Mathematisches Forschungsinstitut Oberwolfach

Report No. 16/2018

DOI: 10.4171/OWR/2018/16

Challenges in Optimal Control of Nonlinear PDE-Systems

Organised by
Michael Hintermüller, Berlin
Karl Kunisch, Graz
Günter Leugering, Erlangen
Elisabetta Rocca, Pavia

8 April – 14 April 2018

Abstract. The workshop focussed on various aspects of optimal control problems for systems of nonlinear partial differential equations. In particular, discussions around keynote presentations in the areas of optimal control of nonlinear/non-smooth systems, optimal control of systems involving nonlocal operators, shape and topology optimization, feedback control and stabilization, sparse control, and associated numerical analysis as well as design and analysis of solution algorithms were promoted. Moreover, also aspects of control of fluid structure interaction problems as well as problems arising in the optimal control of quantum systems were considered.

Mathematics Subject Classification (2010): 35Q93, 35R06, 49J20, 49J21, 49J40, 49J52, 49J53, 49K20, 49K21, 49N45, 49M70, 49Q10, 90C46.

Introduction by the Organisers

Optimal control problems for partial differential equations or variational inequalities nowadays increasingly penetrate the applied sciences and by doing so they are confronted with major new challenges. As a result, besides new mathematical models, novel analytical as well as numerical tools need to be developed. Correspondingly, motivated by optimal control problems for nonlinear partial differential equation (PDE) systems which are related to practical applications, the aims of the workshop were to bring together a group of international experts working at the forefront of research in the field, to foster in-depth-discussions crystallizing around a number of keynote presentations as well as discussion groups on focal topics emerging during the workshop, and to establish an (international) exchange forum for problems, techniques and solutions, both analytically as well as numerically. In particular, the organizers also strived for diversity in the group of invited scientists in order to enable transfer of information from senior to young researchers, and vice versa.

The scientific activity of the workshop developed around several keynote topics with associated keynote presentations, ad hoc presentations, e.g., in the late afternoon or evening, and the organization of discussion groups on emerging focal points. Among the focus topics, the following ones were of particular interest:

- **Control of nonlinear or non-smooth state systems.** Starting points for the discussion were, e.g., state systems of (quasi) variational inequality ((Q)VI) type with applications in thermodynamics or chemotaxis. Specifically, advanced analysis of the control-to-state map and the derivation of proper (sharp) stationarity conditions were focus points. Moreover, local stability analysis (in the spirit of second-order conditions) was considered.
- **Control of state systems with nonlocal operators.** Specific examples which were highlighted are nonlocal convective Cahn-Hilliard systems, systems for describing non-isothermal phase transitions, and nonlocal Cahn-Hilliard-Navier-Stokes systems. Additional complexities came from degenerate mobilities or singular potentials, and connections to non-smooth systems arise whenever non-smooth potentials, such as the double obstacle potential, were considered.
- **Shape and topology optimization.** This is an important branch of optimal design subject to partial differential equations with many applications in engineering and recently also biomedical sciences. Specific topics of interest discussed at the meeting were the establishment of analytical tools for enabling a joint shape and topological derivative (currently, and apart from a very small number of attempts, these two concepts are still considered in separate), second-order analysis, and problems with non-smooth components, either in the data or through considering VI state systems.
- **Feedback control or stabilization.** Feedback stabilization or control are important topics not only in aero-dynamics, but also in other problems involving fluid flow such as stabilization of unsteady flow, flow over surfaces, injection of polymer solutions, mass transport through porous walls, etc. Some of the major research questions discussed during the meeeting involved the type of feedback law (linear vs. nonlinear), the proper choice of Lyapunov functionals, and the treatment of Riccati equations. The latter also play a role for instance in applications of robust optimal placement of sensor networks. This problem class was also considered in this workshop, and it was highlighted that it typically requires to develop suitable solution techniques for ultra-high dimensional Riccati equations upon discretization.

- **Sparse control.** Very recently non-smooth control problems with the aim of computing optimal controls with sparse support set have come into focus. Particular applications are related to the optimal placement of actuators. But there is also a connection to inverse problems with sparsity-promoting priors. The workshop focued on modeling, analysis and numerics for such problems. In particular, as the associated optimal controls are typically measures only, dualization frameworks (including the sound understanding of dense embeddings of classes of convex sets in Sobolev spaces) were studied, proper stationarity and stability concepts were derived and optimal discretization schemes were addressed.
- **Numerical analysis and algorithm design / analysis.** As many of the aforementioned problem classes are either entirely new or have been studied from an analytical point of view only, the workshop also strived for advancing the development of proper discretization and numerical solution schemes. Exemplarily we mention that optimal control problems for VIs cannot be solved by techniques known for the iterative solution of optimal control problems for PDE-systems. This is related to the non-smooth character of the VI problem and the constraint degeneracy which prevents existence of Karush-Kuhn-Tucker-type multipliers. Another example is related to sparse controls which gives rise to questions concerning the discretization of measures and their efficient numerical treatment.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, “US Junior Oberwolfach Fellows”.

Workshop: Challenges in Optimal Control of Nonlinear PDE-Systems

Table of Contents

Abstracts

Sparse optimal control for the heat equation with mixed control-state constraints

Fredi Tröltzsch
(joint work with Eduardo Casas)

We consider the optimal control problem

$$\min J(y,u) := \frac{1}{2}\iint_Q |y-y_Q|^2 + \nu |u|^2 \, dxdt + \kappa \iint_Q |u| \, dxdt$$

subject to

$$\begin{aligned} \partial_t y - \Delta y &= u \quad \text{in } Q := \Omega \times (0,T) \\ \partial_n y &= 0 \quad \text{in } \Sigma := \Gamma \times (0,T) \\ y(x,0) &= 0 \quad \text{in } \Omega, \end{aligned}$$

subject to the mixed control-state constraint

$$u_a \le u(x,t) \le u_d + y(x,t)$$

for a.a. $(x,t) \in Q$.

Here, $y_Q \in L^2(Q)$, $T > 0$, the Tikhonov parameter $\nu > 0$, the sparse parameter $\kappa > 0$, and bounds $u_a < 0$, $u_d > 0$ are given. The state y is defined in $W(0,T) := \{y \in L^2(0,T;H^1(\Omega)) : \partial_t y \in L^2(0,T;H^1(\Omega)')\}$ and the control u is searched in $L^2(Q)$. The L^1-term of the control in the functional J accounts for the effects of sparsity.

Since the celebrated paper of Stadler [4], sparsity has been discussed extensively in the community of PDE-constrained optimization. We refer exemplarily to [1, 3] and the references therein. The main novelty of this presentation is the discussion of sparsity under a mixed pointwise control-state constraint. Special emphasis is laid on existence and boundedness of Lagrange multipliers for the mixed control-state constraints. To this aim, a duality theorem for linear programming problems in Hilbert spaces is proved and applied to the given optimal control problem.

Having a Lagrange multiplier $\bar\mu \in L^\infty(Q)$, we introduce an extended adjoint state $\bar\psi$ by the following adjoint parabolic equation:

$$\begin{aligned} -\partial_t \bar\psi - \Delta \bar\psi &= \bar y - y_Q - \bar\mu \quad \text{in } Q \\ \partial_n \bar\psi &= 0 \quad \text{in } \Sigma \\ \bar\psi(x,T) &= 0 \quad \text{in } \Omega. \end{aligned}$$

A suitable necessary optimality condition is formulated below. There, we use the functional $j(u) := \|u\|_{L^1(Q)}$ and its subdifferential $\partial j(u) \subset L^\infty(Q)$.

Theorem 1 ([2]). *Assume that the control $u = u_a$ satisfies the mixed control state constraints, let $\bar u$ be optimal and $\bar y$ be the associated state. Then a non-negative*

Lagrange multiplier $\bar\mu \in L^\infty(Q)$ *and a function* $\bar\lambda \in \partial j(\bar u)$ *exist such that*

$$\iint_Q (\bar\psi + \nu\,\bar u + \kappa\,\bar\lambda)(u-\bar u)\,dxdt \ge 0 \qquad \forall u \in L^2(Q):\ u_a \le u \le u_d + \bar y$$

holds. The multiplier $\bar\mu$ *can be selected such that*

$$\|\bar\mu\|_{L^\infty(Q)} \le M$$

is satisfied with M *independent of* κ*.*

Notice that the constraint of the variational inequality above is defined upon the fixed optimal state $\bar y$, hence it acts as a pointwise control constraint. The proof of the uniform boundedness of the multiplier with respect to κ is a particular difficulty.

Based on this result, the following theorem on sparsity properties of the optimal control can be proven:

Theorem 2 (Sparsity, [2])**.** *Assume that the control* $u = u_a$ *satisfies the mixed control state constraints and let* $\bar u$ *be optimal. Then a Lagrange multiplier* $\bar\mu \in L^\infty(Q)$ *exists such that a.e. in* Q *the implications*

$$\begin{aligned} |\bar\psi(x,t)| \le \kappa &\implies \bar u(x,t) = 0 \\ \bar u(x,t) = 0 &\implies \bar\psi(x,t) \le \kappa \end{aligned}$$

hold true. For some $\kappa_0 > 0$ *and a.a.* $(x,t) \in Q$*, we have*

$$\bar u(x,t) = 0 \iff |\bar\psi(x,t)| \le \kappa \quad \forall \kappa \ge \kappa_0.$$

There is some $\kappa_1 > 0$ *such that* $\bar u = 0$ *is satisfied for all* $\kappa \ge \kappa_1$*. The element* $\bar\lambda \in \partial j(\bar u)$ *is given by*

$$\bar\lambda(x,t) = \mathbb{P}_{[-1,1]}\left\{-\frac{1}{\kappa}\bar\psi(x,t)\right\},$$

where the projection is defined by $\mathbb{P}_{[-1,1]}(s) = \max\{-1, \min\{1, s\}\}$*.*

We also have a more general version of this theorem, where the constraints $u_a \le u \le u_b$, $u \le u_d + y$ are given. Then the situation is slightly more complex, cf. [2].

References

[1] E. Casas, R. Herzog, and G. Wachsmuth, *Optimality conditions and error analysis of semilinear elliptic control problems with* L^1 *cost functional*, SIAM J. Optim. **22** (2012), 795–820.

[2] E. Casas, F. Tröltzsch, *Sparse optimal control for the heat equation with mixed control-state constraints*, 2018, submitted.

[3] R. Herzog, G. Stadler, and G. Wachsmuth, *Directional sparsity in optimal control of partial differential equations*, SIAM J. Control Optim. **50** (2012), 943–963.

[4] G. Stadler, *Elliptic optimal control problems with* L^1*-control cost and applications for the placement of control devices*, Comput. Optim. Appl. **44** (2009), 159–181.

About the controllability of an advection-diffusion equation with respect to the diffusion coefficient

Arnaud Münch

Let $L > 0$, $T > 0$ and $Q_T := (0, L) \times (0, T)$. The talk is concerned with the scalar advection-diffusion equation

$$(1) \qquad \begin{cases} y^\varepsilon_t - \varepsilon y^\varepsilon_{xx} + M y^\varepsilon_x = 0, & (x,t) \in Q_T, \\ y^\varepsilon(0,t) = v^\varepsilon(t), \; y^\varepsilon(L,t) = 0, & t \in (0,T), \\ y^\varepsilon(x,0) = y^\varepsilon_0(x), & x \in (0,L), \end{cases}$$

where $y^\varepsilon_0 \in H^{-1}(0,L)$ is the initial data. The parameter $\varepsilon > 0$ is the diffusion coefficient while $M \in \mathbb{R}^\star$ is the transport coefficient; $v^\varepsilon = v^\varepsilon(t)$ is the control function in $L^2(0,T)$ and $y^\varepsilon = y^\varepsilon(x,t)$ is the associated state. For any $y^\varepsilon_0 \in H^{-1}(0,L)$ and $v^\varepsilon \in L^2(0,T)$, there exists exactly one solution y^ε to (1), with the regularity $y^\varepsilon \in L^2(Q_T) \cap C([0,T]; H^{-1}(0,L))$. Accordingly, for any final time $T > 0$, the associated null controllability problem at time $T > 0$ is the following: for each $y_0 \in H^{-1}(0,L)$, find $v \in L^2(0,T)$ such that the corresponding solution to (1) satisfies

$$(2) \qquad y(\cdot, T) = 0 \text{ in } H^{-1}(0,L).$$

For any $T > 0$, $M \in \mathbb{R}$ and $\varepsilon > 0$, the null controllability for the parabolic type equation (1) holds true. We therefore introduce the non-empty set of null controls

$$\mathcal{C}(y_0, T, \varepsilon, M) := \{(y, v) : v \in L^2(0,T); y \text{ solves (1) and satisfies (2)}\}.$$

We are mainly concern here with the asymptotic behavior of null controls for (1) when the coefficient ε is small. System (1) can be seen as a simple example of complex models where the diffusion coefficient is very small compared to the others. We have notably in mind the Stokes system where ε stands for the viscosity coefficient. (1) may also be seen as a regularization of conservation law system. Precisely, we are interested with the control of minimal L^2-norm and define for any $\varepsilon > 0$ the cost of control by the following quantity :

$$(3) \qquad K(\varepsilon, T, M) := \sup_{\|y_0\|_{L^2(0,L)} = 1} \left\{ \min_{u \in \mathcal{C}(y_0, T, \varepsilon, M)} \|u\|_{L^2(0,T)} \right\}.$$

$K(\varepsilon, T, M)$ is the norm of the (linear) operator $y^\varepsilon_0 \to v^\varepsilon_{HUM}$ where v^ε_{HUM} is the control of minimal L^2-norm. We also define by T_M the minimal time for which the cost $K(\varepsilon, T, M)$ is uniformly bounded with respect to ε. In other words, (1) is uniformly controllable with respect to ε if and only if $T \geq T_M$.

For $\varepsilon = 0$, the system (1) degenerates into a transport equation and is uniformly controllable as soon as T is large enough, according to the speed $|M|$ of transport, precisely as soon as $T \geq L/|M|$. Indeed, if $T \geq L/|M|$, the zero function is a null control - and so the control of minimal L^2-norm - for the transport equation. According, $K(0, T, M) = 0$ for any $T \geq L/|M|$. On the other hand, the behavior of $K(\varepsilon, T, M)$ is more involved and has been the subject of several recent works. We may expected naively that the cost $K(\varepsilon, T, M)$ goes to zero as $\varepsilon \to 0$ as soon as $T \geq L/|M|$ and therefore that $T_M = 1/|M|$. But, at least for $M < 0$, this is false since the cost $K(\varepsilon, T, M)$ blows up exponentially as $\varepsilon \to 0^+$ for any $T \leq 2(1+\sqrt{3})/|M|$ which is strictly greater that $1/|M|$. This is a surprising and non expected result. There is a kind of balance between the term $-\varepsilon y^\varepsilon_{xx}$ which favors the diffusion (and so the null controllability) for ε large and the term My^ε_x which enhance the complete transport of the solution out of the domain $(0, L)$ for ε small.

One may tackle this problem and the determination of the minimal uniform controllability time T_M using at least two distincts approaches.

A first one consists in approximating numerically the cost $K(\varepsilon, T, M)$ for various values of ε and $T > 0$, the ratio L/M being fixed. This step requires first the reformulation of the cost as the solution of a generalized eigenvalues problem for the control operator. This eigenvalues can be solved iteratively and requires the approximation of the control of minimal L^2-norm, which is a challenging task for small values of ε. This has been done and discussed at length in [3]. Numerical experiments performed in the positive case $M > 0$ suggests that the cost is achieved for the initial condition $y^\varepsilon_0(x) = K_\varepsilon e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)$ and that $T_M = L/M$.

A second approach consists in performing, in the spirit of the book [2], an asymptotic analysis of the optimality system corresponding the control of minimal L^2 norm . However, in spite of the apparent simplicity of the system (1), such analysis is not straightforward because, as ε goes to zero, the direct and adjoint solutions exhibit boundary layers in the transition parabolic-hyperbolic. For example, for $M > 0$, the solution y^ε exhibits a first boundary layer of size $\mathcal{O}(\varepsilon)$ at $x = L$ and a second boundary boundary layer of size $\mathcal{O}(\sqrt{\varepsilon})$ along the characteristic $\{(x,t) \in Q_T, Lx - Mt = 0\}$. A third singular behavior due to the initial condition y^ε_0 occurs for y^ε in the neighborhood of the points $(x_0, t_0) = (0, 0)$ and $(x_1, t_1) = (L, 0)$. A full asymptotic asymptotic analysis is performed in [1] and allows to described very precisely the behavior of the solution of the direct problem (1) with respect to ε, v^ε being assumed of the form $v^\varepsilon = \sum_{k=0}^m \varepsilon^k v^k$. This then allows to determine the v^k's and obtain approximate controllability results for $T > L/M$ and $M > 0$.

The determination of the minimal uniform controllability time T_M remains an open problem.

References

[1] Y. Amirat, A. Münch, *Asymptotic analysis of an advection-diffusion equation and application to boundary controllabilty.* To appear in Asymptotic analysis.

[2] J-L. Lions, *Perturbations singulières dans les problèmes aux limites et en contrôle optimal.* Lecture Notes in Mathematics. Springer 1973.

[3] A. Münch, *Numerical estimations of the cost of boundary controls for the equation $y_t - \varepsilon y_{xx} + My_x = 0$ with respect to ε.* To appear in Sema SIMA Springer Series, (2018).

Cahn–Hilliard systems with general fractional operators

Jürgen Sprekels
(joint work with Pierluigi Colli, Gianni Gilardi)

Let $\Omega \subset \mathbb{R}^3$ denote a bounded and smooth domain, $T > 0$, and $Q := \Omega \times (0, T)$. Moreover, let $H := L^2(\Omega)$, and let $(\cdot, \cdot)$ and $\|\cdot\|$ denote the standard inner product and norm in H. We study in this contribution the following system of equations:

$$\partial_t y + A^{2r}\mu = 0 \quad \text{in } Q, \tag{1}$$

$$\tau\partial_t y + B^{2\sigma} y + f'(y) = \mu + u \quad \text{in } Q, \tag{2}$$

$$y(0) = y_0 \quad \text{in } \Omega, \tag{3}$$

where we postulate that $\tau \geq 0$, $r > 0$ and $\sigma > 0$. Moreover, we generally assume:

(A1) $A : D(A) \subset H \to H$ and $B : D(B) \subset H \to H$ are selfadjoint, unbounded and positive linear operators having compact resolvents.

Under these assumptions, there exist countably many eigenvalues $\{\lambda_j\}$ and $\{\lambda'_j\}$ of A and B, respectively, ordered by their magnitudes, with associated eigenfunctions $\{e_j\}$ and $\{e'_j\}$ such that

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots, \quad 0 \leq \lambda'_1 \leq \lambda'_2 \leq \ldots, \quad \lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \lambda'_j = +\infty,$$

$$Ae_j = \lambda e_j, \quad Be'_j = \lambda'_j e'_j, \quad (e_j, e_k) = (e'_j, e'_k) = \delta_{jk}, \quad \forall j, k \in \mathbb{N},$$

and such that both systems of eigenfunctions form complete subsets of H. We then define for $r > 0$ the fractional order operator

$$A^r v := \sum_{j\in\mathbb{N}} \lambda_j^r (v, e_j) e_j \tag{4}$$

on the domain

$$D(A^r) := \Big\{ v \in H : \sum_{j\in\mathbb{N}} \lambda_j^{2r} |(v, e_j)|^2 < +\infty \Big\}. \tag{5}$$

The fractional order operators B^σ for $\sigma > 0$ are defined accordingly.

We notice that the system (1)–(3) coincides with the classical Cahn–Hilliard system in the special case that $\tau = 0$ and $A = B = -\Delta$ with the domain $D(-\Delta) = \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0 \text{ on } \partial\Omega\}$, where $\partial_{\mathbf{n}}$ denotes the outward normal derivative on the boundary $\partial\Omega$. Recall that the classical Cahn–Hilliard system constitutes a model for the separation of two phases in a container Ω through the process of spinodal decomposition under the action of the nonmonotone thermodynamic force $f'(y)$. In this connection, the unknowns μ and y represent the *chemical potential* and the *order parameter* (usually a scaled density of one of the involved phases) of the phase separation process, respectively, while u stands for a distributed control. Typical forms of the associated nonconvex thermodynamic potential f are given by the expressions

$$f_{\rm reg}(r) = \frac{1}{4}(r^2-1)^2\,, \quad r \in \mathbb{R}, \tag{6}$$

$$f_{\log}(r) = ((1+r)\ln(1+r) + (1-r)\ln(1-r)) - c\,r^2\,, \quad r \in (-1,1), \tag{7}$$

$$f_{\rm obs}(r) = I_{[-1,1]}(r) - c\,r^2\,, \quad r \in \mathbb{R}, \tag{8}$$

which are usually referred to as the *regular, logarithmic*, and *double obstacle* potential, in this order.

Recently, evolutionary processes exhibiting fractional-order diffusive patterns have been observed in various applied fields (see, e.g., the references given in [1]), and first papers (cf. [2, 3]) were devoted to the study of Cahn–Hilliard type equations with fractional orders of the negative Laplacian $-\Delta$ with zero Dirichlet conditions. Since in the case of a phase separation process with mass conservation in a container zero Neumann boundary conditions are more natural, which renders the analysis of the Cahn–Hilliard system considerably more difficult, we have investigated the questions of well-posedness, regularity and stability for the general fractional system (1)–(3) with the aim to include the case of zero Neumann conditions as well. Under suitable assumptions on the initial datum y_0 and on the potential f, which allow for the cases (6)–(8), in [4, Thm. 2.6] for every $r > 0$ and $\sigma > 0$ the existence of a unique weak solution to the system (1)–(3) could be proved that enjoys the regularity

$$y \in H^1(0,T;(V_A^r)^*) \cap L^\infty(0,T;V_B^\sigma), \quad \tau\partial_t y \in L^2(0,T;H), \tag{9}$$

$$\mu \in L^2(0,T;V_A^r). \tag{10}$$

Here, we denote $V_B^\sigma := D(B^\sigma)$, which is a Hilbert space when endowed with the inner product

$$(v,w)_{V_B^\sigma} = (v,w) + (B^\sigma v, B^\sigma w) \qquad \text{for } v,w \in V_B^\sigma. \tag{11}$$

For the entire analysis, and, in particular, for the definition of V_A^r and of its dual space $(V_A^r)^*$, the first eigenvalue λ_1 of A plays a special role. Indeed, if λ_1 is positive, then the operators A and B may be completely unrelated, and we can put $V_A^r = D(A^r)$, which in this case becomes a Hilbert space when endowed with the inner product

$$(v,w)_{V_A^r} = (A^r v, A^r w) \qquad \text{for } v,w \in V_A^r; \tag{12}$$

however, in the case $\lambda_1 = 0$ it must be assumed that λ_1 is a simple eigenvalue and that the corresponding eigenfunction e_1 is constant and belongs to V_B^r (which, for instance, holds true if $B = -\Delta$ with zero Neumann boundary condition). In this case, we put $V_A^r = D(A^r)$ and endow this space with the inner product

$$(v,w)_{V_A^r} = (v,e_1)(w,e_1) + (A^r v, A^r w) \qquad \text{for } v,w \in V_A^r. \tag{13}$$

Better regularity properties can be shown under stronger assumptions on the data. A corresponding result was proved in [4, Thm. 2.8]. Under the conditions given there, one obtains that

(14) $\partial_t y \in L^\infty(0,T;V_A^r) \cap L^2(0,T;V_B^\sigma)$ and $\mu \in L^\infty(0,T;V_A^r)$ if $\tau \geq 0$,

(15) $\partial_t y \in L^\infty(0,T;H)$ and $\mu \in L^\infty(0,T;V_A^{2r})$ if $\tau > 0$.

We conclude with some remarks on the paper [5], which deals with distributed optimal control problems for the system (1)–(3), where it is assumed that $\tau > 0$. It turns out that for such problems it is indispensable to postulate a certain global $L^\infty(Q)$-boundedness of the unknown y and of the quantities $f^{(i)}(y)$, for $i = 1, 2, 3$. Sufficient conditions for this to hold are given. Under the additional assumption that $V_B^\sigma \subset L^4(\Omega)$, it can be shown that the control-to-state operator $\mathcal{S} : u \mapsto (\mu, y)$ is Fréchet differentiable between suitable Banach spaces, which paves the way to derive first-order necessary optimality conditions. For details, we refer the reader to [5].

References

[1] A. Bonito, J. P. Borthagaray, R. H. Nochetto, E. Otárola, and A. J. Salgado, *Numerical methods for fractional diffusion*, Preprint arXiv:1707.01566 [math.NA] (2017), pp. 1–36.

[2] M. Ainsworth, and Z. Mao, *Analysis and approximation of a fractional Cahn–Hilliard equation*, SIAM J. Numer. Anal. **55** (2017), 1689–1718.

[3] G. Akagi, G. Schimperna, and A. Segatti, *Fractional Cahn–Hilliard, Allen–Cahn and porous medium equations*, J. Differ. Equations **261** (2016), 2935–2985.

[4] P. Colli, G. Gilardi, and J. Sprekels, *Well-posedness and regularity for a generalized fractional Cahn–Hilliard system*, in preparation.

[5] P. Colli, G. Gilardi, and J. Sprekels, *Optimal distributed control of a generalized fractional Cahn–Hilliard system*, in preparation.

Some applications & challenges for PDE control and optimization

John Burns

(joint work with Gene Cliff, Terry Herdman)

Thermal management systems (TMS) are complex interconnected systems modeled by coupled ordinary and partial differential equations, algebraic equations and empirical mappings. A fundamental component in the system model are the PDEs that govern thermal fluids in heat exchangers. The development of dynamic mathematical models suitable for control and optimization is a ongoing challenge in this area. In addition, the development of accurate and efficient approximation schemes is needed to construct numerical models for design optimization and control. Some issues are:

- Current models fail at low or zero flows flows.
- Most approximation schemes used today are based on low order finite volume methods.
- Control system properties can be lost in the discretization process and hence these models are rendered unsuitable for optimization & control design.
- Actuator dynamics are rarely included in the models.

- Empirical functions and equations of state are not always smooth.

In this talk we discuss the role of full flux modeling in addressing low flow conditions and higher order numerical methods for approximating composite control systems defined by coupled hyperbolic PDEs and ordinary differential equations. Although the work is motivated by applications to optimization and control of full thermal management systems, we focus on a simple counterflow heat exchanger based on models found in [1], [3], [4] and [5], where one includes the effect of axial conduction and boundary control inputs with actuator dynamics. This "full-flux" model is described by the coupled system

$$\begin{aligned} \frac{\partial T_1(t,x)}{\partial t} &= \mu_1 \frac{\partial^2 T_1(t,x)}{\partial x^2} - v_1(t,x)\frac{\partial T_1(t,x)}{\partial x} + h_1\left[T_2(t,x) - T_1(t,x)\right] \\ \frac{\partial T_2(t,x)}{\partial t} &= \mu_2 \frac{\partial^2 T_2(t,x)}{\partial x^2} + v_2(t,x)\frac{\partial T_2(t,x)}{\partial x} + h_2\left[T_2(t,x) - T_2(t,x)\right] \end{aligned}, \tag{1}$$

where the constants h_1, h_2 are heat transfer coefficients, μ_1, μ_2 are diffusion coefficients and $v_1(t,x)$, $v_2(t,x)$ are flow velocities for channels one and two, respectively. The flow velocities $v_1(t,x)$ and $v_2(t,x)$ are possible control inputs.

For channel one we have the boundary conditions

$$T_1(t,0) = v(t), \quad \mu_1[T_1]_x(t,L) = 0, \tag{2}$$

where $v(\cdot)$ is a "boundary control term" and for channel two we have

$$-\mu_2[T_2]_x(t,0) = 0, \quad T_2(t,L) = 0. \tag{3}$$

The Neumann boundary conditions arise from the assumption that the system is fully developed at the outflow boundaries. Initial conditions for each channel are given by

$$T_1(0,x) = \varphi(x) \ \text{ and } \ T_2(0,x) = \psi(x), \quad 0 < x < L. \tag{4}$$

In addition, we assume the actuator dynamics are described by a finite dimensional system of the form

$$\dot{\mathbf{w}}_a(t) = \mathbf{A}_a \mathbf{w}_a(t) + \mathbf{B}_a \mathbf{u}(t), \tag{5}$$

with output

$$v(t) = \mathbf{H}_a \mathbf{w}_a(t), \tag{6}$$

where we assume that $\mathbf{A}_a$ is an $n \times n$ stable matrix and $\mathbf{B}_a$ is an $n \times m$ matrix.

The combined composite system can be written as

$$\begin{aligned} \dot{\boldsymbol{z}}(t) &= \boldsymbol{\mu}\mathcal{A}\boldsymbol{z}(t) + \boldsymbol{v}(t)\mathcal{H}\boldsymbol{z}(t) + \mathbf{H}\boldsymbol{z}(t) + F\mathbf{w}_a(t) + \mathbf{B}\mathbf{u}(t) \\ \dot{\mathbf{w}}_a(t) &= \mathbf{A}_a\mathbf{w}_a(t) + \mathbf{B}_a\mathbf{u}(t), \end{aligned} \tag{7}$$

which is an abstract bilinear control system of the type found in [2].

This framework allows us to discuss and compare various finite element (FE), finite volume (FV), combined FE-FV and higher order "DG" type methods for simulation, optimization and control of such systems. These schemes are applied to a simple numerical example to illustrate the idea.

References

[1] E. Aulisa, J. A. Burns, and D. S. Gilliam. The effect of viscosity in a tracking regulation problem for a counter-flow heat exchanger. In *2015 54th IEEE Conference on Decision and Control, Osaka, Japan*, pages 561–566, 2015.

[2] Tobias Breiten, Karl Kunisch, and Laurent Pfeiffer. Control strategies for the fokker-planck equation. *ESAIM: Control, Optimisation and Calculus of Variations*, 2017.

[3] John A. Burns and Boris Kramer. Full flux models for optimization and control of heat exchangers. In *2015 American Control Conference, Chicago, IL.*, pages 577–582, 2015.

[4] Ahmed Maidi, Moussa Diaf, and Jean-Pierre Corriou. Boundary geometric control of a counter-current heat exchanger. *Journal of Process Control*, 19(2):297–313, 2009.

[5] Ahmed Maidi, Moussa Diaf, and Jean-Pierre Corriou. Boundary control of a parallel-flow heat exchanger by input–output linearization. *Journal of Process Control*, 20(10):1161–1174, 2010.

Optimal control of an evolution equation with non-smooth dissipation

Daniel Wachsmuth

(joint work with Tobias Geiger)

1. Abstract

We study an optimal control problem subject to an evolution equation with non-smooth dissipation. The solution mapping of this system is non-smooth, and hence the analysis is quite challenging. Our approach is to formulate the equation as a variational inequality of the second kind, which gives us the opportunity to apply known results to obtain optimality conditions.

2. State equation

We consider a quadratic energy $\mathcal{E}$ and a dissipation function $\mathcal{D}$ of the form

$$\mathcal{E} : H_0^1(\Omega) \to \mathbb{R},\ \mathcal{E}(z,g) := \int_\Omega \frac{1}{2}|\nabla z|^2 - zg\,dx$$

$$\mathcal{D} : H_0^1(\Omega) \to \mathbb{R}, \mathcal{D}(\dot{z}) = \int_\Omega |\dot{z}| + \frac{\sigma}{2}|\nabla \dot{z}|\,dx.$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain. Minimization with respect to z motivates the differential inclusion

$$0 \in -\Delta z - g - \sigma\Delta\dot{z} + \partial|\dot{z}| \quad \text{in } H^{-1}(\Omega) \text{ a.e. on } I = (0,T), \tag{1}$$

where $\partial|\dot{z}| := \partial\|\cdot\|_{L^1(\Omega)}(\dot{z})$. The control is g and the state z. This system is inspired by rate-independent systems, which one gets by setting $\sigma = 0$, see also [5]. Optimal control problems of rate independent systems were analyzed in [6]. One drawback in this kind of optimal control problem is the low regularity of the adjoint states. By adding the term $-\sigma\Delta\dot{z}$ with $\sigma > 0$, which can be interpreted as viscosity, we hope to get an optimality system with higher regularity of the adjoint states and hence stronger formulation of the respective equations.

For every control $g \in H^1(I, L^2(\Omega))$ there exists a unique $z \in H^1(I, H_0^1(\Omega))$ that solves the state equation (1) equipped with initial condition $z(0) = 0$. In fact, the state equation can be written as the initial value problem

$$\begin{cases} \dot{z}(t) = \frac{1}{\sigma} \operatorname{prox}_{\|\cdot\|_{L^1(\Omega)}} \big((-\Delta)^{-1}(g) - z\big) & \text{a.e. in } I, \\ z(0) = 0 \end{cases} \tag{2}$$

in the Banach space $H_0^1(\Omega)$, which is uniquely solvable since the prox-operator is a globally Lipschitz continuous mapping.

3. Optimal control problem

Let functions $j_1 : L^2(I, H_0^1(\Omega)) \to \mathbb{R}$, $j_2 : H_0^1(\Omega) \to \mathbb{R}$ be given, which are assumed to be Fréchet differentiable and bounded from below. We equip the state equation with the initial and compatibility conditions $z(0) = 0$, $g(0) = 0$. The optimal control problem, that we consider, is the following. Minimize:

$$J(z, g) := j_1(z) + j_2(z(T)) + \frac{1}{2}\|g\|^2_{H^1(I, L^2(\Omega))} \tag{P}$$

for

$$(z, g) \in H^1(I, H_0^1(\Omega)) \times H^1(I, L^2(\Omega))$$

subject to

$$0 \in \partial|\dot{z}(t)| - \sigma\Delta\dot{z}(t) - \Delta z(t) - g(t) \qquad \text{in} \quad H^{-1}(\Omega) \quad \text{for a.a. } t \in I,$$
$$g(0) = 0,\ z(0) = 0.$$

The existence of global solutions can be shown by standard arguments involving weakly converging subsequences, embeddings, and weak lower semicontinuity of norms.

4. Optimality system

There are several possibilities to get an optimality system for this optimal control problem. One can consider a smoothed version of the state equation by replacing the absolute value function in the $L^1(\Omega)$-norm by smooth functions which depend on a parameter ρ and converge to the absolute value function for $\rho \to 0$, see [4]. Another possibility is to consider a discretization of the time interval, derive optimality conditions for the respective optimal control problem, and driving the discretization parameter to zero. We will present another approach, in which we formulate the state equation (1) as a variational inequality. We define $y(t) := \dot{z}(t)$ and hence $z(t) = \int_0^t y(s)\,ds =: Jy$ since $z(0) = 0$. Moreover, we define the (nonsymmetric) bilinear form

$$a(v, w) := \big(\sigma\nabla v + \nabla(Jv), \nabla w\big)_{L^2(\Omega)}, \qquad v, w \in L^2(I, H_0^1(\Omega)).$$

Then the state equation (1) is equivalent to the variational inequality

$$(3)\quad a(y, v-y) + \|v\|_{L^1(I\times\Omega)} - \|y\|_{L^1(I\times\Omega)} \ge (g, v-y)_{L^2(I\times\Omega)} \quad \forall\, v \in L^2(I, H_0^1(\Omega)).$$

Optimal control problems of variational inequalities of the second kind are analyzed e.g. in [1, 2, 3]. Applying the result from [2] we get an optimality system for our problem. Let $(\bar{z}, \bar{g})$ be locally optimal and define $\bar{y} := \dot{\bar{z}}$ and $\bar{q} \in \partial|\bar{y}|$. Then there exists

$$(p, u) \in L^2(I, H_0^1(\Omega)) \times H^1(I, H^{-1}(\Omega))$$

such that the following system is satisfied.

$$(4a)\quad \begin{cases} \dot{u} = -\Delta p - j_1'(\bar{z}) & \text{in } H^{-1}(\Omega) \text{ a.e. on } I, \\ u(T) = j_2'(\bar{z}(T)) & \text{in } H^{-1}(\Omega), \end{cases}$$

$$(4b)\quad (p, v)_{L^2(I,L^2(\Omega))} + (g, v)_{H^1(I,L^2(\Omega))} = 0, \qquad \forall v \in H^1(I, L^2(\Omega)):\ v(0) = 0,$$

$$(4c)\quad \int_I \langle u + \sigma\Delta p, p\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt \ge 0,$$

$$(4d)\quad \int_I \langle u + \sigma\Delta p, y\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt = 0.$$

The initial value problem (4a) is the adjoint equation. Equation (4b) is obtained by the gradient of the reduced functional and can be interpreted as a weak formulation of

$$-\ddot{\bar{g}} + \bar{g} + p = 0, \qquad \bar{g}(0) = \dot{\bar{g}}(T) = 0.$$

Finally, (4c) and (4d) are complementarity conditions. Unfortunately, one complementarity condition could not be proven, which one expects from formal optimality conditions. This condition reads

$$(5)\qquad \langle p, 1 - |\bar{q}|\rangle = 0.$$

Since this condition is non-linear in $\bar{q}$, and every known approximation scheme only yields weak convergent approximations of $\bar{q}$, it is open whether this condition is a necessary optimality condition. Under structural assumptions on the solution $\bar{y}$, this complementarity condition is part of strong stationarity conditions obtained in [2, Theorem 5.1] and [4, Section 5.5] for optimal control problems subject to variational inequalities of the second kind in the space $H^1(\Omega)$. Hence, any counter-example to the complementarity condition (5) has to violate those structural assumptions, which makes such a construction a tedious task.

References

[1] C. Christof and C. Meyer. Strong stationarity conditions of optimization problems governed by variational inequalities of the second kind. Preprint., 2017.

[2] J. C. De Los Reyes. Optimal control of a class of variational inequalities of the second kind. *SIAM J. Control Optim.*, 49(4):1629–1658, 2011.

[3] J. C. De los Reyes and C. Meyer. Strong stationarity conditions for a class of optimization problems governed by variational inequalities of the second kind. *J. Optim. Theory Appl.*, 168(2):375–409, 2016.

[4] T. Geiger and D. Wachsmuth. Optimal control of an evolution equation with non-smooth dissipation. 2018.

[5] A. Mielke and T. Roubíček. *Rate-independent systems*, volume 193 of *Applied Mathematical Sciences*. Springer, New York, 2015. Theory and application.

[6] U. Stefanelli, D. Wachsmuth, and G. Wachsmuth. Optimal control of a rate-independent evolution equation via viscous regularization. *Discrete and Continuous Dynamical Systems - Series S*, 10(6):1467–1485, 2017.

Hyperbolic Maxwell system of quasi-variational inequality type governed by Bean's critical-state law with temperature effects

Irwin Yousept

Ever since the discovery of superconductivity by Heike Kamerlingh Onnes in 1911, various modern applications and key technologies have been developed, including resonance imaging, magnetic confinement fusion, and magnetic levitation. Such technological advances are made possible by superconductors due to their fundamental properties of vanishing electrical resistance and expulsion of applied magnetic fields (Meissner effect) occurring when the temperature is cooled down below the critical one. A prominent critical-state model describing the irreversible magnetization process in high-temperature (type-II) superconductivity was proposed by Bean [4, 5]. His model postulates a nonlinear and non-smooth constitutive relation between the current density and the electric field through the so-called critical current as follows:

(B1) The current density strength $|\boldsymbol{J}|$ cannot exceed the critical current j_c.
(B2) If $|\boldsymbol{J}|$ is strictly less than j_c, then the electric field $\boldsymbol{E}$ vanishes.
(B3) The electric field $\boldsymbol{E}$ is parallel to the current density $\boldsymbol{J}$.

Under the eddy current approximation for the electromagnetic fields, (B1)-(B3) lead to a parabolic obstacle-type variational inequality (see [1, 2, 3]), whereas it becomes a quasi-variational inequality in the case of $j_c = j_c(\boldsymbol{H})$. If we consider the original Maxwell formulation for the electromagnetic fields, then (B1)-(B3) lead to a hyperbolic variational inequality of the second kind (see [6]): Find $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl}))$ such that

$$
(1)\qquad \begin{cases} \displaystyle\int_\Omega \epsilon \partial_t \boldsymbol{E}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t)) + \mu \partial_t \boldsymbol{H}(t) \cdot (\boldsymbol{w} - \boldsymbol{H}(t)) \, dx \\ \displaystyle + \int_\Omega \mathbf{curl}\, \boldsymbol{E}(t) \cdot \boldsymbol{w} - \mathbf{curl}\, \boldsymbol{H}(t) \cdot \boldsymbol{v} \, dx + \int_\Omega j_c |\boldsymbol{v}(x)| \, dx \\ \displaystyle - \int_\Omega j_c |\boldsymbol{E}(x,t)| \, dx \geq \int_\Omega \boldsymbol{u}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t)) \, dx, \\ \text{for a.e. } t \in (0,T) \text{ and all } (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega), \\ (\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \end{cases}
$$

In real applications of superconductivity, the critical current density j_c depends not only on the magnetic field but also on the temperature. In particular, the temperature dependence cannot be neglected since it strongly influences the superconducting state. We consider the case $j_c = j_c(x, \boldsymbol{H}(x,t), \theta(x,t))$ in (1), which leads to a hyperbolic Maxwell quasi-variational inequality of the second kind with temperature dependence. By means of the implicit Euler scheme, we analyze the resulting nonlinear time-discrete problem and prove its existence using a fixed point argument in combination with techniques from variational inequalities. Afterwards, we study the stability analysis for the solution of the time-discrete problem. Based on the derived stability results along with the Maxwell compactness property and energy balance equalities, we are able to prove a convergence result for the time-discrete solution, which in turn yields a well-posedness result for the original quasi-variational inequality.

References

[1] L. Prigozhin. Variational inequalities in critical-state problems. *Phys. D*, 197(3-4):197–210, 2004.
[2] L. Prigozhin. On the Bean critical-state model in superconductivity. *European J. Appl. Math.*, 7(3):237–247, 1996.
[3] J. W. Barrett and L. Prigozhin. A quasi-variational inequality problem in superconductivity. *Math. Models Methods Appl. Sci.*, 20(5):679–706, 2010.
[4] C. P. Bean. Magnetization of hard superconductors. *Phys. Rev. Lett.*, 8:250–253, 1962.
[5] C. P. Bean. Magnetization of high-field superconductors. *Rev. Mod. Phys.*, 36:31–39, 1964.
[6] I Yousept. Hyperbolic Maxwell Variational Inequalities for Bean's Critical-State Model in Type-II Superconductivity. *SIAM J. Numer. Anal.*, 55(5):2444–2464, 2017.

Solvability and optimal velocity control of a Cahn–Hilliard system with convection and dynamic boundary conditions

Pierluigi Colli
(joint work with Gianni Gilardi, Jürgen Sprekels)

This note is concerned with a PDE system coupling equation and boundary condition both of Cahn–Hilliard type; an additional convective term with a forced velocity field, which may act as a control on the system, is also present in the bulk equation. Either regular or singular potentials are discussed and both the viscous and pure Cahn–Hilliard cases are investigated. Such systems govern phase separation processes between two phases taking place in an incompressible fluid in a container and, at the same time, on the container boundary. The optimal control problem deals with a cost functional of standard tracking type, while the control is exerted by the velocity of the fluid in the bulk. In this way, the coupling between the state (given by the associated order parameter and chemical potential) and the control variables in the governing system of nonlinear partial differential equations is bilinear, then causing some difficulty for the analysis.

The state system reads

$$\partial_t \rho + \nabla\rho \cdot u - \Delta\mu = 0 \quad \text{in } Q\,, \tag{1}$$

$$\tau_\Omega\, \partial_t \rho - \Delta\rho + f'(\rho) = \mu \quad \text{in } Q\,, \tag{2}$$

$$\partial_t \rho_\Gamma + \partial_{\mathbf{n}}\mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{and} \quad \mu_{|\Sigma} = \mu_\Gamma \quad \text{on } \Sigma\,, \tag{3}$$

$$\tau_\Gamma\, \partial_t \rho_\Gamma + \partial_{\mathbf{n}}\rho - \Delta_\Gamma \rho_\Gamma + f_\Gamma'(\rho_\Gamma) = \mu_\Gamma \quad \text{and} \quad \rho_{|\Sigma} = \rho_\Gamma \quad \text{on } \Sigma\,, \tag{4}$$

$$\rho(0) = \rho_0 \quad \text{in } \Omega, \quad \rho_\Gamma(0) = \rho_{0|\Gamma} \quad \text{on } \Gamma\,, \tag{5}$$

where $\Omega \subset \mathbb{R}^3$ is an open, bounded and connected set having smooth boundary Γ and unit outward normal $\mathbf{n}$; $\partial_{\mathbf{n}}$, ∇_Γ, Δ_Γ denote the outward normal derivative, the tangential gradient, and the Laplace–Beltrami operator on Γ, respectively; some final time $T > 0$ is fixed and $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$ are the space-time domain and its lateral boundary.

The variables are ρ, the order parameter, and μ, the chemical potential; according to whether the coefficients τ_Ω and τ_Γ are positive or zero, we speak of viscous Cahn–Hilliard or pure Cahn–Hilliard system; please note in (1) the convection term $\nabla\rho\cdot u$, coupling the gradient of the order parameter with some fixed velocity vector u such that $\operatorname{div} u = 0$ in Q and $u\cdot\nu = 0$ on Σ. We point out that f' is the derivative of a double-well potential f (same f_Γ' of f_Γ), which may have bounded domain and become singular: in particular let us recall the logarithmic potential

$$f_{log}(r) = ((1+r)\ln(1+r) + (1-r)\ln(1-r)) - c\,r^2, \quad r \in (-1,1),$$

where the coefficient c is greater than 1 so that f_{log} is non-convex.

We are interested to state the well-posedness of the system (1)–(5) in the case when f_Γ' dominates f' in terms of the growth: thus, we review the results in [2]. Then, we switch to the optimal control problem:
(CP) minimize the cost functional

$$\begin{aligned} \mathcal{J}(\mu,\mu_\Gamma,\rho,\rho_\Gamma,u) := {} & \frac{\beta_1}{2}\int_Q |\mu - \hat\mu_Q|^2 + \frac{\beta_2}{2}\int_\Sigma |\mu_\Gamma - \hat\mu_\Sigma|^2 \\ & + \frac{\beta_3}{2}\int_Q |\rho - \hat\rho_Q|^2 + \frac{\beta_4}{2}\int_\Sigma |\rho_\Gamma - \hat\rho_\Sigma|^2 \\ & + \frac{\beta_5}{2}\int_\Omega |\rho(T) - \hat\rho_\Omega|^2 + \frac{\beta_6}{2}\int_\Gamma |\rho_\Gamma(T) - \hat\rho_\Gamma|^2 + \frac{\beta_7}{2}\int_Q |u|^2\,, \end{aligned} \tag{6}$$

subject to the state system (1)–(5) and to the control constraint

$$u \in \mathcal{U}_{ad}\,, \tag{7}$$

where $\mathcal{U}_{ad}$ is a suitable closed, convex, and bounded subset of the control space $\mathcal{X}$ defined by

$$\mathcal{X} := L^2(0,T;Z) \cap (L^\infty(Q))^3 \cap H^1(0,T;L^3(\Omega)^3), \tag{8}$$

where

$$Z := \big\{ w \in (L^2(\Omega))^3 :\ \operatorname{div} w = 0 \text{ in } \Omega \ \text{ and } \ w\cdot\nu = 0 \text{ on } \Gamma \big\}\,. \tag{9}$$

In (6), the constants β_i, $1 \le i \le 7$, are nonnegative but not all zero, and $\widehat{\mu}_Q$, $\widehat{\mu}_\Sigma$, $\widehat{\rho}_Q$, $\widehat{\rho}_\Sigma$, $\widehat{\rho}_\Omega$, and $\widehat{\rho}_\Gamma$, are given target functions. We note that the total mass of the order parameter is conserved during the separation process; indeed, integrating (1) for fixed $t \in (0,T]$ over Ω, and using the condition $u(t) \in Z$ and (3), we readily find that

$$\partial_t\Big(\int_\Omega \rho(t) + \int_\Gamma \rho_\Gamma(t) \Big) = 0\,. \tag{10}$$

We also assume that the densities of the local free bulk energy f and the local free surface energy f_Γ are of logarithmic type. The optimal control problem is intensively discussed in the talk, by reporting the results of [3] for the viscous Cahn–Hilliard system ($\tau_\Omega > 0,\ \tau_\Gamma > 0$). A distinguishing feature of our approach is that we use the fluid velocity as the control variable: in practice, this can be realized by placing either a mechanical stirring device or an ultrasound emitter into the container. The related convective term $\nabla\rho \cdot u$ produces some complication in the analysis due to the nonlinear coupling between control and state variables.

In our analysis [3], we deal with controls u which, among other constraints, have to obey the somewhat unusual regularity condition $u \in H^1(0,T;L^3(\Omega)^3)$: this is exactly the kind of regularity that guarantees the existence of a unique solution to the state system having sufficient regularity properties. Under these premises, we are able to show the Fréchet differentiability of the control-to-state operator in suitable Banach spaces. Morever, we can prove the existence of an optimal control and, in a slightly less general setting ($\beta_1 = \beta_2 = 0$), we also derive proper first-order necessary conditions for optimality.

In the case of general potentials and without convective term ($u = 0$), the problem (1)–(5) has been investigated in [1] from the point of view of existence, uniqueness and regularity (see also [6] for a boundary control problem) using an abstract approach.

We also discuss the case in which the bulk and surface free energies are of double obstacle type:

$$f_{2obst}(r) = I_{[-1,1]}(r) + c(1-r^2), \quad r \in \mathbb{R}, \quad c > 0,$$

where I_S = indicator function of S: 0 in S and $+\infty$ outside,

which renders the potentials f and f_Γ nondifferentiable: for such cases standard constraint qualifications are not satisfied so that standard methods do not apply to yield the existence of Lagrange multipliers. In [4], this difficulty can be overcome by taking advantage of the results established for logarithmic nonlinearities, using the so-called "deep quench approximation". The existence of optimal controls is shown and the first-order necessary optimality conditions are stated in terms of a variational inequality and the associated adjoint system.

Finally, the contributions [5] and [7] are mentioned for the study of the long-time behavior and for the results on the optimal control of the pure Cahn–Hilliard system ($\tau_\Omega = \tau_\Gamma = 0$), which are obtained as asymptotic limits of the corresponding results for the viscous Cahn–Hilliard problem, but in the case of everywhere defined smooth potentials f and f_Γ.

References

[1] P. Colli and T. Fukao, *Equation and dynamic boundary condition of Cahn–Hilliard type with singular potentials*, Nonlinear Anal. **127** (2015), 413–433.

[2] P. Colli, G. Gilardi, J. Sprekels, *On a Cahn–Hilliard system with convection and dynamic boundary conditions*, Ann. Mat. Pura Appl. (4), DOI 10.1007/s10231-018-0732-1 (see also preprint arXiv:1704.05337, pp. 1–34).

[3] P. Colli, G. Gilardi, J. Sprekels, *Optimal velocity control of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions*, SIAM J. Control Optim., to appear (2018) (see also preprint arXiv:1709.02335, pp. 1–28).

[4] P. Colli, G. Gilardi, J. Sprekels, *Optimal velocity control of a convective Cahn–Hilliard system with double obstacles and dynamic boundary conditions: a 'deep quench' approach*, J. Convex Anal., to appear (2018) (see also preprint arXiv:1709.03892, pp. 1–30).

[5] P. Colli, G. Gilardi, J. Sprekels, *On the longtime behavior of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions*, preprint arXiv:1803.04318, pp. 1–20.

[6] T. Fukao and N. Yamazaki, *A boundary control problem for the equation and dynamic boundary condition of Cahn–Hilliard type*, in "Solvability, Regularity, Optimal Control of Boundary Value Problems for PDEs", P. Colli, A. Favini, E. Rocca, G. Schimperna, J. Sprekels (ed.), Springer INdAM Series **22**, Springer, Milan, 2017, pp. 255–280.

[7] G. Gilardi, J. Sprekels, *Asymptotic limits and optimal control for the Cahn–Hilliard system with convection and dynamic boundary conditions*, preprint arXiv:1803.05331, pp. 1–37.

Infinite-horizon bilinear optimal control problems

Tobias Breiten, Laurent Pfeiffer

(joint work with Karl Kunisch)

Optimal feedback laws for nonlinear control problems are intimately related to the computation of the optimal value function $\mathcal{V}$ which is known to satisfy the Hamilton-Jacobi-Bellman (HJB) equation. In order to overcome the *curse of dimensionality*, approximation techniques for the HJB equation have received increasing interest over the last years. In the finite-dimensional case, Taylor series approximations of $\mathcal{V}$ have been proposed and numerically investigated, [1, 9]. For nonlinear infinite-dimensional control systems, feedback controls are often designed by a Riccati-based approach for the linearized system, see, e.g., [10].

Polynomial feedback laws

With the intention of constructing higher-order approximations of the optimal feedback law, consider the bilinear infinite-horizon optimal control problem:

$$(P)\qquad \begin{aligned} &\inf_{u\in L^2(0,\infty)} \mathcal{J}(u,y_0) := \frac{1}{2}\int_0^\infty \|Cy(t)\|_Z^2\,\mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u(t)^2\,\mathrm{d}t, \\ &\text{where: } \begin{cases} \dot{y}(t) = Ay(t) + (Ny(t)+B)u(t), & \text{for } t>0 \\ y(0) = y_0 \in Y, \end{cases} \end{aligned}$$

Here, $V \subset Y \subset V^*$ is a Gelfand triple of real Hilbert spaces, $A\colon \mathcal{D}(A) \subset Y \to Y$ is the infinitesimal generator of an analytic C_0-semigroup e^{At} on Y, $B \in Y$, $C \in \mathcal{L}(Y,Z)$, $N \in \mathcal{L}(V,Y)$. Under appropriate assumptions on the operators, in particular stabilizability and detectability, we construct in [5] a polynomial approximation $\mathcal{V}_k$ of the value function $\mathcal{V}$ as follows:

$$\mathcal{V}_k \colon Y \to \mathbb{R}, \quad \mathcal{V}_k(y) = \sum_{j=2}^{k} \frac{1}{j!} \mathcal{T}_j(y, \ldots, y),$$

where $\mathcal{T}_2,\ldots,\mathcal{T}_i,\ldots,\mathcal{T}_k$ are bounded multilinear forms of order $2, \ldots, i, \ldots, k$. The first multilinear form, the bilinear form $\mathcal{T}_2$, is obtained as the solution to an algebraic operator Riccati equation, the other multilinear forms are obtained as the solutions to multilinear generalized Lyapunov operator equations.

A natural polynomial feedback law $\mathbf{u}_k$ based on $\mathcal{V}_k$ is the following:

$$\begin{aligned} \mathbf{u}_k(y) &= -\frac{1}{\alpha} D\mathcal{V}_k(y)(Ny + B) \\ &= -\frac{1}{\alpha}\Big(\sum_{i=2}^{k} \frac{1}{(i-1)!} \mathcal{T}_i(Ny + B, y, \ldots, y)\Big). \end{aligned}$$

It is shown that the associated closed-loop system

$$\dot{y}_k(t) = Ay_k(t) + (Ny_k(t) + B)\mathbf{u}_k(y_k(t)), \quad y_k(0) = y_0,$$

is locally well-posed and asymptotically stable.

Numerical and algorithmic realization

We illustrate in [4] the numerical applicability of the results by means of a bilinear optimal control problem for the controlled Fokker-Planck equation taken from [3]:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \tilde{\nu}\Delta\rho + \nabla\cdot(\rho\nabla G) + u\nabla\cdot(\rho\nabla\alpha) && \text{in } \Omega\times(0,\infty), \\ 0 &= (\tilde{\nu}\nabla\rho + \rho\nabla G)\cdot\vec{n} && \text{on } \Gamma\times(0,\infty), \\ \rho(x,0) &= \rho_0(x) && \text{in } \Gamma, \end{aligned}$$

where $\tilde{\nu} > 0$, $\Omega \subset \mathbb{R}^n$ denotes a bounded domain with smooth boundary $\Gamma = \partial\Omega$, and ρ_0 denotes an initial probability distribution with $\int_\Omega \rho_0(x)\mathrm{d}x = 1$. The Fokker-Planck equation models the evolution of the probability distribution $\rho(\cdot, t)$ of a very large set of particles that are confined by a ground potential G. The uncontrolled system converges to the stationary distribution. The convergence rate depends on the confining potential and the barrier height between adjacent potential wells and can be extremely slow. In order to speed up the rate of convergence, we thus consider an appropriate infinite-horizon stabilization problem. Motivated by the concept of optical tweezing, see e.g., [8], the control interacts with the ground potential along a prescribed control shape function α.

As a model problem, we focus on a spatially discretized two-dimensional Fokker-Planck equation with $n = 50\times 50$ degrees of freedom. The design of the polynomial feedback is based on a reduced-order model obtained by a generalization of the method of balanced truncation as it has been discussed in, e.g., [2]. For the computation of the feedback gain, we exploit the tensor product structure of the generalized Lyapunov equations and approximate the solutions by means of an appropriate tensorial quadrature formula from [7]. We show that this allows to construct polynomial feedback laws up to order 5.

Error analysis

We analyze in [6] the performance of $\mathbf{u}_k$ in comparison with the optimal control. To this purpose, we introduce the mapping Φ defined by

$$\Phi\colon\ (y,u,p) \ \mapsto \ \begin{pmatrix} y(0) \\ \dot{y} - (Ay + (Ny+B)u) \\ -\dot{p} - A^*p - uN^*p - C^*Cy \\ \alpha u + (Ny+B)^*p \end{pmatrix}.$$

For a solution $\bar{u}$ to problem (P) with initial condition y_0 and with associated trajectory $\bar{y}$ and costate $\bar{p}$, the first-order optimality conditions write: $\Phi(\bar{y},\bar{u},\bar{p}) = (y_0,0,0,0)$. Our analysis mainly lies on the fact that the mapping Φ is invertible around $(y,u,p) = (0,0,0)$, with a C^∞ inverse. This enables to show in particular that the value function $\mathcal{V}$ is C^∞ in the neighborhood of the steady state.

For deriving an error estimate for the feedback law $\mathbf{u}_k$, we construct a costate variable p_k satisfying a perturbed costate equation. Denoting by u_k the open-loop control generated by $\mathbf{u}_k$, we show that $\Phi(y_k,u_k,p_k) = (y_0,0,w_k,0)$, where the perturbation term w_k can be conveniently estimated. We finally obtain the following estimate:

$$\begin{aligned} \|(\bar{y},\bar{u},\bar{p}) - (y_k,u_k,p_k)\| &= \|\Phi^{-1}(y_0,0,0,0) - \Phi^{-1}(y_0,0,w_k,0)\| \\ &= \mathcal{O}(\|w_k\|) = \mathcal{O}(\|y_0\|^k), \end{aligned}$$

for appropriate norms, and for an initial condition y_0 sufficiently close to 0.

References

[1] E.G. Al'brekht, *On the optimal stabilization of nonlinear systems*, Journal of Applied Mathematics and Mechanics, 25 (1961), p. 1254.

[2] P. Benner and T. Damm, *Lyapunov equations, energy functionals, and model order reduction of bilinear and stochastic systems*, SIAM Journal on Control and Optimization, 49 (2011), pp. 686–711.

[3] T. Breiten, K. Kunisch, and L. Pfeiffer, *Control strategies for the Fokker-Planck equation*, ESAIM: Control, Optimisation and Calculus of Variations, (2017). to appear.

[4] ———, *Numerical study of polynomial feedback laws for a bilinear control problem*, tech. report, SFB-Report 2017-009, 2017.

[5] ———, *Taylor expansions for the HJB equation associated with a bilinear control problem*, tech. report, SFB-Report 2017-006, 2017.

[6] ———, *Infinite-horizon bilinear optimal control problems: Sensitivity analysis and polynomial feedback laws*, tech. report, SFB-Report, 2018.

[7] L. Grasedyck, *Existence and computation of low Kronecker-rank approximations for large linear systems of tensor product structure*, Computing, 72 (2004), pp. 247–265.

[8] P. H. Jones, M. M. Onofrio, and G. Volpe, *Optical Tweezers: Principles and Applications*, Cambridge University Press, 2015.

[9] D.L. Lukes, *Optimal regulation of nonlinear dynamical systems*, SIAM Journal on Control, 7 (1969), pp. 75–100.

[10] J.-P. Raymond, *Feedback boundary stabilization of the two-dimensional Navier–Stokes equations*, SIAM Journal on Control and Optimization, 45 (2006), pp. 790–828.

Optimal model switching for gas flow in pipe networks

Falk M. Hante
(joint work with Volker Mehrmann, Fabian Rüffler)

Gas flow in a pipeline can be modeled by a hierarchy ranging from nonlinear hyperbolic balance laws

$$\partial_t \varrho + \partial_x(\varrho v) = 0,$$
$$\partial_t(\varrho v) + \partial_x(P + \varrho v^2) = -\theta \varrho v|v| - g\varrho h',$$

to stationary models that can be obtained from the limit $\varepsilon \to 0$ of a semilinear approximation

$$\varepsilon \partial_t \varrho + \partial_x q = 0,$$
$$\varepsilon \partial_t q + c^2 \partial_x \varrho = -\theta \frac{q|q|}{\varrho} - gh'\varrho,$$

where ϱ denotes the density, v the velocity, P the pressure, $q = \varrho v$ the flux of the gas and g is the gravitational constant, h' the slope and θ a friction coefficient for the pipe. A gas network is composed by a coupling of these pipe models on a graph $G = (V, E)$ with nodes $V = (v_1, \ldots, v_m)$ and edges $E = (e_1, \ldots, e_n)$ using transmission conditions for the density

$$\alpha^k_{x(v,e_k)} \varrho^k(t, L_k x(v, e_k)) = \alpha^l_{x(v,e_l)} \varrho^l(t, L_l x(v, e_l)), \quad \forall\, e_k, e_l \text{ incident to } v,$$

and a balance equation for the fluxes

$$\sum_{e_j \text{ ingoing}} q^j(t, L_j) - \sum_{e_j \text{ outgoing}} q^j(t, 0) = q^v(t) \quad \text{at node } v,$$

where α and q^v can be used to model active elements such as valves and compressors as well as in- and outflow to the network. For modeling details and similar hierarchies, for example in water canals, see [3].

For a numerical simulation or optimization of such a network, it can be advantageous to switch the pipe model individually in order to balance numerical efficiency and accuracy [2, 5]. In [8], we consider such model switching exemplary for the semilinear dynamic and stationary pipe model by switching the parameter ε between 1 and a positive value $\bar{\varepsilon} > 0$ close to zero. As a cost function associated to a time dependent choice $\varepsilon \in \{1, \bar{\varepsilon}\}$ for each edge given by a switching sequence (μ, τ) and the corresponding solution $z = (\rho^j, q^j)_j$ for the network dynamics we consider

$$J(\mu, \tau, z) = \sum_{j=1}^{n} \int_0^T \int_0^{L_j} \gamma_1(\varrho^j(t, x) - \varrho^j_d(t, x))^2 + \gamma_2(q^j(t, x) - q^j_d(t, x))^2 \, dx \, dt$$
$$+ \gamma_3 \sum_{k=1}^{N} \sum_{j=1}^{n} \frac{1}{L_j} \int_{\tau_k}^{\tau_{k+1}} (\mu^j_k - \bar{\varepsilon})^2 dt + \gamma_4 N$$

with z_d being the reference obtained from choosing $\varepsilon = 1$ for all edges, L_j being the length of the pipe for e_j, N being the number of switches and $\gamma_1, \ldots, \gamma_4 \geq 0$ being desired weights. Our techniques can also be applied to other cost functions.

In [8], we prove that, for any choice $\varepsilon \in \{1, \bar{\varepsilon}\}$ for each edge, the homogeneous part of the dynamics on the network generates a strongly continuous semigroup with data in L^2 on the edges. This allows us to pose the model switching problem as an optimal switching control problem for hybrid semilinear evolution equations

$$\begin{aligned} \dot{z}(t) &= A^{\mu_k} z(t) + f^{\mu_k}(t, z(t)), \qquad k \in \{1, \dots, N\}, \, t \in (\tau_{k-1}, \tau_k), \\ z(\tau_k) &= g^{\mu_k, \mu_{k+1}}(z^-(\tau_k)), \qquad k \in \{1, \dots, N\}, \end{aligned}$$

with initial condition $z(\tau_0 = 0) = z_0$ as considered in [6]. For $\gamma_4 > 0$, the existence of optimal (τ^*, μ^*) can be obtained as in [4]. In view of optimality conditions, we can define an adjoint state $p^j = (p_1^j, p_2^j)_j$ satisfying

$$\begin{bmatrix} p_1^j \\ p_2^j \end{bmatrix}_t + \begin{bmatrix} 0 & c_j^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1^j \\ p_2^j \end{bmatrix}_x = \theta \begin{bmatrix} 0 & -\frac{q^j |q^j|}{\varrho^j |\varrho^j|} \\ 0 & 2\frac{|q^j|}{\varrho^j} \end{bmatrix} \begin{bmatrix} p_1^j \\ p_2^j \end{bmatrix} + \gamma_1 \begin{bmatrix} \varrho^j - \varrho_d^j \\ q^j - q_d^j \end{bmatrix},$$

$$\begin{bmatrix} p_1^j \\ p_2^j \end{bmatrix}(T, x) = 0,$$

on each edge along with adjoint coupling conditions

$$\alpha^j_{x(v,e_k)} p_1^j(t, L_k x(v, e_k)) = \alpha^k_{x(v,e_l)} p_1^k(t, L_l x(v, e_l)), \quad \forall \, e_k, e_l \text{ incident to } v,$$

$$\sum_{e_j \text{ ingoing}} p_2^j(t, L_j) - \sum_{e_j \text{ outgoing}} p_2^j(t, 0) = 0 \text{ at node } v,$$

so that the derivative for the reduced cost function $\Phi(\mu, \tau) = J(\mu, \tau, z(\mu, \tau))$ with respect to the k-th switching time τ_k satisfies

$$\frac{\partial \Phi}{\partial \tau_k} = \sum_{e_j \in E} \Bigg[\int_0^{L_j} p^j(\tau_k, x) \left[(A^{\mu_k})^j - (A^{\mu_{k-1}})^j \right] z^j(\tau_k, x) \, dx + \gamma_2 \left[(\mu_k(m) - \bar{\varepsilon})^2 - (\mu_{k-1}(m) - \bar{\varepsilon})^2 \right] \Bigg].$$

Moreover, the sensitivity of the cost function with respect to introducing a new mode μ' on an infinitesimal time interval at the point τ' satisfies

$$\frac{\partial \Phi}{\partial \mu'}(\tau') = \sum_{e_j \in E} \Bigg[\int_0^{L_j} p^j(\tau', x) \left[(A^{\mu_k})^j - (A^{\mu'})^j \right] z^j(\tau', x) \, dx + \gamma_2 \left[(\mu_k(m) - \bar{\varepsilon})^2 - (\mu'(m) - \bar{\varepsilon})^2 \right] \Bigg],$$

where μ_k is the original mode at time τ'. These results yield necessary optimality conditions for a switching sequence (μ, τ) of the form that τ is a KKT-point for μ fixed for the ordering condition $\tau_k \leq \tau_{k+1}$ and $\frac{\partial \Phi}{\partial \mu'}(\tau') \geq 0$ for all modes μ' and all times $\tau' \in [0, T]$. Points satisfying these conditions can be found systematically for example using a two-stage projected gradient descent method [1]. The computation of the gradients can be realized very efficiently, because the adjoint state p is the same for all partial derivatives.

In [8], we report numerical results for an ideal gas in a test network of 10 pipes with two cycles and a total length of 340 km with a dynamic in- and outflow scenario. The two-stage projected gradient descent method combined with a 2-step-Richtmyer-method with artificial viscosity and an explicit 4-th order Runge-Kutta-scheme for space and discretization of the forward and adjoint equations then identifies a model where one of the cycles can be switched to a stationary mode for most of the time with a maximal relative error of 6 % with respect to flow and 1 % with respect to density.

Our prototypical approach can be applied in a similar fashion to realistic industrial networks in order to identify reduced models. In [7], we have shown that the same techniques can also be applied to compute optimal compressor and valve switching controls.

Future working directions include extensions to parameterized models and a combination of the methods with the optimization of those parameters, also in the context of receding horizon strategies.

Acknowledgements

The work that has been reported on is funded by the DFG within the projects A03 and B03 in the SFB/Transregio 154 *Mathematical Modelling, Simulation and Optimization using the Example of Gas Networks*.

References

[1] H. Axelsson, Y. Wardi, M. Egerstedt, E. I. Verriest, *Gradient descent approach to optimal mode scheduling in hybrid dynamical systems*, Journal of Optimization Theory and Applications, **136** (2008), 167–186.

[2] P. Domschke, A. Dua, J. J. Stolwijk, J. Lang, V. Mehrmann, *Adaptive Refinement Strategies for the Simulation of Gas Flow in Networks using a Model Hierarchy*, arxiv:1701.09031 (2017), `http://arxiv.org/abs/1701.09031`.

[3] F. M. Hante, G. Leugering, A. Martin, L. Schewe, M. Schmidt, *Challenges in optimal control problems for gas and fluid flow in networks of pipes and canals: From modeling to industrial applications*, Industrial Mathematics and Complex Systems, P. Manchanda, R. Lozi, A.H. Siddiqi Eds., Springer Singapore (2017), 77–122.

[4] F. M. Hante, G. Leugering, T.I. Seidman, *Modeling and Analysis of Modal Switching in Networked Transport Systems*, Applied Mathematics and Optimization, **59**:2 (2009), 275–292.

[5] V. Mehrmann, M. Schmidt, J. J. Stolwijk, *Model and Discretization Error Adaptivity within Stationary Gas Transport Optimization*, arxiv:1712.02745 (2017), `http://arxiv:1712.02745`.

[6] F. Rüffler, F. M. Hante, *Optimal Switching for Hybrid Semilinear Evolutions*. Nonlinear Analysis: Hybrid Systems, **22** (2016), 215–227.

[7] F. Rüffler, F. M. Hante, *Optimality Conditions for Switching Operator Differential Equations*, Proc. Appl. Math. Mech., **17** (2017), 777–778.

[8] F. Rüffler, V. Mehrmann, F. M. Hante, *Optimal Model Switching for Gas Flow in Pipe Networks*, TRR154 Preprint **232** (2018), `https://opus4.kobv.de/opus4-trr154/files/232/`.

On an optimal control problem with quasilinear parabolic PDE

Ira Neitzel
(joint work with Lucas Bonifacius)

In this talk we present optimality conditions for optimal control problems governed by quasilinear parabolic PDEs on rough domains with control q and state u. Special emphasis is placed on deriving second order sufficient conditions without two-norm discrepancy and minimal gap to the associated necessary ones. A typical model problem in two or three space dimensions is given by

$$\text{Minimize } J(u,q) := \frac{1}{2}\|u-\hat{u}\|^2_{L^2((0,T)\times\Omega)} + \frac{\lambda}{2}\|q\|^2_{L^2(\Lambda,\varrho)}, \tag{1a}$$

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= Bq \quad \text{in } (0,T)\times\Omega,\\ u(0) &= u_0 \qquad\ \text{in } \Omega, \end{aligned} \tag{1b}$$

$$q \in Q_{ad} \subset Q, \tag{1c}$$

where

$$\mathcal{A}(u) = -\nabla\cdot\xi(u)\mu\nabla$$

with ξ being a scalar function, and μ is a spatially dependent coefficient function. Boundary conditions are implicitly included in the definition of the differential operator in (1b).

Our theory also covers Neumann boundary control in two dimensions, or Neumann boundary control only depending on time, with fixed shape functions in space, in three dimensions. Details regarding the precise functional analytic setting as well as the full analysis can be found in [2].

Optimal control problems with quasilinear parabolic PDEs have been considered mainly with respect to existence of solutions and first order necessary otpimality conditions, see e.g. [3, 10, 1, 14, 15] or [12, 13, 8]. Regarding second order sufficient conditions, we are only aware of one other recent publication, [4], where the authors prove first and second order optimality conditions for quasilinear parabolic control problems with distributed control with possibly unbounded nonlinearity, yet on smooth domains.

The starting point for our work are recent uniform Hölder estimates for linear parabolic equations subject to mixed boundary conditions and rough domains, established in [11]. These imply that the state belongs to

$$W^{1,s}((0,T);W^{-1,p}_D) \cap L^s((0,T);W^{1,p}_D) \hookrightarrow_c C^\alpha((0,T);C^\kappa(\Omega)),$$

for any right-hand side in $L^s((0,T);W^{-1,p}_D)$.

Adapting the ideas of Casas and Tröltzsch from [5, 6], we prove second order necessary as well as sufficient optimality conditions. A key challenge lies in the fact that the first and second derivative of the reduced objective functional have to be extended to $L^2(\Lambda,\varrho)$, but the linearized state equation contains an additional term involving the gradient of u. This difficulty can be treated with a careful regularity analysis of linear equations based on works by [7].

In order to eventually apply the second order conditions to e.g. stability analysis with respect to perturbations in the nonlinear operator, we show improved regularity results for the state and adjoint state for right-hand-sides in $L^s((0,T); H_D^{-\zeta,p})$. In fact, then the state belongs to

$$W^{1,s}((0,T); H_D^{-\zeta,p}(\Omega)) \cap L^s((0,T); \mathcal{D}_{H_D^{-\zeta,p}(\Omega)}(-\nabla\cdot\mu\nabla)) \hookrightarrow_c C^\alpha((0,T); W_D^{1,p}(\Omega)),$$

where $H_D^{-\zeta,p}(\Omega)$ is the Bessel-potential space that can be obtained by complex interpolation between $L^p(\Omega)$ and $W_D^{-1,p}(\Omega)$, and $\mathcal{D}_{H_D^{-\zeta,p}(\Omega)}(-\nabla\cdot\mu\nabla)$ is the domain of $-\nabla\cdot\mu\nabla$ considered on $H_D^{-\zeta,p}(\Omega)$. This functional analytic setting has been proposed in [9] and covers rough domains and mixed boundary conditions.

References

[1] N. U. Ahmed, Optimal control of a class of strongly nonlinear parabolic systems, *J. Math. Anal. Appl.*, **61** (1977), 188–207.

[2] L. Bonifacius, I. Neitzel, Second order otpimality conditions for optimal control of quasilinear parabolic equations *Mathematical Control & Related Fields*, **8** (2018), 1–34.

[3] E. Casas, L. A. Fernández and J. Yong, Optimal control of quasilinear parabolic equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **125** (1995), 545–565.

[4] E. Casas and K. Chrysafinos, Analysis and optimal control of some quasilinear parabolic equations, Submitted.

[5] E. Casas and F. Tröltzsch, First- and second-order optimality conditions for a class of optimal control problems with quasilinear elliptic equations, *SIAM J. Control Optim.*, **48** (2009), 688–718.

[6] E. Casas and F. Tröltzsch, Second order analysis for optimal control problems: improving results expected from abstract theory, *SIAM J. Optim.*, **22** (2012), 261–279.

[7] K. Disser, A. F. M. ter Elst and J. Rehberg, Hölder estimates for parabolic operators on domains with rough boundary, *Ann. Sc. Norm. Sup. Pisa.*

[8] L. A. Fernández, Integral state-constrained optimal control problems for some quasilinear parabolic equations, *Nonlinear Anal.*, **39** (2000), 977–996.

[9] R. Haller-Dintelmann and J. Rehberg, Maximal parabolic regularity for divergence operators including mixed boundary conditions, *J. Differential Equations*, **247** (2009), 1354–1396.

[10] J.-L. Lions, Optimisation pour certaines classes d'équations d'évolution non linéaires, *Ann. Mat. Pura Appl. (4)*, **72** (1966), 275–293.

[11] H. Meinlschmidt and J. Rehberg, Hölder-estimates for non-autonomous parabolic problems with rough data, *Evolution Equations and Control Theory*, **5** (2016), 147–184.

[12] S. Nababan and E. S. Noussair, Existence theorems for optimal control of quasilinear parabolic partial differential equations, *J. Austral. Math. Soc. Ser. B*, **21** (1979/80), 90–101.

[13] E. S. Noussair, S. Nababan and K. L. Teo, On the existence of optimal controls for quasilinear parabolic partial differential equations, *J. Optim. Theory Appl.*, **34** (1981), 99–115.

[14] N. S. Papageorgiou, On the optimal control of strongly nonlinear evolution equations, *J. Math. Anal. Appl.*, **164** (1992), 83–103.

[15] D. Tiba, Optimality conditions for distributed control problems with nonlinear state equation, *SIAM J. Control Optim.*, **23** (1985), 85–110.

Recent results for the 3D Cahn-Hilliard-Brinkman system with unmatched viscosities

Andrea Giorgini
(joint work with Monica Conti)

Many applications in Engineering rely on the interaction between multicomponent fluid mixture. Due to the complexity of the dynamics at the interface, the mathematical description of the motion of globally immiscible materials, such as alloys and viscous fluids, is a long-standing problem in Material Science and Fluid Dynamics starting at the beginning of the 19th century. Since then, a vast literature has been devoted to find accurate models which comply the physical laws and lead to efficient numerical calculations. The mutual interaction between the interface dynamics and the surrounding fluid motion is indeed a complex phenomenon, depending on surface tension effects, temperature gradients and viscosity ratios. The common goal among these investigations has been understanding the nature of the interface. In the classical attempt, the interface is assumed to be an evolving surface with zero thickness, across which physical quantities must satisfy suitable boundary conditions. A radical change of view in the theory dates back to Van der Waals which postulated in [20] the notion of diffuse interface. This idea inspired many physicists during the last century leading to the development of the so-called phase field method. The origin of the first equations can be attributed to Cahn and Hilliard in [6], whose aim was to describe the spinodal decomposition in alloy mixtures. Later on this approach has been employed in many areas of Materials Science such as solidification of pure and binary materials, grain boundary, nucleation, solid-solid or liquid-liquid phase transition and crystallization (see [8]).

The key concept of diffuse interface methods is treating the interface as a finite-width region in which physical quantities have a rapid but smooth variation. The evolution of thick interfaces is taken into account by means of an additional variable. This is the so-called order parameter (or phase field) φ which distinguishes one constituent from the other. The order parameter is ruled by an additional equation based on the mass balance of the mixture, assuming Fick's law and free energy from Statistical Mechanics. The resulting equation is the so-called (convective) Cahn–Hilliard equation

$$\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi = \Delta(-\Delta \varphi + \Psi'(\varphi)). \tag{1}$$

Here $\mathbf{u}$ is the volume-averaged velocity and Ψ' is the first derivative of a double well potential Ψ. The thermodynamically relevant energy density Ψ is

$$\Psi(s) = \frac{\theta}{2}\Big((1+s)\ln(1+s) + (1-s)\ln(1-s)\Big) - \frac{\theta_0}{2}s^2, \quad \forall s \in (-1,1), \tag{2}$$

where the parameters θ and θ_0 satisfy the physical relations $0 < \theta < \theta_0$. To model the dynamics of two incompressible Newtonian fluids, (1) is coupled with a system of PDEs ruling the velocity $\mathbf{u}$. Let us report two important cases:

(i) **Model H.** This is a Cahn–Hilliard–Navier–Stokes (CHNS) system where $\mathbf{u}$ satisfies the Navier–Stokes equations (see, e.g., [4, 12, 15])

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \operatorname{div}(\nu(\varphi) D\mathbf{u}) - \nabla \pi + \mu \nabla \varphi, \quad \operatorname{div} \mathbf{u} = 0. \tag{3}$$

(ii) **Hele–Shaw flows.** The flow confined in a Hele–Shaw cell is modelled by the so-called Cahn–Hilliard–Hele–Shaw (CHHS) system, where the velocity field fulfils the Darcy's law (see [14, 10, 16])

$$\nu(\varphi)\mathbf{u} + \nabla \pi = \mu \nabla \varphi, \quad \operatorname{div} \mathbf{u} = 0. \tag{4}$$

The viscosity term ν is a linear combinations of the components (see, e.g., [13]) $\nu(s) = \nu_1 \frac{1+s}{2} + \nu_2 \frac{1-s}{2}$, where ν_1, ν_2 are the viscosities of the two fluids. In this approach topological changes of the interface are naturally allowed by the formulation of the system. This is one of the main reasons that made this approach widespread in numerical simulations. The Cahn-Hilliard-Brinkman (CHB) model arises in this context as a model for phase separation phenomena in porous media (see [18, 19]) and as a regularized system of the models mentioned above. In a bounded domain $\Omega \subset \mathbb{R}^3$, the system reads as

$$\begin{cases} -\operatorname{div}(\nu(\varphi) D\mathbf{u}) + \eta(\varphi)\mathbf{u} + \nabla \pi = \mu \nabla \varphi, \quad \operatorname{div} \mathbf{u} = 0, \\ \partial_t \varphi + \mathbf{u} \cdot \nabla \varphi = \Delta(-\Delta \varphi + \Psi'(\varphi)), \end{cases} \quad \text{in } \Omega \times (0,T), \tag{5}$$

completed with $\mathbf{u} = \mathbf{0}$ and $\partial_{\mathbf{n}} \varphi = \partial_{\mathbf{n}} \mu = 0$ on $\partial\Omega \times (0,T)$, and $\varphi(0) = \varphi_0$.

The mathematical analysis of these diffuse interface systems, in particular the uniqueness issue and the existence of global-in-time strong solutions, is quite challenging. This is due to the intrinsic difficulty of handling the equations for the velocity field and the singular behavior of $\Psi'(s)$ and its derivatives as s approaches ± 1. To simplify the analysis, most of the papers addressed the case of regular potentials, namely, (2) is replaced with polynomial functions. However, this choice cannot ensure that φ ranges in the physical interval $[-1,1]$. Under such restrictions, the CHB system has been investigated in [2]. In the same framework, among a vast literature, we refer the reader to [3, 9, 15] for the CHNS system and to [17, 21, 22] for the CHHS system. On the other hand, the are only few papers concerning the above-mentioned systems with the physically relevant logarithmic potential. The sole contribution on the CHNS model is [1]. Notably, the well-posedness of strong solutions is established: the order parameter φ is global in any dimensions, while the velocity field $\mathbf{u}$ is global in dimension two and local in dimension three. The CHHS model with logarithmic potential and matched viscosities has been studied in [11]. The uniqueness of weak solutions and their instantaneous regularization have been achieved in dimension two. Besides, in dimension three the existence of global strong solution is shown provided that the initial state φ_0 is sufficiently close to any local minimizer of the Ginzburg–Landau free energy. The result of our investigation on the CHB system is a comprehensive mathematical theory in dimension three. More precisely, our main results in a smooth domain $\Omega \subset \mathbb{R}^3$ are the following: uniqueness of weak solutions, global well-posedness of strong solutions, further regularity properties and validity of the separation property. In accordance with the previous discussion, the completeness of these results is a validation of the CHB system as a robust diffuse interface model for the description of three dimensional two-component flows.

References

[1] H. Abels, *On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Ration. Mech. Anal. **194** (2009), 463–506.

[2] S. Bosia, M. Conti, M. Grasselli, *On the Cahn-Hilliard-Brinkman system*, Commun. Math. Sci. **13** (2015), 1541–1567.

[3] F. Boyer, *Mathematical study of multi-phase flow under shear through order parameter formulation*, Asymptot. Anal. **20** (1999), 175–212.

[4] F. Boyer, *Nonhomogeneous Cahn-Hilliard fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), 225–259.

[5] H.C. Brinkman, *A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles*, Appl. Sci. Res. **A1** (1947), 27–36.

[6] J.W. Cahn, J.E. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys. **28** (1958), 258–267.

[7] C.Y. Chen, Y.S. Huang, J.A. Miranda, *Diffuse-interface approach to rotating Hele-Shaw flows*, Phys. Rev. E **84** (2011), 046302.

[8] K. Elder, N. Provatas, *Phase-filed methods in material science and engineering*, John Wiley & Sons, Weinheim, 2010.

[9] C.G. Gal, M. Grasselli, *Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 401–436.

[10] H. Garcke, K.F. Lam, E. Sitka, V. Styles, *A Cahn-Hilliard-Darcy model for tumour growth with chemotaxis and active transport*, Math. Models Methods Appl. Sci. **26** (2016), 1095–1148.

[11] A. Giorgini, M. Grasselli, H. Wu, *The Cahn-Hilliard-Hele-Shaw system with singular potential*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **35** (2018), 1079–1118.

[12] M.E. Gurtin, D. Polignone, J. Viñals, *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Models Methods Appl. Sci. **6** (1996), 815–831.

[13] J.S. Kim, *Phase-field models for multi-component fluid flows*, Commun. Comput. Phys. **12** (2012), 613–661.

[14] H-G. Lee, J.S. Lowengrub, J. Goodman, *Modeling pinchoff and reconnection in a Hele-Shaw cell. I. The models and their calibration*, Phys. Fluids **14** (2002),492–513.

[15] C. Liu, J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier spectral method*, Phys. D **179** (2003), 211–228.

[16] J.S. Lowengrub, H.B. Frieboes, F. Jin, Y.-L. Chuang, X. Li, P. Macklin, S.M. Wise, V. Cristini, *Nonlinear modelling of cancer: bridging the gap between cells and tumours*, Nonlinearity **23** (2010), R1–R91.

[17] J. Lowengrub, E. Titi, K. Zhao, *Analysis of a mixture model of tumor growth*, European J. Appl. Math. **24** (2013), 691–734.

[18] W. Ngamsaad, J. Yojina, W. Triampo, *Theoretical studies of phase-separation kinetics in a Brinkman porous medium*, J. Phys. A **43** (2010), 202001.

[19] M. Schmuck, M. Pradas, G.A. Pavliotis, S. Kalliadasis, *Derivation of effective macroscopic Stokes-Cahn-Hilliard equations for periodic immiscible flows in porous media*, Nonlinearity **26** (2013), 3259–3277.

[20] J.D. Van der Waals, *The thermodynamic theory of capillary under the hypothesis of a continuous variation of density*, Verhandel Konink. Akad. Weten. Amsterdam **1** (1892), 1–56.

[21] X. Wang, H. Wu, *Long-time behavior for the Hele-Shaw-Cahn-Hilliard system*, Asymptot. Anal. **78** (2012), 217–245.

[22] X. Wang, Z. Zhang, *Well-posedness of the Hele-Shaw-Cahn-Hilliard system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), 367–384.

Multiscale optimal control of collective behavior phenomena

Dante Kalise

Multi-agent dynamical systems (MAS) naturally arise in the mathematical modeling of social dynamics in a wide spectrum of applications such as animal behavior, cellular aggregation, opinion dynamics and human crowd motion, among many others (see [10, 12] and references therein). So far, it has been of utmost interest to study different collective behavior phenomena such as clustering or consensus emergence without external forces. Depending on the degree of cohesiveness of the initial configuration of the agents and their interaction strength, dynamical patterns may arise naturally by self-organization. However, if self-organization is not sufficient to enforce a stable pattern, collective behavior can be induced by means of exogenous interventions. Our goal is to study the design of external control actions which are able to steer a MAS towards prescribed stable consensus patterns. We will address this challenge by means of optimal control techniques, thus minimizing an energy measure of both the control and the state of the system, constrained to the multi-agent dynamics.

MAS are naturally represented as particle sets dynamically interacting under simple rules such as attraction, repulsion, or alignment. We are interested in an optimal control formulation for MAS, so we will be concerned with a large-scale system of coupled nonlinear stochastic differential equations of the type

$$dx_i = \left(\frac{1}{N} \sum_{j=1}^{N} P(x_i, x_j)(x_j - x_i) + u_i \right) dt + \sqrt{2\sigma}\, dB_i^t, \qquad i = 1, \cdots, N, \quad t > 0,$$

where the kernel $P : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ encodes the interaction forces between agents, and the control $u = (u_1, \ldots, u_N)$ minimizes a given functional $J(x, u)$. As an example we can consider the following variational formulation

$$u^* = \arg\min_{u \in \mathcal{U}} J(x, u) := \mathbb{E}\left[\int_0^T \frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{2} |x_i - x_d|^2 + \gamma \Lambda(u_i) \right) dt \right],$$

where x_d represents a target reference point, γ is the penalization parameter of the control u, which is chosen among the admissible controls in $\mathcal{U}$, and $\Lambda : \mathbb{R}^d \to \mathbb{R}_+ \cup \{0\}$ is a convex function. The choice of this particular cost function is absolutely arbitrary.

As the number of agents increases, the complexity of the associated optimal control problem becomes prohibitively expensive, a phenomenon often referred as Bellman's curse of dimensionality. In order to circumvent this difficulty, we follow a multiscale (MS) approach. By borrowing a leaf from statistical mechanics, the mean field approximation of a multi-agent system replaces the microscopic representation of the state by an agent density function which evolves according to a nonlinear, nonlocal transport equation

$$\partial_t \rho + \nabla \cdot ((\mathcal{P}[\rho] + u)\, \rho) = \sigma \Delta \rho,$$

where the interaction force $\mathcal{P}$ is given by

$$\mathcal{P}[\rho](x) = \int P(x,y)(y-x)\rho(y,t)\,dy$$

and the solution ρ is controlled by the minimizer of the cost functional

$$J(\rho,u) = \int_0^T \left(\frac{1}{2} \int |x-x_d|^2 \rho(x,t)\,dx + \gamma \int \Lambda(u)\rho(x,t)\,dx \right) dt.$$

Overall, our research is focused around the following three challenges:

(1) The derivation of MS models for the control of collective behavior and its analysis.
(2) The numerical analysis of the solution of multiscale optimal control problems (MSOC), and the quantification of the closed-loop MSOC performance over the original microscopic models.
(3) The analytical and computational study of applications in animal and human crowd motion, and opinion dynamics.

Previous research

The topic of emergent collective behavior in multi-agent systems has been linked the study of pattern formation and self-organization phenomena [13, 14], and to recent developments covered within the area of Mean Field Games (MFG) [17, 6, 9]. In the case of self-organization, focus is put on the study of system dynamics and the internal structures which can lead to consensus. On the other hand, the approach based on MFG does include a decision process, but its design is based on the optimization of individual goals, as for instance in the financial market, and the emphasis is in the characterization of Nash equilibria. We follow a different approach, enforcing consensus by optimizing the intervention of an external policy maker endowed with limited resources. This approach has been already studied for microscopic dynamics in [11, 8, 7, 5], and at the mean-field level in [6, 15, 16] among others. In [1], we have developed an analytical and computational MSOC approach for the control of mean-field dynamics through the inclusion microscopic leaders, with applications in crowd motion evacuation. More recently, in [2, 3, 4] we developed a mean-field control hierarchy, where optimal feedback controllers are computed for a binary system of particles, and its action is inserted in the mean-field dynamics, regulating the density evolution towards a target. Starting from these works, we expect to further develop the advancement of this framework by deriving new MS mean-field models, by studying improved optimal control formulations and by quantifying the performance of the MS optimal controllers.

References

[1] G. Albi, M. Bongini, E. Cristiani and D. Kalise. *Invisible control of self-organizing agents leaving unknown environments*, SIAM J. Appl. Math. **76**(4)(2016), 1683–1710.

[2] G. Albi, Y.-P. Choi, M. Fornasier and D. Kalise. *Mean field control hierarchy*, Appl. Math. Optim. **76**(1)(2017) 93–175.

[3] G. ALbi, M. Fornasier, and D. Kalise. *A Boltzmann approach to mean-field sparse feedback control*, IFAC-PapersOnLine **50**(1)(2017) 2898–2903.

[4] G. Albi and D. Kalise *(Sub)Optimal feedback control of mean field multi-population dynamics*, arXiv:1803.00301 (2018).

[5] R. Bailo, M. Bongini, J.A. Carrillo and D. Kalise. *Optimal consensus control of the Cucker-Smale model*, arXiv:1802.01529 (2018).

[6] A. Bensoussan, J. Frehse and P. Yam. *Mean field games and mean field type control theory*, SpringerBriefs in Mathematics, Springer-Verlag, New York (2013).

[7] M. Bongini, M. Fornasier and D. Kalise. *(Un)conditional consensus emergence under perturbed and decentralized feedback controls*, Discrete Contin. Dyn. Syst. **35**(9)(2015), 4071–4094.

[8] A. Borzì and S. Wongkaew. *Modeling and control through leadership of a refined flocking system*, Math. Methods Appl. Sci. **25**(2)(2015), 255–282.

[9] L.M. Briceño-Arias, D. Kalise and F.J. Silva. *Proximal methods for stationary Mean Field Games with local couplings*, SIAM J. Control Optim. **56**(2)(2018), 801–836.

[10] S. Camazine, J.-L. Deneubourg, N. R. Franks, J. Sneyd, G. Theraulaz, and E. Bonabeau. *Self-organization in biological systems*, Princeton Studies in Complexity, Princeton University Press (2003).

[11] M. Caponigro, M. Fornasier, B. Piccoli and E. Trélat. *Sparse stabilization and control of alignment models*, Math. Methods Appl. Sci. **25**(3)(2015), 521–564.

[12] E. Cristiani, B. Piccoli and A. Tosin. *Multiscale modeling of granular flows with application to crowd dynamics*, Multiscale Model. Simul., **9**(1)(2011), 155–182.

[13] F. Cucker and S. Smale. *Emergent behavior in flocks*, IEEE Trans. Automat. Control, **52**(5)(2007), 852–862.

[14] J. A. Carrillo, M. Fornasier, J. Rosado, and G. Toscani. *Asymptotic flocking dynamics for the kinetic Cucker–Smale model*, SIAM J. Math. Anal. **42**(1)(2010), 218–236.

[15] M. Fornasier and F. Solombrino. *Mean-field optimal control*, ESAIM Control Optim. Calc. Var. **20**(4)(2014), 1123–1152.

[16] M. Fornasier, B. Piccoli and F. Rossi. *Mean-field sparse optimal control*, Philos. Trans. A Math. Phys. Eng. Sci. **372**(2028): 20130400 (2014).

[17] J.-M. Lasry and P.-L. Lions. *Mean field games* Jpn. J. Math. **2**(1)(2007), 229–260.

Necessary optimality conditions for infinite dimensional state constrained control problems

Helene Frankowska

(joint work with Elsa Maria Marchini, Marco Mazzola)

The maximum principle for optimal control problems can be considered as a milestone in the theory of first order necessary optimality conditions. Due to its importance, an extensive literature has been devoted to this subject, both in finite and in infinite dimensions. The main interest of the infinite dimensional setting is due to the fact that many physical models can be formulated in this framework, as for instance heat conduction, reaction-diffusion processes, properties of elastic materials, to mention only a few of them. To optimize a measure of best performance is indeed a natural need in concrete problems. For this reason optimal control governed by PDEs is a very active field of research, see e.g. the classical books, [3, 5, 6, 12, 18, 19, 21], containing also rich bibliographies. In the literature two strategies can be found : the abstract semigroup approach, and a direct one relying on PDEs methods. The advantage of the second approach is that many fine properties of the solutions, as regularity, can be used. In contrast, the first more general framework directly applies to a variety of models.

We discuss here the first approach, relying on some results derived in the abstract setting in [13, 14], suitable to deal with state constrained problems. In particular, we have proved some neighbouring feasible trajectory (NFT) theorems allowing to estimate the distance between a given trajectory of an evolution system and its trajectories lying in the interior of the state constraint. (NFT) theorems have been studied in depth in the finite dimensional setting, see for instance [15] and the references contained therein. Let us underline that control systems under state constraints are of crucial importance in applied sciences because constraints do appear naturally in many models. The most frequently used tool to deal with state constrained optimization problems is the penalization technique that replaces a given optimization problem by a family of penalized problems without state constraints. Then the idea is to get optimal solutions and multipliers for the penalized problems and to pass to a limit. In infinite dimensions this technique should be used with caution, because weak limits of multipliers may be trivial. In contrast, (NFT) theorems are kind of implicit function theorems allowing to replace any given trajectory violating state constraints by a feasible trajectory satisfying some estimates. Such feasible trajectory is obtained in a constructive way making this approach more efficient than penalization.

In an infinite dimensional separable Banach space X, consider the solutions $x : I = [0, 1] \to X$ of the control system

$$\dot{x}(t) = \mathbb{A}\, x(t) + f(t, x(t), u(t)), \qquad \text{a.e. } t \in I\,, \tag{1}$$

that satisfy state constraints of the form

$$x(0) \in Q_0, \qquad x(t) \in K, \quad \forall\, t \in I\,. \tag{2}$$

Here, u is a measurable selection of a given measurable set valued map $U : I \rightsquigarrow Z$ with closed non-empty images, and Z is a complete separable metric space modeling the control set. The densely defined unbounded linear operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$, the map $f : I \times X \times Z \to X$ is Fréchet differentiable with respect to the second variable x, Q_0 and K are closed subsets of X. The trajectories of (1) are understood in the mild sense. Notice that we allow nonsmooth constraints, that are important in the applications (industrial, medical, economical...).

Given a differentiable map $g : X \to \mathbb{R}$, consider the Mayer optimization problem

$$\text{minimize} \quad \big\{ g(x(1)) \,:\, x \;\text{ is a solution of } (1), (2) \;\text{ for some control } u(\cdot) \big\}\,. \tag{3}$$

Recall that optimal control problems involving the integral cost can be reduced to (3) by adding an extra variable. Our main result is a direct proof of the following constrained Pontryagin Maximum Principle (PMP): for any locally optimal trajectory/control pair $(\bar{x}, \bar{u})$ for problem (3) and any nonempty convex cone C_0 contained in the contingent cone $T_{K \cap Q_0}(\bar{x}(0))$ to $K \cap Q_0$ at $\bar{x}(0)$, there exist a multiplier $\lambda \in \{0, 1\}$ and a countably additive regular measure of bounded variation γ, not vanishing simultaneously, such that the solution z (in the sens of [12]) to the measure-driven adjoint variational equation

$$(4) \qquad \begin{cases} \mathrm{d}z(t) = -\big(\mathbb{A}^* + \partial_x f(t, \bar{x}(t), \bar{u}(t))^*\big) z(t)\mathrm{d}t - \gamma(\mathrm{d}t)\,, & t \in I \\ z(1) = \lambda \nabla g(\bar{x}(1))\,, \end{cases}$$

satisfies the optimality condition

$$(5) \qquad \langle z(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle = \min_{u \in U(t)} \langle z(t), f(t, \bar{x}(t), u) \rangle, \qquad \text{for a.e. } t \in I$$

together with the transversality condition $-z(0) \in C_0^-$.

The maximum principle in Banach spaces has been studied in the 60ies, see for instance [9, 10]. Since then, many authors have contributed to extend it to the state constrained case, both in the abstract semigroup setting and in the PDEs framework: among many others we mention [1, 2, 4, 7, 17, 20], the classical books quoted above, and the references contained therein. The novelty of our approach relies in the fully general state constrained evolutionary systems considered. For these problems we are able to provide a simple proof of the constrained (PMP) together with sufficient conditions implying the validity of the optimality condition in a qualified (normal) form.

To derive this result, instead of using Ekeland's principle, as it was done in many papers dealing with necessary optimality conditions, both in the abstract semigroup setting and in the direct approach to PDEs, see e.g. [12], we apply a variational technique based on our results from [14]. The main idea is to linearize the constrained control problem, using convexified variational differential inclusions and a convex linearization of the state constraints and to prove the generalized Fermat rule thanks to the (NFT)-theorem. Then, the duality arguments lead to necessary conditions for optimality in a straightforward way. Moreover, by exploiting inward pointing conditions, we guarantee that (PMP) holds in the normal form, that is with $\lambda = 1$ (see [16] for an overview of the existing results in finite dimension). Normality of the maximum principle plays a crucial role in necessary optimality conditions since it allows to deduce qualitative properties of the optimal trajectories.

References

[1] N. Arada, J.-P. Raymond, *Dirichlet boundary control of semilinear parabolic equations, part 2: problems with pointwise state constraints*, Appl. Math. Optim., **45** (2002), 145–167.

[2] V. Barbu, *Constrained control problems with convex cost in Hilbert space*, J. Math. Anal. Appl., **56** (1976), 502–528.

[3] V. Barbu, *Analysis and control of nonlinear infinite dimensional systems*, Academic Press, Boston, New York, (1993).

[4] V. Barbu, G. Wang, *State constraint optimal control problems governed by semilinear equations*, Numer. Funct. Anal. Optim., **21** (2000), 411–424.

[5] A. Bensoussan, G. Da Prato, M. Delfour, S. K. Mitter, *Representation and control of infinite dimensional systems*, vol.1, Birkhäuser, Boston, (1992).

[6] A. Bensoussan, G. Da Prato, M. Delfour, S. K. Mitter, *Representation and control of infinite dimensional systems*, vol.2, Birkhäuser, Boston, (1993).

[7] E. Casas, *Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations*, SIAM J. Control Optim., **35** (1997), 1297–1327.

[8] A. Cernea, H. Frankowska, *A connection between the maximum principle and dynamic programming for constrained control problems*, SIAM J. Control Optim., **4** (2005), 673–703.

[9] Yu. B. Egorov, *Optimal control in Banach spaces*, Dokl. Akad. Nauk SSSR, **150** (1962), 241–244.

[10] Yu. B. Egorov, *Some necessary conditions for optimality in Banach spaces*, Mat. Sbornik, **64** (1964), 79–101.

[11] H. O. Fattorini, *Optimal control problems with state constraints for semilinear distributed-parameter system*, J. Optim. Theory Appl., **88** (1996), 25–59.

[12] H. O. Fattorini, *Infinite-dimensional optimization and control theory*, Cambridge University Press, Cambridge, (1999).

[13] H. Frankowska, E.M. Marchini, M. Mazzola, *A relaxation result for state constrained inclusions in infinite dimension*, Math. Control Relat. Fields, **6** (2016), 113–141.

[14] H. Frankowska, E.M. Marchini, M. Mazzola, *Distance estimates for state constrained trajectories of infinite dimensional differential inclusions*, ESAIM: COCV, in press, doi: https://doi.org/10.1051/cocv/2017032

[15] H. Frankowska, M. Mazzola *On relations of the adjoint state to the value function for optimal control problems with state constraints*, NODEA, **20** (2013), 361–383.

[16] H. Frankowska, D. Tonon, *Inward pointing trajectories, normality of the maximum principle and the non occurrence of the Lavrentieff phenomenon in optimal control under state constraints*, J. Convex Anal., **20** (2013), 1147–1180.

[17] B. Hu, J. Yong, *Pontryagin maximum principle for semilinear and quasi-linear parabolic equations with pointwise state constraints*, SIAM J. Control Optim., **33** (1995), 1857–1880.

[18] I. Lasiecka, R. Triggiani, *Control theory for partial differential equations: continuous and approximation theories*, Cambridge University Press, Cambridge, (2000).

[19] X. Li, J. Yong, *Optimal control theory for infinite-dimensional systems*, Birkhäuser, Boston, (1995).

[20] H. Liu, J. Yang, *Optimal control of semilinear parabolic systems with state constraint*, J. Math. Anal. Appl., **417** (2014), 787–803.

[21] F. Tröltzsch, *Optimal control of partial differential equations: theory, methods and applications*, AMS, Providence, (2010).

A parallel-in-time gradient-type method for optimal control of nonlinear systems

Matthias Heinkenschloss

(joint work with Xiaodi Deng)

A new parallel-in-time gradient-type method for the solution of time dependent optimal control problems is introduced. Each iteration of the classical gradient method requires the solution of the forward-in-time state equation followed by the solution of the backward-in-time adjoint equation to compute the gradient. To introduce parallelism, the time steps are split into N time subintervals. If correct state and adjoint information was available at the time subinterval boundaries, the time subdomain problems could be solved in parallel to compute the gradient. However, the state and adjoint information at the time subinterval boundaries must now be computed as part of the optimization problem. Unlike in multiple shooting formulations, were continuity conditions of the state equation at time subinterval boundaries are introduced as constraints and states at time subinterval boundaries are introduced as additional optimization variables, we simply use state and adjoint information at time subinterval boundaries from the previous iteration. Given this information, on each time subinterval the forward-in-time state equation is solved, the backward-in-time adjoint equation is solved, gradient-type information is generated, and the controls are updated. These computations can be performed in parallel across time subintervals. State and adjoint information at time subinterval boundaries is then exchanged with neighboring subintervals and the process is repeated. The overall structure of the algorithm is nearly identical to the classical gradient method, and therefore relatively easy to implement. Since the state and adjoint equations are not satisfied at time subdomain boundaries the algorithm is not a gradient method, and a new convergence analysis is needed.

Applied to a convex linear quadratic optimal control problem with controls in $\mathbb{R}^m$, this method can be interpreted as a so-called $(2N-1)$-part iteration scheme. Convergence of this iteration scheme can be proven by analyzing the spectrum of the corresponding iteration matrix, a block $(2N-1)\times(2N-1)$ companion matrix. We prove that the spectral radius of this companion matrix is less than one for sufficiently small, positive step sizes. Therefore, the parallel-in-time gradient-type method converges for sufficiently small, positive step sizes. The step size typically depends on the number N of time subintervals, and a better characterization of suitable step-sizes is work in progress.

For general nonlinear, smooth problems basic first-order convergence is proven for sufficiently small positive step sizes, by estimating the difference between the gradient-type direction and the gradient direction. Currently, these results are established for problems with finite dimensional controls, but it seems possible to extend the proofs to problems with controls in Hilbert spaces.

The convergence of the new method is illustrated on two examples, a 3D linear-quadratic parabolic advection diffusion control problem, and a well rate optimization problem for a two-phase immiscible reservoir. Nearly perfect speed-up is observed for a small to moderate number of time subdomains. For the second example two-level parallelism is employed. Within the computations on a single time subdomain parallel solvers were used for nonlinear and linear systems in the state and adjoint computations. It is demonstrated that the speed-up due to time decomposition multiplies the speed-up due to parallelization in the solution of state and adjoint equations.

References

[1] X. Deng and M. Heinkenschloss, *A Parallel-in-Time Gradient-Type Method for Discrete Time Optimal Control Problems*, Technical Report, Department of Computational and Applied Mathematics, Rice University, 2016.

Second order optimality conditions and applications for some bang-bang control problems of semilinear parabolic equations

Eduardo Casas
(joint work with Mariano Mateos, Arnd Rösch)

We consider the following optimal control problem

$$\text{(P)} \qquad \min_{u\in\mathcal{U}_{ad}} J(u) = \int_Q L(x,t,y_u(x,t))\,dx\,dt,$$

where

$$\mathcal{U}_{ad} = \{u = (u_j)_{j=1}^m \in L^\infty(0,T)^m : \alpha_j \le u_j(t) \le \beta_j \ \text{ for a.a. } t\in(0,T),\ 1\le j\le m\}$$

with $-\infty < \alpha_j < \beta_j < +\infty$ for $j = 1,\dots,m$; and y_u is the solution of the equation

$$(1)\quad \left\{\begin{array}{rcll} \dfrac{\partial y}{\partial t}(x,t) - \Delta y(x,t) + f(x,t,y(x,t)) &=& \displaystyle\sum_{j=1}^m u_j(t)g_j(x) & \text{in } Q = \Omega\times(0,T),\\ y(x,t) &=& 0 & \text{on } \Sigma = \Gamma\times(0,T),\\ y(x,0) &=& y_0(x) & \text{in } \Omega. \end{array}\right.$$

We assume that $\Omega \subset \mathbb{R}^n$ $(1 \le n \le 3)$ is a bounded domain with a Lipschitz boundary, $y_0 \in L^\infty(\Omega)$ and the functions $\{g_j\}_{j=1}^m \subset L^{\hat p}(0,T;L^{\hat q}(\Omega)) \setminus \{0\}$ ($\hat p, \hat q \in [2,\infty]$ and $\dfrac{1}{\hat p} + \dfrac{n}{2\hat q} < 1$) have pairwise disjoint supports $\omega_j = \operatorname{supp}(g_j)$. We also assume that $f, L : Q\times\mathbb{R} \longrightarrow \mathbb{R}$ are Borel functions, of class C^2 with respect to the last variable. The function f satisfies for almost all $(x,t)\in Q$

$$(2)\qquad f(\cdot,\cdot,0) \in L^{\hat p}(0,T;L^{\hat q}(\Omega)),$$

$$(3)\qquad \exists\Lambda\in\mathbb{R} : \frac{\partial f}{\partial y}(x,t,y) \ge \Lambda \quad \forall y\in\mathbb{R},$$

$$(4)\qquad \forall M>0\ \exists C_M : \left|\frac{\partial f}{\partial y}(x,t,y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x,t,y)\right| \le C_M \quad \forall |y|\le M,$$

$$(5)\qquad \left\{\begin{array}{l} \forall M>0 \text{ and } \forall\rho>0\ \exists\varepsilon>0 \text{ such that}\\ \left|\dfrac{\partial^2 f}{\partial y^2}(x,t,y_2) - \dfrac{\partial^2 f}{\partial y^2}(x,t,y_1)\right| \le \rho \text{ if } |y_2-y_1|<\varepsilon \text{ and } |y_1|,|y_2|\le M. \end{array}\right.$$

Finally, L satisfies for almost all $(x,t) \in Q$

(6) $L(\cdot,\cdot,0) \in L^1(Q)$,

(7) $\forall M > 0\, \exists \psi_M \in L^{\hat{p}}(0,T;L^{\hat{q}}(\Omega)) : \left|\frac{\partial L}{\partial y}(x,t,y)\right| \le \psi_M(x,t) \quad \forall |y| \le M$,

(8) $\forall M > 0\, \exists C_M : \left|\frac{\partial^2 L}{\partial y^2}(x,t,y)\right| \le C_M \quad \forall |y| \le M$,

(9) $\begin{cases} \forall M > 0 \text{ and } \forall \rho > 0\, \exists \varepsilon > 0 \text{ such that} \\ \left|\frac{\partial^2 L}{\partial y^2}(x,t,y_2) - \frac{\partial^2 L}{\partial y^2}(x,t,y_1)\right| \le \rho \text{ if } |y_2 - y_1| < \varepsilon \text{ and } |y_1|, |y_2| \le M. \end{cases}$

Following [1], we say that $\bar{u}$ is a strong local minimum of (P) with a local quadratic growth if there exist $\varepsilon > 0$ and $\kappa > 0$ such that

(10) $$J(\bar{u}) + \frac{\kappa}{2}\|y_u - \bar{y}\|^2_{L^2(Q)} \le J(u) \quad \forall u \in U_{ad} : \ \|y_u - \bar{y}\|_{L^\infty(\Omega)} < \varepsilon.$$

We prove that $\bar{u} \in \mathcal{U}_{ad}$ satisfies (10) if besides the first order optimality condition, the following second order condition holds

(11) $$\exists \tau > 0,\ \exists \delta > 0\ :\ J''(\bar{u})v^2 \ge \delta \|z_v\|^2_{L^2(Q)} \quad \forall v \in C^\tau_{\bar{u}},$$

where

$$C^\tau_{\bar{u}} = \{v \in L^2(0,T)^m :\ v_j(t) \begin{cases} \ge 0 & \text{if } \bar{u}_j(t) = \alpha_j \\ \le 0 & \text{if } \bar{u}_j(t) = \beta_j \\ = 0 & \text{if } \left|\int_\Omega \bar{\varphi}(t) g_j\, dx\right| > \tau \end{cases} \quad \text{for a.a. } t \in (0,T)\},$$

Above, $\bar{\varphi}$ denotes the adjoint state associated with $\bar{u}$. Then, under the assumption (11), we can prove stability of optimal states of the control problem with respect to some perturbations of the data, and we can prove some error estimates corresponding to the optimal states for the numerical discretization of the control problem. In order to deduce estimates for the optimal controls, an additional hypothesis is assumed:

(12) $$\exists K > 0 : \forall \varepsilon > 0 \quad \left|\left\{t \in (0,T) : \left|\int_\Omega \bar{\varphi}(t) g_j\, dx\right| < \varepsilon\right\}\right| \le K\varepsilon \quad \forall 1 \le j \le m.$$

If $\bar{u} \in \mathcal{U}_{ad}$ satisfies the first order optimality conditions, (11) and (12), then there exist $\varepsilon > 0$, $\nu > 0$, and $\kappa > 0$ such that

$$J(\bar{u}) + \frac{\nu}{2}\|u - \bar{u}\|^{1+\frac{1}{\gamma}}_{L^1(Q)} + \frac{\kappa}{2}\|y_u - \bar{y}\|^2_{L^2(Q)} \le J(u) \quad \forall u \in U_{ad} :\ \|y_u - \bar{y}\|_{L^\infty(\Omega)} < \varepsilon.$$

This inequality is used to derive error estimates for the discretization of the controls.

References

[1] T. Bayen, J. Bonnans, and F. Silva, *Characterization of local quadratic growth for strong minima in the optimal control of semi-linear elliptic equations*, Trans. Amer. Math. Soc., **366** (2014), 2063–2087.

Explicit exponential stabilization of nonautonomous linear parabolic-like systems by a finite number of internal actuators

Sérgio S. Rodrigues
(joint work with Karl Kunisch)

The aim is to construct an explicit finite-dimensional feedback controller for stabilization of linear parabolic equations, with a time-dependent reaction-convection operator. The reason for looking for finite-dimensional controls is motivated by applications where only a finite number of actuators can be tuned. Feedback controls are also important for applications due to their robustness against small measurement/disturbance errors. We also look for an explicit feedback law/operator which is easy to compute, because as it is well known, computing the classical Riccati feedback operator can be a quite demanding numerical task.

The range of the controller is modeled by indicator functions of small subdomains. Its dimension M depends polynomially on a suitable norm of the reaction-convection operator. A sufficient condition for stabilizability is given, which involves the asymptotic behavior of the eigenvalues of the (time-independent) diffusion operator, the norm of the reaction-convection operator, and the norm of the nonorthogonal projection onto the controller's range along a suitable infinite dimensional (higher-modes) eigenspace. To construct the explicit feedback, the essential step consists in computing the nonorthogonal projection. Numerical simulations are presented, in 2D, showing the practicability of the controller and its response to measurement errors, where the actuators are indicator functions of suitable small subsets.

We consider a nonautonomous parabolic system in a smooth bounded domain $\Omega \subset \mathbb{R}^d$

$$\dot{y}(t) - \nu\Delta y(t) + a(t)y(t) + \nabla\cdot(b(t)y(t)) - \sum_{i=1}^{M} u_i(t)1_{\omega_i} = 0, \quad y(0) = y_0, \tag{1}$$

where $y = y(t,x)$ is the state, y_0 is given in $L^2(\Omega)$, $u(t) = (u_1,\ldots,u_M)(t)$ is a control function at our disposal, taking values in $\mathbb{R}^M$, and and $1_{\omega_i} = 1_{\omega_i}(x)$, $i \in \{1, 2, \ldots, M\}$, are indicator functions of some small domains $\omega_i \subset \Omega$.

Here, we assume Dirichlet boundary conditions, $y|_{\partial\Omega} = 0$.

Let $0 < \alpha_i$, $i \in \mathbb{N}_0$, be the increasing sequence of (repeated) eigenvalues of $-\nu\Delta$ and let E_M be the span of the eigenfunctions associated with its first M eigenvalues. We will assume that our actuators satisfy $L^2(\Omega) = \mathcal{U} \oplus E_M^\perp$, which allow us to define the (nonorthogonal) projection $P_{\mathcal{U}}^{E_M^\perp} : L^2(\Omega) \to \mathcal{U}$ onto $\mathcal{U}$ along $E_M^\perp$.

Let us denote the operator $y \mapsto A_{\mathrm{rc}}y = a(t)y + \nabla\cdot(b(t)y)$. The main result is as follows: if

$$L^2(\Omega) = \mathcal{U} \oplus E_M^\perp, \tag{2a}$$

$$\overline{\mu}_M \coloneqq \alpha_{M+1} - \left(6 + 4\left|P_{\mathcal{U}}^{E_M^\perp}\right|^2_{\mathcal{L}(L^2(\Omega))}\right)|A_{\mathrm{rc}}|^2_{L^\infty((0,+\infty),\mathcal{L}(L^2(\Omega),H^{-1}(\Omega)))} > 0, \tag{2b}$$

then, for any given constant $\lambda > 0$, a feedback stabilizing control is given by

$$y \mapsto \mathcal{K}(t)y \coloneqq P_{\mathcal{U}}^{E_M^\perp}\left(-\nu\Delta y + A_{\mathrm{rc}}(t)y - \lambda y\right). \tag{3}$$

More precisely, the system

$$\dot{y}(t) - \nu\Delta y(t) + A_{\mathrm{rc}}(t)y(t) - \mathcal{K}(t)y(t) = 0, \quad y(0) = y_0, \tag{4}$$

is exponentially stable: there exist suitable constants $\mu > 0$ and $D \geq 1$ such that

$$|y(t)|^2_{L^2(\Omega)} \leq D\mathrm{e}^{-\mu(t-s)}\,|y(s)|^2_{L^2(\Omega)}, \quad \text{for all} \quad t \geq s \geq 0.$$

Observe that (4) is exactly (1), with u given by

$$\sum_{i=1}^{M} u_i 1_{\omega_i} = P_{\mathcal{U}}^{E_M^\perp}(-\nu\Delta y + A_{\mathrm{rc}}y - \lambda y). \tag{5}$$

Once we have an estimate $\widehat{y}(t)$ for $y(t)$, we can compute an estimate of our control as $\mathcal{K}(t)\widehat{y}(t)$. Feedback controls are demanded in applications, because they are able to respond to (small) measurement/estimation errors. In Figure 1 we confirm that our proposed feedback is robust against such errors. As the magnitude of the noise (measurement error), $\eta \coloneqq \widehat{y}(t) - y(t)$, gets smaller the solution goes to a smaller neighborhood of zero.

We present the results corresponding to simulations of the system (1) with a perturbed feedback (i.e., with an estimated feedback control $\mathcal{K}(t)\widehat{y}(t)$):

$$\dot{y} - \nu\Delta y + ay + \nabla\cdot(by) - P_{\mathcal{U}}^{E_M^\perp}(-\nu\Delta\widehat{y} + a\widehat{y} + \nabla\cdot(b\widehat{y}) - \lambda\widehat{y}) = 0, \tag{6}$$

$$y(0) = y_0. \tag{7}$$

where $\widehat{y} \coloneqq y + \eta$. We have taken

$$a(t, x_1, x_2) = -0.1 - 0.2|\sin(t + x_1)|_{\mathbb{R}}, \quad b(t, x_1, x_2) = \begin{pmatrix} 0.1(x_1 + x_2) \\ 0.1\cos(t)x_1x_2 \end{pmatrix}, \tag{8}$$

$$\begin{cases} \nu = 0.1, \\ y_0 = 0.01e_1. \end{cases} \tag{9}$$

and we have taken 4 actuators 1_{ω_i}, $i \in \{1, 2, 3, 4\}$, whose regions are as in Figure 1.

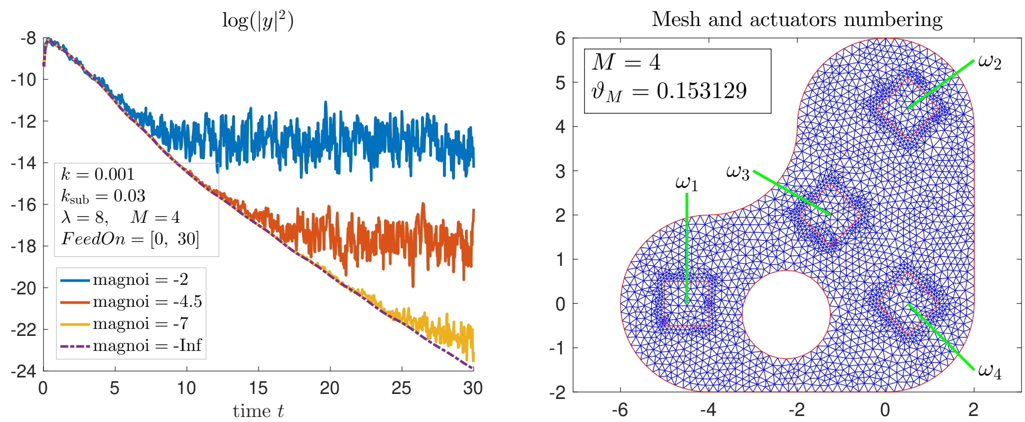

FIGURE 1. Response to measurement errors and actuators regions.

The value ϑ_M in Figure 1 defines the norm of the nonorthogonal projection as $\left|P_{\mathcal{U}}^{E_M^\perp}\right|^2_{\mathcal{L}(L^2(\Omega))} = \vartheta_M^{-1}$.

We took the noise in the form $\eta = \eta_1 y + \eta_2$ having a component $\eta_1 y$ which is proportional to the state y and a component η_2 which is independent of the state.

We tested with a hypothetical noise η ("typnoi" in figures) as

$$\eta(x,t) = \mathrm{rndn}(t,x,\zeta)$$
$$:= \mathrm{e}^\zeta\Big((\min\{1,\max\{-1,\mathrm{v}_{\mathrm{ran}1}(x,t)\}\}-1)y(x,t)+\min\{1,\max\{-1,\mathrm{v}_{\mathrm{ran}2}(x,t)\}\}\Big),$$

The function rndn is "random" and is to be understood as follows: once we have solved our system up to time $t_m = mk$, say we have just found $y(t_m)$, then we generate random vectors $\mathrm{v}_{\mathrm{ran}_i}(t_m) \in \mathbb{R}^{N+1}$, from which we construct the noise function rndn at time $t = t_m$.

The vectors $\mathrm{v}_{\mathrm{ran}_i}(t_m)$, were generated by the Matlab function `randn`.

Further details may be found in the preprint [1], where we can also find further numerical results, including the 1D case.

References

[1] K. Kunisch and S.S. Rodrigues, *Explicit Exponential Stabilization of Nonautonomous Linear Parabolic-like Systems by a Finite Number of Locally Supported Actuators*, RICAM-Report No. 2017-44 (2017). https://www.ricam.oeaw.ac.at/publications/ricam-reports/

Optimization problems with quasi-variational inequality constraints

Carlos N. Rautenberg

(joint work with Amal Alphonse, Michael Hintermüller)

We consider a non-standard optimization problem with Quasi-Variational Inequalities (QVIs) constraints. The objective functional of the optimization problem takes set valued arguments, and the reduced formulation can be posed in terms of minimal and maximal solutions to the QVI. The stability of minimal and maximal elements of the solution set with respect to perturbations of the forcing term is considered, and a well-posedness result for the optimization problem is provided.

The optimization problem

The QVI problem class under consideration is given by

Problem $(\mathrm{P}_{\mathrm{QVI}})$: Given $f \in V'$

$$(\mathrm{P}_{\mathrm{QVI}}) \qquad \text{Find } y \in \mathbf{K}(y) : \langle A(y) - f, v - y\rangle \geq 0, \quad \forall v \in \mathbf{K}(y),$$

where $\mathbf{K}(y) \in 2^V$ for each $y \in V$.

In this setting, we assume that V is the state space on a Hilbert space Gelfand triple (V, H, V'), of real valued maps on a certain domain $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$, where $L^\infty(\Omega)$ embeds continuously into H. We further suppose that the operator

$A : V \to V'$ is homogenous of degree one, Lipschitz continuous, and uniformly and T-monotone. We further assume that there exists a closed convex cone H_+ in H satisfying

$$H_+ = \{v \in H : (v, y) \geq 0, \quad \text{for all} \quad y \in H^+\}.$$

The cone H_+ defines the cone of non-negative elements inducing the vector ordering: $x \leq y$ if and only if $y - x \in H_+$, and we assume that V is compatible with this order in the following sense: $y \in V \Rightarrow y^+ \in V$ and there exists $\mu > 0$ such that $\|y^+\|_V \leq \mu \|y\|_V$, for all $y \in V$. Note that the ordering in H induces one in V': if $f, g \in V'$, we write $f \leq g$ if $\langle f, \phi \rangle \leq \langle g, \phi \rangle$ for all $g \in V_+ := V \cap H_+$ and define $V'_+ := \{f \in V' : f \geq 0\}$.

The general structure of $\mathbf{K}$ is of obstacle type and given by

$$\mathbf{K}(v) := \{w \in V : w \leq \Phi(v)\},$$

where $\Phi : H \to H$ is increasing: if $v \leq w$, then $\Phi(v) \leq \Phi(w)$, and non-negative above zero, that is, $\Phi(0) \geq 0$. Additionally, we consider the set of admissible forcing terms $\mathcal{A}_{\mathrm{ad}} = \{g \in U_{\mathrm{ad}} : 0 \leq g \leq f_{\max}\}$, for some $f_{\max} \in U$, where $U_{\mathrm{ad}} \subset U \subset H$, where U is a subspace of H.

The optimization problem of interest is given by

Problem ($\mathbb{P}$):

$$(\mathbb{P}) \qquad \begin{aligned} &\text{minimize } J(\mathbf{O}, f) := J_1(T_{\sup}(\mathbf{O}), T_{\inf}(\mathbf{O}), f) \\ &\text{over} \quad (\mathbf{O}, f) \in 2^H \times U, \\ &\text{subject to } f \in \mathcal{A}_{\mathrm{ad}}, \\ &\qquad\qquad y \in \mathbf{O}, \quad \mathbf{O} = \{z \in V : z \text{ solves } (\mathrm{P_{QVI}}) \cap [\underline{y}, \overline{y}]\}. \end{aligned}$$

In the above problem we consider $J_1 \colon H \times H \times U \to \mathbb{R}$ and for $\underline{y}, \overline{y} \in H$ we define the set map $T_{\sup}$

$$T_{\sup}(\mathbf{O}) := \begin{cases} \sup_{z \in \mathbf{O} \cap [\underline{y}, \overline{y}]} z, & \mathbf{O} \cap [\underline{y}, \overline{y}] \neq \emptyset\ ; \\ \underline{y}, & \text{otherwise.} \end{cases}$$

The map $T_{\inf}$ defined analogously as

$$T_{\inf}(\mathbf{O}) := \begin{cases} \inf_{z \in \mathbf{O} \cap [\underline{y}, \overline{y}]} z, & \mathbf{O} \cap [\underline{y}, \overline{y}] \neq \emptyset\ ; \\ \overline{y}, & \text{otherwise.} \end{cases}$$

In our setting, the supremum of an arbitrary subset of H that is bounded above (in the order) is also correctly defined since H is Dedekind complete, which shows that $T_{\inf}$ and $T_{\sup}$ are well defined in our setting.

Thermoforming. In the industrial production of plastics, the technique of thermoforming is usually used. In this procedure, a plastic sheet is heated to pliable temperature and then forced towards a mold, commonly made of metal and associated to some cooling mechanism.

The time asymptotic behaviour of the thermoforming process leads to an elliptic problem. We consider a plastic membrane y over the domain Ω, assume that the temperature T of the membrane remains constant (neglecting changing rheological properties), and denote Φ to the position of the mold.

The problem is then given by: Find $(y, \Phi, T) \in V \times V \times W$ such that

$$\begin{aligned} y \le \Phi, \quad \langle Ay - f, y - v\rangle &\le 0, && \forall v \in V : v \le \Phi \\ \langle kT - \Delta T, w\rangle &= (g(\Phi - u), w) && \forall w \in W \\ \Phi &= \Phi_0 + LT, && \text{in } V \end{aligned}$$

where $f \in H^+$, $k > 0$ is a constant, $\Phi_0 \in V$, and $L\colon W \to V$ is a (locally) order preserving bounded linear operator, $g\colon \mathbb{R} \to \mathbb{R}$ is decreasing with $g(0) = M > 0$ a constant, $0 \le g \le M$ and g' bounded.

The reduced problem, well-posedness, and open questions

Solutions to the QVI of interest can be posed as fixed points of an increasing map. In fact, the Tartar-Birkhoff fixed point theorem establishes that we have the operators

$$\mathsf{m}\colon \mathcal{A}_{\mathrm{ad}} \to V \qquad \text{and} \qquad \mathsf{M}\colon \mathcal{A}_{\mathrm{ad}} \to V,$$

that take elements of $\mathcal{A}_{\mathrm{ad}}$ to the minimal and maximal solutions to $(\mathrm{P}_{\mathrm{QVI}})$ on the interval $[\underline{y}, \overline{y}] = [0, A^{-1} f_{\max}]$. Then the reduced version of problem $(\mathbb{P})$ is formulated in terms of m and M as

$$(\mathbb{P}_{\mathrm{red}}) \qquad \begin{aligned} &\text{minimize } J_1(\mathsf{M}(f), \mathsf{m}(f), f) \\ &\text{subject to } f \in \mathcal{A}_{\mathrm{ad}}. \end{aligned}$$

An important class of examples is given when we want to force the solution set to be a singleton and at the same time close to some desired state y_d. Here, a possible choice for J_1 is given by

$$J_1(\mathsf{M}(f), \mathsf{m}(f), f) = \frac{1}{2}\int_\Omega |\mathsf{M}(f) - \mathsf{m}(f)|^2 dx + \frac{\sigma}{2}\int_\Omega |y_d - \mathsf{m}(f)|^2 dx.$$

In the setting of the thermoforming example this is the expected problem to solve. If we consider the solutions to $(\mathrm{P}_{\mathrm{QVI}})$ to be limits as $t \to \infty$ of an underlying evolutionary process, such control problem can be translated as an attempt to force production to be as uniform as possible, together with forcing it to be close to y_d.

The well-posedness result that can be obtained is: Suppose that $U_{\mathrm{ad}} \subset H$ is bounded, $U_{\mathrm{ad}} \subset L^\infty_\nu(\Omega) = \{h \in L^\infty(\Omega) : h \ge \nu > 0 \text{ a.e. in } \Omega\}$, and that the embedding $U \hookrightarrow L^\infty(\Omega)$ is compact. Then, there exists a solution to problem $(\mathbb{P})$, provided structural assumptions on the map Φ, and assuming that $\Phi : V \to V$ is completely continuous. The structural results mentioned are satisfied if the map $\mathbb{R}^+ \ni \lambda \mapsto \lambda\Phi(y) - \Phi(\lambda y)$ is increasing for any $y \in V \cap H_+$.

Such a result hinges on the stability of the maps $f \mapsto \mathsf{m}(f)$ and $f \mapsto \mathsf{M}(f)$, and a full characterization of such has not been obtained yet. Specifically, if $\{f_n\}$ is in $\mathcal{A}_{\mathrm{ad}}$, what conditions are necessary and sufficient so that

$$\mathsf{m}(f_n) \to \mathsf{m}(f) \quad \text{and} \quad \mathsf{M}(f_n) \to \mathsf{M}(f)$$

in H and in V?

References

[1] A. Alphonse, M. Hintermüller, C. N. Rautenberg, *Directional differentiability for elliptic quasi-variational inequalities of obstacle type*, WIAS Preprint No. 2492, (2018) .
[2] A. Alphonse, M. Hintermüller, C. N. Rautenberg, *Stability of the Solution Set and Optimal Control of Quasi-variational Inequalities*, to appear.

On the Barzilai-Borwein step-sizes for optimization problems in Hilbert spaces

Behzad Azmi
(joint work with Karl Kunisch)

Due to simplicity and numerical efficiency, the Barzilai and Borwein (BB) method [1] has received a considerable amount of attention in different fields of optimization. In this method, to incorporate the quasi-Newton property, the Hessian is approximated by a scalar multiple of the identity in such a manner that the secant condition holds. In this talk, we discuss the convergence of this method for problems posed in infinite-dimensional Hilbert spaces. First, based on the spectral analysis, the R-linear global convergence of this method for strictly convex quadratic problems is presented. Then this result is extended to the local convergence for twice continuously Fréchet differentiable functions. Next, aiming at problems governed by partial differential equations, the results concerning the mesh-independent principle for the BB-method are presented. Numerical experiments are also given.

References

[1] J. Barzilai and J. M. Borwein, *Two-point step size gradient methods*, IMA Journal of Numerical Analysis, 8 (1988), pp. 141–148.

Phase field systems with maximal monotone nonlinearities related to SMC problems

Michele Colturato
(joint work with Pierluigi Colli)

We investigate phase field systems and Cahn-Hilliard systems perturbed by maximal monotone nonlinearities and singular terms, proving existence, uniqueness and longtime behavior of the solution. Then, we consider the related sliding mode control (SMC) problem: the main idea behind this scheme is first to identify a manifold of lower dimension (called the sliding manifold) where the control goal is fulfilled and such that the original system restricted to this sliding manifold has a desired behavior, and then to act on the system through the control in order to constrain the evolution on it, that is, to design a SMC-law that forces the trajectories of the system to reach the sliding surface and maintains them on it.

Phase-field systems. In [1] we consider a phase-field system of Caginalp type arising from a recent study of a sliding mode control problem. The two-phase–system is written in terms of a rescaled balance of energy and of a balance law for the microforces that govern the phase–transition. Moreover, the first equation of the system is perturbed by the presence of an additional maximal monotone nonlinearity. The unknowns of the problem are the *absolute temperature* ϑ and a *phase–parameter* φ which may represent the local proportion of one of the two phases. To ensure thermomechanical consistency, suitable physical constraints on φ are introduced: if it is assumed, e.g., that the two phases may coexist at each point with different proportions and it turns out to be reasonable to require that φ lies between 0 and 1, with $1-\varphi$ representing the proportion of the second phase. In particular, the values $\varphi = 0$ and $\varphi = 1$ may correspond to the pure phases, while φ is between 0 and 1 in the regions when both phases are present. Clearly, the following system should provide an evolution for ϑ and φ that complies with the previous physical constraint:

$$\partial_t(\vartheta + \ell\varphi) - k\Delta\vartheta + \zeta = f \quad \text{a.e. in } Q := (0,T)\times\Omega, \tag{1}$$

$$\partial_t\varphi - \upsilon\Delta\varphi + \xi + \pi(\varphi) = \gamma\vartheta \quad \text{a.e. in } Q, \tag{2}$$

$$\zeta(t) \in A(\vartheta(t) + \alpha\varphi(t) - \eta^*) \text{ for a.e. } t \in (0,T), \tag{3}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{4}$$

where Ω is the domain in which the evolution takes place, T is some final time, ℓ, k, υ, γ and α are positive constants, η^* is a function in $H^2(\Omega)$ with null outward normal derivative on the boundary of Ω and f is a source term.

The term $\xi + \pi(\varphi)$, appearing in (2), represents the derivative of a double-well potential $\mathcal{W}$ associated with the phase–configuration. $\mathcal{W}$ can be defined as the sum $\mathcal{W} = \widehat{\beta} + \widehat{\pi}$, where $\widehat{\beta} : \mathbb{R} \to [0,+\infty]$ is a proper, l.s.c. and convex function and $\widehat{\pi} : \mathbb{R} \to \mathbb{R}$ is a function in $C^1(\mathbb{R})$ such that $\pi := \widehat{\pi}'$ is Lipschitz continuous. Due to the properties of $\widehat{\beta}$, the subdifferential $\partial\widehat{\beta} =: \beta$ is well defined and is a maximal monotone graph. In the problem (1)–(4) a maximal monotone operator $A : L^2(\Omega) \to L^2(\Omega)$ also appears. We assume that A is the subdifferential of a proper, convex and lower semicontinuous function $\Upsilon : L^2(\Omega) \to \mathbb{R}$ which takes its minimum in 0, and A is linearly bounded in $L^2(\Omega)$. The above system is complemented by homogeneous Neumann boundary conditions for both ϑ and φ, that is, $\partial_\nu\vartheta = 0$, $\partial_\nu\varphi = 0$ on $\Sigma := (0,T)\times\Gamma$, where Γ is the boundary of Ω and ∂_ν denotes the outward normal derivative. Finally, we prescribe the initial conditions $\vartheta(0) = \vartheta_0$ and $\varphi(0) = \varphi_0$ in Ω. The paper [1] will focus only on analytical aspects: we prove existence and uniqueness of strong solutions for problem (1)–(4) and show the continuous dependence on the initial data.

Cahn–Hilliard systems. In the paper [2] we prove existence and regularity for the solutions to a Cahn–Hilliard system describing the phenomenon of phase–separation for a material contained in a bounded and regular domain. Using the same assumptions and notations listed in the previous paragraph, the system under study is

$$\partial_t(\vartheta + \ell\varphi) - \Delta\vartheta + \zeta = f \quad \text{a.e. in } Q, \tag{5}$$

$$\partial_t\varphi - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{6}$$

$$\mu = -\upsilon\Delta\varphi + \xi + \pi(\varphi) - \gamma\vartheta \quad \text{a.e. in } Q, \tag{7}$$

$$\zeta(t) \in A(a\vartheta(t) + b\varphi(t) - \eta^*) \quad \text{for a.e. } t \in (0,T), \tag{8}$$

$$\xi \in \beta(\varphi) \quad \text{a.e. in } Q, \tag{9}$$

$$\partial_\nu\vartheta = \partial_\nu\varphi = \partial_\nu\mu = 0 \quad \text{on } \Sigma, \tag{10}$$

$$\vartheta(0) = \vartheta_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega, \tag{11}$$

where a, b are positive constants and μ denotes the chemical potential. We point out that here ϑ does not represent the absolute temperature Θ, but it is related to it by $\vartheta = \Theta - \Theta_c$ where Θ_c denotes a critical temperature. We observe that the system (5)–(11) contains a fourth-order equation and it is constructed as the conserved version of the phase-field system (1)–(4), thoroughly discussed in [1]. The second part of the paper [2] is devoted to the sliding mode control (SMC) problem: the main idea behind this scheme is first to identify a manifold of lower dimension (called the sliding manifold) where the control goal is fulfilled and such that the original system restricted to this sliding manifold has a desired behavior, and then to act on the system through the control in order to constrain the evolution on it, that is, to design a SMC-law that forces the trajectories of the system to reach the sliding surface and maintains them on it. The main advantage of sliding mode control is that it allows the separation of the motion of the overall system in independent partial components of lower dimensions, and consequently it reduces the complexity of the control problem. In particular, we prove the existence of sliding modes for the solutions of our system (5)–(11) for a suitable choice of the operator A and the coefficients a and b.

Singular phase–field systems. In [3] we consider a singular phase–field system located in a smooth and bounded three-dimensional domain. This system is constructed with the help of an entropy balance equation, which includes a logarithmic nonlinearity and additionally shows an extra term involving a possibly nonlocal maximal monotone operator. In order to explain the role of this further nonlinearity, we refer to [1, 2], where a class of sliding mode control problems is considered. The second equation of the system accounts for the phase–dynamics, and it is deduced from a balance law for the microscopic forces that are responsible for the phase–transition process. We also prescribe a no-flux condition on the boundary for both variables, while initial conditions are stated for $\ln\vartheta$ and φ:

$$\partial_t(\ln\vartheta + \ell\varphi) - k\Delta\vartheta + \zeta = f \quad \text{a.e. in } Q, \tag{12}$$

$$\partial_t\varphi - \Delta\varphi + \beta(\varphi) + \pi(\varphi) \ni \ell\vartheta \quad \text{a.e. in } Q, \tag{13}$$

$$\zeta(t) \in A(\vartheta(t) - \vartheta^*) \quad \text{for a.e. } t \in (0,T). \tag{14}$$

$$\partial_\nu \vartheta = 0, \quad \partial_\nu \varphi = 0 \quad \text{on } \Sigma, \tag{15}$$

$$\ln \vartheta(0) = \ln \vartheta_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{16}$$

The resulting system is highly nonlinear. The main difficulties lie in the treatment of the doubly nonlinear equation (12): the expert reader can realise that it is not trivial to recover some coerciveness and regularity for ϑ from (12), (14) and (15); moreover, the presence of both $\ln \vartheta$ under time derivative and the selection ζ from $A(\vartheta - \vartheta^*)$ complicates possible uniqueness arguments. The paper [4] focuses on the study of well-posedness and longtime behavior for a singular phase–field system with perturbed phase–dynamics, i.e., a problem similar to (12)–(16) characterized by the presence of a nonlocal maximal monotone nonlinearity in the second equation (13).

References

[1] M. Colturato, Solvability of a class of phase–field systems related to a sliding mode control problem, Appl. Math. **6** (2016), 623–650.

[2] M. Colturato, *On a class of conserved phase–field systems with a maximal monotone perturbation*, Appl. Math. Optim. **4** (2017), 1–35.

[3] P. Colli, M. Colturato, *Global existence for a singular phase–field system related to a sliding mode control problem*, Nonlinear Anal. Real World Appl. **41** (2018), 128–151.

[4] M. Colturato, *Well-posedness and longtime behavior for a singular phase–field system with perturbed phase–dynamics*, (2018), 1–34 [Accepted paper].

Optimal control of the 3D thermistor problem

Hannes Meinlschmidt

(joint work with Christian Meyer, Joachim Rehberg)

This talk is about the optimal control of the three-dimensional so-called thermistor problem, a coupled system of PDEs modeling the heating of a 3D workpiece Ω by inducing electrical current flow and using the Joule heating effect. The equations under consideration are as follows:

- a quasilinear potential equation for the potential φ with a temperature-dependent electrical conductivity $\sigma(\theta)\varepsilon$ subject to mixed boundary conditions, corresponding to grounding (on the part D), insulation, and the area where the current u is induced (on $\partial\Omega \setminus D = N$), in weak form:

$$-\nabla \cdot \sigma(\theta)\varepsilon\nabla\varphi = \mathcal{B}u \quad \text{in } W_D^{-1,q}(\Omega)$$

  with $\mathcal{B}$ being the adjoint of the trace operator onto N, and
- an also quasilinear heat equation with heat conductivity $\eta(\theta)\kappa$, subject to Robin boundary conditions $\eta(\theta)\nabla\theta \cdot \nu = \alpha(\theta - \theta_\ell)$, where the heat source is given by the electrical field strength $\sigma(\theta)\nabla\varphi \cdot \nabla\varphi$, in weak form:

$$\theta' - \nabla \cdot \eta(\theta)\kappa\nabla\theta = \sigma(\theta)\nabla\varphi \cdot \nabla\varphi + \mathcal{B}_\alpha(\theta - \theta_\ell) \quad \text{in } W_\emptyset^{-1,q}(\Omega)$$

with $\mathcal{B}_\alpha$ the α-weighted adjoint of the trace operator onto $\partial\Omega$.

Here, we allow Ω to be nonsmooth (in the class of Lipschitz domains), and require only L^∞-coefficient functions ε and κ as well as locally Lipschitz continuous conductivities σ and η; of course the assumptions for σ and η have to be improved to continuous differentiability with Lipschitz continuous derivatives for first-order optimality theory.

The goal is to control the induced current u in such a way that after some fixed time T a given temperature profile θ_d is achieved as close as possible in the $L^2(\Omega)$-sense. Thereby, it is necessary to also include control and state constraints in the form $0 \le u \le u_{\max}$ and $\theta \le \theta_{\max}$, the former corresponding to a maximal heating power, the latter assuring that a certain melting point $\theta_{\max}$ of the material of Ω is not surpassed. A similar problem without the quasilinear structure in the heat equation in two spatial dimensions was treated in [1].

Due to the very nonlinear structure in the state system and the quite general assumptions, the optimal control problem exhibits quite some difficulties such as in general no global solutions to the PDE system in the maximal regularity class—which we use to obtain continuous solutions in view of the state constraints—and very little *a priori* bounds on the solutions (θ, φ) in terms of u. This makes in particular the proof of existence of optimal controls quite difficult. Using maximal parabolic regularity techniques [4, 5], we were able to show that the set of controls whose associated state (θ, φ) exists globally in the maximal regularity space is nonempty and open, which allows to restrict the optimal control problem to such controls in a meaningful way. Further "minimal" bounds on the states were enforced using the objective functional, which allowed to derive a satisfactory theory for the optimal control of this nonlinear system of PDEs [2, 3].

References

[1] D. Hömberg, C. Meyer, J. Rehberg, and W. Ring, *Optimal control for the thermistor problem*, SIAM J. Control Optim. **48**(5) (2010), 3449–3481.

[2] H. Meinlschmidt, C. Meyer, and J. Rehberg, *Optimal control of the thermistor problem in three spatial dimensions, part 1: Existence of optimal solutions*, SIAM J. Control Optim. **55**(5) (2017), 2876–2904.

[3] H. Meinlschmidt, C. Meyer, and J. Rehberg, *Optimal control of the thermistor problem in three spatial dimensions, part 2: Optimality conditions*, SIAM J. Control Optim. **55**(4) (2017), 2368–2392.

[4] H. Meinlschmidt and J. Rehberg, *Hölder-estimates for non-autonomous parabolic problems with rough data*, Evol. Equ. Control Theory **5**(1) (2016), 147–184.

[5] J. Prüss, *Maximal regularity for evolution equations in L^p-spaces*, Conf. Semin. Mat. Univ. Bari **285** (2002), 1–39.

On Optimal Control Problems in Thermoelastoplasticity

Roland Herzog
(joint work with Christian Meyer, Ailyn Stötzner)

We consider the following quasistatic, thermoelastoplastic model at small strains with linear kinematic hardening and von Mises yield condition:

(1) stress-strain relation: $\boldsymbol{\sigma} = \mathbb{C}\big(\varepsilon(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)\big)$,

(2) conjugate forces: $\boldsymbol{\chi} = -\mathbb{H}\,\boldsymbol{p}$,

(3) viscoplastic flow rule: $\epsilon\,\dot{\boldsymbol{p}} + \partial_{\dot{\boldsymbol{p}}} D(\dot{\boldsymbol{p}}, \theta) \ni [\boldsymbol{\sigma} + \boldsymbol{\chi}]$,

(4) balance of momentum: $-\operatorname{div}\big(\boldsymbol{\sigma} + \gamma\,\varepsilon(\dot{\boldsymbol{u}})\big) = \boldsymbol{\ell}$,

(5) heat equation:
$$\varrho\, c_p\, \dot\theta - \operatorname{div}(\kappa\,\nabla\theta) = r + \gamma\,\varepsilon(\dot{\boldsymbol{u}}) : \varepsilon(\dot{\boldsymbol{u}}) + (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}} \\ -\theta\,\boldsymbol{t}'(\theta) : \mathbb{C}(\varepsilon(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}}).$$

The unknowns are the stress $\boldsymbol{\sigma}$, back-stress $\boldsymbol{\chi}$, plastic strain $\boldsymbol{p}$, displacement $\boldsymbol{u}$ and temperature θ. The first three are functions with values in the symmetric 3×3-matrices. Further, $\mathbb{C}$ and $\mathbb{H}$ denote the elastic and hardening moduli, respectively. $\varepsilon(\boldsymbol{u})$ denotes the symmetrized gradient, or linearized strain, associated with $\boldsymbol{u}$. The temperature dependent term $\boldsymbol{t}(\theta)$ expresses thermally induced strains. The dissipation function D is given by

$$D(\boldsymbol{q}, \theta) := \sqrt{2/3}\,\sigma_0(\theta)\,|\boldsymbol{q}|.$$

Here σ_0 denotes the temperature dependent yield stress, and $|\boldsymbol{q}|$ is the Frobenius norm, associated with the inner product $A : B = \operatorname{trace}(A^\top B)$.

The right hand sides $\boldsymbol{\ell}$ and r in (4) and (5) represent mechanical and thermal volume and boundary loads, respectively. ϱ, c_p and κ describe the density, specific heat capacity and thermal conductivity of the material. Finally, the positive parameters ϵ and γ represent viscous effects in the evolution of the plastic strain and in the balance of momentum.

The existence of a unique solution, given $(\boldsymbol{\ell}, r)$ in suitable function spaces, has been established in [4] under appropriate assumptions on the data. A major challenge in this analysis is the low integrability of the nonlinear terms on the right hand side of the heat equation (5). Several approaches for similar models have been previously discussed in [1, 2, 3, 6].

A second result proved in [4] is the weak sequential continuity of the map $(\boldsymbol{\ell}, r) \mapsto (\boldsymbol{u}, \boldsymbol{p}, \theta)$. Consequently, when $(\boldsymbol{\ell}, r)$ serve as optimization variables in an associated optimal control problem, the existence of a global minimizer can be proved by standard techniques.

In a subsequent paper [5] we consider the differentiability of the control-to-state map. To this end, it turns out to be useful to equivalently reformulate the plastic flow rule (3) as the following Banach space-valued ordinary differential equation,

$$\dot{\boldsymbol{p}} = -\epsilon^{-1} \min\left(\frac{\sqrt{2/3}\,\sigma_0(\theta)}{\big|[\boldsymbol{\sigma} + \boldsymbol{\chi}]^D\big|} - 1,\ 0 \right) [\boldsymbol{\sigma} + \boldsymbol{\chi}]^D, \tag{6}$$

where $[\,\cdot\,]^D$ denotes the deviatoric part of a matrix. Due to the non-smoothness of this equation, the control-to-state map will in general not be differentiable. However we obtain in [5] its directional, and in fact Hadamard differentiability. The directional derivative is characterized by the corresponding linearized system. As a consequence, we can obtain first-order necessary optimality conditions in primal form for associated optimal control problems. Sufficient conditions for the linearity of the directional derivative at a certain control pair $(\boldsymbol{\ell}, r)$ can be established as well. At such a point the control-to-state map is Gâteaux differentiable and the derivative can be evaluated via an adjoint approach.

References

[1] S. Bartels, T. Roubíček, *Thermoviscoplasticity at small strains*, Journal of Applied Mathematics and Mechanics **88** (2008), 735–754.

[2] S. Bartels, T. Roubíček, *Thermo-visco-elasticity with rate-independent plasticity in isotropic materials undergoing thermal expansion*, ESAIM. Mathematical Modelling and Numerical Analysis **45** (2011), 477–504.

[3] K. Chełmiński, R. Racke, *Mathematical analysis of thermoplasticity with linear kinematic hardening*, Journal of Applied Analysis **12** (2006), 37–57.

[4] R. Herzog, C. Meyer, A. Stötzner, *Existence of solutions of a thermoviscoplastic model and associated optimal control problems*, Nonlinear Analysis: Real World Applications **35** (2017), 75–101.

[5] R. Herzog, A. Stötzner, *Hadamard differentiability of the solution map in thermoviscoplasticity*, accepted for publication in Pure and Applied Functional Analysis; available at `https://spp1962.wias-berlin.de/preprints/044.pdf`.

[6] L. Paoli, A. Petrov, *Global existence result for thermoviscoelastic problems with hysteresis*, Nonlinear Analysis: Real World Applications **13** (2012), 524–542.

Directional differentiability for elliptic quasi-variational inequalities

Amal Alphonse

(joint work with Michael Hintermüller, Carlos N. Rautenberg)

We consider the differentiability of the solution map associated to the quasi-variational inequality (QVI)

$$\begin{aligned} y \in &\mathbf{K}(y) : \quad \langle Ay - f, v - y\rangle \geq 0 \quad \forall v \in \mathbf{K}(y) \\ &\mathbf{K}(y) := \{v \in V : v \leq \Phi(y)\}, \end{aligned} \tag{1}$$

in particular, the multi-valued mapping taking the source term f into the set of solutions y. Showing that this map is differentiable (in a suitable sense) is not only an interesting analytical problem in its own right but is also of use for optimal control, numerics and applications. We give a first result for the directional differentiability for QVIs in the infinite-dimensional setting (the corresponding theory for variational inequalities (VIs) has been thoroughly investigated [2, 3, 4]).

1. Functional framework and background

Let X be a locally compact topological space, countable at infinity, with ξ a Radon measure on X. Suppose $V \subset L^2(X;\xi) =: H$ is a Hilbert space with the embedding continuous and dense and such that $|u| \in V$ whenever $u \in V$, and let $A\colon V \to V'$ be a bounded, linear, coercive and T-monotone operator. Assume also that

$$V \cap C_c(X) \subset C_c(X) \quad \text{and} \quad V \cap C_c(X) \subset V \quad \text{are dense embeddings.}$$

Define $V'_+ := \{g \in V' : \langle g, v\rangle \geq 0 \quad \forall v \in V_+\}$ which arises from the cone V_+ which is the set of almost everywhere (a.e.) non-negative elements of V. Precisely, we study the directional differentiability of the map $\mathbf{Q}\colon V'_+ \rightrightarrows V$ taking $f \mapsto y$ in (1).

Before proceeding, let us recall the sensitivity result for VIs. Given an obstacle $\phi \in V_+$, define the set

$$\mathbf{K} := \{w \in V : w \leq \phi\},$$

and given a source term $f \in V'$, define by $S\colon V' \to V$ the mapping such that $S(f)$ solves the inequality in (1) with $\mathbf{K}(y)$ replaced by $\mathbf{K}$ (and hence the inequality simplifies into a VI). The *tangent cone* and the *critical cone* associated to $\mathbf{K}$ are given respectively by

$$T_{\mathbf{K}}(y) := \{\varphi \in V : \varphi \leq 0 \text{ q.e. on } \{y = \phi\}\} \text{ and } \mathcal{K}_{\mathbf{K}}(y) := T_{\mathbf{K}}(y) \cap [f - Ay]^{\perp},$$

where the notation 'q.e.' means quasieverywhere. Given $f \in V'$ and $d \in V'$, Theorem 3.3 of [3] yields that the map S is directionally differentiability in the sense that there exists a function $S'(f)(d) \in V$ such that

$$S(f + td) = S(f) + tS'(f)(d) + o(t) \quad \forall t > 0$$

where $t^{-1}o(t) \to 0$ as $t \to 0^+$ in V and $\delta := S'(f)(d)$ is positively homogeneous in d and it satisfies the VI

$$\delta \in \mathcal{K}_{\mathbf{K}}(y) : \langle A\delta - d, v - \delta\rangle \geq 0 \quad \forall v \in \mathcal{K}_{\mathbf{K}}(y), \text{ where } y = S(f).$$

2. Directional differentiability for QVIs

To formulate the QVI case, consider (1) with $\Phi\colon V \to V$ an increasing map with $\Phi(0) \geq 0$. The idea in [1] is the following: approximate a QVI solution $q(t) \in \mathbf{Q}(f + td)$ by a sequence $q_n(t)$ of solutions of VIs, obtain suitable differential formulae for those $q_n(t)$ and then pass to the limit in those formulae to obtain an expansion formula relating elements of $\mathbf{Q}(f + td)$ to elements of $\mathbf{Q}(f)$. There are some delicacies in this procedure:

(1) **derivation of the expansion formulae** for the above-mentioned VI iterates $q_n(t)$; they must relate $q(t)$ to a $y \in \mathbf{Q}(f)$, and recursion plays a highly nonlinear role in the relationship between the iterates
(2) **obtaining uniform bounds on the directional derivatives of the iterates**; even though the derivatives satisfy a VI, one has to handle a recurrence inequality (unless some regularity is available [1, §4.3] which allows some simplification)

(3) **identifying the limit of the higher-order terms as a higher-order term**; this procedure involves two limits: one as $t \to 0^+$ and one as $n \to \infty$, and commutation of limits in general requires an additional uniform convergence.

The main difficulty is indeed the final point here. The iteration scheme alluded to above requires some further restrictions on the data f and the direction d, namely $f, d \in V'_+$. Since we study the differentiability of implicit obstacle problems defined through the obstacle mapping Φ, it is clear that at least some differentiability is required of Φ and we introduce these further assumptions below. First, first let us fix some notation. Define $\bar{y} \in V$ as the weak solution of the unconstrained problem $A\bar{y} = f$. In a similar fashion, define $\bar{q}(t) \in V$ as the solution of the unconstrained problem with right hand side $f + td$: $A\bar{q}(t) = f + td$. The following set can be thought as a *translated* critical cone:[1]

$$\mathcal{K}_{\mathbf{K}(y)}(y, \alpha) := \Phi'(y)(\alpha) + \mathcal{K}_{\mathbf{K}(y)}(y).$$

The main result of [1] is the following.

Theorem 3 (Theorem 1.6 of [1])**.** *Let $f, d \in V'_+$. Given $y \in \mathbf{Q}(f) \cap [0, \bar{y}]$, assume the following:*

(1) *the map $\Phi\colon V \to V$ is Hadamard directionally differentiable*
(2) *either*
   (a) *$\Phi\colon V \to V$ is completely continuous, or*
   (b) *$V = H^1(\Omega)$, $X = \overline{\Omega}$ where Ω is a bounded Lipschitz domain, $\Phi\colon L^\infty_+(\Omega) \to L^\infty_+(\Omega)$ and is concave with $\Phi(0) \geq c > 0$, and $f, d \in L^\infty_+(\Omega)$*
(3) *the map $\Phi'(v)\colon V \to V$ is completely continuous (for fixed $v \in V$)*
(4) *for any $b \in V$, $h\colon (0, T) \to V$ and $\lambda \in [0, 1]$,*

$$\frac{\|\Phi'(y + tb + \lambda h(t))h(t)\|_V}{t} \to 0 \text{ as } t \to 0^+ \text{ if } \frac{h(t)}{t} \to 0 \text{ as } t \to 0^+$$

(5) *given $T_0 \in (0, T)$ small, if $z\colon (0, T_0) \to V$ satisfies $z(t) \to y$ as $t \to 0^+$, then*

$$\|\Phi'(z(t))b\|_V \leq C_\Phi \|b\|_V \quad \text{where } C_\Phi < \frac{1}{1 + c^{-1}C}$$

for all $t \in (0, T_0)$, where C and c are (respectively) the constants of boundedness and coercivity of A.

Then there exists $q(t) \in \mathbf{Q}(f + td) \cap [y, \bar{q}(t)]$ and $\alpha = \alpha(d) \in V_+$ such that

$$q(t) = y + t\alpha + o(t) \quad \forall t > 0$$

holds where $t^{-1}o(t) \to 0$ as $t \to 0^+$ in V and α satisfies the QVI

$$\alpha \in \mathcal{K}_{\mathbf{K}(y)}(y, \alpha) : \langle A\alpha - d, v - \alpha \rangle \geq 0 \quad \forall v \in \mathcal{K}_{\mathbf{K}(y)}(y, \alpha)$$

The directional derivative $\alpha = \alpha(d)$ is positively homogeneous in d.

---

[1] Explicitly this set is $\{\varphi \in V : \varphi \leq \Phi'(y)(w) \text{ q.e. on } \{y = \Phi(y)\} \text{ and } \langle Ay - f, \varphi - \Phi'(y)(w)\rangle = 0\}$.

It should be emphasized that the assumptions 4 and 5 depend on the specific function y, i.e., these are *local* conditions. The result in the general multi-valued setting given in Theorem 3 is a differentiability result for a specific selection mechanism that associates to a function $y \in \mathbf{Q}(f)$ a function $q(t) \in \mathbf{Q}(f + td)$. A useful variant of the theorem would be to obtain the result for the mapping that selects the minimal or maximal solution to the QVI, i.e., if $\mathsf{M}(f) \in \mathbf{Q}(f)$ is the maximal solution of the QVI with source term f, is M directionally differentiable?

Theorem 4 (Theorem 1.7 of [1]). *In the context of Theorem 3, if the set $\mathcal{K}_{\mathbf{K}(y)}(y, w)$ simplifies to*

$$\mathcal{S}_{\mathbf{K}(y)}(y, w) := \{\varphi \in V : \varphi = \Phi'(y)(w) \text{ q.e. on } \{y = \Phi(y)\}\},$$

then the derivative α satisfies

$$\alpha \in \mathcal{S}_{\mathbf{K}(y)}(y, \alpha) : \langle A\alpha - d, \alpha - v\rangle = 0 \quad \forall v \in \mathcal{S}_{\mathbf{K}(y)}(y, \alpha).$$

In this case, if $h \mapsto \Phi'(v)(h)$ is linear, $\alpha = \alpha(d)$ satisfies $\alpha(c_1 d_1 + c_2 d_2) = c_1\alpha(d_1) + c_2\alpha(d_2)$ for constants $c_1, c_2 > 0$ and directions $d_1, d_2 \in V'_+$.

Current and future work involves deriving strong stationarity conditions for optimal control problems with QVI constraints and sensitivity for the parabolic QVI case.

References

[1] A. Alphonse, M. Hintermüller, C.N. Rautenberg, *Directional differentiability for elliptic quasi-variational inequalities of obstacle type*, arXiv preprint (2018).

[2] A. Haraux, *How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities*, J. Math. Soc. Japan **29**(4) (1977), 615–631.

[3] F. Mignot, *Contrôle dans les inéquations variationelles elliptiques*, J. Functional Analysis **22**(2) (1976), 130–185.

[4] E. Zarantonello, *Projections on convex sets in Hilbert space and spectral theory. I. Projections on convex sets*, Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971) Academic Press, New York (1971), 237–341.

An optimal control approach for determining optical flow

Gabriela Marinoschi

(joint work with Viorel Barbu)

A fundamental problem in the processing of image sequences is the measurement of the image velocity. A sequence of ordered images provide information about the arrangement of objects in space, the distances between them and the rate at which these change, enabling thus the perception of the motion. The optical flow is the field velocity of the apparent motion of an object image between two consecutive frames. The optical flow problem under discussion in this report consists in determining, by a variational method, the velocity field on the basis of the observation of the brightness pattern of the object in a sequence of images at fixed sampling times. Apart from the previous models in the literature (see [2-6]) in which the brightness intensity of the moving object was considered to be constant in time along a moving pattern defined by the deterministic linear transport equation, the model considered in this report is derived from a new assumption, that is, the brightness intensity is conserved on a trajectory driven by a Gaussian stochastic process (see [1]). Thus, the main assumption is that the image trajectory obeys the stochastic equation

$$\begin{aligned} dX(t) &= U(t, X(t))dt + dW(t),\ 0 \le t \le T, \\ X(0) &= x \end{aligned}$$

where $W(t)$ is a Gaussian (Wiener) process of the form $W(t) = \left\{ \sum_{i=1}^{d} a_{ij}\beta_i(t) \right\}_{j=1}^{d}$, $\{\beta_i\}_{i=1}^{d}$ is a system of independent Brownian motions in a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, and the real matrix $(a_{ij})_{i,j=1}^{d}$ is not singular. Here, b_{ij} will be taken of the form $2\mu\delta_{ij}$, where $\mu > 0$ and δ_{ij} is the Kronecker symbol. Imposing that the brightness $I(t,x)$ is constant along this trajectory, that is, $dI(t, X(t)) = 0$, by performing a straightforward stochastic calculus involving Itô's formula, followed by the computation of the expectation $J(t,x) = E[I(t, X(t,x))]$ in the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, one deduces the equation

$$\frac{\partial}{\partial t} J(t,x) + U(t,x) \cdot \nabla J(t,x) + \frac{1}{2} L(J(t,x)) = 0,$$

with $L(y) = \sum_{i,j=1}^{d} b_{ij} \frac{\partial^2 y}{\partial x_i \partial x_j}$ the second order elliptic operator. This equation is completed by a final condition $J(T,x) = 0$ and by natural homogeneous Neumann boundary conditions.

In this way, the determination of the optical flow $U(t,x)$ which transports an initially observed image of intensity J_0 to a final image of intensity J_1 within an interval of time T is reduced to the minimization problem

$$\text{minimize } \left\{ \frac{1}{2} \int_O (J(0,x) - J_0(x))^2 dx + \frac{\sigma}{2} \int_0^T \int_O |U(t,x)|_d^2 \, dx dt \right\}$$

subject to

$$\begin{aligned} \frac{\partial J}{\partial t} + \mu\Delta J + U \cdot \nabla J = 0 \text{ in } Q = (0,T) \times O, \\ J(T,x) = J_1(x) \text{ in } O,\ \frac{\partial J}{\partial \nu} = 0 \text{ on } \Sigma = (0,T) \times \partial O, \end{aligned}$$

with U such that $\nabla \cdot U(t,x) = 0$, a.e. $(t,x) \in Q$, and $U(t,x) \cdot \nu(x) = 0$, a.e. $(t,x) \in \Sigma$. The divergence free constraint was chosen to preserve the conservation of the flow volume and to prevent it not to vary too much inside a non-deforming moving object, while the tangential velocity field was assumed as a natural condition for the pattern dynamics. Here, $O \subset \mathbb{R}^d$, $d = 2$, ν is the unit outer normal to ∂O and σ is a positive constant. The functions J_0 and J_1 represent known data and $|\cdot|_d$ denotes the Euclidian norm in $\mathbb{R}^d$.

Changing t to $T-t$ and setting $u(t,x) = U(T-t,x)$, $y(t,x) = J(T-t,x)$, $y_0(x) = J_1(x)$ and $y_1(x) = J_0(x)$ we reformulate the previous optimal control problem as the following problem, called (P),

$$\underset{u\in L^2(0,T:H)}{\text{minimize}} \left\{ \frac{1}{2}\int_O (y(T,x)-y_1(x))^2 dx + \frac{\sigma}{2}\int_0^T \int_O |u(t,x)|_d^2 \, dxdt \right\}$$

subject to the forward parabolic equation

$$\frac{\partial y}{\partial t} - \mu\Delta y - u\cdot\nabla y = 0 \text{ in } Q,$$
$$y(0,x) \quad = \quad y_0(x) \text{ in } O, \;\; \frac{\partial y}{\partial \nu} = 0 \text{ on } \Sigma,$$

where H is the free divergence tangential vector space

$$H = \{u \in (L^2(O))^d;\; \nabla\cdot u = 0 \text{ in } O,\; u\cdot\nu = 0 \text{ on } \partial O\}.$$

Existence, uniqueness and other properties of a weak solution to the state system and the existence of at least a solution to (P) are proved. Since the optimal controller has not enough regularity to enhance the rigorous computation of the first order conditions of optimality, we introduce, for $\varepsilon > 0$, the approximating control problem (P_ε),

$$\underset{u\in L^2(0,T:H)}{\text{minimize}} \left\{ \frac{1}{2}\int_O (y(T,x)-y_1(x))^2 dx + \frac{\sigma}{2}\int_0^T \int_O |u(t,x)|_d^2 \, dxdt \right\}$$

subject to the approximating state system

$$\frac{\partial y}{\partial t} - \mu\Delta y - \left(\int_0^T ((I+\varepsilon A)^{-1}u(s))\rho_\varepsilon(t-s)ds\right)\cdot\nabla y = 0, \text{ in } Q,$$
$$y_0^\varepsilon \quad = \quad (I-\varepsilon\Delta)^{-1}y_0 \text{ in } O, \;\; \frac{\partial y}{\partial \nu} = 0 \text{ on } \Sigma,$$

where $-\Delta : H^2(O)\cap H_0^1(O) \subset L^2(O) \to L^2(O)$ and $A = \Pi(-\Delta u)$ is the Stokes operator. We recall that

$$A : D(A) \subset H \to H, \;\; D(A) = (H^2(O))^d \cap (H_0^1(O))^d \cap H,$$

where $\Pi : (L^2(O))^d \to H$ is the orthogonal projection of $(L^2(O))^d$ on H (the Leray projector) and that A is self-adjoint and m-accretive on H.

After proving the existence of at least a solution $(u_\varepsilon^*, y_\varepsilon^*)$ to (P_ε), the convergence to (P_ε) to (P) concludes that $u_\varepsilon^* \to u^*$ strongly in $L^2(0,T;H)$, $y_\varepsilon^* \to y^*$ strongly in $L^2(0,T;H^1(O))$, $y_\varepsilon^*(t) \to y^*(t)$ strongly in $L^2(O)$, uniformly on $[0,T]$, and establishes that (u^*, y^*) is optimal in (P). Next, the approximating optimality conditions are determined and they provide at limit the system of optimality conditions for (P), formed by the state system for y^*, together with the relation for u^* and the adjoint system,

$$u^*(t) = \frac{1}{\sigma}\Pi(p(t)\nabla y^*(t)), \text{ for all } t\in[0,T],$$

$$\frac{\partial p}{\partial t} + \mu\Delta p - u^* \cdot \nabla p = 0 \text{ in } Q,$$
$$p(T,x) \quad = \quad -(y^*(T) - y_1(x)) \text{ in } O, \quad \frac{\partial p}{\partial \nu} = 0 \text{ on } \Sigma.$$

A discussion regarding the controller uniqueness reveals that for smooth initial and observed data, under certain conditions involving a relation between the L^∞-norms of y^*, ∇y^*, p, σ and μ, the controller u^* is unique in the class of L^∞-functions, on a time interval bounded by a constant depending on $\|y^*\|_{L^\infty(Q)}$ and $\|p\|_{L^\infty(Q)}$.

Continuation of this work will aim at studying new models for other types of Gaussian processes W and the construction of a rigorous algorithm for the image reconstruction, based on the optimality conditions system.

References

[1] V. Barbu, G. Marinoschi, *An optimal control approach to the optical flow problem*, Systems and Control Lett. **87** (2016), 1-9.
[2] J.L. Barron, D.J. Fleet, S.S. Beauchemin, *Performance of optical flow techniques*, Int. J. Comput. Vis. **12** (1994), 43-77.
[3] A. Borzi, K. Ito, K. Kunisch, *Optimal control formulation for determining optical flow*, SIAM J. Sci. Comput. **24** (2002), 818-847.
[4] K. Chen, K., D.A. Lorenz, *Image sequence interpolation using optimal control*, J. Math. Imaging Vis. **41** (2011), 222-238.
[5] W. Hinterberger, O. Scherzer, *Models for image interpolation based on the optical flow*, Computing **66** (2001), 231-247.
[6] B.P.K. Horn, B.G. Schunck, *Determining optical flow*, Artificial Intelligence **17** (1981), 185-203.

Shape optimization for unsteady fluid-structure interaction

Michael Ulbrich
(joint work with Johannes Haubner)

Fluid-structure interaction problems are challenging both theoretically and computationally. On the one hand, the not fully compatible regularity properties of solutions to the elasticity equations and the Navier-Stokes equations require particular care such as hidden regularity results for hyperbolic systems. On the other hand, the canonical frameworks for the fluid and solid equations are different. A well-known approach for a unified framework is the Arbitrary Lagrangian-Eulerian (ALE) framework [2]. We apply ALE in the way that the solid's displacement on the Lagrangian reference domain $\hat{\Omega}_s$ is suitably extended to a fluid reference domain $\hat{\Omega}_f$. From this, a transformation $\hat{X}(\cdot,t) : \hat{\Omega}_f \to \check{\Omega}_f(t)$ with inverse $\check{Y}$ that maps the time-dependent fluid domain to a fixed reference domain is obtained. The transformed Navier-Stokes equations contain highly nonlinear terms involving the transformation and the state. For the theory, we choose a fully Lagrangian approach and the ALE transformation then depends on the primitive of the velocity w.r.t. time. For the numerical implementation, the ALE displacement on

the fluid reference domain is introduced as a new state variable that extends the solid displacement by solving a biharmonic equation. Based on this formulation we perform shape optimization. An approach that fits well to the concept of domain transformations is the method of mappings [1, 6]. This method allows for a reformulation of the shape optimization problem in an optimal control setting. To this end, a shape reference or nominal domain $\tilde{\Omega}$ is introduced. Instead of optimizing over the admissible domains $\hat{\Omega} \in \hat{\mathcal{O}}_{ad}$, we optimize over the admissible transformations $\tilde{\tau} \in \tilde{\mathcal{T}}_{ad}$, which is a suitable subset of bi-Lipschitz transformations defined on $\tilde{\Omega}$. This additional transformation of the FSI system from the ALE domain $\hat{\Omega}$ to the shape reference domain $\tilde{\Omega}$ of the partial differential equations on $\tilde{\Omega}$ results in further nonlinear terms involving $\tilde{\tau}$ and the state.
The resulting optimization problem that we consider is of the form

$$\begin{aligned} &\min_{\tilde{\tau} \in \tilde{\mathcal{T}}_{ad}} \tilde{J}((\tilde{v}, \tilde{p}, \tilde{w}), \tilde{\tau}) \\ &\text{s.t. } (\tilde{v}, \tilde{p}, \tilde{w}) \text{ fulfills the transformed monolithic formulation} \\ &\qquad \text{of the FSI problem on the shape reference domain } \tilde{\Omega}, \end{aligned}$$

where $\tilde{J}$ denotes an appropriate objective function. The monolithic formulation of the fluid-structure interaction problem on $\tilde{\Omega}$ with fluid-solid interface $\tilde{\Gamma}_s$ is given by

$$\begin{aligned} \partial_t \tilde{v} - \nu \Delta \tilde{v} + \nabla \tilde{p} &= \tilde{\mathcal{F}}(\tilde{v}, \tilde{p}, \tilde{\tau}) && \text{in } \tilde{\Omega}_f \times (0,T), \\ \operatorname{div}(\tilde{v}) &= \tilde{\mathcal{G}}(\tilde{v}, \tilde{\tau}) && \text{in } \tilde{\Omega}_f \times (0,T), \\ \tilde{v} &= \partial_t \tilde{w} && \text{on } \tilde{\Gamma}_s \times (0,T), \\ \sigma_f(\tilde{v}, \tilde{p}) \tilde{n}_f &= \sigma_s(\tilde{w}, \tilde{\tau}) \tilde{n}_f + \tilde{\mathcal{H}}(\tilde{v}, \tilde{p}, \tilde{\tau}) && \text{on } \tilde{\Gamma}_s \times (0,T), \\ \partial_{tt} \tilde{w} - \operatorname{div}(\sigma_{s,z}(\tilde{w}, \tilde{\tau})) &= 0 && \text{in } \tilde{\Omega}_s \times (0,T), \end{aligned}$$

and additional initial and boundary conditions. Here, $\tilde{v}$ denotes the fluid velocity, $\tilde{p}$ the fluid pressure and $\tilde{w}$ the solid displacement. The fluid and solid stress tensors are denoted by σ_f and σ_s. The right hand sides $\tilde{\mathcal{F}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{H}}$ collect all nonlinear terms.
In an appropriate setting that considers the Navier-Stokes equations coupled to the Lamé system, local-in-time existence and uniqueness results [7] can be adapted to show that a unique solution

$$(\tilde{v}, \tilde{p}, \tilde{w}) \in \tilde{E}_T \times \tilde{P}_T \times (\mathcal{C}^0([0,T], H_{\#}^{\frac{7}{4}+\frac{\ell}{2}}(\tilde{\Omega}_s)) \cap \mathcal{C}^1([0,T], H_{\#}^{\frac{3}{4}+\frac{\ell}{2}}(\tilde{\Omega}_s)))$$

exists if $T > 0$ is sufficiently small. Here, the velocity space is given by $\tilde{E}_T = H_{\#}^{2+\ell, 1+\frac{\ell}{2}}(\tilde{\Omega}_f \times (0,T))$ with $\ell \in (\frac{1}{2}, 1)$ and the pressure space P_T is chosen as an appropriate subspace of $L^2(\tilde{\Omega}_f \times (0,T))$.
Based on these results, which are shown using a fixed point argument, we develop a general framework for showing continuity and differentiability for parametric unsteady PDE systems. In order to apply it, the nonlinearities on the right hand

side must be Lipschitz with sufficiently small modulus as well as Fréchet differentiable. The latter is, e.g., the case if these terms are Lipschitz continuous with a Lipschitz constant that is bounded by CT^α for some constants $\alpha > 0$ and $C > 0$ independent of T.

We apply this approach to the FSI problem on the shape reference domain and derive local-in-time Fréchet differentiability of the states with respect to shape variations:

$$||\tilde{v}(\tilde{\tau} + \tilde{h}) - \tilde{v}(\tilde{\tau}) - \delta_h \tilde{v}(\tilde{\tau})||_{\tilde{E}_T} \leq C||\tilde{h}||^2_{H^{2+\ell}(\tilde{\Omega})},$$
$$||\tilde{p}(\tilde{\tau} + \tilde{h}) - \tilde{p}(\tilde{\tau}) - \delta_h \tilde{p}(\tilde{\tau})||_{\tilde{P}_T} \leq C||\tilde{h}||^2_{H^{2+\ell}(\tilde{\Omega})}.$$

Here, $\delta_h \tilde{v}(\tilde{\tau})$ and $\delta_h \tilde{p}(\tilde{\tau})$ are solutions of a linearized state equation, $\tilde{\tau}, \tilde{\tau} + \tilde{h} \in \tilde{\mathcal{T}}_{ad}$ are sufficiently close to the identity, and

$$\tilde{\tau} \in \tilde{\mathcal{T}}_{ad} := \{\tilde{\tau} \in H^{2+\ell}_{\#}(\tilde{\Omega}_f) \ ; \ \nabla\tilde{\tau} \text{ is invertible a.e.}, \quad ||\nabla\tilde{\tau}||_{H^{1+\ell}_{\#}(\tilde{\Omega}_f)} \leq \alpha_1, \\ ||(\nabla\tilde{\tau})^{-1}||_{H^{1+\ell}_{\#}(\tilde{\Omega}_f)} \leq \alpha_1, \quad \tilde{\tau}|_{\tilde{\Omega}_s} = \text{id}, \quad \operatorname{supp}\tilde{\tau} \cap \operatorname{supp}\tilde{v}_0 = \emptyset\},$$

with an appropriate constant $\alpha_1 > 0$ and initial fluid velocity $\tilde{v}_0$. A paper containing the details of our analysis will be finished soon [4].

The numerical implementation can handle a more general setting. and our numerical tests are based on the FSI benchmark II problem. It models fluid flow in a two dimensional pipe around an obstacle to which an elastic flag is attached. The fluid is modeled by the Navier-Stokes equations and the solid by the nonlinear elasticity equations with St.Venant-Kirchhoff material. The goal is to minimize the drag along the obstacle and the flag plus a suitable regularization term by optimizing the shape of the obstacle subject to suitable constraints. The numerical discretization of the ALE formulation of the FSI problem is similar to [9] and uses a biharmonic extension to obtain the ALE mapping. The shape variations are represented by design boundary normal displacements which are extended to the domain by again using a biharmonic equation. Exact discrete gradients are evaluated via the adjoint and application of the chain rule. Our implementation couples FEniCS [5], dolfin-adjoint [3], and IPOPT [8] and uses parallelization. A transformation of variables is used to achieve that IPOPT implicitly works with the correct inner product. The numerical results show the viability of the approach. In comparison to the initial configuration, the drag is reduced by more than 40 %.

This project is part of the DFG/FWF-funded international research training group IGDK 1754 "Optimization and Numerical Analysis for Partial Differential Equations with Nonsmooth Structures".

References

[1] C. Brandenburg, F. Lindemann, M. Ulbrich, S. Ulbrich *A continuous adjoint approach to shape optimization for Navier-Stokes flow*, In *Optimal control of coupled systems of partial differential equations*, vol 158 of *Internat. Ser. Numer. Math.*, Birkhäuser Verlag, Basel, 35–56, 2009.

[2] J. Donea, A. Huerta, J.-Ph. Ponthot, A. Rodríguez-Ferran, *Arbitrary Lagrangian-Eulerian Methods*, Encyclopedia of Computational Mechanics, John Wiley & Sons, Ltd, 413–437, 2004.

[3] P.E. Farrell, D.A. Ham, S.W. Funke, M.E. Rognes, *Automated derivation of the adjoint of high-level transient finite element programs*, SIAM Journal on Scientific Computing **35(4)** (2013), C369–C393.

[4] J. Haubner, M. Ulbrich, *Analysis of shape optimization problems for unsteady fluid-structure interaction*, Preprint (draft), Technical University of Munich, 2018.

[5] A. Logg, K.-A. Mardal, G.N. Wells (Eds.), *Automated solution of differential equations by the finite element method: The FEniCS book*, Vol.84, Springer Science & Business Media, 2012.

[6] F. Murat, S. Simon, *Etudes de problemes d'optimal design*, *Lecture Notes in Computer Science* **41** (1976), 54–62, Springer.

[7] J.P. Raymond, M. Vanninathan, *A fluid-structure model coupling the Navier-Stokes equations and the Lamé system*, Journal de Mathématiques Pures et Appliquées **102(3)** (2014), 546–596.

[8] A. Wächter, L.T. Biegler, *On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming*, Mathematical programming **106(1)** (2006), 25–57.

[9] T. Wick, *Fluid-structure interactions using different mesh motion techniques*, Computers & Structures **89(13-14)** (2011), 1456–1467.

Shape optimization from an optimal control perspective

Volker Schulz

Shape optimization is a practically relevant and theoretically challenging field of mathematical research. Recent results on the treatment of shape optimization problems as optimization on shape manifolds are reported, where particular emphasis is laid on the proper choice of the respective Riemannian metric for usage in PDE constrained shape optimization problems [1, 2, 3]. A Steklov-Poincaré type metric is reported to be of practically good use [1]. In the final part of the talk, the perspective is shifted from optimization of shapes to optimization of deformation vector fields like $\mathcal{S}(\Omega_\circ) := \{W(\Omega_\circ) \,|\, W \in H^2(D, \mathbb{R}^d)\}, \Omega_0 \subset D$. The respective vector fields have two algebraic structures, a group structure (concatenation, $V_2 \circ V_1$) and a vector space structure ($tV_1 + sV_2$). It is shown that a second order derivative of shape functionals $J(\Omega)$ based on the group structure, i.e.,

$$d^2_{gr}J(W)[V_1, V_2] = \frac{d}{ds}\bigg|_{s=0+} \frac{d}{dt}\bigg|_{t=0+} J((id + sV_2) \circ (id + tV_1)(W(\Omega_\circ)))$$

leads to the (nonsymmetric) classical shape Hessian. On the other hand, the Hessian based on the vector space structure, i.e.,

$$d^2_{vs}J(W)[V_1, V_2] = \frac{d}{ds}\bigg|_{s=0+} \frac{d}{dt}\bigg|_{t=0+} J((W + tV_1 + sV_2)(\Omega_\circ))$$

is inherently symmetric and easily provides a Taylor series expansion. However, the vector space Hessian has a huge kernel, which means that increments ΔW in Newton-like methods have to use pseudo inverses ideally based on an appropriate bilinear form like the variational form of the elasticity equation ($a(.,.)$) in the manner

$$\min_{\Delta W} a(\Delta W, \Delta W)$$
$$\text{s.t. } d^2_{vs}J(W)[\Delta W, V] = d_{vs}J(W)[V]\,,\ \forall V\,.$$

In this way a new approach to PDE constrained shape optimization is opening up

References

[1] V. Schulz and M. Siebenborn. Computational comparison of surface metrics for PDE constrained shape optimization. *Computational Methods in Applied Mathematics*, 16(3):485–496, 2016.

[2] V. Schulz, M. Siebenborn, and K. Welker. Structured inverse modeling in parabolic diffusion problems. *SIAM Journal on Control and Optimization*, 53(6):3319–3338, 2015.

[3] V. Schulz, M. Siebenborn, and K. Welker. Efficient PDE constrained shape optimization based on Steklov-Poincaré type metrics. *SIAM Journal on Optimization*, 26(4):2800–2819, 2016.

Tomographic reconstruction with few views

Maitine Bergounioux

(joint work with Isabelle Abraham, Romain Abraham, Guillaume Carlier, Emmanuel Trélat)

The project we report concerns a specific application of tomographic reconstruction for a physical experiment whose goal is to study the behavior of a material under a shock. The experiment consists in causing the implosion of the hull of some material (usually, a metal) whose features are well known, using surrounding explosives. The problem is to determine the shape of the interior interface at a specific moment of the implosion. For this purpose, at most two or three radiographs (projections on detectors) are acquired, and the shape of the object must then be reconstructed using a tomographic approach.

When enough projections of the object, taken from different angles, are available, several techniques exist for tomographic reconstruction, providing an analytic formula for the solution. There is a huge literature about theoretical and practical aspects of the problem of reconstruction from projections. When only few projections are known, these methods cannot be used directly, and some alternative methods have been proposed to reconstruct partially the densities.

As in any tomographic reconstruction process, this problem leads to an ill-posed inverse problem. Since we only have few radiographs at our disposal, data are not redundant and the ill-posed character is even more accurate. Moreover, the flash has to be very brief (several nanoseconds) due to the imploding movement of the hull. Such X-rays cannot be provided by standard sources, and hence numerous drawbacks appear, for instance the X-rays beam is not well focused and the X-rays source is not punctual. This causes a blur on the radiograph. Furthermore, contrarily to medical radiography where photons are absorbed by bones, here X-rays must cross a very dense object and therefore must be of high energy. Most of the photons are actually absorbed by the object and only a few number of them arrive at the detector. It is therefore necessary to add some amplification devices and very sensitive detectors, which cause a high noise level and another blur.

(1) To simplify the problem we first assumed that all components of the initial physical setup are axially symmetric, and are assumed to remain as such during the implosion process. Therefore a single radiograph of the cross section suffices in theory to reconstruct the 3D object.Based on a single X-ray radiograph which is at our disposal, it is our aim to perform a tomographic reconstruction to reconstruct the whole axially symmetric object. We first proposed in [1] a variational method adapted to the tomographic reconstruction of blurred and noised binary images, based on a minimization problem in the space of bounded variation functions, using the concept of total variation, prove existence and uniqueness results. The binary structure of the material under consideration is modeled as a binary constraint: the intensity function is either equal to 0 or 255 (normalized to 0 and 1). Due to this binary constraint, deriving an optimality system is not straightforward, and we propose a penalization method for which we establish some properties and derive an optimality system.
(2) Later, in [5], we provide a refined functional analysis of the Radon operator restricted to axisymmetric functions, and show that it enjoys strong regularity properties in fractional order Hilbert spaces. We proposed a variational approach to handle this problem, consisting in solving a minimization problem settled in adapted fractional order Hilbert spaces. We showed the existence of solutions, and then derived first order necessary conditions for optimality in the form of optimality systems. Numerical experimentation was achieved (to appear).
(3) We also investigated an active curve method for this problem in [2]. The model lives in the BV space and leads to a non local Hamilton-Jacobi equation, via a local set strategy.
(4) Next, we abandoned the axisymmetry assumption and investigated a different modeling using an optimal transport approach [3]. Indeed, we want to move an object (given by its density ρ_0 which has been computed by the former methods for example) toward an unknown object ρ_T, with an optimal transport mapping. We made the optimal transport model precise, gave a dual formulation and proved existence and uniqueness of the solution. However, the numerical computation of the problem may costful (many inf-convolution process to compute). So, we gave an equivalent multi-marginal formulation of the same problem. We presented numerical hints based on the dual formulation of the multi-marginal problem and some preliminary results.

(5) More recently, we still use the optimal transport framework but rather use a PDE formulation. This leads to an optimal control problem driven by a PDE. Indeed,using Benamou-Brenier [4], we interpret the Wasserstein distance as the kinetic energy associated to a transport equation. Precisely, a time interval $[0,T]$ and $\Omega \subset \mathbb{R}^d$ are given and we introduce a density function $\rho : [0,T] \times \Omega \to \mathbb{R}$ (with compact support) and a velocity field $v : [0,T] \times \Omega \to \mathbb{R}^d$ that satisfy:

$$\partial_t \rho + v \nabla \rho = 0 \text{ on } (0,T) \times \mathbb{R}^d, \ \rho(0,\cdot) = \rho_0 \tag{1}$$

with $\rho_0 \in L^p(\mathbb{R}^d)$ with $p \in [0,+\infty]$, $v \in L^1_{loc}(0,T;(L^1(\mathbb{R}^d)^d \cap (L^q(\mathbb{R}^d))^d)$ and div $v = 0$. The (unique) solution to (1) is denoted $\rho[v]$. Then the Wassertein distance $\mathcal{W}_2$ satisfies

$$\mathcal{W}_2^2(\rho_0,\rho_T) = \inf_{(\rho,v)\text{solution to }(1)-\rho(T)=\rho_T} T \int_{\mathbb{R}^d}\int_0^T \rho[v](t,x)|v(t,x)|^2\,dx\,dt\ . \tag{2}$$

As ρ_T is unknown, we use the projection information (arising from the tomographic process). The model writes

$$(\mathcal{P}) \qquad \min_{v \in U_{ad}} \mathcal{J}(v)\ ,$$

where

- the velocity field v plays the role of the control. We include constraints as div $v = 0$ in the admissible set U_{ad}.
- the cost functional $\mathcal{J}$ has the following form $\mathcal{J} = J_1 + \alpha J_2 + \beta J_3$ with

$$qJ_1(v) := \int_{\mathbb{R}^d}\int_0^T \rho[v]|v|^2\,dx\,dt. \tag{3}$$

$$J_2(v) := \int_{\mathbb{R}^{d-1}} \sum_{i=1}^{k} \|\Pi_i(\rho[v])(T) - \pi_i\|^2\,dx. \tag{4}$$

Here Π_i is the projector operator on the ith detector and π_i is the data.
The third term J_3 is an ad-hoc regularization of v that should indicate that we do not transport anything when there is nothing. One could choose a sparsity term for example as

$$J_3(v) := \int_{\mathbb{R}^d}\int_0^T |v(t,x)|\,dx\ dt. \tag{5}$$

Under classical assumptions for transport equations, we can prove existence of a solution and optimality conditions.

References

[1] R. Abraham, M. Bergounioux and E. Trélat, *A penalization approach for tomographic reconstruction of binary radially symmetric objects*, **Applied Mathematics and Optimization**, Vol. 58, no. 3, pp. 345-371, 2008
[2] I. Abraham, R. Abraham and M. Bergounioux, *An Active Curve Approach for Tomographic Reconstruction of Binary Radially Symmetric Objects* , **Numerical Mathematics and Advanced Applications**, Kunisch K. , Of G., Steinbac O. (Eds.), pp. 663-670, 2008
[3] I. Abraham, R. Abraham, M. Bergounioux and G. Carlier, *Tomographic reconstruction from a few views: a multi-marginal optimal transport approach*, **Appl Math Optim** (2017) 75: 55.
[4] J. D. Benamou and Y. Brenier *A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem* **Numerische Mathematik**, 84(3), 375-393, (2000).
[5] M. Bergounioux and E. Trélat, *A variational method using fractional Sobolev spaces for tomographic reconstruction of blurred and noised binary images*, **Journal of Functional Analysis, 259** (2010), 2296–2332.

Optimal control methods in shape optimization

Dan Tiba

A typical example of shape optimization problems has the form:

$$\operatorname*{Min}_{\Omega\in\mathcal{O}} \int_\Lambda j(x, y_\Omega(x), \nabla y_\Omega(x))dx, \tag{1}$$

$$-\Delta y_\Omega = f \quad in\ \Omega, \tag{2}$$

$$y_\Omega = 0 \quad on\ \partial\Omega \tag{3}$$

with other supplementary constraints (on y, Ω, etc.), if necessary. Here, $\Omega \subset D$ is an (unknown) domain, D is some given bounded Lipschitzian domain, $f \in L^2(D)$, $j(.,.,.)$ is a Caratheodory mapping and Λ is either Ω or some fixed subdomain $E \subset D$. The cases of boundary observation, Neumann boundary conditions are not discussed here and we refer to [11], [12].

The problem (1) - (3) has the structure of an optimal control problem: cost functional, state system, constraints. The main difference and difficulty is that the minimization parameter is the domain Ω itself, where the state equation is defined. Consequently, it is a natural approach to study (1) - (3) by using optimal control methods. This is a rather old research direction: even the pioneering "mapping method" of Murat and Simon enters already in this category (control by the coefficients), see Pironneau [10]. Its drawbacks are given by the high regularity assumptions on the class of admissible domains Ω, their prescribed topological type, the control parameter T_Ω (the involved diffeomorphism) appears together with its derivatives in the transformed problem, etc.

Let me also mention that many geometric optimization problems arising in mechanics (for plates, beams, arches, curved rods or shells), are expressed, as well, as optimal control problems by the coefficients, due to the special form of their models. See [6], [5].

Recently, more direct approaches, under low regularity hypotheses and with a large range of applications, have been introduced in optimal design problems, with relevance both at the computational and theoretical levels. The admissible family of domains $\mathcal{O}$ is represented as the subgraphs of a family of functions $\mathcal{F}$ satisfying appropriate assumptions. Then, the characteristic functions of the unknown domains may be represented by applying the Heaviside operator to the elements in $\mathcal{F}$. This also allows their regularization via the Moreau-Yosida approximation and standard smoothing techniques. One procedure going back to Kawarada [2] gives the approximation of (2), working in the given domain D and this may be extended to the shape optimization problem (1) - (3). Notice that such a technique allows simultaneous boundary and topological variations of the unknown geometry, [8], [7], [5]. Recent developments have a wide range of applications [11], [12].

We remark that such ideas are also useful in free boundary problems and variational inequalities, where geometric unknowns play an outstanding role as well, [3], [1], [4].

The presentation will discuss in detail two cases: optimization of a plate with holes and a penalization approach to a general shape optimization problem.

An essential ingredient in these developments is the new implicit parametrization method that allows an advantageous description of implicitly defined manifolds via iterated Hamiltonian systems, [9], [13].

References

[1] A. Halanay, C.M. Murea, D. Tiba, Existence of a steady flow of Stokes fluid past a linear elastic structure using fictitious domain, J. Math. Fluid Mech. 18, pp. 397–413 (2016).

[2] H. Kawarada, Numerical methods for free surface problems by means of penalty, LNM 704, pp.282-291, Springer-Verlag, New York (1979).

[3] C. M. Murea, D. Tiba A direct algorithm in some free boundary problems, Journal of Numerical Mathematics; vol.24, iss.4, pp. 253-271, (2016) DOI 10.1515/ jnma-2015-0048.

[4] C.M. Murea, D. Tiba , Approximation of a simply supported plate with obstacle, Math. and Mech. of Solids, Vol. 23(3) 348–358, DOI: https://doi.org/10.1177/1081286517718228 (2018).

[5] C. M. Murea, D. Tiba, Optimization of a plate with holes, submitted to Comp. & Math. Appl. (2017).

[6] P. Neittaanmäki, J. Sprekels, D. Tiba, Optimization of elliptic systems. Theory and applications, Springer, New York (2006).

[7] P. Neittaanmaki, D. Tiba, Fixed domain approaches in shape optimization problems, Inverse Problems, vol.28, p.1-35, (2012) doi:10.1088/0266-5611/28/9/093001.

[8] P. Neittaanmaki, A. Pennanen, D. Tiba, Fixed domain approaches in shape optimization problems with Dirichlet boundary conditions, Inverse Problems, vol.25, p.1-18, (2009), doi:10.1088/0266-5611/25/5/055003.

[9] M.R. Nicolai and D. Tiba, Implicit functions and parametrizations in dimension three: generalized solutions, DCDS - A vol. 35, no.6, pp.2701 - 2710 (2015). doi:10.3934/dcds.2015.35.2701.

[10] O. Pironneau, Optimal shape design for elliptic systems, Springer, Berlin, 1984.

[11] D.Tiba, Boundary Observation in Shape Optimization, in "'New trends in differential equations, control theory, and optimization"', V. Barbu, C. Lefter, I. Vrabie (Eds.), World Scientific Publishing, Singapore, pp.301 - 314 (2016).
[12] D. Tiba, Neumann boundary conditions in shape optimization, Pure Appl. Funct. Anal., Vol. 3, No. 1, (2018) 241-253.
[13] D.Tiba, Iterated Hamiltonian type systems and applications. J. Differential Equations, 264 (2018) 5465–5479, https://doi.org/10.1016/j.jde.2018.01.003.

Numerical analysis of sparse initial data identification for parabolic problems

Boris Vexler
(joint work with Dmitriy Leykekhman, Daniel Walter)

In this talk we discuss the problem of identification of initial data for a homogeneous heat equation from an observation of the terminal state. This problem is known to be exponentially ill-conditioned. Under the assumption that the unknown initial state is sparse, we formulate the problem as a PDE-constrained optimal control problem on a measure space for the control variable as follows:

$$\text{minimize}\ \ \frac{1}{2}\|u(T)-u_d\|^2_{L^2(\Omega)}+\alpha\|q\|_{\mathcal{M}(\Omega)},$$

for $q\in\mathcal{M}(\Omega)$, subject to

$$\begin{aligned} \partial_t u-\Delta u &= 0 && \text{in } (0,T)\times\Omega,\\ u &= 0 && \text{on } (0,T)\times\partial\Omega,\\ u(0) &= q && \text{in } \Omega. \end{aligned}$$

Here, $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ is a convex polygonal / polyhedral domain, $\mathcal{M}(\Omega)$ is the space of regular Borel measures, which can be identified with the dual space of continuous functions, i.e $\mathcal{M}(\Omega)=C_0(\Omega)^*$, $u_d\in L^2(\Omega)$ is the desired terminal state. The terminal time is denoted by $T>0$ and the cost parameter by $\alpha>0$.

A similar problem, which is equivalent to the problem described above, is analyzed in [1]. There, optimality conditions and structural properties of the optimal control are derived, and finite element discretization is considered. However, only plain convergence result (without rates) is shown. The goal of my talk is to present numerical analysis with convergence rates for a space-time finite element discretization.

The optimal control problem under consideration possesses a unique solution consisting of the optimal control $\bar{q}\in\mathcal{M}(\Omega)$ and the corresponding optimal state $\bar{u}\in L^r(0,T;W^{1,p}(\Omega))$ with $\bar{u}(T)\in H^2(\Omega)\cap H^1_0(\Omega)$. It is characterized by the following optimality system involving the adjoint state $\bar{z}\in W(0,T)$ with $\bar{z}(0)\in H^2(\Omega)\cap H^1_0(\Omega)\hookrightarrow C_0(\Omega)$,

$$\text{(1a)}\qquad \begin{aligned} \partial_t \bar u - \Delta \bar u &= 0 && \text{in } (0,T)\times\Omega,\\ \bar u &= 0 && \text{on } (0,T)\times\partial\Omega,\\ \bar u(0) &= \bar q && \text{in } \Omega, \end{aligned}$$

$$\text{(1b)}\qquad \begin{aligned} -\partial_t \bar z - \Delta \bar z &= 0 && \text{in } (0,T)\times\Omega,\\ \bar z &= 0 && \text{on } (0,T)\times\partial\Omega,\\ \bar z(T) &= \bar u(T) - u_d && \text{in } \Omega, \end{aligned}$$

$$\text{(1c)}\qquad -\langle q-\bar q, \bar z(0)\rangle \le \alpha\left(\|q\|_{\mathcal{M}(\Omega)} - \|\bar q\|_{\mathcal{M}(\Omega)}\right) \quad \text{for all } \ q\in\mathcal{M}(\Omega).$$

This optimality system implies that $\|z(0)\|_{C_0(\Omega)} \le \alpha$ and the following support condition for the optimal control $\bar q = \bar q^+ - \bar q^-$ holds:

$$\operatorname{supp}\bar q^+ \subset \Omega^+ = \{\, x\in\Omega \mid \bar z(0,x) = -\alpha \,\}, \operatorname{supp}\bar q^- \subset \Omega^- = \{\, x\in\Omega \mid \bar z(0,x) = \alpha \,\},$$

see [1] and [2] for the elliptic case. This condition leads to sparsity of $\bar q$ since the sets $\Omega^\pm$ are the sets of measure zero. This is due to the fact that $\Omega^\pm$ lie in the interior of Ω and $z(0)$ is analytic there.

We discretize the problem using discontinuous Galerkin method dG(r) of order r in time and usual conforming cG(1) finite elements in space. The corresponding discrete space is called X_{kh} with k being the maximal time step and h the maximal mesh size, see, e.g., [4] for details of this notation in the context of optimal control problems. The control variable is discretized using the space $M_h \subset \mathcal{M}(\Omega)$ being the span of Dirac functionals δ_{x_i} corresponding to all interior nodes of the underlying finite element mesh. This results in the discrete problem

$$\text{minimize } \ \frac{1}{2}\|u_{kh}(T) - u_d\|^2_{L^2(\Omega)} + \alpha\|q_{kh}\|_{\mathcal{M}(\Omega)},$$

for $q_{kh}\in M_h$, subject to $u_{kh}\in X_{kh}$ and

$$B(u_{kh},\varphi_{kh}) = \langle q_{kh}, \varphi_{kh}(0)\rangle \quad \text{for all } \varphi_{kh}\in X_{kh},$$

where B is the standard bilinear form used for formulation of dG(r) discretization in time. This discrete problem possesses a unique solution $(\bar q_{kh}, \bar u_{kh})$. For the error in the optimal state we prove the following error estimate

$$\|\bar u(T) - \bar u_{kh}(T)\|_{L^2(\Omega)} \le c(T)|\ln h||\ln k|^{\frac{1}{2}}\left(k^{r+\frac{1}{2}} + h\right).$$

For the optimal control no convergence of $\|\bar q - \bar q_{kh}\|_{\mathcal{M}(\Omega)}$ can be expected. We show (cf. also [1]) that

$$\bar q_{kh} \overset{*}{\rightharpoonup} \bar q \ \text{ in } \ \mathcal{M}(\Omega) \qquad \text{and} \qquad \|\bar q_{kh}\|_{\mathcal{M}(\Omega)} \to \|\bar q\|_{\mathcal{M}(\Omega)}, \qquad (k,h)\to 0.$$

Under the following additional structural assumption we can provide further information on the convergence of support points.

Assumption. We assume that

(1) $\operatorname{supp}\bar q = \{\, x\in\Omega \mid |\bar z(0,x)| = \alpha \,\} = \{x_1, x_2, \dots, x_N\}$,
(2) For x_i with $\bar z(0,x_i) = -\alpha$ the Hessian matrix $\nabla^2_x \bar z(0,x_i)$ is positive definite,

(3) For x_i with $\bar{z}(0,x_i)=\alpha$ the Hessian matrix $\nabla_x^2\bar{z}(0,x_i)$ is negative definite.

This assumption states that the minima and maxima of $\bar{z}(0)$ fulfill standard second order sufficient optimality conditions. A similar assumption can be found in the literature in the context of optimal control problems with state constraints, see, e. g., [5].

Under this assumption we know that the optimal control $\bar{q}$ is a linear combination of Dirac delta functions, i.e.

$$\bar{q}=\sum_{i=1}^{N}\beta_i\delta_{x_i}$$

with

$$\beta_i>0 \quad \text{for} \quad \bar{z}(0,x_i)=-\alpha \quad \text{and} \quad \beta_i<0 \quad \text{for} \quad \bar{z}(0,x_i)=\alpha.$$

For the discrete control $\bar{q}_{kh}$ we can prove the following: There are $\varepsilon>0$, $k_0,h_0>0$ such that for all $k<k_0$ and $h<h_0$

(1) $\operatorname{supp}\bar{q}_{kh}\cap B_\varepsilon(x_i)\neq\emptyset, \quad i=1,2,\dots,N,$

(2) $\operatorname{supp}\bar{q}_{kh}\subset\cup_i B_\varepsilon(x_i)$. Moreover, we can estimate the distance between any support point $x_{i,kh}\in B_\varepsilon(x_i)$ of $\bar{q}_{kh}$ and x_i by

$$|x_i-x_{i,kh}|\le c(T)|\ln h||\ln k|\left(k^{\frac{r}{2}+\frac{1}{4}}+h^{\frac{1}{2}}\right).$$

The main tool used in the proof are sharp smoothing type pointwise finite element error estimates for homogeneous parabolic equations, which are based on smoothing estimates and discrete maximal parabolic regularity from [3].

References

[1] E. Casas, B. Vexler, E. Zuazua, *Sparse initial data identification for parabolic PDE and it's finite element approximations*, Math. Control Related Fields 5(3), pp. 377-399, 2015

[2] C. Clason, K. Kunisch, *A duality-based approach to elliptic control problems in non-reflexive Banach spaces*, ESAIM Control Optim. Calc. Var. 17(1), pp. 243–266, 2011.

[3] D. Leykekhman, B. Vexler, *Discrete maximal parabolic regularity for Galerkin finite element methods*, Numer. Mathematik, 135(3), pp. 923-952, 2017

[4] D. Meidner, B. Vexler, *A priori error estimates for space-time finite element discretization of parabolic optimal control problems. Part I: Problems without control constraints*, SIAM J Control Optim. 47(3), pp. 1150-1177, 2008

[5] P. Merino, I. Neitzel, F. Tröltzsch, *On linear-quadratic elliptic control problems of semi-infinite type*, Appl. Anal., 90(6), pp. 1047-1074, 2011.

Unique global solutions in optimal control with PDEs and VIs

Michael Hinze

(joint work with Ahmad Ahmad Ali, Klaus Deckelnick)

In [1] we consider an optimal control problem subject to a semilinear elliptic PDE together with its variational discretization, where we provide a condition which allows to decide whether a solution of the necessary first order conditions is a global minimum. This condition can be explicitly evaluated at the discrete level.

Furthermore, we prove that if this condition holds uniformly with respect to the discretization parameter the sequence of discrete solutions converges to a global solution of the corresponding limit problem. Moreover, in [2] we prove error estimates for those discrete global solutions, which are confirmed by numerical experiments. Our approach can be modified and adapted in order to derive corresponding conditions for the optimal control of the obstacle problem. With this talk we present an overview of our achievements obtained so far.

References

[1] A. Ahmad Ali, K. Deckelnick, M. Hinze, *Global minima for semilinear optimal control problems*, Comput. Optim. Appl. **65** (2016), 261–288.

[2] A. Ahmad Ali, K. Deckelnick, M. Hinze, *Error analysis for global minima of semilinear optimal control problems*, Mathematical Control and related Fields **8** (2018), 195–215.

Convergence of the SQP method for problems with low regularity

Arnd Rösch

SQP methods are often used to solve optimal control problems numerically. We study here the convergence of the Lagrange-Newton SQP method. The first convergence proof for optimal control problems governed by ODEs was given by Alt [1] and for PDEs by Alt, Sontag and Tröltzsch [2].

Nowadays the convergence of the Lagrange-Newton SQP is well studied even for coupled system of semilinear or quasilinear partial differential equations. However, a series of new challenges appear if one changes the objective in a simple way. We demonstrate these effects on two problems. The first one is the most simple nonlinear optimal control problem.

We aim to minimize the functional

$$\min \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$

subject to the state equation

$$(\nabla y, \nabla v) + (d(y), v) = (u, v) \quad \forall v \in H^1_0(\Omega)$$

and the control constraint

$$u \leq \psi \quad \text{a.e. in } \Omega.$$

The second example differs in the norm of the control in the objective and in the box constraints. Here we aim to minimize

$$\min \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{H^1(\Omega)}$$

subject to the state equation

$$(\nabla y, \nabla v) + (d(y), v) = (u, v) \quad \forall v \in H^1_0(\Omega)$$

and the box constraints

$$u_a \leq u \leq u_b \quad \text{a.e. in } \Omega.$$

The standard approach for proving convergence of the Lagrange-Newton SQP method reformulates the necessary optimality condition of these optimal control problems into generalized equations of the form

$$0 \in F(w) + N(w)$$

where the vector w contains the state y, control u, the adjoint state p, and the Lagrange multiplier μ. The normal cone is denoted by $N(w)$. The Lagrange-Newton iterate can be written in a similar form

$$0 \in F(w_k) + F'(w_k)(w - w_k) + N(w).$$

Again, the generalized equation can be interpreted as a solution of a generalized equation. For convenience we write the Newton iteration in a different way

$$\delta_{k+1} \in F(\bar{w}) + F'(\bar{w})(w_{k+1} - \bar{w}) + N(w_{k+1})$$

with

$$\delta_{k+1} = F(\bar{w}) - F(w_k) + F'(\bar{w})(w_{k+1} - \bar{w}) - F'(w_k)(w_{k+1} - w_k)$$

where $\bar{w}$ denotes the solution of the generalized equation. To show local quadratic convergence, two estimates of the perturbation δ_{k+1} are important

$$\|\delta_{k+1}\|_Z \leq L\|w_k - \bar{w}\|_W \leq Lr$$

and

$$\|w_{k+1} - \bar{w}\|_W \leq L_\delta\|\delta_{k+1}\|_Z$$

with appropriate function spaces Z and W. For the first example we find

$$\begin{aligned} Y &= H_0^1(\Omega) \cap C(\bar{\Omega}), \\ W &= Y \times L^\infty(\Omega) \times Y \times L^\infty(\Omega), \\ Z &= L^2(\Omega) \times L^\infty(\Omega) \times L^2(\Omega) \times L^\infty(\Omega). \end{aligned}$$

The convergence analysis for a specific problem requires now the derivation of these two inequalities. The first inequality can often be obtained by standard methods. Boundedness results and Lipschitz properties of the involved partial differential equations yield the desired first inequality.

The derivation of the second inequality is very different. One has to show Lipschitz stability of a generalized equation with respect to a perturbation term. Here, the following steps are important. First, testing the weak formulations of the elliptic equations with suitable terms, a basic estimate is established. Second, a careful discussion of the multiplier term is needed to get an estimate of the form

$$\begin{aligned} L''(\bar{w})(\delta y, \delta u)^2 \quad \leq \quad & \|\delta p\|\|\delta_1 - \delta_1'\| + \|\delta y\|\|\delta_2 - \delta_2'\| \\ & + \|\delta u\|\|\delta_3 - \delta_3'\| + \|\delta\mu\|\|\delta_4 - \delta_4'\|. \end{aligned}$$

where L denotes the Lagrangian of the optimal control problems. In a third step a sufficient second-order optimality condition is applied. Until now, only the primal variables can be estimated by this inequality. The insertion of the dual variables has to be done in a fourth step. This leads to a relation of the form

$$\|\delta w\|_{L^2(\Omega)^4} \leq c\|\delta - \delta'\|_{L^2(\Omega)^4}.$$

In a last step, the $L^2(\Omega)$-norms are replaced by W- and Z-norms using the mapping properties of the optimality system. At the end one has shown local quadratic convergence of the SQP method for the first problem.

The second example cannot be handled in the same way. The optimality condition yields an obstacle problem as optimality condition. The choice of the space for the Lagrange multiplier becomes crucial. The choice $L^2(\Omega)$ cannot be used since the solution mapping of the obstacle problem is not Lipschitz continuous. The Lipschitz continuity holds for the choice $H^{-1}(\Omega)$. Because of the measure nature of the multiplier, new techniques have to be developed to get local quadratic convergence for the second example, too.

References

[1] W. Alt, *Stability of Solutions and the Lagrange-Newton Method for Nonlinear Optimization and Optimal Control Problems*, habilitation thesis, University of Bayreuth, 1991.

[2] W. Alt, R. Sontag and F. Tröltzsch *An SQP method for optimal control of weakly singular Hammerstein integral equations*, Appl. Math. Optim. **33** (1996), 227–252.

Reporter: Amal C. Alphonse

Participants

Dr. Amal C. Alphonse
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
GERMANY

Dr. Behzad Azmi
Johann Radon Institute
Austrian Academy of Sciences
Altenberger Strasse 69
4040 Linz
AUSTRIA

Prof. Dr. Maitine Bergounioux
Département de Mathématiques
Université d'Orleans
B.P. 6759
45067 Orléans Cedex 2
FRANCE

Dr. Tobias Breiten
Institute for Mathematics and
Scientific Computing
Graz University
Heinrichstrasse 36/III
8010 Graz
AUSTRIA

Prof. Dr. Martin Brokate
Zentrum Mathematik
Technische Universität München
85747 Garching bei München
GERMANY

Prof. Dr. John Allen Burns
ICAM
Virginia Tech
Blacksburg, VA 24061-0531
UNITED STATES

Prof. Dr. Eduardo Casas
Dept. de Matematica Aplicada
E.T.S.I. Industriales y Telecom.
Universidad de Cantabria
Avda de los Castros, s/n
39005 Santander
SPAIN

Prof. Dr. Pierluigi Colli
Dipartimento di Matematica
Universita di Pavia
Via Ferrata, 1
27100 Pavia
ITALY

Dr. Michele Colturato
Dipartimento di Matematica
Universita di Pavia
Via Ferrata, 5
27100 Pavia
ITALY

Prof. Dr. Hélène Frankowska
Institut de Mathématiques de Jussieu -
Paris Rive Gauche
Sorbonne Université
Campus Pierre et Marie Curie, Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Andrea Giorgini
Dipartimento di Matematica
Politecnico di Milano
Via Bonardi 9
20133 Milano
ITALY

Dr. Falk M. Hante
Department Mathematik
FAU Erlangen-Nürnberg
Cauerstrasse 11
91058 Erlangen
GERMANY

Prof. Dr. Matthias Heinkenschloss
Department of Computational Sciences
and Applied Mathematics
Rice University; MS-134
6100 S. Main Street
Houston, TX 77005-1892
UNITED STATES

Prof. Dr. Roland Herzog
Fakultät für Mathematik
Technische Universität Chemnitz
Reichenhainer Strasse 41
09126 Chemnitz
GERMANY

Prof. Dr. Michael Hintermüller
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
GERMANY

Prof. Dr. Michael Hinze
Fachbereich Mathematik
Universität Hamburg
Bundesstrasse 55
20146 Hamburg
GERMANY

Prof. Dr. Dietmar Hömberg
Weierstrass Institute for Applied
Analysis and Stochastics
Mohrenstrasse 39
10117 Berlin
GERMANY

Dr. Olivier Huber
The Wisconsin Institute for Discovery
University of Wisconsin-Madison
330 North Orchard Street
Madison WI 53715
UNITED STATES

Prof. Dr. Kazufumi Ito
Department of Mathematics
North Carolina State University
Campus Box 8205
Raleigh, NC 27695-8205
UNITED STATES

Dr. Dante Kalise
Department of Mathematics
South Kensington Campus
Imperial College London
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Karl Kunisch
Institut für Mathematik und
wissenschaftliches Rechnen
Karl-Franzens-Universität Graz
Heinrichstrasse 36
8010 Graz
AUSTRIA

Prof. Dr. Günter Leugering
Department Mathematik
FAU Erlangen-Nürnberg
Cauerstraße 11
91058 Erlangen
GERMANY

Dr. Caroline Löbhard
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
GERMANY

Dr. Gabriela G. Marinoschi
Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy of Sciences (ISMMA)
Calea 13 Septembrie 13
050711 Bucharest
ROMANIA

Dr. Hannes Meinlschmidt
Johann Radon Institute for Computational
and Applied Mathematics (RICAM)
Austrian Academy of Sciences
Altenberger Straße 69
4040 Linz
AUSTRIA

Prof. Dr. Christian Meyer
Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund
GERMANY

Prof. Dr. Gisèle M. Mophou
Head of the German Research Chair
in AIMS Cameroon
South West Region
Crystal Garden
Limbe
CAMEROON

Prof. Dr. Arnaud Diego Münch
Laboratoire de Mathématiques
Campus Universitaire des Cezeaux
3, place Vasarely
63178 Aubière Cedex
FRANCE

Prof. Dr. Ira Neitzel
Institut für Numerische Simulation
Universität Bonn
Wegelerstrasse 6
53115 Bonn
GERMANY

Dr. Laurent Pfeiffer
Institut für Mathematik
Karl-Franzens-Universität Graz
Heinrichstrasse 36
8010 Graz
AUSTRIA

Dr. Carlos N. Rautenberg
Fachbereich Mathematik
Humboldt Universität Berlin
Unter den Linden 6
10099 Berlin
GERMANY

Prof. Dr. Jean-Pierre Raymond
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Elisabetta Rocca
Dipartimento di Matematica
Universita di Pavia
Via Ferrata, 1
27100 Pavia
ITALY

Dr. Sérgio S. Rodrigues
Johann Radon Institute for Computational
and Applied Mathematics (RICAM)
Austrian Academy of Sciences
Altenberger Strasse 69
4040 Linz
AUSTRIA

Prof. Dr. Arnd Rösch
Fakultät für Mathematik
Universität Duisburg-Essen
Thea-Leymann-Strasse 9
45127 Essen
GERMANY

Prof. Dr. Ekkehard Sachs
Abteilung Mathematik
Fachbereich IV
Universität Trier
54286 Trier
GERMANY

Prof. Dr. Volker Schulz
Fachbereich IV - Mathematik, Numerik, Optimierung und partielle Differentialgleichungen
Universität Trier
54286 Trier
GERMANY

Prof. Dr. Jürgen Sprekels
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
GERMANY

Steven-Marian Stengl
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
GERMANY

Prof. Dr. Thomas M. Surowiec
Fachbereich Mathematik und Informatik
Philipps-Universität Marburg
Hans-Meerwein-Straße 6
35043 Marburg
GERMANY

Dr. Marita Thomas
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
GERMANY

Prof. Dr. Dan Tiba
Institute of Mathematics "Simion Stoilow"
of the Romanian Academy of Sciences
P.O. Box 1-764
014700 Bucharest
ROMANIA

Prof. Dr. Fredi Tröltzsch
Institut für Mathematik
Technische Universität Berlin
Sekr. MA 4-5
Strasse des 17. Juni 136
10623 Berlin
GERMANY

Prof. Dr. Marius Tucsnak
Mathématiques et Informatique
Université Bordeaux I
351, cours de la Liberation
33405 Talence Cedex
FRANCE

Prof. Dr. Michael Ulbrich
Zentrum Mathematik
Technische Universität München
85747 Garching bei München
GERMANY

Prof. Dr. Stefan Ulbrich
Fachbereich Mathematik
Fachgebiet: Nichtlineare Optimierung
Technische Universität Darmstadt
Dolivostraße 15
64293 Darmstadt
GERMANY

Prof. Dr. Boris Vexler
Zentrum Mathematik
Technische Universität München
Boltzmannstrasse 3
85748 Garching bei München
GERMANY

Prof. Dr. Daniel Wachsmuth
Mathematisches Institut
Universität Würzburg
Am Hubland
97074 Würzburg
GERMANY

Prof. Dr. Irwin Yousept
Fakultät für Mathematik
Universität Duisburg-Essen
Thea-Leymann-Straße 9
45127 Essen
GERMANY