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Mini-Workshop: Superexpanders and Their Coarse Geometry

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Abstract. It is a deep open problem whether all expanders are superexpanders. In fact, it was already a major challenge to prove the mere existence of superexpanders. However, by now, some classes of examples are known: Lafforgue's expanders constructed as sequences of finite quotients of groups with strong Banach property (T), the examples coming from zigzag products due to Mendel and Naor, and the recent examples coming from group actions on compact manifolds. The methods which are used to construct superexpanders are typically functional analytic in nature, but also rely on arguments from geometry and combinatorics. Another important aspect of the study of superexpanders is their (coarse) geometry, in particular in order to distinguish them from each other.

The aim of this workshop was to bring together researchers working on superexpanders and their coarse geometry from different perspectives, with the aim of sharing expertise and stimulating new research.

Mathematics Subject Classification (2010): 05C10, 05C12, 20F65, 46B85.

Introduction by the Organisers

Expanders are sequences of finite highly connected sparse graphs with an increasing (and unbounded) number of vertices. It is a non-trivial fact that expanders exist, but by now several constructions, coming from different areas of mathematics, are known. Expanders have found striking applications in various areas of mathematics and in theoretical computer science.

A coarse embedding of a metric space into another one is an embedding that preserves the large-scale metric structure. By the seminal and by now well-known result of Gromov, we know that expanders do not coarsely embed into a Hilbert space, which shows that even when considering their large-scale geometry, they are incompatible with Euclidean structure.

Recall that a Banach space is superreflexive if and only if it admits an equivalent uniformly convex norm. A superexpander is a sequence of finite d-regular graphs $\mathcal{G}_n = (V_n, E_n)$ ($n \in \mathbb{N}$) such that $\lim_{n\to\infty} |V_n| = \infty$ and such that for every superreflexive Banach space X there exists a constant $\gamma > 0$ such that for all $n \in \mathbb{N}$ and every $f : V_n \to X$, the inequality

$$\frac{1}{|V_n|^2} \sum_{v,w \in V_n} \|f(v) - f(w)\|^2 \leq \frac{\gamma}{d|V_n|} \sum_{v \sim w} \|f(v) - f(w)\|^2$$

holds. As a consequence, superexpanders do not coarsely embed into any superreflexive Banach space. Note that the above condition, known as a Poincaré inequality, is equivalent to such a condition with the squares of the norms replaced by a pth power for any $1 \leq p < \infty$.

It is a fundamental, deep open problem whether all expanders are superexpanders. By now, some classes of examples of superexpanders are known. In his seminal work on the Baum-Connes conjecture, Lafforgue introduced strong Banach property (T) and proved that certain sequences of finite quotients of groups with this property are superexpanders. Superexpanders coming from zigzag products were constructed in the groundbreaking work of Mendel and Naor. In recent years, a new source of superexpanders, constructed from group actions on compact manifolds, has been investigated, by Sawicki, by de Laat and Vigolo, and by Fisher, Nguyen and van Limbeek.

To prove that an expander is a superexpander, one typically employs methods from functional analysis, and expertise in the geometry of Banach spaces appears to be inevitable. So far, there is no theory or framework that allows us to consider the aforementioned classes of examples of superexpanders in a unified way, and the problem whether all expanders are superexpanders remains wide open.

An important problem that is intimately related to the superexpander problem is the classification of expanders (and superexpanders) up to coarse equivalence. A priori, it could very well be the case that different constructions of (super)expanders yield equivalent objects, and distinguishing between expanders is a new and challenging topic. Only in 2014, Mendel and Naor initiated the study of non-equivalence of expanders. One of the many fruitful discussions during the mini-workshop was about the question of what the right notion of coarse equivalence (or coarse distinction) of expanders is.

The aim of this mini-workshop was to bring together researchers working on superexpanders and their coarse geometry from different perspectives, in order to share expertise and explore new directions of research. The workshop consisted of five lectures (each 60 minutes), which covered some of the recent and earlier breakthroughs in the topic, 19 research talks (each 50 minutes), and several very stimulating discussion sessions. Additionally, there was also much time for informal discussions.

It is the pleasure of the organizers to thank all participants for their lectures, research talks and very fruitful contributions to the discussions.

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Workshop: Mini-Workshop: Superexpanders and Their Coarse Geometry

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Asymptotic structure and nonlinear geometry of Banach spaces

Florent Baudier

1. Super-expander graphs and geometry of Banach spaces

Problem 1. *Let Y be an arbitrary super-reflexive Banach space and $(G_n)_{n\in\mathbb{N}}$ be a sequence of expander graphs. Is it true that $(G_n)_{n\in\mathbb{N}}$ does not equi-coarsely embed into Y?*

Ozawa's localization technique [17] (see also [13]) shows that a large class of super-reflexive Banach spaces cannot equi-coarsely contain sequences of expander graphs.

Theorem 2. *Let Y be a Banach space and $(G_n)_{n\in\mathbb{N}}$ be a sequence of expander graphs. If the unit ball of Y uniformly embeds into Hilbert space then $(G_n)_{n\in\mathbb{N}}$ does not equi-coarsely embed into Y.*

We refer to [3, Chapter 9] for Banach spaces satisfying the condition in Theorem 2. In particular, Theorem 2 applies to all the spaces below since one can construct Mazur-type maps.

(1) ℓ_p and $L_p[0,1]$ for $p \in [1,\infty)$ [12].
(2) noncommutative L_p spaces for $p \in [1,\infty)$ [19].
(3) Banach spaces with an unconditional basis and non-trivial cotype [15].
(4) Banach lattices with non-trivial cotype [4].

Therefore, Problem 1 has a positive answer for super-reflexive Banach lattices. The following Banach spaces do not satisfy the condition in Theorem 2.

(1) c_0 (more generally Banach spaces with trivial cotype) [5]
(2) non-reflexive Banach spaces with non-trivial type [18].

Problem 1 would follow from the following long-standing open problem in non-linear Banach space theory.

Problem 3. *Let X be a separable super-reflexive Banach space. Is the unit sphere of X uniformly homeomorphic to the unit sphere of ℓ_2?*

The problem of the coarse minimality of Hilbert space shares some similitudes with Problem 1 since the existence of Mazur-type maps are pivotal as well. Using the Mazur maps, Nowak [14] proved that ℓ_2 coarsely embeds into ℓ_p for all $p \in [1,\infty)$. This result was generalized by Ostrovskii [16].

Theorem 4. *Let Y be a Banach space with an unconditional basis such that its unit sphere is uniformly homeomorphic to the unit sphere of ℓ_2. Then ℓ_2 coarsely embeds into Y.*

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It is unclear whether the unconditionality assumption is needed. Whether Hilbert space coarsely embeds into every infinite-dimensional Banach space was recently solved negatively in [2].

Theorem 5. *Hilbert space does not coarsely embed into Tsirelson's original space T^*.*

T^* is a reflexive Banach space with an unconditional basis and trivial cotype. It follows from Theorem 4 that Hilbert space coarsely embeds into every super-reflexive Banach space with an unconditional basis. The following problem seems challenging.

Problem 6. *Is there a super-reflexive Banach space that does not coarsely contain Hilbert space?*

By Gowers' dichotomy, such a space must be hereditarily indecomposable. The only known super-reflexive Banach space that is hereditarily indecomposable is Ferenczi's space [6]. A key concept in the proof of Theorem 5 is the concept of asymptotic structure of Maurey, Milman, and Tomczak-Jaegermann [11], and is touched upon in the second part of this report. It would certainly be interesting to exhibit new subclasses of super-reflexive Banach spaces that cannot equi-coarsely contain sequences of expander graphs.

Problem 7. *Let Y be a weak Hilbert space and $(G_n)_{n\in\mathbb{N}}$ be a sequence of expander graphs. Is it true that $(G_n)_{n\in\mathbb{N}}$ does not equi-coarsely embed into Y?*

Problem 8. *Let Y be an asymptotically Hilbertian Banach space and $(G_n)_{n\in\mathbb{N}}$ be a sequence of expander graphs. Is it true that $(G_n)_{n\in\mathbb{N}}$ does not equi-coarsely embed into Y?*

2. Stability of reflexivity under smoothness assumptions

In this second part we discuss to what extend reflexivity is preserved under nonlinear embeddability. Heinrich and Mankiewicz [7] proved that if a Banach space X bi-Lipschitzly embeds into a reflexive Banach space Y then X linearly embeds into Y. Therefore reflexivity is preserved under bi-Lipschitz embeddability. A version of Ribe's rigidity theorem [20] says that if a Banach space X coarse Lipschitzly embeds into a Banach space Y then X is (crudely) finitely representable into Y, and thus super-reflexivity is preserved under coarse Lipschitz embeddability. However, in 1984 Ribe [21] showed that the Banach spaces $\left(\sum_{n=1}^{\infty}\ell_{1+\frac{1}{n}}\right)_{\ell_2}$ and $\left(\sum_{n=1}^{\infty}\ell_{1+\frac{1}{n}}\right)_{\ell_2}\oplus\ell_1$ are uniformly homeomorphic (and in particular coarse Lipschitzly embeddable into each other). It follows that reflexivity is not stable under coarse Lipschitz embeddability. It turns out that reflexivity can be stable under nonlinear embeddability if we impose some smoothness condition. For a Banach space X the modulus of asymptotic uniform smoothness $\bar{\rho}_X$ is given for $t>0$ by

$$\bar{\rho}_X(t)=\sup_{x\in S_X}\inf_{Y\in\operatorname{cof}(X)}\sup_{y\in S_Y}\|x+ty\|-1,$$

where $\mathrm{cof}(X)$ denotes the collection of all closed finite codimensional subspaces. X is called asymptotically uniformly smooth if $\lim_{t\to 0^+}\bar{\rho}_X(t)/t = 0$, and if $\bar{\rho}_X(t) \leq ct^p$ for some $c > 0$ and $p \in (1,\infty)$, we say that X is asymptotically uniformly smooth of power type p. It is a deep result of Knaust, Odell and Schlumprecht [10] that every separable asymptotically uniformly smooth Banach space admits an equivalent norm that is asymptotically uniformly smooth of power type p.

We now define a collection of inequalities that are crucial for our purposes. Let $d_{\mathbb{H}}$ be the Hamming metric on $[\mathbb{N}]^k$ (the k-subsets of $\mathbb{N}$).

Definition. Let $p \in (1,\infty]$. We say that a metric space (Y, d_Y) has property $\mathcal{K}_p^{\mathbb{H}}$ if there exists $C > 0$ such that for every $k \in \mathbb{N}$ and every $f\colon ([\mathbb{N}]^k, d_{\mathbb{H}}) \to Y$ with $\mathrm{Lip}(f) < \infty$, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ such that

$$\sup_{\bar{m},\bar{n}\in[\mathbb{M}]^k} d_Y(f(\bar{m}), f(\bar{n})) \leq C\mathrm{Lip}(f)k^{1/p}, \tag{1}$$

where we adopt the convention $k^{1/\infty} = 1$.

For Banach spaces, property $\mathcal{K}_p^{\mathbb{H}}$ is easily seen to imply reflexivity. Indeed, assume by contradiction that Y is not reflexive. Then, by James' characterization of reflexive spaces [8], there exists a sequence $(y_n)_{n=1}^\infty \subset B_Y$ such that for all $k \geq 1$ and $\bar{m} = \{m_1, m_2, \ldots, m_{2k}\} \in [\mathbb{N}]^{2k}$,

$$\Big\| \sum_{i=1}^{k} y_{m_i} - \sum_{i=k+1}^{2k} y_{m_i} \Big\| \geq \frac{k}{2}. \tag{2}$$

For every $k \in \mathbb{N}$ and $\bar{m} = \{m_1, \ldots, m_k\} \in [\mathbb{N}]^k$, define $\varphi_k(\bar{m}) = \sum_{i=1}^k y_{m_i}$. Then $\varphi_k\colon [\mathbb{N}]^k \to Y$ is clearly 2-Lipschitz with respect to $d_{\mathbb{H}}$, and by property $\mathcal{K}_p^{\mathbb{H}}$ there is $\mathbb{M} \in [\mathbb{N}]^\omega$ such that $\mathrm{diam}(\varphi_k([\mathbb{M}]^k)) \leq 2Ck^{1/p}$ for some universal constant $C > 0$. Since by (2) $\mathrm{diam}(\varphi_k([\mathbb{M}]^k)) \geq k/2$, we obtain a contradiction.

Kalton and Randrianarivony [9] showed that every separable reflexive Banach space that is asymptotically uniformly smooth of power type p satisfies property $\mathcal{K}_p^{\mathbb{H}}$. The following consequence was observed in [1].

Theorem 9. *Let Y be a reflexive Banach space that admits an equivalent norm that is asymptotically uniformly smooth. If a Banach space X coarse Lipschitzly embeds into Y then X is reflexive.*

The core of the proof of Theorem 5 is to show that T^* satisfy property $\mathcal{K}_\infty^{\mathbb{H}}$. The proof can be upgraded to show that every reflexive asymptotic-c_0 Banach space also satisfy property $\mathcal{K}_\infty^{\mathbb{H}}$.

Definition. A Banach space X is said to be an asymptotic-c_0 Banach space if for all $k \geq 1$, and all $\varepsilon > 0$

$$\exists X_1 \in \mathrm{cof}(X),\ \forall x_1 \in S_{X_1}, \exists X_2 \in \mathrm{cof}(X),\ \forall x_2 \in S_{X_2},\ \ldots, \exists X_k \in \mathrm{cof}(X),\ \forall x_k \in S_{X_k}$$

$$\text{and for all } (a_i)_{i=1}^k \subset \mathbb{R},\ \frac{1}{(1+\varepsilon)} \sup_{i=1,\ldots,k} |a_i| \leq \Big\| \sum_{i=1}^{k} a_i x_i \Big\| \leq (1+\varepsilon) \sup_{i=1,\ldots,k} |a_i|.$$

Asymptotic-c_0 Banach spaces are extremely smooth in the sense that they are asymptotically uniformly smooth of power type p for every $p \in (1, \infty)$. In [2] the following theorem was proved.

Theorem 10. *Let Y be a reflexive asymptotic-c_0 Banach space. If a Banach space X coarsely embeds into Y then X is reflexive.*

Theorem 9 and Theorem 10 exhibit an interesting phenomenon and shed some light on the subtle tension between the degree of smoothness and the degree of faithfulness of the embedding needed in order to preserve reflexivity.

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Superexpanders from zigzag products

Assaf Naor

In [1] the zigzag product of graphs was introduced as a way to combine two graphs so as to obtain a larger graph while maintaining control of both its degree and its spectral gap. Using the zigzag product, an iterative procedure was devised in [1] to construct arbitrarily large bounded degree expander graphs. In [2] it was shown how this approach can be used to control nonlinear spectral gaps, and the iterative procedure of [1] was adapted to construct super-expanders. In [3] it was shown that this approach yields expanders relative to certain "wild CAT(0) spaces and random graphs. This tutorial-style presentation is an introduction to the zigzag product and the associated iterative construction of expanders relative to certain metric spaces. It explains geometric and analytic issues that need to be overcome in order to implement this strategy in the absence of tools from linear algebra, and presents some open problems.

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Existence of superexpanders, after Vincent Lafforgue

Mikael de la Salle

The aim of this lecture was to present Lafforgue's proof of the existence of superexpanders, that is a sequence of bounded degree graphs which are expanders with respect to every superreflexive Banach spaces. This statement is a very particular case of the results from [4, 5], so I took the opportunity to simplify as much as possible the statements and proofs.

Let p be a prime number (for example 2). For every integer $n \geq 1$, denote by $\mathbf{F}_{p^n}$ the field with p^n elements, consider $t_n \in \mathbf{F}_{p^n}^*$ a generator of the multiplicative group of $\mathbf{F}_{p^n}$ and denote by $\mathcal{G}_n$ the Cayley graph of $\mathrm{SL}_3(\mathbf{F}_{p^n})$ with respect to the elementary matrices $\cup_{i\neq j}\{E_{i,j}(1), E_{i,j}(t_n)\}$.

Theorem. [4,5] *The sequence $\mathcal{G}_n = (V_n, E_n)$ is an expander with respect to every uniformly convex Banach space X (and even every Banach space not containing uniformly the ℓ_1^n's): there is $\gamma > 0$ such that, for every n and every $f\colon V_n \to X$,*

$$\inf_{x\in X}\left(\frac{1}{|V_n|}\sum_{s\in V_n}\|f(s)-x\|^2\right)^{\frac{1}{2}} \le \gamma\left(\frac{1}{|E_n|}\sum_{(s,t)\in E_n}\|f(s)-f(t)\|^2\right)^{\frac{1}{2}}.$$

The inequality in the theorem is called X-valued Poincaré inequality, as it can be written as $\|f\|_{L^p(V;X)/\text{constants}} \le \gamma\|\nabla f\|_{L^p(E,V)}$.

For the non-experts of Banach space geometry, let me simply recall that not containing uniformly the ℓ_1^n's has many equivalent forms and names: it is the same as having type > 1, being B-convex, K-convex etc.

Consider Γ_n, the subgroup of $\mathrm{SL}_3(\mathbf{F}_p[t])$ whose reduction modulo $t^{p^n}-1$ is the identity, that is the kernel of the reduction morphism

$$\mathrm{SL}_3(\mathbf{F}_p[t]) \to \mathrm{SL}_3(\mathbf{F}_p[t]/(t^{p^n}-1)).$$

One can regard Γ_n as a lattice in the locally compact group $G = \mathrm{SL}_3(\mathbf{F}_p(\!(t^{-1})\!))$ over the locally compact field of Laurent series $\mathbf{F}_p(\!(t^{-1})\!)$. One shows (induction) that, for a fixed Banach space X, the sequence $\mathcal{G}_n$ is an expander with respect to X if and only if the sequence of representations $\lambda_{n,X}$ of G on $L^2(G/\Gamma_n;X)$ has uniform spectral gap in the sense that there is a compact subset $Q \subset G$ and $\varepsilon > 0$ such that for every n, and every $f \in L^2(G/\Gamma_n;X)$,

$$\sup_{g\in Q}\|\lambda_{n,X}(g)f - f\| \ge \varepsilon \inf_{x\in X}\|f-x\|. \tag{1}$$

So far no assumption on X is made, and it is likely that (1) (equivalently the Theorem) holds for every Banach space X which does not contains the family $(\ell_\infty^n)_n$ uniformly, see Question d in [5]. However the way (1) is proven is by proving another stronger form of spectral gap, namely that there is a probability measure μ on G such that, for every $f \in L^2(G/\Gamma_n;X)$ of mean 0,

$$\|\int \lambda_{n,X}(g) f d\mu(f)\| \le \frac{1}{2}\|f\|. \tag{2}$$

By the triangle inequality, (2) implies (1). These two forms of spectral gap are equivalent when X is uniformly convex ([3, Proposition 5.1] or [2, Theorem 1.1]), but (2) cannot hold if X contains the ℓ_1^n's uniformly. Indeed, otherwise by duality it would also hold for ℓ_∞, which would imply that the Theorem holds from ℓ_∞. This is clearly impossible as ℓ_∞ contains isometrically every finite (even separable) metric space.

To prove (2), we can forget about the specific form of the $\lambda_{n,X}$ and work with an arbitrary isometric representation without invariant vectors of G on a space not containing uniformly the ℓ_1^n's. The argument is too long to be explained in details in this extended abstract, but let me at least say that it is obtained in two steps.

Step 1: Understanding representations of the maximal compact subgroup $K = \mathrm{SL}_3(\mathbf{F}_p[\![t^{-1}]\!])$ of G. This is done by exploiting the existence of large nipoltent (upper-triangular) subgroups of K, and by using Bourgain's version [1] of the Hausdorff-Young inequality for X-valued harmonic analysis of abelian groups together with ideas from the fast Fourier transform [5].

Step 2: Applying Step 1 to the restriction to K of the representation of G, and using a thorough and very fast exploration of the Weyl chamber $K\backslash G/K$ first obtained in [4].

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Aut($\mathbb{F}_5$) has property (T)

Piotr W. Nowak

(joint work with M. Kaluba and N. Ozawa)

Property (T) is a fundamental rigidity property for groups. It was introduced by Kazhdan in 1966. The simplest way to define it is through one of its characterizations: a finitely generated group G has property (T) if every action of G on a Hilbert space by affine isometries has a fixed point.

Property (T) is now known to hold for a few classes of groups, nevertheless it should certainly be considered a rare property. There are two main reasons behind property (T). The first is an algebraic structure of the group rigid enough to force this kind of behavior. This is the case for $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, and more generally for higher rank Lie groups and their lattices. The second is a spectral property, that can be described as possesing positive curvature with respect to Euclidean geometry. More precisely, the condition is that the link of a generating set, or of a contractible simplicial complex on which G acts cocompactly, has the first positive eigenvalue greater than $1/2$. This method yields property (T) for automorphism groups of thick buildings, such as $\widetilde{A}_2$ buildings, and certain random hyperbolic groups in the Gromov density model. We refer to [1] for an excellent overview of the topic.

The aim of our work is to prove property (T) for a new group. Denote by $\mathrm{Aut}(\mathbb{F}_n)$ the group of automorphisms of the free group $\mathbb{F}_n$ on n generators, and by $\mathrm{Out}(\mathbb{F}_n)$ its quotient by the subgroup of inner automorphisms. The abelianization $\alpha : \mathbb{F}_n \to \mathbb{Z}^n$ induces a surjection

$$\alpha_* : \mathrm{Aut}(\mathbb{F}_n) \to \mathrm{GL}_n(\mathbb{Z}),$$

which factors through $\mathrm{Out}(\mathbb{F}_n)$. For $n = 2$, the map $\mathrm{Out}(\mathbb{F}_2) \to \mathrm{GL}_2(\mathbb{Z})$ is in fact an isomorphism. The special automorphism group $\mathrm{SAut}(\mathbb{F}_n) \subseteq \mathrm{Aut}(\mathbb{F}_n)$ is the preimage $\alpha_*^{-1}(\mathrm{SL}_n(\mathbb{Z}))$. This subgroup has index 2 in $\mathrm{Aut}(\mathbb{F}_n)$ and has a particularly convenient set of generators within $\mathrm{Aut}(\mathbb{F}_n)$: it is generated by the Nielsen transformations. We fix this generating set for the group $\mathrm{SAut}(\mathbb{F}_n)$. Our main result is the following

Theorem 1 ([4])**.** *The group* $\mathrm{SAut}(\mathbb{F}_5)$ *has property* (T) *with Kazhdan constant* $\kappa \geq 0.176$.

The proof relies on the following characterization of property (T) due to Ozawa.

Theorem 2 ([6])**.** *A finitely generated group has property* (T) *if and only if there exists a finite collection* $\xi_i \in \mathbb{R}G$ *and* $\kappa > 0$ *such that*

$$\Delta^2 - \kappa\Delta = \sum_{i=1}^{n} \xi_i^* \xi_i. \tag{1}$$

The condition (1) can easily be translated into the existence of a positive definite matrix P, indexed by some finite subset $E \subseteq G$, such that

$$\Delta^2 - \kappa\Delta = \mathbf{x} P \mathbf{x}^T, \tag{2}$$

where $\mathbf{x} = [\delta_{g_1}, \ldots, \delta_{g_n}]$ and the g_i run through the elements of E.

From now on we fix E to be the ball of radius 2 in the word length metric on G. The strategy to prove Theorem 1 is to find a solution of the equation (2) with the assistance of a computer. There are solvers appropriate for semidefinite programming, i.e. software designed to solve systems of linear equations such as (2), with the restriction that P is positive semidefinite. This approach was used successfully to reprove property (T) for $\mathrm{SL}_n(\mathbb{Z})$ by Netzer and Thom for $n = 3$ [5], Fujiwara and Kabaya for $n = 3, 4$ [2] and Kaluba and Nowak for $n = 3, 4, 5$ [3]. In these cases the new proof included new, drastically improved, estimates of Kazhdan constants for $\mathrm{SL}_n(\mathbb{Z})$, $n = 3, 4, 5$.

In the case of $\mathrm{Aut}(\mathbb{F}_5)$ the ball $B(e, 2)$ has 4 641 elements, and consequently the matrix P depends on 10 771 761 variables, too many for a solver to handle. In order to simplify the problem and reduce the number of variables we symmetrize the problem. Consider an action of a finite group Σ of automorphisms of G which preserves the set E and the Laplacian Δ. We show that if a solution P of (2) exists then there exists a solution P invariant under the action of Σ, and thus it suffices to find such a Σ-invariant P. This significantly reduces the number of variables in the matrix, however poses a new problem: solvers are not able to handle group-invariant matrices.

A classical theorem of Wedderburn implies that the algebra of Σ-invariant matrices is isomorphic to a direct sum of matrix algebras. The next step is thus a construction of an explicit isomorphism

$$\mathbb{M}_E^\Sigma \simeq \oplus_{\pi \in \hat{\Sigma}} 1_{\dim \pi} \otimes \mathbb{M}_{m_\pi},$$

where $\hat{\Sigma}$ is the unitary dual of Σ and m_π denotes the multiplicity of $\pi \in \hat{\Sigma}$. For $G = \mathrm{Aut}(\mathbb{F}_5)$ we choose $\Sigma = \mathbb{Z}_2 \wr S_5$ and the above isomorphism is constructed using a system of minimal projections. The resulting reduction in complexity is indeed significant: from the previously mentioned 10 771 761 variables to 13 232 variables in 36 blocks. Then using a solver we obtain a solution P with accuracy of the order 10^{-9} and $\kappa = 1.2$.

The last and crucial step is the certification of the solution. The solution provided by the solver is by definition approximate. However, since the solution is obtained with very good accuracy an additional argument, based on the fact that the Laplacian Δ is an order unit for self-adjoint elements of the augmentation ideal in $\mathbb{R}G$, allows to deduce the existence of a mathematically precise solution. This argument turns the above reasoning into a rigorous proof of property (T) for $\mathrm{Aut}(\mathbb{F}_5)$ and in the process gives an explicit estimate on the Kazhdan constant of G.

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Generalized expanders and coarse fundamental groups

Thiebout Delabie

Generalized expanders. A well-known property of expanders is that they do not coarsely embed into a Hilbert space. So any metric space that coarsely contains an expander does not coarsely embed into a Hilbert space. However these are not the only metric spaces that do not coarsely embed into a Hilbert space. In order to find such an equivalence we introduce generalized expanders.

Definition 1 (Tessera [Tes])**.** *A sequence of bounded metric spaces X_n is a generalised expander if there exists a sequence $r_n > 0$, a sequence of probability measures μ_n on $X_n \times X_n$ and a constant C such that the following conditions are met:*

- *The sequence r_n tends to infinity as $n \to \infty$.*
- *We have that $\mu_n(D) = 0$ for $D = \{(x, y) : d(x, y) < r_n\}$.*
- *For every $\varphi\colon X_n \to \ell^2$ that is 1-Lipschitz we have*

$$\sum_{x,y \in X_n} \|\varphi(x) - \varphi(y)\|^2 \mu_n(x, y) \leq C.$$

For this property we do indeed find an equivalence between containing a generalized expander and non-embeddability into a Hilbert space.

Theorem 2 (Tessera [Tes]). *A metric space does not coarsely embed into a Hilbert space if and only if it has a coarsely-embedded sequence of generalized expanders.*

In order to find examples more easily we introduce relative expanders. For a sequence of Cayley graphs we can use the group structure to create the measures μ_n.

Definition 3 (Arzhantseva and Tessera [AT]). *A sequence of finite groups G_n is an expander relative to a sequence of subgroups $H_n < G_n$ if there exists a constant C such that the following conditions are met:*

- *The diameter of H_n in the Cayley graph of G_n goes to infinity as n goes to infinity.*
- *For every $\varphi\colon X_n \to \ell^2$ that is 1-Lipschitz we have*

$$\sum_{x\in X_n}\sum_{z\in H_n} \|\varphi(x)-\varphi(xz)\|^2 \leq C|G_n||H_n|.$$

As H_n is a group, at least half of the elements in H_n are not contained in the ball of radius $\frac{1}{2}\operatorname{diam}(H_n)$ centred at e. Therefore every relative expander is a generalized expander, because we can take μ_n as the uniform measure on the set of pairs (x, xz) with $x \in G_n$, $z \in H_n$ and $d(e, z) \geq \frac{1}{2}\operatorname{diam}(H_n)$.

Box spaces. Most examples of relative expander are box spaces.

Definition 4. *Let G be a group with a finite generating set S and let N_n be filtration, i.e. a sequence of finite index normal subgroups such that this sequence is decreasing, i.e. $N_{n+1} < N_n$ and their intersection is trivial, i.e. $\bigcap_{n=1}^{\infty} N_n = \{1\}$. Then the box space $\square_{(N_n)}G$ is the disjont union $\bigsqcup_{n=1}^{\infty}\operatorname{Cay}(G/N_n, \bar{S})$, where $\bar{S}$ is the image of S under the quotient map $G \to G/N_n$.*

This disjoint union can be seen as either a sequence of graphs or as a single metric space. In the latter case the distance between elements of the same graph is the distance within that graph and the distance between elements of different graphs is the sum of the diameters of those graphs.

Box spaces are useful to provide examples. For example Arzhantseva, Guentner and Špakula in [AGS] show that $\square_{(M_n)}F_S$ embeds into a Hilbert space, where F_S is a finitely generated free group, $M_1 = F_S$ and M_n is the group generated by squares of elements in M_{n-1}.
There also exists a box space $\square_{(K_n)}F_S$ that is an expander. In [Lub] Lubotzky shows that $\mathrm{SL}_2(\mathbb{Z}/2^n\mathbb{Z})$ is an expander. As a corollary $\square_{(K_n)}F_2$ is an expander with K_n the kernel of $F_2 \to \mathrm{SL}_2(\mathbb{Z}/2^n\mathbb{Z})$, where the generators are mapped to $\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$.

These two sequences of sets can be used to construct generalized expanders that do not coarsely contain expanders.
In [AT] Arzhantseva and Tessera show that a box space of $\mathbb{Z}^2 \rtimes F_3$ is such an

example, where the generators of F_3 act on $\mathbb{Z}^2$ as $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Here the quotients are $(\mathbb{Z}/2^n\mathbb{Z})^2 \rtimes F_3/M_n$.
In the same paper Arzhantseva and Tessera show that $\mathbb{Z} \wr_{F_2/K_n} F_2/M_n = \bigoplus_{F_2/K_n} \mathbb{Z} \rtimes F_2/M_n$ is generalized expander, but not a relative expander as defined in Definition 3. This is the most promising example of a metric space that coarsely embeds into some ℓ^p with $p > 2$, but not in ℓ^2. The other examples of relative expanders do not embed into any ℓ^p with $p < \infty$.
A last example is given by Khukhro and the author. In [DK] they show that $\square_{(N_n)} F_3$ is a relative expander, for $N_n = M'_{k_n} \cap K'_n$ where M'_n is similar to M_n, K'_n is similar to K_n and $k_n \to \infty$ as $n \to \infty$, but k_n should be small compared to n.

Coarse fundamental groups. In order to coarsely distinguish different box spaces we define a coarse version of a fundamental group. These are fundamental groups relative to some constant $r > 0$.
An r-path p in a metric space X is an r-Lipschitz map $p\colon \{0, \ldots, \ell(p)\} \to X$. Here $\ell(p)$ is the length of p. Coarse homoptopy between such paths was first defined by Barcelo, Kramer, Laubenbacher and Weaver in [BKLW].

Definition 5. *Let $r > 0$ be a constant, let X be an r-connected metric space and let p and q be two r-paths in X. We say that p and q are r-homotopic if there exists a sequence $p_0 = p$, $p_1, \ldots, p_n = q$ such that p_i is r-close to p_{i-1} for every $i \in \{1, 2, \ldots, n\}$.*
Two r-paths p and q are r-close if one of the two following cases is satisfied:

(a) *For every $i \le \min(\ell(p), \ell(q))$ we have that $p(i) = q(i)$ and for bigger i we either have $p(i) = p(\ell(q))$ or $q(i) = q(\ell(p))$, depending on which path is defined at i.*
(b) *We have that $\ell(p) = \ell(q)$ and for every $0 \le i \le \ell(p)$ we have $d(p(i), q(i)) \le r$.*

The coarse fundamental group $\pi_{1,r}(X)$ is the set of r-loops at a certain base point up to r-homotopy with the usual composition as a group action.

Note that $\pi_{1,r}(X)$ is independent of the base point, because X is r-connected. In [DK2] Khukhro and the author show that if G is a finitely presented group with $G = \langle S|R\rangle$, then for r big compared to the length of the relators in R and $N \lhd G$ such that $N \cap B_G(e, 4r) = \{e\}$ we have that $\pi_{1,r}(\mathrm{Cay}(G/N))$ is isomorphic N. As a consequence we can use coarse fundamental groups to recall the filtration that is used to create the box space.

Theorem 6. *Let G and H be finitely presented groups with respective filtrations N_i and M_i such that $\square_{(N_i)} G \simeq_{CE} \square_{(M_i)} H$. Then there exists an almost permutation with bounded displacement f of $\mathbb{N}$ such that $N_i \cong M_{f(i)}$ for every i in the domain of f.*

Note that a map $f\colon A \to B$ is an almost permutation if it is a bijection between co-finite sets of A and B.

A first corollary of this theorem is that if two box spaces of finitely presented groups are coarsely equivalent, then the groups are commensurable.
This theorem can also be used to give an example of two box spaces $\Box_{N_i}G$ and $\Box_{M_i}G$ that are not coarsely equivalent such that $G/N_i \twoheadrightarrow G/M_i$ with $[M_i : N_i]$ bounded.

Finally Theorem 6 can be used to show that there exist infinitely many Ramanujan expanders with different coarse structures.

Corollary 7. *There exist infinitely many coarse equivalence classes of box spaces of the free group F_3 that contain Ramanujan expanders.*

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Superexpanders from random graphs and actions on compact spaces

Cornelia Druţu

This talk has been a survey of some topics on superexpanders, in particular in connection with random graphs and with actions on compact spaces.

Superexpanders have interesting connections with John Roe's construction of warped cones. This construction associates to an action of an infinite group on a compact space an unbounded metric space. John Roe introduced it in the hope of producing a metric counter-example to the Baum–Connes conjecture, and his expectations were confirmed recently by work of Damian Sawicki. This made it essential to understand when do warped cones contain families of expanders: families of expanders are another metric counter-example to Baum–Connes, by work of Higson-Lafforgue-Skandalis, thus the question is equivalent to asking when do warped cones represent a really new counter-example. The question has been answered by Federico Vigolo, who gave necessary and sufficient conditions on the action, emphasizing the fact that quite often expanders do appear in the warped cone. In the process, Tim de Laat and Federico Vigolo construct new families of super-expanders and explain that these families are not coarsely equivalent to V. Lafforgue's super-expanders in any way.

Another interesting aspect of warped cones is that they provide a metric invariant of a group action. This has been emphasized recently by work of Fisher-Limbeek-Nguyen and opened another compelling direction of research.

Random graphs are presumably sources of superexpanders even though for the moment this is not known. Some results are known, about expansion features of random graphs considered with respect to particular type of Banach spaces such as the L^p–spaces.

Expanders and coarse non-universality of CAT(0) spaces

Manor Mendel

A Banach space X is called 2-convex if there exists $K > 0$ such that for any probability distribution μ in $W_2(X)$ (the Wasserstein space over X), and for every $x_0 \in X$,

$$d_X(x_0, \mathcal{B}(\mu))^2 + K^{-2}\mathbb{E}_{x\sim\mu} d_X(\mathcal{B}(\mu), x)^2 \leq \mathbb{E}_{x\sim\mu} d(x_0, x)^2, \tag{1}$$

where $\mathcal{B}(\mu) = \int_X x d\mu(x)$.

The CAT(0) property is a natural generalization of 2-convexity to metric spaces, where $\mathcal{B} : W_2(X) \to X$ is a barycenter map, mapping the Dirac delta measure at point x to point x, and with $K = 1$.

Expander relative to a metric space X is a family of constant degree graphs $G = (V, E)$ for which there exists $\Gamma > 0$ satisfying for any map $f : V \to X$,

$$\mathbb{E}_{x,y\in V} d_X(f(x), f(y))^2 \leq \Gamma\, \mathbb{E}_{x\sim y} d_X(f(x), f(y))^2. \tag{2}$$

Expander relative to a metric space X implies that the expander does not embed in X, not even (uniformly) coarsely.

Following the discovery of super-expanders by Lafforgue [2], it is natural to ask whether there exist "CAT(0)-expanders", i.e., expander relative to all CAT(0) spaces. While this question is still unresolved, in a joint work with A. Naor [5] we have proved the existence of expanders relative to interesting CAT(0) spaces that contain the Euclidean cones over the (properly scaled) metric of most high-girth graphs. In particular it shows:

- The existence of a CAT(0) space X and two different families of expanders: one that is not an expander relative to X, and one that is.
- The existence of an expander relative to the metric of random regular graphs.

The proof uses the zigzag expander construction of Reingold et. al. [6] as applied relative to 2-convex spaces in [4].

While it is unknown whether there exists an expander that does not coarsely embed in any CAT(0) space, in a joint work with A. Eskenazis and A. Naor [1] we showed another family of finite metric spaces, the discrete tori with the ℓ_∞ distance, that do not coarsely embed in any CAT(0) space. The proof uses the (sharp) metric cotype property, that was first studied as a metric characterization of the cotype property of Banach spaces [3].

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Coarse geometry of approximating graphs: discrete fundamental group

Federico Vigolo

Given an action by diffeomorphisms of a finitely generated group $\Gamma = \langle S \rangle$ on a compact riemannian manifold M and a parameter $r > 0$, we can "approximate" this action up to error $\approx r$ with a finite graph. The idea for doing so is to choose a finite partition $\mathcal{P}$ of M into many (regular) regions $R \subset M$ that have comparable sizes and so that the diameter of these regions is approximately r. We then define an *approximating graph at scale r* as the graph $\mathcal{G}_r(\Gamma \curvearrowright M)$ whose vertex set is the set of regions in the partition $\mathcal{P}$, and so that two regions $R, R' \in \mathcal{P}$ are linked by an edge in $\mathcal{G}_r(\Gamma \curvearrowright M)$ if there exists a generator $s \in S$ so that the intersection $s(R) \cap R'$ is *not* trivial.

Mimicking the notion of Cheeger constant in the dynamical setting, we say that the action $\Gamma \curvearrowright M$ is *expanding action in measure* if there exists some constant $\alpha > 0$ such that for every measurable subset $A \subset M$ with $\operatorname{Vol}(A) \leq \frac{1}{2}\operatorname{Vol}(M)$ the measure of the union of the images of A under the generators of Γ is at least α-times larger than the measure of A:

$$\operatorname{Vol}\left(\bigcup_{s\in S} s(A)\right) \geq (1+\alpha)\operatorname{Vol}(A).$$

It turns out that for any sequence of parameters $r_k \to 0$, the sequence of approximating graphs $\mathcal{G}_{r_k}(\Gamma \curvearrowright M)$ is a family of expanders if and only if the action $\Gamma \curvearrowright M$ is expanding in measure [14].

This construction is very flexible, and allows us to construct a wealth of new examples of families of expanders. In fact, any measure preserving action with a spectral gap is expanding in measure [14], and there are a number of examples of such actions. Moreover, if a measure preserving action has a strong Banach-valued spectral gap then any sequence of graphs approximating it is actually a family of *superexpanders* [4, 11] (see also [7]). An example that is worth mentioning is the following: let $\Gamma_d \coloneqq \mathrm{SO}(d, \mathbb{Z}[\frac{1}{5}])$ and consider the action by isometries on the sphere $\Gamma_d \curvearrowright \mathbb{S}^{d-1}$. Then the graphs approximating this action are superexpanders for every $d \geq 5$ [4].

Another feature of this construction is that the graphs thus obtained are well suited to a geometric study. This is because they relate very nicely with another metric construction: John Roe's warped cone (see [8–10]). Given an action by diffeomorphisms of a finitely generated group $\Gamma = \langle S \rangle$ on a compact Riemannian manifold (M, ϱ), the *warped cone* associated with it is the infinite metric space $\big(\mathcal{O}_\Gamma(M), \delta_\Gamma\big)$ obtained from the infinite Riemannian cone $(M \times [1,\infty), t^2\varrho + dt^2)$ by warping the metric. That is, the metric δ_Γ is obtained from the cone metric $t^2\varrho + dt^2$ by imposing the condition that for every element s in the generating set S the distance between any two points of the form (x,t) and $(s \cdot x, t)$ be at most 1. It turns out [14] that the approximating graphs $\mathcal{G}_r(\Gamma \curvearrowright M)$ are uniformly quasi-isometric to the level sets $M \times \{1/r\} \subset \mathcal{O}_\Gamma(M)$ (*i.e.* the quasi-isometry constants do not depend on r).

It follows that in order to study the geometry of the expanders obtained via this approximation procedure it is enough to study the geometry of the associated warped cone. As an example, it is proved in [4] (see also [12, 13]) that the graphs approximating a free action by isometries $\Gamma \curvearrowright M$ are 'locally' quasi-isometric to $\Gamma \times \mathbb{Z}^{\dim(M)}$. Using this result, it can be proved that the superexpanders $\mathcal{G}_{r_k}(\Gamma_d \curvearrowright \mathbb{S}^{d-1})$ above mentioned are pairwise not coarsely equivalent when $d \geq 5$ varies. Moreover, they are never coarsely equivalent to superexpanders obtained as box spaces of a cocompact lattice of an algebraic group of higher rank.[1]

A rather different approach to coarse rigidity for approximating graphs is inspired by [2] and makes use of discrete fundamental groups. The *discrete fundamental group at scale* θ of a metric space (X, d) was defined by Barcelo, Capraro and White [1] as the group $\pi_{1,\theta}(X)$ which is the analogue of the fundamental group of X where continuous loops are replaced by closed θ-paths (*i.e.* finite sequences of points with $d(x_i, x_{i+1}) \leq \theta$) which are considered up to θ-homotopies.

From our perspective, the usefulness of the discrete fundamental groups is that the study of the groups $\pi_{1,\theta}(X)$ for (families of) metric spaces can provide some strong coarse invariants. Indeed, even if it is not true in general that $\pi_{1,\theta}(X)$ is invariant under coarse equivalences, it is easy to show that a coarse equivalence $X \to Y$ induces a homomorphism of $\pi_{1,\theta}(X)$ to $\pi_{1,\theta'}(Y)$ where the parameter θ' is explicitly bounded in term of θ and the constants of the coarse equivalence. This information can sometimes be enough to prove that such a coarse equivalence cannot exist.

It turns out that it is possible to explicitly compute the discrete fundamental groups of the level sets of warped cones [3, 15], and this allows one to prove that a number of families of expanders and superexpanders are not pairwise coarsely equivalent (uncountably many, actually [3]). As an interesting byproduct, it turns out that—once θ is fixed—the discrete fundamental group of the approximating graphs $\mathcal{G}_r(\Gamma \curvearrowright M)$ does not depend on the approximation scale r (as long as it is small enough). Moreover, the superexpanders $\mathcal{G}_{r_k}(\Gamma_d \curvearrowright \mathbb{S}^{d-1})$ are coarsely simply

[1] This is one of the two previously known constructions of superexpanders, due to V. Lafforgue [5] (the other one being via zig-zag products [6]).

connected; from which it follows that they are not coarsely equivalent to any box space [2, 15].

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BiLipschitz vs Quasi-isometric equivalence

Tullia Dymarz

A finitely generated group G can be considered as a metric space when endowed with a **word metric**

$$d_S(g,h) := \|g^{-1}h\|_S$$

where $\|\cdot\|_S$ measures the minimal word length with respect to a fixed finite generating set S. The metric d_S, however, can vary depending on the chosen generating set S. In geometric group theory we are interested in properties that can be detected by any word metric on G. The notion of *biLipschitz equivalence* captures this relation although usually we study the geometry of groups up to *quasi-isometric equivalence* which allows one to study groups not only via their word metrics but also via geodesic 'model' spaces on which these groups act properly discontinuously and cocompactly.

A **quasi-isometry** between metric spaces (X, d_X) and (Y, d_Y) is a map $f : X \to Y$ that is coarsely biLipschitz and whose image is coarsely dense: i.e. there exist $K, C \geq 0$ such that for all $x, y \in X$

$$-C + 1/K d_X(x,y) \leq d_Y(f(x), f(y)) \leq K d_X(x,y) + C$$

and if the C neighbourhood of $f(X)$ is all of Y.

If $C = 0$ then f is a **biLipschitz equivalence**.

Question 1. *Is biLipschitz equivalence the same as quasi-isometric equivalence for finitely generated groups?*

In choosing to study finitely generated groups up to quasi-isometric equivalence instead of simply biLipschitz equivalence we are allowed to import a rich theory of techniques from differential geometry. It is easy to see that biLipschitz equivalence is not the same as quasi-isometric equivalence when comparing arbitrary metric spaces. When restricting to finitely generated groups themselves, however, this was an open question.

Indeed in [G], one of the fundamental monographs in geometric group theory, Gromov proposed the problem of understanding which quasi-isometries are close to bijective quasi-isometries. (For finitely generated groups a biLipschitz equivalence is the same as a bijective quasi-isometry.) Later in [W], Whyte proved that any quasi-isometry between *nonamenable* finitely generated groups (see Definition 1 below) is a bounded distance from a bijection. In other words, he proved that any quasi-isometry gives rise to a biLipschitz equivalence. In [D] I answered the above question by giving the first examples of finitely generated groups that are quasi-isometric but not biLipschitz equivalent; necessarily these groups are *amenable.* (See definition of amenability below).

Theorem 2 (Dymarz)**.** *Let F and G be finite groups with $|F| = n$ and $|G| = n^k$ where $k > 1$. Then there does not exist a bijective quasi-isometry between the lamplighter groups $F \wr \mathbf{Z}$ and $G \wr \mathbf{Z}$ if k is not a product of prime factors appearing in n.*

These first examples given by the above theorem are finitely generated but not finitely presented. More recently in [DPT] as a follow up to the above theorem, with Taback and Peng we prove the following:

Theorem 3 (Dymarz-Peng-Taback)**.** *For each n there exist families of examples of groups of type F_n that are quasi-isometric but not biLipschitz equivalent.*

In particular this theorem provides the first finitely presented counterexamples.

Separated nets. More generally, the question of quasi-isometric versus biLipschitz equivalence can be asked for *separated nets.* Given a metric space (X, d), a coarsely dense subset $\mathcal{D} \subset X$ is a separated net if there is some $\epsilon > 0$ such that for all $x, y \in \mathcal{D}$ we have $d(x, y) \geq \epsilon$. Any two such nets in X are quasi-isometric to X and hence to each other but it is not always the case that are biLipschitz equivalent. Indeed Burago-Kleiner [BK] and McMullen [M] constructed separated nets in $\mathbf{R}^n$ for $n \geq 2$ that are not biLipschitz equivalent to $\mathbf{Z}^n$.

With Kelly, Li and Lukyanenko in [DKLL] we were able to extend these results to all connected, simply connected nilpotent Lie groups, showing that each such group has separated nets that are not biLipschitz equivalent. Additionally with Navas in [DN] we give examples of non-nilpotent solvable Lie groups containing separated nets that are not biLipschitz equivalent.

Definition 1. *A finitely generated group or more generally a uniformly discrete space with bounded geometry is* **amenable** *if it contains a sequence of finite sets* $\{S_i\}$ *with the property that for all* $r > 0$

$$\lim_{i \to \infty} \frac{|\partial_r S_i|}{|S_i|} = 0$$

where $\partial_r S$ *is the set of points not in* S *but at distance at most* r *away from* S*. It is* **non-amenable** *otherwise.*

By Whyte's theorem all quasi-isometries between nonamenable groups (spaces) are bounded distance from a biLipschitz equivalence so counter examples can only occur in amenable spaces.

Question 4. *Does any finitely generated amenable group (space)* Γ *always contain a* coarsely dense net $\mathcal{D} \subset \Gamma$ *that is not biLipschitz equivalent to* Γ*?*

In fact this is the case for lattices in $\mathbf{R}^n$ with $n \geq 2$ and lattices in all Lie groups covered by [DKLL] and [DN]. In [DN] we also show this is true for solvable Baumslag-Solitar groups.

Question 5. *In general (non-nilpotent) solvable Lie groups are there always separated nets that are not biLipschitz equivalent?*

The techniques used in [DN] to prove results for non-nilpotent solvable Lie groups rely on quasi-isometric rigidity theorems but such rigidity theorems are currently not available for many solvable Lie groups.

The most interesting problem is the following:

Question 6. *Is there a solvable or nilpotent Lie group that contains two cocompact lattices that are not biLipschitz equivalent?*

A positive answer to this question would add to the very short list of counter examples we provide in Theorems 2 and 3 of finitely generated groups that are quasi-isometric but not biLipschitz equivalent. It would also show that even in the 'nicest' of groups these equivalences are not the same.

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Rigidity of warped cones' coarse geometry

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(joint work with David Fisher and Wouter van Limbeek)

Warped cones were first defined by John Roe ([6]) for the purpose of constructing metric spaces without property (A) and counterexamples to the coarse Baum-Connes conjecture. The geometry of warped cones is determined by both dynamics of the action and geometry of the space and group. Warped cones can also be a source of expanders and super-expanders. The main goal of this project is to understand how much coarse geometry of warped cones tells us about dynamics of actions.

Let us first recall the warped cone construction. Let Γ be a finitely generated group acting on a closed Riemannian manifold M. For $t > 0$, define the metric space $M_t := (tM \times \Gamma)/\Gamma$, where tM is a copy of M scaled by a factor t and Γ is equipped with a fixed word metric. Each M_t is called a level set of the warped cone for Γ action on M. And a warped cone can be either uncountable or countable disjoint union of the M_t. We note that when the action has spectral gap, by discretizing M_t by approximating graphs we obtain (super-)expanders ([5, 7, 9]).

We study coarse geometry of warped cones in terms of quasi-isometry. Suppose that we have two isometric actions $\Gamma \curvearrowright M$ and $\Lambda \curvearrowright N$ of finitely generated groups on closed manifolds. We say $(M_{t_n})_{n=1}^{\infty}$ and $(N_{s_n})_{n=1}^{\infty}$ are quasi-isometric if there are quasi-isometries $f_n : M_{t_n} \to N_{s_n}$ with uniform constants for all n. It can be seen that if $\Gamma \curvearrowright M$ and $\Lambda \curvearrowright N$ are conjugate then (M_t) are quasi-isometric with (N_t). We also have quasi-isometry between warped cones in a slightly more general setting as follows.

Definition 1. Two group actions $\Gamma_0 \curvearrowright M_0$ and $\Gamma_1 \curvearrowright M_1$ are commensurable if

- M_i is a finite cover of M_{i-1} for one of $i = 0, 1$ and
- for the same index i we have that Γ_i is the group of lifts of Γ_{i-1} to M_i.

In the definition the indices should be interpreted modulo 2.

Definition 2. A *commensuration* of the actions $\Gamma \curvearrowright M$ and $\Lambda \curvearrowright N$ consists of the following data:

- an action $\Gamma' \curvearrowright M'$ commensurable to $\Gamma \curvearrowright M$ and an action $\Lambda' \curvearrowright N'$ commensurable to $\Lambda \curvearrowright N$,
- a bi-Lipschitz map $f : M' \to N'$,

- an isomorphism $\varphi : \Gamma' \to \Lambda'$,

such that f is φ-equivariant.

It is not hard to see that two actions having a commensuration give rise to quasi-isometric warped cones. A natural question is whether the converse holds. The answer turns out to be negative by examples of $\mathbb{Z}$-action by rotations on S^1 of Kim's [4] and $\mathbb{Z}$-action by irrational rotations on S^1 versus trivial action on the torus $S^1 \times S^1$ of Kielak–Sawicki [8]. The absence of a $\mathbb{Z}$ factor in acting groups is necessary for the converse to hold. In fact, we obtain a rigidity statement when we assume this necessary condition together with an extra technical condition (FSL). Our main result is

Theorem 3. *Let Γ and Λ be finitely presented groups acting isometrically, freely and minimally on closed manifolds M and N (respectively). Assume that*

- *Neither of Γ and Λ is commensurable to a group with a nontrivial free abelian factor, and Γ has Property* FSL, *and*
- *There exist uniform quasi-isometries $f_n : M_{t_n} \to N_{s_n}$, where $t_n, s_n \to \infty$.*

Then there is a commensuration of the action of Γ on M and the action of Λ on N.

Roughly speaking, groups with property FSL are groups that have only finitely many automorphisms satisfying any fixed constraint on distortion with respect to the stable length of group elements. Sources of FSL groups are rich, including lattices in semisimple Lie groups, hyperbolic groups, some free products and direct products.

Independently, de Laat-Vigolo ([1]) and shortly after Sawicki ([8]) study local geometry of warped cones, deducing quasi-isometry of $\Gamma \times \mathbb{R}^{dim(M)}$ and $\Lambda \times \mathbb{R}^{dim(N)}$ from quasi-isometry of warped cones. Our result is not only able to tell us coarse equivalence of geometry of Γ and Λ but also the conjugation between the actions. We also mention here that there are also studies about quasi-isometry of box spaces, objects that share lots of similarity with warped cones, in [2, 3].

We briefly mention here intuition and how to build such a conjugation in our result. Because of the Riemannian metric in N is rescaled by s_n in the level set N_{s_n}, images of close points under f_n are close if the metric in N_{s_n} were not distorted by Λ-action. If this were the case, taking limits of quasi-isometries should give us a Lipschitz limit. But because the metric is distorted by Λ-jumps, we first need to correct the quasi-isometries by canceling all $\Lambda-$jumps. This can be done locally and then extended by monodromy. The obstruction for doing this is $\Lambda-$jumps along image of a loop in $\pi_1(M)$ is not trivial. Studying coarse fundamental groups helps us to show that the obstruction is finite. Using structure of homogeneous spaces, we can pass to finite quotients of manifolds and quotients by finite groups to obtain commensurable actions without obstruction for correcting the quasi-isometries. The property FSL then helps us to be able to take limit with a resulted Lipschitz map. We do the same for reverse direction to have inverse of Lipschitz limit map.

An application of our main result, we construct uncountably many non-quasi-isometric (super-)expanders:

Theorem 4. *Let K be a compact Lie group and fix $r \geq 2$. Then free, rank r subgroups of K with spectral gap yield quasi-isometrically disjoint expanders unless they are commensurable. In particular, there exist explicit quasi-isometrically disjoint continuous families of expanders.*

To obtain families as in the theorem, we just note that from the existing literature we can produce many free groups with spectral gap: For example, set $K = \mathrm{SU}(n)$ with $n \geq 2$ and fix a free dense subgroup $\Lambda \subseteq \mathrm{SU}(n)$ of rank $r-1$ and with spectral gap. Then consider the family $\{\langle \Lambda, k\rangle\}_{k\in K}$. For generic k, we have $\Gamma := \langle \Lambda, k\rangle \cong F_r$, and the spectral gap for Λ implies that Γ has spectral gap. For super-expander construction, we take Γ of the form $\Delta \times F_n$ and do similar construction, where Δ has strong (T) and F_n is free.

To end this, we also mention here some open questions. We first note that the condition FSL is technical that is necessary for this proof. We do not know if it is necessary for our theorem or if all finitely generated subgroups of compact Lie groups have Property FSL.

Question 5. Can the technical condition FSL be removed?

It is also natural to study quasi-isometric embeddings or coarse embeddings.

Question 6. Is there a rigidity statement for the case of QI-embeddings or even coarse embedding?

In our current proof, we use quasi-isometry to obtain isomorphism of coarse fundamental groups and to obtain inverse of limit map. The former one seems to be a more serious issue in the embedding case than the latter one. So to study embedding question by using same idea in our proof, we may need to understand the algebraic property of induced maps on coarse fundamental groups.

There is another notion of equivalence, proposed by A. Naor, that could be more natural in study of graphs and expanders. We say two sequences of metric spaces (A_n) and (B_n) are *weak-equivalent* if there are subsequences of indexes (t_n) and (s_n) such that there are coarse embeddings $f_n : (A_n) \to (B_{s_n})$ and $f_n : (B_n) \to (A_{t_n})$.

Question 7. How much does weak-equivalence of warped cones tell us about dynamics of actions?

This question is more subtle than Question 6. To get some rigidity statement, this may need more conditions than Question 6.

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The Baum-Connes conjecture and Oka's principle

Maria Paula Gomez-Aparicio

Let G be a locally compact group and denote by $C^*_r(G)$ the reduced C^*-algebra associated to G. If A is a C^*-algebra endowed with an action of G, denote by $C^*_r(G,A)$ the reduced crossed product associated to this action. In [1], Baum, Connes and Higson defined an assembly map,

$$\mu_r^B : K^{\mathrm{top}}(G,A) \to K(C^*_r(G,A)),$$

where $K^{\mathrm{top}}(G,A)$ is the G-equivariant K-homology with compact support and with values in A, of $\underline{E}G$, the universal classifiying space for proper G-actions. Recall that $K^{\mathrm{top}}(G,A)$ is given by the following formula :

$$K^{\mathrm{top}}(G,B) = \lim_{\rightarrow} KK_G(C_0(X),B),$$

where the inductive limit is taken among all G-invariant and G-compact close subsets X of $\underline{E}G$. In particular, for $B=\mathbb{C}$, we denote $\mu_r^{\mathbb{C}}$ by μ_r.
This morphism is defined using Kasparov's KK-theory (cf. [6]) and is called *The Baum-Connes assembly map.* The Baum-Connes conjecture is then stated as follows:

Conjecture 1 (The Baum-Connes conjecture)**.** *For all locally compact groups the assembly map μ_r is an isomophism.*

And the Baum-Connes conjecture with coefficients as follows :

Conjecture 2 (The Baum-Connes conjecture with coefficients)**.** *For all locally compact groups G and for all G-C^*-algebras B, the assembly map μ_r^B is an isomorphism.*

Both conjectures, with and without coefficients, have been proven for a large class of groups that includes for example all a-T-menable groups. For those groups in particular, the proof is due to Higson and Kasparov (see [4]) and it is based on a method known as "Dirac-dual Dirac". introduced by Kasparov in [6]. This method does not work for groups having property (T).

However, Lafforgue proved Conjecture 1 for all semi-simple Lie groups and for some of its closed subgroup, precisely those having property (RD). For example, Conjecture 1 is true for all cocompact lattices in $SL_3(\mathbb{R})$ but it is still open for $SL_3(\mathbb{Z})$.

On the other hand, Conjecture 2 has been proven for all hyperbolic groups (see [11]), but it still open for higher rank semi-simple Lie groups and their closed subgroups.

Moreover, Higson, Lafforgue and Skandalis gave a counterexample to Conjecture 2 using the existence of some groups that contain expanders coming from property (T) (see [5]). Furthermore, Lafforgue proved that a strong version of property (T) prevents the methods that have been used so far to succeed (see [10]). Nonetheless, a direction that is still open concerns applying the ideas of Bost, who defined a version of Oka principle in Noncommutative Geometry (see [2]).

Let us mention that in regards to Lie groups, injectivity of both maps μ_r^A and μ_r is known thanks to the work of Kasparov and Skandalis (see [6], [7], [8]), hence the problem that is still open for those groups is surjectivity.

Before explaining what is Oka principle in his context, let us mention that for semi-simple Lie groups, Lafforgue also proved an analogue conjecture known as Bost's conjecture intended to compute the K-theory of $L^1(G)$. More precisely, Lafforgue defined a morphism

$$\mu_{L^1}^B : K^{\mathrm{top}}(G,B) \to K(L^1(G,B)),$$

for all locally compact groups G and all G-C^*-algebra B. The Bost conjecture is stated as follows:

Conjecture 3 (Bost)**.** *For all locally compact groups G and all G-C^*-algebra B the map $\mu_{L^1}^B$ is an isomorphism.*

Conjecture 3 is true for all semi-simple Lie groups and all their closed subgroups (see [9]). Moreover, the Baum-Connes map μ_r^B factors through $\mu_B^{L^1}$, hence, if G is a semi-simple Lie group or a closed subgroup in a semi-simple Lie group, the question that remains open is to prove that the inclusion map $i : L^1(G,B) \hookrightarrow C_r^*(G,B)$ induces an isomorphism i_* at the level of K-theory. This implies Conjecture 2 and Conjecture 1 if $B = \mathbb{C}$ for G. Unfortunately, to prove that i_* is a isomorphism is not an easy task. As an example, for $G = SL_2(\mathbb{R})$, it is a known fact that $L^1(G)$ is not stable under holomorphic calculus in $C_r^*(G)$.

Let ρ be a finite representation of G on a complex hermitian finite dimensional vector space V. Then $\|\rho\|$ can be used as a weight to define exponential decay subalgebras of crossed product algebras. In the case of $L^1(G,A)$, this are easy to define : let $L^{1,\rho}(G,A)$ be the completion of $C_c(G,A)$ by the norm

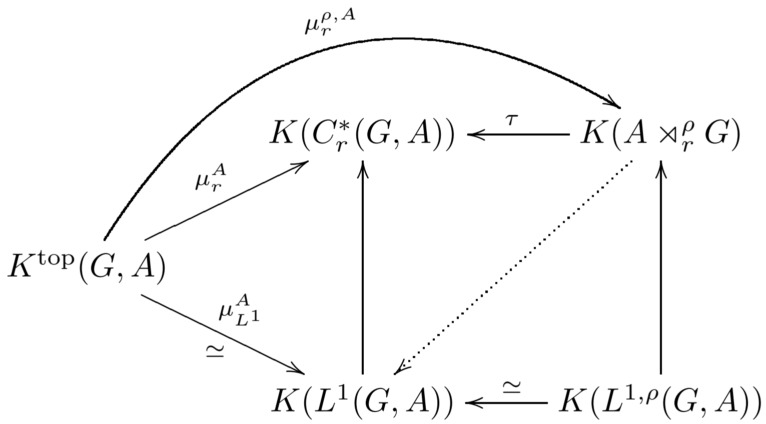
$$\|f\|_{1,\rho} = \int_G \|f(g)\|_A \|\rho(g)\|_{\mathrm{End}(V)} dg.$$

It is clear that $L^{1,\rho}(G,A)$ is a dense subalgebra of $L^1(G,A)$. Following the work of Bost (see [2]) an Oka principle applied to this case should say that this two algebras have the same K-theory. This statement is true, by the work of Lafforgue for all semi-simple Lie groups and all their closed subgroups.

In [3], some twisted crossed products, denoted by $A \rtimes_r^\rho G$ were defined as well as morphisms

$$\mu_r^{\rho,A} : K^{\mathrm{top}}(G,A) \to K(A \rtimes_r^\rho G).$$

Taking ρ very large this algebras, play the same role in $C_r^*(G,A)$ as $L^{1,\rho}(G,A)$ in $L^1(G,A)$; they are constructed to be some kind of exponential decay "subalgebras" of $C_r^*(G,A)$. Suppose now that G is a group for which Conjecture 3 is known to be true. Then, taking ρ very large, allows us to write the following diagram :



Oka principle applied to this crossed products states that the twisted group algebras, $A \rtimes_r^\rho G$, have the same K-theory as $C_r^*(G,A)$, i.e. τ is an isomorphism. This would then imply the surjectivity of μ_r^B, hence the Baum-Connes conjecture with coefficients for G.

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Warped cones violating the coarse Baum–Connes conjecture

Damian Sawicki

A version of the Baum–Connes conjecture for metric spaces was proposed by Roe in 1993. Similarly to other versions, it predicts that a certain assembly map μ is an isomorphism, in this case between the coarse K-homology of the metric space in question and the K-theory of its Roe algebra.

In particular, if our metric space is a finitely generated group, the conjecture is equivalent to the classical Baum–Connes conjecture with certain coefficients and it implies injectivity of the classical assembly map, which is known as the strong Novikov conjecture.

In 2014 Druţu and Nowak [1] predicted that the coarse assembly map is not surjective for warped cones over actions with a spectral gap, see Oberwolfach Report 3/2015. Warped cones are unbounded metric spaces whose geometry encodes the dynamics of a group action on a compact space and its interplay with metric properties of this space.

By a celebrated result of G. Yu, the conjecture holds for bounded geometry metric spaces coarsely embeddable into a Hilbert space. As a first step towards the prediction of Druţu and Nowak, in a joint work [4] with Nowak we proved that indeed warped cones over actions with a spectral gap do not satisfy the assumptions of Yu's result, namely they do not admit coarse embeddings into the Hilbert space and in fact also into more general Banach spaces under the appropriate spectral gap assumptions.

Kasparov and Yu [3] showed that in order to conclude injectivity of the coarse assembly map, it suffices to assume coarse embeddability into a super-reflexive Banach space. This sparked a lot of interest in the search for super-expanders (the topic of the Mini-Workshop summarised in the present Oberwolfach Report), that is, expanders with respect to super-reflexive Banach spaces.

While warped cones over spectral gap actions need not be coarsely equivalent to graphs, under assumptions guaranteeing that they are, we strengthened the result of [4] about the non-embeddability into Banach spaces to the expansion with respect to these spaces [5]. It includes expansion with respect to all Banach spaces of non-trivial type—expanders of such strength were earlier constructed by V. Lafforgue.

Work [5] generalises to the setting of Banach spaces a seminal result for classical expanders by Vigolo [8], who was the first to study graphs quasi-isometric to warped cones. Quasi-isometric rigidity of such (super-)expanders was later studied by de Laat–Vigolo, the author, Fisher–Nguyen–van Limbeek, and Vigolo [9] (details can be found in other abstracts of the present Oberwolfach Report). In particular, paper [9] provides examples of warped cones whose levels are not quasi-isometric to any box space.

Until recently, the following counterexamples to the coarse Baum–Connes conjecture were known:

- μ is not injective for $\bigsqcup_n n\mathbb{S}^{2n}$—the coarse disjoint union of spheres of dimension $2n$ and radius n (Yu, 1998);
- μ is not surjective for expanders obtained as box spaces of certain groups (Higson–Lafforgue–Skandalis [2], 2002);
- μ is not surjective for expanders with large girth and of not necessarily bounded degree (Willett–Yu, 2012).

As a topological space, the warped cone $\mathcal{O}_\Gamma Y$ over an action $\Gamma \curvearrowright Y$ is the product $[0,\infty) \times Y$ (quotiented by $\{0\} \times Y$). It turns out that $\mathcal{O}_\Gamma Y$ may satisfy the coarse Baum–Connes conjecture even if $\Gamma \curvearrowright Y$ has a spectral gap, so, in order to prove the prediction of Druţu and Nowak, it is necessary to pass to a subspace $T \times Y$ such that the inclusion $T \subseteq [0,\infty)$ is *not* a quasi-isometry.

Denote any such subspace by $\mathcal{O}'_\Gamma Y$. Then, the main result of [6] is the following.

Theorem 1. *Assume that an action $\Gamma \curvearrowright Y$ by Lipschitz homeomorphisms is free and has a spectral gap and that Γ has property A. Then the surjectivity part of the coarse Baum–Connes conjecture fails for $\mathcal{O}'_\Gamma Y$.*

By the above-mentioned work of D. Fisher, T. Nguyen, and W. van Limbeek, there are continuum actions yielding pairwise coarsely non-equivalent spaces $\mathcal{O}'_\Gamma Y$ satisfying the assumptions of Theorem 1. By [9] some of them are not coarsely equivalent to earlier counterexamples to the coarse Baum–Connes conjecture obtained as box spaces, and they are also not coarsely equivalent to counterexamples obtained as large-girth graphs (e.g. by Theorem 4 below).

As a result of the following, Theorem 1 provides first counterexamples to the coarse Baum–Connes conjecture that are not coarse disjoint unions of graphs (moreover, they have bounded geometry).

Proposition 2. *Let $\Gamma \curvearrowright Y$ be an action on a Cantor set Y and equip Y with an ultrametric satisfying a mild condition and such that the action is by Lipschitz homeomorphisms (such a metric always exists). Then, for any T as above, the space $\mathcal{O}'_\Gamma Y$ is not coarsely equivalent to a family of graphs.*

Example 3. *The action $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \varprojlim_n \mathrm{SL}_m(\mathbb{Z}/k^n\mathbb{Z})$ (with a product metric) satisfies the assumptions of Theorem 1 and Proposition 2 for any $m, k \geq 2$.*

Since any countable group Γ admits a free, measure-preserving action on a Cantor set, Proposition 2 yields plenty of examples, in particular it can be applied to actions of simple groups, where the theory of box spaces is not available. If such Γ has Kazhdan property (T), then its action on an ergodic component has a spectral gap, opening the possibility to apply Theorem 1.

The proof of Theorem 1 follows a strategy of Higson. We start with a projection $\mathfrak{P}$ constructed by Druţu and Nowak, which is supposed to yield a K-theory class not in the image of μ. One of the fundamental difficulties was to prove that $\mathfrak{P}$ belongs to the Roe algebra because it naturally occurs as a limit of finite propagation operators that are yet not locally compact.

Then we need the following result.

Theorem 4 (S–Jianchao Wu [7]). *Consider $\Gamma \times \mathcal{O}Y$ with a certain metric d_1 and a map $\pi\colon \Gamma \times \mathcal{O}Y \to \mathcal{O}_\Gamma Y$ given by $\pi(\gamma, x) = \gamma x$. Then the metric on $\mathcal{O}_\Gamma Y$ is the quotient metric of d_1 under π. Moreover, π is asymptotically faithful if and only if $\Gamma \curvearrowright Y$ is free.*

If $\Gamma \curvearrowright Y$ is isometric, then d_1 is simply the product metric of the word metric on Γ and the metric on the *infinite* (or: Euclidean) cone $\mathcal{O}Y$ (the warped cone is obtained by modifying ('warping') the metric on the infinite cone). Asymptotic faithfulness of Willett and Yu means that π is isometric on balls of increasing radius. In particular, the fact that for isometric actions on manifolds the pointed Gromov–Hausdorff limit of levels of the warped cone is the Cartesian product $\Gamma \times \mathbb{R}^m$ is a corollary of the above structural result.

The core of the argument is a construction of two tracial maps on the K-theory of the Roe algebra of $\mathcal{O}'_\Gamma Y$ that are equal after composing with μ by the Atiyah Γ-index theorem but take different values on $[\mathfrak{P}]$. The value of the first trace on $[\mathfrak{P}]$ is non-zero (it is explicit). In order to construct the second trace, we lift—using Theorem 4—operators in the Roe algebra of $\mathcal{O}'_\Gamma Y$ to operators on $\Gamma \times \mathcal{O}'Y$. This involves the operator norm localisation property of Chen, Tessera, Wang, and Yu, which we prove for $\Gamma \times \mathcal{O}'Y$.

The Kazhdan projection $\mathfrak{P}$ of Druţu and Nowak is defined using the group action as the limit of increasing powers of a Markov operator. After lifting to $\Gamma \times \mathcal{O}'Y$, the action involves only the first coordinate and—by the non-amenability of Γ—powers of the Markov operator converge to zero. Consequently, the value of the second trace (defined via the lifting procedure) on $[\mathfrak{P}]$ is zero, which concludes the proof.

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Expanders, fixed point properties on Banach spaces and random graphs

Cornelia Druţu

This talk has been based on joint work with J. Mackay. In it, further details have been provided on how random graphs have expansion properties stronger than the classical ones, and therefore random groups have stronger versions of property (T). Moreover these stronger fixed point properties have an interesting connection with a geometric feature of a random group: the conformal dimension of its boundary. In particular, in the triangular model random groups have the property that the supremmum of the set of parameters $p \in (0, \infty)$ for which the group has property FL^p is at most $\delta =$ the conformal dimension of the boundary of the group and at least $\frac{\delta^{1/2}}{\log \delta}$. Using completely different methods and work of Pisier, de la Salle and de Laat improved the lower bound to $\delta^{1/2}$. It would be very interesting to have better estimates than these, in particular to investigate if indeed there is anything special about the power $\delta^{1/2}$ of the conformal dimension.

With J. Mackay we also proved a weaker version of the theorem, for random groups in the Gromov density model.

Diameters in box spaces

Alain Valette

It is well known that the diameter in a graph sequence of d-regular graphs, is at least logarithmic in the number of vertices in the corresponding graphs: this is obtained by comparing a ball in a d-regular graph to a ball of same radius in the d-regular tree. If the graph sequence is an expander, the diameter is exactly logarithmic in the number of vertices, see e.g. [Lub]. So having a diameter growing faster than logarithmic can be viewed as a strong form of non-expansion. To quantify this, fix $\alpha \in]0,1]$, say that the graph sequence $(X_n)_{n>0}$ has *property* D_α if the diameter of X_n is larger than a constant times $|X_n|^\alpha$. We study property D_α for box spaces of finitely generated, residually finite groups G.

In the first part of the talk, we report on joint work with Ana Khukhro, published in [KV]. We first prove that G admits a box space with property D_1 if and only if G is virtually cyclic (this appeals to the fact that a group has linear growth if and only if it is virtually cyclic, a result due to J. Justin [Jus]). For $0 < \alpha < 1$, a remarkable result by E. Breuillard and M. Tointon [BT] (invoking the theory of approximate groups) says that if some box space has D_α, then G virtually maps onto $\mathbb{Z}$. We prove that the converse also holds. It has to be emphasized that, since non-abelian free groups have box spaces with D_α and other box spaces which are expanders, we cannot replace "some box space" by "any box space" in the above characterization.

In the second part of the talk, we consider groups G with a given embedding into $SL_N(\mathbb{Z})$. This allows to define a congruence subgroup of G as the intersection of G with a congruence subgroup of $SL_N(\mathbb{Z})$. An interesting open question is whether that notion of congruence subgroups of G, depends on the choice of the embedding into $SL_N(\mathbb{Z})$. (Compare with a remarkable result by A. Garrido [Gar]: for a branch group, the congruence subgroups - as defined by the levels of the rooted tree on which the group acts - do not depend on the choice of a branch action). In work in progress with Etienne Grezet, we show that, for semi-direct products $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$, with $A \in SL_2(\mathbb{Z})$ a hyperbolic matrix, if such an arithmetic box space has D_α, then $\alpha \leq 2/3$. Using deep results by M. Aka and U. Shapira [AS], we show that the bound can be lowered to $\alpha \leq 1/3$ for congruence box spaces modulo families of integers supported on finitely many primes.

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Upgrading fixed points

Masato Mimura

Let X be a metric space and G be a locally compact group (mainly, countable discrete group). By $\alpha\colon G \curvearrowright X$, we mean a (continuous) G-action by isometries. (If $X = E$ is a Banach space, then we in addition assume that $\alpha\colon G \curvearrowright E$ is by affine isometries.) For a subgroup $M \leqslant G$ and for $\alpha\colon G \curvearrowright X$, set $X^{\alpha(M)}$ as the set of $\alpha(M)$-fixed points in X. Let R be a unital and associative ring, *possibly non-commutative.*

We address the following problem.

Problem 1 (Upgrading Problem)**.** *Let* $\alpha\colon G \curvearrowright X$. *Let* $(M_j)_{j\in J}$ *be a family of subgroups of* G *such that* $\langle \bigcup_{i\in J} M_j\rangle = G$. *Assume that for every* $j \in J$, *it holds that* $X^{\alpha(M_j)} \neq \emptyset$.

Under which condition can we conclude that $X^{\alpha(G)} \neq \emptyset$*?*

Note that the conclusion above is *not* always true: Consider a natural action $D_\infty \curvearrowright \mathbb{R}$ of the infinite dihedral group $D_\infty = \langle a, b | a^2 = b^2 = e\rangle$, and set $M_1 = \langle a\rangle$ and $M_2 = \langle b\rangle$.

In this report, we will concentrate on the following setting of G and $(M_j)_{j\in J}$. Before proceeding to our setting, we explain the definition of *elementary groups* in algebraic K-theory. Take R as above. Let $n \in \mathbb{N}_{\geq 2}$. By $[n]$, we mean the set $\{1, 2, \ldots, n\}$. Recall that $\mathrm{GL}(n, R)$ is defined as the group of all invertible elements in the matrix ring $\mathrm{Mat}_{n\times n}(R)$. For $i, j \in [n]$ with $i \neq j$ and $r \in R$, we define an *elementary matrix* $e^r_{i,j}$ by

$$(e^r_{i,j})_{k,l} = \begin{cases} 1, & \text{for} \quad k = l, \\ r, & \text{for} \quad (k, l) = (i, j), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2. The *elementary group* $\mathrm{E}(n, R)$ is the subgroup of $\mathrm{GL}(n, R)$ generated by $\{e^r_{i,j} : i \neq j \in [n], r \in R\}$.

By Gaussian elimination, if R is a Euclidean domain, then for every $n \geq 2$, it holds that $\mathrm{E}(n, R) = \mathrm{SL}(n, R)$. Hence, we may regard $\mathrm{E}(n, R)$ as a generalization of $\mathrm{SL}(n, \mathbb{Z})$ to other coefficient rings R from $\mathbb{Z}$. (Unlike $\mathrm{SL}(n, R)$, the elementary group $\mathrm{E}(n, R)$ is defined even over a non-commutative ring.)

Note that for every $i \neq j \in [n]$, $e^{r_1}_{i,j} e^{r_2}_{i,j} = e^{r_1+r_2}_{i,j}$ $(r_1, r_2 \in R)$. Moreover, the following commutator relation

$$(\#) \qquad [e^{r_1}_{i,j}, e^{r_2}_{j,k}] = e^{r_1 r_2}_{i,k} \quad \text{for } i \neq j \neq k \neq i \text{ and for } r_1, r_2 \in R$$

holds true, where our commutator convention is $[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$. They imply that if R is a finitely generated ring and $n \geq 3$, then $\mathrm{E}(n, R)$ is a *finitely generated group.*

Here is our setting: Fix R as above and $n \in \mathbb{N}_{\geq 3}$.

- $G_R(= G^{(n)}_R) = \mathrm{E}(n, R)$.
- $J = [2]$, and $M_R(= M^{(n)}_{1,R}) = \langle e^r_{i,n} : i \in [n-1], r \in R\rangle$ and $L_R(= M^{(n)}_{2,R}) = \langle e^r_{n,j} : j \in [n-1], r \in R\rangle$.

Abstractly, M_R and L_R above are both isomorphic to the additive group $(R^{n-1}, +)$. By (#), it holds that $\langle M_R \cup L_R\rangle = G_R$. Throughout this report, we use the symbols G_R, M_R, L_R for these three groups above.

The main motivation of us to study this example of (G_R, M_R, L_R) is the following theorem. Recall that we assume R that is unital and associative.

Theorem 3 (Relative fixed point properties)**.** *Assume that* R *is finitely generated.*

1. (*Kassabov* [3]) *For every* $n \geq 3$, $G_R \geqslant M_R$ *and* $G_R \geqslant L_R$ *have relative property* (T). *This property is equivalent to relative property* $(\mathrm{F}_{\mathcal{H}\mathrm{ilbert}})$.

(2) (*M.* [5]*, together with results of Olivier* [8] *and Tanaka* [11]) *For* $n \geq 4$, $G_R \geqslant M_R$ *and* $G_R \geqslant L_R$ *have relative property* $(\mathrm{F}_{\mathcal{E}})$. *Here* $\mathcal{E}$ *is either the class of all non-commutative* L_q*-spaces* (*associated to every von Neumann algebra*) *for all* $q \in [1, \infty)$, *or the class of all reflexive Orlicz spaces over* $[0, 1]$ (*with the Lebesgue measure*) *over* $\mathbb{C}$.

Here for a (non-empty) class $\mathcal{E}$ of Banach spaces, we say that $G \geqslant M$ has *relative property* $(\mathrm{F}_{\mathcal{E}})$ if for every $E \in \mathcal{E}$ and for every $\alpha\colon G \curvearrowright E$, it holds that $E^{\alpha(M)} \neq \emptyset$. The (full) fixed point property, *property* $(\mathrm{F}_{\mathcal{E}})$, is defined as relative property $(\mathrm{F}_{\mathcal{E}})$ for $G \geqslant G$. (Similarly, (relative) property (F_E) is defined for a single Banach space E.) In (1) above, by $\mathcal{H}$ilbert, we denote the class of all Hilbert spaces. Therefore, the Upgrading Problem in our setting asks whether we can upgrade the relative fixed properties as in Theorem 3 to the full fixed point property for G_R.

Our main theorem gives a new answer to Problem 1. It is stated in terms of "(Game)-condition" on G and $(M_j)_{j \in J}$. We do not exhibit the condition in this report. See the forthcoming version of our preprint [6] or the expository article [7]; the latter focuses only on property $(\mathrm{F}_{\mathcal{H}\mathrm{ilbert}})$. Nevertheless, we emphasize that our "(Game)-condition" is *intrinsic*, more precisely, it is a purely algebraic condition on G and $(M_j)_{j \in J}$ (described in terms of inclusions and group generation) with no dependence on X or α. Our main theorem asserts that under this "(Game)-condition" on G and $(M_j)_{j \in J}$, some weak conditions on X and α suffice to deduce that $X^{\alpha(G)} \neq \emptyset$. In particular, it enables us to upgrade relative properties $(\mathrm{F}_{\mathcal{E}})$ for $G_R \geqslant M_R$ and $G_R \geqslant L_R$ to property $(\mathrm{F}_{\mathcal{E}})$ for G_R if either of the following two conditions on the class $\mathcal{E}$ is satisfied.

- Each element in $\mathcal{E}$ is super-reflexive, and $\mathcal{E}$ is closed under taking metric ultraproducts (with respect to a fixed non-principal ultrafilter on $\mathbb{N}$).
- Each element in $\mathcal{E}$ is separable and reflexive, and property (T) implies property $(\mathrm{T}_{\mathcal{E}})$.

Here we do not give the definition of property $(\mathrm{T}_{\mathcal{E}})$; see Bader–Furman–Gelander–Monod [1].

For every fixed $q \in (1, \infty)$, the class $\mathcal{B}_{NCL_q}$ of all non-commutative L_q-spaces satisfies the former condition (Raynaud). The class of all reflexive Orlicz spaces over $[0, 1]$ over $\mathbb{C}$ meets the latter one ([11]). Therefore, we obtain the following corollary. (To prove (2) below, we need some extra work.)

Theorem 4 (Full fixed point properties, M. [6]). *Assume that* R *is finitely generated.*

(0) (*Simpler alternative proof of the work of Ershov and Jaikin-Zapirain* [2]) *For every* $n \geq 3$, $G_R = \mathrm{E}(n, R)$ *has property* (T) (*which is equivalent to property* $(\mathrm{F}_{\mathcal{H}\mathrm{ilbert}})$).

(1) *For every* $n \geq 4$, $G_R = \mathrm{E}(n, R)$ *has property* $(\mathrm{F}_{\mathcal{E}})$. *Here* $\mathcal{E}$ *is either the class of all non-commutative* L_q*-spaces for all* $q \in (1, \infty)$, *or the class of all reflexive Orlicz spaces over* $[0, 1]$ *over* $\mathbb{C}$.

(2) *For every* $n \geq 4$, $G_R = \mathrm{E}(n, R)$ *has property* (F_{C_1}). *Here* C_1 *denotes the Banach space of trace class operators acting on* ℓ_2.

Item (0) was first proved in [2]. In their proof, the upgrading process was done by *ϵ-orthogonality argument*. This argument moreover provided them with an estimate of Kazhdan constants. Our alternative proof, on the other hand, does not supply any estimation of them. However, their upgrading condition is *extrinsic*: Their condition is a numerical comparison of some spectral quantity to a certain threshold; later Oppenheim [9] extended this framework, but both of the spectral quantity and the threshold are sensitively affected if we change the target Banach space E. In particular, their methods may not seem to be able to prove the statement in (1) in Theorem 4. Our proof is based on the intrinsic upgrading mentioned above. Some outcomes of it are that proofs are simpler (see a 9-page expository article [7] for the full proof of our upgrading for property $(\mathrm{F}_{\mathcal{H}\mathrm{ilbert}})$) and that we can obtain (0)–(2) in Theorem 4 in a unified approach.

The fixed point property with respect to non-commutative L_1-space is mysterious. For ordinary L_1-spaces E, it is showed that property (T) implies property (F_E). However, the proof is based on the fact that these spaces admit radial coarse embeddings into a Hilbert space; it completely *breaks down* for non-commutative L_1-spaces. Indeed, C_1 is an example of non-commutative L_1-spaces, but it is known that C_1 even *fails* to admit a coarse embedding into a Hilbert space. Non-commutative L_1-spaces have trivial (linear-)type, and hence the approach by V. Lafforgue [4] (or Oppenheim [9]) via strong property $(\mathrm{T}_{\mathcal{E}})$ (or robust property $(\mathrm{T}_{\mathcal{E}})$) *totally* does *not* work. Our approach does not give the full result (because these spaces are not super-reflexive) at the present. However, as stated in (2) of Theorem 4, we managed to prove the fixed point property if our non-commutative L_1-space is C_1.

To close up this report, we discuss some application to expanders. Let $(p_m)_{m\in\mathbb{N}_{\geq 1}}$ be a sequence of primes. Then, there is a natural isomorphism

$$\mathrm{SL}(4m,\mathbb{F}_{p_m})(=\mathrm{E}(4m,\mathbb{F}_{p_m})) \quad \simeq \quad \mathrm{E}(4,\mathrm{Mat}_{m\times m}(\mathbb{F}_{p_m})).$$

Here $\mathbb{F}_p$ denotes the finite filed of order p for a prime p. The key observation here is that regardless of values of m (and p_m), the ring $\mathrm{Mat}_{m\times m}(\mathbb{F}_{p_m})$ is generated (as a ring) by unit and two elements (for instance, $e^1_{1,2}$ and a cyclic permutation matrix). Hence, $\mathrm{E}(4,\mathbb{Z}\langle s,t\rangle)$ *maps onto* $\mathrm{SL}(4m,\mathbb{F}_{p_m})$ *for every* $m\in\mathbb{N}_{\geq 1}$, where $\mathbb{Z}\langle s,t\rangle$ means the *non-commutative* polynomial ring over $\mathbb{Z}$ with two indeterminates s and t. By (1) of Theorem 4, the group $\mathrm{E}(4,\mathbb{Z}\langle s,t\rangle)$ has property $(\mathrm{F}_{\mathrm{B}_{NCL_q}})$ for all $q\in(1,\infty)$. By combining this with results of de la Salle [10] and Oppenheim [9] (which show that property $(\mathrm{F}_{\mathrm{B}_{NCL_q}})$ is equivalent to "robust property $(\mathrm{T}_{\mathrm{B}_{NCL_q}})$" ([9]) if $q\in(1,\infty)$) and a forthcoming work of Gomez-Aparicio, Liao and de la Salle on "geometric robust property $(\mathrm{T}_{\mathcal{E}})$", we obtain the following byproduct.

Corollary 5. *For every sequence $(p_m)_{m\in\mathbb{N}_{\geq 1}}$ of primes, there exists a system $(S_m)_{m\in\mathbb{N}_{\geq 1}}$ of generators of cardinality 4 for finite groups $(\mathrm{SL}(4m,\mathbb{F}_{p_m}))_{m\in\mathbb{N}_{\geq 1}}$ such that he following holds true: The sequence of Cayley graphs*

$$(\mathrm{Cay}(\mathrm{SL}(4m,\mathbb{F}_{p_m}),S_m)))_m$$

forms an expander family with "geometric robust property ($T_{\mathcal{B}_{NCL_q}}$) *for all* $q \in (1, \infty)$*."*

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Problem Session

Assaf Naor, Florent Baudier, Piotr Nowak, Masato Mimura

The most fundamental problem related to this workshop is the question of whether all expanders are superexpanders, as recalled in the introduction. In several extended abstracts, interesting open problems were mentioned. In this section, we record some additional open problems, which were suggested by participants or came up during discussion sessions.

Problem 1 (Naor)**.** *Do there exist two sequences of bounded degree graphs that are incomparable?*

Two sequences of bounded degree graphs are said to be *incomparable* if each is an expander with respect to the other, i.e., each satisfies a Poincaré inequality with respect to the other. This is a stronger notion than that of the two sequences not admitting a coarse embedding into one another. Note that being incomparable in this way forces both sequences to be expanders.

Problem 2. *Does there exist a bounded geometry metric space that embeds into* ℓ_p *for some* $2 < p < \infty$ *but not into a Hilbert space?*

The problem dates back to the work of Johnson and Randrianarivony [JR], which proves that ℓ_p does not coarsely embed into the Hilbert space for $p > 2$. Note that there exists an example of a space that does not embed into a Hilbert space and for which it is unknown whether it embeds into ℓ^p for all p (see the extended abstract by Thiebout Delabie above).

Problem 3 (Baudier). *Let $2 < p < \infty$. Does $L_p([0,1])$ embed coarsely into ℓ_p?*

From the perspective of locally finite metric spaces (e.g. discrete bounded geometry spaces), the two spaces are indistinguishable: a locally finite subset of L_p embeds in a bi-Lipschitz way in ℓ_p by a work of Baudier [Bau]. Note also that for $1 \leq p, q \leq 2$ there always is a coarse embedding $L_p([0,1]) \hookrightarrow \ell_q$, which can be obtained by composing the classical embedding $L_p([0,1]) \hookrightarrow L_2([0,1])$ of [BDCK] with Nowak's embedding $L_2([0,1]) \hookrightarrow \ell_q$ [Now06]. Further details can be found in [AB], where the above problem is stated as Question 6.5.

Problem 4 (Nowak). *Is there a free and measure-preserving action of a free group on a manifold such that the associated warped cone admits a wall structure?*

Note that such a warped cone would not satisfy property A. The first examples of locally finite metric spaces without property A yet coarsely embeddable into the Hilbert space were found by Nowak [Now07] and the first examples among box spaces by Arzhantseva, Guentner, and Špakula [AGŠ]. Works of Khukhro [Khu12, Khu14] provide many further examples as well as permanence properties for groups admitting such box spaces.

By considering warped cones over the respective completions, one can construct coarsely embeddable warped cones without property A [Saw], however the case of actions on manifolds remains open. The construction of [Saw] relates to Sawicki's question whether there is a 'slowly-growing' box space $\square_{(N_n)}G$ as above, that is, whether one can require $[N_n : N_{n+1}]$ to be bounded.

Problem 5 (Mimura). *For a prime power q, let $\mathbb{F}_q$ be the finite field of order q. Let $(p_m)_{m\in\mathbb{N}}$ be a sequence of primes. Let $(n_m)_{m\in\mathbb{N}}$ be a sequence of integers at least 3. Assume that either $\lim_{m\to\infty} p_m = \infty$ or $\lim_{m\to\infty} n_m = \infty$ holds true. Fix such $(p_m)_m$ and $(n_m)_m$.*

Can the sequence of finite groups $(\mathrm{SL}(n_m, \mathbb{F}_{p_m}))_m$ form a family of super-expanders? More precisely, does there exist a system $(S_m)_m$ of generating sets of a fixed cardinality for $(\mathrm{SL}(n_m, \mathbb{F}_{p_m}))_m$ such that the sequence of the resulting Cayley graphs $((\mathrm{Cay}(\mathrm{SL}(n_m, \mathbb{F}_{p_m}), S_m))_{m\in\mathbb{N}}$ is a super-expander family?

The condition of $[\lim_{m\to\infty} p_m = \infty$ or $\lim_{m\to\infty} n_m = \infty]$ is imposed in order to ensure $\lim_{m\to\infty} |\mathrm{SL}(n_m, \mathbb{F}_{p_m})| = \infty$.

This problem consists of two cases differing by the boudedness of (n_m): one may be called "bounded rank case" and the other "unbounded rank case." (Recall that the local rank of $\mathrm{SL}(n,k)$ is $n-1$ for a local field k.) The "bounded rank case" will be resolved in the affirmative if $\mathrm{SL}(3,\mathbb{Z})$ has property $(\mathrm{T}_{\mathcal{B}_{\mathrm{sr}}})$. Here $\mathcal{B}_{\mathrm{sr}}$ is the class of all super-reflexive Banach spaces.

We remark that for "bounded rank case", if we consider $(q_m)_m$ a sequence of prime powers, instead of the sequence $(p_m)_m$ of primes, then there exists $(q_m)_m$ such that the problem is settled in the affirmative. Indeed, fix a prime p and let $q_m = p^{r_m}$, $r_m \in \mathbb{N}_{\geq 1}$, such that $\lim_{m\to\infty} r_m = \infty$. For simplicity, we discuss the case where $n_m = 3$ for all m. The key here is that every group of the form $\mathrm{SL}(3, \mathbb{F}_{p^{r_m}})$ is a group quotient of $\mathrm{SL}(3, \mathbb{F}_p[t])$, where $\mathbb{F}_p[t]$ denotes the polynomial ring over $\mathbb{F}_p$ with indeterminate t. To see this, employ a basic result in elementary number theory that the multiplicative group $\mathbb{F}_{p^{r_m}}^{\times}$ is cyclic. Fix a generator $t_{r_m} \in \mathbb{F}_{p^{r_m}}^{\times}$ of it. Then, the map which sends $t \in \mathbb{F}_p[t]$ to $t_{r_m} \in \mathbb{F}_{p^{r_m}}$ induces a surjective group homomorphism from $\mathrm{SL}(3, \mathbb{F}_p[t])$ to $\mathrm{SL}(3, \mathbb{F}_{p^{r_m}})$ (to see that the group homomorphism above is surjective, observe that both of the special linear groups above coincide with the elementary groups respectively over the same rings). The celebrated work of V. Lafforgue [Laf] implies that $\mathrm{SL}(3, \mathbb{F}_p[t])$ has property $(\mathrm{T}_{\mathcal{B}_{\mathrm{sr}}})$. Therefore, a Margulis-type argument indicates a way to find a system of generators of $(\mathrm{SL}(3, \mathbb{F}_{p^{r_m}}))_m$ to obtain a super-expander family.

Problem 6 (Naor)**.** *Give a description of the unique discrete Hilbert space.*

Any two nets in an infinite-dimensional Banach space are bi-Lipschitz equivalent by [LMP], and if a net in a Banach space X is bi-Lipschitz equivalent to a net in ℓ^2, then X is linearly isomorphic to ℓ^2 by Theorem 10.21 of [BL]. There is thus a unique discrete Hilbert space in the bi-Lipschitz category. Note that in finite dimensions, nets are not unique ([BK], [McM]).

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