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Geometrie

Organised by
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Abstract. The workshop *Geometrie*, organized by John Lott (Berkeley), André Neves (London), Iskander Taimanov (Novosibirsk) and Burkhard Wilking (Münster) was well attended with over 53 participants with broad geographic representation from all continents, and held in a very active atmosphere. During the meeting, various interesting topics in geometry were discussed, such as geometric flows, Kähler geometry, manifolds with nonnegative or positive curvature, and minimal surfaces.

Mathematics Subject Classification (2010): 53–xx.

Introduction by the Organisers

The workshop consisted of 17 one hour talks, 2 half hour talks (Friday) and 4 half hour after dinner talks (Monday–Thursday). The after dinner talks were given by PhD students and very recent PhD's. All the speakers did an excellent job, which were the main contributions to the good atmosphere at the workshop.

Among all the talks, seven were related to geometric flows. Gerhard Huisken introduced a fully non-linear flow for 2-convex hypersurfaces in a Riemannian manifold, and extended the surgery algorithm of Huisken–Sinestrari to this fully nonlinear flow by employing an induction-on-scales argument, which relies on a combination of several ingredients, including the almost convexity estimate, the inscribed radius estimate, as well as a regularity result for radial graphs. Lu Wang studied the space of asymptotically conical self-expanders of mean curvature flow. For instance, she proved that the space of asymptotically conical self-expanders of a fixed diffeomorphism type has a smooth Banach manifold structure. Anusha M.

Krishnan talked on Ricci flow on cohomogeneity one manifolds and exhibited the first examples of compact 4-manifolds with metrics of nonnegative sectional curvature which lose this property when evolved by the Ricci flow. Robert Haslhofer proved two conjectures on intrinsic diameter bound and sharp curvature estimate for mean curvature flow of two-convex closed embedded hypersurfaces. Jason D. Lotay answered a question of Joyce and Neves by proving that the Clifford torus is unstable for Lagrangian mean curvature flow under arbitrarily small Hamiltonian perturbations. On the other hand, he showed that the Clifford torus is locally unique as a self-shrinker for mean curvature flow. Franziska Beitz generalized Hamilton's maximum principle for Ricci flow by weakening the notion of convexity to Bianchi-convexity, and obtained several applications. Alix Deruelle focused on the uniqueness question for expanding solution of the harmonic map flow and introduced a relative entropy.

Olivier Biquard, Tamás Darvas and Jian Song talked on Kähler geometry. Olivier Biquard explained how to construct complete Kähler Ricci flat metrics on complex symmetric spaces, at least in certain rank 2 cases. Tamás Darvas reported the result on solving complex Monge-Ampère equations with added constraint on the singularity type of the solutions, and, as the main application, resolved the log-concavity conjecture of Boucksom-Eyssidieux-Guedj-Zeriahi related to the intersection number of positive currents. Jian Song studied the compactness of the moduli space of Kähler-Einstein manifolds of negative scalar curvature.

Martin Kerin, Anand Dessai and Lee Kennard talked on manifolds with nonnegative or positive curvature. Until now, all known examples of 2-connected 7-manifolds admitting a metric of non-negative curvature have been at least homeomorphic to an S^3-bundle over S^4. However, Martin Kerin showed there exist infinitely many non-negatively curved, mutually homotopy inequivalent, 2-connected 7-manifolds which are not even homotopy equivalent to an S^3-bundle over S^4. Anand Dessai proved the moduli space of metrics with nonnegative sectional curvature on a closed 5-manifold homotopy equivalent to the real projective space has infinitely many path components. Lee Kennard proved the Hopf conjecture under weak symmetry condition, namely, an even-dimensional compact manifold admitting a Riemannian metric with positive sectional curvature has positive Euler characteristic under the additional assumption that the isometry group has rank at least five. Under the additional assumption that the odd Betti numbers vanish, which would follow if the Bott-Grove-Halperin ellipticity conjecture holds, he recovered the rational cohomology ring of the manifold if the isometry group has rank at least ten. As an application, he proved that no compact, simply connected even dimensional Riemannian symmetric space of rank greater than one admits a positively curved metric with isometry group containing a ten-dimensional torus, except possibly when the space is the Grassmannian of oriented two-planes.

For spaces with upper curvature bounds, Alexander Lytchak obtained a weak analog of Perelman's stability theorem in the theory of Alexandrov spaces, and answered a folklore question in the field about the infinitesimal characterization of topological manifolds. Artem Nepechiy talked on the construction of canonical convex functions in Alexandrov spaces, and answered a question of Anton Petrunin. Nina Lebedeva introduced a new type of metric comparison, which is closely related to the continuity of optimal transport between regular measures. Anton Petrunin explained two results on generalized saddle maps, where the first gives a partial answer to the conjecture of Samuil Shefel, stating that a saddle disc equipped with the intrinsic metric is CAT(0), and the second is an analog of the Schoen–Yau univalentness theorem for harmonic maps.

There were four talks related to the theory of minimal surfaces. Antoine Song talked on the equidistribution of minimal hypersurfaces for generic metrics on closed manifolds of dimension $n+1$ ($2 \leq n \leq 6$). Henrik Matthiesen studied the systole of large genus minimal surfaces in a three-manifold with positive Ricci curvature. Christos Mantoulidis considered solutions to the Allen–Cahn equation on 3-manifolds with uniform Allen–Cahn functional bounds and uniform Morse index bounds, and resolved a strong form of the "multiplicity one" conjecture of Marques–Neves for Allen–Cahn. For all ambient dimensions, he provided a resolution of the "index lower bound" conjecture of Marques–Neves. As an important consequence, he resolved a conjecture due to Yau in the case of bumpy metrics. Marco Méndez Guaraco surveyed results that explore the analogy between the phase transition theory of the Allen-Cahn equation and the Almgren-Pitts theory of minimal hypersurfaces, and particularly talked on his recent result.

The remaining two talks were given by Ailana Fraser and Carla Cederbaum. Ailana Fraser considered the Steklov eigenvalue problem on high dimensional manifolds with boundary, and showed that some of the refined results that are true for surfaces do not hold in general. Carla Cederbaum talked on CMC foliations of asymptotically flat manifolds, and presented a new foliation by constant spacetime mean curvature surfaces (STCMC), which remedies some deficiencies of the center of mass notion suggested by Huisken and Yau.

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Workshop: Geometrie

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Abstracts

Kähler Ricci flat metrics on rank 2 complex symmetric spaces

Olivier Biquard
(joint work with Thibault Delcroix)

Let G/H be a (real) symmetric space of compact type, and dimension n. Its complexification $M = G^{\mathbb{C}}/H^{\mathbb{C}}$ is a complex symmetric space, which can also be identified topologically with $T^*(G/H)$. Such a complex symmetric space has a canonical homogeneous holomorphic n-form Ω, so it is natural to search complete Kähler metrics ω on M whose volume form satisfies

$$\omega^n = i^{n^2} \Omega \wedge \bar{\Omega}, \tag{1}$$

which implies in particular that they are Ricci flat; it is also natural to require the metric to be invariant under the compact group G of automorphisms of M.

Known examples include:

- Stenzel's metrics on rank one complex symmetric spaces [10];
- Biquard-Gauduchon's explicit hyperKähler metrics on the complexification of Hermitian symmetric spaces [1].

Of course the second family includes the famous Eguchi-Hanson metric on T^*S^2 (or, more appropriately in our context, on the complexification of S^2, that is a complex quadric in $\mathbb{C}^3$).

On the other hand, Tian and Yau [12, 13] developed a general abstract method to solve a Monge-Ampère equation like (1) on the complement of a smooth divisor supporting the anticanonical divisor in a Fano manifold (or more generally orbifold). Since this foundational work several mathematicians gave more precise versions in specific settings, in particular recently Conlon and Hein gave complete answers in the Asymptotically Conical (AC) case [2, 3], in the case of smooth cones at infinity. Even more recently, a number of authors provided examples of constructions of complete Kähler Ricci flat metrics, which are asymptotically conical but with singular cone at infinity [4, 9, 11]. In particular, for $n \geq 3$ there are Kähler Ricci flat metrics on $\mathbb{C}^n$ which are asymptotically conical, with a singular cone at infinity rather than the standard Euclidean cone.

In this family of AC Kähler Ricci flat metrics with non smooth cone at infinity, we also have the Biquard-Gauduchon's examples, where the cone at infinity is actually a nilpotent coadjoint orbit of $G^{\mathbb{C}}$ with a conical hyperKähler metric.

In this work, we prove the existence of complete Kähler Ricci flat metrics on the complex symmetric spaces $M = G^{\mathbb{C}}/H^{\mathbb{C}}$, at least in certain rank 2 cases. Actually, as we will explain below, on a rank 2 complex symmetric space one can hope to construct with our methods two different Kähler Ricci flat AC metrics. The results are the following:

Theorem.

(1) The following complex symmetric spaces of rank 2 have at least two complete Kähler Ricci flat AC metrics (with non smooth cone at infinity), one is a hyperKähler Biquard-Gauduchon metric, and the other one is not hyperKähler:

$$SO(n)/S(O(2) \times O(n-2)), \quad SO(8)/GL(4), \quad SL(5)/S(GL(2) \times GL(3)).$$

(2) The following complex symmetric spaces of rank 2 have at least one complete Kähler Ricci flat AC metric:

$$SO(5) \times SO(5)/SO(5), \quad Sp(8)/Sp(4) \times Sp(4), \quad G_2/SO(4), \quad G_2 \times G_2/G_2.$$

(3) In a certain sense which will be made precise later, the second expected Kähler Ricci flat AC metric on $G_2/SO(4)$ and $G_2 \times G_2/G_2$ does not exist.

There remains a number of cases not covered by the theorem, for example the simplest rank 2 symmetric space $SL(3)/SO(3)$. At the moment, we are not able to prove the theorem in these cases, but we expect the general result to be the existence of two different Kähler Ricci flat AC metric on any rank 2 complex symmetric space, with the only exception of $G_2/SO(4)$ and $G_2 \times G_2/G_2$ which carry only one such metric.

More generally, on a rank r complex symmetric space, we expect up to r different Kähler Ricci flat metrics.

Idea of the construction. Let $M = G^{\mathbb{C}}/H^{\mathbb{C}}$ be a rank r complex symmetric space. Then De Concini and Procesi [5] constructed a wonderful compactification $\bar{M}$ of M, which is a smooth $G^{\mathbb{C}}$-equivariant complex compact manifold such that

$$\bar{M} \setminus M = \cup_1^r D_i$$

is a simple normal crossing divisor; the orbit closures of $G^{\mathbb{C}}$ in $\bar{M}$ are precisely the partial intersections $\cap_{j \in J} D_j$ for all subsets $J \subset \{1, \ldots, r\}$.

Restricting to the rank 2 case, each divisor D_i is a fibration

$$X_i \longrightarrow D_i \setminus (D_1 \cap D_2) \longrightarrow G^{\mathbb{C}}/P_i,$$

where the P_i are the maximal parabolic subgroups of $G^{\mathbb{C}}$ and the X_i are rank one complex symmetric spaces, and

$$D_1 \cap D_2 = G^{\mathbb{C}}/P_{\min},$$

where $P_{\min}$ is a minimal parabolic (a Borel subgroup) in $G^{\mathbb{C}}$. The space $\bar{M}$ is actually the complexification of the Furstenberg-Satake compactification of the noncompact dual symmetric space of G/H.

For the construction, we make a choice of indexing of the divisors (D_1, D_2): this is why the other indexing will provide another, different, Kähler Ricci flat metric on M, except in the case when M has an automorphism exchanging D_1 and D_2 (for example for $SL(3)/SO(3)$).

It turns out that there is a singular Fano manifold $\hat{D}_2$ with a desingularization $D_2 \to \hat{D}_2$ which is an isomorphism over $D_2 \setminus (D_1 \cap D_2)$. Roughly speaking one can describe the steps of the construction in the following way:

(1) Construct a (singular) Kähler-Einstein metric on $\hat{D}_2$.
(2) Use the Tian-Yau ansatz to construct an approximate solution of (1) on M near $D_2 \setminus (D_1 \cap D_2)$; this is an AC metric with singular cone at infinity (a line bundle over $\hat{D}_2$).
(3) Use a new ansatz near D_1 to find a model metric which will desingularize the singularity of the cone, and glue it to the previous metric to get a global asymptotic solution on M near $D_1 \cup D_2$.
(4) Solve the Monge-Ampère equation (1) keeping the same asymptotic of the metric.

One of our main technical tools in this work is the extension of the toric formalism to the study of "horosymmetric manifolds" by the second author [7]. This enables to reduce the complex Monge-Ampère equation (1) to a real Monge-Ampère equation in two variables, which can be written in terms of the root theory of the symmetric space M. The various ansatz that we use can be written explicitly in these terms.

For the first step, there is no general existence theorem for Kähler-Einstein metrics on singular Fano manifolds. In our horosymmetric formalism, this is a second order ODE. Nevertheless we use a continuity method approach to solve the equation, in which the main difficulty is the C^0-estimate which is familiar in Monge-Ampère problems. Here there is an obstruction which is similar to that found in [6], and generalizes the well-known obstruction for Kähler-Einstein metrics on Fano manifolds in terms of barycenters of the Delzant polytope. In our problem, this obstruction turns out to cancel except when the restricted root system if of type G_2. This explains the non existence part of the Theorem for $G_2/SO(4)$ and $G_2 \times G_2/G_2$.

For the resolution of the Monge-Ampère equation once we have an asymptotic model, we rely on general results like the version in [8] of the Tian-Yau theorem. Here a new difficulty appears: in some cases, our models have injectivity radius going to zero and unbounded holomorphic bisectional curvatures when one goes to infinity in M. Then the usual techniques to solve the Monge-Ampère equation can not be applied: in particular, there is no known method for the C^2-estimate without a lower or an upper bound on the holomorphic bisectional curvatures. This phenomenon happens depending on the combinatorics of the root system of the symmetric space, and we have to exclude these cases in our Theorem. This explains why we can construct the Kähler Ricci flat metrics only in the cases listed in the Theorem.

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A fully non-linear flow with surgery for 2-convex hypersurfaces in a Riemannian manifold

Gerhard Huisken
(joint work with Simon Brendle)

We consider a one-parameter family of closed, embedded hypersurfaces moving with normal velocity $G_\kappa = (\sum_{i<j} \frac{1}{\lambda_i+\lambda_j-2\kappa})^{-1}$, where $\lambda_1 \leq ... \leq \lambda_n$ denote the curvature eigenvalues and κ is a nonnegative constant. This defines a fully nonlinear parabolic equation, provided that $\lambda_1+\lambda_2 > 2\kappa$. In contrast to mean curvature flow, this flow preserves the condition $\lambda_1 + \lambda_2 > 2\kappa$ in a general ambient manifold. Our main goal in this paper is to extend the surgery algorithm of Huisken–Sinestrari to this fully nonlinear flow. This is the first construction of this kind for a fully nonlinear flow. As a corollary, we show that a compact Riemannian manifold satisfying $\overline{R}_{1313} + \overline{R}_{2323} \geq -2\kappa^2$ with non-empty boundary satisfying $\lambda_1 + \lambda_2 > 2\kappa$ is diffeomorphic to a 1–handlebody. The main technical advance is the pointwise curvature derivative estimate. The proof of this estimate requires a new argument, as the existing techniques for mean curvature flow due to Huisken–Sinestrari, Haslhofer–Kleiner, and Brian White cannot be generalized to the fully nonlinear setting. To establish this estimate, we employ an induction-on-scales argument; this relies on a combination of several ingredients, including the almost convexity estimate, the inscribed radius estimate, as well as a regularity result for radial graphs.We expect that this technique will be useful in other situations as well.

Non-negative curvature and the linking form

Martin Kerin
(joint work with Sebastian Goette, Krishnan Shankar)

Closed Riemannian manifolds with non-negative sectional curvature are little understood. Few constructions are known and, to this day, almost all known examples arise by exploiting bi-invariant metrics on compact Lie groups and Riemannian submersions. Nevertheless, many interesting classes of examples have been discovered in this manner and the hope is that, by reducing symmetry assumptions, one may gain some understanding about non-negatively curved manifolds in more generality. This philosophy of symmetry reduction has driven much research activity and can already be seen at work in the passage from classical homogeneous spaces to biquotients, which, for example, led to the discovery of the first exotic sphere known to admit non-negative curvature [2], as well as the revelation that there are infinitely many rational homotopy types of non-negatively curved manifolds in each dimension at least six [6].

Another demonstration of the symmetry-reduction philosophy in action can be found in the groundbreaking work on manifolds of cohomogeneity one by Grove and Ziller [3]. They studied manifolds M admitting an isometric action by a Lie group G such that the orbit space M/G is diffeomorphic to a closed interval (as opposed to a point in the case of homogeneous spaces). The orbits corresponding to the end points of M/G are called the singular orbits and it was shown in [3] that M admits a G-invariant metric of non-negative curvature whenever the singular orbits are of codimension 2 in M. By demonstrating that all principal $(S^3 \times S^3)$-bundles over S^4 admit a cohomogeneity-one structure with codimension-2 singular orbits, Grove and Ziller were then able to conclude that all S^3-bundles over S^4 admit a metric of non-negative curvature. In particular, all Milnor exotic 7-spheres admit such a metric. As the Milnor spheres achieve only 11 of the 15 possible unoriented diffeomorphism types of homotopy 7-spheres, it remained unknown whether all exotic 7-spheres admit non-negative curvature.

In around 2007, Wilking made the observation that one can replace the orbits of a cohomogeneity-one action as above with biquotients and, in so doing, obtain families of manifolds admitting a codimension-1 singular Riemannian foliation by biquotients and having few obvious symmetries. In particular, just as for cohomogeneity-1 manifolds, these manifolds can be decomposed as the union of two disk-bundles over the singular leaves of the foliation and a metric of non-negative curvature again exists whenever the singular leaves are of codimension 2. This observation lay unexploited until recently, when in [1] it was used to achieve the following generalisation of the work of Grove and Ziller.

Theorem A. *There exists a six-parameter family $\mathcal{F}$ of 2-connected, non-negatively curved 7-manifolds, each of which admits a Seifert fibration with generic fibre S^3. In particular, the family $\mathcal{F}$ contains all (oriented) exotic 7-spheres.*

The members of the family $\mathcal{F}$ are already interesting in view of the work of Totaro [5], wherein it was demonstrated that there are only finitely many diffeomorphism types of 2-connected biquotients in each dimension. Therefore, a generic manifold in $\mathcal{F}$ cannot be a biquotient.

In this case, however, the existence of a Seifert fibration by S^3 is interesting in its own right. Recall that a Seifert fibration of a manifold M is a regular Riemannian foliation with compact leaves. The leaf space B then naturally inherits the structure of a smooth orbifold, while the generic leaves form an open, dense set in M and are each diffeomorphic to some fixed manifold F (in this case S^3). Finally, the exceptional leaves are each finitely covered by F and the projection map $\pi : M \to B$ is endowed with the structure of an orbi-bundle.

Until now, all known examples of 2-connected 7-manifolds admitting a metric of non-negative curvature have been at least homeomorphic to an S^3-bundle over S^4. For example, even though the existence of a Seifert S^3-fibration of the 4 unoriented non-Milnor exotic 7-spheres is new, these manifolds are obviously still homeomorphic to an S^3-bundle over S^4. This begs the natural question of whether there exist 2-connected 7-manifolds which admit non-negative curvature and are not homeomorphic to an S^3-bundle over S^4. In this joint work in progress with Sebastian Goette and Krishnan Shankar, this question is answered in the affirmative, even up to homotopy equivalence, further demonstrating the unexpected richness of the family $\mathcal{F}$, as well as the potential of the construction observed by Wilking a decade ago.

Theorem B. *Within the family $\mathcal{F}$ there exist infinitely many non-negatively curved, mutually homotopy inequivalent, 2-connected 7-manifolds which are not even homotopy equivalent to an S^3-bundle over S^4.*

The key to obtaining this result is the computation of the linking form for those manifolds in $\mathcal{F}$ which are rational 7-spheres. Indeed, in [4] Kitchloo and Shankar showed that a 2-connected rational 7-sphere is homotopy equivalent to an S^3-bundle over S^4 if and only if its linking form is equivalent to a standard linking form up to sign. In other words, a rational 7-sphere $M \in \mathcal{F}$ is homotopy equivalent to an S^3-bundle over S^4 if and only if there is some generator $\mathbf{1}$ of the (cyclic) torsion group $H^4(M;\mathbb{Z})$ of order $n = n(M)$ such that

$$\pm \ell k : H^4(M;\mathbb{Z}) \times H^4(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}\,;\ (a \cdot \mathbf{1}, b \cdot \mathbf{1}) \mapsto \tfrac{ab}{n} \mod 1\,.$$

The linking form is, in general, difficult to compute. In the case of rational 7-spheres, one might hope to exploit the fact that it may be rewritten in terms of the Bockstein homomorphism arising from the short exact coordinate sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. By taking advantage of the decomposition of the manifolds in $\mathcal{F}$ as the union of two disk-bundles, it turns out to be possible in this setting to compute the Bockstein homomorphism explicitly, leading to a closed formula for the linking form. With this formula in hand, it is now a non-trivial exercise involving elementary number theory and quadratic reciprocities to obtain infinitely many homotopy types of manifolds in $\mathcal{F}$ which are not even homotopy equivalent to an S^3-bundle over S^4.

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Spaces with upper curvature bounds

Alexander Lytchak

(joint work with Koichi Nagano)

In the talk I have discussed the following results obtained jointly with Koichi Nagano (Tsukuba).

Theorem 1 *Let M_i be a sequence of smooth Riemannian manifolds with uniform upper bounds on dimension, volume and sectional curvature and uniform lower bound on the injectivity radius. If M_i converge in the Gromov–Hausdorff topology to a metric space X then X is topological manifold homeomorphic to M_i for all i, large enough.*

The result is a weak analog of Perelman's stability theorem in the theory of Alexandrov spaces. Our stability theorem follows from the so called α-homotopy theorem in geometric topology, once one can verify that the limit space is a topological manifold. The verification of this statement is based on the following result, which answers a folklore question in the field about the infinitesimal characterization of topological manifolds:

Theorem 2 *Let X be a locally compact space with an upper curvature bound in the sense of Alexandrov. Then the following are equivalent:*

(1) X is an n-dimensional topological manifold.
(2) Every tangent space T_xX in X is homeomorphic to $\mathbb{R}^n$.
(3) Every space of directions Σ_xX in X is homotopy equivalent to $\mathbb{S}^{n-1}$.

The double suspension of the Poincaré sphere, which is a topological sphere, by a theorem of Cannon and Edwards, is an example showing that in (3) above the term “homotopy equivalent” cannot be changed to “homeomorphic”.

Towards canonical convex functions in Alexandrov spaces

Artem Nepechiy

Alexandrov spaces are complete, intrinsic metric spaces with a synthetic lower curvature bound in the sense of Toponogov. Such spaces with finite Hausdorff dimension turn out to be geodesic (i.e for every two points there exists a shortest path connecting them) and having curvature $\geq \kappa$ is equivalent to the following statement: Distance functions are more concave than distance functions in the comparison space of constant sectional curvature κ.

One would like to speak about concavity and convexity of functions even though these might not be differentiable. For this reason one introduces the following definition.

Definition (λ-concavity, λ-convexity)**.** *Denote by A an n-dimensional Alexandrov space without boundary and by $\Omega \subset A$ an open set. A locally Lipschitz function*

$$f : \Omega \to \mathbb{R}$$

is called λ-concave (λ-convex) on Ω, if for all $x, y \in \Omega$ and each unit-speed shortest path γ lying in Ω and connecting x and y the function

$$f \circ \gamma(t) - \frac{\lambda}{2} t^2$$

is concave (convex) on its domain of definition.

If A is an n-dimensional Alexandrov space with boundary and $\Omega \subset A$ is open, then a locally Lipschitz function $f : \Omega \to \mathbb{R}$ is called λ-concave (λ-convex) on Ω if $p \circ f$ is λ-concave (λ-convex) on $p^{-1}(\Omega)$, where $p : A \amalg_{\partial A} A \to A$ denotes the canonical projection. Notice that $A \amalg_{\partial A} A$ is an Alexandrov space without boundary.

Denote by A an n-dimensional Alexandrov space of curvature ≥ 0. By the above for every $p \in A$ the function dist_p^2 is 2-concave (for arbitrary curvature bounds $\text{dist}_p^2 : B_r(p) \to \mathbb{R}$ is still $(2 + O(r^2))$-concave). However, there are examples, where dist_p^2 is not λ-convex for any λ in any neighborhood around p.

The aim of this talk is to present the results obtained in [1], where a map is constructed, which approximates dist_p^2 up to second order and has convexity properties as in the Euclidean space. In particular, the following theorem is proven

Theorem 1. *Let A be a finite-dimensional Alexandrov space and $p \in A$ a point. Then there exist $r > 0$ and a locally Lipschitz 2-convex function $f : B_r(p) \to \mathbb{R}$ satisfying*

$$\lim_{x \to p} \frac{f(x) - \text{dist}_p^2(x)}{\text{dist}_p^2(x)} = 0.$$

The starting point in proving theorem 1 is the following result:

Theorem 2. *Let A be a finite-dimensional Alexandrov space and $p \in A$ a point. Then for any $\varepsilon > 0$ there exist an $r > 0$ and a map $f_\varepsilon : B_r(p) \to \mathbb{R}$ satisfying the following conditions:*

(1) The function f_ε is $(-2+\varepsilon)$-concave and Lipschitz continuous on $B_r(p)$.
(2) The function f_ε has an isolated maximum at p and satisfies $f_\varepsilon(p) = 0$.
(3) For all $x \in B_r(p)$ one has $f_\varepsilon(x) \geq -\operatorname{dist}_p^2(x)$.

In particular these properties imply: For all $x \in B_r(p)$ one has

$$\|f_\varepsilon(x) - (-\operatorname{dist}_p^2(x))\| \leq \varepsilon \cdot |px|^2,$$

where $|px|$ denotes the distance from x to p.

Although theorem 2 looks like a corollary of theorem 1, it is the other way round. By slightly strengthening an argument in the proof of theorem 2 one obtains theorem 1. Moreover, the result above gives an affirmative answer to question 7.3.6 in [2].

Question. *Is it true that for any $p \in A$ and any $\varepsilon > 0$, there is a $(-2+\varepsilon)$-concave function f_p defined in a neighborhood of p, such that $f_p(p) = 0$ and $f_p \geq -\operatorname{dist}_p^2$?*

In [3] Perelman constructed functions satisfying condition 1 in theorem 2, which were stable under Gromov-Hausdorff convergence. The latter means that the functions can be lifted to Gromov-Hausdorff close spaces of the same dimension without loosing their concavity properties. In [4] Kapovitch refined this argument and was additionally able to ensure condition 2 in theorem 2. However, the last property could not been obtained using the methods mentioned above.

As a final remark the functions in theorem 1 and theorem 2 are also stable under Gromov-Hausdorff convergence in the sense stated above.

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The space of asymptotically conical self-expanders of mean curvature flow

Lu Wang
(joint work with Jacob Bernstein)

A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is a *self-expander* if it satisfies

$$\mathbf{H}_\Sigma - \frac{\mathbf{x}^\perp}{2} = \mathbf{0}.$$

Here

$$\mathbf{H}_\Sigma = \Delta_\Sigma \mathbf{x} = -H_\Sigma \mathbf{n}_\Sigma = -(\operatorname{div}_\Sigma \mathbf{n}_\Sigma)\mathbf{n}_\Sigma,$$

$\mathbf{n}_\Sigma$ is a choice of unit normal of Σ and $\mathbf{x}^\perp$ is the normal part of position vector $\mathbf{x}$.

A mean curvature flow is a one-parameter family of hypersurfaces, $\Sigma_t \subset \mathbb{R}^{n+1}$ that satisfy

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)^{\perp} = \mathbf{H}_{\Sigma_t}.$$

Self-expanders are a special class of solutions to the flow, in which a later time slice is a scale-up copy of an earlier one. That is,

$$\left\{\sqrt{t}\,\Sigma\right\}_{t>0}$$

is a mean curvature flow. Self-expanders are expected to model behaviors of a mean curvature flow as it emerges from a conical singularity. They are also expected to model the long time behavior of the flow.

In recent work [1] and [3], Bernstein and myself show:

Theorem 1. *The following is true:*

(1) The space of asymptotically conical self-expanders of a fixed diffeomorphism type has a smooth Banach manifold structure.
(2) The natural projection Π that maps any asymptotically conical self-expander to its link of the asymptotic cone is a smooth map.
(3) If Π is proper, then it has a well-defined integer degree.

We also show, in [3], that for several natural classes of self-expanders, the projection Π is indeed proper. Our theorem may be thought of as an extension of work of White for compact minimal surfaces to a non-compact and weighted case.

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Complex Monge–Ampère equations with prescribed singularity

Tamás Darvas
(joint work with E. Di Nezza, C.H. Lu)

Suppose (X, ω) is a compact connected Kähler manifold of complex dimension n. Let θ be a smooth $(1,1)$-form on X such that $\{\theta\}$ represents a big cohomology class. By $\mathrm{PSH}(X, \omega)$ we denote the space of all θ-psh functions, i.e., upper semi-continuous potentials u on X such that $\theta + i\partial\bar{\partial}u \geq 0$ in the sense of currents. We say that two potentials $u, v \in \mathrm{PSH}(X, \theta)$ have the same singularity class iff there exists a constant $C > 0$ such that $u - C \leq v \leq u + C$. This relation induces an equivalence class on $\mathrm{PSH}(X,\theta)$ whose equivalence classes $[u]$ are called *singularity types.*

The purpose of this note is to report the main results of ongoing work with E. Di Nezza and C.H. Lu, that aims to solve complex Monge-Ampère equations with added constraint on the singularity type of the solutions [DDL2, DDL3]: given $\phi \in \mathrm{PSH}(X,\theta)$, let $f \in L^p(X,\omega^n),\ p > 1$ with $f \geq 0$. We would like to solve the following system for $u \in \mathrm{PSH}(X,\theta)$:

$$\begin{cases} (\theta + i\partial\bar\partial u)^n = f\omega^n, \\ [u] = [\phi], \\ \int_X f\theta^n = \int_X (\theta + i\partial\bar\partial\phi)^n. \end{cases} \tag{1}$$

Here $(\theta + i\partial\bar\partial u)^n$ is the non-pluripolar Radon measure of u, as introduced in [BEGZ10]. When θ is a Kähler form, f is smooth, and ϕ is the zero potential, this system reduces to solving the Calabi-Yau equation, in which case solutions are actually smooth [Yau]. As a first question, one may ask: when is (1) well posed? It is easy to see that for generically chosen ϕ, solutions may not exists. However, we are able to show that (1) is solvable for all $f \in L^p(X,\omega), p > 1$ and $\int_X f\theta^n = \int_X (\theta + i\partial\bar\partial\phi)^n$ if and only if ϕ and $P[\phi]$ have the same singularity type. Here $P[\phi]$ is the envelope of the singularity type $[\phi]$, defined as follows:

$$P[\phi] = \mathrm{usc}\big[\sup\{\psi \in \mathrm{PSH}(X,\theta) \ \text{ s.t. } \ \psi \leq 0 \text{ and } [\psi] = [\phi]\}\big].$$

Summarizing, we state our main existence and uniqueness result:

Theorem 2. *For any $f \in L^p(\omega^n), p > 1$ with $f \geq 0$ and $\int_X f\theta^n = \int_X (\theta + i\partial\bar\partial\phi)^n$ there exists a $u \in \mathrm{PSH}(X,\theta)$, unique up to a constant, solving* (1) *if and only if $[\phi] = [P[\phi]]$.*

The proof of this result requires the development of relative pluripotential theory, as carried out in [DDL1, DDL2], along with qualitative improvements to Kolodziej's L^∞ estimates [DDL3] and supersolution techniques. Analogous results hold for Aubin-Yau type equations as well, given potential applications to Kähler–Einstein metrics.

As the main application of the above theorem, we resolve the log-concavity conjecture of Boucksom-Eyssidieux-Guedj-Zeriahi [BEGZ10] related to the intersection number of positive currents:

Theorem 3. *Let $T_1, ..., T_n$ be closed positive* $(1,1)$*-currents on X. Then*

$$\int_X \langle T_1 \wedge \cdots \wedge T_n\rangle \geq \Big(\int_X \langle T_1^n\rangle\Big)^{\frac{1}{n}} \cdots \Big(\int_X \langle T_n^n\rangle\Big)^{\frac{1}{n}}. \tag{2}$$

In particular, the function $T \mapsto \log\big(\int_X \langle T^n\rangle\big)$ on the set of all positive currents is concave.

In connection with the above theorem, a number of partial results have been obtained in the past. When $T_1, \ldots, T_n$ are smooth this result is due to Demailly [De93]. As pointed out in [BEGZ10, Page 223], in case the potentials of $T_1, \ldots, T_n$

have analytic singularity type, after passing to a log-resolution, the above result reduces to the nef version of an inequality of Khovanski-Teissier (see [De93, Proposition 5.2]). In addition to this, in [BEGZ10, Corollary 2.15] the above result is proved in the special case when $\{T_1\} = \ldots = \{T_n\}$ and $T_1, \ldots, T_n$ have full mass. In [DDL2] we generalized this to the case when $\{T_1\}, \ldots, \{T_n\}$ are possibly different, but $T_1, \ldots, T_n$ have small unbounded locus. Here we finally obtain the general form of the conjecture. What is more, following our method of proof, it is clear that generalizations of Theorem 2 to k-Hessian type equations will pave the way to other types of Khovanskii-Teissier type inequalities (see [La04, Section 1.6]) in the context of big cohomology classes.

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Equidistribution of minimal hypersurfaces for generic metrics

Antoine Song

(joint work with Fernando Codá Marques, André Neves)

In the early 80's, S.-T. Yau [1] conjectured that in any 3-dimensional closed manifold, there should exist infinitely many immersed minimal surfaces. Recently, K. Irie, F. C. Marques and A. Neves [2] settled this conjecture in the generic case by showing that generically, a much stronger property holds: in a closed manifold of dimension $n+1$ $(2 \leq n \leq 6)$ endowed with a generic C^∞ metric in the sense of Baire, the union of embedded closed minimal hypersurfaces is dense. Their proof relies on the Weyl law for the volume spectrum proved by Y. Liokumovich, F. C. Marques and A. Neves [3]. In a joint work with F. C. Marques and A. Neves [4], we are able to quantify this result in the following way. In a closed manifold M of dimension $n+1$ $(2 \leq n \leq 6)$ endowed with a generic C^∞ metric, there exists a sequence $\{\Sigma_j\}_{j\in\mathbb{N}}$ of closed, smooth, embedded, connected minimal hypersurfaces that is equidistributed in M: for any $f \in C^\infty(M)$ we have

$$\lim_{q\to\infty} \frac{1}{\sum_{j=1}^{q} \mathrm{vol}_g(\Sigma_j)} \sum_{j=1}^{q} \int_{\Sigma_j} f \, d\Sigma_j = \frac{1}{\mathrm{vol}_g M} \int_M f dM.$$

Even more, for any symmetric $(0,2)$-tensor h on M we have:

$$\lim_{q\to\infty} \frac{1}{\sum_{j=1}^{q} \mathrm{vol}_g(\Sigma_j)} \sum_{j=1}^{q} \int_{\Sigma_j} \mathrm{Tr}_{\Sigma_j}(h)\, d\Sigma_j = \frac{1}{\mathrm{vol}_g M} \int_M \frac{n \mathrm{Tr}_M h}{n+1} dM.$$

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Compactness of Kähler-Einstein manifolds of $c_1 < 0$

Jian Song

Let $\nu(n,\kappa) = \{(M,g) \mid (M,g)$ is Kähler, $dim_{\mathbb{C}} M = n, Ric(g) = -g, Vol(M,g) \le \kappa\}$. We discuss the compactness for $\nu(n,\kappa)$. Any sequence in $\nu(n,\kappa)$ converges (after passing to a subsequence) to a disjoint compact or complete metric spaces whose regular part is an n–$dim_{\mathbb{C}}$ open Kähler-Einstein manifold and each component is a quasi-projective variety.

The systole of large genus minimal surfaces in positive Ricci curvature

Henrik Matthiesen

(joint work with Anna Siffert)

We fix an ambient three-manifold M and consider the space

$$\mathcal{M} := \{\Sigma \subset M \text{ closed, embedded minimal surface}\}$$

its natural subspaces, e.g.

$$\mathcal{M}_\gamma := \{\Sigma \in \mathcal{M} \text{ orientable, genus}(\Sigma) = \gamma\}.$$

A by now classical result of Choi and Schoen states that $\mathcal{M}_\gamma$ is compact in the C^k-topology for any $k \ge 2$ if M has positive Ricci curvature, [CS85]. Combined with more recent work by Colding and Minicozzi, it follows that for generic metrics of positive Ricci curvature $\mathcal{M}_\gamma$ is in fact finite. On the other hand, Marques and Neves proved recently that $\mathcal{M}$ is infinite for any metric of positive Ricci curvature, [MN17]. The combination of these results gives motivation to investigate properties of minimal surfaces in M that have very large genus.

Recall that the *systole* of a closed surface Σ is given by

$$\mathrm{sys}(\Sigma) := \inf\{\mathrm{length}(c) \ : \ c\colon S^1 \to \Sigma \ \text{ non-contractible}\}.$$

We can now state our main result in a slightly simplified version.

Theorem 1. *Assume that M is a three-manifold with positive Ricci curvature and consider a sequence $\Sigma_j \subset M$ of closed, embedded minimal surfaces with $-\chi(\Sigma_j) \to \infty$, as $j \to \infty$. Then we have for the systole that*

$$\mathrm{sys}(\Sigma_j) \to 0,$$

as $j \to \infty$.

In fact, it is possible to get some more information on the pinching curve that we find. We would like to point out that Eq. (1) does not hold without any curvature assumption, but we do not know about any counterexample with a metric of positive scalar curvature.

The proof uses Colding–Minicozzi lamination theory, which describes how a sequence of minimal surfaces converges to a limit lamination in the presence of a genus bound, [CM04a, CM04b, CM04c, CM04d, CM08, CM15]. A main step of the proof is to show that a contradicting sequence for Eq. (1) can be dealt with in this framework, which is not obvious since the systole is a global invariant.

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The geometry of an extremal eigenvalue problem on manifolds with boundary

Ailana Fraser

(joint work with Richard Schoen)

One of the fundamental problems in spectral geometry is to determine sharp eigenvalue bounds. Determining sharp eigenvalue bounds is related to finding extremal metrics for the eigenvalue problem. This is a subject with a long history, particularly in the case of the Laplace operator on closed surfaces. The focus of this talk is on an eigenvalue problem on manifolds with boundary, and the main theme of the talk is to show that some of the refined results that are true for surfaces, don't hold in higher dimensions. If we take a compact Riemannian manifold (M, g) with boundary, the most standard eigenvalue problems are the Dirichlet and Neumann eigenvalue problems. But there is another important eigenvalue problem, that turns out to lead to a geometrically interesting variational problem, and that is the Steklov problem:

$$\begin{cases} \Delta_g u &= 0 \quad \text{on } M \\ \frac{\partial u}{\partial \eta} &= \sigma u \quad \text{on } \partial M, \end{cases}$$

where g is a Riemannian metric on M, η is the outward unit normal vector to ∂M, $\sigma \in \mathbb{R}$, and $u \in C^\infty(M)$. Steklov eigenvalues are eigenvalues of the Dirichlet-to-Neumann map, which sends a given smooth function on the boundary of M to the normal derivative of its harmonic extension to the interior. The Dirichlet-to-Neumann map is a nonnegative, self-adjoint operator with discrete spectrum $\sigma_0 = 0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_k \leq \ldots \to \infty$.

For surfaces, there are upper bounds on the Steklov eigenvalues that are independent of the Riemannian metric and depend only on the topology of the surface. If we fix a surface M of genus γ with k boundary components, define

$$\sigma^*(\gamma, k) = \sup_g \sigma_1(g)\, L_g(\partial M)$$

where the supremum is over all smooth metrics on M. By a result of Weinstock [7], $\sigma^*(0,1) = 2\pi$, and the supremum is achieved by the Euclidean disk. In general, there is a coarse upper bound ([3], [6]) $\sigma^*(\gamma, k) \leq \min\{2\pi(\gamma + k),\ 8\pi[(\gamma+3)/2]\}$. For simply-cnnected surfaces, this reduces to Weinstock's estimate and is sharp. In all other cases we expect this to be a non-sharp bound. A basic question is, for other surfaces, assuming we fix the boundary length to be 1, what is the metric that maximizes the first eigenvalue? More specifically: Does such a maximizing metric exist? And if so, what can we say about its geometry?

It turns out that there is a close connection between maximizing metrics, and minimal surfaces in the Euclidean unit ball $\mathbb{B}^n$ that are proper in the ball and that meet the boundary of the ball orthogonally. Such surfaces are referred to as *free boundary minimal surfaces* since they arise variationally as critical points of the area among surfaces in the ball whose boundaries lie on $\partial\mathbb{B}^n$ but are free to vary on $\partial\mathbb{B}^n$. Classical examples include the equatorial plane disk and the critical catenoid, the unique portion of a suitably scaled catenoid which defines a free boundary surface in $\mathbb{B}^3$. Free boundary minimal surfaces Σ in $\mathbb{B}^n$ are characterized by the condition that the coordinate functions are Steklov eigenfunctions with eigenvalue 1; that is, $\Delta x_i = 0$ on Σ and $\frac{\partial x_i}{\partial \eta} = x_i$ on $\partial\Sigma$. Moreover, if we assume that we have a smooth maximizing metric g, then there are independent first eigenfunctions $u_1, \ldots, u_n$ such that the map $u = (u_1, \ldots, u_n)$ defines a proper conformal map from M into $\mathbb{B}^n, n \geq 3$, [4]. The image $\Sigma = u(M)$ is a free boundary minimal surface in $\mathbb{B}^n$, and the maximizing metric can be realized by the induced metric.

The question of existence of a maximizing metric is extremely difficult, and the main result of [4] is:

Theorem 1. *For any $k \geq 1$ there exists a smooth metric g on the surface of genus 0 with k boundary components with the property $\sigma_1(g)L_g(\partial M) = \sigma^*(0,k)$.*

A key step in the proof is to show that if $\sigma^*(0,k-1)$ is achieved, then $\sigma^*(0,k) > \sigma^*(0,k-1)$. Using this, we show that the conformal structure does not degenerate for a maximizing sequence. Finally, we use a canonical regularization procedure to produce a special maximizing sequence.

In the case of the annulus, in [4] we explicitly characterize the maximizing metric as the induced metric on the critical catenoid. In general, while we can't expect to explicitly characterize the maximizing metrics, we show that $\sigma^*(0,k)$ is strictly increasing in k and converges to 4π as $k \to \infty$. For each k, $\sigma^*(0,k)$ is achieved by a free boundary minimal surface Σ_k in $\mathbb{B}^3$, and for k large, Σ_k is approximately a pair of nearby parallel plane disks joined by k boundary bridges.

In higher dimensions, for manifolds of dimension $n \geq 3$, in general there is no upper bound on the first normalized Steklov eigenvalue that is independent of the metric. However, for bounded domains $\Omega \subset \mathbb{R}^n$ there is an upper bound, $\sigma_1(\Omega)|\partial\Omega|^{\frac{1}{n-1}} \leq C(n)$. A result of Brock [1] from 2001 proves a weaker sharp upper bound for arbitrary bounded domains Ω in $\mathbb{R}^n$,

$$\sigma_1(\Omega) \leq \sigma_1(\Omega^*)$$

where Ω^* is a ball of the same *volume* as that of Ω. Equality holds if and only if Ω is a ball. When $n = 2$ this bound is $\sigma_1\sqrt{A} \leq \sqrt{\pi}$. It is implied by the Weinstock [7] bound $\sigma_1 L \leq 2\pi$ by applying the isoperimetric inequality $\sqrt{A} \leq \frac{L}{2\sqrt{\pi}}$. On the other hand, the bound of Brock holds for arbitrary plane domains and domains in higher dimensions. This leads to the question of whether there is an analog of Weinstock's estimate in higher dimensions. Recently, Bucur-Ferone-Nitsch-Trombetti [2] proved such an estimate for bounded *convex* domains $\Omega \subset \mathbb{R}^n$. The Weinstock Theorem suggests that for *contractible* domains in $\mathbb{R}^n$ with fixed boundary volume, the ball might maximize σ_1. We show in [5] that this is not true for $n \geq 3$.

Theorem 2. *For $n \geq 3$ there exist smooth contractible domains $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| = |\partial\mathbb{B}^n|$ but $\sigma_1(\Omega) > \sigma_1(\mathbb{B}^n)$.*

To prove Theorem 2 we first consider the annular domain $\Omega_\epsilon = \mathbb{B}_1 \backslash \mathbb{B}_\epsilon$. We show that the first Steklov eigenvalue is decreased by approximately a positive constant times ϵ^n, and it follows that when ϵ is small the normalized first Steklov eigenvalue $\sigma_1(\Omega_\epsilon)|\partial\Omega_\epsilon|^{\frac{1}{n-1}}$ is strictly larger than that of $\mathbb{B}_1$. For $n \geq 3$, we then show that we can modify the domain Ω_ϵ to make it contractible while changing the normalized first Steklov eigenvalue by an arbitrarily small amount. This is accomplished by adding a thin tube joining the boundary components and showing that the construction can be done keeping the normalized eigenvalue nearly unchanged.

It is straightforward to give an explicit upper bound on $\sigma_1(\Omega)$ for any smooth domain in $\mathbb{R}^n$ in terms of its boundary volume. This leaves open the question of finding the sharp value for this upper bound. Theorem 2 shows that it is strictly larger than its value for a ball.

Open Question 1. *On which domain $\Omega \subset \mathbb{R}^n$ (or in the limit of which sequence of domains) is the supremum of $\sigma_1(\Omega)|\partial\Omega|^{\frac{1}{n-1}}$ realized?*

The surgery construction of the proof of Theorem 2 leads to a more general question about boundary connectedness. Recall that for surfaces we showed that adding boundary components increases the value of σ_1 normalized by boundary length [4]. In contrast, in higher dimensions, we show in [5] that the number of boundary components does not affect the supremum of the normalized first Steklov eigenvalue:

Theorem 3. *Given any compact Riemannian manifold Ω^n with non-empty boundary and $n \geq 3$, and given any $\epsilon > 0$ there exists a smooth subdomain Ω_ϵ of Ω with connected boundary such that*

$$|\Omega| - |\Omega_\epsilon| < \epsilon, \ ||\partial\Omega| - |\partial\Omega_\epsilon|| < \epsilon, \ \textit{and} \ |\sigma_1(\Omega) - \sigma_1(\Omega_\epsilon)| < \epsilon.$$

The idea of the proof is similar to that of Theorem 2; we consider the effect of adding thin tubes connecting boundary components.

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Synthetic property of metric spaces related to continuity of optimal transport

Nina Lebedeva

(joint work with Anton Petrunin, Vladimir Zolotov)

We introduce a new type of metric comparison, which is closely related to the continuity of optimal transport between regular measures.

We say that a metric space X satisfies the (k,l)-*bipolar comparison* if for any $a_0, a_1, \dots, a_k; b_0, b_1,$
$\dots, b_l \in X$ there are points $\bar{a}_0, \bar{a}_1, \dots, \bar{a}_n, \bar{b}_0, \bar{b}_1, \dots, \bar{b}_n$ in the Hilbert space $\mathbb{H}$ such that

$$|\bar{a}_0 - \bar{b}_0|_{\mathbb{H}} = |a_0 - b_0|_X, \quad |\bar{a}_i - \bar{a}_0|_{\mathbb{H}} = |a_i - a_0|_X, \quad |\bar{b}_i - \bar{b}_0|_{\mathbb{H}} = |b_i - b_0|_X$$

for any i, j and

$$|\bar{x} - \bar{y}|_{\mathbb{H}} \geq |x - y|_X$$

for any $x, y \in \{a_0, a_1, \dots, a_k, b_0, b_1, \dots, b_l\}$.

These comparisons are in general more strong then Alexandrov comparison. It turns out that some of these comparisons are closely related to certain properties of Riemannian manifolds, arising in optimal transport theory. These are Ma-Trudinger-Wang (MTW) condition and convex tangent injectivity domain property (CTIL). These two properties are in some sense almost equivalent to the transport continuity property (TCP). We show that a Riemannian manifold satisfies $(4,1)$-bipolar comparison if and only if it is (CTIL) and ($\mathrm{MTW}^{\not\perp}$), where ($\mathrm{MTW}^{\not\perp}$) is some stronger version of (MTW). We conjecture that $(4,1)$-bipolar comparison implies (TCP) or may be even equivalent to (TCP).

The spaces satisfying bipolar comparisons for all k, l include all subspaces of quotients of Hilbert spaces by groups of isometries. The class of such quotients includes all double quotients of compact Lie groups with bi-invariant metrics (by Terng–Thorbergsson-95).

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Minimal surfaces and the Allen–Cahn equation on 3-manifolds

Christos Mantoulidis

(joint work with Otis Chodosh)

Fix (M^3, g) to be a closed Riemannian 3-manifold. The Allen–Cahn equation

$$\varepsilon^2 \Delta_g u = W'(u) \tag{1}$$

is a semilinear PDE which is deeply linked to the theory of minimal hypersurfaces. For instance, it is known that the Allen–Cahn functional

$$E_\varepsilon[u] := \int_M \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) d\mu_g,$$

whose critical points satisfy (1), Γ-converges as $\varepsilon \to 0$ to the perimeter functional [12, 14] and the level sets of E_ε-minimizing solutions to (1) converge as $\varepsilon \to 0$ to area-minimizing boundaries. When u is not E_ε-minimizing, the limit may occur with high multiplicity. Together with Otis Chodosh we studied solutions to (1) on 3-manifolds with uniform E_ε-bounds and uniform Morse index bounds and showed that multiplicity does *not* occur when the metric g is "bumpy," i.e., when no immersed minimal surface carries nontrivial Jacobi fields; bumpy metrics are generic in the sense of Baire category—see White [16]. This resolves a strong form of the "multiplicity one" conjecture of Marques–Neves [10] for Allen–Cahn. Our main theorem is:

Theorem 1 ([1])**.** *Suppose that u_i are critical points of E_{ε_i} with $\varepsilon_i \to 0$ and*

$$E_{\varepsilon_i}[u_i] \le E_0, \ \operatorname{ind}(u_i) \le I_0 \ \textit{for all } i = 1, 2, \ldots$$

Passing to a subsequence, for each $t \in (-1, 1)$, $\{u_i = t\}$ converges in the Hausdorff sense and in $C^{2,\alpha}_{\text{loc}}$ away from $\le I_0$ points to a smooth closed minimal surface Σ. For any connected component $\Sigma' \subset \Sigma$, either:

- *Σ' is two-sided and occurs as a multiplicity one graphical $C^{2,\alpha}$ limit; or,*
- *Σ' is a two-sided stable minimal surface with a positive Jacobi field and which occurs as multiplicity two limit or higher; or*
- *Σ' is one-sided and its two-sided double cover is a stable minimal surface with a positive Jacobi field.*

The case $I_0 = 0$ of Theorem 1 is largely a consequence of Theorem 2 below:

Theorem 2 ([1])**.** *Let u be a stable critical point of E_ε. As $\varepsilon \to 0$ we have*

$$\exp(-\sqrt{2}\varepsilon^{-1} \operatorname{dist}_g(x, x')) = o(\varepsilon^2 |\log \varepsilon|), \tag{2}$$

for any x, x' that belong to different connected components of $\{u = 0\}$, as well as

$$\|H\|_{C^0(\{u=t\})} = o(\varepsilon |\log \varepsilon|), \tag{3}$$

$$\|A\|_{C^{0,\alpha}(\{u=t\})} = O(1) \tag{4}$$

for the mean curvature H and the second fundamental form A of $\{u = t\}$, $|t| \le 1 - \beta$. Here α, $\beta \in (0, 1)$, and the constants depend on α, β, $E_\varepsilon[u]$, M, g.

This theorem is based on sharpening some recent novel work of Wang–Wei [15] that resolved the "finite index implies finitely many ends" Allen–Cahn conjecture without energy bounds in $\mathbf{R}^2$. Their remarkable insight was the reduction of the question of regularity to a question about the distance among the sheets comprising $\{u = 0\}$. We remark that (2)-(3) are stronger than the bounds obtained in [15] (on the other hand, we have dependence on $E_\varepsilon[u]$). Though bounds on the order of those in [15] suffice for obtaining $C^{2,\alpha}$ estimates for *some* $\alpha \in (0, 1)$, the stronger bounds stated in (2)-(3) ensure that the level sets are mean curvature dominated and play a crucial role for our geometric applications.

Our other result, which is proved for all ambient dimensions, is a resolution of the "index lower bound" conjecture of Marques–Neves [10] (for Allen–Cahn):

Theorem 3 ([1]). *Suppose that u_i are critical points of E_{ε_i} with $\varepsilon_i \to 0$ and $\{u_i = t\}$ converging to a smooth two-sided minimal hypersurface Σ with multiplicity one. Then, after possibly passing to a subsequence,*

$$\operatorname{ind}(\Sigma) \leq \lim_i \operatorname{ind}(u_i) \tag{5}$$

$$\leq \lim_i [\operatorname{ind}(u_i) + \operatorname{nul}(u_i)] \leq \operatorname{ind}(\Sigma) + \operatorname{nul}(\Sigma). \tag{6}$$

Only the last inequality, (6), is new to [1]; (5) was previously shown to be true by Gaspar [4] without two-sidedness or multiplicity one assumptions. (See also the works of Hiesmayr [7], Le [13].) To prove (6), one needs to obtain a very precise understanding of how $\{u_i = t\}$ tend to Σ. To do so, we borrow ideas from [2] where the authors studied the index and nullity for solutions of (1) that they constructed; an added difficulty is that in our case the u_i are essentially arbitrary.

Finally we point out an important consequence of this work to the study of minimal surfaces in closed Riemannian 3-manifolds (M^3, g). Recall the following minmax construction of Gaspar–Guaraco (which works for ambient dimensions $3 \leq n \leq 7$):

Theorem 4 (Gaspar–Guaraco [5]). *Let $p \in \{1, 2, 3, \ldots\}$. For all sufficiently small $\varepsilon > 0$, there exists a critical point $u_{\varepsilon,p}$ of E_ε with*

$$E_\varepsilon[u_{\varepsilon,p}] \sim p^{1/3}, \ \textit{and} \tag{7}$$

$$\operatorname{ind}(u_{\varepsilon,p}) \leq p \leq \operatorname{ind}(u_{\varepsilon,p}) + \operatorname{nul}(u_{\varepsilon,p}). \tag{8}$$

Let us now assume that the metric g on M is bumpy, i.e., that there are no closed immersed minimal surfaces that carry nontrivial Jacobi fields. Invoking Theorems 1, 3, 4, together with the bumpiness condition and Ejiri–Micallef [3], we obtain a closed embedded minimal hypersurface Σ_p with

$$|\Sigma_p| \sim p^{1/3}, \ \operatorname{ind}(\Sigma) = p, \ \operatorname{genus}(\Sigma_p) \geq \tfrac{1}{6}p - O(p^{1/3}). \tag{9}$$

In particular, we resolve a conjecture due to Yau [17] in the case of bumpy metrics:

Corollary 1. *Any closed (M^3, g) with a bumpy metric g contains infinitely many closed embedded minimal surfaces. They satisfy* (9).

Irie–Marques–Neves [8] previously resolved Yau's conjecture in a Baire-generic sense using the Liokumovich–Marques–Neves Weyl law for the Almgren–Pitts width spectrum [9]. See also the more recent work of Gaspar–Guaraco [6] who proved the Weyl law in the Allen–Cahn setting and obtained similar conclusions as [8]. Our corollary also carries through when $\operatorname{Ric}_g > 0$; see also the previous work of Marques–Neves [11] using Almgren–Pitts in this same setting.

Remark. Theorems 2-3 are stated for closed manifolds, i.e., those without boundary, but there are analogs in case $\partial M \neq \emptyset$. See [1].

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Ricci Flow on Cohomogeneity one manifolds

Anusha M. Krishnan

We study the Ricci flow in the setting of cohomogeneity one manifolds, i.e. a Riemannian manifold (M, g) with a group G acting isometrically such that the orbit space M/G is one-dimensional. Since isometries are preserved under the flow, the evolving metrics are also invariant. Heuristically this means the Ricci flow reduces to a system of coupled PDEs in 2 variables, one for time and one for space. This makes its analysis much more tractable. In several works including [1], [2], [4] and [5], this structure has been utilized to obtain new information about the Ricci flow. In joint work with R. G. Bettiol [3] we used this framework to exhibit the first examples of compact 4-manifolds with metrics of nonnegative sectional curvature which immediately lose this property when evolved by the Ricci flow, thereby demonstrating certain limitations of the Ricci flow in dimensions above 3.

An essential step in gainfully studying Ricci flow on cohomogeneity one manifolds is imposing a special ansatz known as a "diagonal metric", and showing that this ansatz is preserved under the flow. It is a subtle point that this ansatz does not follow merely by the assumption of invariance under the group that acts by cohomogeneity one. In fact all of the above cited works explicitly or implicitly assume a larger isometry group of the initial metrics, which causes the diagonal ansatz to be preserved under the flow. On the other hand, this assumption of extra isometries significantly restricts the class of metrics that can be studied.

We introduce an algebraic condition necessary for the diagonal metric ansatz to be preserved under the flow, and we prove that in low dimensions (n less than 5) it is also sufficient. The proof involves proving existence of solutions to an overdetermined initial-boundary value problem (IBVP). We also conjecture that this condition is sufficient in all higher dimensions.

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The bounded diameter conjecture for two-convex mean curvature flow

Robert Haslhofer

(joint work with Panagiotis Gianniotis)

Given any closed embedded initial hypersurface $M_0^n \subset \mathbb{R}^{n+1}$, there exists a unique smooth solution $\{M_t^n \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ of the mean curvature flow defined on a maximal time interval $[0,T)$. One naturally wonders whether one can control the intrinsic diameter as one approaches the first singular time T. Another related question is whether one can obtaining sharp integral bounds for the mean curvature, e.g. it has been proved by Topping [8] that

$$\operatorname{diam}(M_t, d_t) \le C_n \int_{M_t} |H|^{n-1}\, d\mu\,. \tag{1}$$

Note that for three-convex hypersurfaces, e.g. for $M_0 = S_r^{n-2} \times S_R^2$, it can happen that

$$\lim_{t\nearrow T} \int_{M_t} H^{n-1}\, d\mu = \infty.$$

We thus assume that M_0 is two-convex, i.e. that the sum of the smallest two principal curvatures is positive. It has been proved by Head [5] and Cheeger-Haslhofer-Naber [1] that for the mean curvature flow of two-convex hypersurfaces one has

$$\int_{M_t} |H|^{n-1-\varepsilon}\, d\mu \leq C(M_0,\varepsilon). \tag{2}$$

Motivated by this result, it is natural to conjecture:[1]

Conjecture (Bounded diameter conjecture). *If $\{M_t \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ is a mean curvature flow of two-convex closed embedded hypersurfaces, then*

$$diam(M_t, d_t) \leq C(M_0). \tag{3}$$

Conjecture (L^{n-1}-curvature conjecture). *If $\{M_t \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ is a mean curvature flow of two-convex closed embedded hypersurfaces, then*

$$\int_{M_t} H^{n-1}\, d\mu \leq C(M_0). \tag{4}$$

The question of whether or not one can actually get rid of the ε in estimates like (2) depends on the fine structure of singularities and high curvature regions. For comparison, in a recent breakthrough [6], Naber-Valtorta improved the known $L^{3-\varepsilon}$-estimates for the gradient of minimizing harmonic maps to sharp L^3_{weak}-estimates. The simple example $u(x) = x/|x|$ in dimension three, shows that in their case the L^3_{weak}-estimate actually *cannot* be replaced by an L^3-estimate.

In joint work with Panagiotis Gianniotis, we proved the above two conjectures. More precisely, we proved the following two theorems:

Theorem (Intrinsic diameter bound [3]). *If $\{M_t \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ is a mean curvature flow of two-convex embedded hypersurfaces, then*

$$diam(M_t, d_t) \leq C, \tag{5}$$

for a constant $C = C(\mathcal{A},\alpha,\beta,\gamma) < \infty$, which only depends on certain geometric parameters of the initial hypersurface M_0 (area bound, noncollapsing constant, two-convexity constant, initial mean curvature bound).

Theorem (Sharp curvature estimate [3]). *If $\{M_t \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ is a level set flow with smooth two-convex initial data, then we have the sharp estimate*

$$\int_{M_t} H^{n-1} d\mu \leq C, \tag{6}$$

where $C = C(\alpha,\beta,\gamma,\mathcal{A}) < \infty$ only depends on the geometry of M_0.

Our proofs rely on a detailed analysis of cylindrical regions (ε-tubes) under mean curvature flow. In particular, we use the Łojasiewicz inequality from Colding-Minicozzi [2] and the canonical neighborhood theorem from Haslhofer-Kleiner [4].

[1] I thank John Head for introducing me to these conjectures in 2011. While we unfortunately don't know the precise history, the conjectures have certainly been discussed among experts well before 2011, c.f. Perelman's bounded diameter conjecture for 3d Ricci flow [7].

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Allen Cahn approach to variational theory of minimal hypersurfaces

Marco Méndez Guaraco

I surveyed recent results that explore the analogy between the phase transition theory of the Allen-Cahn equation and the Almgren-Pitts theory of miniaml hypersurfaces, in particular, my joint work with Pedro Gaspar, the Weyl law for the phase transition spectrum and its applications to the denseness and equidistribution of minimal hypersurfaces in the spirit of Marques-Neves min-max theory.

Remarks on the self-shrinking Clifford torus

Jason D. Lotay

(joint work with Christoper G. Evans, Felix Schulze)

The Clifford torus in the 3-sphere is a simple and important example of a self-shrinker for Lagrangian mean curvature flow in $\mathbb{C}^2$. On the one hand, we prove that the Clifford torus is unstable for Lagrangian mean curvature flow under arbitrarily small Hamiltonian perturbations, even though it is Hamiltonian F-stable and locally area minimising under Hamiltonian variations, thus answering a question of Joyce and Neves. On the other hand, we show that the Clifford torus is rigid: it is locally unique as a self-shrinker for mean curvature flow, despite having infinitesimal deformations which do not arise from rigid motions. The proofs rely on analysing higher order phenomena: specifically, showing that the Clifford torus is not a local entropy minimiser even under Hamiltonian variations, and demonstrating that infinitesimal deformations which do not generate rigid motions are genuinely obstructed.

Moduli space of nonnegatively curved metrics on real projective spaces

Anand Dessai
(joint work with David González-Álvaro)

In my talk I first gave an overview of results on the topology of the moduli space of metrics of nonnegative sectional curvature for closed manifolds and then explained how to prove the following recent

Theorem. [DG18] *Let M be a smooth 5-dimensional manifold homotopy equivalent to the real projective space $\mathbb{R}P^5$. Then the moduli space of metrics on M of nonnegative sectional curvature has infinitely many path components.*

In [GZ00] Grove and Ziller constructed invariant metrics of nonnegative sectional curvature on certain cohomogeneity one manifolds. Their construction leads to infinitely many different metrics on each homotopy $\mathbb{R}P^5$ which are of nonnegative sectional curvature ($sec \geq 0$) and of positive scalar curvature (psc).

In the proof of the theorem above we use reduced eta-invariants for twisted $Spin^c$-Dirac operators to show that these metrics represent infinitely many path components of the moduli space of $sec \geq 0$-metrics of M, thereby answering a question asked in [GZ00]. This line of reasoning can also be applied to study the moduli space of metrics of positive Ricci curvature ($Ric > 0$) for higher dimensional homotopy real projective spaces and to study the moduli space of $sec \geq 0$- or $Ric >$ 0-metrics for many other manifolds including certain free quotients of homotopy spheres by finite groups. In my talk I focused on the theorem above.

If one restricts to closed manifolds of nonnegative sectional curvature with finite fundamental group the theorem above seems to be the first result on the topology of the moduli space of $sec \geq 0$-metrics in dimension $\neq 4k+3$. In fact, all other results we know of use either the Kreck-Stolz invariant or the Gromov-Lawson relative index construction and are confined to dimensions $4k + 3$, see [KS93, KPT05, DKT18, De17a, De17b, Go17] (for $Ric > 0$ see [Wr11], for infinite fundamental groups see [TW17]).

The idea of the proof is as follows: One starts with Brieskorn's description of homotopy spheres as links of singularities (see [Br66]).

The polynomial $f_d : \mathbb{C}^4 \to \mathbb{C}, z := (z_1, z_2, z_3, z_4) \mapsto z_1^2 + z_2^2 + z_3^2 + z_4^d$, $d \geq 3$ odd, has 0 as an isolated singularity. Its link $\Sigma_0(d)$, i.e. the intersection of $f_d^{-1}(0)$ with a small sphere centered in 0, is diffeomorphic to S^5 and comes with an action of $SO(2) \times SO(3)$ of cohomogeneity one. The involution τ in $SO(2)$ acts freely on $\Sigma_0(d)$. The quotient is a homotopy $\mathbb{R}P^5$. From surgery theory one knows that the oriented diffeomorphism type of $\Sigma_0(d)/\tau$ only depends on d mod 8 (see [Lo71]). By the work of Grove and Ziller each quotient $\Sigma_0(d)/\tau$ has an invariant $sec \geq 0$-metric. This leads to infinitely many elements in the moduli space of metrics of nonnegative sectional curvature for each of the four homotopy $\mathbb{R}P^5$s.

Next one moves away from the singular fiber and considers the intersection of $f_d^{-1}(\epsilon)$ with the sphere for some small $\epsilon \neq 0$. The intersection $\Sigma_\epsilon(d)$ is a smooth manifold diffeomorphic to $\Sigma_0(d)$ which bounds a smooth manifold $W(d)$. The group $\mathbb{Z}_{2d} \times SO(3)$ still acts on $\Sigma_\epsilon(d)$. The involution τ acts freely on $\Sigma_\epsilon(d)$ and acts holomorphically with d isolated fixed points on the complex manifold $W(d)$. The quotient $M_d := \Sigma_\epsilon(d)/\tau$ is diffeomorphic to $\Sigma_0(d)/\tau$.

Next consider the $Spin^c$-structure on M_d which is induced from the canonical equivariant $Spin^c$-structure on $W(d)$ and consider the reduced eta-invariant $\tilde{\eta}_\alpha(M_d, g(d))$ for the $Spin^c$-Dirac operator twisted with the non-trivial complex line bundle, where the line bundle (which is also the line bundle associated to the $Spin^c$-structure) is equipped with a flat connection. Here $g(d)$ denotes a $sec \geq 0$-metric which corresponds to the Grove-Ziller metric on $\Sigma_0(d)/\tau$.

Eta-invariants were introduced by Atiyah, Patodi and Singer in connection with their index theorem for manifolds with boundary. As pointed out in [APS75] reduced eta-invariants are constant on path components of the space of psc-metrics for operators like the one above and, hence, can be used to distinguish path components. This idea has been used (and refined) to study the space and moduli space of psc-metrics for manifolds with non-trivial fundamental group starting with the work of Botvinnik and Gilkey in [BoGi95].

The reduced eta-invariant $\tilde{\eta}_\alpha(M_d, g(d))$ can be computed using a Lefschetz fixed point formula for manifolds with boundary (see [Do78]) in terms of local topological data at the τ-fixed points in $W(d)$ and certain indices. The latter vanish if $W(d)$ has an invariant psc-metric which extends the metric on $\Sigma_\epsilon(d)$ and is of product form near the boundary. This can be ensured, for example, by a Cheeger-type deformation argument.

Now computations show that the eta-invariants $\tilde{\eta}_\alpha(M_{d+8l}, g(d+8l))$ for $l \geq 0$ take infinitely many distinct values. Since M_d is diffeomorphic to M_{d+8l} it then follows that the moduli space of metrics on $M_d \cong \Sigma_0(d)/\tau$ of nonnegative sectional curvature has infinitely many path components.

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Bianchi-convexity and applications to Ricci flow

Franziska Beitz

Convexity and weaker forms of convexity play a crucial role in many areas of mathematics. In the analysis of the Ricci flow, where Riemannian metrics on a manifold evolve according to

$$\frac{\partial}{\partial t} g_t = -2\mathrm{ric}_{g_t}$$

(here ric_{g_t} denotes the Ricci curvature of the metric g_t), convexity is of particular interest via Hamilton's maximum principle [1, Theorem 4.3]. The latter states that an $O(n)$-invariant, closed and convex subset of the space of algebraic curvature tensors $\mathcal{A}_n := S^2_B(\Lambda^2(\mathbb{R}^n)^*)$, which is invariant under the ordinary differential equation

$$R'(t) = R(t)^2 + R(t)^{\#}, \tag{1}$$

is already invariant under the Ricci flow, i.e. for all n-dimensional compact manifolds M and solutions g_t, $t \in [0,T)$, to the Ricci flow on M with g_0 satisfying Ω, we have that g_t satisfies Ω for all $t \in [0,T)$. Here, $\#$ is a certain $O(n)$-equivariant quadratic map, and a Riemannian metric g satisfies Ω, if the Riemannian curvature operator Rm_g is contained in $\Omega^g \subseteq S^2_B(\Lambda^2 T^*M)$ (i.e. Ω transfered to the fibres via g-isometries) at all points in M. Moreover, a set being invariant under a differential equation means that solutions of this differential equation which start in the set stay in it for all times. It turns out that this theorem can be generalized by weakening the notion of convexity to what we call *Bianchi-convexity*. We define a closed subset $\Omega \subseteq \mathcal{A}_n$ with smooth boundary to be Bianchi-convex, if for all $R \in \partial\Omega$ and tuples $(T_1, \dots, T_n) \in (T_R\partial\Omega)^n$ which satisfy a certain second Bianchi identity, we have that

$$\sum_{i=1}^{n} \mathrm{II}_R^{\partial\Omega}(T_i, T_i) \leq 0,$$

where $\mathrm{II}_R^{\partial\Omega}$ denotes the second fundamental form of $\partial\Omega$ in R. (In the general case that the boundary of Ω is not smooth, one can give a definition involving supporting submanifolds.) This leads to the following generalization of Hamilton's maximum principle for Bianchi-convex sets.

Theorem A. Let $\Omega \subseteq \mathcal{A}_n$ be an $O(n)$-invariant, closed, Bianchi-convex and uniformly transversally star-shaped set. If Ω is invariant under the ordinary differential equation (1), then Ω is invariant under the Ricci flow.

In dimension three, using the maximum principle, Theorem A, one can show that for $a \in (\frac{1}{3}, \frac{2}{5})$ and $c > 0$ the non-convex but Bianchi-convex set

$$\left\{R \in \mathcal{A}_3 \;\middle|\; \|R\|^2 - a\mathrm{scal}(R)^2 \leq c \;\text{ and }\; \mathrm{scal}(R) \geq b_{a,c}\right\}$$

is invariant under (1), thus invariant under the Ricci flow, where

$$b_{a,c} := \sqrt{\frac{3c}{3a-1}} \sinh\left(\frac{3}{2}\right) > 0.$$

The second part of the talk treats Bianchi-convex functions, i.e. smooth functions $F : U \to \mathbb{R}$, where $U \subseteq \mathcal{A}_n$ is open, such that for all $R \in U$ and tuples $(T_1, \ldots, T_n) \in (T_R U)^n$ that satisfy the afore-mentioned second Bianchi identity, we have that

$$\sum_{i=1}^{n} \mathrm{Hess}_R F(T_i, T_i) \geq 0.$$

If the inequality above is strict unless $T_i = 0$ for each $i = 1, \ldots, n$, then F is called strictly Bianchi-convex. In dimension $n \geq 3$, a non-constant smooth function F on an open cone $\Omega \subseteq \mathcal{A}_n \setminus \{0\}$, the sublevel sets of which are strictly convex cones, can never be convex. However, up to an appropriate reparametrization and restriction such a function is Bianchi-convex:

Theorem B. For each open cone U with $\overline{U} \subset \Omega \cap \mathcal{B}_n$ and such that $\mathrm{Hess}_R F|_{R^\perp}$ is positive definite for all $R \in \overline{U}$ with $dF_R = 0$, there exists a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi' > 0$ such that $\varphi \circ F$ restricted to U is strictly Bianchi-convex.

Here, the closure of U is taken in $\mathcal{A}_n \setminus \{0\}$ and we define the cone

$$\mathcal{B}_n := \left\{R \in \mathcal{A}_n \mid R|_{\Lambda^2(v^\perp)} \not\equiv 0 \text{ for all } v \in \mathbb{R}^n \setminus \{0\}\right\}.$$

The concept of a Bianchi-convex function introduced above leads to the following rigidity result for shrinking gradient Ricci solitons, i.e. tuples (M, g, f, λ) consisting of a Riemannian manifold (M, g), a smooth function $f : M \to \mathbb{R}$ and a positive real number λ which satisfy

$$\mathrm{ric}_g + \mathrm{Hess}_g f = \lambda g.$$

Theorem C. Let $\Omega \subseteq \mathcal{A}_n \backslash \{0\}$ be an open and $O(n)$-invariant cone and $F : \Omega \to \mathbb{R}$ a scale- and $O(n)$-invariant, smooth, bounded and strictly Bianchi-convex function, the sublevel sets of which are invariant under the ordinary differential equation (1). Then all n-dimensional complete shrinking gradient Ricci solitons (M, g, f, λ) such that g satisfies Ω are locally symmetric.

By [2], such a Ricci soliton (M, g, f, λ) is either Einstein or a finite quotient of $E \times \mathbb{R}^k$, where $k > 0$, E is an $(n-k)$-dimensional Einstein manifold and $\mathbb{R}^k$ is the Gaussian shrinking soliton.

Keeping the reparametrization theorem, Theorem B, in mind, a first step into the direction of finding functions that satisfy the assumption of Theorem C is to find a one-parameter family of strictly convex cones in $\mathcal{A}_n$ which are invariant under the ordinary differential equation (1). Given a conjecture of Böhm-Wilking which states that the cones

$$\Omega_a := \left\{ R \in \mathcal{A}_n \;\middle|\; \left(\frac{n-2}{4} + a\right) \|R\|^2 \leq \|\mathrm{ric}(R)\|^2 \;\text{ and }\; \mathrm{scal}(R) > 0 \right\}$$

are invariant under (1) for $n \geq 12$ and $a \in [0, \frac{n}{4}]$, one obtains the scale- and $O(n)$-invariant, bounded and smooth function

$$\Omega \to \mathbb{R} : R \mapsto \frac{\|R\|^2}{\|\mathrm{ric}(R)\|^2},$$

where $\Omega := \cup_{a>0} \Omega_a$. The sublevel sets of this function are strictly convex and invariant under (1). With the help of the reparametrization theorem, Theorem B, this yields the following applications of Theorem C.

Theorem D. Let $n \geq 12$. Then all n-dimensional complete shrinking gradient Ricci solitons (M, g) with g satisfying Ω_a for some $a > \frac{1}{2}$ are locally symmetric.

Theorem E. Let $n \geq 12$. Then all n-dimensional complete Einstein manifolds (M, g) with g satisfying Ω_a for some $a > 0$ are locally symmetric.

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Positive curvature and torus symmetry

Lee Kennard
(joint work with Michael Wiemeler and Burkhard Wilking)

In the 1930s, H. Hopf conjectured that an even-dimensional compact manifold admitting a Riemannian metric with positive sectional curvature has positive Euler characteristic. We prove this under the additional assumption that the isometry group has rank at least five.

Under the additional assumption that the odd Betti numbers vanish, which would follow if the Bott-Grove-Halperin ellipticity conjecture holds, we recover the rational cohomology ring of the manifold if the isometry group has rank at least ten. The only rational types that appear coincide with the three known families in large dimensions known to admit positive sectional curvature:

- spheres,
- complex projective spaces, and
- quaternionic projective spaces.

As an application, we consider the question of whether any compact, simply connected Riemannian symmetric space of rank greater than one admits a metric with positive sectional curvature. Conjecturally this is impossible, and the first case of such a problem is the well known conjecture of Hopf that a product of two-dimensional spheres cannot admit a positively curved metric. Our results imply that no such space of even dimension admits a positively curved metric with isometry group containing a ten-dimensional torus, except possibly when the space is the Grassmannian of oriented two-planes in Euclidean space.

Tools include previous results in this setting (e.g., the connectedness lemma of Wilking [Wil03] and the periodicity theorem in [Ken13]), equivariant cohomology computations (applying results of Bredon [Bre64], Chang and Skjelbred [CS74], and Hsiang and Su [HS75]), and crucially a reduction to, and some structural results concerning, a representation theoretic problem involving torus representations all of whose isotropy groups are connected. The classification of such representations remains an open problem.

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Generalized saddle maps

Anton Petrunin

(joint work with Stephan Stadler)

A map s from the disc $\mathbb{D}$ to a Euclidean space is called *saddle* if for any point $p \in \mathbb{D}$ and any curve γ that surrounds p we have

$$s(p) \in \operatorname{Conv}[s(\gamma)],$$

where $\operatorname{Conv} X$ denotes the convex hull of X.

Examples of saddle maps include harmonic maps defined on $\mathbb{D}$, and more generally metric minimizing discs which we are about to define. A Lipschitz embedding s of the disc $\mathbb{D}$ into the Euclidean space is called *metric minimizing* if its intrinsic metric is minimal. Explicitly, this means that if a map $s'\colon \mathbb{D} \to \mathbb{R}^3$ agrees with s on $\partial\mathbb{D}$ and fulfills

$$\operatorname{length}(s \circ \gamma) \geq \operatorname{length}(s' \circ \gamma)$$

for all curves γ in $\mathbb{D}$, then equality holds for all curves γ.

We discuss the following two baby cases of the theorems in [1, 2]:

(1) *Any metric minimizing disc, equipped with induced intrinsic metric is* CAT(0).

(2) *Any saddle map from disc to itself that is identical on the boundary can be arbitrary well approximated by a homeomorphism* (equivalently, any point has connected inverse image and connected complement).

The first result gives a partial answer to the conjecture of Samuil Shefel [3, 4], stating that saddle disc equipped with intrinsic metric is CAT(0).

The second is an analog of Schoen–Yau univalentness theorem for harmonic maps [5].

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On CMC-foliations of asymptotically flat manifolds

Carla Cederbaum

(joint work with Anna Sakovich)

In 1996, Huisken and Yau [4] proved existence and (almost) uniqueness of, as well as asymptotic decay estimates for foliations by constant mean curvature (or *CMC*) 2-spheres in the asymptotic end of any asymptotically spherically symmetric Riemannian 3-manifold of positive mass. Their work has inspired the study of various other foliations in asymptotic ends of asymptotically flat Riemannian 3-manifolds of positive mass, most notably the foliations by Willmore 2-spheres (Lamm, Metzger, Schulze [5]) and by constant expansion/null mean curvature 2-spheres in the context of "initial data sets" in General Relativity (Metzger [6]). We suggest a new foliation by constant spacetime mean curvature (or *STCMC*) 2-spheres, also in the context of "initial data sets" in General Relativity.

Here, an *initial data set* (M, g, K) in General Relativity is a smooth Riemannian 3-manifold (M, g) equipped with a smooth symmetric $(0, 2)$-tensor field K to be thought of as the second fundamental form of the initial data set inside some ambient Lorentzian spacetime. An initial data set (M, g, K) is called *asymptotically flat* if there is a compact set $I \subset M$ called the "interior", a compact ball $B \subset \mathbb{R}^3$, and a smooth diffeomorphism $\Phi\colon M \setminus I \to \mathbb{R}^3 \setminus B$ such that $((\Phi_* g)_{ij})$ is uniformly positive definite and uniformly continuous on $\mathbb{R}^3 \setminus B$, and such that there are $\varepsilon > 0$ and $k \in \mathbb{N}$, $k \geq 2$, so that, for all $i, j = 1, 2, 3$,

$$(\Phi_* g)_{ij} = \delta_{ij} + \mathcal{O}_k(r^{-\frac{1}{2}-\varepsilon}) \tag{1}$$

$$(\Phi_* K)_{ij} = \mathcal{O}_{k-1}(r^{-\frac{3}{2}-\varepsilon}) \tag{2}$$

$$\Phi_* \left[\mathrm{R}_g - |K|_g + (\mathrm{tr}_g K)^2\right] = \mathcal{O}_{k-2}(r^{-3-\varepsilon}) \tag{3}$$

$$(\Phi_* \left[\mathrm{div}_g(K - (\mathrm{tr}_g K) g)\right])_i = \mathcal{O}_{k-2}(r^{-3-\varepsilon}) \tag{4}$$

on $\mathbb{R}^3 \setminus B$ as the coordinate radius $r := \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} \to \infty$. Here, (δ_{ij}) denotes the Euclidean metric on $\mathbb{R}^3 \setminus B$, R_g, tr_g, $|\cdot|_g$, and div_g the scalar curvature, the trace, the tensor norm, and the divergence with respect to g, respectively, and $f = \mathcal{O}_k(r^{-\tau})$ on $\mathbb{R}^3 \setminus B$ as $r \to \infty$ for some *rate* τ is an abbreviation for

$$\left\| \sum_{i=0}^{k} \sum_{|\alpha|=i} |D^\alpha f| \, r^{\tau+i} \right\|_{C^0(\mathbb{R}^3 \setminus B)} \leq C \tag{5}$$

for some constant $C > 0$.

The last two conditions (3) and (4) are related to the Gauss–Mainardi–Codazzi equations and the (asymptotically vacuum) Einstein equation satisfied by the ambient spacetime. The case when $K \equiv 0$ is called the *Riemannian case*; in this Riemannian case, (4) becomes trivial and (3) reduces to a decay condition on the scalar curvature.

The *energy* $E \in \mathbb{R}$ of an asymptotically flat initial data set (M, g, K) was defined by Arnowitt, Deser, and Misner [1] and is given as a surface integral expression over the coordinate sphere at infinity, $\mathbb{S}^2_{r\to\infty}$, where the integrand is computed from partial coordinate derivatives of $(\Phi_* g)_{ij}$. It is well-known that this expression is well-defined and independent of the choice of diffeomorphism Φ under the asymptotic flatness conditions described above. This notion of energy in fact coincides with the well-known notion of mass for asymptotically flat Riemannian manifolds referred to in the work on CMC-foliations by Huisken and Yau [4] or in other words in the Riemannian case.

Given an asymptotically flat initial data set (M, g, K) as above with energy $E \neq 0$, we prove existence and uniqueness of, as well as asymptotic decay estimates for a foliation of $M \setminus I$ by 2-spheres of *constant spacetime mean curvature (STCMC)*, that is, by 2-spheres Σ with

$$\sqrt{H^2 - (\mathrm{tr}_\Sigma K)^2} \equiv \mathrm{const} \tag{6}$$

along Σ, where H denotes the mean curvature of $\Sigma \hookrightarrow (M, g)$ with respect to the unit normal ν pointing outward toward infinity ($r \to \infty$) and $\operatorname{tr}_\Sigma K$ denotes the partial trace of K tangential to Σ. Indeed, if the initial data set (M, g, K) is embedded in an ambient spacetime (read: Lorentzian manifold) $(\mathfrak{L}, \mathfrak{g})$ such that g is the induced metric and K is the induced second fundamental form with respect to the future pointing unit normal η, then the codimension 2 mean curvature vector of $\Sigma \hookrightarrow (\mathfrak{L}, \mathfrak{g})$ is given by $\vec{\mathcal{H}} = -H\nu - (\operatorname{tr}_\Sigma K)\eta$ and thus its Lorentzian length is given by

$$\sqrt{\mathfrak{g}(\vec{\mathcal{H}}, \vec{\mathcal{H}})} = \sqrt{H^2 - (\operatorname{tr}_\Sigma K)^2}, \tag{7}$$

where we implicitly use that $\sqrt{H^2 - (\operatorname{tr}_\Sigma K)^2}$ is spacelike in this situation.

In the Riemannian case $K \equiv 0$, this foliation – including the asymptotic decay estimates – coincides with the CMC-foliation suggested by Huisken and Yau in the optimal decay version established in a series of works by a number of people including Ye [9], Metzger [6], and finally Nerz [7].

The proof of existence, uniqueness and asymptotic decay for the STCMC-foliation builds upon ideas of Metzger [6] and Nerz [7, 8]. It is based on a method of continuity argument, exploiting established existence, uniqueness, and asymptotic decay properties of CMC-foliations in asymptotically flat Riemannian manifolds. In particular, we introduce a non-selfadjoint STCMC-stability operator and analyze the asymptotic behavior of its lowest (a priori real) eigenvalues and eigenfunctions. This operator resembles the CMC-stability operator except for an additional non-selfadjoint term which turns out to have sufficiently good decay properties.

We prove furthermore that the STCMC-foliation has many properties that are relevant for the definition of a total relativistic center of masssuch as equivariant transformation under the asymptotic Poincaré-group and in particular time-evolution under the Einstein evolution equations in accordance with a point particle in Special Relativity. We also show that it remedies some deficiencies of the relativistic center of mass notion suggested by Huisken and Yau (which were pointed out in joint work with Nerz [3]). Moreover, the STCMC-foliation lends itself to the construction of center of mass coordinates (joint work in progress with Metzger). Finally, the leaves of the STCMC-foliation turn out to be "independent" of the initial data set in the following sense: Given two asymptotically flat initial data sets (M_I, g_I, K_I), $I = 1, 2$ in the same ambient spacetime, both with non-vanishing energy; if (M_I, g_I, K_I) both contain the same surface $\Sigma \hookrightarrow M_I$, then Σ will be STCMC with respect to (M_1, g_1, K_1) if and only if it is STCMC with respect to (M_2, g_2, K_2).

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A relative entropy for self-similarities of the harmonic map flow

Alix Deruelle

In this short note, we consider the Harmonic map flow coming out of a 0-homogeneous map $u_0 : \mathbb{R}^n \to N$, $n \geq 3$, where N is a closed Riemannian manifold assumed to be isometrically embedded in some Euclidean space $\mathbb{R}^m$. Formally speaking, we are looking for a solution to the following system of partial differential equations:

$$(1) \qquad \begin{cases} \partial_t u - \Delta u \perp T_u N, & \text{on } \mathbb{R}^n \times \mathbb{R}_+, \\ u|_{t=0} = u_0, \end{cases}$$

where the solution attains its initial condition u_0 in the $C^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$ topology if u_0 is
in $C^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$. Equation (1) is equivalent to the following parabolic flow:

$$(2) \qquad \begin{cases} \partial_t u = \Delta u + A(u)(\nabla u, \nabla u), & \text{on } \mathbb{R}^n \times \mathbb{R}_+, \\ u|_{t=0} = u_0, \end{cases}$$

where $A(u)(\cdot,\cdot) : T_u N \times T_u N \to (T_u N)^\perp$ denotes the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^m$ evaluated at u.

As the initial map u_0 is 0-homogeneous, the gradient ∇u_0 decays quadratically at infinity only, i.e. $|\nabla u_0| \in L^2_{loc}(\mathbb{R}^n)$ but $|\nabla u_0|$ is not in $L^2(\mathbb{R}^n)$ unless u_0 is a constant map. Consequently, one cannot use the work of Struwe [Str88] and Chen-Struwe [CS89] to get existence of a solution of (2).

To circumvent this issue, we consider expanding solutions of the Harmonic map flow, i.e. solutions that are invariant under parabolic rescalings,

$$(3) \qquad u_\lambda(x,t) := u(\lambda x, \lambda^2 t) = u(x,t), \quad \lambda > 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+.$$

Equation (2) coupled with condition (3) is equivalent to a time-independent partial differential equation:

$$(4) \qquad \begin{cases} \left(\Delta + \frac{x}{2} \cdot \nabla\right) U + A(U)(\nabla U, \nabla U) = 0, & \text{on } \mathbb{R}^n, \\ \lim_{|x| \to +\infty} U(x) = u_0(x/|x|), \end{cases}$$

where the convergence at infinity holds in the smooth sense if u_0 is. Therefore, the initial condition u_0 is interpreted as a boundary data at (spatial) infinity in the setting of expanding solutions. One should notice that the operator $\Delta + \frac{x}{2} \cdot \nabla$ is symmetric on $L^2(e^{|x|^2/4}dx)$ and it is unitarily conjugate to a harmonic oscillator $\Delta - |x|^2/16$ which implies in particular that the $L^2(e^{|x|^2/4}dx)$-spectrum is discrete, a fact that is in sharp contrast with the spectral behavior of the Laplacian on $\mathbb{R}^n$.

Let us give some examples of expanding solutions.

(1) Constant maps are expanding solutions obviously.

(2) Harmonic maps $u_0 : \mathbb{S}^{n-1} \to N$ extended radially are static expanding solutions.

(3) Germain and Rupflin [GR11] have investigated the existence of expanding solutions in a corotational setting. Partial differential equation (4) is reduced to an ordinary differential equation. This fact lets them to draw a clear picture of what the flow does: see also [BB11] for numerical evidences regarding the existence of such solutions.

(4) In a joint work with Lamm [DL18], we proved the existence of weak expanding solutions coming out of Lipschitz maps $u_0 : \mathbb{S}^{n-1} \to N$ homotopic to a constant. These solutions are shown to be smooth outside a ball whose radius is controlled by the L^2_{loc} energy of the map u_0. However, if u_0 is already harmonic, our approach does not guarantee our solution to be harmonic or not.

Before stating the main results, we would like to advertise the importance of expanding solutions of the Harmonic map flow (or more generally geometric flows) for at least two reasons. On one hand, they are likely to be the best candidates to smooth 0-homogeneous maps out instantaneously in case the initial condition seen as a map $u_0 : \mathbb{S}^{n-1} \to N$ is homotopic to a constant. On the other hand, 0-homogeneous harmonic maps appear naturally as blow-ups of the Harmonic map flow in case there is a finite time singularity. In order to restart the flow in a canonical way , one has to understand the number of expanding solutions coming out of such 0-homogeneous harmonic maps.

The main questions we address here are twofolds:

(1) How can one detect an expanding solution among other solutions coming out of a 0-homogeneous map ?
(2) In which terms can uniqueness be stated and expected ?

It turns out that these two questions can be linked with the help of a suitable entropy designed for expanding solutions only. Indeed, an expanding solution u of the Harmonic map flow is a formal critical point of the following entropy:

$$\mathcal{E}^+(u) := \int_{\mathbb{R}^n} |\nabla u|^2 e^{\frac{|x|^2}{4}} dx.$$

Unfortunately, the quantity $\mathcal{E}^+(u)$ diverges unless u is constant. To remedy to this issue, one can consider a relative entropy: this is Tom Ilmanen's idea. More precisely, let u_b be a background expanding solution coming out of a 0-homogeneous map u_0 and let u be a solution to (1) coming out of the same map u_0. Assume for simplicity that both u and u_b are smooth and regular at infinity, i.e. for each nonnegative integer k, there exists a positive constant C_k such that

$$|\nabla^k u_b|(x,t) + |\nabla^k u|(x,t) \leq \frac{C_k}{(|x|^2+t)^{\frac{k}{2}}}, \quad k \geq 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ .$$

Then the relative entropy of u with respect to u_b is:

$$\mathcal{E}(u,u_b)(t) := \lim_{R\to+\infty} t\int_{B(0,R)} \left(|\nabla u|^2(x,t) - |\nabla u_b|^2(x,t)\right) \frac{e^{\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}dx, \quad t>0.$$

A similar relative entropy had been previously considered by Ilmanen, Neves and Schulze [INS14] for the Network flow for regular networks.

The first result regarding this relative entropy for solutions of the Harmonic map flow is:

Theorem 1. *Let u and u_b be defined as above. Then the function $t\in\mathbb{R}_+ \to \mathcal{E}(u,u_b)(t)\in\mathbb{R}$ is well-defined and is non-increasing. Moreover:*

$$\frac{d}{dt}\mathcal{E}(u,u_b) = -2t\int_{\mathbb{R}^n} \left|\partial_t u + \frac{x}{2t}\cdot\nabla u\right|^2 \frac{e^{\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}dx, \quad t>0, \tag{5}$$

$$-\infty < \lim_{t\to+\infty}\mathcal{E}(u,u_b)(t) \leq \lim_{t\to 0}\mathcal{E}(u,u_b)(t) < +\infty. \tag{6}$$

In particular, the family of rescaled maps $(u_\lambda)_{\lambda>0}$ defined in (3) subconverges smoothly locally to an expanding solution coming out of the same initial map u_0 as the parameter λ tends to 0 or $+\infty$.

Theorem 1 reduces the uniqueness issue for general solutions coming out of 0-homogeneous maps to the uniqueness issue for expanding solutions: in case one knows a priori that there is a unique expanding solution coming out of a given 0-homogeneous map, Theorem 1 ensures that all suitable solutions coming out of this same 0-homogeneous map coincide.

The first variation formula (5) shows that the obstruction tensor $\partial_t u + \frac{x}{2t}\cdot\nabla u$ belongs to the weighted space $L^2(e^{|x|^2/4t})$: this is not straightforward since a priori, each term decays quadratically at infinity only, i.e. $\partial_t u = O((|x|^2+t)^{-1}) = \frac{x}{2t}\cdot\nabla u$.

Theorem 2. *(1) Assume the target N is non-positively curved and let $u_0 : \mathbb{R}^n \to N$ be a 0-homogeneous map assumed to be in $C^\infty(\mathbb{R}^n\setminus\{0\})$. Then there exists a unique smooth solution smoothly coming out of u_0: this solution must be expanding.*

(2) (Generic uniqueness) The set of maps $u_0:\mathbb{R}^n\to N$ admitting at least two expanding solutions with 0 relative entropy is of first category in the Baire sense. Moreover, the set of such maps leading to non-degenerate expanding solutions is of codimension 1.

Theorem [2, (1)] echoes Hamilton's Theorem [Ham75] on the existence of harmonic maps from a compact manifold with boundary with values into a non-positively curved target with Dirichlet boundary data: in our case, the boundary data is pushed at infinity. We use a continuity method to prove existence and uniqueness of expanding solutions first: since the target is non-positively curved, it is aspherical by Hadamard's Theorem. Consequently, it gives us a path of initial maps $(u_0^\tau)_{0\leq\tau\leq 1}$ from $\mathbb{S}^{n-1}$ to N connecting the map $u_0^0 := u_0$ to a constant map u_0^1. Existence and uniqueness are essentially due to the invertibility of the Jacobi operator defined between suitable function spaces. We notice that both the proof and the statement of Theorem [2, (1)] find their analogous statement for expanding solutions of the Ricci flow coming out of metric cones over spheres with curvature operator larger than 1: [Der16], [Der17].

Theorem [2, (2)] is inspired by the work of Hardt and Mou [HM92] on harmonic maps from a domain of Euclidean space with values into a closed Riemannian manifold N. Their work is in turn based on the seminal work of White on minimal surfaces [Whi87]. The first key ingredient to prove Theorem [2, (2)] is based on an analysis of Fredholm properties of the Jacobi operator between suitable function spaces.

If $N = \mathbb{S}^{m-1}$ is a Euclidean sphere then the Jacobi operator associated to an expanding solution u is defined by:

$$L_u\kappa := \left(\Delta + \frac{x}{2}\cdot\nabla\right)\kappa + 2\langle\nabla u, \nabla\kappa\rangle u + |\nabla u|^2\kappa, \quad \kappa \in C_0^\infty(\mathbb{R}^n, T_u\mathbb{S}^{m-1}).$$

Our approach is very close in spirit to the analysis of the Moduli space of expanding solutions of the Mean curvature flow by Bernstein and Wang: [BW17]. Once the local properties of the Moduli space of smooth expanding solutions of the Harmonic map flow are established, a unique continuation at infinity result is needed to conclude the proof of Theorem [2, (2)]. Our strategy is based on Carleman estimates and these estimates are adapted from corresponding ones for the Ricci flow: [Der17b].

Theorem 3. *(Unique continuation at infinity) Let u_1 and u_2 be two smooth expanding solutions coming out of the same 0-homogeneous map u_0. Assume u_1 and u_2 are regular at infinity. Then there exists a smooth map $\kappa_{12} \in C^\infty(\mathbb{S}^{n-1}, T_{u_0}N)$ such that the limit,*

$$\lim_{r\to+\infty} r^n e^{\frac{r^2}{4}}(u_2 - u_1)(r,\omega) =: \kappa_{12}(\omega), \quad \omega \in \mathbb{S}^{n-1}, \tag{7}$$

exists and holds in the smooth topology. Moreover, $u_1 = u_2$ if and only if $\kappa_{12} = 0$.

As a last remark, we notice that Bernstein and Wang [BW17] have proved a similar unique continuation at infinity for expanding solutions of the Mean curvature flow by different means: their method is based on a suitable frequency function associated to the difference of two expanding solutions coming out of the same cone.

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