contribution to the self-energy of a droplet δ behaves (for the purpose of a lower bound) as $-2J_c(p)|\partial\delta|$, where $|\partial\delta|$ is the length (if $d = 2$, or area, if $d = 3$) of the boundary of the droplet, plus a positive constant times the number of corners, that is, the points where the domain walls bend by 90^o. In this respect, the corners look like the elementary excitations of the system.

(2) We localize the droplet energy in bad boxes, characterized by a local 'a-typical' configuration (which either has corners, or too large uniformly magnetized regions – called 'holes'), and good regions, which are the connected components of the complement of the union of the bad boxes. By 'localizing', we mean here that the original energy is bounded from below in terms of a sum of local energy functionals, each depending only on the local droplet configuration (supported either in a bad box or in a good region). By construction, the configuration in a good region is quasi-one-dimensional, and consists of stripes all in the same direction, but not necessarily all of the same width.

(3) We use our lower bound on the self-energy of the droplets, to infer that the localized energy in a bad box is much larger than the energy of an optimal striped configuration in the same box. The energy difference scales like the number of corners contained in the bad box, plus the volume of the holes. We shall refer to this energy difference as the energy gain associated with each bad box.

(4) We use a slicing procedure, combined with block RP and an optimal control of the boundary errors, to derive an optimal lower bound on the localized energy in a good region. Such a lower bound scales like the energy of the optimal striped configuration in the same region, minus a boundary error, which is so small that it can be over- compensated by the energy gains of the bad boxes at the boundary of the good region (note that every boundary portion of a good region borders on a bad box).

Our result provides the first rigorous proof of the formation of mesoscopic periodic structures in $d \geq 2$ systems with competing interactions. It leaves a number of important problems open:

(1) Extend the result of [6] to smaller decay exponents. In particular, prove that the ground states of (1) with $d = 2$ and $p = 3$ are periodic and striped, for all sufficiently large J.

(2) Prove that there are at least d infinite volume Gibbs states at low temperatures, which are translationally invariant in $d - 1$ coordinate directions. Depending on the dimension, prove the existence of Long-Range Striped Order (LRSO), or of quasi-LRSO a'la Kosterlitz-Thouless, in the last coordinate direction.

(3) Extend these results to the continuum setting, for an effective free energy functional that is rotationally invariant. In particular, prove the onset of continuous symmetry breaking, both in the ground state and in the low-temperature Gibbs states.

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Reflection positivity: an operator algebraic approach to the representation theory of the Lorentz group

Christian Jäkel
(joint work with Jens Mund)

The space-time symmetry group of the two-dimensional de Sitter space

$$dS \doteq \{x \in \mathbb{R}^{1+2} \mid x_0^2 - x_1^2 - x_2^2 = -r^2\} , \quad r > 0 ,$$

is the Lorentz group $SO_0(1,2)$. A wedge

$$W = \Lambda W_1 \subset dS , \qquad W_1 \doteq \{x \in dS \mid x_2 > |x_0|\} , \quad \Lambda \in SO_0(1,2) ,$$

is a space-time region, which is invariant under the action of the Lorentz boosts

$$\Lambda_W(t) = \Lambda\Lambda_1(t)\Lambda^{-1}, \quad \Lambda_1(t) \doteq \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} .$$

The reflection at the edge of the wedge W,

$$\Theta_{\Lambda W_1} = \Lambda(P_1 T)\Lambda^{-1}, \qquad \Lambda \in SO_0(1,2) , \quad P_1 T = \Lambda_1(i\pi) ,$$

maps W to its space-like complement, the *opposite wedge* W'.

Now, let $\Lambda \mapsto u(\Lambda)$ be a (anti-)unitary irreducible representation of the Lorentz group $O(1,2)$ on some Hilbert space $\mathcal{H}$. Let ℓ_W be the self-adjoint generator of the one-parameter subgroup $t \mapsto u\big(\Lambda_W\big(\frac{t}{r}\big)\big)$. Set

$$\delta_W \doteq e^{-2\pi r \ell_W}, \qquad j_W \doteq u(\Theta_W).$$

δ_W is a densely defined, closed, positive non-singular linear operator on $\mathcal{H}$; j_W is an anti-unitary operator on $\mathcal{H}$. These properties allow one to introduce the operator

$$s_W \doteq j_W \delta_W^{1/2} , \qquad W = \Lambda W_1 . \tag{1}$$

s_W is a densely defined, antilinear, closed operator on $\mathcal{H}$ with range $\mathcal{R}(s_W) = \mathcal{D}(s_W)$ and $s_W^2 \subset 1$. Moreover,

$$u(\Lambda) s_W u(\Lambda)^{-1} = s_{\Lambda W} , \quad \Lambda \in SO_0(1,2).$$

The *modular localisation map* $W \mapsto \mathcal{H}(W)$, introduced by Brunetti, Guido and Longo [4], associates a closed $\mathbb{R}$-linear subspace

$$\mathcal{H}(W) \doteq \{h \in \mathcal{D}(s_W) \mid s_W h = h\}$$

of $\mathcal{H}$ to a wedge W. Each $\mathcal{H}(W)$ is a standard subspace in $\mathcal{H}$, *i.e.*,

$$\mathcal{H}(W) \cap i\mathcal{H}(W) = \{0\} , \qquad \overline{\mathcal{H}(W) + i\mathcal{H}(W)} = \mathcal{H} .$$

The operator s_W introduced in (1) is the Tomita operator of $\mathcal{H}(W)$, *i.e.*,

$$\begin{aligned} s_W \colon \mathcal{H}(W) + i\mathcal{H}(W) &\to \mathcal{H}(W) + i\mathcal{H}(W) \\ h + ik &\mapsto h - ik. \end{aligned}$$

In particular, $\delta_W^{it}\mathcal{H}(W) = \mathcal{H}(W)$ and $j_W\mathcal{H}(W) = \mathcal{H}(W)'$, with $\mathcal{H}(W)'$ the symplectic complement of $\mathcal{H}(W)$ in $\mathcal{H}$. By construction,

$$u(\Lambda)\mathcal{H}(W) = \mathcal{H}(\Lambda W) , \qquad \Lambda \in SO_0(1,2) .$$

We now turn from the one-particle picture to quantum field theory. It is convenient to use the *coherent vectors*

$$\Gamma(h) = \oplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \underbrace{h \otimes_s \cdots \otimes_s h}_{n-times}$$

to define operator algebras on the Fock space $\mathcal{F} \equiv \Gamma(\mathcal{H}) \doteq \oplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s^n}$: for $h, g \in \mathcal{H}$, the relations

$$V(h)V(g) = e^{-i\Im\langle h,g\rangle} V(h+g) , \quad V(h)\Omega_\circ = e^{-\frac{1}{2}||h||^2}\Gamma(ih) ,$$

define unitary operators, called the *Weyl operators*. They satisfy $V^*(h) = V(-h)$ and $V(0) = 1$. The one-parameter group $\Lambda \mapsto u(\Lambda)$ induces a group of automorphisms

$$\alpha_\Lambda^\circ(V(h)) \doteq V\big(u(\Lambda)h\big) , \qquad h \in \mathcal{H} , \quad \Lambda \in SO_0(1,2) ,$$

representing the free dynamics.

The modular localization available on the one-particle Hilbert space can now be used to associate von Neumann algebras to space-time regions in dS:

i.) for the wedge W_1, we set $\mathcal{A}_\circ(W_1) \doteq \{V(h) \mid h \in \mathcal{H}(W_1)\}''$;
ii.) for an arbitrary wedge $W = \Lambda W_1$, we set $\mathcal{A}_\circ(W) \doteq \alpha_\Lambda^\circ\big(\mathcal{A}_\circ(W_1)\big)$;
iii.) for an arbitrary bounded, causally complete, convex region $\mathcal{O} \subset dS$, set $\mathcal{A}_\circ(\mathcal{O}) = \bigcap_{\mathcal{O}\subset W} \mathcal{A}_\circ\big(W\big)$.

The map $\mathcal{O} \mapsto \mathcal{A}_\circ(\mathcal{O})$ preserves inclusions, the algebras $\mathcal{A}_\circ(\mathcal{O})$ are hyperfinite type III_1 factors, and $\alpha_\Lambda^\circ(\mathcal{A}_\circ(\mathcal{O})) = \mathcal{A}_\circ(\Lambda\mathcal{O})$.

It remains to justify that the Fock zero-particle vector $\Omega_\circ$ induces the *physically relevant de Sitter vacuum state*. This is not completely obvious: there is no global time evolution on de Sitter space and hence no natural notion of energy. However, stability of matter against spontaneous collapse has to be ensured somehow, and also the energy-momentum currents should not fluctuate in an uncontrollable manner. The *geodesic KMS condition* (proposed by Borchers and Buchholz [2]) ensures such stability properties. It requires that the restriction of the *de Sitter vacuum state* to the wedge W_1 is a thermal state with respect to the dynamics provided by the one-parameter group $t \mapsto \exp(itL_\circ)$ of boosts which leave the wedge W_1 invariant. The *unique* state satisfying this condition is the one induced by the Fock vacuum vector $\Omega_\circ$.

Somewhat surprisingly the $\mathcal{P}(\varphi)_2$ model can be formulated on Fock space too. In fact, it can be reconstructed from the vector representing the interacting de Sitter vacuum state. The latter is given by Araki's perturbation theory of modular automorphisms:

$$(2) \qquad \Omega = \frac{\mathrm{e}^{-\pi H}\Omega_\circ}{\|\mathrm{e}^{-\pi H}\Omega_\circ\|} \, , \qquad H := L_\circ + \int_0^\pi r\cos\psi \, \mathrm{d}\psi \ :\!\mathcal{P}(\varphi(0,\psi))\!: \, ,$$

with $\mathcal{P}$ a real valued polynomial, bounded from below.

The modular group for the pair $\big(\mathcal{A}_\circ(W_1), \Omega\big)$ provides a one-parameter group which leaves the algebra $\mathcal{A}_\circ(W_1)$ invariant. Since Ω lies in the natural positive cone $\mathcal{P}^\sharp(\mathcal{A}_\circ(W_1), \Omega_\circ)$, we have

$$J_\circ \Delta^{1/2} A\Omega = A^*\Omega \, , \qquad A \in \mathcal{A}_\circ(W_1) \, ,$$

with $J_\circ$ the modular conjugation for the pair $(\mathcal{A}_\circ(W_1), \Omega_\circ)$. We can thus interpret $t \mapsto \Delta^{it}_{W_1}$ as a new group of Lorentz boosts and, in fact, together with the (free) rotations $U_\circ(R_0(\alpha))$, $\alpha \in [0, 2\pi)$, they generate a new representation $U(\Lambda)$ of $SO_0(1,2)$.

Form a more technical perspective, several mathematical challenges have to be overcome in order to guarantee that the operator sum in the second equation in (2) is well-defined. Local symmetric semi-groups techniques [6, 10] are used to establish the operator sums

$$(3) \qquad L := \overline{L_\circ + V} \, , \qquad V = \int_{S^1} r\cos\psi \, \mathrm{d}\psi \ :\!\mathcal{P}(\varphi(0,\psi))\!: \, .$$

The theory of virtual representations [7] is used to prove that the newly defined one-parameter unitary groups actually give rise to a representation of $SO(1,2)$. The resulting unitary representation $\Lambda \mapsto U(\Lambda)$ induces a group of automorphisms

$$\alpha_\Lambda(V(h)) \doteq U(\Lambda)V(h)U(\Lambda)^{-1} \, , \qquad h \in \mathcal{H} \, , \quad \Lambda \in O(1,2) \, ,$$

representing the *interacting dynamics*. The sum given in (3) provides the crucial link between the free and the interacting quantum field theory, as L is the generator of the modular group which leaves the von Neumann algebra $\mathcal{A}_\circ(W_1)$ for the free field associated to the wedge W_1 invariant. We can now proceed just as before:

i.) for the wedge W_1, set $\mathcal{A}(W_1) \doteq \mathcal{A}_\circ(W_1)$;
ii.) for an arbitrary wedge $W = \Lambda W_1$, set $\mathcal{A}(W) \doteq \alpha_\Lambda\big(\mathcal{A}(W_1)\big)$;
iii.) for a causally complete, convex region $\mathcal{O} \subset dS$, set $\mathcal{A}(\mathcal{O}) = \bigcap_{\mathcal{O}\subset W} \mathcal{A}(W)$.

The map $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ is the net of local von Neumann algebras for the $\mathcal{P}(\varphi)_2$ model. The de Sitter vacuum state, induced by the vector Ω, is uniquely characterised by the geodesic KMS condition. Due to thermalisation effects introduced by the curvature of space-time, it is unique even for large coupling constants, despite the fact that different phases occur in the limit of curvature to zero (*i.e.*, the Minkowski limit).

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Reflection positivity for parafermions

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(joint work with Arthur Jaffe)

We describe reflection positivity for parafermions or *anyons*. Whereas bosons and fermions receive a plus or minus sign upon exchanging particles, parafermions receive a factor $q = \exp(2\pi i/p)$, where $p \in \mathbb{N}$ is the *order*. Note that parafermions of order 1 are bosons, and parafermions of order 2 are fermions. Parafermions of order 3 or more behave in a qualitatively different fashion, as in this case q is not the same as q^{-1}.

In a discrete setting, parafermions are described by the *-algebra $\mathcal{A}(\mathbb{Z}, p)$ with generators c_i labelled by $i \in \mathbb{Z} + \frac{1}{2}$, satisfying the parafermion relations

$$c_i c_j = q c_j c_i \quad \text{for} \quad i < j \tag{1}$$

$$c_i^p = 1 \tag{2}$$

$$c_i^* = c_i^{-1} . \tag{3}$$

If $A = c_{i_1}^{n_1} \cdots c_{i_k}^{n_k}$, then the number of parafermions $n_1 + \ldots + n_k$ is well-defined only modulo p. It is called the *degree* of A, and denoted by $|A| \in \mathbb{Z}_p$.

1. The setting

Abstracting from the discrete situation, we move to the more general setting where parafermions are described by a $\mathbb{Z}_p$-graded, unital algebra $\mathcal{A}$. In order for the exponential series to make sense, we will assume that $\mathcal{A}$ is a locally convex topological algebra, for which multiplication is separately continuous. Further, we will take the Hamiltonian H to be a neutral (degree zero) element of $\mathcal{A}$. We require that the exponential series

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

converges, and that it defines a continuous map $\exp\colon \mathcal{A} \to \mathcal{A}$. Note that we do *not* require that $\mathcal{A}$ is a *-algebra, and we do *not* require H to be hermitean.

A *reflection* is an antilinear homomorphism $\theta\colon \mathcal{A} \to \mathcal{A}$ that squares to the identity, and reverses the grading. We assume that our algebra $\mathcal{A}$ is the *q-double* of a $\mathbb{Z}_p$-graded subalgebra $\mathcal{A}_+$. This means that $\mathcal{A} = \mathcal{A}_-\mathcal{A}_+$ with $\mathcal{A}_- := \theta(\mathcal{A}_+)$, and that the two halves of the system *paracommute*, in the sense that

$$A_- A_+ = q^{|A_-||A_+|} A_+ A_- \tag{4}$$

for all $A_+ \in \mathcal{A}_+$ and $A_- \in \mathcal{A}_-$. For example, in the algebra $\mathcal{A}(\mathbb{Z}, p)$ introduced above, we can take $\theta(c_i) = c_{-i}^{-1}$, with $\mathcal{A}_+(\mathbb{Z}, p)$ the algebra generated by the parafermion operators c_i with $i > 0$.

2. Reflection positivity

Let τ_0 be a *neutral* continuous linear functional on $\mathcal{A}$, meaning that $\tau_0(A) = 0$ if A is of pure degree $|A| \neq 0$. We consider τ_0 as a 'background state', and we are interested in the Bolzmann functionals $\tau_{\beta H}(A) := \tau_0(e^{-\beta H}A)$, where $\beta \geq 0$. In the context of parafermions, we define $\tau_{\beta H}$ to be *reflection positive* if

$$\zeta^{|A_+|^2} \tau_{\beta H}(\theta(A_+)A_+) \geq 0 \tag{5}$$

for all $A_+ \in \mathcal{A}_+$, where $\zeta \in \mathbb{C}$ is a square root of q with $\zeta^{p^2} = 1$. Note that if $\tau_{\beta H}$ is reflection positive, then the Hermitian form

$$\langle A_+, B_+ \rangle := \zeta^{|A_+|^2} \tau_{\beta H}(\theta(A_+)B_+)$$

is positive definite on $\mathcal{A}_+$. Since the above expression is zero for $|A_+| \neq |B_+|$, the closure $\mathcal{H}_+$ of $\mathcal{A}_+$ is a $\mathbb{Z}_p$-graded Hilbert space.

Note that alternatively, reflection positivity can be formulated in terms of the convex cone $\mathcal{K}$ spanned by elements of the form $\zeta^{|A_+|^2}\theta(A_+)A_+$ with $A_+ \in \mathcal{A}_+$. By definition, $\tau_{\beta H}$ is reflection positive if and only if $\tau_{\beta H}(\mathcal{K}) \subseteq \mathbb{R}^{\geq 0}$. Since $\tau_{\beta H}$ is continuous, this is equivalent to $\tau_{\beta H}(\overline{\mathcal{K}}) \subseteq \mathbb{R}^{\geq 0}$, where $\overline{\mathcal{K}}$ is the closure of $\mathcal{K}$.

The following theorem [10, 11] gives sufficient conditions on H in order for $\tau_{\beta H}$ to be reflection positive, extending well-known results in the bosonic and fermionic case [1, 2, 3, 4, 5, 6, 7, 8, 9]. Under mild additional assumptions, one can prove that these conditions are not just sufficient, but also *necessary* [11].

Theorem 1. *Suppose that $H = H_- + H_0 + H_+$, where $H_\pm \in \mathcal{A}_\pm$ with $\theta(H_+) = H_-$, and where $-H_0$ is in the closure $\overline{\mathcal{K}}$ of $\mathcal{K}$. Then reflection positivity of τ_0 implies reflection positivity of $\tau_{\beta H}$ for all $\beta > 0$.*

The cornerstone of the proof – and the reason for the appearance of $\zeta^{|A_+|^2}$ in equation (5) – is the fact that $\mathcal{K}$ is closed under multiplication. Indeed, the product of $\zeta^{|A_+|^2}\theta(A_+)A_+$ and $\zeta^{|B_+|^2}\theta(B_+)B_+$ is equal to $\zeta^{|A_+B_+|^2}\theta(A_+B_+)A_+B_+$, and hence an element of $\mathcal{K}$ again. To see this, use the paracommutation relation (4) in the expression $\theta(A_+)A_+\theta(B_+)B_+$ to exchange $A_+ \in \mathcal{A}_+$ and $\theta(B_+) \in \mathcal{A}_-$. Since $|\theta(B_+)| = -|B_+|$, this yields a factor $q^{|A_+||B_+|}$, which combines with $\zeta^{|A_+|^2+|B_+|^2}$ to form $\zeta^{|A_+B_+|^2}$.

Since this argument extends to convex combinations, one finds that the cone $\mathcal{K}$ is multiplicatively closed. The same conclusion for its closure $\overline{\mathcal{K}}$ can be derived from separate continuity of the multiplication. Indeed, since left multiplication by $K \in \mathcal{K}$ is continuous and $K \cdot \mathcal{K} \subseteq \mathcal{K}$, we find that $\mathcal{K} \cdot \overline{\mathcal{K}} \subseteq \overline{\mathcal{K}}$. Similarly, since right multiplication by $K' \in \overline{\mathcal{K}}$ is continuous, we find that $\mathcal{K} \cdot K' \subseteq \overline{\mathcal{K}}$ for all $K' \in \overline{\mathcal{K}}$, and hence that $\overline{\mathcal{K}} \cdot \overline{\mathcal{K}} \subseteq \overline{\mathcal{K}}$.

To prove Theorem 1, consider first the special case $H = H_0$, where H_- and H_+ vanish. Since $\overline{\mathcal{K}}$ is multiplicatively closed, $-\beta H_0 \in \overline{\mathcal{K}}$ implies that $\sum_{k=0}^{n} \frac{1}{k!}(-\beta H_0)^k$ is in $\overline{\mathcal{K}}$ for every n, and hence that $e^{-\beta H_0} \in \overline{\mathcal{K}}$. As $\overline{\mathcal{K}}$ is multiplicatively closed, we have $e^{-\beta H_0}\overline{\mathcal{K}} \subseteq \overline{\mathcal{K}}$, and hence $\tau_{\beta H}(\overline{\mathcal{K}}) = \tau_0(e^{-\beta H_0}\overline{\mathcal{K}}) \subseteq \mathbb{R}^{\geq 0}$. It follows that $\tau_{\beta H}$ is reflection positive for every $\beta > 0$.

Next, consider the case where H_+ and H_- are nonzero. Let

$$H_\varepsilon := H - \varepsilon^2 H_- H_+ - \tfrac{1}{\varepsilon^2}\mathbf{1}\,.$$

Since $H_0 \in -\overline{\mathcal{K}}$, and since H_ε can be written as $H_\varepsilon = H_0 - \theta(\frac{1}{\varepsilon}\mathbf{1} - \varepsilon H_+)(\frac{1}{\varepsilon}\mathbf{1} - \varepsilon H_+)$, we have $H_\varepsilon \in -\overline{\mathcal{K}}$. By the above reasoning, we thus find that the functional $\tau_{\beta H_\varepsilon}$ is reflection positive for all $\beta > 0$. Since $H'_\varepsilon := H - \varepsilon^2 H_- H_+$ differs from H_ε by an additive constant, reflection positivity of $\tau_{\beta H_\varepsilon}$ implies reflection positivity of $\tau_{\beta H'_\varepsilon}$. It follows that $\tau_{\beta H}(K) = \lim_{\varepsilon \downarrow 0} \tau_{\beta H'_\varepsilon}(K) \geq 0$ for all $K \in \mathcal{K}$, so that $\tau_{\beta H}$ is reflection positive.

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Reflection positivity, Lax-Phillips theory, and spectral theory

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(joint work with Karl-Hermann Neeb, Gestur Olafsson, and Feng Tian)

We review reflection-positivity (Osterwalder-Schrader positivity, O.S.-p.) as it is used in the study of renormalization questions in physics. In concrete cases, this refers to specific Hilbert spaces that arise before and after the reflection. Our focus is a comparative study of the associated spectral theory, now referring to the canonical operators in these two Hilbert spaces.

We analyze in detail a number of geometric and spectral theoretic properties connected with axiomatic reflection positivity, as well as their probabilistic counterparts; especially the role of the Markov property. In rough outline: It is possible to express OS-positivity purely in terms of a triple of projections in a fixed Hilbert space, and a reflection operator. For such three projections, there is a related property, often referred to as the Markov property; and it is well known that the latter implies the former; i.e., when the reflection is given, then the Markov property implies O.S.-p., but not conversely. In this paper we shall prove two theorems which flesh out a much more precise relationship between the two. We show that for every OS-positive system (E_+, θ), the operator $E_+\theta E_+$ has a canonical and universal factorization.

Our second focus is a structure theory for all admissible reflections. Our theorems here are motivated by Phillips' theory of dissipative extensions of unbounded operators (see e.g., [21]). The word "Markov" traditionally makes reference to a random walk process where the Markov property in turn refers to past and future: Expectation of the future, conditioned by the past. By contrast, our present initial definitions only make reference to three prescribed projection operators, and associated reflections. Initially, there is not even mention of an underlying probability space. This in fact only comes later.

The notion "*reflection-positivity*" came up first in a renormalization question in physics: "How to realize observables in relativistic quantum field theory (RQFT)?" This is part of the bigger picture of quantum field theory (QFT); and it is based on a certain analytic continuation (or reflection) of the Wightman distributions (from the Wightman axioms). In this analytic continuation, Osterwalder-Schrader (OS) axioms induce Euclidean random fields; and Euclidean covariance. (See, e.g., [25, 26, 5, 6, 13, 8, 9].) For the unitary representations of the respective symmetry groups, we therefore change these groups as well: OS-reflection applied to the Poincaré group of relativistic fields yields the Euclidean group as its reflection.

The starting point of the OS-approach to QFT is a certain positivity condition called "reflection positivity."

Now, when it is carried out in concrete cases, the initial function spaces change; but, more importantly, the inner product which produces the respective Hilbert spaces of quantum states changes as well. Before reflection we may have a Hilbert space of functions, but after the time-reflection is turned on, then, in the new inner product, the corresponding completion, magically becomes a *Hilbert space of distributions*.

The motivating example here is derived from a certain version of the Segal-Bargmann transform. For more detail on the background and the applications, we refer to two previous joint papers [15] and [16], as well as [17, 18, 19, 11, 12, 22, 14, 10, 3].

Our present purpose is to analyze in detail a number of geometric properties connected with the axioms of reflection positivity, as well as their probabilistic counterparts; especially the role of the Markov property.

In rough outline: It is possible to express Osterwalder-Schrader positivity (O.S.-p.) purely in terms of a triple of projections in a fixed Hilbert space, and a reflection operator. For such three projections, there is a related property, often referred to as the Markov property. It is well known that the latter implies the former; i.e., when the reflection is given, then the Markov property implies O.S.-p., but not conversely.

For the readers benefit we have included the following citations [23, 24, 20] on Markov random fields.

We begin by recalling the fundamentals in the subject.

1. A characterization of the Markov property: Markov vs O.-S. positivity

In the classical case of Gaussian processes, the question of *reflection symmetry* and *reflection positivity* is of great interest; see, e.g., [7, 14, 8], and also [17, 18, 19].

Let $\mathscr{H}$ be a given (fixed) Hilbert space; e.g., $\mathscr{H} = L^2(\Omega, \mathscr{F}, \mathbb{P})$, square integrable random variables, where Ω is a set (sample space) with a σ-algebra of subsets $\mathscr{F}$ (information), and $\mathbb{P}$ a given probability measure on $(\Omega, \mathscr{F})$. But the question may in fact be formulated for an arbitrary Hilbert space $\mathscr{H}$, and possible inseparable generally.

Recall that $\theta : \mathscr{H} \to \mathscr{H}$ is a *reflection* if it satisfies $\theta^* = \theta$, and $\theta^2 = I_{\mathscr{H}}$.

Definition 1.1. Given a Hilbert space $\mathscr{H}$, let $Ref(\mathscr{H})$ be the set of all reflections in $\mathscr{H}$, i.e., $Ref(\mathscr{H}) = \left\{\theta : \mathscr{H} \to \mathscr{H}\,;\, \theta^* = \theta,\ \theta^2 = I_{\mathscr{H}}\right\}$.

Question. *(1) Given $\varepsilon = \{E_\pm\}$, what is $\mathscr{R}(\varepsilon)$? (2) Given θ, what is $\mathscr{E}(\theta)$?*

Definition 1.2. Suppose $\varepsilon = (E_0, E_\pm)$ is given, and $\theta \in \mathscr{R}(\varepsilon)$.

(1) We say that *reflection positivity* holds iff (Def.)

$$E_+ \theta E_+ \geq 0, \tag{1}$$

also called *Osterwalder-Schrader positivity* (O.S.-p).

(2) Given ε, we say that it satisfies the *Markov property* iff (Def.)

(2) $$E_{+}E_{0}E_{-}=E_{+}E_{-}.$$

(3) We set $\mathscr{E}_{OS}\left(\theta\right)=\left\{\left(E_{0},E_{\pm}\right);E_{+}\theta E_{+}\geq0\right\}$.

Lemma 1.3. *Suppose (2) holds (the Markov property), and $\theta\in\mathscr{R}\left(\varepsilon\right)$, then*

(3) $$E_{+}\theta E_{+}\geq0,$$

i.e., the O.S.-positivity condition (1) follows.

Recall the definition of $\mathscr{R}\left(\varepsilon\right)$ and $\mathscr{R}\left(\varepsilon,U\right)$. Lemma 1.3 can be reformulated as:

Lemma 1.4. *For all $\theta\in\mathscr{R}\left(\varepsilon\right)$, we have*

(4) $$\mathscr{E}\left(Markov\right)\cap\mathscr{E}\left(\theta\right)\subseteq\mathscr{E}_{OS}\left(\theta\right).$$

(See Definitions 1.1 and eq. (3).)

Question. *Let $\varepsilon=\left(E_{0},E_{\pm}\right)$ be given, and suppose $E_{+}\theta E_{+}\geq0$, for all $\theta\in\mathscr{R}\left(\varepsilon\right)$, then does it follow that $E_{+}E_{0}E_{-}=E_{+}E_{-}$ holds?*

Theorem 1.5. *Given an infinite-dimensional complex Hilbert space $\mathscr{H}$, let the setting be as above, i.e., reflections, Markov property, and O.S.-positivity defined as stated. Then*

(5) $$\bigcap_{\theta\in\mathscr{R}\left(\varepsilon\right)}\mathscr{E}_{OS}\left(\theta\right)=\mathscr{E}\left(Markov\right).$$

Remark 1.6. If $\left(E_{\pm},E_{0},U\right)$ is Markov, then (5) also holds with θ, U. The idea in (5) is that when a system ε of projections is fixed as specified on the RHS in the formula, then on the LHS, we intersect only over the subset of reflections θ subordinated to this ε-system. And similarly when both ε and U are specified, we intersect over the smaller set of jointly ε, U subordinated reflections θ.

Example 1.7 (Markov property)**.** Let $\mathscr{H}=L^{2}\left(\Omega,\mathscr{F},\mathbb{P}\right)$, where

- Ω: sample space;
- $\mathscr{F}$: total information;
- $\mathscr{F}_{-}$: information from the past (or inside);
- $\mathscr{F}_{+}$: information from the future (predictions), or from the outside;
- $\mathscr{F}_{0}$: information at the present.

Let $\mathbb{E}\left(\cdot\mid\mathscr{F}_{0}\right)$, $\mathbb{E}\left(\cdot\mid\mathscr{F}_{\pm}\right)$ be the corresponding conditional expectations, and the Markov property (2) then takes the form $\mathbb{E}_{0}\mathscr{H}=\mathscr{H}_{0}$, $\mathbb{E}_{\pm}\mathscr{H}=\mathscr{H}_{\pm}$.

The Markov process is a probability system:

(6) $$\mathbb{E}\left(\mathbb{E}\left(\psi_{+}\mid\mathscr{F}_{-}\right)\mid\mathscr{F}_{0}\right)=\mathbb{E}\left(\psi_{+}\mid\mathscr{F}_{-}\right),$$

for $\forall\psi_{+}$ (random variables conditioned by $\mathscr{F}_{+}=$ the future); or, if $\mathscr{F}_{0}\subseteq\mathscr{F}_{-}$, it simplifies to:

(7) $$\mathbb{E}\left(\psi_{+}\mid\mathscr{F}_{-}\right)=\mathbb{E}\left(\psi_{+}\mid\mathscr{F}_{0}\right),\ \forall\psi_{+}\in\mathscr{H}_{+}.$$

For more details on this point, see Section 2 below.

Question. *Do we have analogies of O.S.-positivity (see (1)) in the free probability setting? That is, in the setting of free probability and non-commuting random variables.*

2. Markov processes and Markov reflection positivity

In the above, we considered systems $\mathscr{H}$, E_0, $E_\pm$, θ, and U, where $\mathscr{H}$ is a fixed Hilbert space; E_0, $E_\pm$ are then three given projections in $\mathscr{H}$, θ is a reflection, and U is a unitary representation of a Lie group G.

The axioms for the system are as follows:

(1) $\theta E_0 = E_+$;
(2) $E_+ \theta E_- = \theta E_-$;
(3) $E_- \theta E_+ = \theta E_+$;
(4) the O.S.-positivity holds, i.e.,

$$E_+ \theta E_+ \geq 0; \tag{8}$$

(5) $\theta U \theta = U^*$, or $\theta U(g)\theta = U(g^{-1})$.

It is further assumed that, for some sub-semigroup $S \subset G$, we have $U(s)\mathscr{H}_+ \subset \mathscr{H}_+$, $\forall s \in S$; or equivalently,

$$E_+ U(s) E_+ = U(s) E_+, \ s \in S. \tag{9}$$

2.1. **Probability Spaces.** By a probability space we mean a triple $(\Omega, \mathscr{F}, \mathbb{P})$, where Ω is a set (the sample space), $\mathscr{F}$ is a σ-algebra of subsets (information), and $\mathbb{P}$ is a probability measure defined on $\mathscr{F}$. Measurable functions ψ on $(\Omega, \mathscr{F})$ are called *random variables*. If ψ is a random variable in $L^2(\Omega, \mathscr{F}, \mathbb{P})$, we say that it has finite second moment. An indexed family of random variables is called a *stochastic process*, or a *random field*.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a fixed probability space. The expectation will be denoted

$$\mathbb{E}(\psi) = \int_\Omega \psi \, d\mathbb{P}, \tag{10}$$

if ψ is a given random variable on $(\Omega, \mathscr{F}, \mathbb{P})$.

We shall be primarily interested in the $L^2(\Omega, \mathscr{F}, \mathbb{P})$ setting.

If ψ is a random variable (or a random field) then

$$\psi^{-1}(\mathscr{B}) \subseteq \mathscr{F}, \tag{11}$$

where $\mathscr{B}$ is the Borel σ-algebra of subsets of $\mathbb{R}$.

For every sub-σ-algebra $\mathscr{G} \subset \mathscr{F}$, there is a unique conditional expectation

$$\mathbb{E}(\cdot \mid \mathscr{G}) : L^2(\Omega, \mathscr{F}, \mathbb{P}) \longrightarrow L^2(\Omega, \mathscr{F}, \mathbb{P}). \tag{12}$$

In fact $\mathscr{G}$ defines a closed subspace in $L^2(\Omega, \mathscr{F}, \mathbb{P})$, the closed span of the indicator functions $\{\chi_S \,;\, S \in \mathscr{G}\}$, and $\mathbb{E}(\cdot \mid \mathscr{G})$ in (12) will then be the projection onto this subspace.

If $\mathscr{G} \subset \mathscr{F}$ is as in (11) then, for random variables $\psi_1 \in L^2(\mathscr{G}, \mathbb{P})$, and $\psi_2 \in L^2(\mathscr{F}, \mathbb{P})$, we have $\mathbb{E}(\psi_1\psi_2) = \mathbb{E}(\psi_1 \mathbb{E}(\psi_2 \mid \mathscr{G}))$. If ψ_1 is also in $L^\infty(\mathscr{G}, \mathbb{P})$, then $\mathbb{E}(\psi_1\psi_2 \mid \mathscr{G}) = \psi_1 \mathbb{E}(\psi_2 \mid \mathscr{G})$.

The following property is immediate from this: If $\mathscr{G}_i$, $i = 1, 2$, are two sub-σ-algebras with $\mathscr{G}_1 \subseteq \mathscr{G}_2$, then for all $\psi \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ we have $\mathbb{E}(\mathbb{E}(\psi \mid \mathscr{G}_2) \mid \mathscr{G}_1) = \mathbb{E}(\psi \mid \mathscr{G}_1)$.

Let $\{\psi_t\}_{t\in\mathbb{R}}$ be a random process in the given probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For $t \in \mathbb{R}$, set $\mathscr{F}_t :=$ the σ-algebra $(\subseteq \mathscr{F})$ generated by the random variables $\{\psi_s \,;\, s \le t\}$. When t is fixed, we set $\mathscr{B}_t :=$ the σ-algebra generated by the random variable ψ_t. We say that $\{\psi_t\}_{t\in\mathbb{R}}$ is a *Markov-process* iff (Def.), for every $t > s$, and every measurable function f, we have

$$\mathbb{E}(f \circ \psi_t \mid \mathscr{F}_s) = \mathbb{E}(f \circ \psi_t \mid \mathscr{B}_s) \tag{13}$$

where $\mathbb{E}(\cdot \mid \mathscr{F}_s)$, and $\mathbb{E}(\cdot \mid \mathscr{B}_s)$, refer to the corresponding conditional expectations. It is well known that the Markov property is equivalent to the following *semigroup property*: Set

$$(S_t f)(x) := \mathbb{E}(f \circ \psi_t \mid \psi_0 = x), \tag{14}$$

then, for all $t, s \ge 0$, we have

$$S_{t+s} = S_t S_s. \tag{15}$$

So the semigroup law (15) holds if and only if the Markov property (13) holds.

2.2. **The covariance operator.** Now let V be a real vector space; and assume that it is also a LCTVS, locally convex topological vector space. Let G be a Lie group, U a unitary representation of G; and let $\{\psi_{v,g}\}_{(v,g)\in V\times G}$ be a real valued stochastic process s.t. $\psi_{v,g} \in \mathscr{H} = L^2(\Omega, \mathscr{F}, \mathbb{P})$, and $\mathbb{E}(\psi_{v,g}) = 0$, $(v, g) \in V \times G$. We further assume that a reflection θ is given, and that $\theta(\psi_{v,g}) = \psi_{v,g^{-1}}$, $(v, g) \in V \times G$.

Let (v_i, g_i), $i = 1, 2$, be given, and set

$$\mathbb{E}(\psi_{v_1,g_1} \psi_{v_2,g_2}) = \langle v_1, r(g_1, g_2) v_2 \rangle \tag{16}$$

where $\langle \cdot, \cdot \rangle$ is a fixed positive definite Hermitian inner product on V. Hence (16) determines a function r on $G \times G$; it is operator valued, taking values in operators in V. This function is called the *covariance operator*.

To sketch the setting for the Markov property, we shall make two specializations (these may be removed!):

(i) $G = \mathbb{R}$, $S = \mathbb{R}_+ \cup \{0\} = [0, \infty)$, and
(ii) the process is stationary; i.e., referring to (16) we assume that the covariance operator r is as follows:

$$\mathbb{E}(\psi_{v_1,t_1} \psi_{v_2,t_2}) = \langle v_1, r(t_1 - t_2) v_2 \rangle, \quad \forall t_1, t_2 \in \mathbb{R}, \forall v_1, v_2 \in V. \tag{17}$$

In this case, the O.S.-condition (8) is considered for the following three sub-σ-algebras $\mathscr{A}_0$, $\mathscr{A}_\pm$ in $\mathscr{F}$:

$\mathscr{A}_0 =$ the σ-algebra generated by $\{\psi_{v,0}\}_{v\in V}$,
$\mathscr{A}_+ =$ the σ-algebra generated by $\{\psi_{v,t}\}_{v\in V, t\in[0,\infty)}$, and
$\mathscr{A}_- =$ the σ-algebra generated by $\{\psi_{v,t}\}_{v\in V, t\in(-\infty,0]}$.

The corresponding conditional expectations will be denoted as follows:

$$\mathbb{E}_0(\psi) = \mathbb{E}(\psi \mid \mathscr{A}_0), \text{ and } \mathbb{E}_\pm(\psi) = \mathbb{E}(\psi \mid \mathscr{A}_\pm). \tag{18}$$

The corresponding closed subspaces in $\mathscr{H} = L^2(\Omega, \mathscr{F}, \mathbb{P})$ will be denoted $\mathscr{H}_0$, $\mathscr{H}_\pm$, respectively, and we consider the positivity conditions (8) O.S.-p, and Markov, in this context.

3. Extension from stationary (Gaussian) to stationary-increment processes

A novelty is a new extension of known results from stationary Gaussian processes X_t indexed by $t \in \mathbb{R}$ to a much more realistic class of Gaussian processes, the stationary-increment processes X_t, i.e., satisfying $\mathbb{E}(X_t) = 0$, $t \in \mathbb{R}$, and

$$\mathbb{E}(X_t X_s^*) = \frac{r(|t|) + r(|s|) - r(|s-t|)}{2};$$

in particular, $\mathbb{E}\big(|X_t - X_s|^2\big) = r(|t-s|)$, where r is a function on $[0, \infty)$. This part is based on a joint work with D. Alpay et al. (see [2, 4, 1]). In particular, we recall that the stationary-increment processes are indexed by tempered measures ν on $(\mathbb{R}, \mathscr{B})$ via $r = r_{(\nu)}$, where

$$r(t) = \int_{\mathbb{R}} \left(1 - e^{itx} + \frac{itx}{1+x^2}\right) \frac{d\nu(x)}{x^2}, \ t \in \mathbb{R}.$$

In this case $X^{(\nu)}$ is first constructed, and indexed by the Schwartz-space $\mathcal{S}$, with

$$\mathbb{E}\big(\big|X^{(\nu)}(\varphi)\big|^2\big) = \int_{\mathbb{R}} |\hat{\varphi}(x)|^2 \, d\nu(x), \ \varphi \in \mathcal{S},$$

where $\hat{\varphi}$ denotes the usual Fourier transform.

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Conformal field theory and operator algebras

Yasuyuki Kawahigashi

We give a review of recent progress of operator algebraic studies of chiral conformal field theory. See [1] and [2] for more details and references.

Chiral conformal field theory arises from a decomposition of a 2-dimensional conformal field theory. A local conformal net is an operator algebraic object describing a chiral conformal field theory. A local conformal net is an assignment of a von Neumann algebra $A(I)$ to each interval I contained in the circle S^1 which plays the role of "spacetime". The "spacetime symmetry" group is $\mathrm{Diff}(S^1)$, the group of orientation preserving diffeomorphisms of S^1. Then we impose physically natural axioms such as isotony, locality, conformal covariance, positivity of the conformal Hamiltonian and existence of the vacuum on the family $\{A(I)\}$.

Our important tool to study local conformal nets is a representation theory in the style of Doplicher-Haag-Roberts. The notion of a dimension of a representation is given by the Jones index of a subfactor. We are often interested in the situation where we have only finitely many irreducible representations. Such a case is called rational. An operator algebraic characterization of rationality was given by Kawahigashi-Longo-Müger in terms of finiteness of certain Jones index. In the rational case, we have a modular tensor category for representations.

We have a machinery of α-induction introduced by Longo-Rehren. Works of Xu and Ocneanu were unified by Böckenhauer-Evans-Kawahigashi and general properties of modular invariance were proved. Using these methods, Kawahigashi-Longo gave a complete classification of local conformal nets with central charge less than 1. (A central charge is a numerical invariant arising from a representation of the Virasoro algebra.) The classification list contains four exceptionals and one of them does not seem to arise from any other known construction.

A vertex operator algebra is an algebraic axiomatization of Fourier expansions of operator-valued distributions on S^1. It arose from studies of the celebrated Moonshine conjecture of Borcherds and Frenkel-Lepowsky-Meurman. Since a local conformal net and a vertex operator algebra both give mathematical axiomatizations of chiral conformal field theory, it is natural to expect a direct relation between the two. We now present such a relation due to Carpi-Kawahigashi-Longo-Weiner.

First, we need so-called unitarity of a vertex operator algebra. Then we impose a physically natural condition called strong locality. We do not know any example of a unitary vertex operator algebra which does not satisfy strong locality and we have a simple sufficient condition for strong locality. Note that if there should exist a unitary vertex operator algebra without strong locality, it would not correspond to physical chiral conformal field theory.

If we have strong locality, we can construct a corresponding local conformal net through a construction of smeared vertex operators. We can also come back to the original vertex operator algebra from the local conformal net we construct. This is due to an idea of Fredenhagen-Jörss and the Tomita-Takesaki theory.

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Modular Localization and Constructive Algebraic QFT

Gandalf Lechner

This talk was the second in a series of three (by R. Longo, myself, and Y. Tanimoto, respectively) on applications of Tomita-Takesaki modular theory in algebraic quantum field theory. The focus of my talk was the construction of examples of algebraic quantum field theories.

A central notion in this context is that of a *Borchers triple*. In the simplest two-dimensional situation, this consists of a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$, a unitary strongly continuous positive energy representation $U(x_+, x_-) = e^{ix_+P_+}e^{ix_-P_-}$ of $\mathbb{R}^2$ such that $\mathrm{ad}U(x_+, x_-)(\mathcal{M}) \subset \mathcal{M}$ for $x_+ \geq 0$, $x_- \leq 0$, and a unit vector $\Omega \in \mathcal{H}$ that is invariant under U and cyclic and separating for $\mathcal{M}$ (see [2], and [5] for a review).

Given a Borchers triple $(\mathcal{M}, U, \Omega)$, the representation U extends from $\mathbb{R}^2$ to the proper Poincaré group $\mathcal{P}$ by a theorem of Borchers [1]. The algebra $\mathcal{M}$ can be interpreted as being localized in the wedge region $W = \{x \in \mathbb{R}^2 : x_+ > 0, x_- < 0\}$, and by a canonical procedure, the triple defines a map from open subsets $\mathcal{O} \subset \mathbb{R}^2$ to von Neumann algebras $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ that is inclusion-preserving, local, covariant under the representation U, and fixed by $\mathcal{A}(W) = \mathcal{M}$.

If this map has also the property that Ω is cyclic for $\mathcal{A}(\mathcal{O})$ for every non-empty $\mathcal{O}$, these data describe a quantum field theory in its vacuum representation. One is therefore interested in finding examples of Borchers triples. Free field theory examples of Borchers triples are well-known.

In this talk, I reviewed two procedures for constructing examples of Borchers triples. The first [2] is related to Rieffel deformations of C^*-algebras [7] and is based on the notion of a *warped convolution*, a deformation procedure for operators in $\mathcal{B}(\mathcal{H})$. Let $(\mathcal{M}, U, \Omega)$ be a Borchers triple (for example, one given by a free field theory), and let $A \in \mathcal{M}$ be an operator that is smooth w.r.t. U, i.e. such that $x \mapsto \mathrm{ad}(U(x))(A)$ is smooth in norm. As a deformation parameter, consider a 2×2 matrix Q antisymmetric w.r.t. the Minkowski inner product. Then the warped convolution of A is defined as

$$A_Q = (2\pi)^{-2} \int_{\mathbb{R}^2} dp \int_{\mathbb{R}^2} dx \, e^{-ip\cdot x} \, \mathrm{ad}(U(Qp))(A) \, U(x).$$

This integral exists in an oscillatory sense on the dense domain of smooth vectors in $\mathcal{H}$ and extends to a bounded operator on all of $\mathcal{H}$. Denote the von Neumann algebra generated by all A_Q, $A \in \mathcal{M}$ smooth, by $\mathcal{M}_Q$.

Then the main theorem in this context, a consequence of various properties of the map $A \mapsto A_Q$, is that the triple $(\mathcal{M}_Q, U, \Omega)$ is again a Borchers triple if Q satisfies a positivity condition related to the spectrum of U. One thus obtains a family of Borchers triples, indexed by Q, from a given one. Triples with different parameters are inequivalent, but they all have the special property that the modular data of $\mathcal{M}_Q$ are independent of Q, i.e. coincide with the modular data of the original von Neumann algebra $\mathcal{M}_0 = \mathcal{M}$. It is an open problem whether the "local" von Neumann algebras $\mathcal{A}_Q(\mathcal{O})$ generated from $(\mathcal{M}_Q, U, \Omega)$ are non-trivial or not.

The second procedure for constructing Borchers triples is based on the notion of a crossing-symmetric R-matrix. One starts from the single particle Hilbert space $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}$, where $\mathcal{K}$ is a separable (often times finite-dimensional) Hilbert space for internal degrees of freedom, and $L^2(\mathbb{R}, d\theta)$ carries the usual realization of the unitary massive irreducible positive energy representation U_1 of $\mathcal{P}$. In this context, a crossing-symmetric R-matrix is a function R from $\mathbb{R}$ to unitaries on $\mathcal{K} \otimes \mathcal{K}$ satisfying a number of properties. In particular, $R(\theta)$ is required to satisfy the Yang-Baxter equation with spectral parameter θ and a symmetry condition that ensures that R generates a unitary representation of the symmetric group S_n on $\mathcal{H}_1^{\otimes n}$. Denoting by $\mathcal{H}_n \subset \mathcal{H}_1^{\otimes n}$ the subspace on which this representation acts trivially, the Hilbert space of the Borchers triple to be constructed is then $\mathcal{H} := \bigoplus_n \mathcal{H}_n$. The representation U is defined by second quantization of U_1, and Ω is defined as the Fock vacuum of $\mathcal{H}$.

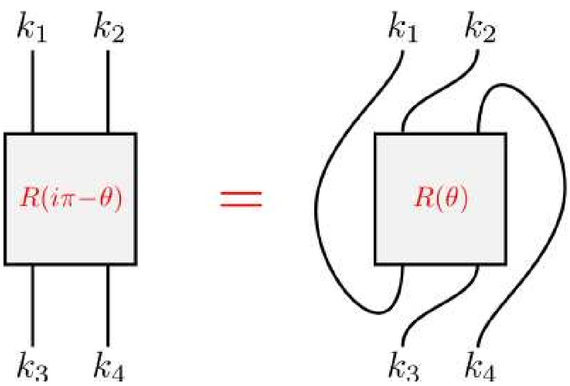
Similar to a usual Bose Fock space, also $\mathcal{H}$ carries canonical creation / annihilation operators, and their sum defines a quantum field φ [6]. These field operators generate a von Neumann algebra $\mathcal{M}$ such that $(\mathcal{M}, U, \Omega)$ is a Borchers triple if R extends to a bounded analytic function on the complex strip $0 < \mathrm{Im}\theta < \pi$ and the crossing symmetry

$$\langle k_1 \otimes k_2,\, R(i\pi - \theta)\, k_3 \otimes k_4 \rangle = \langle k_2 \otimes \Gamma k_4,\, R(\theta)\, \Gamma k_1 \otimes k_3 \rangle, \qquad \theta \in \mathbb{R},$$

reminiscent of the KMS condition. Here $k_1, \ldots, k_4$ are arbitrary vectors in $\mathcal{K}$, and Γ is an antiunitary involution on $\mathcal{K}$ related to the modular conjugation of $\mathcal{M}$ (see [3]). The local algebras are non-trivial if R satisfies further regularity conditions.

To make connections to other talks given at this meeting, let us make the following two observations.

(1) The crossing symmetry of R can be depicted graphically as



which suggests a relation to the string Fourier transform presented in the talk by A. Jaffe.

(2) The domain of the one-particle component of the modular operator $\Delta^{1/2}$ becomes a complex Hilbert space when closed in its graph norm. For the setting of the second example, this Hilbert space coincides with the classical Hardy space on the strip $0 < \mathrm{Im}\theta < \pi$ [4], and thus connects to the reproducing kernel Hilbert space setting discussed by P. Jorgensen.

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Reflection positivity and operator theoretic correlation inequalities

Tadahiro Miyao

This talk consists of the following three sections:

Section 1: Operator theoretic correlation inequalities
Section 2: Spin-reflection positivity
Section 3: Universality in the Hubbard model

1. Operator theoretic correlation inequalities

In Section 1, operator theoretic correlation inequalities [7, 8] are introduced as follows. Let $\mathfrak{H}$ be a complex Hilbert space. By a *convex cone*, we understand a closed convex set $\mathfrak{P} \subset \mathfrak{H}$ such that $t\mathfrak{P} \subseteq \mathfrak{P}$ for all $t \geq 0$ and $\mathfrak{P} \cap (-\mathfrak{P}) = \{0\}$.

Definition 1.1.

- The *dual cone of* $\mathfrak{P}$ is defined by $\mathfrak{P}^\dagger = \{\eta \in \mathfrak{H} \,|\, \langle \eta|\xi\rangle \geq 0 \ \forall \xi \in \mathfrak{P}\}$. We say that $\mathfrak{P}$ is *self-dual* if $\mathfrak{P} = \mathfrak{P}^\dagger$.
- A vector ξ is said to be *positive w.r.t.* $\mathfrak{P}$ if $\xi \in \mathfrak{P}$. We write this as $\xi \geq 0$ w.r.t. $\mathfrak{P}$.
- A vector $\eta \in \mathfrak{P}$ is called *strictly positive w.r.t.* $\mathfrak{P}$ whenever $\langle \xi|\eta\rangle > 0$ for all $\xi \in \mathfrak{P}\backslash\{0\}$. We write this as $\eta > 0$ w.r.t. $\mathfrak{P}$.

We denote by $\mathscr{B}(\mathfrak{H})$ the set of all bounded linear operators on $\mathfrak{H}$.

Definition 1.2. Let $A, B \in \mathscr{B}(\mathfrak{H})$. Let $\mathfrak{P}$ be a self-dual cone in $\mathfrak{H}$.

- If $A\mathfrak{P} \subseteq \mathfrak{P}$, we then write this as $A \trianglerighteq 0$ w.r.t. $\mathfrak{P}$. In this case, we say that *A preserves the positivity w.r.t.* $\mathfrak{P}$.
- Let $\mathfrak{H}_\mathbb{R}$ be a real subspace of $\mathfrak{H}$ generated by $\mathfrak{P}$. Suppose that $A\mathfrak{H}_\mathbb{R} \subseteq \mathfrak{H}_\mathbb{R}$ and $B\mathfrak{H}_\mathbb{R} \subseteq \mathfrak{H}_\mathbb{R}$. If $(A - B)\mathfrak{P} \subseteq \mathfrak{P}$, then we write this as $A \trianglerighteq B$ w.r.t. $\mathfrak{P}$.
- Let $A \in \mathscr{B}(\mathfrak{H})$. We write $A \triangleright 0$ w.r.t. $\mathfrak{P}$, if $A\xi > 0$ w.r.t. $\mathfrak{P}$ for all $\xi \in \mathfrak{P}\backslash\{0\}$. In this case, we say that *A improves the positivity w.r.t.* $\mathfrak{P}$.

We give an important example as follows: The set of all Hilbert–Schmidt class operators on $\mathfrak{H}$ is denoted by $\mathscr{L}^2(\mathfrak{H})$, i.e., $\mathscr{L}^2(\mathfrak{H}) = \{\xi \in \mathscr{B}(\mathfrak{H}) \,|\, \mathrm{Tr}[\xi^*\xi] < \infty\}$.We regard $\mathscr{L}^2(\mathfrak{H})$ as a Hilbert space equipped with the inner product $\langle \xi|\eta\rangle_{\mathscr{L}^2} = \mathrm{Tr}[\xi^*\eta]$, $\xi, \eta \in \mathscr{L}^2(\mathfrak{H})$.

Definition 1.3. For each $A \in \mathscr{B}(\mathfrak{H})$, the *left multiplication operator* is defined by

$$\mathcal{L}(A)\xi = A\xi, \quad \xi \in \mathscr{L}^2(\mathfrak{H}). \tag{1}$$

Similarly, the *right multiplication operator* is defined by

$$\mathcal{R}(A)\xi = \xi A, \quad \xi \in \mathscr{L}^2(\mathfrak{H}). \tag{2}$$

Note that $\mathcal{L}(A)$ and $\mathcal{R}(A)$ belong to $\mathscr{B}(\mathscr{L}^2(\mathfrak{H}))$.

Let ϑ be an antiunitary operator on $\mathfrak{H}$. Let Φ_ϑ be an isometric isomorphism from $\mathscr{L}^2(\mathfrak{H})$ onto $\mathfrak{H} \otimes \mathfrak{H}$ defined by

$$\Phi_\vartheta(|x\rangle\langle y|) = x \otimes \vartheta y \quad \forall x, y \in \mathfrak{H}. \tag{3}$$

Then,

$$\mathcal{L}(A) = \Phi_\vartheta^{-1} A \otimes 1 \Phi_\vartheta, \quad \mathcal{R}(\vartheta A^* \vartheta) = \Phi_\vartheta^{-1} 1 \otimes A \Phi_\vartheta \tag{4}$$

for each $A \in \mathscr{B}(\mathfrak{H})$. We write these facts simply as

$$\mathfrak{H} \otimes \mathfrak{H} = \mathscr{L}^2(\mathfrak{H}), \quad A \otimes 1 = \mathcal{L}(A), \quad 1 \otimes A = \mathcal{R}(\vartheta A^* \vartheta), \tag{5}$$

if no confusion arises.

Recall that a bounded linear operator ξ on $\mathfrak{H}$ is said to be *positive* if $\langle x|\xi x\rangle_{\mathfrak{H}} \ge 0$ for all $x \in \mathfrak{H}$. We write this as $\xi \ge 0$.

Definition 1.4. A canonical cone in $\mathscr{L}^2(\mathfrak{H})$ is given by

$$\mathscr{L}^2(\mathfrak{H})_+ = \Big\{\xi \in \mathscr{L}^2(\mathfrak{H}) \,\Big|\, \xi \text{ is self-adjoint and } \xi \ge 0 \text{ as an operator on } \mathfrak{H}\Big\}. \tag{6}$$

$\mathscr{L}^2(\mathfrak{H})_+$ is a self-dual cone in $\mathscr{L}^2(\mathfrak{H})$.

The following theorem is an abstract form of spin-reflection positivity.

Theorem 1.5. *Let $A, B_n \in \mathscr{B}(\mathfrak{H})$. Suppose that A is self-adjoint. Let H be a self-adjoint operator on $\mathscr{L}^2(\mathfrak{H})$ defined by*

$$H = \mathcal{L}(A) + \mathcal{R}(A) - \sum_{n=1}^{N} \mathcal{L}(B_n)\mathcal{R}(B_n^*). \tag{7}$$

We have $e^{-\beta H} \trianglerighteq 0$ w.r.t. $\mathscr{L}^2(\mathfrak{H})_+$ for all $\beta \ge 0$.

Remark that the framework here is closely related with [1, 3].

2. Spin-reflection positivity

Spin-reflection positivity was introduced by Lieb [6]. It originates from reflection positivity in the axiomatic quantum field theory [2, 9]. In this section, we explain Lieb's spin-reflection positivity in terms of operator theoretic correlation inequalities. Let Λ be a finite lattice. The Hamiltonian of the Hubbard model on Λ is given by

$$H_{\mathrm{H}} = \sum_{x,y\in\Lambda}\sum_{\sigma=\uparrow,\downarrow} t_{xy} c_{x\sigma}^* c_{y\sigma} + \sum_{x,y\in\Lambda} \frac{U_{xy}}{2}(n_x-1)(n_y-1), \tag{8}$$

where $c_{x\sigma}$ is the electron annihilation operator at site x; n_x is the electron number operator at site $x\in\Lambda$ given by $n_x = \sum_{\sigma=\uparrow,\downarrow} n_{x\sigma}$, $n_{x\sigma} = c_{x\sigma}^* c_{x\sigma}$. t_{xy} is the hopping matrix, U_{xy} is the energy of the Coulomb interaction. We suppose that $\{t_{xy}\}$ and $\{U_{xy}\}$ are real symmetric $|\Lambda|\times|\Lambda|$ matrices.

H_{H} acts in the half-filled space

$$\mathfrak{E} = \bigwedge^{|\Lambda|}(\ell^2(\Lambda)\oplus\ell^2(\Lambda)), \tag{9}$$

where $\bigwedge^n \mathfrak{h}$ indicates the n-fold antisymmetric tensor product of $\mathfrak{h}$.

Let $\mathfrak{E}[M]$ be the M-subspace. We set $H_{\mathrm{H}}[M] = H_{\mathrm{H}} \restriction \mathfrak{E}[M]$. By the hole-particle transformation W, we obtain the following identifications:

- $W\mathfrak{E}[M] = \mathcal{E}^{M^\dagger}\otimes\mathcal{E}^{M^\dagger} = \mathscr{L}^2(\mathcal{E}^{M^\dagger})$, where $\mathcal{E}^{M^\dagger} = \bigwedge^{M^\dagger}\ell^2(\Lambda)$ with $M^\dagger = M+|\Lambda|/2$
- The transformed Hamiltonian $\tilde{H}_{\mathrm{H}}[M] = WH_{\mathrm{H}}[M]W^{-1}$ becomes

$$\tilde{H}_{\mathrm{H}}[M] = \mathcal{L}(T)+\mathcal{R}(T) - \sum_{x,y\in\Lambda} U_{xy}\mathcal{L}(\mathsf{n}_x)\mathcal{R}(\mathsf{n}_y), \tag{10}$$

where

$$T = \sum_{x,y\in\Lambda} t_{xy}\mathsf{c}_x^*\mathsf{c}_y + \sum_{x,y\in\Lambda}\frac{U_{xy}}{2}\mathsf{n}_x\mathsf{n}_y, \quad \mathsf{n}_x = \mathsf{c}_x^*\mathsf{c}_x. \tag{11}$$

c_x is the annihilation operator on $\mathcal{E}^{M^\dagger}$.

Applying Theorem 1.5, we obtain the following:

Proposition 2.1. Assume that $\{U_{xy}\}$ is positive semi-definite. Assume that Λ is bipartite. Then $e^{-\beta\tilde{H}_{\mathrm{H}}[M]} \trianglerighteq 0$ w.r.t. $\mathscr{L}^2(\mathcal{E}^{M^\dagger})_+$ for all $\beta\ge 0$.

We can prove a stronger property as follows:

Theorem 2.2 ([7]). *Assume that $\{U_{xy}\}$ is positive definite. Assume that Λ is connected and bipartite. Then $e^{-\beta\tilde{H}_{\mathrm{H}}[M]} \rhd 0$ w.r.t. $\mathscr{L}^2(\mathcal{E}^{M^\dagger})_+$ for all $\beta>0$.*

Remark that, by the Perron-Frobenius-Faris theorem, this theorem implies Lieb's theorem [6], as we will explain the next section.

3. Universality in the Hubbard model

A new viewpoint of universality is introduced in strongly correlated electron system in this section; Our description relies on the operator theoretic correlation inequalities. We explain the Marshall-Lieb-Mattis theorem [4, 5] and Lieb's theorem [6] from a viewpoint of universality; in addition, from the new perspective, we prove that Lieb's theorem still holds true even if the electron-phonon and electron-photon interactions are taken into account. The details are given in [8].

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The role of positivity in generalized coherent state transforms

José Mourão
(joint work with T. Baier, W. Kirwin, J. P. Nunes and T. Thiemann)

From the rather messy quantization ambiguity of a symplectic manifold (M, ω) on a real polarization one gets a very nice, infinite dimensional, geodesically convex space of Kähler quantizations. This is done, in geometric quantization, by allowing the preferred local observables defining a *polarization* to be complex–valued while restricting them to a class satisfying adequate positivity conditions.

For systems with one degree of freedom this is illustrated by changing from the (infinite–dimensional) family of real quantizations with quantum Hilbert–spaces of L^2–functions of the variables,

$$z_{tf} = q + t\, f(p)\,, \qquad t \in \mathbb{R}$$

to the family of complex observables obtained by letting t to enter the upper half plane

$$z_{\tau f} = q + \tau\, f(p)\,, \qquad \tau \in \mathbb{C}, \Im(\tau) > 0\,.$$

It turns out that if $f'(p) > 0$ then several simplifying facts occur:

1. **Complex Structure:** There is a unique complex structure $J_{\tau f}$ on $\mathbb{R}^2$ for which $z_{\tau f}$ is a global holomorphic coordinate.

2. **Kähler Metric:** The (standard) symplectic form together with the complex structure J_{if} define on $\mathbb{R}^2$ the Kähler metric

$$\gamma_f = \frac{1}{f'(p)}\, dq^2 + f'(p)\, dp^2 \ , \tag{1}$$

with scalar curvature,

$$S(\gamma_f) = -\left(\frac{1}{f'(p)}\right)'' \ .$$

3. **Quantum Hilbert space:** much better defined than in the case of quantizations based on real observables:

$$\mathcal{H}_f^Q = \left\{\Psi(q,p) = \psi(z_{if})\, e^{-k_f(p)/2},\ ||\Psi|| < \infty\right\}$$

where ψ is a J_{if}–holomorphic function and $k_f(p) = pf(p) - \int f(p)dp$ is the Kähler potential for (1). Things improve even further if one takes into account the half–form correction.

4. **Reality conditions:** The inner product is fixed by geometric quantization and resolves the problem of "reality conditions". With the half–form correction, in our example, takes the form,

$$< \Psi_1, \Psi_2 >= \int_{\mathbb{R}^2} \overline{\psi_1(z_{if})}\psi_2(z_{if})\, e^{-k_f(p)}\, (f'(p))^{1/2}\, dqdp\, .$$

In this formalism the space of quantizations becomes the space of Kähler structures on a given symplectic manifold. In the complex picture, in which one fixes a complex structure and lets the symplectic form to change, on a compact manifold, the space of Kähler forms with fixed cohomology class is

$$\mathcal{H}/\mathbb{R} = \left\{\omega_\varphi = \omega + i\partial\bar{\partial}\,\varphi\,, \varphi \in C^\infty(M)\ :\ \gamma_\varphi = \omega_\varphi(\cdot, J\cdot) > 0\right\},$$

and can be interpreted as (part of the) space of Kähler quantizations of (M, ω). Despite being an open subset of $C^\infty(M)$, the space of Kähler potentials $\mathcal{H}$ with Mabuchi metric,

$$\langle f_1, f_2\rangle_\varphi = \int_M f_1\, f_2\, \frac{\omega_\varphi^n}{n!},$$

has the very rich geometry of an infinite dimensional symmetric space. In particular its geodesics, described by the complex homogeneous Monge–Ampère (CHMA) equation are given by one–parameter "groups" of *imaginary time* canonical transformations [4, 3, 14],

$$e^{isX_H}\ . \tag{2}$$

In [14] a method, based on the Gröbner theory of Lie series, to construct these imaginary time flows has been proposed, which effectively reduces the analytic Cauchy problem for the CHMA equation to analytically continuing to complex time the corresponding Hamiltonian flow. The coherent state transforms (CST) correspond to appropriate liftings of e^{isX_H} to the quantum bundle,

$$V_{is}^H\ :\ \mathcal{H}_0^Q \longrightarrow \mathcal{H}_{is}^Q\,. \tag{3}$$

Examples are given by Hall generalizations [7] of the Coherent State Segal–Bargmann transforms to complexifications $G_{\mathbb{C}}$ of compact Lie groups G (see also [8, 10, 6, 9]),

$$\begin{array}{rcl} U \,:\, L^2(G, dx) & \longrightarrow & \mathcal{H}L^2(G_{\mathbb{C}}, d\nu(g)) \\ U & = & \mathcal{C} \circ e^{\frac{\Delta}{2}}\,, \end{array} \tag{4}$$

where $G_{\mathbb{C}}$ is the unique complexification of G, $\mathcal{H}L^2$ means holomorphic L^2 functions, ν is the averaged heat kernel measure on $G_{\mathbb{C}}$.

Following the introduction of the concept of "complexifiers" in [15, 16] the CST (4) has been shown to be equivalent to geometric quantization transforms of the form (3) in [11, 12, 5, 13] with the complexifier H being the norm square of the moment map,

$$H = \frac{||\mu||^2}{2}\,. \tag{5}$$

The CST (3) with Hamiltonians H given by (5) (or other convex function of μ) play an important in tropical geometry [1] and in representation theory and algebraic geometry [2].

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Generalized transfer matrices and automorphic functions

Anke Pohl

As known for a long time, transfer matrix techniques prove to be powerful in the study of lattice spin systems, for example for deriving exact solutions of one- and two-dimensional systems such as the Onsager solution. Reflection positivity in lattice spin systems is intimately related to the existence of self-adjoint positive definite transfer matrices.

Also known for a long time, the correspondence principle of quantum mechanics suggests close relations between geometric and spectral entities of Riemannian manifolds (and, more generally, of Riemannian orbifolds). In particular, one expects strong interdependencies between geodesics (classical mechanical objects) on the one side and L^2-eigenfunctions and L^2-eigenvalues of the Laplacian, and more generally, resonances and resonant states (quantum mechanical objects) on the other side.

Over the last century much effort was spend on establishing instances of such interdependencies in a mathematically rigorous way. An ever increasing number of results were found and were seen to be of great importance for various areas of mathematics, including dynamical systems, spectral theory, harmonic analysis, representation theory, number theory, and mathematical physics. However, the full scope and depth of the relation between geometric and spectral objects of Riemannian orbifolds is still mysterious.

In the talk we restricted to the case of non-elementary hyperbolic surfaces $X = \Gamma\backslash\mathbb{H}$ with at most finitely many ends (of finite and infinite area). Here, $\mathbb{H}$ denotes the hyperbolic plane, and Γ is a discrete non-cyclic geometrically finite subgroup of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$. For these spaces, a relation between periodic geodesics and resonances is shown by the Selberg zeta function which is the dynamical zeta function given by

$$(1)\qquad Z_X(\beta) := \prod_{\ell\in L(X)} \prod_{k=0}^{\infty} \left(1 - e^{-(\beta+k)\ell}\right) \qquad (\beta \in \mathbb{C},\ \mathrm{Re}(\beta) \gg 1)$$

where $L(X)$ denotes the primitive geodesic length spectrum of X, counted with multiplicities. The infinite product in (1) converges if $\mathrm{Re}\,\beta$ is sufficiently large, and it has a meromorphic continuation to all of $\mathbb{C}$. The zeros of the Selberg zeta function consist of the resonances of X and some well-understood 'trivial' zeros (of rather topological nature). Thus, the Selberg zeta function Z_X establishes a relation between the geodesic length spectrum and the Laplace spectrum of X, or, in other words, a relation between geodesics and resonant states on the spectral level.

We discussed a construction of generalized transfer matrices and showed that these allow us to establish a relation between periodic geodesics and L^2-eigenfunctions beyond the spectral level, thereby improving on the connection provided by means of the Selberg zeta function. The construction of the generalized transfer matrices rely on a good choice of a discretization for the geodesic flow on X. We took advantage of the discretizations provided in [16], which are particularly well-suited for our purposes. Each such discretization provides a discrete dynamical system $F\colon D \to D$ on a union of certain intervals in $\mathbb{R}$ that is semi-conjugate to the geodesic flow on X and that branches into finitely many 'submaps' given by the Möbius action of some element in Γ. The associated generalized transfer matrix with parameter $\beta \in \mathbb{C}$ (transfer operator in the sense of Ruelle and Mayer) is

$$\mathcal{L}_\beta f(x) := \sum_{y \in F^{-1}(x)} e^{-\beta \ln |F'(y)|} f(y),$$

acting on functions $f\colon D \to \mathbb{C}$.

Some of the major results regarding the role of these transfer operators in the study of the interdependencies of geodesics and eigenfunctions and resonant states of X are roughly as follows:

- If X has finite area and at least one cusp, that is, an end of finite area, and if $\operatorname{Re}\beta \in (0,1)$ then the space of rapidly decaying L^2-eigenfunctions on X (Maass cusp forms) is isomorphic to the space of sufficiently regular eigenfunctions with eigenvalue 1 of $\mathcal{L}_\beta$ [12, 15, 14, 18, 13]. The isomorphism is given by an explicit integral transform. Up to date, transfer operator techniques are the only tool known to provide such a deep relation between geometric and spectral entities of hyperbolic surfaces.
- If X has finite or infinite area and at least one cusp then an induction procedure of the discretization of the geodesic flow used for the construction of $\mathcal{L}_\beta$ provides a uniformly expanding, infinitely branched discrete dynamical system. The associated transfer operator $\widetilde{\mathcal{L}}_\beta$ acts on a certain Banach space of holomorphic functions. As such it is nuclear of order zero and hence admissible for the thermodynamic formalism. Its Fredholm determinant equals the Selberg zeta function

  $$Z_X(\beta) = \det\left(1 - \widetilde{\mathcal{L}}_\beta\right).$$

  The possibility to represent Z_X as a Fredholm determinant of a transfer operator family suggests that many results obtained with the help of the Selberg zeta function and the Selberg trace formula should follow as a 'shadow' from results obtained via transfer operators. Moreover, transfer operator techniques provide an alternative proof of meromorphic extendability of the Selberg zeta function. See [12, 18, 17, 19] for all of these results.

- Eigenfunctions with eigenvalue 1 of $\mathcal{L}_\beta$ and $\widetilde{\mathcal{L}}_\beta$ are isomorphic, see [1] for Hecke triangle groups and forthcoming manuscripts for general Γ. This result together with the previously mentioned allows us to recover already a part of the spectral interpretations of the zeros of the Selberg zeta function without relying on the Selberg trace formula.
- Twists by finite-dimensional unitary representations can easily be accommodated by the transfer operators as additional weights. The results on the connection between Selberg zeta functions and $\widetilde{\mathcal{L}}_\beta$ as well as on the relation between $\mathcal{L}_\beta$ and $\widetilde{\mathcal{L}}_\beta$ extend to the twisted objects [19, 1].
- Also twists by finite-dimensional representations χ with non-expanding cusp monodromies (representations which are not necessarily unitary but have controlled behavior in cusps) can be accommodated by transfer operators. Transfer operator techniques are currently the only known method to prove meromorphic extendability of the χ-twisted Selberg zeta functions [8].
- These results recover, illuminate and refine the seminal transfer operator techniques for the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ by Mayer [10, 11], Chang–Mayer [3], Efrat [7], Lewis–Zagier [9], Bruggeman [2], and its extension to certain finite-index subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ [4, 5, 6].

It is expected that the mentioned results, in particular the isomorphism between eigenfunctions of transfer operators and Maass cusp forms, can be generalized to eigenfunctions of other regularity, to (Γ,χ)-twisted and vector-valued eigenfunctions, and to general resonant states. Moreover, generalizations to more general locally symmetric spaces are expected. The relation between these generalized transfer matrices and reflection positivity remains to be understood.

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Local nets of von Neumann algebras in the sine-Gordon model

Kasia Rejzner

In my talk I presented recent results on construction of local nets of von Neumann algebras using perturbative algebraic quantum field theory (pAQFT). The talk is based on [BR17, BFR17].

1. Perturbative AQFT

Algebraic quantum field theory (AQFT) is a convenient framework to investigate conceptual problems in QFT. It started as the axiomatic framework of Haag-Kastler [HK64]: a model is defined by associating to each region $\mathcal{O}$ of Minkowski spacetime an operator algebra $\mathfrak{A}(\mathcal{O})$ of observables that can be measured in $\mathcal{O}$. The physical notion of subsystems is realized by the condition of *Isotony*, i.e.: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$. Other axioms include: *Einstein causality* and the *Time-slice axiom.*

Perturbative algebraic quantum field theory (pAQFT) is a mathematically rigorous framework that allows to build interacting AQFT models, where $\mathfrak{A}(\mathcal{O})$ is now a formal power series in topological $*$-algebras. It combines Haag's idea of local quantum physics with methods of perturbation theory. Main ingredients:

- Free theory obtained by the formal *deformation quantization* of Poisson (Peierls) bracket: $\star$-product [DF01, BF00, BDF09].
- Interaction introduced in the causal approach to renormalization due to Epstein and Glaser [EG73].
- Generalization to curved spacetime [BFV03, HW01, FV12].

2. sine-Gordon model in pAQFT

In pAQFT we need the following physical input:

- A *globally hyperbolic* spacetime M. For the sine-Gordon model in two dimensions, $M = \mathbb{M}_2$, the 2-dimensional Minkowski spacetime.
- *Configuration space* $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...). For the scalar field: $\mathcal{E}(M) \equiv \mathcal{C}^\infty(M, \mathbb{R})$.
- *Dynamics*: we use a modification of the Lagrangian formalism.

Observables are functionals on $\mathcal{E}(M)$. For the free massless scalar field the equation of motion is $P\varphi = 0$, where $P = -\Box$ is (minus) the wave operator. For M globally hyperbolic, P admits retarded and advanced Green's functions Δ^R, Δ^A. They satisfy: $P \circ \Delta^{R/A} = \mathrm{id}_{\mathcal{D}(M)}$, $\Delta^{R/A} \circ (P\big|_{\mathcal{D}(M)}) = \mathrm{id}_{\mathcal{D}(M)}$ and

$$\mathrm{supp}(\Delta^R) \subset \{(x,y) \in M^2 | y \in J_-(x)\}\,, \quad \mathrm{supp}(\Delta^A) \subset \{(x,y) \in M^2 | y \in J_+(x)\}\,.$$

For the massless scalar field in 2D:

$$\Delta^R(x) = -\frac{1}{2}\theta(t - |\boldsymbol{x}|) \qquad \Delta^A(x) = -\frac{1}{2}\theta(-t - |\boldsymbol{x}|)\ , x = (t, \boldsymbol{x}) \in \mathbb{M}_2\ .$$

Their difference is the Pauli-Jordan "function": $\Delta \doteq \Delta^R - \Delta^A$.

The Poisson bracket of the free theory is

$$\{F, G\} \doteq \left\langle F^{(1)}, \Delta G^{(1)} \right\rangle .$$

We define the $\star$-product (deformation of the pointwise product):

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), W^{\otimes n} G^{(n)}(\varphi) \right\rangle \ ,$$

where W is the 2-point function of a Hadamard state and it differs from $\frac{i}{2}\Delta$ by a symmetric bidistribution, denoted by H. For the massless scalar field in 2D it is convenient to use the Hadamard parametrix

$$W(x) = \frac{i}{2}(\Delta^R(x) - \Delta^A(x)) + H(x) = -\frac{1}{4\pi} \ln\left(\frac{-x \cdot x + i\varepsilon t}{\mu^2}\right)$$

where $\mu > 0$ is the scale parameter that we need to fix. The Hadamard Parametrix W differs from the 2-point function of a Hadamard state by a smooth symmetric function v, $W_v = \frac{i}{2}(\Delta^R - \Delta^A) + H_v = W + v$. Define $\star_v$ as the star product induced by W_v, while $\star$ denotes the star product induced by the parametrix W. These products are intertwined by a "gauge transformation" $\alpha_H \doteq e^{\frac{\hbar}{2}\mathcal{D}_H}$, where $\mathcal{D}_H \doteq \left\langle H, \frac{\delta^2}{\delta\varphi^2} \right\rangle = \int H(x,y) \frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)} dxdy$. Hence $\star$, and $\star_v$ are equivalent products.

The free QFT is defined as $\mathfrak{A}_0(M) \doteq (\mathcal{F}(M)[[\hbar]], \star, *)$, where $F^*(\varphi) \doteq \overline{F(\varphi)}$ and $\mathcal{F}(M)$ is an appropriate functional space (some wavefront set conditions on $F^{(n)}(\varphi)$s induced by W).

For construction of interacting fields we need the time-ordered product. We define time-ordering operator $\mathcal{T}$ as:

$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^F)^{\otimes n} \right\rangle \ ,$$

where $\Delta^F = \frac{i}{2}(\Delta^A + \Delta^R) + H$ and $H = W - \frac{i}{2}\Delta$. Formally $\mathcal{T}$ corresponds to the operator of convolution with the oscillating Gaussian measure "with covariance $i\hbar\Delta^F$",

$$\mathcal{T}F(\varphi) \stackrel{\text{formal}}{=} \int F(\varphi - \varphi)\, d\mu_{i\hbar\Delta_F}(\varphi) \ .$$

We define the time-ordered product $\cdot_{\mathcal{T}}$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

In the sine-Gordon model, prominent role is played by the vertex operators. These are defined as $V_a(g) \doteq \int \exp(ia\Phi_x)g(x)dx$, where $\Phi_x(\varphi) \doteq \varphi(x)$ is the evaluation functional at x. Note that we are constructing the abstract algebra first, *with no reference to Fock space*. Using the commutative product $\cdot_{\mathcal{T}}$ we define the *S-matrix*:

$$\mathcal{S}(V) \doteq e_{\mathcal{T}}^{iV/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(iV/\hbar)})\,.$$

Interacting fields are defined by the formula of Bogoliubov:

$$R_V(F) \doteq (e_{\mathcal{T}}^{iV/\hbar})^{\star -1} \star (e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(V)^{-1} \mathcal{S}(V + \mu F)\big|_{\mu=0}$$

Passing from $\star$ to $\star_v$ means changing the Wick ordering. Denote $\alpha_{H_v}^{-1}F \doteq {:}F{:}_v$. The expectation value of the product of two normally-ordered observables F, G in the quasi-free Hadamard state with 2-point function W_v is:

$$\omega_v({:}F{:}_v \star {:}G{:}_v) \doteq \alpha_v({:}F{:}_v \star {:}G{:}_v)(0) = (F \star_v G)(0)\,.$$

Similar for the S-matrix: $\omega_v(\mathcal{S}(\lambda{:}V{:}_v)) \doteq \alpha_v \left(e_{\mathcal{T}}^{i\lambda{:}V{:}_v/\hbar} \right)(0) = e_{\mathcal{T}_v}^{i\lambda V/\hbar}(0)$. Here $\cdot_{\mathcal{T}_v}$ is the time-ordered product corresponding to $\star_v$.

Theorem 2.1. *The formal S-matrix $\alpha_v \circ \mathcal{S}(\lambda{:}V{:}_v) = e_{\mathcal{T}_v}^{i\lambda V/\hbar}$ in the-sine Gordon model with $V = \frac{1}{2}(V_a(f) + V_{-a}(f))$ and $0 < \beta = \hbar a^2/4\pi < 1$, $f \in \mathcal{D}(M)$, converges as a functional on the configuration space in the appropriate topology (related to Hörmander topology on distribution spaces).*

In [BFR17] it was shown that by choosing appropriate Hadamard states (based on the construction of [DM06]) one can show that the interacting fields form a net of von Neumann algebras. These Hadamard states are, moreover, locally normal to vacuum states for scalar field theories with non-zero mass.

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Half-sided modular inclusions (and free products in AQFT)

Yoh Tanimoto

(joint work with Roberto Longo, Yoshimichi Ueda)

1. Conformal nets on the circle

In two-dimensional conformal field theory on the Minkowski space, important observables such as currents and stress-energy tensor decompose into the so-called chiral components, which are quantum fields living on one of the lightrays. These chiral components extends further to the circle (the one-point compactification of the lightray) (the Lüscher-Mack theorem, see [FST89, Section 3.2]). *It turns out that operator algebras are useful in constructing such chiral components.*

In the operator-algebraic approach, such a chiral component is realized as a **Möbius covariant net** on S^1: a family $\{\mathcal{A}(I)\}_{I\subset S^1}$ of von Neumann algebras parametrized by open, proper, connected and non empty intervals in S^1, a unitary representation U of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ and the "vacuum" vector Ω invariant for U, which satisfy the Haag-Kastler axioms, namely, isotony, locality, Möbius covariance, positivity of energy and cyclicity of the vacuum [GF93, Definition 2.5].

It follows from these axioms that Ω is cyclic and separating for each algebra $\mathcal{A}(I)$ (Reeh-Schlieder property), hence we can define the modular objects:

$$S_I : \mathcal{A}(I)\Omega \ni x\Omega \longmapsto x^*\Omega,$$

whose closure is still denoted by S_I and $S_I = J_I \Delta_I^{\frac{1}{2}}$ is the polar decomposition.

Let us recall that the lightray $\mathbb{R}$ is identified as a subset of S^1 by the stereographic projection. With this identification, the positive half-line $\mathbb{R}_+$ is an interval in S^1. For a Möbius covariant net, the Bisognano-Wichmann property holds [GF93, Theorem 2.19]: $\Delta_{\mathbb{R}_+}^{it} = U(D_{\mathbb{R}_+}(2\pi t))$, where $D_{\mathbb{R}_+}$ is the one-parameter subgroup of dilations of $\mathrm{PSL}(2,\mathbb{R})$, which clearly preserves $\mathbb{R}_+$. Now, observe that, by covariance, it holds that $\mathrm{Ad}\,\Delta^{it}(\mathcal{A}(\mathbb{R}_+ + 1)) \subset \mathcal{A}(\mathbb{R}_+ + 1)$ for $t \geq 0$. This inclusion of algebras, called a half-sided modular inclusion, turns out to contain much information of the given net.

2. Half-sided modular inclusions

Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras and Ω be a vector cyclic and separating for both $\mathcal{N}$ and $\mathcal{M}$. We can then define the modular group $\Delta_{\mathcal{M}}^{it}$ of $\mathcal{M}$ with respect to Ω. This triple $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is said to be a **half-sided modular inclusion** (HSMI) if $\mathrm{Ad}\,\Delta_{\mathcal{M}}^{it}(\mathcal{N}) \subset \mathcal{N}$ for $t \geq 0$. From this simple object, many interesting properties follow, among which is that $\Delta_{\mathcal{M}}^{it}$ and $\Delta_{\mathcal{N}}^{is}$ generate a positive-energy representation of the translation-dilation group [Wie93] [AZ05, Theorem 2.1]. Moreover, if Ω is cyclic for $\mathcal{N}' \cap \mathcal{M}$ ($\mathcal{N}'$ is the set of all bounded operators commuting with $\mathcal{N}$), this inclusion is said to be **standard**.

Let $(\mathcal{A}, U, \Omega)$ be a Möbius covariant net. It is said to be **strongly additive** if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ holds, where I_1 and I_2 are two intervals made from an interval I by removing one point, and $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ denotes the von Neumann algebra generated by $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$. There is a one-to-one correspondence between

- Standard half-sided modular inclusions $(\mathcal{N} \subset \mathcal{M}, \Omega)$
- Strongly additive Möbius covariant nets $(\mathcal{A}, U, \Omega)$

and the correspondence is given by $\mathcal{N} = \mathcal{A}(\mathbb{R}_+ + 1), \mathcal{M} = \mathcal{A}(\mathbb{R}_+)$ [GLW98, Corollary 1.9].

This correspondence allows one to construct new Möbius covariant nets from standard HSMIs. For example, consider the Virasoro nets Vir_c with $c > 1$ (the nets generated by the stress-energy tensor only). They are known not to be strongly additive [BSM90, Section 4], therefore, from the standard HSMI $(\mathcal{N} = \mathrm{Vir}_c(\mathbb{R}_+ + 1), \mathcal{M} = \mathrm{Vir}_c(\mathbb{R}_+), \Omega)$ one obtains a Möbius covariant net $\mathcal{A}$ which is different from Vir_c. These nets have not been identified with any known Möbius covariant net.

Another construction goes as follows: for a given Möbius covariant net $\mathcal{A}$, one considers its restriction $\mathcal{A}|_{\mathbb{R}}$ to the real line $\mathbb{R}$. If one takes a KMS state φ on $\mathcal{A}|_{\mathbb{R}}$ and makes the GNS representation π_φ with the GNS vector Ω_φ, with respect to the translation group, then the inclusion $(\pi_\varphi(\bigcup_{I \Subset \mathbb{R}_+} \mathcal{A}(I)) \subset \pi_\varphi(\bigcup_{I \Subset \mathbb{R}} \mathcal{A}(I))'', \Omega_\varphi)$ is a standard HSMI, and hence one can construct a Möbius covariant net [Lon01, Proposition 3.2]. We have constructed continuously many differerent KMS states on $\mathrm{Vir}_c, c \geq 1$ [CLTW12, Section 5], therefore, from these standard HSMIs one can construct (possibly new) Möbius covariant nets.

Problem 2.1. Determine whether these Möbius covariant nets are isomorphic to any known net.

3. Non standard HSMI from free products

If $\mathcal{A}$ is a Haag-Kastler net in $1+1$ or more dimensions with the Bisognano-Wichmann property (meaning that the modular group is the Lorentz boost), then $(\mathcal{A}(W+a) \subset \mathcal{A}(W), \Omega)$ is a half-sided modular inclusion, where W is a wedge region and a is a past lightlike vector in W. It is in general a difficult problem to determine whether such a given HSMI is standard or not, if $\mathcal{A}$ is not the free field net. Some successful cases are those coming from two-dimensional interacting nets [Tan14], and they are some fixed point subnets of the tensor product of the so-called U(1)-current net [BT15, Section 5.3]. Many other cases are open.

Problem 3.1. Determine whether the HSMIs coming from the interacting nets of [Lec08] are standard.

Now it is a natural question whether there is a HSMI $(\mathcal{N} \subset \mathcal{M}, \Omega)$ where $\mathcal{N}' \cap \mathcal{M}$ is trivial, i.e. containing only the multiples of the identity operator. We answer this positively by constructing an example from free products [LTU17].

A **free product** of a family of von Neumann algebras $\{\mathcal{M}_k\}_{k\in K}$ with respect to cyclic and separating vectors $\{\Omega_k\}_{k\in K}$ is a large von Neumann algebra containing isomorphic images of all $\mathcal{M}_k$'s which are highly non commutative, and equipped with a cyclic separating vector Ω [Voi85]. One can determine the modular objects of $(\mathcal{M}, \Omega)$ in terms of those of $(\mathcal{M}_k, \Omega_k)$ [Bar95, Lemma 1].

Let $(\mathcal{N}_0 \subset \mathcal{M}_0, \Omega_0)$ be a standard HSMI and $\{(\mathcal{N}_k \subset \mathcal{M}_k, \Omega_k)\}_{k\in K}$ isomorphic copies of $(\mathcal{N}_0 \subset \mathcal{M}_0, \Omega_0)$. We can then construct an inclusion of the free product von Neumann algebras $(\mathcal{N} \subset \mathcal{M}, \Omega)$. As the modular objects are known, it is immediate to see that this is a HSMI.

Theorem 1. *If $|K| = \infty$, then for the free product HSMI $(\mathcal{N} \subset \mathcal{M}, \Omega)$, $\mathcal{N}' \cap \mathcal{M}$ is trivial.*

Problem 3.2. Determine $\mathcal{N}' \cap \mathcal{M}$ when $|K| < \infty$, and if it is nontrivial, study the corresponding Möbius covariant net.

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Wick rotation for quantum field theories on degenerate Moyal space/-time

Rainer Verch
(joint work with Harald Grosse, Gandalf Lechner, Thomas Ludwig)

This talk is based on joint a article of Harald Grosse, Gandalf Lechner, Thomas Ludwig and Rainer Verch [1]. The main result is an extension of the Osterwalder-Schrader theorem [2, 3] which establishes a (bijective) correspondence between Euclidean quantum field theories on d-dimensional Euclidean space $\mathbb{R}^d$ and Wightman quantum field theories on d-dimensional Minkowski spacetime $\mathbb{R}^{1,d-1}$, to the case where the underlying space/-time is non-commutative as the result of a Moyal deformation. For the methods used to be applicable, it is important that the Moyal deformation does not act on the time coordinate, or the Euclidean coordinate with respect to which the condition of reflection positivity is imposed for the Euclidean quantum field theory. In establishing this result, we rely on a generalization, or reformulation, of the Osterwalder-Schrader theorem to the operator-algebraic framework of quantum field theory due to Schlingemann [4] which in turn relies substantially on the theory of virtual group representations based on a certain type of reflection positivity as another branch of generalization of the Osterwalder-Schrader theorem, developed in works by Fröhlich, Osterwalder and Seiler [5], Klein and Landau [6, 7], and Jorgensen and Olafsson [8].

Schlingemann's result states that, if $\mathcal{M}$ denotes a Minkowski spacetime quantum field theory in the operator algebraic setting (with properties such as Poincaré covariance, locality, spectrum condition and with a vacuum state), and if $\mathcal{E}$ denotes a Euclidean quantum field theory in the operator algebraic setting (characterized by Euclidean covariance, commutativity, and existence of a reflection-positive functional), and if the theories satisfy a property referred to as time-zero condition (reminiscent of the condition that solutions to a hyperbolic field equation are determined by their Cauchy data), then a Euclidean quantum field theory $\mathcal{E}$ uniquely determines a Minkowski spacetime quantum field theory $\mathcal{M} = \mathcal{M}(\mathcal{E})$; this is an abstract form of what in physics is called "Wick rotation".

For any operator algebra $\mathcal{A}$ that carries a group action τ_x, $x \in \mathbb{R}^d$, of $\mathbb{R}^d$ by automorphisms, one can obtain a new operator algebra $\mathcal{A}_\theta$ by endowing $\mathcal{A}$ with a new operator product $\times_\theta$, the Moyal-Rieffel product [9], given by

$$A \times_\theta B = \frac{1}{(2\pi)^d} \int dp\, dx\, e^{i(p,x)} \tau_{\theta p}(A) \tau_x(B) \qquad (A, B \in \mathcal{A}) .$$

where $\theta = (\theta_{\mu\nu})_{\mu,\nu=1}^d$ is a real, anti-symmetric matrix and (p, q) is the Euclidean scalar product (or the Minkowski metric product if the underlying $\mathbb{R}^d$ is interpreted as Minkowski spacetime).

For concreteness, focus on the case $d = 4$ (generalizations to other dimensions d

are obvious), and take

$$\theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the operator algebras $\mathcal{M}$ and $\mathcal{E}$, both carrying automorphic actions of the translation group, can be Moyal-Rieffel deformed into $\mathcal{M}_\theta$ and $\mathcal{E}_\theta$. It turns out that $\mathcal{E}_\theta$ still fulfills the conditions of Schlingemann's result (although with a smaller covariance group) and thus one can assign a Minkowski spacetime theory $\mathcal{M}(\mathcal{E}_\theta)$ to $\mathcal{E}_\theta$. On the other hand, one can also Moyal-Rieffel deform the Minkowski spacetime quantum field theory $\mathcal{M} = \mathcal{M}(\mathcal{E})$ into $\mathcal{M}(\mathcal{E})_\theta$. One of the important results of [1] is that $\mathcal{M}(\mathcal{E}_\theta)$ and $\mathcal{M}(\mathcal{E})_\theta$ are isomorphic. (For a commutative diagram depicting this, see [1].)

We mention that there are various motivations to consider quantum field theories on non-commutative spaces and spacetimes. In approaches to quantum gravity, there is some motivation for uncertainty relations between spacetime coordinates akin to the uncertainty relations between position and momentum in quantum mechanics [10, 11]. On the other hand, the Moyal-Rieffel deformation (and other ways of making spacetime non-commutative) has a delocalizing effect and may be helpful in "softening" the short-distance/ultraviolet problems encountered in the construction of interacting quantum field theories. In fact there are some promising results in that direction [12, 13, 14]. Actually, this was the original motivation for introducing quantum fields on deformed Minkowski spacetime by Snyder in 1947 [15].

It is at present unclear if, or how, the result can be extended to the case that θ is invertible and hence there are noncommutativity relations between space and time coordinates. It is known that Wick rotation cannot be done naively in this situation [16].

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Reflection positivity in large-deformation limits of noncommutative field theories

Raimar Wulkenhaar
(joint work with Harald Grosse, Akifumi Sako)

1. Quantum field theory of matrix models

We investigate the possibility to construct quantum field theories as limits of models defined on some Euclidean noncommutative space. Such models are essentially matrix models with action $S(\Phi) = \mathrm{tr}(E\Phi^2 + \mathrm{pol}(\Phi))$, understood as limit of finite matrices. Here E is a positive selfadjoint operator which defines a dimension $D = \inf\{p > 0 \ : \ \mathrm{tr}((1+E)^{-p/2}) < \infty\}$, and $\mathrm{pol}(\Phi) = \sum_{k=1}^{r} \lambda_k \Phi^k$. The task could be to give a meaning to the limit measure $\frac{1}{\mathcal{Z}} e^{-V\,S(\Phi)} d\Phi$, where $V > 0$ represents the volume.

We do not suppose that the limit can be constructed. Instead we derive (for $\mathcal{N} \times \mathcal{N}$-matrices) equations between moments of the measure, simplify them by further Ward-Takahashi identities [1] resulting from the $U(\mathcal{N})$-group action, take the limit of the equations (which requires renormalisation $\Phi \mapsto \sqrt{Z}\Phi$ and suitable dependence $Z(\mathcal{N}), \lambda_k(\mathcal{N})$ on $\mathcal{N}, D$) and look for exact solutions of these Schwinger-Dyson equations.

This strategy was developed and investigated first for $\mathrm{pol}(\Phi) = \frac{\lambda}{4}\Phi^4$ in $D = 4$ and in a special limit $\mathcal{N}, V \to \infty$, with $\frac{\mathcal{N}}{V^{2/D}} = \Lambda^2$ fixed, followed by $\Lambda \to \infty$ [2]. In examples, this limit corresponds to a large-deformation limit of noncommutative geometries. We proved that this approach collapsed the tower of Schwinger-Dyson equations into a closed non-linear integral equation for the matricial 2-point function and a hierarchy of affine integral equations for all higher correlation functions. In fact higher functions were algebraically expressable in terms of fundamental building blocks, which in particular proved that the β-function in this matricial

$\lambda\Phi^4$-model is identically zero (perturbatively proved in [1]). The equation for the 2-point function was reduced to a fixed point problem (for which we proved existence of a solution) for the boundary 2-point function $\mathbb{R}_+ \ni x \mapsto G(x,0)$.

Recent highlight is the $\lambda\Phi^3$ model in $D = 2$ [3] and $D \in \{4,6\}$ [4] where renormalisation requires (for $D = 6$) to consider

$$S(\Phi) = \mathrm{tr}\big(ZE\Phi^2 + (\kappa + \nu E + \zeta E^2)\Phi + \tfrac{1}{2}\mu_{bare}^2\Phi^2 + \tfrac{1}{3}\lambda_{bare}\Phi^3\big). \tag{1}$$

BPHZ normalisation conditions were directly implemented in the Schwinger-Dyson equation, leading to exact formulae for $Z, \kappa, \nu, \zeta, \mu_{bare}^2, \lambda_{bare}$ as function of $\mathcal{N}, V$ and the given spectrum of E. It turns out that λ_{bare} is $Z^{\frac{3}{2}}$ times a running coupling constant which corresponds to positive β-function for real λ, λ_{bare}. Nevertheless there is no Landau ghost; the model can be solved up to any scale Λ. After renormalisation we obtain a closed non-linear equation for the 1-point function. This equation is exactly solvable similar to the Makeenko-Semenoff solution [5] of $f^2(x) + \lambda^2 \int_a^b dt\ \rho(t)\frac{f(x)-f(t)}{x-t} = x$ by $f(x) = \sqrt{x+c} + \frac{\lambda^2}{2}\int_a^b \frac{dt\ \rho(t)}{(\sqrt{x+c}+\sqrt{t+c})\sqrt{t+c}}$ (together with a consistency condition on c). We prove: Let the spectrum of E converge in the considered limit to a positive function $e(x)$, and let $X(x) := (2e(x)+1)^2$. Then for $D = 6$ the 1-point function reads in the considered limit

$$\begin{aligned} G(x) = {}& \frac{\sqrt{(X+c)(1+c)} - c - \sqrt{X}}{2\lambda} \\ &+ \frac{\lambda}{4}\int_1^\infty \frac{dT\ (e^{-1}(\frac{\sqrt{T}-1}{2}))^2\ (\sqrt{X+c}-\sqrt{1+c})^2}{\sqrt{T}e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{X+c}+\sqrt{T+c})(\sqrt{1+c}+\sqrt{T+c})^2\sqrt{T+c}}, \end{aligned} \tag{2}$$

with $c(\lambda)$ the implicit solution of $-c = \lambda^2 \int_1^\infty \frac{dT\ (e^{-1}(\frac{\sqrt{T}-1}{2}))^2}{\sqrt{T}e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{1+c}+\sqrt{T+c})^3\sqrt{T+c}}$. We checked that Taylor expansion to $\mathcal{O}(\lambda^3)$ agrees with renormalised Feynman graph computation. See [4].

Higher N-point functions can be viewed as representations of the permutation group. Every permutation is a product of cycles. Collecting permutations of the same cycle lengths $(N_1, \dots, N_B)$ leads to a decomposition of (total) N-point functions into $N_1 + \dots + N_B$-point functions $G(x_1^1, \dots, x_{N_1}^1|\dots|x_1^B, \dots, x_{N_B}^B)$. It was straightforward [3] to reduce them to $1 + \dots + 1$-point functions:

$$\begin{aligned} &G(x_1^1, \dots, x_{N_1}^1|\dots|x_1^B, \dots, x_{N_B}^B) \\ &= \frac{1}{\lambda^B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B}\underbrace{G(x_{k_1}^1|\dots|x_{k_B}^B)}_{(*)}\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{4\lambda}{(2e(x_{k_\beta}^\beta)+1)^2 - (2e(x_{l_\beta}^\beta)+1)^2}, \end{aligned} \tag{3}$$

where for $B = 1$ one should read $e(x_{k_1}^1) + \frac{1}{2} + \lambda G(x_{k_1}^1)$ instead of $(*)$. Solving the equations for the $1 + \dots + 1$-functions is a difficult combinatorial problem. Making essential use of Bell polynomials we proved (with $X(x^i) = (2e(x^i)+1)^2$):

$$G(x^1|x^2) = \frac{4\lambda^2}{\sqrt{X(x^1)+c}\sqrt{X(x^2)+c}(\sqrt{X(x^1)+c}+\sqrt{X(x^2)+c})^2}$$

$$G(x^1|\dots|x^B) = \frac{d^{B-3}}{dt^{B-3}}\Big(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}}\frac{1}{\sqrt{X(x^1)+c-2t}^3}\cdots\frac{1}{\sqrt{X(x^B)+c-2t}^3}\Big)\Big|_{t=0}$$

for $B=2$ and $B\geq 3$, respectively, where $R(t)$ is an explicit integral [3, eq. (4.9)], which depends on $D, \lambda, e(.)$.

2. Schwinger functions and reflection positivity

It was speculated that space-time might be a noncommutative manifold. In its Euclidean formulation, a scalar field would be an element of a noncommutative algebra $\mathcal{A}$ which in many cases is approximated by matrices. A convenient example is the D-dimensional Moyal space, the Rieffel deformation of Schwartz functions by translation, $(f\star g)(\xi) = \int_{\mathbb{R}^{2D}} \frac{d(k,y)}{(2\pi)^D} f(\xi+\frac{1}{2}\Theta k)g(\xi+y)e^{\mathrm{i}\langle k,y}$, where Θ is a skew-symmetric real $D\times D$-matrix. We describe the transition to matrices in $D=2$ with $\Theta = \left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0\end{smallmatrix}\right)$. The functions $f_{mn}(z) = 2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}z\Big)^{n-m}L_m^{n-m}\Big(\frac{2|z|^2}{\theta}\Big)e^{-\frac{|z|}{\theta}}$, with $z\in\mathbb{C}\equiv\mathbb{R}^2$, satisfy $(f_{mn}\star f_{kl})(z) = \delta_{nk}f_{ml}(z)$ and $\int_{\mathbb{C}} dz\, f_{mn}(z) = 2\pi\theta\delta_{mn}$. Therefore, expanding an action functional for a scalar field $\varphi = \sum_{\underline{m},\underline{n}}\Phi_{\underline{mn}}f_{\underline{mn}}$ on Moyal space, where $\underline{m} = (m_1,\dots,m_{D/2})$ and $f_{\underline{mn}}(\underline{z}) = \prod_{i=1}^{D/2} f_{m_in_i}(z_i)$, in this matrix basis leads back to the starting point (1) of a matricial QFT model. Connected Schwinger N-point functions on Moyal space can thus be obtained via

(4)

$$S_c(\underline{z}_1,\dots,\underline{z}_N) := \lim_{\mathcal{N},V\to\infty}\sum_{\underline{m}_i,\underline{n}_i} f_{\underline{m}_1\underline{n}_1}(\underline{z}_1)\cdots f_{\underline{m}_N\underline{n}_N}(\underline{z}_N)\frac{(-\mathrm{i})^N\partial^N\log\hat{\mathcal{Z}}(J)}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0},$$

where formally $\hat{\mathcal{Z}}(J) = \int D\Phi\exp(-S(\Phi)+\mathrm{i}V\sum_{\underline{ab}}\Phi_{\underline{ab}}J_{\underline{ab}})$. We proved that only the diagonals $G(x^1,\dots,x^1|\dots|x^B,\dots,x^B)$ of the rigorously constructed matricial correlation functions (3) contribute to the limit.

Inserting the explicit formulae we were able to check reflection positivity [6]. It should be known that reflection positivity implies the following for the momentum space Schwinger functions $\hat{S}$: *the temporal Fourier transform from independent energies* p_j^0 *to time differences* $\tau_j > 0$ *is, for all spatial momenta* $\vec{p}_j$*, a positive definite function on* $\mathbb{R}_+^m$. By the Hausdorff-Bernstein-Widder theorem, (i) positive definiteness is equivalent to (ii) being Laplace transform of a positive measure and to (iii) being a completely monotonic function. The latter property is what we check. We find that the 2-point function of $\lambda\Phi_D^3$ on Moyal space is reflection positive iff $D=4,6$ (not $D=2$!) and $\lambda\in\mathbb{R}$ (where the partition function does not define a measure). The Källén-Lehmann measure was explicitly computed; it consists of a 'broadened mass shell' of width $2\mu^2\sqrt{-c}$ centred at $p^2=\mu^2$ (with

c given after (2), $|\lambda| \leq \lambda_c$ expressed in terms of the Lambert W-function) and a 'scattering part' supported on $p^2 \geq 2\mu^2$. See [4, Thm 6.1+6.2].

In unpublished work we prove that the projection to diagonal matricial correlation functions violates reflection positivity in higher Schwinger N-point functions. Hence, the above limit procedure needs modification. A natural suggestion would be to replace in (4) the pointwise product $f_{\underline{m}_1\underline{n}_1}(\underline{z}_1)\cdots f_{\underline{m}_N\underline{n}_N}(\underline{z}_N)$ by a state $\omega_{z_1,\dots,\underline{z}_N}(f_{\underline{m}_1\underline{n}_1}\otimes\cdots\otimes f_{\underline{m}_N\underline{n}_N})$ on $\mathcal{A}^{\otimes N}$. It would be interesting to study whether the choice of state permits enough flexibility to rescue reflection positivity.

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