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Calculus of Variations

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Abstract. The Calculus of Variations is at once a classical subject, and a very modern one. Its scope encompasses a broad range of topics in geometric analysis, and deep questions about PDE. New frontiers are constantly emerging, where problems from mechanics, physics, and other applications introduce new challenges. The 2018 Calculus of Variations workshop reflected this breadth and diversity.

Mathematics Subject Classification (2010): 49-xx [Calculus of Variations] 35Jxx [Elliptic equations and systems] 53Cxx [Global differential geometry] 58xx [Global analysis, analysis on manifolds].

Introduction by the Organisers

This Calculus of Variations workshop was organized by Alessio Figalli (Zurich), Robert Kohn (New York), Tatiana Toro (Seattle), and Neshan Wickramasekera (Cambridge). It gathered an outstanding group of 47 participants, among them many PhD students and postdocs.

The workshop's scope was very broad. The diversity of the topics represented encouraged scientific cross-fertilization and was a key to the workshop's success. To capture the diversity – and coherence – of the workshop's 24 talks, we shall organize them into groups.

Three talks discussed **free boundary problems**. Two speakers addressed thin obstacle problems: *Emanuele Spadaro* presented work with M. Focardi about graphs that minimize surface area subject to the constraint of being above a flat, lower-dimensional obstacle; the work establishes sharp regularity results for the solutions by reducing the question to previously known results on branched minimal graphs. *Angkana Rüland* presented work with H. Koch and W. Shi concerning variable-coefficient scalar analogues of the classic Signorini problem. A third talk, by *Charles Smart*, presented work with Will Feldman on an evolutionary free boundary problem modeling the motion of the contact line where a spreading liquid drop meets a periodically patterned surface. Starting from a discrete Hele-Shaw-type formulation, Smart discussed its scaling limit and explained why the solution has facets (a phenomenon also seen experimentally).

Quite a few talks discussed problems from geometric analysis. One thread involved the **asymptotic character of solutions near singularities**. *Nick Edelen* discussed work with M. Colombo and L. Spolaor establishing an asymptotic decay estimate for stationary varifolds close to an integrable multiplicity 1 polyhedral cone. *Max Engelstein* described work with L. Spolaor and B. Velichkov establishing log-epiperimetric inequalities and asymptotic decay results for area minimizers and energy minimizing free boundaries at strongly isolated singularities. A key step of the proof involves the use of the classical (finite dimensional) Lojasiewicz inequality. *Brian Krummel* described work with N. Wickramasekera establishing asymptotic decay of a Dirichlet energy minimizing multivalued function to a unique tangent function a.e. along its branch locus. A key idea in this work is the use of the Almgren frequency function to classify homogeneous "Jacobi fields" produced by sequences of minimizers converging to a homogeneous minimzer, even though these Jacobi fields themselves need not be energy minimizing or even stationary in the usual sense.

Another thread from geometric analysis was the use of **min-max techniques** for constructing and analyzing minimal surfaces. *Christos Mantoulidis* described work with O. Chodosh giving a PDE-based proof of regularity of two-dimensional Allen-Cahn min-max minimal surfaces in 3-manifolds; for generic metrics the resulting solutions surfaces have multiplicity 1. *Alessandro Pigati* described work with T. Rivière showing regularity of the "parameterised stationary 2-varifolds" in arbitrary co-dimension; these arise from a novel min-max construction (due to Rivière) for a certain perturbation of the mapping area.

A third thread from geometric analysis was the study of **geometric flows**. *Brian White* presented work with D. Hoffman, T. Ilmanen, and F. Martín concerning "translator" solutions of the mean curvature flow (in other words: hypersurfaces M such that the translating surface $M - te_{n+1}$ is a solution of the mean curvature). *Jacob Bernstein* described recent progress with L. Wang concerning asymptotically conical self-expanders for the mean curvature flow; the work adapts global analysis techniques used previously in minimal surface theory to self-expanders, and establishes a certain non-degeneracy property of the expanders asymptotic to a generic regular cone. *Melanie Rupflin* presented work with P. Topping and with C. Robertson describing finite time degeneration of Teichmüller harmonic map flow.

There were also talks on **other aspects of geometric analysis**. One, by *Francesco Maggi*, described work with M. Delgadino, C. Mihailia, and R. Neumayer establishing that unions of spheres are the only finite-volume Caccioppoli sets that are stationary for the isoperimetric problem in Euclidean space. Another, by *Guy David*, discussed various notions of a "solution" to the classical Plateau problem, focusing especially on a definition involving the "sliding boundary condition" and on boundary regularity of the associated class of almost minimal sets. *Antonio De Rosa*'s talk presented work with G. De Philippis and F. Ghiraldin giving an extension of the Allard rectifiability theorem to anisotropic integrands. *Dali Nimer*'s talk discussed uniformly distributed measures, presenting new examples and characterisations of conical 3-uniform measures.

About half the talks were on topics other than geometric analysis. **Lower semicontinuity of functionals** and the **regularity of minimizers** are familiar topics in the calculus of variations, and three speakers discussed problems of this type. *Yury Grabovsky* presented a new example of a variational problem that is rank-one convex but not quasiconvex, obtained by using connections between homogenization, optimal design, and quasiconvexity, combined with an algebraic approach to "exact relations" for polycrystalline composite materials. *Connor Mooney* presented new results on the regularity of solutions to elliptic and parabolic systems; in the parabolic setting, a key idea was to look for a "spiraling self-similar" solution. *Felix Otto* presented a new variational approach to regularity for the Monge-Ampere equation (work with M. Goldman and M. Huesmann); it uses the connection between Monge-Ampere and optimal transport, and well-chosen test functions in the Benamou-Brenier variational formulation of optimal transport.

In many physical applications, variational problems must be solved in settings where the spatial environment is highly heterogeneous; there were four talks of this type, where the calculus of variations interacted with **homogenization**. Two (by Charles Smart and by Yury Grabovsky) were already discussed above. A third, by *Ken Golden*, discussed how homogenization is critical to our understanding of sea ice in the arctic and antarctic, improving our understanding of how global warming will affect the climate and raise the level of the sea. Another talk, by *Caterina Zeppieri*, discussed work with F. Cagnetti, G. Dal Maso, and L. Scardia on a family of stochastic homogenization problems motivated by the modeling of fracture.

Another application-driven frontier is the modeling of **thin elastic sheets**; there were three talks in this area. *Heiner Olbermann* discussed this topic's deep connections to the Nash embedding theorem, then presented recent results explaining why a sheet with a conical singularity is rather rigid. *Marta Lewicka* discussed work with D. Lucic, using methods from Γ-convergence to see how prestrain (due, for example, to growth or thermal expansion) affects the mechanical behavior of a thin elastic sheet. *Ian Tobasco* discussed the wrinkling seen when a piece of a thin spherical shell is forced to be (approximately) flat by putting it on water, identifying a regime where the energy of the sheet (suitably renormalized)

Γ-converges to a convex variational problem, and drawing conclusions about the associated wrinkling patterns.

Yet another application-driven frontier is **energy-driven pattern formation**, where complex patterns emerge from the solutions of variational problems. Tobasco's talk (just discussed) had this character. So did the one by *Benedikt Wirth*, who discussed the numerical approximation of "branched transport" problems – identifying, for a broad class of such problems, a family of diffuse-interface approximations that are well-suited to numerical minimization.

The diversity of the workshop's topics was an important element in its success. Experts in one area enjoyed looking for and finding connections to the others. A comprehensive list of examples is beyond the scope of this Introduction, but here are a few examples: (i) some geometric analysts wondered whether Yury Grabovsky's rank-one-convex but non-quasiconvex integrand might have a geometric interpretation; (ii) experts in the mean curvature flow were interested to see self-similarity (or something very close to it) playing a role not only in the analysis of topological change but also in Connor Mooney's results on singular solutions of other parabolic systems; and (iii) people working on steady-state homogenization problems took great interest in the homogenization-like evolutionary free boundary problem discussed by Charles Smart.

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Workshop: Calculus of Variations

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Abstracts

Space of asymptotically conical self-expanders of the mean curvature flow

Jacob Bernstein
(joint work with Lu Wang)

The mean curvature flow (MCF) is the flow of hypersurfaces $\{\Sigma_t\}_{t\in I}$ that satisfies the evolution equation

$$\left(\frac{d\mathbf{x}}{dt}\right)^{\perp} = \mathbf{H}_{\Sigma_t}.$$

That is, a point on the flow moves normally to the surfaces with speed given by the mean curvature. This is a natural geometric flow and arises variationally as the negative gradient flow of the area functional.

A standard fact in the theory of MCF is that a closed initial surfaces possess a mean curvature flow that develops a singularity in finite times. Simple examples show that the flow need not vanish at this first singular time. Hence, for suitable weak notions of the flow, it becomes possible, and interesting, to "flow through" singularities.

By Huisken's monotonicity formula, singularity formation is well modeled by self-shrinking solutions of MCF. An interesting, and large, class of such singularity models are those associated to self-shrinkers that are asympototic to some cone. For such singularities, the way the singularity is "resolved" is modeled, in some sense, by a self-expander, Γ, that is asympototic to C – here C is the cone asymptotic to one of the self-shrinkers modeling the way the singularity forms. Here Γ is a hypersurface so that $\{\sqrt{t}\Gamma\}_{t>0}$ is a MCF and so that $\lim_{\rho\to 0}\rho\Gamma = C$ in an appropriate sense. It follows from the equation for MCF that $\sqrt{t}\Gamma$ is a MCF if and only if Γ satisfies

$$\mathbf{H}_\Gamma = \frac{\mathbf{x}^\perp}{2}.$$

It is worth mentioning that, due to the current lack of an appropriate forward in time analogue of Huisken's monotonicity formula, this picture of singularity resolution is more heuristic than it is for self-shrinkers and singularity formation.

As such, it is interesting to study what sorts of asymptotically self-expanders exist. It turns out that there are a great many such self-expanders – indeed, modulo regularity issues, any sufficiently regular cone admits at least one self-expander asymptotic to the cone. In joint work with Lu Wang, we bring ideas from global analysis to bear on the study the number self-expanders asymptotic to a given cone and also to properties of self-expanders asymptotic to generic cones. Specifically, we adapt ideas first introduced by Tomi-Tromba (and expanded on by Brian White) for minimal surfaces with boundary to show the following:

Theorem 1. *For* $\Gamma \in \mathcal{ACH}_n^{k,\alpha}$ *let*

$$\mathcal{ACE}_n^{k,\alpha}(\Gamma) = \left\{ [\mathbf{f}] \colon \mathbf{f} \in \mathcal{ACH}_n^{k,\alpha}(\Gamma) \text{ and } \Sigma = \mathbf{f}(\Gamma) \text{ a self-expander} \right\}.$$

Then the following statements hold:

1. *(1)* $\mathcal{ACE}_n^{k,\alpha}(\Gamma)$ *is a smooth Banach manifold modeled on* $C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$, *with a countable cover by coordinate domains* $\mathcal{O}_i$.
2. *(2) The projection map* $\Pi \colon \mathcal{ACE}_n^{k,\alpha}(\Gamma) \to C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$ *which assigns to* $[\mathbf{f}]$ *the trace at infinity,* $\mathrm{tr}^1_\infty[\mathbf{f}]$, *of* $\mathbf{f}$ *is a smooth map of Fredholm index* 0.
3. *(3) Each* $\Pi|_{\mathcal{O}_i}$ *has a coordinate representation given by the map* $(z,\kappa) \mapsto (z,\psi_i(z,\kappa))$ *from a neighborhood of* $0 \in \mathcal{Z}_i \oplus \mathcal{K}_i$ *to itself, where* $\mathcal{K}_i$ *is the kernel of* $D\Pi([\mathbf{f}_i])$ *for some* $[\mathbf{f}_i] \in \mathcal{O}_i$ *and* $\mathcal{Z}_i$ *is the complement of* $\mathcal{K}_i$ *in* $C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$.
4. *(4) The kernel of* $D\Pi([\mathbf{f}])$ *is isomorphic to the space of normal Jacobi fields of* $\mathbf{f}(\Gamma)$ *that fix the asymptotic cone.*

Here $\mathcal{ACE}_n^{k,\alpha}(\Gamma)$ is the space of asymptotically conical self-expanding embeddings of Γ. Roughly, speaking these are embeddings of Γ whose images are self-expanders asymptotic to a $C^{k,\alpha}$ regular cone and which encode (in a natural way) a parameterization the asymptotic cone of the image by the asymptotic cone of Γ, modulo an equivalence relation that identifies parameterizations that parameterize the same hypersurface and which have the same parameterizations of the asymptotic cone. The trace at infinity is just the restriction of the asymptotic parametrization of the asymptotic cone to the link, $\mathcal{L}(\Gamma)$ of Γ.

An immediate consequence is the following "bumpy cone" result:

Corollary 1. *Given* $\Gamma \in \mathcal{ACH}_n^{k,\alpha}$ *there is a nowhere dense set* $\mathcal{S} \subset C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$ *so that if* $\varphi \in C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})\backslash\mathcal{S}$ *and* $[\mathbf{f}] \in \mathcal{ACE}_n^{k,\alpha}(\Gamma)$ *has* $\Pi([\mathbf{f}]) = \varphi$, *then the space of normal Jacobi fields of* $\Sigma = \mathbf{f}(\Gamma)$ *that fix the asymptotic cone of* Σ *is trivial.*

There are number of other interesting consequences, including the existence of a $\mathbb{Z}_2$-degree for the map Π restricted to suitable domains.

Regularity for almost minimal sets near the boundary

Guy David

First a motivation for the sliding almost minimal sets that are at the center of this lecture.

In its simplest forms, the Plateau problem (say, for 2-dimensional objects in $\mathbb{R}^3$) consists in taking a closed curve $\Gamma \subset \mathbb{R}^3$ and asking for a surface "bounded by Γ" and with minimal area.

In fact there are a lots of ways to formalize all this, and some part of the lecture consists in giving a few examples, and the concentrating on a special type of minimal and almost minimal sets. Then only we try to describe attempts to get a C^1 description, if possible, of sliding almost minimal sets near a simple boundary (a curve or even a line).

The new results are taken from two long papers by the author, a first one with general results ("Local regularity properties of almost- and quasiminimal sets with a sliding boundary condition", available on arXiv and hopefully soon to be published, and "A local description of 2-dimensional almost minimal sets bounded by a curve near some cones" which should have been available this summer but the author is late again).

So the lecture starts with a few examples of Plateau problem, with different ways to present matters. For instance, Radó and Douglas use parameterizations and get nice results but limited to 2-dimensional sets and that maybe do not model soap films so well when the surface crosses itself.

Then there is the theory of Mass minimizing integral currents (Federer, Fleming, De Giorgi, and many others), which is very successful in terms of existence and regularity boundary results, but again fails to give a good representation of soap films and their singularities. Recall that in this context we take the algebra from differential geometry and write the boundary condition as $\partial T = S$, where T is the minimizing current that we seek and S is a given current, for instance the current of integration on the given curve Γ.

Recall that soap films exhibit singular sets of dimension 1.

In this respect size minimizers, where one counts the Hausdorff measure of the support (where the integer multiplicity is nonzero), and not integrated against multiplicity. But then regularity results become much weaker and existence is not known in general, even for 2-dimensional currents .

Much closer to the spirit of the lecture are the homology minimizers of Reifenberg. He starts from a boundary set Γ (think, of dimension $d-1$ but any compact set would do) a group G to compute homology (think about $\mathbb{Z}$ but some times compact is needed), and a subgroup H of the Čech homology group $H_{d-1}(\Gamma, G)$ (of dimension $d-1$, on Γ, sorry for the notation). Then say that a set $E \supset \Gamma$ is bounded by Γ if the image of H by the homology mapping induced by the inclusion $i : \Gamma \to E$ is $\{0\}$. In simple cases, this corresponds to our idea of filling the curve. And we look for a set $E \supset \Gamma$ that is bounded by Γ and has minimal Hausdorf measure $\mathcal{H}^d(E)$.

The good news are that this describes soap films much better, and in addition there are good existence results (initially by Reifenberg, then other authors, and recently a quite general one by Yangqin Fang. The regularity is still quite hard though.

Finally here is another Plateau problem that the author likes, but which has neither general existence nor good regularity theory yet, even for 2-dimensional sets bounded by a smooth curve.

We give the definition for 2-dimensional sets. Again let $\Gamma \subset \mathbb{R}^n$ be given. Let $E \subset \mathbb{R}^n$ be a closed set. A deformation for E in a compact ball B is a (continuous) one parameter family of continuous mappings $\varphi_t : E \to \mathbb{R}^n$, such that φ_0 is the identity, φ_1 is Lipschitz (condition added by Almgren, not necessarily needed, but we keep it because it does not disturb in general), $\varphi_t(x) = x$ when $x \in E \setminus B$, $\varphi_t(E \cap B) \subset B$ (for the moment, nothing surprising I hope!), and finally the sliding condition

$$\varphi_t(x) \in \Gamma \text{ when } x \in E \cap \Gamma. \tag{1}$$

Think about a shower curtain attached to its bar. Pinching is allowed, but not tearing apart. The sliding version seems recent, but without it the definition essentially comes from Almgren.

A competitor for E (in B) is a set $F = \varphi_1(E)$, where $\{\varphi_t\}$ is as above.

A simple Plateau problem (but with no general solution so far) consists in taking E and Γ as above, say compact, and asking for a competitor F of E (in a large ball) that minimizes $\mathcal{H}^d(F)$.

A sliding almost minimal set is a set E such that

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(F \cap B) + h(r)r^d \tag{2}$$

whenever F is a sliding competitor for E in any compact ball B of radius r, where $h : (0, +\infty) \to [0, +\infty]$ is a given gauge function, i.e., a nondecreasing function such that $\lim_{r \to 0} h(r) = 0$ (often $h(r) \leq Cr^\alpha$).

And a sliding minimal set is just a sliding almost minimal set with $h(r) \equiv 0$ (no error term). But almost minimal set give a welcome flexibility to our definitions.

We want to study the local regularity of the sliding almost minimal set, in particular near a point of the boundary set Γ (this is the new part). Concerning general (or should we say vague) properties, we know that (if the boundary is essentially as nice as a Lipschitz graph), the reduced (remove the part which is not in the support of $\mathcal{H}^d_{|E}$) almost minimal sets are locally Ahlfors regular (strongly d-dimensional), rectifiable, that limits of almost minimal sets with uniform bounds on $h(r)$ are rectifiable, and are almost minimal sets with the same function h. Also, $\mathcal{H}^d$ goes to the limit well along such sequences.

But for more precise results, we are for the moment limited to two-dimensional sets. The best result for inside regularity (away from Γ or with no Γ) is Jean Taylor's theorem, which says two things. First there are only three types of minimal cones (same definition as above, without the sliding condition) of dimension 2 in $\mathbb{R}^3$: the planes, the sets Y composed of three half planes bounded by a same line and that make $120\circ$ angles along that line, and the sets T, obtained as the cone over the union of the edges of a regular tetrahedron centered at the origin. And then for each point x of a (reduced) almost minimal set of dimension 2 in $\mathbb{R}^3$, x has a neighborhood where E is equivalent to one of the cones above, through a C^{1+a} diffeomorphism of $\mathbb{R}^3$. Provided that $h(r) \leq Cr^\alpha$ for some $\alpha > 0$, for instance.

For almost minimal sets of dimension 2 in $\mathbb{R}^n$, $n \geq 4$, there is a similar result (G. D., Ann. Fac. Sci. Toulouse Math. 2009), but the list of minimal cones is not known, and maybe there is only a biHölder equivalence near some points with "bad" blow-up limits.

We would like a similar result near the boundary. Such a result exist (recent preprint of Yangqin Fang) for 2 dimensional sets bounded by smooth oriented surfaces $\Gamma \subset \mathbb{R}^3$, with the constraint of containing Γ. But when Γ is a nice curve (or a line), this is may depend on a blow-up limit of E at the given point, and the author did not complete the study (by far).

When E is close enough to one of the J. Taylor cones that crosses Γ transversally, we see nothing and J. Taylor's theorem holds as before. Not in contradiction with soap experiments that can be done.

When $0 \in E \cap \Gamma$, Γ is a line, and E is close enough to a half plane, then there is a full J. Taylor theorem near 0, as above.

When instead E is close enough to a generic set V, composed of two half planes that meet along Γ with an angle $\beta(V) \in (0, 2\pi/3)$, we have a similar result.

More interestingly, when E is close to a plane that contains Γ, E may have a small flat crease of V-type along a part of Γ, and otherwise leave Γ tangentially.

For a sharp set V, with angle $\beta(V) = 2\pi/3$, E may also leave Γ in a nice smooth way, and leave behind a thin "vertical" face that connects a singularity set of $\mathbb{R}^3 \setminus \Gamma$ to Γ. Think about taking a V-set and pinching it near Γ.

All these configurations (and a few similar ones) are under control. In all these cases, the proof is a complicated variant of a proof for Taylor's theorem, with a new monotonicity formula.

But unfortunately, when E is close to a Y-set whose singular set coincides with Γ the methods above, and in particular the adapted monotonicity formula, seem to fail so far. This is unfortunate because this seems to be the only really difficult case left, and the ensuing putative regularity should also lead to existence results.

Anisotropic counterpart of Allard's rectifiability theorem and applications

Antonio De Rosa
(joint work with Guido De Philippis, Francesco Ghiraldin)

Allard's rectifiability theorem, [1], asserts that every d-varifold in $\mathbb{R}^n$ with locally bounded (isotropic) first variation is d-rectifiable when restricted to the set of points in $\mathbb{R}^n$ with positive lower d-dimensional density.

We recall that a d-dimensional varifold $V \in \mathbb{V}_d(\mathbb{R}^n)$ is said d-rectifiable if there exist a d-rectifiable set K and a Borel function $\Theta : \mathbb{R}^n \to (0, +\infty)$ such that

$$V = \Theta \mathcal{H}^d \llcorner (K \cap \Omega) \otimes \delta_{T_x K}. \tag{1}$$

It is a natural question whether Allard's rectifiability theorem holds for varifolds whose first variation with respect to an anisotropic integrand is locally bounded.

More specifically, for an open set $\Omega \subset \mathbb{R}^n$ and a positive C^1 integrand

$$F : \Omega \times G(n,d) \to (0, +\infty),$$

where $G(n,d)$ denotes the Grassmannian of d-planes in $\mathbb{R}^n$, we define the *anisotropic energy* of a d-varifold $V \in \mathbb{V}_d(\Omega)$ as

$$\mathbf{F}(V, \Omega) := \int_{\Omega \times G(n,d)} F(x,T)\, dV(x,T).$$

We also define its *anisotropic first variation* as the order one distribution whose action on $g \in C_c^1(\Omega, \mathbb{R}^n)$ is given by

$$\begin{aligned} \delta_F V(g) :&= \frac{d}{dt}\mathbf{F}\big(\varphi_t^{\#} V\big)\Big|_{t=0} \\ &= \int_{\Omega\times G(n,d)} \Big[\langle d_x F(x,T), g(x)\rangle + B_F(x,T) : Dg(x)\Big] dV(x,T), \end{aligned}$$

where $\varphi_t(x) = x + tg(x)$, $\varphi_t^{\#} V$ is the image varifold of V through φ_t, $B_F(x,T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is an explicitly computable $n \times n$ matrix and $\langle A, B\rangle := \operatorname{tr} A^T B$ for $A, B \in \mathbb{R}^n \otimes \mathbb{R}^n$.

We have then the following:

Question 1. *Is it true that for every $V \in \mathbb{V}_d(\Omega)$ such that $\delta_F V$ is a Radon measure in Ω, the associated varifold V_* defined as*[1]

$$V_* := V \llcorner \{x \in \Omega : \Theta_*^d(x, V) > 0\} \times G(n,d) \tag{2}$$

is d-rectifiable?

In a joint work with G. De Philippis and F. Ghiraldin, [7], we prove that this is true if and only if F satisfies the following *atomic condition* at every point $x \in \Omega$.

Definition 1. *For a given integrand $F \in C^1(\Omega \times G(n,d))$, $x \in \Omega$ and a Borel probability measure $\mu \in \mathcal{P}(G(n,d))$, let us define*

$$A_x(\mu) := \int_{G(n,d)} B_F(x,T)\, d\mu(T) \in \mathbb{R}^n \otimes \mathbb{R}^n.$$

We say that F verifies the atomic condition *at x if the following two conditions are satisfied:*

(i) $\dim \operatorname{Ker} A_x(\mu) \le n - d$ *for all* $\mu \in \mathcal{P}(G(n,d))$,
(ii) if $\dim \operatorname{Ker} A_x(\mu) = n - d$, *then* $\mu = \delta_{T_0}$ *for some* $T_0 \in G(n,d)$.

Since the atomic condition is essentially necessary to a positive answer to Question 1, see Theorem 2, it is relevant to relate it to the previously known notions of *ellipticity* (or *convexity*) of F with respect to the "plane" variable T. For $d = (n-1)$ we can completely characterize the integrands satisfying the atomic condition. In this case F can be equivalently thought as a positive one-homogeneous even function $G : \Omega \times \mathbb{R}^n \to (0,\infty)$ via the identification

$$G(x, \lambda\nu) := |\lambda| F(x, \nu^{\perp}) \qquad \text{for all } \lambda \in \mathbb{R} \text{ and } \nu \in \mathbb{S}^{n-1}. \tag{3}$$

The atomic condition then turns out to be equivalent to the *strict convexity* in the second variable of G in all but the radial directions, more precisely:

Theorem 1. *An integrand $F \in C^1(\Omega \times G(n, n-1), (0,\infty))$ satisfies the atomic condition at x if and only if the function $G(x,\cdot)$ defined in (3) satisfies:*

$$G(x,\nu) > \langle d_\nu G(x,\bar\nu), \nu\rangle \qquad \textit{for all } \bar\nu, \nu \in \mathbb{S}^{n-1} \textit{ and } \nu \neq \pm\bar\nu.$$

[1]Here $\Theta_*^d(x, V)$ is the lower d-dimensional density of V at the point x.

The following theorem positively answers Question 1:

Theorem 2. *Let $F \in C^1(\Omega \times G(n,d),(0,+\infty))$ be a positive integrand and let us define*

$$\mathscr{V}_F(\Omega) = \Big\{V \in \mathbb{V}_d(\Omega) : \delta_F V \text{ is a Radon measure}\Big\}.$$

Then we have the following:

(i) If F satisfies the atomic condition at every $x \in \Omega$, then for every $V \in \mathscr{V}_F(\Omega)$ the associated varifold V_* defined in (2) is d-rectifiable.

(ii) Assume that F is autonomous, i.e. that $F(x,T) \equiv F(T)$: then every V_* associated to a varifold $V \in \mathscr{V}_F(\Omega)$ is d-rectifiable if and only if F satisfies the atomic condition.

In particular Theorem 2 provides a new independent proof of Allard's rectifiability theorem for the area integrand, $F(x,T) \equiv 1$.

We apply Theorem 2 in [9] to prove an anisotropic counterpart of Allard's compactness for integral varifolds. A d-rectifiable varifold V is said to be integral if in the representation (1) the density function Θ is integer valued. In [1, Section 6.4], Allard proves that every sequence of rectifiable (resp. integral) varifolds enjoying a uniform bound on the mass and on the isotropic first variation is precompact in the space of rectifiable (resp. integral) varifolds. This proof relies on the monotonicity formula for the mass ratio of stationary varifolds. Although the monotonicity formula is deeply linked to the isotropic case, in [9] we can get the following integrality theorem for elliptic integrands:

Theorem 3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F \in C^1(\Omega \times G(n,d),(0,+\infty))$ be a positive integrand satisfying the atomic condition at every $x \in \Omega$. Let $(V_j)_{j\in\mathbb{N}} \subseteq \mathbb{V}_d(\Omega)$ be a sequence of integral varifolds converging to a varifold V. Assume that V enjoys the density lower bound*

$$\Theta^d_*(x,V) > 0 \quad \text{for } \|V\|\text{-a.e. } x \in \Omega$$

and that the sequence $(V_j)_{j\in\mathbb{N}}$ satisfies

$$\sup_{j\in\mathbb{N}} |\delta_F V_j|(W) < \infty, \qquad \forall\, W \subset\subset \Omega;$$

then $V \llcorner \Omega \times G(n,d)$ is an integral varifold.

In the joint work [10] with G. De Philippis and F. Ghiraldin, we apply Theorem 2 to get a compactness principle for the set theoretical formulation of the anisotropic Plateau problem in any codimension, extending the previous results proved in [6, 8, 5]. In particular, we perform a new strategy for the proof of the rectifiability of the minimal set, based on Theorem 2. Given a closed subset $H \subset \mathbb{R}^n$, which will denote the *boundary*, and prescribed a class $\mathcal{P}(H)$ of relatively closed d-rectifiable subsets K of $\mathbb{R}^n \setminus H$, we can formulate the anisotropic Plateau problem by asking whether the infimum

$$m_0 := \inf\left\{\mathbf{F}(K) := \int_K F(x, T_xK)\, d\mathcal{H}^d(x) \,:\, K \in \mathcal{P}(H)\right\} \tag{4}$$

is achieved by some set, which should be a suitable limit of a minimizing sequence. We say that a sequence $(K_j)_{j\in\mathbb{N}} \subset \mathcal{P}(H)$ is a *minimizing sequence* if $\mathbf{F}(K_j) \to m_0$. Under suitable conditions on F and under the assumption that the class $\mathcal{P}(H)$ is (roughly speaking) closed by Lipschitz deformations which are the identity on the boundary H, we can prove the following:

Theorem 4. *Assume that $m_0 < \infty$ and let $(K_j)_{j\in\mathbb{N}} \subset \mathcal{P}(H)$ be a minimizing sequence. Then, up to subsequences,*

$$F(\cdot, T_{(\cdot)}K_j)\mathcal{H}^d \llcorner K_j \rightharpoonup^\star F(\cdot, T_{(\cdot)}K)\mathcal{H}^d \llcorner K =: \mu$$

in the sense of measures in $\mathbb{R}^n \setminus H$,

where $K = \operatorname{spt}\mu \setminus H$ is a d-rectifiable set. Furthermore, the integral varifold naturally associated to μ is F-stationary in $\mathbb{R}^n \setminus H$. In particular, $\liminf_j \mathbf{F}(K_j) \geq \mathbf{F}(K)$ *and if $K \in \mathcal{P}(H)$, then K is a minimum for* (4).

Corollaries of Theorem 4 are the solutions to three formulations of the Plateau problem: one introduced by Reifenberg in [12], one proposed by Harrison and Pugh in [11] and another one studied by David in [4] (and inspired by Almgren's $(\mathbf{M}, 0, \infty)$-minimal sets in [3]).

To conclude, we combine Theorem 2 and Theorem 3 in [9] to extend Theorem 4 to the minimization of anisotropic energies in classes of rectifiable varifolds in any dimension and codimension, in the spirit of the existence and regularity theorem for homological boundary constraints achieved by Almgren in [2]. We prove that the limit of a minimizing sequence of varifolds with density uniformly bounded from below is rectifiable. Moreover, with the further assumption that the minimizing sequence is made of integral varifolds with uniformly locally bounded anisotropic first variation, the limiting varifold turns out to be integral.

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The structure of minimal surfaces near polyhedral cones

Nick Edelen
(joint work with M. Colombo, L. Spolaor)

Let us call a 2-dimensional stationary cone $\mathbf{C}_0^2 \subset \mathbb{R}^{2+k}$ polyhedral if its link with the sphere is an equiangular geodesic net, in the sense that every junction consists of precisely three geodesics meeting at 120^0 degrees. Three basic examples are $\mathbb{R}^2$, $\mathbf{Y} \times R$ consisting of three half-planes meeting at 120^0, and the tetrahedral cone $\mathbf{T}$, which is the cone of the 1-skelaton of a regular tetrahedron. These cones arise naturally as singularity models for soap bubbles.

In this paper, with Maria Colombo and Luca Spolaor, we a prove $C^{1,\alpha}$-regularity theorem for minimal varifolds M (or varifolds with bounded mean curvature) which weakly resemble polyhedral cones $\mathbf{C}_0$. For varifolds which additionally admit a kind of "no hole" condition on the singular set, we establish a $C^{1,\alpha}$-regularity for M near $\mathbf{C}_0^2 \times \mathbb{R}^{n-2}$, for $\mathbf{C}_0$ polyhedral.

In general, a tangent cone of M may gain symmetries not seen in M – in fact, it is an open question whether or not an isolated singularity can admit a tangent with a line of symmetry. For this reason, to establish a C^1-type regularity of M near a cone like $\mathbf{C}_0 \times \mathbb{R}^{n-2}$, one must know (or prove) that "lots" of singular points are present in M near the spine $\{0\} \times \mathbb{R}^{n-2}$. Simon [4] was the first to consider regularity near a general cone of the form $\mathbf{C}_0 \times \mathbb{R}^m$, for $\mathbf{C}_0$ smooth, and he established a dichotomy of the form: either there is a "gap" in the singular set, or the excess decays to one scale.

Our main result builds on the seminal results of Simon, extending this dichotomy to polyhedral cones of the form $\mathbf{C}_0^2 \times \mathbb{R}^{n-2}$.

Theorem 1. *Suppose M is an integral stationary n-varifold in $\mathbb{R}^{n+k}$, and $\mathbf{C}_0^2$ is an integrable polyhedral cone (e.g., if $\mathbf{C}_0$ is contained in an $\mathbb{R}^3$). There are ε and θ, so that: if $0 \in M$, $|M \cap B_1| \leq (3/2)|\mathbf{C}_0 \cap B_1|$, and*

$$\int_{M \cap B_1} dist(x, \mathbf{C}_0^2 \times \mathbb{R}^{n-2})^2 < \varepsilon^2,$$

then either *there is a "gap" in the singular set:*

$$B_\varepsilon(X) \cap \{Y : \theta_M(Y) \geq \theta_{\mathbf{C}_0}(0)\} = \emptyset \quad \textit{for some } X \in \{0\} \times \mathbb{R}^{n-2},$$

or *there is an excess decay:*

$$\theta^{-n-2}\int_{M\cap B_\theta} dist(x,q(\mathbf{C}_0\times\mathbb{R}^{n-2}))^2 \leq (1/2)\int_{M\cap B_1} dist(x,\mathbf{C}_0\times\mathbb{R}^{n-2})^2,$$

for some rotation q.

Here $\theta_M(X)=\lim_{r\to 0} r^{-n}|M\cap B_r(X)|$ is the density of M at X.

In certain important circumstances, we can deduce the presence of singularities, and then by iterating the above excess decay, we get $C^{1,\alpha}$ regularity. For example, if M is sufficiently weakly close to $\mathbf{Y}\times\mathbb{R}^{n-1}$, then M must have "lots" of singular points of the type $\mathbf{Y}$. In this case, proven by Simon [4], M is a $C^{1,\alpha}$ perturbation of the $\mathbf{Y}\times\mathbb{R}^{n-1}$.

Our main Corollary is that, for $\mathbf{C}_0=\mathbf{T}$, and when M has a "cycle structure," then we can deduce a no holes condition, and thereby establish $C^{1,\alpha}$-regularity. We say M has a cycle structure if the following holds: there are countably many integral currents T_i, with $\partial T_i=0$, so that writing

$$T_i(\omega)=\int_{M_{T_i}} < \xi_{T_i},\omega > \theta_{T_i} d\mathcal{H}^n$$

for M_{T_i} rectifiable, then $M=\mathcal{H}^n \llcorner \cup_i M_{T_i}$. In other words, M is supported on the union of rectifiable sets, each arising from a current with zero boundary.

Corollary 1. *There is an* $\varepsilon(n)$ *so that the following holds. Let* M^n *be a stationary integral varifold in* $\mathbb{R}^{n+1}$*, such that*

$$0\in M,\quad |M\cap B_1|\leq(3/2)|\mathbf{T}\cap B_1|,\quad \int_{M\cap B_1} dist(x,\mathbf{T}\times\mathbb{R}^{n-2})^2<\varepsilon^2,$$

and M *has an associated cycle structure. Then* $M\cap B_{1/2}$ *is a* $C^{1,\alpha}$*-perturbation of* $\mathbf{T}\times\mathbb{R}^{n-2}$.

Here is a neat application of our result. A cluster minimizer is a solution to the problem: find a partition $E_0\cup E_1\cup\cdots\cup E_N=\mathbb{R}^{n+1}$ minimizing the perimeter

$$P=(1/2)\sum_{i=0}^{N}|\partial^* E_i|,$$

subject to the restraints $E_1=m_1,\ldots,E_N=m_N$, for specified volumes $m_1,\ldots,m_N>0$. In other words, we find the least area for a multiply-constrained volume problem. Almgren [2] has shown existence of cluster minimizers, for any n, N, and choice of $m_1,\ldots,m_N$, and moreoever proved that the underlying varifold $M=\mathcal{H}^n\llcorner\cup_i\partial_i^* E_i$ has bounded mean curvature, and the support of M is $(\mathbf{M},\varepsilon,\delta)$-minimizing.

A beautiful theorem of Taylor [5] implies that if $n=2$, then M is locally $C^{1,\alpha}$ equivalent to either $\mathbb{R}^2$, $\mathbf{Y}\times\mathbb{R}$, or $\mathbf{T}$. By combining our result with results of Allard [1], Simon [4], and Naber-Valtorta [3], we obtain the following generalization:

Theorem 2. *Let* M *be the support of a cluster minimizer. Then* M *decomposes as* $M=M_n\cup M_{n-1}\cup M_{n-2}\cup M_{n-3}$*, where* M_n, M_{n-1}, M_{n-2} *are locally* $C^{1,\alpha}$

manifolds, near which M is locally diffeomorphic to $\mathbb{R}^n, \mathbf{Y}\times\mathbb{R}^{n-1}, \mathbf{T}\times\mathbb{R}^{n-2}$ (resp), and M_{n-3} is closed, locally finite, $(n-3)$-rectifiable.

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An epiperimetric approach to isolated singularities

Max Engelstein
(joint work with Luca Spolaor, Bozhidar Velichkov)

We present a new technique for studying the infinitesimal behavior of energy minimizers near the points where the minimizer exhibits non-smooth behavior. To avoid further vaguaries, we focus our attention on minimizers to the functional,

$$\int |\nabla u|^2 + \chi_{\{u>0\}}, \tag{1}$$

though the technique described below is quite general and has yielded similar results in the setting of (almost-)area minimizing currents (c.f. [7]) .

We are interested in what is called the "free boundary", $\partial\{u > 0\}$. In [2], Alt and Caffarelli proved the following dichotomy; let $x_0 \in \partial\{u > 0\}$, then either the free boundary in a neighborhood of x_0 can be written as the graph of an analytic function, or there is no $r > 0$ such that $B_r(x_0) \cap \partial\{u > 0\}$ is contained in a δr-neighborhood of an $(n-1)$-plane (for some $\delta > 0$). The former points are called **regular**, and the latter, **singular**.

We can rephrase this result in terms of parameterization; the free boundary near regular points is parameterized over an $(n-1)$-plane by a smooth function. The long-term goal of our investigation (and a central problem in the subject of regularity theory) is to extend this parameterization to singular points. The main theorem of this talk does this for a class of singular points:

Theorem 1. *[Main Theorem in [6]] Let $b \in W^{1,2}(B_1)$ be a 1-homogenous minimizer to (1), such that $\partial\{b > 0\}$ is smooth away from 0. Assume that u is a minimizer and $r_j \downarrow 0, x_0 \in \partial\{u > 0\}$ are such that $\frac{\partial\{u>0\}-x_0}{r_j} \to \partial\{b > 0\}$. Then $\lim_{r\downarrow 0} \frac{\partial\{u>0\}-x_0}{r} = \partial\{b > 0\}$ and there exists some $r_0 > 0$ such that $\partial\{u > 0\} \cap B_{r_0}(x_0)$ can be written as the $C^{1,\log}$ image of $B_{r_0}(0) \cap \partial\{b > 0\}$.*

Let us make three quick remarks; first if $\frac{\partial\{u>0\}-x_0}{r_j} \to S$ for some set S, we call S a **blow-up** of $\partial\{u > 0\}$ at x_0. Alternatively, we can examine $u_{r_j,x_0}(x) \equiv \frac{u(r_jx+x_0)}{r_j}$ and refer to $\lim_j u_{r_j,x_0} = v(x)$ as the blowup. Note that $S = \partial\{v > 0\}$ in this scenario. The theorem above is an example of a "uniqueness of blow-ups" result, more on this below. Second, the assumption that b is 1-homogeneous is redundant; it is a result of Weiss [15] that if b is a blowup of a minimizer, then b must be 1-homogenous. The final remark is that the regularity of the parameterization depends on the "symmetries" of the cone b. Imprecisely, if the only deformations of b which preserve the energy (1) to second order are given by ambient isometries of the space, then we call b **integrable through rotations** and the parameterization given by Theorem 1 is $C^{1,\alpha}$. Otherwise, the stated $C^{1,\log}$ regularity is optimal. We note that the only known one-homogenous minimizers to (1) were constructed by De Silva and Jerison [5] and each of these are integrable through rotations.

As mentioned above, a central question is the uniqueness of blow-ups; the limit $\frac{u(r_jx+x_0)}{r_j}$ exists up to subsequence by compactness, but in order to parameterize the free boundary over the blow-up it must be the case that the limit is independent of the subsequence $r_j \downarrow 0$. Theorem 1 is the only known uniqueness of blowups result in the setting of (1), but similar questions have been investigated for obstacle problems ([14], [9]), harmonic maps ([12]) and minimal surfaces ([12], [1], [13]). Uniqueness of blow-ups is not always true; Brian White constructs harmonic maps from $\mathbb{R}^4 \to N$ where N is a C^∞ four-manifold such that there is an isolated critical point with a continuum of blow-ups at that point, see [17].

The main tool in proving Theorem 1 is what is called an epiperimetric inequality. In [15], Weiss proved that

$$W(u, x_0, r) \equiv \frac{1}{r^n}\int_{B_r(x_0)} |\nabla u|^2 + \chi_{\{u>0\}}dx - \frac{1}{r^{n+1}}\int_{\partial B_r(x_0)} u^2 d\sigma,$$

is monotone increasing in r as long as u is a minimizer to (1) and that the difference, $W(u, x_0, r) - W(u, x_0, s)$, measures how far u is from being one-homogeneous in the annulus $B_r(x_0)\backslash B_s(x_0)$. Thus to prove a uniqueness of blowups result like Theorem 1, it suffices to bound the growth of $r \mapsto W(u, x_0, r)$ from above. This is the role of an epiperimetric inequality (see Theorem 2 below), which says, roughly, that the difference in energy between the one-homogeneous and minimizing extensions of a trace, $c \in L^2(\partial B_1)$, is proportional to the gap between c and the "closest" trace of a one-homogeneous minimizer (where the gap is measured by W).

Epiperimetric inequalities have been used to prove uniqueness of blow-ups and regularity in minimal surfaces [11, 13, 16] and free boundary problems [14, 8, 10]. Recently, the second and third authors, with Maria Colombo [3, 4], have pioneered the concept of a log-epiperimetric inequality, in which the gap (alluded to above) has non-linear dependence, which in turn gives a $C^{1,\log}$ rate of blow-up.

Before we state our epiperimetric inequality, let us briefly outline one critical way in which ours differs from those mentioned above. Our epiperimetric inequality is the first to treat blow-ups which are not integrable through rotations. In order for the minimizing extension to be quantitatively better than the homogeneous one, one often needs to identify which trace of a homogeneous minimizer is "closest" to the given trace. The condition of being integrable through rotations means that all the "nearby" traces of homogeneous minimizers are simply rotations of each other, which makes it easier to find the closest one through an implicit function theorem argument (see [16]).

In order to prove a log-epiperimetic inequality at singularities which are not integrable through rotations, we had to find nearby problematic traces by hand. To do so, we borrowed a powerful idea from L. Simon [12], and used a Lyaponov-Schmidt reduction to identify the closest "problematic" trace and used gradient flow to improve its energy. We then invoked the Łojasiewicz inequality to show that this energy improvement was quantitative.

Let us end with a statement of our epiperimetric inequality. For space considerations we take $x_0 = 0$ and $r = 1$ and refer to $W(f, x_0, r)$ simply as $W(f)$:

Theorem 2. *[Epiperimetric inequality in [6]] Let $b \in H^1(B_1)$ be a one-homogeneous minimizer of (1) with an isolated singularity at the origin. There exist constants $\epsilon = \epsilon(d, b) > 0, \gamma = \gamma(d, b) \in [0, 1)$ and $\delta_0 = \delta_0(d, b) > 0$, depending on b and on the dimension d, such that the following holds.*

If $c \in H^1(\partial B_1, \mathbb{R}_+)$ is such that there exists $\zeta \in C^{2,\alpha}(\partial\{b > 0\} \cap \mathbb{S}^{n-1})$ such that $\partial\{c > 0\}$ is the graph (in the sphere) of ζ over $\partial\{b > 0\} \cap \mathbb{S}^{n-1}$ and

$$\|\zeta\|_{C^{2,\alpha}} \leq C_d \|\zeta\|_{L^2} < \delta\,, \quad \text{and} \quad \|c - b\|_{L^2(\partial B_1)} < \delta\,, \tag{2}$$

then there exists a function $h \in H^1(B_1, \mathbb{R}_+)$ such that $h = c$ on ∂B_1 and

$$W(h) - W(b) \leq \Big(1 - \epsilon \big|W(z) - W(b)\big|^{\gamma}\Big)\big(W(z) - W(b)\big), \tag{3}$$

where z is the 1-homogeneous extension of c to B_1.

In the case where b is integrable through rotations, we can take $\gamma = 0$ in (3) above.

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Homogenization, random matrices, and integral representations for transport in composite materials: how can studies of sea ice and its role in the climate system help advance variational analysis?

Kenneth M. Golden

(joint work with Elena Cherkaev, Ben Murphy, Noa Kraitzman, Rebecca Hardenbrook, Christian Sampson, and Huy Dinh)

The analytic continuation method for two phase composite media [15] provides Stieltjes integral representations for the effective or homogenized transport parameters, such as effective conductivity, complex permittivity, or diffusion coefficent. The method was developed independently by Bergman and Milton [3, 13] in the late 1970's, and yields rigorous bounds on the effective properties under constraints on the composite geometry. Subsequently a mathematical formulation of this approach was given by Golden and Papanicolaou [6], and extended to multicomponent media [7, 5, 14]. The mixture geometry of the phases is incorporated into the Stieltjes integral for the effective parameter through the spectral measures of a self-adjoint operator involving the characteristic function of one of the phases. A resolvent representation for the local field in the problem involves this operator, and is the key analytic step in this approach. From the point of view of the calculus of variations, the analytic continuation method yields rigorous bounds that do *not* rely on variational principles. Instead, bounds are obtained by extreme points of appropriate sets of measures, such as Dirac point masses. Classical bounds for real parameters, such as the Wiener and Hashin-Shtrikman bounds based on energy variational principles [15], can then be extended to the complex case.

Figure 1. **Multiscale structure of sea ice.** From left to right: Sub−millimeter scale brine inclusions form the porous microstructure of sea ice [22]. When the temperature exceeds a critical value, the brine phase percolates enabling fluid flow, which mediates a broad range of key climatological and biological processes. The centimeter scale polycrystalline structure of sea ice [1] helps determine its strength and fluid flow properties. Decimeter to meter scale pancakes forming in a wave field in the Southern Ocean (Golden). Wave−ice interactions have historically been important in the Southern Ocean, and have become increasingly significant in the Arctic Ocean with receding sea ice and larger fetch. Meter to kilometer scale melt ponds on the surface of Arctic sea ice in late spring and summer (Perovich) determine the albedo of sea ice, a key parameter in climate modeling. Kilometer scale (and smaller) floes in the Arctic sea ice pack (Perovich) exhibit local diffusion in interacting with other floes and are advected in larger scale wind and current fields.

There have been significant advances in applying the analytic continuation method - and the bounding procedure - to important classes of composite materials, such as matrix-particle composites [4], and in extending the method to other transport problems, such as the advection diffusion equation [2]. In our work we have been interested in using the ideas of the analytic continuation method to address problems in linking scales in the polar sea ice system and its role in Earth's climate [9] (see Figure 1 above). That is, we are developing methods to relate behavior and structure on small scales, such as the sub-millimeter scale brine inclusions in a pure ice host, to effective or homogenized behavior on larger scales [8], and the inverse problem of estimating parameters controlling small scale processes from large scale observations [10, 20]. We have also been working to extend the method to related transport problems on length scales larger than the brine inclusions, such as its polycrystalline structure [11], advection diffusion procecesses [18, 19, 12], and waves in the Marginal Ice Zone [21]. For example, we have developed the first rigorous theory for thermal conduction through sea ice in the presence of a convective flow field, and the first bounds on the complex viscoelasticity for waves propagating through the sea ice pack.

Finally, by focusing on computations of the spectral measure for discretizations of images of sea ice structures [17], where the key self adjoint operator becomes a random matrix, we made an unexpected discovery [16]. Anderson transitions, such as the metal-insulator transition where electronic wave functions become localized with sufficient disorder in the system, have been observed throughout the physics of waves in solids, optics, acoustics, and fluids. We recently uncovered the hallmarks of the Anderson transition for classical homogenized transport coefficients in two phase composite materials, such as the effective thermal conductivity or complex permittivity, without wave interference or scattering effects. As one of the phases in a composite becomes connected and develops long range order, such as the brine phase in sea ice forming channels through which fluid can flow, we observe striking transitional behavior. The eigenvalues of the random matrix governing effective transport undergo a transition from uncorrelated Poisson statistics to universal Wigner-Dyson statistics with replulsion, and the field eigenvectors become delocalized with the appearance of mobility edges as in quantum mechanics.

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Construction of rank-one convex, non-quasiconvex functions via nonreflexive representations of Jordan multialgebras in the theory of exact relations for effective tensors of composite materials

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This extended abstract is an update of OWR-33/2017 (Vol. 14, No. 3, pp. 2063-2065).

A central concept in modern Calculus of Variations is quasiconvexity, introduced by C.B. Morrey [10] as a criterion of sequential weak-* lower semi-continuity. The same condition is also a necessary condition for a C^1 vector field $\boldsymbol{u}_0(\boldsymbol{x})$ to be a strong local minimizer of the energy functional

$$E[\boldsymbol{u}] = \int_\Omega W(\nabla \boldsymbol{u}) d\boldsymbol{x}.$$

Indeed, using $\boldsymbol{u}_\epsilon = \boldsymbol{u}_0(\boldsymbol{x}) + \epsilon\boldsymbol{\phi}((\boldsymbol{x}-\boldsymbol{x}_0)/\epsilon)$ as a competitor, we compute

$$\delta E[\boldsymbol{\phi}] = \lim_{\epsilon\to 0^+} \frac{E[\boldsymbol{u}_\epsilon] - E[\boldsymbol{u}_0]}{\epsilon^d} = \int_{\mathbb{R}^d} \{W(\nabla \boldsymbol{u}(\boldsymbol{x}_0) + \nabla\boldsymbol{\phi}) - W(\boldsymbol{F})\} d\boldsymbol{x}$$

Quasiconvexity condition at $\boldsymbol{F} \in \mathbb{R}^{m\times d}$ can be regarded as the Jensen's inequality restricted to gradients:

$$\int_{\mathbb{R}^d} \{W(\boldsymbol{F} + \nabla\boldsymbol{\phi}) - W(\boldsymbol{F})\} d\boldsymbol{x} \geq 0$$

for every $\boldsymbol{\phi} \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^m)$.

An early attempt to understand the true meaning of this concept was to decide if it was equivalent to rank-one convexity (convexity along all rank-one lines). The question remained opened until Šverák settled it [11], giving an example of a Lagrangian defined on 3x2 matrices, that is rank-one convex, but not quasiconvex. The example is a polynomial function with no apparent symmetries. For any function $W(\boldsymbol{F})$ its quasiconvex envelope (the largest quasiconvex function that does not exceed $W(\boldsymbol{F})$) is given by Dacorogna's formula [2]

$$QW(\boldsymbol{F}) = \inf_{\boldsymbol{\phi}:[0,1]^d\text{-periodic}} \int_{[0,1]^d} W(\boldsymbol{F} + \nabla\boldsymbol{\phi}) d\boldsymbol{x} \tag{1}$$

There is also a parallel question about the effective behavior of composite materials. A periodic (for simplicity) composite material is described by a $Q = [0,1]^d$-periodic tensor of local material properties $\mathsf{L} : Q \to U \subset \mathrm{Sym}^+(\mathbb{R}^{m\times d})$, where U is a set of constituent materials. The effective tensor $\mathsf{L}^* \in \mathrm{Sym}^+(\mathbb{R}^{m\times d})$ of a composite with period cell properties $\mathsf{L}(\boldsymbol{x})$ is defined via a variational principle [8]

$$\langle \mathsf{L}^*\boldsymbol{F}, \boldsymbol{F}\rangle = \inf_{\boldsymbol{\phi}:Q\text{-periodic}} \int_Q \langle \mathsf{L}(\boldsymbol{x})(\boldsymbol{F} + \nabla\boldsymbol{\phi}), \boldsymbol{F} + \nabla\boldsymbol{\phi}\rangle d\boldsymbol{x} \tag{2}$$

The main problem in the theory of composites is computation of the G-closure set $G(U)$—the set of effective tensors of periodic composites made with materials from U. Now if we define

$$W_U(\boldsymbol{F}) = \inf_{\mathsf{L}\in U} \langle \mathsf{L}\boldsymbol{F}, \boldsymbol{F}\rangle,$$

then, combining (1) and (2), we have a link between quasiconvexification and G-closure:

$$QW_U(\boldsymbol{F}) = W_{G(U)}(\boldsymbol{F}). \tag{3}$$

The analog of rank-one convexity on the composites side is a laminate—a composite made by alternating layers of two given materials. We note that if U contains effective tensor of every laminate of two materials from U, then $W_U(\boldsymbol{F})$ is rank-one convex. One can therefore ask whether every composite can be mimicked by an iterated laminate made with the same set of constituent materials. This question has also been answered in the negative, by Milton [9, Sections 31.8–9], who exploited Šverák's construction and a well-known connection between G-closed sets and quasiconvex functions and L-closed sets (sets containing effective tensors of all laminates made from materials taken from that set) and rank-one convex functions.

Exploiting the relation with the theory of composites in the opposite direction is seemingly more difficult, as we restrict the set of possible integrands to $W_U(\boldsymbol{F})$. The idea is to first find U, stable under lamination and therefore generating the rank-one convex W_U, such that it is not G-closed. We will then need to obtain just enough information about $G(U)$ to conclude that $W_{G(U)}(\boldsymbol{F}_0) < W_U(\boldsymbol{F}_0)$ at a particular $\boldsymbol{F}_0$. Such an example comes from the theory of exact relations—formulas that hold for effective tensors of all composites made with a given set of materials, regardless of the microstructure [5, 3]. The theory identifies submanifolds of $\mathrm{Sym}^+(\mathbb{R}^{m\times d})$ that are stable under lamination. According to the theory all such submanifolds are diffeomorphic images[1] of convex subsets of Jordan multialgebras—subspaces of $\mathrm{Sym}(\mathbb{R}^{m\times d})$ closed with respect to a family of

[1]With a completely explicit diffeomorphism.

Jordan multiplications

$$\mathsf{K}_1 *_{\mathsf{A}} \mathsf{K}_2 = \frac{1}{2}(\mathsf{K}_1\mathsf{A}\mathsf{K}_2 + \mathsf{K}_2\mathsf{A}\mathsf{K}_1), \tag{4}$$

parametrized by $\mathsf{A} \in \mathfrak{A} = \boldsymbol{I}_m \otimes \mathcal{A}$, where $\mathcal{A} = \{\boldsymbol{A} \in \mathrm{Sym}(\mathbb{R}^d) : \mathrm{Tr}\,\boldsymbol{A} = 0\}$. The strategy for finding the desired example was inspired by understanding the algebraic properties of all subspaces of $\mathrm{Sym}(\mathbb{R}^n)$ that are closed with respect to the symmetrized product $X * Y = (XY + YX)/2$ [1]. All such subspaces are representations of formally real Jordan algebras characterized by Jordan, von Neumann and Wigner [7] for the purposes of building quantum mechanics on an axiomatic foundation. While their effort had ultimatley failed, the work marked the beginning of systematic study of Jordan algebras.

The example is based on the observation that the multiplication of quaternions (a number system invented by W. R. Hamilton [6]) gives rise to a linear map $Q(h) : \mathbb{R}^4 \to \mathbb{R}^4$, if we make a natural identification between $\mathbb{R}^4$ and the algebra of quaternions $\mathbb{H}$, via $Q(h)\boldsymbol{q} = hq$. In order to simplify notation we denote by h the 4×4 matrix of $Q(h)$. Then

$$U = \left\{ \mathsf{L} = \begin{bmatrix} \lambda & h \\ \overline{h} & \mu \end{bmatrix} \in \mathfrak{H}^+(\mathbb{H}^2) : \det\nolimits_{\mathbb{H}} \mathsf{L} = \lambda\mu - |h|^2 = 1 \right\}$$

is the desired set of materials. We then compute [4]

$$W_U(\boldsymbol{F}) = \sqrt{\det\nolimits_{\mathbb{H}}(\boldsymbol{F}^T\overline{\boldsymbol{F}})} \tag{5}$$

Using quaternions a function $\boldsymbol{u} : \mathbb{R}^2 \to \mathbb{R}^8$ will be represented by $\boldsymbol{u} : \mathbb{R}^2 \to \mathbb{H}^2$. Then

$$W_U(\nabla \boldsymbol{u}) = \sqrt{\left\|\frac{\partial \boldsymbol{u}}{\partial x}\right\|_{\mathbb{H}^2}^2 \left\|\frac{\partial \boldsymbol{u}}{\partial y}\right\|_{\mathbb{H}^2}^2 - \left|\left(\frac{\partial \boldsymbol{u}}{\partial x}, \frac{\partial \boldsymbol{u}}{\partial y}\right)_{\mathbb{H}^2}\right|^2}, \tag{6}$$

is rank-one convex, but not quasiconvex: $QW(\boldsymbol{I}_2) < W(\boldsymbol{I}_2)$, where $\boldsymbol{I}_2$ is the quaternionic 2×2 identity matrix. This beautiful function has a very large group of symmetries

- $W(\lambda \boldsymbol{F}) = \lambda^2 W(\boldsymbol{F}), \quad \forall \lambda \in \mathbb{R}$
- $W(\boldsymbol{F}\boldsymbol{R}) = W(\boldsymbol{F}) \quad \forall \boldsymbol{R} \in O(2, \mathbb{R})$
- $V(\boldsymbol{G}\boldsymbol{Q}) = V(\boldsymbol{G}) \quad \forall \boldsymbol{Q} \in Sp(2) \cong U(2, \mathbb{H}) = \{\boldsymbol{Q} \in \mathbb{H}^{2\times 2} : \boldsymbol{Q}\boldsymbol{Q}^H = \boldsymbol{I}_2\}$

For full discussion see [4].

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Fine structure of branch sets of Dirichlet energy minimizing multi-valued functions

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(joint work with Neshan Wickramasekera)

We consider the structure of singularities for multi-valued Sobolev functions which minimize Dirichlet energy. Dirichlet energy minimizing multi-valued functions were introduced by Almgren [Alm83] as part of his monumental work showing that the singular set of an n-dimensional area minimizing submanifold (i.e. integral current) of a smooth Riemannian manifold has Hausdorff dimension at most $n-2$ (also see [DeLSpa14, DeLSpa16a, DeLSpa16b]). Let M be an n-dimensional singular submanifold of a Riemannian manifold which is a critical point of area (i.e. let M be an area stationary integral varifold). The *singular set* $\operatorname{sing} M$ is the set of points $p \in M$ for which there is no $\delta > 0$ such that $M \cap B_\delta(p)$ is a smoothly embedded submanifold (with constant integer multiplicity). For general n-dimensional area stationary submanifolds M, it is unknown whether $\mathcal{H}^n(\operatorname{sing} M) = 0$. This is precisely due to the presence of *branch point singularities*, at which at least one tangent cone to M is an n-dimensional plane with integer multiplicity ≥ 2. When M is codimension one and area minimizing, De Giorgi [DG61] showed that branch point singularities of M do not occur and thus by [SJ68, Fed69] $\dim_{\mathcal{H}}(\operatorname{sing} M) \leq n-7$. When M is an area minimizing submanifold of higher codimension, branch point singularities do occur; for instance, the holomorphic variety $M = \{(z,w) \in \mathbb{C}^2 : w^2 = z^3\}$ has a branch point at the origin. Almgren argued that at a branch point of M at which M is tangent to a multiplicity q plane P, M can be approximated by the graph of a Dirichlet energy minimizing q-valued function over P. Almgren developed a theory of Dirichlet energy minimizing q-valued functions on domains in $\mathbb{R}^n$, including existence of solutions to the Dirichlet problem, interior Hölder continuity of Dirichlet energy minimizers, and optimal dimension bound for the singular set of $\dim_{\mathcal{H}}(\operatorname{sing} u) \leq n-2$. He then used this in an intricate blow-up method to deduce that $\dim_{\mathcal{H}}(\operatorname{sing} M) \leq n-2$ for an n-dimensional area minimizing submanifold M.

Let $n, m \geq 1$ and $q \geq 2$ be integers. We let $\mathcal{A}_q(\mathbb{R}^m)$ denote the space of all sums $a = \sum_{i=1}^q [\![a_i]\!]$ of q Dirac point masses $[\![a_i]\!]$ at points $a_i \in \mathbb{R}^m$, where the points a_i are unordered and may repeat. We equip $\mathcal{A}_q(\mathbb{R}^m)$ with the metric $\mathcal{G}$ defined by

$$\mathcal{G}(a,b) = \min_{\sigma} \left(\sum_{i=1}^q |a_i - b_{\sigma(i)}|^2 \right)^{1/2}$$

for all $a = \sum_{i=1}^q [\![a_i]\!]$, $b = \sum_{i=1}^q [\![b_i]\!] \in \mathcal{A}_q(\mathbb{R}^m)$, where the infimum is over all permutations σ of $\{1, 2, \ldots, q\}$. A *q-valued function* $u : \Omega \to \mathcal{A}_q(\mathbb{R}^m)$ is a map from a domain $\Omega \subset \mathbb{R}^n$ into $\mathcal{A}_q(\mathbb{R}^m)$. Since $(\mathcal{A}_q(\mathbb{R}^m), \mathcal{G})$ is a metric space, we may define the spaces of continuous, Hölder continuous, and L^p $(1 \leq p \leq \infty)$ q-valued functions in the usual way and we may define the space of Sobolev q-valued functions $W^{1,2}(\Omega, \mathcal{A}_q(\mathbb{R}^m))$ via theory of Sobolev functions taking values in a metric space as independently developed by [Amb90] and [Res04], see [DeLSpa11]. The *Dirichlet energy* of $u \in W^{1,2}(\Omega, \mathcal{A}_q(\mathbb{R}^m))$ over the domain Ω is given by

$$\int_{\Omega} \sum_{i=1}^q |Du_i(Z)|^2 \, dZ$$

where $X \mapsto \sum_{i=1}^q [\![u_i(Z) + Du_i(Z) \cdot (X - Z)]\!]$ is the q-valued affine approximation of u at $\mathcal{L}^n$-a.e. $Z \in \Omega$ (see [DeLSpa11, Definition 2.6, Corollary 2.7]). We say the q-valued Sobolev function $u \in W^{1,2}(\Omega, \mathcal{A}_q(\mathbb{R}^m))$ is *Dirichlet energy minimizing* if

$$\int_K \sum_{i=1}^q |Du_i(Z)|^2 \, dZ \leq \int_K \sum_{i=1}^q |Dv_i(Z)|^2 \, dZ$$

for all $v \in W^{1,2}(\Omega, \mathcal{A}_q(\mathbb{R}^m))$ such that $u = v$ in $\Omega \setminus K$ for some compact subset $K \subset \Omega$.

The *singular set* Σ_u of a Dirichlet energy minimizing function $u \in W^{1,2}(\Omega, \mathcal{A}_q(\mathbb{R}^m))$ is the set of all points $Y \in \Omega$ such that there is no $\delta > 0$ such that $u(X) = \sum_{i=1}^q [\![u_i(X)]\!]$ on $B_\delta(Y)$ for some single-valued harmonic functions $u_i : B_\delta(Y) \to \mathbb{R}^m$ such that for each $i \neq j$ either $u_i \equiv u_j$ on $B_\delta(Y)$ or $u_i(X) \neq u_j(X)$ for all $X \in B_\delta(Y)$. We let

$$\Sigma_{u,q} = \{Y \in \Sigma_u : u(Y) = q[\![u_1(Y)]\!] \text{ for some } u_1(Y) \in \mathbb{R}^m\}.$$

Define $u_a : \Omega \to \mathbb{R}^m$ and $u_f : \Omega \to \mathcal{A}_q(\mathbb{R}^m)$ by

$$u_a(X) = \frac{1}{q} \sum_{i=1}^q u_i(X), \quad u_f(X) = \sum_{i=1}^q [\![u_i(X) - u_a(X)]\!].$$

Since u is Dirichlet energy minimizing, u_a is a single-valued harmonic function and u_f is a q-valued Dirichlet energy minimizing function with zero average. Moreover, $\Sigma_{u,q} = \Sigma_{u_f,q}$. Thus to study the local structure of $\Sigma_{u,q}$, it suffices to assume that

u is *average-free*, i.e. $u_a \equiv 0$ on Ω. If u is average-free and not identically zero, then

$$\Sigma_{u,q} = \{Y \in \Omega : u(Y) = q[\![0]\!]\}.$$

To each average-free, Dirichlet energy minimizing, q-valued function $u : \Omega \to \mathcal{A}_q(\mathbb{R}^m)$ and each $Y \in \Sigma_{u,q}$ we associate the *frequency function* $N_{u,Y}$ defined by

$$N_{u,Y}(\rho) = \frac{\rho^{2-n} \int_{B_\rho(Y)} \sum_{i=1}^q |Du_i|^2}{\rho^{1-n} \int_{\partial B_\rho(Y)} \sum_{i=1}^q |u_i|^2}$$

for each $0 < \rho < \operatorname{dist}(Y, \partial\Omega)$. Almgren's fundamental observation was that $N_{u,Y}(\rho)$ is monotone nondecreasing as a function of ρ. Consequently, we can define the *frequency* $\mathcal{N}_u(Y)$ of u at Y by

$$\mathcal{N}_u(Y) = \lim_{\rho \downarrow 0} N_u(Y).$$

We know that there exists at least one nonzero, average-free, Dirichlet energy minimizing, homogeneous degree $\mathcal{N}_u(Y)$, q-valued function $\varphi : \mathbb{R}^n \to \mathcal{A}_q(\mathbb{R}^m)$, called a *tangent function*, such that

$$\frac{u(Y + \rho_j X)}{\rho^{-n/2} \|u\|_{L^2(B_{\rho_j}(Y))}} \to \varphi(X)$$

uniformly on compact subsets of $\mathbb{R}^n$ for some $\rho_j \to 0^+$. It is open whether in general φ is unique independent of the sequence (ρ_j). Using tangent functions in a dimension reduction argument, Almgren showed that $\dim_{\mathcal{H}}(\operatorname{sing} u) \leq n - 2$. One can further show that for $\mathcal{H}^{n-2}$-a.e. $Y \in \Sigma_{u,q}$ there is at least one tangent function φ which is translation invariant along an $(n-2)$-dimensional linear subspace and thus after an orthogonal change of coordinates takes the form

$$\varphi(X) = m_0[\![0]\!] + \sum_{j=1}^N m_j [\![\operatorname{Re}(c_j(x_1 + ix_2)^{k_0/q_0})]\!] \tag{1}$$

where $m_0 \geq 0$ and $m_i \geq 1$ $(1 \leq j \leq N)$ are positive integers such that $\sum_{i=0}^N m_i = q$, $k_0 \geq 1$ and $q_0 \in \{1, 2, \ldots, q\}$ are relatively prime integers, and $c_j \in \mathbb{C}^m$. We call the functions 0 and $\operatorname{Re}(c_j(x_1 + ix_2)^{k_0/q_0})$ *components* of φ. Whenever after an orthogonal change of coordinates φ takes the form (1), we say that φ is *cylindrical.*

In [KrumWic-1], we show that for each nonzero, average-free, Dirichlet energy minimizing $u \in W^{1,2}(\Omega, \mathcal{A}_q(\mathbb{R}^m))$ on a domain $\Omega \subseteq \mathbb{R}^n$, $\Sigma_{u,q}$ is countably $(n-2)$-rectifiable and u has a unique cylindrical tangent function at $\mathcal{H}^{n-2}$-a.e. $Y \in \Sigma_{u,q}$. It then follows by induction on q and continuity that for each Dirichlet energy minimizing $u \in W^{1,2}(\Omega, \mathcal{A}_q(\mathbb{R}^m))$, the singular set Σ_u is countably $(n-2)$-rectifiable. Our approach involves a blow-up method of L. Simon [Sim93], which was initially applied to multiplicity one classes of minimal submanifolds. We apply L. Simon's blow-up method in the higher multiplicity setting of Dirichlet energy minimizing functions and introduce several new ideas to accomplish this.

Let us consider an average-free Dirichlet energy minimizing function $u \in W^{1,2}(B_1(0), \mathcal{A}_q(\mathbb{R}^m))$ which is close to a homogeneous degree α cylindrical function φ. We want to show that if u is sufficiently close to φ and u has a high concentration of points of frequency $\geq \alpha$ along the axis of φ, then the L^2-distance of u to homogeneous degree α cylindrical functions is decaying. Our argument proceeds in four steps. In Step 1, we express $\operatorname{graph} u$ has the graph of an appropriate multi-valued function over each of the components of φ. For this to hold true, we must assume that u is much closer to φ than u is to any cylindrical function with the same axis as φ and fewer components than φ, much like was previously assumed in [Wic14]. Step 2 is to use the monotonicity formula for frequency functions to obtain various L^2 estimates for u and φ much like in [Sim93], including an estimate showing that u cannot concentrate near the axis of φ. Step 3 involves taking an appropriate sequence of Dirichlet energy minimizers $u^{(\nu)}$ and cylindrical function $\varphi^{(\nu)}$ converging to a fixed cylindrical function $\varphi^{(0)}$ and "blowing up" $u^{(\nu)}$ with respect to $\varphi^{(\nu)}$. Then we do an analysis of the blow-ups, based on [Sim93], in order to show that the L^2-distance of the blow-ups to a homogeneous degree α blow-up is decaying. This requires rephrasing certain arguments of [Sim93] as blow-up arguments so that they apply in the higher multiplicity setting. This also requires introducing a new estimate for u which was obtained via a comparison argument and implies that the blow-ups satisfy the inner variational formula (see [DeLSpa11, Section 3.1]) for variations of the form $\zeta(X)\,(X - Z)$, where $\zeta \in C^1_c(B_1(0))$ and Z is a point on the axis of $\varphi^{(0)}$. Step 4 involves using a blow-up argument to show that the L^2-distance of u to homogeneous degree α cylindrical functions is decaying. Finally, the rectifiability of $\Sigma_{u,q}$ and a.e. uniqueness of cylindrical tangent functions follows by iteratively applying the excess decay lemma like in [Sim93].

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Models for thin prestrained structures

Marta Lewicka

(joint work with Danka Lučić)

We report on our work concerned with the analysis of thin elastic films exhibiting residual stress at free equilibria. Examples of the prestrained structures and their actuations include: plastically strained sheets, swelling or shrinking gels, growing tissues such as leaves, flowers or marine invertebrates, nanotubes, atomically thin graphene layers, etc. Motivated by the idea of imposing and controlling the *prestrain* (or *"misfit"*) field in order to cause the plate to achieve a desired shape, with study the forward problem based on the minimisation of the elastic energy with incorporated inelastic effects.

0.1. **The set-up.** Let $\omega \subset \mathbb{R}^2$ be an open, bounded, connected set with Lipschitz boundary. We consider a family of thin hyperelastic sheets occupying the domains:

$$\Omega^h = \omega \times \Big(-\frac{h}{2}, \frac{h}{2}\Big) \subset \mathbb{R}^3, \qquad 0 < h \ll 1.$$

A typical point in Ω^h is denoted by $x = (x', x_3)$. For $h = 1$ we use the notation $\Omega = \Omega^1$ and view Ω as the referential rescaling of each Ω^h. We study the singular limit behaviour, as $h \to 0$, of the following energy functionals defined on vector fields $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$, that are interpreted as deformations of Ω^h:

$$\mathcal{E}^h(u^h) = \frac{1}{h}\int_{\Omega^h} W\big(\nabla u^h(x) G^h(x)^{-1/2}\big)\,\mathrm{d}x, \tag{1}$$

The films are characterized by smooth incompatibility (Riemann metric) tensors G^h, satisfying the structure assumption, referred to as *"oscillatory"*:

$$\text{(O)}\quad \left[\begin{array}{c} \textsc{Oscillatory case:} \\ G^h(x) = \mathcal{G}^h(x', \frac{x_3}{h}) \quad \text{for all} \quad x = (x', x_3) \in \Omega^h, \\ \mathcal{G}^h(x',t) = \bar{\mathcal{G}}(x') + h\mathcal{G}_1(x',t) + \frac{h^2}{2}\mathcal{G}_2(x',t) + o(h^2) \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{3\times 3}_{\mathrm{sym,pos}}), \\ \text{where } \int_{-1/2}^{1/2}\mathcal{G}_1(x',t)\,\mathrm{d}t = 0 \text{ for all } x' \in \bar{\omega}. \end{array}\right.$$

The requirement of $\bar{\mathcal{G}}$ being independent of the transversal variable $t \in (-1/2, 1/2)$ is essential for the energy scaling order: $\inf \mathcal{E}^h \leq Ch^2$. The zero mean requirement on $\mathcal{G}_1$ can be relaxed to requesting that $\int_{-1/2}^{1/2}\mathcal{G}_1(x',t)_{2\times 2}\,\mathrm{d}t$ be a linear strain with respect to the leading order midplate metric $(\bar{\mathcal{G}}_1)_{2\times 2}$ or it can be removed altogether (which will be the content of future work).This set-up includes a subcase of a single metric $G^h = G$, upon taking: $\mathcal{G}_1(x',t) = t\bar{\mathcal{G}}_1(x')$, $\mathcal{G}_2(x',t) = t^2\bar{\mathcal{G}}_2(x')$. We refer to this special case as *"non-oscillatory"*:

(NO)

$$\left[\begin{array}{c}\text{Non-oscillatory case :}\\ G^h = G_{|\bar{\Omega}^h} \quad \text{for some} \quad G\in\mathcal{C}^\infty(\bar{\Omega},\mathbb{R}^{3\times 3}_{\mathrm{sym,pos}}),\\ G^h(x) = \bar{\mathcal{G}}(x') + x_3\partial_3 G(x',0) + \frac{x_3^2}{2}\partial_{33}G(x',0) + o(x_3^2) \quad \text{for all} \quad x = \in \Omega^h.\end{array}\right.$$

Mechanically, the assumption (NO) describes thin sheets that have been cut out of a single specimen block Ω, prestrained according to a fixed (though arbitrary) tensor G. The general case (O) can be reduced to (NO) via the following:

(EF)

$$\left[\begin{array}{l}\qquad\qquad\text{Effective non-oscillatory case :}\\ \bar{G}^h(x) = \bar{G}(x) = \bar{\mathcal{G}}(x') + x_3\bar{\mathcal{G}}_1(x') + \frac{x_3^2}{2}\bar{\mathcal{G}}_2(x') \quad \text{for all} \quad x = (x',x_3)\in\Omega^h,\\ \text{where: } \bar{\mathcal{G}}_1(x')_{2\times 2} = 12\int_{-1/2}^{1/2} t(\mathcal{G}_1))_{2\times 2}\,\mathrm{d}t,\ \bar{\mathcal{G}}_1(x')e_3 = -60\int_{-1/2}^{1/2}(2t^3 - \\ \frac{1}{2}t)\mathcal{G}_1 e_3\,\mathrm{d}t,\ \bar{\mathcal{G}}_2(x')_{2\times 2} = 30\int_{-1/2}^{1/2}(6t^2-\frac{1}{2})(\mathcal{G}_2)_{2\times 2}\,\mathrm{d}t.\end{array}\right.$$

0.2. **Singular energies in the non-oscillatory case.**

0.2.1. *Kirchhoff scaling regime.* In the setting of (NO), the Γ-limit of $\frac{1}{h^2}\mathcal{E}^h$ is:

$$\begin{aligned}\mathcal{I}_2(y) &= \frac{1}{2}\big\|Tensor_2\big\|^2_{\mathcal{Q}_2} = \frac{1}{2}\big\|x_3\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}x_3\partial_3 G(x',0)_{2\times 2}\big\|^2_{\mathcal{Q}_2}\\ &= \frac{1}{24}\big\|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)_{2\times 2}\big\|^2_{\mathcal{Q}_2}.\end{aligned}$$

Above, $\|\cdot\|_{\mathcal{Q}_2}$ is a weighted L^2 norm on the space $\mathbb{E}$ of $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$-valued tensor fields on Ω. The weights are determined by the density W and the leading order metric coefficient $\bar{\mathcal{G}}$. The functional $\mathcal{I}_2$ is defined on the set of isometric immersions $\{y \in W^{2,2}(\omega,\mathbb{R}^3);\ (\nabla y)^{\mathsf{T}}\nabla y = \bar{\mathcal{G}}_{2\times 2}\}$; each such immersion generates the corresponding Cosserat vector $\vec{b}$, uniquely given by requesting: $\big[\partial_1 y,\ \partial_2 y,\ \vec{b}\big] \in SO(3)\bar{\mathcal{G}}^{1/2}$ on ω.

The energy $\mathcal{I}_2$ measures the bending quantity $Tensor_2$, linear in x_3 and resulting in its reduction to the single nonlinear bending term, that equals the difference of the curvature form $\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}}$ from the preferred curvature $\frac{1}{2}\partial_3 G(x',0)_{2\times 2}$. We then identify the necessary and sufficient conditions for $\min\mathcal{I}_2 = 0$, in terms of the vanishing of the Riemann curvatures $R_{1212}, R_{1213}, R_{1223}$ of G at $x_3 = 0$. In this case, it follows that $\inf\mathcal{E}^h \leq Ch^4$.

0.2.2. *Von Kármán scaling regime.* We derive the Γ-limit of $\frac{1}{h^4}\mathcal{E}^h$, which is:

$$\mathcal{I}_4(V,\mathbb{S}) = \frac{1}{2}\big\|Tensor_4\big\|^2_{\mathcal{Q}_2},$$

defined on the spaces:

$$\mathcal{S}_{y_0} = \{\mathbb{S} = \lim_{n\to\infty, L^2}\big((\nabla y_0)^{\mathsf{T}}\nabla w_n\big)_{\mathrm{sym}};\ w_n \in W^{1,2}(\omega,\mathbb{R}^3)\}$$

and $\mathcal{V}_{y_0} = \{V \in W^{2,2}(\omega,\mathbb{R}^3);\ \big((\nabla y_0)^{\mathsf{T}}\nabla V\big)_{\mathrm{sym}} = 0\}$ on the deformed midplate $y_0(\omega) \subset \mathbb{R}^3$. Here, y_0 is the unique smooth isometric immersion of $\bar{\mathcal{G}}_{2\times 2}$ for which $\mathcal{I}_2(y_0) = 0$. The expression in $Tensor_4$ is quite complicated but it has the structure of a quadratic polynomial in x_3. A key tool for identifying this expression, also in the general case (O), involves the subspaces $\{\mathbb{E}_n \subset \mathbb{E}\}_{n\geq 1}$ consisting of the tensorial polynomials in x_3 of order n. The bases of $\{\mathbb{E}_n\}$ are then naturally given in terms of the Legendre polynomials $\{p_n\}_{n\geq 0}$ on $(-\frac{1}{2},\frac{1}{2})$. Since $Tensor_4 \in \mathbb{E}_2$, we write the decomposition:

$$Tensor_4 = p_0(x_3)Stretching_4 + p_1(x_3)Bending_4 + p_2(x_3)Curvature_4,$$

which results in:

$$\begin{aligned}\mathcal{I}_4(V,\mathbb{S}) &= \frac{1}{2}\Big(\big\|Stretching_4\big\|^2_{\mathcal{Q}_2} + \big\|Bending_4\big\|^2_{\mathcal{Q}_2} + \big\|Curvature_4\big\|^2_{\mathcal{Q}_2}\Big)\\ &= \frac{1}{2}\big\|\mathbb{S} + \frac{1}{2}(\nabla V)^{\mathsf{T}}\nabla V + \frac{1}{24}(\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 - \frac{1}{48}\partial_{33}G(x',0)_{2\times 2}\big\|^2_{\mathcal{Q}_2}\\ &\qquad + \frac{1}{24}\big\|\big[\langle\nabla_i\nabla_j V,\vec{b}_0\rangle\big]_{i,j=1,2}\big\|^2_{\mathcal{Q}_2} + \frac{1}{1440}\big\|\big[R_{i3j3}(x',0)\big]_{i,j=1,2}\big\|^2_{\mathcal{Q}_2}.\end{aligned}$$

Above, ∇_i denotes the covariant differentiation with respect to the metric $\bar{\mathcal{G}}$ and R_{i3j3} are the potentially non-zero curvatures of G on ω at $x_3 = 0$.

The necessary and sufficient conditions for having $\min\mathcal{I}_4 = 0$ are precisely that $R_{ijkl} \equiv 0$ on $\omega\times\{0\}$, for all $i,j,k,l = 1\ldots 3$. In that case, we show that $\inf\mathcal{E}^h \leq Ch^6$ and also identify the curvature term that will be present in the corresponding decomposition of $Tensor_6$; it is $\big[\partial_3 R_{i3j3}(x',0)\big]_{i,j=1,2} = \big[\nabla_3 R_{i3j3}(x',0)\big]_{i,j=1,2}$ which in view of the second Bianchi identity carries the only potentially non-vanishing components of the covariant gradient $\nabla Riem(x',0)$. This finding is consistent with analyzing the conformal non-oscillatory metric $G = e^{2\phi(x_3)}Id_3$, where different orders of vanishing of ϕ at $x_3 = 0$ correspond to different even orders of scaling of $\mathcal{E}^h$ as $h\to 0$, together with the lower bound: $\inf\mathcal{E}^h \geq c_n h^n\big\|\big[\partial_3^{(n-2)}R_{i3j3}(x',0)\big]_{i,j=1,2}\big\|^2_{\mathcal{Q}_2}$.

0.3. **Singular energies in the oscillatory case.** The analysis in the general case (O) may follow a similar procedure, where we first project the limiting quantity $Tensor^O$ on an appropriate polynomial space and then decompose the projection along the respective Legendre basis. For the Γ-limit of $\frac{1}{h^2}\mathcal{E}^h$, we write:

$$\begin{aligned}&Tensor_2^O\\ &= x_3\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}(\mathcal{G}_1)_{2\times 2} = p_0(x_3)Stretching_2^O + p_1(x_3)Bending_2^O\\ &+ Excess_2, \quad \text{with } Excess_2 = Tensor_2^O - \mathbb{P}_1(Tensor_2^O).\end{aligned}$$

Consequently:

$$\begin{aligned}\mathcal{I}_2^O(y) &= \frac{1}{2}\Big(\big\|Stretching_2^O\big\|^2_{\mathcal{Q}_2} + \big\|Bending_2^O\big\|^2_{\mathcal{Q}_2} + \big\|Excess_2\big\|^2_{\mathcal{Q}_2}\Big)\\ &= \frac{1}{24}\big\|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}(\bar{\mathcal{G}}_1)_{2\times 2}\big\|^2_{\mathcal{Q}_2} + \frac{1}{8}\mathrm{dist}^2_{\mathcal{Q}_2}\big((\mathcal{G}_1)_{2\times 2},\mathbb{E}_1\big),\end{aligned}$$

where again $Stretching_2^O = 0$ in view of the assumed $\int_{-1/2}^{1/2} \mathcal{G}_1 \, \mathrm{d}x_3 = 0$. For the same reason: $\mathbb{P}_1\big((\mathcal{G}_1)_{2\times 2}\big) = x_3(\bar{\mathcal{G}}_1)_{2\times 2}$ with $(\bar{\mathcal{G}}_1)_{2\times 2}$ defined in (EF). The limiting oscillatory energy $\mathcal{I}_2^O$ consists thus of the bending term that coincides with $\mathcal{I}_2$ for the effective metric $\bar{G}$, plus the purely metric-related excess term.

It is easy to observe that: $\min \mathcal{I}_2^O = 0$ if and only if $(\mathcal{G}_1)_{2\times 2} = x_3(\bar{\mathcal{G}}_1)_{2\times 2}$ on $\omega \times \{0\}$, which automatically implies: $\inf \mathcal{E}^h \leq Ch^4$. The Γ-limit of $\frac{1}{h^4}\mathcal{E}^h$ is further derived by considering the decomposition:

$$\begin{aligned} &Tensor_4^O \\ &= p_0(x_3) Stretching_4^O + p_1(x_3) Bending_4^O + p_2(x_3) Curvature_4^O + Excess_4, \\ &\text{with } \ Excess_4 = Tensor_4^O - \mathbb{P}_2(Tensor_4^O). \end{aligned}$$

It follows that:

$$\begin{aligned} \mathcal{I}_4^O(V,\mathbb{S}) &= \frac{1}{2}\Big(\big\|Stret._4^O\big\|_{\mathcal{Q}_2}^2 + \big\|Bend._4^O\big\|_{\mathcal{Q}_2}^2 + \big\|Curv._4^O\big\|_{\mathcal{Q}_2}^2 + \big\|Excess_4\big\|_{\mathcal{Q}_2}^2\Big) \\ &= \frac{1}{2}\big\|\mathbb{S} + \frac{1}{2}(\nabla V)^{\mathsf{T}}\nabla V + B_0\big\|_{\mathcal{Q}_2}^2 + \frac{1}{24}\big\|\big[\langle \nabla_i\nabla_j V, \vec{b}_0\rangle\big]_{i,j=1,2} + B_1\big\|_{\mathcal{Q}_2}^2 \\ &\quad + \frac{1}{1440}\big\|\big[R_{i3j3}(x',0)\big]_{i,j=1,2}\big\|_{\mathcal{Q}_2}^2 \\ &\quad + \frac{1}{2}\mathrm{dist}_{\mathcal{Q}_2}^2\Big(\frac{1}{4}(\mathcal{G}_2)_{2\times 2} - \int_0^{x_3} \big[\nabla_i\big((\mathcal{G}_1 e_3) - \frac{1}{2}(\mathcal{G}_1)_{33}e_3\big)\big]_{i,j=1,2,\mathrm{sym}} \, \mathrm{d}t, \mathbb{E}_2\Big), \end{aligned}$$

where $R_{1313}, R_{1323}, R_{2323}$ are the respective Riemann curvatures of the effective metric $\bar{G}$ in (EF) at $x_3 = 0$. The corrections B_0 and B_1 coincide with the same expressions written for $\bar{G}$ under two extra constraints that can be seen as the h^4-order counterparts of the h^2-order condition $\int_{-1/2}^{1/2} \mathcal{G}_1 \, \mathrm{d}x_3 = 0$ that has been assumed throughout. In case these conditions are valid, the functional $\mathcal{I}_4^O$ is the sum of the effective stretching, bending and curvature in $\mathcal{I}_4$ for $\bar{G}$, plus the additional purely metric-related excess term.

0.4. **Coercivity of $\mathcal{I}_2$ and $\mathcal{I}_4$.** We additionally analyze the derived limiting functionals by identifying their kernels, when nonempty. The kernel of $\mathcal{I}_2$ consists of the rigid motions of a single smooth deformation y_0 that solves:

$$(\nabla y_0)^{\mathsf{T}}\nabla y_0 = \bar{\mathcal{G}}_{2\times 2}, \qquad \big((\nabla y_0)^{\mathsf{T}}\nabla \vec{b}_0\big)_{\mathrm{sym}} = \frac{1}{2}\partial_3 G(x',0)_{2\times 2}.$$

Further, $\mathcal{I}_2(y)$ bounds from above the squared distance of an arbitrary $W^{2,2}$ isometric immersion y of the midplate metric $\bar{\mathcal{G}}_{2\times 2}$, from the indicated kernel of $\mathcal{I}_2$.

For the case of $\mathcal{I}_4$, we first identify the zero-energy displacement-strain couples $(V,\mathbb{S})$ and show that the minimizing displacements are exactly the linearised rigid motions of the referential y_0. We then prove that the bending term in $\mathcal{I}_4$, which is solely a function of V, bounds from above the squared distance of an arbitrary $W^{2,2}$ displacement obeying $\big((\nabla y_0)^{\mathsf{T}}\nabla V\big)_{\mathrm{sym}} = 0$, from the indicated minimizing set in V. On the other hand, the full coercivity result involving minimization in both V and $\mathbb{S}$ is false. We exhibit an example in the setting of the classical von Kármán functional, where $\mathcal{I}_4(V_n,\mathbb{S}_n) \to 0$ as $n\to\infty$, but the distance of $(V_n,\mathbb{S}_n)$

from the kernel of $\mathcal{I}_4$ remains uniformly bounded away from 0. We note that this lack of coercivity is not prevented by the fact that the kernel is finite dimensional.

Critical and almost-critical points in isoperimetric problems

Francesco Maggi
(joint work with M. Delgadino, C. Mihaila, R. Neumayer)

De Giorgi's isoperimetric inequality [1] states that if Ω is a measurable set with finite volume $|\Omega| < \infty$, then the distributional perimeter $P(\Omega)$ of Ω satisfies

$$P(\Omega) \geq (n+1)\,|B_1|^{1/(n+1)}\,|\Omega|^{n/(n+1)} \qquad B_1 = \{x : |x| < 1\}\,, \tag{1}$$

with equality if and only if Ω is a equivalent to a ball. In other words, *among sets of finite perimeter (SFP) with fixed volume, balls are the unique minimizers of perimeter.* Looking more generally at critical points, rather than at minimizers, for a variation $f_t(x) = x + t\,X(x) + O(t^2)$ with $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ we have that

$$\frac{d}{dt}\bigg|_{t=0} |f_t(\Omega)| = \int_\Omega \mathrm{div} X = \int_{\partial^*\Omega} X \cdot \nu_\Omega\,, \qquad \frac{d}{dt}\bigg|_{t=0} P\big(f_t(\Omega)\big) = \int_{\partial^*\Omega} \mathrm{div}^{\partial^*\Omega} X\,.$$

Thus a critical point of perimeter must satisfy, for a constant λ,

$$\int_{\partial^*\Omega} \mathrm{div}^{\partial^*\Omega} X = \lambda \int_{\partial^*\Omega} X \cdot \nu_\Omega\,, \qquad \forall X \in C^1_c\,. \tag{2}$$

Here $\partial^*\Omega$ is the reduced boundary of Ω, ν_Ω the outer unit normal to Ω in the measure theoretic sense, and, necessarily, $\lambda = H^0_\Omega$, where

$$H^0_\Omega = \frac{n\,P(\Omega)}{(n+1)\,|\Omega|}\,.$$

When $\partial\Omega \in C^2$, then (2) is equivalent to $H_\Omega \equiv H^0_\Omega$ along $\partial\Omega$, where H_Ω denotes the mean curvature of Ω (w.r.t. ν_Ω); moreover, in this case, $|\Omega| < \infty$ implies that Ω is bounded (by area monotonicity), and so the moving planes method of Alexandrov [2] can be applied to conclude that *among C^2-sets with fixed volume, balls are the unique critical points of perimeter.* The gap in the characterization of critical points between C^2-sets and finite perimeter sets is addressed in a joint paper with Delgadino [3], where we prove the following theorem.

Theorem [Alexandrov's theorem revisited] *Among sets of finite perimeter with fixed volume, finite unions of balls are the unique critical points of perimeter.*

Wente's torus is a non-shperical example of a 2-dimensional stationary unit density integer rectifiable varifold in $\mathbb{R}^3$ with constant mean curvature (CMC). As an immediate corollary,

Compactness I: *If $\{\Omega_j\}_j$ is a sequence of sets of finite perimeter and finite volume with $\Omega_j \to \Omega$ in L^1, such that there exists a constant λ with*

$$(3)\qquad \lim_{j\to\infty} P(\Omega_j) = P(\Omega) \qquad \lim_{j\to\infty} \int_{\partial^*\Omega_j} \Big\{ \mathrm{div}^{\partial^*\Omega_j} X - \lambda\, X\cdot \nu_{\Omega_j} \Big\} = 0\,,$$

whenever $X \in C^1_c$, then Ω is a finite union of balls.

This compactness statement is interesting in view of the many variational problems where almost-CMC boundaries arise. Examples include capillarity type problems, CMC-foliations in general relativity, and long-time behavior of weak solutions to mean curvature flows (MCF). Weak solutions to the volume preserving MCF are generally constructed as families of SFP $\{\Omega(t)\}_{t\ge 0}$ with distributional mean curvature $H_t \in L^2(\mathcal{H}^n \llcorner \partial^*\Omega(t))$, and satisfy the dissipation inequality

$$(4)\qquad \int_0^\infty dt \int_{\partial^*\Omega(t)} (H_t - \langle H_t\rangle)^2\, d\mathcal{H}^n \le P(\Omega(0)) < \infty\,.$$

Now assume, for a sequence of times $t_j \to \infty$, that: (i) the averages $\langle H_{t_j}\rangle$ are bounded; (ii) there exists Ω such that $\Omega(t_j)\to\Omega$ in L^1 and $P(\Omega(t_j)) \to P(\Omega)$; and (iii) exploiting (4), and up to extracting subsequences, that

$$\lim_{j\to\infty} \int_{\partial^*\Omega(t_j)} (H_{t_j} - \langle H_{t_j}\rangle)^2\, d\mathcal{H}^n = 0\,.$$

Then Ω is necessarily a finite union of balls, thanks to Compactness I. The assumption $P(\Omega_j)\to P(\Omega)$ in Compactness I can be dropped if working with smooth sets with H_{Ω_j} converging to a constant in L^2, and satisfying a uniform mean convexity bound. More precisely, in a joint paper with Delgadino, Mihaila and Neumayer [4] we proved:

Compactness II: *If $\{\Omega_j\}_j$ is a sequence of open sets with smooth boundary and finite volume, normalized by scaling so to have $H^0_{\Omega_j} = n = H_{B_1}$ and such that for a constant $\kappa > 0$*

$$H_{\Omega_j} \ge \kappa \text{ on } \partial\Omega_j$$

then

$$\Omega_j \to \Omega \text{ in } L^1 \qquad \text{and} \qquad \lim_{j\to\infty}\int_{\partial\Omega_j} |H_{\Omega_j} - n|^2 = 0$$

imply that Ω is a finite union of unit balls and that $P(\Omega_j)\to P(\Omega)$.

This second compactness statement is a particular case of a more general compactness result, related to the anisotropic version of Alexandrov's theorem. Define a *geometric integrand* to be a convex, one-homogenous function $F : \mathbb{R}^{n+1} \to [0,\infty)$, positive on the sphere. The *Wulff shape of F* is the bounded open convex set W_F,

$$W_F = \bigcap_{\nu\in\mathbb{S}^n} \Big\{ x\in\mathbb{R}^{n+1} : x\cdot\nu < F(\nu) \Big\}\,.$$

The isoperimetric inequality (1) holds with W_F in place of B_1, and with the anisotropic energy

$$\mathcal{F}(\Omega) = \int_{\partial^*\Omega} F(\nu_\Omega)\, d\mathcal{H}^n$$

in place of $P(\Omega)$. In particular, *among SFP with fixed volume, F-Wulff shapes are the unique minimizers of $\mathcal{F}$* [7]. Given a variation $f_t(x) = x + t\,X(x) + O(t^2)$, the convexity of F implies the existence of the first variation

$$\delta\mathcal{F}|_\Omega(X) = \lim_{t\to 0^+} \mathcal{F}(f_t(\Omega))\,.$$

If Ω is a local minimizer of $\mathcal{F}$ at fixed volume, then $\delta\mathcal{F}|_\Omega(X) \geq 0$ on every $X \in C^1_c$ s.t. $\int_{\partial^*\Omega} X \cdot \nu_\Omega = 0$. A set of finite perimeter satisfying this property is *a critical point of $\mathcal{F}$ at fixed volume.*

Conjecture: *F-Wulff shapes are the unique sets of finite perimeter and finite volume that are critical points of $\mathcal{F}$ at fixed volume.*

This anisotropic version of Alexandrov's theorem is open even when Ω is assumed to be an open set with Lipschitz boundary rather than to be merely of finite perimeter. To the best of our knowledge this question seems to have been considered for the first time in a paper of Morgan [6], where it is affirmatively solved in the planar case, and, actually, in the most general case of immersed closed rectifiable curves. When F is smooth and λ-uniformly elliptic (i.e., $\lambda\,\mathrm{Id} \leq \nabla^2 F(\nu) \leq \mathrm{Id}/\lambda$ on $\nu^\perp$ for every ν), and Ω has a C^2-boundary, then the condition of being a critical point of $\mathcal{F}$ at fixed volume translates into

$$H^F_\Omega = \mathrm{div}^{\partial\Omega}(\nabla F(\nu_\Omega))$$

being constant. (By construction, we always have $H^F_{W_F} = n$.) Assuming that Ω is bounded, as proved by He, Li, Ma and Ge [5], H^F_Ω is constant if and only if Ω is an F-Wulff shape. From the physical viewpoint, the most significant case would however be that of *crystalline* integrands F, obtained as maxima of finitely many linear functions. In the joint paper [4] with Delgadino, Mihaila and Neumayer, we proved the following result, pointing to the validity of the above conjecture for *every* integrand F.

Compactness III: *If $\{F_j\}_j$ is a sequence of smooth, λ_j-elliptic integrands with $m \leq F_j \leq M$ for uniformly-in-j positive constants m and M; and if $\{\Omega_j\}_j$ is a sequence of open sets with smooth boundary and finite volume, normalized by scaling so to have $n\,\mathcal{F}_j(\Omega_j)/[(n+1)|\Omega_j|] = n$ and such that for a constant $\kappa > 0$*

$$H^{F_j}_{\Omega_j} \geq \kappa \text{ on } \partial\Omega_j\,; \tag{5}$$

then

$$F_j \to F \text{ on } \mathbb{R}^{n+1} \qquad \Omega_j \to \Omega \text{ in } L^1 \qquad \lim_{j\to\infty} \frac{1}{\lambda_j^2} \int_{\partial\Omega_j} |H^{F_j}_{\Omega_j} - n|^2 = 0 \tag{6}$$

imply that Ω is a finite union of F-Wulff shapes, with $\mathcal{F}_j(\Omega_j) \to \mathcal{F}(\Omega)$.

The conjecture would follow from Compactness III by solving the following:

Approximation problem: *Given a geometric integrand F, consider a bounded open set Ω with boundary at most as regular as that of W_F, and which is a critical* *point of $\mathcal{F}$ at fixed volume. Construct sequences $\{F_j\}_j$ of smooth λ_j-elliptic integrands, and $\{\Omega_j\}_j$ of bounded smooth sets satisfying* (5) *for some uniform $\kappa > 0$, in such a way that* (6) *holds.*

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Minimal surfaces and the Allen–Cahn equation on 3-manifolds

Christos Mantoulidis

(joint work with Otis Chodosh)

Fix (M^3, g) to be a closed Riemannian 3-manifold. The Allen–Cahn equation

$$\varepsilon^2 \Delta_g u = W'(u) \tag{1}$$

is a semilinear PDE which is deeply linked to the theory of minimal hypersurfaces. For instance, it is known that the Allen–Cahn functional

$$E_\varepsilon[u] := \int_M \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) d\mu_g,$$

whose critical points satisfy (1), Γ-converges as $\varepsilon \to 0$ to the perimeter functional [12, 14] and the level sets of E_ε-minimizing solutions to (1) converge as $\varepsilon \to 0$ to area-minimizing boundaries. When u is not E_ε-minimizing, the limit may occur with high multiplicity. Together with Otis Chodosh we studied solutions to (1) on 3-manifolds with uniform E_ε-bounds and uniform Morse index bounds and showed that multiplicity does *not* occur when the metric g is "bumpy," i.e., when no immersed minimal surface carries nontrivial Jacobi fields; bumpy metrics are generic in the sense of Baire category—see White [16]. This resolves a strong form of the "multiplicity one" conjecture of Marques–Neves [10] for Allen–Cahn. Our main theorem is:

Theorem 1 ([1]). *Suppose that u_i are critical points of E_{ε_i} with $\varepsilon_i \to 0$ and*

$$E_{\varepsilon_i}[u_i] \leq E_0, \ \operatorname{ind}(u_i) \leq I_0 \textit{ for all } i = 1, 2, \ldots$$

Passing to a subsequence, for each $t \in (-1,1)$, $\{u_i = t\}$ converges in the Hausdorff sense and in $C^{2,\alpha}_{\text{loc}}$ away from $\leq I_0$ points to a smooth closed minimal surface Σ. For any connected component $\Sigma' \subset \Sigma$, either:

- *Σ' is two-sided and occurs as a multiplicity one graphical $C^{2,\alpha}$ limit; or,*
- *Σ' is a two-sided stable minimal surface with a positive Jacobi field and which occurs as multiplicity two limit or higher; or*
- *Σ' is one-sided and its two-sided double cover is a stable minimal surface with a positive Jacobi field.*

The case $I_0 = 0$ of Theorem 1 is largely a consequence of Theorem 2 below:

Theorem 2 ([1]). *Let u be a stable critical point of E_ε. As $\varepsilon \to 0$ we have*

$$\exp(-\sqrt{2}\varepsilon^{-1}\operatorname{dist}_g(x,x')) = o(\varepsilon^2|\log\varepsilon|), \tag{2}$$

for any x, x' that belong to different connected components of $\{u=0\}$, as well as

$$\|H\|_{C^0(\{u=t\})} = o(\varepsilon|\log\varepsilon|), \tag{3}$$

$$\|A\|_{C^{0,\alpha}(\{u=t\})} = O(1) \tag{4}$$

for the mean curvature H and the second fundamental form A of $\{u=t\}$, $|t| \leq 1-\beta$. Here α, $\beta \in (0,1)$, and the constants depend on α, β, $E_\varepsilon[u]$, M, g.

This theorem is based on sharpening some recent novel work of Wang–Wei [15] that resolved the "finite index implies finitely many ends" Allen–Cahn conjecture without energy bounds in $\mathbf{R}^2$. Their remarkable insight was the reduction of the question of regularity to a question about the distance among the sheets comprising $\{u=0\}$. We remark that (2)-(3) are stronger than the bounds obtained in [15] (on the other hand, we have dependence on $E_\varepsilon[u]$). Though bounds on the order of those in [15] suffice for obtaining $C^{2,\alpha}$ estimates for *some* $\alpha \in (0,1)$, the stronger bounds stated in (2)-(3) ensure that the level sets are mean curvature dominated and play a crucial role for our geometric applications.

Our other result, which is proved for all ambient dimensions, is a resolution of the "index lower bound" conjecture of Marques–Neves [10] (for Allen–Cahn):

Theorem 3 ([1]). *Suppose that u_i are critical points of E_{ε_i} with $\varepsilon_i \to 0$ and $\{u_i = t\}$ converging to a smooth two-sided minimal hypersurface Σ with multiplicity one. Then, after possibly passing to a subsequence,*

$$\operatorname{ind}(\Sigma) \leq \lim_i \operatorname{ind}(u_i) \tag{5}$$

$$\leq \lim_i [\operatorname{ind}(u_i) + \operatorname{nul}(u_i)] \leq \operatorname{ind}(\Sigma) + \operatorname{nul}(\Sigma). \tag{6}$$

Only the last inequality, (6), is new to [1]; (5) was previously shown to be true by Gaspar [4] without two-sidedness or multiplicity one assumptions. (See also the works of Hiesmayr [7], Le [13].) To prove (6), one needs to obtain a very precise understanding of how $\{u_i = t\}$ tend to Σ. To do so, we borrow ideas from

[2] where the authors studied the index and nullity for solutions of (1) that they constructed; an added difficulty is that in our case the u_i are essentially arbitrary.

Finally we point out an important consequence of this work to the study of minimal surfaces in closed Riemannian 3-manifolds (M^3, g). Recall the following minmax construction of Gaspar–Guaraco (which works for ambient dimensions $3 \leq n \leq 7$):

Theorem 4 (Gaspar–Guaraco [5])**.** *Let* $p \in \{1, 2, 3, \ldots\}$. *For all sufficiently small* $\varepsilon > 0$, *there exists a critical point* $u_{\varepsilon,p}$ *of* E_ε *with*

$$E_\varepsilon[u_{\varepsilon,p}] \sim p^{1/3}, \text{ and} \tag{7}$$

$$\operatorname{ind}(u_{\varepsilon,p}) \leq p \leq \operatorname{ind}(u_{\varepsilon,p}) + \operatorname{nul}(u_{\varepsilon,p}). \tag{8}$$

Let us now assume that the metric g on M is bumpy, i.e., that there are no closed immersed minimal surfaces that carry nontrivial Jacobi fields. Invoking Theorems 1, 3, 4, together with the bumpiness condition and Ejiri–Micallef [3], we obtain a closed embedded minimal hypersurface Σ_p with

$$|\Sigma_p| \sim p^{1/3}, \ \operatorname{ind}(\Sigma) = p, \ \operatorname{genus}(\Sigma_p) \geq \tfrac{1}{6}p - O(p^{1/3}). \tag{9}$$

In particular, we resolve a conjecture due to Yau [17] in the case of bumpy metrics:

Corollary 1. *Any closed* (M^3, g) *with a bumpy metric* g *contains infinitely many closed embedded minimal surfaces. They satisfy* (9).

Irie–Marques–Neves [8] previously resolved Yau's conjecture in a Baire-generic sense using the Liokumovich–Marques–Neves Weyl law for the Almgren–Pitts width spectrum [9]. See also the more recent work of Gaspar–Guaraco [6] who proved the Weyl law in the Allen–Cahn setting and obtained similar conclusions as [8]. Our corollary also carries through when $\operatorname{Ric}_g > 0$; see also the previous work of Marques–Neves [11] using Almgren–Pitts in this same setting.

Remark. Theorems 2-3 are stated for closed manifolds, i.e., those without boundary, but there are analogs in case $\partial M \neq \emptyset$. See [1].

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Regularity vs. singularity for elliptic and parabolic systems

Connor Mooney

We consider the regularity of minimizers and gradient flows of variational integrals of the form

$$E(\mathbf{u}) := \int_{B_1} F(D\mathbf{u})\, dx. \tag{1}$$

Here $\mathbf{u}$ is a map from $B_1 \subset \mathbb{R}^n \to \mathbb{R}^m$, and F is smooth and uniformly convex on $M^{m\times n}$ with bounded second derivatives.

We first discuss minimizers. By a minimizer we mean that $\mathbf{u} \in H^1(B_1)$ and $E(\mathbf{u}+\varphi) \geq E(\mathbf{u})$ for all maps $\varphi \in C_0^1(B_1)$. Minimizers solve the Euler-Lagrange system $\partial_i(F_{p_i^\alpha}(D\mathbf{u})) = 0$, $1 \leq \alpha \leq m$. Morrey showed that minimizers are smooth when $n = 2$ [Mo]. De Giorgi [DG1] and Nash [Na] showed the smoothness of minimizers in the scalar case $m = 1$. In each case, the approach is to differentiate the Euler-Lagrange system and treat the problem as a linear system for the derivatives of $\mathbf{u}$ with measurable coefficients. Counterexamples of De Giorgi [DG2], Giusti-Miranda [GM], Maz'ya [Ma], and later Frehse [F] show that this technique fails when $n \geq 3$, $m \geq 2$. Singular minimizers of functionals of the type (1) were first constructed by Nečas in dimension $m = n^2$ large [Ne]. Šverák-Yan later constructed counterexamples in the cases $n = 3$, $m = 5$ and $n = 4$, $m = 3$ [SY1], [SY2]. The cases $n = 3$, $m = 2$ or 3, which are particularly interesting for applications, remained open (see e.g. [Gi]). In recent joint work with O. Savin we prove [MS]:

Theorem 1. *There exists a one-homogeneous singular minimizer to a functional of the type (1) with* $n = 3$, $m = 2$.

By the previous results, these dimensions are the lowest possible. The example is based on the construction of a singular minimizer in the scalar case with degenerate convex integrand. By considering a pair of such minimizers, we obtain a map into $\mathbb{R}^2$ which solves a de-coupled Euler-Lagrange system of two equations with degenerate convex integrand on $M^{2\times 3}$. However, we arrange that at least one of the components solves a uniformly elliptic equation at each point, which allows us to remove the degeneracy of the integrand in the vectorial case.

We now discuss the parabolic case. The gradient flow of E is a map $\mathbf{w} \in L^2_t H^1_x$ from $B_1 \times (-1, 0] \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ that solves the system $\partial_t \mathbf{w} = \partial_i (F_{p_i^\alpha}(D\mathbf{w}))$, $1 \leq \alpha \leq m$. Nash proved the smoothness of gradient flows in the scalar case $m = 1$ [Na]. Nečas-Šverák later proved the smoothness of gradient flows in the case $n = 2$ [NS]. Unlike the elliptic case, the result in [NS] relies on a higher-integrability result for linear parabolic systems with measurable coefficients rather than a continuity result. The continuity of solutions to such systems in the case $n = 2$ remained open for some time ([SJM], [SJ1], [SJ2]). In recent work we construct a counterexample to continuity for solutions to linear uniformly parabolic systems in $2D$ [M1]:

Theorem 2. *There exists a solution* $\mathbf{v} : \mathbb{R}^2 \times (-\infty, 0] \to \mathbb{R}^2$ *to a linear, uniformly parabolic system of the form* $\partial_t \mathbf{v} = div(A(x, t)D\mathbf{v})$ *with measurable coefficients, such that* $\mathbf{v}$ *is globally bounded and smooth up to* $t = 0$ *away from* $(0, 0)$ *but discontinuous at* $(0, 0)$.

To construct the example we impose the self-similarity $\mathbf{v} = \mathbf{V}(x/\sqrt{-t})$. This reduces the problem to finding a global bounded solution $\mathbf{V}$ to an elliptic system with a gradient term. An important feature of our construction is that $|\mathbf{V}|$ is radially decreasing near ∞, and $\mathbf{V}(0) = 0$. Each component of $\mathbf{V}$ solves a scalar equation, except for in an annulus where it takes a local maximum. By introducing coupling coefficients in this annulus, we can cancel the error in each equation without breaking the ellipticity of the system.

The result in [M1] is surprising at first because solutions to linear uniformly elliptic systems are continuous in $2D$. In recent work we clarify the picture by showing that solutions to linear uniformly parabolic systems are in general only slightly better than parabolic energy estimates allow [M2]:

Theorem 3. *For all* $n \geq 2$ *and* $\delta > 0$, *there exists a solution* $\mathbf{v}$ *to a linear, uniformly parabolic system of two equations on* $\mathbb{R}^n \times \mathbb{R}$ *such that*

$$\lim_{t\to 0^-} \|\mathbf{v}\|_{L^{2+\delta}_x(B_1\times\{-t\})} = \infty, \quad \lim_{t\to 0^-} \|\nabla \mathbf{v}\|_{L^{2+\delta}(B_1\times(-1,\,-t))} = \infty.$$

The examples we construct are in fact $\mathbb{C}$-valued solutions to uniformly parabolic equations with $\mathbb{C}$-valued coefficients, and can be viewed as parabolic analogues of elliptic examples previously constructed by Frehse [F]. They are spiraling-homogeneous of negative degree under parabolic rescalings. The key observation is that the asymptotic shape of such solutions on the time slice $\{t = -1\}$ distinguishes parabolic energy estimates from elliptic ones in $\mathbb{R}^n$. Our examples in fact tend to elliptic counterexamples from two dimensions higher at $t = 0$. In [M2] we prove Liouville theorems that make precise the connection between parabolic systems in $\mathbb{R}^n \times \mathbb{R}$ and elliptic systems in $\mathbb{R}^{n+2}$, and also explain why previous approaches ([SJM], [FM]) only produced "elliptic" discontinuities.

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n-uniform measures: examples and characterizations

Abdalla Dali Nimer

n-uniform measures are measures that satisfy the following property: there exists $c > 0$ such that for every x in the support of μ, for every $r > 0$, we have:

$$\mu(B_r(x)) = cr^n.$$

They were first studied by David Preiss in the proof of his seminal theorem on the rectifiability of measures. Since his work on the classification of codimension 1 n-uniform measures with Kowalski, only one non-flat such measure was known.

We present our results on n-uniform measures that include a new family of examples, a characterization of the geometry of their support and a classification result in codimension 2.

We show that the support of a conical 3-uniform measure is a cone over a finite union of disjoint 2-spheres. We isolate the condition of distance symmetry of the centers of those 2-spheres as a sufficient condition to guarantee that a measure is 3-uniform.

We finally present a family of examples arising from a family of distance symmetric configuration of centers.

Almost conically deformed thin elastic sheets

Heiner Olbermann

In this contribution, we are concerned with nearly conical deformations of thin elastic sheets from the variational point of view. The elastic sheet is modeled as the two-dimensional ball $B_1 \subset \mathbb{R}^2$, and to a deformation $y : B_1 \to \mathbb{R}^3$ we associate the free energy

$$I_{h,\Delta}(y) = \int_{B_1} \left(|g_y - g_0|^2 + h^2|D^2y|^2\right) \mathrm{d}x \,, \tag{1}$$

where $g_y = Dy^T Dy$ is the metric that is induced on B_1 by pulling back the Euclidean metric on $\mathbb{R}^3$ by y, g_0 is some given reference metric, and h is some small scalar parameter that can be thought of as the thickness of the sheet.

A single disclination. The first variational problem that we are going to consider arises by setting

$$g_0^\Delta = \mathrm{d}r^2 + (1 - \Delta^2)r^2\mathrm{d}\theta^2 \,,$$

with $0 < \Delta < 1$, and then considering the energy functional I_h^Δ that is obtained in (1) when inserting $g_0 \equiv g_0^\Delta$. This corresponds to shortening the two-dimensional Euclidean metric in the angular direction. The metric g_0^Δ is the metric on a cone of height Δ. This cone is the image of the deformation $y^\Delta(x) = \sqrt{1 - \Delta^2}x + |x|e_3\Delta$, which has vanishing membrane energy (the first term on the right hand side of (1)) but infinite bending energy (the second term on the right hand side of (1)). We are not going to impose any constraints on the set of allowed deformations, and will first be interested in deriving an energy scaling law, i.e., upper and lower bounds for the minimum of energy that come along as expansions in the small parameter h. Our first result is

Theorem 1 ([3, 5]). *There exists a constant C_1 that only depends on Δ such that for $h < \frac{1}{2}$, we have that*

$$2\pi\Delta^2 h^2 \left(\log\frac{1}{h} - C_1\right) \leq \min_{y\in W^{2,2}(B_1;\mathbb{R}^3)} I_h^\Delta(y) \leq 2\pi\Delta^2 h^2 \left(\log\frac{1}{h} + C_1\right).$$

The upper bound is a straightforward construction that consists in smoothing y^Δ on the ball B_h. The lower bound is proved by observing that the quantity $\sum_{i=1}^3 \det D^2 y$ can be considered as a linearization of the Gauss curvature of the graph of y. The membrane term of I_h^Δ forces this quantity to be close to a δ-type distribution. This yields bounds from below for integrals of the linearized Gauss curvature over balls B_ρ, with $h \lesssim \rho < 1$. By a certain Sobolev-type inequality, this yields lower bounds on $\sum_i \mathcal{H}^1(Dy_i(\partial B_\rho))$, which in turn can be translated in a lower bound for the bending term in I_h^Δ. This idea has been first put forward in [2].

Even more can be said about (almost-) minimizers of I_h^Δ:

Corollary 1 ([5]). *For $h \to 0$, let $\tilde{y}_h$ be a sequence that satisfies $I_h^\Delta(\tilde{y}_h) \leq 2\pi\Delta^2 h^2 \left(\log\frac{1}{h} + C_1\right)$. Then up to subsequences and up to Euclidean motions, for every $0 < \rho < 1$, we have that*

$$\tilde{y}_h \rightharpoonup y \quad \text{in } W^{2,2}(B_1 \setminus B_\rho;\mathbb{R}^3).$$

The proof of the corollary works by combining the energy estimates from the theorem with rigidity results on $W^{2,2}$ isometric immersions [6, 1].

A boundary value problem for conically constrained sheets. The idea to use Gauss curvature (or linearizations thereof) as the decisive control variable can also be transferred to settings of flat sheets, i.e., sheets where the reference metric is the Euclidean metric. For our result, we have to change the setting: Instead of (1), we now consider the so-called von Kármán approximation. This consists in writing $y(x) = x + u(x) + v(x)e_3$, where $u : B_1 \to \mathbb{R}^2$ is the in-plane-deformation and $v : B_1 \to \mathbb{R}$ is the out-of-plane deformation, and then treating these components differently in the energy. Namely, the energy reads

$$I_{h,p}(u,v) = \int_{B_1} \left|\mathrm{sym}Du + \frac{1}{2}Dv\otimes Dv\right|^2 + h^2|D^2v|^p \mathrm{d}x\,. \tag{2}$$

The standard von Kármán energy is the case $p = 2$. Here, we are going to assume $p \in (2, \frac{8}{3})$, which is due to our method of proof for Theorem 2 that is based on a certain Gagliardo-Nirenberg inequality, only valid for that range. In the sequel, we write p' for the dual exponent fulfilling $\frac{1}{p} + \frac{1}{p'} = 1$.

We consider the boundary condition of a so-called *d-cone* (short for "developable cone"). Before carrying out the von Kármán approximation, the d-cone is described by the deformation $y^\gamma(x) = |x|\gamma(x/|x|)$, where $\gamma : \partial B_1 \to S^2$ is a unit speed curve, not contained in a hyperplane. Taking the boundary conditions for

y^γ and Dy^γ and translating them into the von Kármán approximation yields our class of allowed deformations $(u,v) \in \mathcal{A}_\gamma \subset W^{1,2}(B_1;\mathbb{R}^3) \times W^{2,2}(B_1)$. Our second result is

Theorem 2 ([4]). *Let $p \in (2, \frac{8}{3})$. Then there exists a constant $C = C(\gamma, p)$ such that*

$$C^{-1} h^{p'} \le \min_{(u,v)\in\mathcal{A}_\gamma} I_{h,p}(u,v) \le C h^{p'} .$$

The upper bound is again straightforward; the proof of the lower bound works by observing that the boundary conditions for Dv allow for a lower bound of the (von Kármán) Gauss curvature $\det D^2 v$ in the negative Sobolev space $W^{-1,p}$. By an appropriate Gagliardo-Nirenberg inequality, this translates to a lower bound of a certain product of the norms of $\det D^2 v$ in the spaces $W^{-2,2}$ and $L^{p/2}$. These norms in turn are lower bounds for the membrane and bending term in the energy (2).

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A variational approach to regularity for the Monge-Ampère equation, and an application to the matching problem

Felix Otto
(joint work with Michael Goldman, Martin Huesmann)

We present a One-step Improvement Lemma for the Monge-Ampère equation with rough right hand side. Its proof is purely variational, passing via the Optimal Transport formulation in its Eulerian version (Benamou-Brenier), and is orthogonal to the maximum principle-based approach (Caffarelli). In fact, the lemma states that if the local transportation cost is small and the given measures weakly close to the uniform distribution, then the displacement is close to a harmonic gradient, with homogeneities as for a linear problem. On the one hand, by Campanato iteration, this gives rise to an ϵ-regularity result and partial regularity (Figalli-Kim, De Philippis-Figalli), in the spirit of the corresponding results for minimal surfaces, see [1]. On the other hand, it can be used as a large-scale regularity theory for the problem of matching the Lebesgue measure to the Poisson measure in the thermodynamic limit, in the spirit of quantitative stochastic homogenization. More precisely, it can be shown that in the critical dimension two (Ambrosio-Stra-Trevisan), increments of the displacement are still stationary.

Let π be the optimal transference plan between two measures μ_0 and μ_1 in $\mathbb{R}^d$. Consider the local energy in a (centered) ball of radius 6 ("E" for energy)

$$E := \int_{(B_6\times\mathbb{R}^d)\cup(\mathbb{R}^d\times B_6)} |x_1 - x_0|^2 \pi(dx_0 dx_1)$$

and the squared local distance of μ_0 and μ_1 to the Lebesgue measure ("D" for data term)

$$D := W_{B_6}^2(\mu_0, c_0) + |c_0 - 1|^2 + W_{B_6}^2(\mu_1, c_1) + |c_1 - 1|^2,$$

where W_{B_6} denotes the Wasserstein metric between measures on B_6.

Proposition [One-Step-Improvement]. For any $0 < \tau \leq 1$ there exists a constant $C = C(d,\tau) < \infty$ such that provided $E + D \leq \frac{1}{C}$ there exists a harmonic gradient $\nabla\phi$ in B_1 such that

$$\begin{aligned}(1)\qquad &\int_0^1 \int_{(B_1\times\mathbb{R}^d)\cup(\mathbb{R}^d\times B_1)} |x_1 - x_0 - \nabla\phi(tx_1 + (1-t)x_0)|^2 \pi(dx_0 dx_1) dt \\ &\qquad\qquad \leq \tau E + CD, \\ &\int_{B_1} |\nabla\phi|^2 \leq C(E + D).\end{aligned}$$

This proposition states that if the transportation cost is locally small ($E \ll 1$) and the initial and target measures are locally close to uniform ($D \ll 1$) then the displacement $x_1 - x_0$ is close to a harmonic gradient $\nabla\phi$. It does so using the right topology: Closeness (to uniform and to a harmonic gradient, respectively) is measured in terms of transportation itself. More importantly, the homogeneity of all three terms match: a volume integral of a squared length.

The well-known connection between Optimal Transportation and the Monge-Ampère equation is as follows: From soft convex analysis we learn that there exists a convex ψ with sub-gradient $\partial\psi$ such that $x_1 = \partial\psi(x_0)$ for all (x_0, x_1) in the support on π. For μ_0 absolutely continuous w. r. t. Lebesgue, μ_1 is the push-forward of μ_0 under the Lebesgue-a. e. existing gradient $\nabla\psi$, that is, $\mu_1 = \nabla\psi\#\mu_0$. For μ_0, μ_1 continuous and ψ twice continuously differentiable with positive definite Hessian, this relation $\mu_1 = \nabla\psi\#\mu_0$ takes the form of the Monge-Ampère type equation $\mu_1(\nabla\psi(x_0))\det\nabla^2\psi(x_0) = \mu_0(x_0)$.

The proof of the proposition relies on the Benamou-Brenier Eulerian formulation of Optimal Transportation in terms of density/flux measures

$(\rho, j) = (\rho_t dt, j_t dt)$ defined through

$$\int \zeta d\rho_t = \int \zeta(tx_1 + (1-t)x_0)\pi(dx_0 dx_1), \ \forall \zeta \in C_0^0(\mathbb{R}^d)$$
$$\int \xi \cdot dj_t = \int \xi(tx_1 + (1-t)x_0) \cdot (x_1 - x_0)\pi(dx_0 dx_1), \ \forall \xi \in C_0^0(\mathbb{R}^d)^d$$

for all $t \in [0,1]$. In its localized version, the formulation states: For every radius R, the (normal) flux boundary data $f := \nu \cdot j$ on ∂B_R exist and (ρ, j) minimizes

$$\int_{B_R \times (0,1)} \frac{1}{2\tilde{\rho}} |\tilde{j}|^2 := \sup_{\xi \in C_0^0(B_R \times (0,1))^d} \{ \int \xi \cdot d\tilde{j} - \frac{1}{2} \int |\xi|^2 d\tilde{\rho} \}$$

among all $(\tilde{\rho}, \tilde{j})$ satisfying the continuity equation with initial/terminal and flux boundary conditions

$$\left.\begin{array}{rcll} \partial_t \tilde{\rho} + \nabla \cdot \tilde{j} & = & 0 & \text{in } B_R \times (0,1), \\ \tilde{\rho} & = & \mu_0 & \text{on } B_R \times \{0\}, \\ \tilde{\rho} & = & \mu_1 & \text{on } B_R \times \{1\}, \\ \nu \cdot \tilde{j} & = & f & \text{on } \partial B_R \times (0,1) \end{array}\right\} \quad \text{distributionally.} \tag{2}$$

With these notions, (1) follows from

$$\int_{B_2 \times (0,1)} \frac{1}{\rho} |j - \rho \nabla \phi|^2 \le \tau E + CD$$

in conjunction with an L^∞-bound on the displacement (see the lemma below), which ensures that

$$\{tx_1 + (1-t)x_0 | (x_0, x_1) \in \operatorname{supp}\pi \cap ((B_1 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_1))\} \subset B_2.$$

In the same vein, we note that

$$\int_{B_5 \times (0,1)} \frac{1}{\rho} |j|^2 \le E.$$

Our $\nabla\phi$ will be the solution of the Neumann problem

$$-\triangle \phi = \text{const. in } B_R, \quad \nu \cdot \nabla \phi = \bar{\rho} \text{ on } \partial B_R, \tag{3}$$

for suitable $R \in (3,4)$, where $\bar{\rho}$ is a suitable approximation of $\int_0^1 f dt$. The above proposition is a consequence of the following two lemmas.

Lemma [Construction]. Suppose $E + D \ll 1$. For every $0 < \tau \ll 1$ there exists a radius $R \in (3,4)$, a density/flux pair $(\tilde{\rho}, \tilde{j})$ satisfying (2) and flux boundary data $\bar{\rho}$ such that for the $\nabla\phi$ defined through (3) we have

$$\int_{B_R \times (0,1)} \frac{1}{\tilde{\rho}} |\tilde{j}|^2 - \int_{B_R} |\nabla \phi|^2$$
$$\lesssim \tau E + \frac{1}{\tau} D + \text{super-linear terms}_\tau \text{ in } E + D.$$

Lemma [Orthogonality]. Under the assumptions of the above lemma we have for any length scale $0 < r \ll 1$

$$\int_{B_2\times(0,1)} \frac{1}{\rho}|j-\rho\nabla\phi|^2 - \Big(\int_{B_R\times(0,1)} \frac{1}{\rho}|j|^2 - \int_{B_R} |\nabla\phi|^2\Big)$$
$$\lesssim rE + \frac{1}{r}D + \text{super-linear terms}_r \text{ in } E+D.$$

The only place where we use the Euler-Lagrange equation in form of the monotonicity (not even the cyclical none) of the support suppπ of π is the following lemma, which is also one of the sources of the above super-linear terms.

Lemma [L^∞-estimate]. Provided $E + D \ll 1$ we have

$$|x_1 - x_0| \lesssim (E+D)^{\frac{1}{d+2}}$$
$$\text{for } (x_0, x_1) \in \operatorname{supp}\pi \cap \big((B_5 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_5)\big).$$

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Parametrized stationary varifolds and the multiplicity one conjecture

Alessandro Pigati
(joint work with Tristan Rivière)

In the last fifty years many ways of constructing unstable minimal submanifolds Σ^k, in a given closed Riemannian ambient manifold $(\mathcal{M}^m, g)$, have been proposed. These approaches deal with the existence of min-max minimal hypersurfaces ($k = m-1$) or minimal surfaces ($k = 2$).

We mention for instance the work by Colding–Minicozzi for minimal spheres obtained from a mountain pass min-max, the main technique being the harmonic replacement; this was recently generalized to arbitrary genus by Zhou. Almgren–Pitts theory allows to construct embedded minimal hypersurfaces from a min-max in a suitable space of cycles; when $m = 3$, a simpler and more effective version of this theory was developed by Almgren, Meeks, Pitts, Simon, Smith and Yau. Still in codimension one, another approach based on the link between minimal surfaces and phase transitions was recently proposed, starting from contributions by Hutchinson, Tonegawa, Wickramasekera, Guaraco and others.

This talk focuses on a new penalization approach devised by Rivière [4], which provides existence of immersed minimal surfaces ($k = 2$) without a priori restrictions on the genus, on the codimension $m-2$ or on the number of parameters in the min-max.

A viscous relaxation of the area functional. The method proposed in [4] is based on a penalization of the area functional involving the second fundamental form A. More specifically, for a fixed parameter $\sigma > 0$, one first finds an immersion $\Phi : \Sigma \to M$ which is critical for the perturbed area functional

$$A^\sigma(\Phi) := \int_\Sigma d\mathrm{vol}_{g_\Phi} + \sigma^2 \int_\Sigma (1 + |A|^2_{g_\Phi})^2 \, d\mathrm{vol}_{g_\Phi},$$

where Σ is a fixed closed oriented surface and g_Φ is the metric induced by Φ, with volume form vol_{g_Φ}. This functional A^σ enjoys a sort of Palais–Smale condition up to diffeomorphisms.

Considering any sequence $\sigma_j \downarrow 0$, one gets a sequence $\Phi_j : \Sigma_j \to M$ of conformal immersions, where Σ_j denotes Σ endowed with the induced conformal structure. Assuming for simplicity that we are dealing with a constant conformal structure, the sequence Φ_j is then bounded in $W^{1,2}$ and we can consider its weak limit Φ_∞, up to subsequences.

At this stage of the theory, it is still not clear whether the strong $W^{1,2}$-convergence holds, even away from a finite bubbling set. However, in [4] the second author shows that, if the sequence σ_j is carefully chosen so as to satisfy a certain *entropy condition*, then the surfaces $\Phi_j(\Sigma_j)$ converge to a *parametrized stationary varifold.*

Parametrized stationary varifolds and their regularity. Parametrized stationary varifolds, introduced in [4, 2], are two-dimensional varifolds admitting a *parametrization* in the following sense: they are induced by a weakly conformal map $\Phi \in W^{1,2}(\Sigma, \mathcal{M}^m)$ (where Σ is a closed Riemann surface and $\mathcal{M}^m \subseteq \mathbb{R}^q$ is either a closed Riemannian manifold or $\mathbb{R}^q$ itself), together with a multiplicity function $N \in L^\infty(\Sigma, \mathbb{N} \setminus \{0\})$ *on the domain.*

They are required to satisfy a natural stationarity property: namely, we assume that, for almost all domains $\omega \subseteq \Sigma$, the varifold induced by the map $\Phi\big|_\omega$ with the multiplicity function $N\big|_\omega$ is stationary in the complement of the compact set $\Phi(\partial\omega)$. In [2] the following optimal regularity result was obtained.

Theorem 1. *The triple* (Σ, Φ, N) *is a parametrized stationary varifold, in a compact manifold* $\mathcal{M}^m \subseteq \mathbb{R}^q$ *or in* $\mathbb{R}^q$ *itself, if and only if* Φ *is a smooth conformal harmonic map and* N *is a.e. constant. In this case,* Φ *is a minimal branched immersion.*

The work [2] relies on the previous treatment of the simpler situation where N is assumed to be constant [5], which in turn is based on Allard's ϵ-regularity theorem. However, the strategy of [5] breaks down in the general situation, which requires completely new ideas.

Everywhere regularity for general integer stationary varifolds fails without additional assumptions, even in low dimension (consider e.g. three half-lines in the plane emanating from the origin, forming equal angles $\frac{2\pi}{3}$). Pitts was able to obtain the optimal regularity for varifolds arising via min-max in codimension one by introducing the stronger concept of *almost minimizing* varifold. We also mention Allard's almost everywhere regularity theorem for the case of constant multiplicity and, for the *stable, codimension one* case, Schoen–Simon regularity result under the assumption that the singular set has locally finite $\mathcal{H}^{n-2}$-measure, which was recently reduced to an optimal assumption by Wickramasekera. The almost everywhere regularity in full generality is still an open problem.

In the present situation, regularity stems from a subtle interaction between stationarity and the topological information of being parametrized. Also, the simplest examples of singular varifolds, namely *classical singularities*, are automatically ruled out in this setting (whereas they can appear among stable, codimension one varifolds).

Corollary 1. *Given a family $\mathcal{A} \subseteq \mathcal{P}(Imm(\Sigma, \mathcal{M}^m))$ invariant under isotopies of $Imm(\Sigma, \mathcal{M}^m)$, its* width

$$W_{\mathcal{A}} := \inf_{A \in \mathcal{A}} \sup_{\Phi \in A} \operatorname{area}(\Phi)$$

is the area of a (possibly branched) minimal immersion $\Phi : S \to \mathcal{M}^m$ with a locally constant integer multiplicity N.

Multiplicity one. The result in [2], which is optimal for the class of parametrized stationary varifolds, leaves nonetheless open the question whether one can have $N > 1$ on some connected component of S. This question should be compared with the *multiplicity one conjecture* by Marques and Neves. Roughly speaking, it asks whether a minimal hypersurface obtained from some min-max method should always have multiplicity one, at least for generic metrics.

Marques and Neves were able to prove this conjecture for one-parameter sweepouts. It was also recently established by Chodosh and Mantoulidis for bumpy metrics in 3-manifolds, in the setting of Allen–Cahn level set approach.

The importance of this conjecture in relation to the Morse index of Σ is twofold. First of all, there is no satisfactory definition for the Morse index of an embedded minimal hypersurface with multiplicity bigger than one: such an object could be thought as the limit of many qualitatively different sequences of multiplicity one hypersurfaces. Also, if one can establish a lower bound on the Morse index like

$$k \leq \textstyle\sum_i n_i \big(\operatorname{index}(\Sigma_i) + \operatorname{nullity}(\Sigma_i) \big), \qquad \Sigma = \textstyle\bigsqcup_i n_i \Sigma_i,$$

k being the number of parameters in the min-max, then the multiplicity one conjecture gives infinitely many *geometrically distinct* minimal hypersurfaces, provided there exists at least one for every value of k.

In [3] the natural counterpart of this conjecture in the viscosity approach is established, namely we have the following result in *arbitrary codimension* and *without any genericity assumption.*

Theorem 2. *We have $N \equiv 1$.*

Corollary 2. *If there is no bubbling or degeneration of the underlying conformal structure, we have strong $W^{1,2}$-convergence $\Phi_k \to \Phi_\infty = \Phi$. In general we have a bubble tree convergence $\Phi_k \to \Phi$.*

The last corollary paves the way to obtain meaningful Morse index bounds. Indeed, although Φ could still be a multiple cover, a crucial advantage of having a parametrization at our disposal is that we have a good definition of Morse index and nullity. The natural expected inequalities would be

$$\text{index}(\Phi) \leq k \leq \text{index}(\Phi) + \text{nullity}(\Phi).$$

Using the results in [1] and [6], we are able to reach the following.

Corollary 3. *Given* $\mathcal{A} \subseteq \mathcal{P}(Imm(\Sigma, \mathcal{M}^m))$ *as above,* $\Phi : S \to \mathcal{M}^m$ *satisfies*

(i) $W_{\mathcal{A}} = \text{area}(\Phi)$,
(ii) $\text{genus}(S) \leq \text{genus}(\Sigma)$,
(iii) $\text{index}(\Phi) \leq k$*, the dimension of* $\mathcal{A}$*.*

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Regularity and higher regularity for the thin obstacle problem with Hölder continuous coefficients

Angkana Rüland
(joint work with Herbert Koch and Wenhui Shi)

The thin obstacle problem arises in a number of applications ranging from the modelling of osmosis to the analysis of American options. It can also be viewed as a scalar toy model of the (vector valued) Signorini problem from elasticity and as a degenerate version of the classical obstacle problem in which the obstacle degenerates into a co-dimension one object.

In the presence of an anisotropic medium and after normalization of the underlying geometry it can be phrased as the following minimization problem:

$$\text{Minimize} \int_{B_1^+} a^{ij} \partial_i u \partial_j u dx \tag{1}$$

with $u \in \mathcal{K} := \{v \in H^1_g(B_1^+) : \ v \geq \psi \text{ on } B_1' := B_1^+ \cap \{x_{n+1} = 0\}\}$. Here $B_1^+ := \{x \in \mathbb{R}^{n+1}_+ : \ |x| \leq 1\}$, $a^{ij} : B_1^+ \to \mathbb{R}^{(n+1)\times(n+1)}$ is a symmetric, uniformly elliptic matrix modelling the anisotropic environment and $\psi : B_1' \to \mathbb{R}$ denotes the *obstacle*. Moreover, $H^1_g(B_1^+) := \{u \in H^1(B_1^+) : u = g \text{ on } \partial B_1^+ \cap \{x_{n+1} > 0\}\}$. The convex constraint $u \geq \psi$ gives rise to three different domains: The *contact set* $\Lambda(u) := \{x \in B_1' : u = \psi\}$, the *non-coincidence set* $\Omega(u) := \{x \in B_1' : u > \psi\}$ and the *free boundary* $\Lambda(u) := \partial_{B_1'}\Omega(u)$. In their strong form the associated Euler-Lagrange equations read

$$
\begin{aligned}
\partial_i a^{ij}\partial_j u &= 0 \text{ in } B_1^+,\\
u &\geq 0 \text{ on } B_1',\\
a^{n+1,i}\partial_i u &\leq 0 \text{ on } B_1',\\
(u-\psi)(a^{n+1,i}\partial_i u) &= 0 \text{ on } B_1'.
\end{aligned} \tag{2}
$$

As the main objectives of the talk I presented optimal regularity results on the solution and the regular free boundary of the variable coefficient thin obstacle problem with Hölder continuous coefficients.

For simplicity of presentation, in the sequel we focus on the situation $\psi = 0$ but remark that this can be substantially generalized. As a first main result, we show that solutions to (2) enjoy similar interior regularity properties as in the isotropic thin obstacle problem [CSS08, ACS08] if the coefficients a^{ij} are Hölder continuous:

Theorem 1 ([RS17]). *Let $u : B_1^+ \to \mathbb{R}$ be a weak solution to (2) with $a^{ij} \in C^{0,\alpha}(B_1^+, \mathbb{R}^{(n+1)\times(n+1)})$, $\alpha \in (0,1)$, a uniformly elliptic tensor field and $\psi = 0$. Then, $u \in C^{1,\min\{\alpha,\frac{1}{2}\}}(B_{1/2}^+)$.*

We remark that these results can be extended to the setting of the interior thin obstacle problem and to the presence of more general obstacles and inhomogeneities. Further it can be complemented by a result on the (regular) free boundary.

Comparing (1) to the regularity theory for linear elliptic equations in divergence form, this result appears natural. However, in proving it, major technical obstructions have to be overcome: The "classical" strategy of proving optimal regularity is based on a blow-up argument and a classification of low frequency global solutions to (2). However, in our situation of only Hölder regular coefficients the two main tools which have been used in this context – Almgren's monotonicity formula [GSVG14] and Carleman estimates [KRS16] – fail. As a consequence, we resort to a linearization technique, which had been introduced by Andersson [And16] in the context of the full vectorial Signorini problem with constant coefficients. A key ingredient here consists of the Liouville theorem for low-frequency solutions to the thin obstacle problem. More precisely, the argument is based on two steps: First we derive *almost* optimal $C^{1,\min\{\alpha,\frac{1}{2}\}-}$ regularity estimates. This relies on a blow-up argument, where compactness is obtained through a contradiction argument. Invoking the Liouville theorem then implies the desired regularity. In a second step, we then bound the difference

$$
\inf_{p\in\mathcal{E}} \|u-p\|_{L^2(\partial B_r)}, \tag{3}
$$

where $\mathcal{E} := \{cRe(Qx' + ix_{n+1})^{3/2} : \ Q \in SO(n), \ c \in \mathbb{R}_+\}$ by an appropriate power of $r > 0$. This is substantially more delicate as there might be concentration effects due to the nonlinearity of the problem at hand. In deriving decay estimates, for which we again rely on compactness arguments, we hence invoke tools such as the Weiss energy and the epiperimetric inequality [FS16, GPSVG16]. As a direct consequence of the decay estimate (3) we also infer the $C^{1,\alpha}$ regularity of the regular free boundary in $B'_{1/2}$, where

$$\Gamma_{3/2}(u) \cap B'_{1/2} := \left\{ x_0 \in \Gamma(u) \cap B'_{1/2} : \liminf_{r \to 0} \frac{\ln(r^{-\frac{n+1}{2}} \|w - \ell_{x_0}\|_{L^2(B_r^+)})}{\ln(r)} < 1 + \alpha \right\},$$

with ℓ_{x_0} being the best affine approximation of u at x_0 and $\alpha \in (\frac{1}{2}, 1)$ being the regularity exponent of the metric.

Based on these optimal regularity results for Hölder coefficients, we further presented optimal higher regularity results, displaying regularity improvement for the regular free boundary.

Theorem 2 ([KRS17]). *Let $u : B_1^+ \to \mathbb{R}$ be a weak solution to the thin obstacle problem* (2) *with $\psi = 0$ and a^{ij} a uniformly elliptic tensor field. Then the following holds:*

- *Assuming that $a^{ij} \in W^{1,p}(B_1^+, \mathbb{R}^{(n+1)\times(n+1)})$ with $p \in (n+1, \infty)$, then $\Gamma_{3/2}(u) \cap B'_{1/2} \in C^{1,1-\frac{n+1}{p}}$.*
- *If $a^{ij} \in C^{k,\alpha}(B_1^+, \mathbb{R}^{(n+1)\times(n+1)})$ with $p \in (n+1, \infty)$, then $\Gamma_{3/2}(u) \cap B'_{1/2} \in C^{k+1,\alpha}$.*
- *If a^{ij} is analytic, then $\Gamma_{3/2}(u) \cap B'_{1/2}$ is analytic.*

Hence, there is a gain of a full derivative with respect to the metric. Similar results can further be obtained in the presence of obstacles and inhomogeneities (yielding a gain of *three halves* of a derivative with respect to the inhomogeneity).

The derivation of these results is based on a partial Legendre-Hodograph transform. Due to the only $C^{1,\frac{1}{2}}$ regularity of the solution to the thin obstacle problem, this however requires careful estimates. It leads to the analysis of a fully nonlinear degenerate elliptic operator in a corner domain. Interpreting this operator as a perturbation of the Baouendi-Grushin Laplacian and setting up an implicit functions argument in appropriate Hölder spaces adapted to the underlying operator through a Carnot-Caratheodory metric and the corner domain finally implies the desired results.

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Finite time degeneration of Teichmüller harmonic map flow

Melanie Rupflin
(joint work with P. M. Topping and C. Robertson)

Teichmüller harmonic map flow is a geometric flow, first introduced in the joint work [1] with Topping, that is designed to flow maps towards parametrisations of minimal surfaces. As we explain in this talk this flow succeeds in its task and decomposes every initial map from a closed surface into an arbitrary compact manifold into a (union of) branched minimal immersion(s).

Given a Riemannian manifold (N, g_N) of arbitrary dimension and a domain surface M, we evolve pairs (u, g) of maps $u : M \to N$ and constant curvature metrics g on the domain by a natural gradient flow of the Dirichlet energy E in order to try to flow to a map that is both harmonic and (weakly) conformal and hence a branched minimal immersion (or constant). For closed surfaces this flow evolves the map component by the full gradient of E, but changes the metric only orthogonally to the symmetries and is hence described by

$$\partial_t u = \tau_g(u), \qquad \partial_t g = \mathrm{Re}(P_g(\Phi(u, g))) \tag{1}$$

where $\tau_g(u)$ is the tension of u, $\Phi(u, g)$ the Hopf-differential and P_g denotes the projection onto the space of holomorphic quadratic differentials.

As long as $\mathrm{inj}(M, g(t))$ is bounded away from zero, the metric component of a solution of (1) remains regular and hence the map component behaves as well as comparable solutions of the harmonic map flow, i.e. remains smooth away from finitely many times, with all of the 'lost energy' at such singularities accounted for in terms bubbles, i.e. maps $\omega^j : S^2 \to N$ that are harmonic and non-constant and hence themselves (branched) minimal immersions.

One of the main challenges in the analysis of the flow, and in the proof that the flow decomposes maps into minimal surfaces, is thus to control its behaviour if $\text{inj}(M, g(t)) \to 0$, which is equivalent to the collapse of some simple closed geodesics $\sigma^j(t) \subset (M, g(t))$ and hence allows for a change of the topology of the domain. If such a degeneration occurs at a finite time T, which is impossible for closed surfaces of genus less than 2, then the results of [3] ensure that we can flow all the way to the singular time and obtain a canonical continuation by restarting the flow with the obtained limit map on the resulting collection of lower genus domain surfaces. While proving convergence on the δ-thick part of the surface for every $\delta > 0$ would be sufficient to obtain such a canonical continuation, this would not exclude the possibility that 'unstructured energy' can be lost down the degenerating collar. To obtain the desired result that the flow decomposes the initial map into minimal immersions we hence prove in [3] that at times $t < T$ close to the singular time T both the metric and map component will have essentially settled down to their limits (up to bubbles for the map) on the $(T - t)$-thick part, which allows us to show that the parts of the domain that can be lost are sufficiently collapsed that along $t_n \nearrow T$ the maps $u(t_n)$ on this part of the domain are almost harmonic with respect to the flat metric and thus map close to a collection of (branched) minimal spheres and curves.

Despite this precise understanding of the flow at potential finite time degenerations it was unclear until recently whether one should expect such singularities to occur at all, with a previous joint result [2] with Topping excluding this for targets that support no bubbles. In the final part of this talk we discuss recent joint work [4] with Robertson which establishes that if the image of a collar $\mathcal{C}(\sigma(t))$ around a geodesic $\sigma(t)$ of sufficiently small length $\ell(t)$ stretches out at a rate of at least $\ell(t)^{-(\frac{1}{4}+\delta)}$, $\delta > 0$, then the solution of (1) must degenerate in finite time. This result applies both for the flow from closed domains and the corresponding flow from cylinders and allowed us to construct examples of solutions of Teichmüller harmonic map flow from cylinders which indeed degenerate in finite time. For the rescaled flow

$$\tau_g(u) = 0, \qquad \partial_t g = \text{Re}(\Phi(u, g))$$

introduced by Huxol we indeed obtain the more precise result that solutions cannot degenerate in finite time if the image of collars stretches out at a rate of no more than $|\log(\ell)|^{\frac{1}{2}}$, but must degenerate in finite time if it stretches out at a rate of at least $|\log(\ell)|^{\frac{1}{2}+\delta}$ for some $\delta > 0$.

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A free boundary problem with facets

Charles Smart
(joint work with William Feldman)

This is joint work with William Feldman. We study a variational problem on the lattice $\mathbb{Z}^d$ whose scaling limit is a free boundary problem of the form

$$\begin{cases} Lu = 0 & \text{in } \{u > 0\} \\ H(\nabla u) = 1 & \text{on } \partial\{u > 0\}, \end{cases}$$

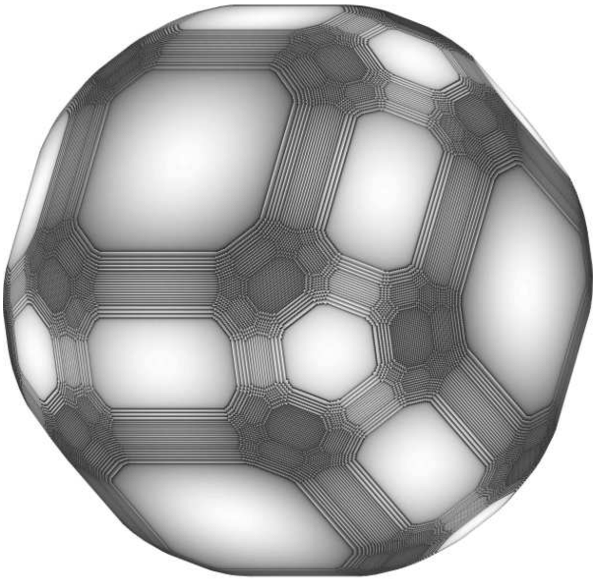
where L is the Laplacian and H is a lower semicontinuous Hamiltonian. We study viscosity solutions of this problem for general H, and prove existence and uniqueness of solutions for certain boundary value problems. We exactly compute our limiting Hamiltonian for the lattice problem, prove that it is not continuous, and show that the scaling limit has facets.

The main motivation for our study is to explain the appearance of facets in the contact line of liquid droplets wetting rough surfaces or spreading in a porous medium. This phenomenon has been observed in physical experiments. While it is easy enough to construct a problem of the above form with facets in the free boundary, we are able to derive such a problem as a scaling limit of a simple microscopic model for the liquid droplet problem. Furthermore we find solutions which can be reliably obtained by a natural flow at the level of the microscopic problem, advancing the contact line from a small initial wetted set as was done in the experiments.

For large $L > 0$, let $u_L : \mathbb{Z}^d \to \mathbb{R}$ be least such that

$$u \geq L 1_{B_L} \quad \text{and} \quad \Delta u \leq 1_{\{u=0\}}.$$

Here Δ denotes the discrete Laplacian. For $d = 3$ and $L = 128$, the function Δu_L restricted to the free boundary $\{\Delta u_L > 0\}$ is:



For $p \in \mathbb{R}^d$, let $H(p) = \sup \Delta v$, where $v : \mathbb{Z}^d \to \mathbb{R}$ is least such that

$$v \geq \max\{0, p \cdot x\} \quad \text{and} \quad \Delta v \leq 0 \quad \text{in } \{v > 0\}.$$

First, we have a formula for the Hamiltonian:

Theorem 1 (Feldman-S)**.** *For $p \in \mathbb{R}^d \setminus \{0\}$,*

$$H(p) = 2d|p|^2 \exp\left(\sum_{\substack{q \in \mathbb{Z}^d \\ q \cdot p = 0}} \hat{L}(q)\right)$$

where $L \in L^2(\mathbb{R}^d/\mathbb{Z}^d)$ is given by

$$L(\theta) = \log\left(1 + \frac{1}{d}\sum_{k=1}^{d} \cos(\theta_k)\right)$$

and satisfies

$$0 > \hat{L}(q) \geq C(1 + |q|)^{-d} \log(2 + |q|).$$

Second, we have the scaling limit:

Theorem 2 (Feldman-S)**.** *The rescalings*

$$\bar{u}_L(x) = L^{-1} u_L(Lx)$$

converge uniformly as $L \to \infty$ to the least $\bar{u} \in C(\mathbb{R}^d)$ satisfying

$$\begin{cases} \bar{u} \geq 1 & \text{in } B_1 \\ \Delta \bar{u} \leq 0 & \text{in } \{u > 0\} \setminus B_1 \\ H(D\bar{u}) \leq 1 & \text{on } \partial\{u > 0\} \setminus B_1 \end{cases}$$

in the sense of viscosity.

Third, we can describe some features of the limit:

Theorem 3 (Feldman-S)**.** *Let $\bar{u} \in C(\mathbb{R}^d)$ be as in the previous result. The support $\{\bar{u} > 0\}$ is open, bounded, and convex and its boundary $\partial\{\bar{u} > 0\}$ has facets in every rational direction. That is, for all $p \in \mathbb{Z}^d \setminus \{0\}$, the set*

$$F_p = \{x \in \overline{\{\bar{u} > 0\}} : p \cdot x = \sup_{y \in \{\bar{u}>0\}} p \cdot y\}$$

is convex and has positive $(d-1)$-dimensional Hausdorff measure.

How a minimal surface leaves a thin obstacle

Emanuele Spadaro
(joint work with M. Focardi)

In this talk I present some of the recent results in [2] on the nonparametric thin obstacle problem for the area functional.
The setting of the problem is the one proposed by J. Nitsche in his paper [7] *"[...] how to fashion a cheap hat for Giacometti's brother"*: we are given an obstacle function $\psi : \{x_{n+1} = 0\} \subset \mathbb{R}^{n+1} \to \mathbb{R}$, $\psi \in C^2(\{x_{n+1} = 0\})$, ψ is assumed to describe the profile of the head of Giacometti's sculpture; and we are given a boundary datum $g \in C^2(\mathbb{R}^{n+1})$ satisfying $g|_{\mathbb{R}^n \times \{0\}} \geq \psi$ and $g(x', x_{n+1}) = g(x', -x_{n-1})$, describing the external profile of the hat; we seek the solution to the following variational problem

$$\min_{v \in \mathcal{A}_g} \int_{B_1} \sqrt{1 + |\nabla v|^2}\, dx \tag{1}$$

in the class $\mathcal{A}_g := \big\{v \in g|_{B_1} + W^{1,\infty}_0(B_1) : \ v|_{B_1'} \geq \psi,\, v(x', x_{n+1}) = v(x', -x_{n+1})\big\}$. Here $B_1' = B_1 \cap \{x_{n+1} = 0\}$; and in addition we set $B_1^+ := B_1 \cap \{x_{n+1} > 0\}$.

This variational problem is sometimes described as a *thin obstacle problem*, because the unilateral constrain $v \geq \psi$ is prescribed on a thin set $B_1 \cap \{x_{n+1} = 0\}$. It gives arise to a *free boundary problem*: indeed the solution u satisfies the following boundary conditions on B_1'

$$(u - \psi)\, \partial_{n+1} u = 0 \text{ on } \ B_1',$$

which defines two subsets of B_1', the one where u coincides with the obstacle $\{u = \psi\}$ and that complementary set $\{u > \psi\}$, separated by a boundary region that is not prescribed apriopri but is an outcome of the minimization problem, the so called free boundary $\Gamma(u) := \partial_{\mathbb{R}^n \times \{0\}}\{u = \psi\}$.

The problem (1) is a nonlinear version of the scalar Signorini problem in elasticity, for which the area functional is replaced by the Dirichlet energy $\int_{B_1} |\nabla u|^2 dx$. The existence and the uniqueness of a solution u in the class $g|_{B_1} + W^{1,\infty}_0(B_1)$ is a classical issue that has been investigated by E. Giusti [5, 6]. The Lipschitz continuity for u is the best possible global regularity in B_1, as simple examples show. Nevertheless, the solution is expected to be more regular on both sides of the obstacle, thus leading to the investigation of the one-sided regularity on $B_1^+ \cup B_1'$. This is a central question in understanding the qualitative properties of the solutions to variational inequalities with thin obstacles and several important questions remain still unanswered.

1. The main results

Differently from the case of the scalar Signorini problem, for which many results have been recently proven, for the nonlinear case of minimal surfaces only fewer results are available. Apart from the Lipschitz regularity by E. Giusti [5, 6], the most general result is due to J. Frehse [3, 4] where the the continuity of the first derivatives of u taken along tangential directions to B_1' in any dimension and one-sided continuity (up to B_1') for the normal derivative in two dimensions (i.e. $n = 1$) is established for the solutions to a very general class of nonlinear variational inequalities.

In the paper in collaboration with M. Focardi of the University of Firenze [2], we establish a first general result on the optimal $C^{1,\frac{1}{2}}$ regularity and provide a detailed analysis of the free boundary of the solutions to the thin obstacle problem for nonparametric minimal surfaces (1).
The following are the main results (more refined conclusions are shown in the paper).

Theorem 1. *Let u be a solution to the thin obstacle problem* (1) *with $\psi \equiv 0$ and let $\Gamma(u)$ be its free boundary, namely the boundary of $\{(x', 0) \in B_1' : u(x', 0) = 0\}$ in the relative topology of B_1'. Then,*

(i) $u \in C^{1,\frac{1}{2}}_{\mathrm{loc}}(B_1^+ \cup B_1')$;
(ii) *$\Gamma(u)$ has locally finite $(n-1)$-dimensional Hausdorff measure and is $\mathcal{H}^{n-1}$-rectifiable.*

Remarks. (1) The one-sided $C^{1,\frac{1}{2}}$ regularity is optimal: one can indeed construct explicit solutions that enjoys no better regularity, using minimal surfaces with suitable branch points.
(2) The conclusion (ii) of the theorem establishes a structure theorem for the free boundary, saying that it can be covered almost everywhere (with respect to the right measure) with C^1 submanifolds.
(3) More refined conclusions for the free boundary are available, by distinguishing among *regular* and *singular* points, cf. [2] for more details.

2. Concerning the proof

2.1. **Optimal $C^{1,\frac{1}{2}}$ regularity.** Building upon the results by J. Frehse [4], the proof is given in several steps:

(A) one-sided C^1 regularity, obtained via a barrier argument;
(B) the analysis of a penalized problem:

$$\min_{v|_{\partial B_1}=g} \mathcal{E}_\varepsilon(v),$$

where

$$\mathcal{E}_\varepsilon(v) := \int_{B_1} \Big(\sqrt{1+|\nabla v|^2} + \chi(|\nabla v|)\Big)\, dx + \int_{B_1'} F_\varepsilon(v(x',0))\, dx',$$

where $\varepsilon > 0$, $F_\varepsilon(t) := \int_0^t \beta_\varepsilon(s)\, ds$, $\beta_\varepsilon(t) := \varepsilon^{-1}\beta(\frac{t}{\varepsilon})$ and $\beta, \chi \in C^\infty(\mathbb{R})$ are such that

$$|t| - 1 \le |\beta(t)| \le |t| \quad \forall\, t \le 0, \quad \beta(t) = 0 \quad \forall\, t \ge 0,$$

$$\beta'(t) \ge 0, \quad \chi''(t) \ge 0 \quad \forall\, t \in \mathbb{R},$$

$$\chi(t) = \begin{cases} 0 & \text{for } t \le \mathrm{Lip}(u), \\ \frac{1}{2}\,(t - 2\mathrm{Lip}(u))^2 & \text{for } t > 3\mathrm{Lip}(u); \end{cases}$$

(C) the De Giorgi method, as pioneered by N. Uraltseva [9] for variational inequalities;

(D) the identification of the graph of the multiple-valued map $U = \{u, -u\}$ with a stationary varifold and the use of the regularity result by Simon and Wickramasekera [8].

2.2. **The structure of the free boundary.** As far as the second part of the theorem is concerned with, it turns out that u minimizes the nonhomogeneous quadratic form

$$\mathcal{A}_g \ni v \longmapsto \frac{1}{2}\int_{B_1} \vartheta(x)|\nabla v(x)|^2\, dx, \tag{2}$$

with $\vartheta(x) := \big(1+|\nabla u(x)|^2\big)^{-\frac{1}{2}}$. Note that the above functional is coercive because in view of the Lipschitz continuity of u we have that

$$0 < \big(1 + \mathrm{Lip}(u)^2\big)^{-\frac{1}{2}} \le \vartheta(x) \le 1 \quad \forall\, x. \tag{3}$$

Moreover, $\vartheta \in W^{1,\infty}(B_1^+)$: indeed, setting $d(x) := \mathrm{dist}(x, \Gamma(u))$, by the regularity result in Theorem 1 (i) and the curvature estimates for minimal surfaces we deduce that

$$|u(x)| \le C\, d^{\frac{3}{2}}(x), \quad |\nabla u(x)| \le C\, d^{\frac{1}{2}}(x) \quad \text{and} \quad |D^2 u(x)| \le C\, d^{-\frac{1}{2}}(x),$$

for some constant $C > 0$, and therefore

$$|\nabla\vartheta| = \big(1 + |\nabla u|^2\big)^{-\frac{3}{2}}\, |D^2 u\, \nabla u| \le C.$$

The proof of the theorem follows then from a generalization to quadratic energies with Lipschitz coefficients of the arguments based on the frequency function developed in our paper [1]. Note that the Lipschitz continuity of ϑ, and hence the optimal regularity of u in Theorem 1 (i), seems essential for this approach.

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The Cost of Crushing: Curvature-Driven Wrinkling of Thin Elastic Sheets

Ian Tobasco

We discuss the wrinkling and folding patterns that form when a thin elastic shell is floated onto a water bath. Our work is motivated by the recent experiments reported in [1, 2]. We develop a mathematically rigorous framework for proving the existence of "non-wrinkling" or "stable" directions throughout the shell. Such directions are stable in that any asymptotically persistent oscillations or concentrations parallel to them are ruled out. Depending on the application, such directions can be globally defined, defined only on certain sub-domains, or fail to exist altogether. These alternatives are consistent with the experimental results. In what follows, we briefly describe our model and state our results. A more complete presentation is under preparation at the time of this writing.

We consider the minimization of a singularly perturbed, non-convex energy functional to describe the deformation of a thin, weakly curved, floating elastic shell. If the undeformed mid-shell is given by the graph S of a function $p:\Omega\subset\mathbb{R}^2\to\mathbb{R}$, a deformation of the shell corresponds to a map $\Phi:S\to\mathbb{R}^3$. Introducing "in-plane" and "out-of-plane" displacements $u:\Omega\to\mathbb{R}^2$ and $w:\Omega\to\mathbb{R}$ (the plane being referenced is that of the initial water bath) we obtain

$$\Phi(x,p(x))=(x_1+u_1(x),x_2+u_2(x),w(x)),\quad x\in\Omega.$$

Provided the shell is weakly curved, its equilibrium shape can be determined by minimization of the geometrically linearized energy functional

$$E_{b,k}(u,w)=\int_\Omega |e(u)+\frac{1}{2}\nabla w\otimes\nabla w-\frac{1}{2}\nabla p\otimes\nabla p|^2+b|\nabla\nabla w|^2+k|w|^2\,dx.$$

The notation $|\cdot|$ indicates the Frobenius norm, and $e(u)$ denotes the symmetric gradient of u. The first two terms account for the internal elastic energy of the shell, and are called the "stretching" and "bending" terms. The third term accounts for the work done by buoyancy forces. The positive parameters b and k are proportional to the squared thickness of the shell and the gravitational acceleration of the water, respectively, but are actually non-dimensional. Their inverses are analogous to the "bendability" and "deformability" parameters from [3]. Taking $b \ll 1$ permits the shell to oscillate more and more rapidly, while taking $k \gg 1$ ensures that it must lie nearly flat. To complete the description we must explain our choice of strain,

$$\varepsilon = e(u) + \frac{1}{2}\nabla w \otimes \nabla w - \frac{1}{2}\nabla p \otimes \nabla p.$$

Such a formula arises from a geometric linearization procedure applied to the nonlinear strain induced by Φ, in which the leading order terms in ∇u, ∇w, and ∇p are retained. Similar Föppl–von Kármán-type models have been used across the literature on pattern formation in thin elastic structures (see, e.g., [3, 4]).

Before stating our results, we introduce the space of *bounded deformation* maps

$$BD(\Omega) = \left\{u \in L^1(\Omega;\mathbb{R}^2) : e(u) \in \mathcal{M}(\Omega;\mathrm{Symm}_{2\times 2})\right\},$$

where $\mathcal{M}(\Omega;\mathrm{Symm}_{2\times 2})$ denotes the two-by-two symmetric matrix-valued Radon measures on Ω. A map $u \in BD(\Omega)$ satisfies $e(u) = 0$ if and only if it belongs to

$$\mathcal{R} = \{x \mapsto \omega x + b : \omega \in \mathrm{Skew}_{2\times 2}, b \in \mathbb{R}^2\}.$$

Each of $BD(\Omega)$ and $BD(\Omega)/\mathcal{R}$ turns out to be a Banach space which is also a dual. For our purposes, it suffices to note that a sequence $\{u_n\} \subset BD(\Omega)/\mathcal{R}$ converges weakly-$*$ to u if and only if there exists a sequence $\{r_n\} \subset \mathcal{R}$ so that

$$\begin{aligned} u_n + r_n &\to u \quad \text{strongly in } L^1(\Omega)\\ e(u_n) &\overset{*}{\rightharpoonup} e(u) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega;\mathrm{Symm}_{2\times 2}). \end{aligned}$$

We refer the reader to [5] for further discussion.

We come now to our results. Throughout, we understand that $p \in W^{2,\infty}(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is open, bounded, Lipschitz, and strictly star-shaped. This last requirement is satisfied if there exists $x_0 \in \Omega$ such that for every $x \in \partial\Omega$ the open line segment from x_0 to x is contained in Ω. We also fix a sequence $\{(b_n, k_n)\}$ which satisfies

$$\frac{1}{k_n^{3/2}} \ll b_n \ll \frac{1}{k_n} \quad \text{as } n \to \infty,$$

though we hide the subscript n in the remainder. Our first result establishes the Γ-convergence of rescaled versions of $E_{b,k}$ in a topology where they are equi-coercive.

Theorem 1. *We have that*

$$\frac{E_{b,k}}{4\sqrt{bk}} \xrightarrow{\Gamma} \begin{cases} \int_\Omega \frac{1}{2}|\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{\nu}\, ds & e(u) \leq \frac{1}{2}\nabla p \otimes \nabla p\, dx, w = 0 \\ +\infty & \textit{otherwise} \end{cases}$$

with respect to the weak-$*$ topology on $BD(\Omega)/\mathcal{R} \times H^1(\Omega)$. Moreover, any admissible $\{(u_{b,k}, w_{b,k})\}$ that satisfies $E_{b,k}(u_{b,k}, w_{b,k}) = O(\sqrt{bk})$ is weakly-$*$ precompact.

The following corollary is immediate. We say that $\{(u_{b,k}, w_{b,k})\}$ is a sequence of *almost minimizers* provided that

$$E_{b,k}(u_{b,k}, w_{b,k}) = \min E_{b,k} + o(\sqrt{bk}).$$

Corollary 1. *The minimum energies satisfy*

$$\lim \frac{\min E_{b,k}}{4\sqrt{bk}} = \min_{\substack{u \in BD(\Omega)/\mathcal{R} \\ e(u) \le \frac{1}{2}\nabla p \otimes \nabla p\, dx}} \int_\Omega \frac{1}{2}|\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{\nu}\, ds.$$

Moreover, $(u_{\text{eff}}, w_{\text{eff}}) \in BD(\Omega)/\mathcal{R} \times H^1(\Omega)$ is the weak-$*$ limit of a sequence of almost minimizers if and only if u_{eff} solves the limiting problem appearing above and $w_{\text{eff}} = 0$.

Next, we present a duality theory for the limiting problem. Let $(\cdot)_\delta$ denote a standard non-negative mollifier.

Theorem 2. *We have the duality*

$$\min_{\substack{u \in BD(\Omega)/\mathcal{R} \\ e(u) \le \frac{1}{2}\nabla p \otimes \nabla p\, dx}} \int_\Omega \frac{1}{2}|\nabla p|^2 - \int_{\partial\Omega} u \cdot \hat{\nu}\, ds = \max_{\substack{\varphi : \mathbb{R}^2 \to \mathbb{R} \\ \varphi \text{ is convex} \\ \varphi = \frac{1}{2}|x|^2 \text{ on } \mathbb{R}^2 \setminus \Omega}} \int_\Omega \left(\varphi - \frac{1}{2}|x|^2\right) \det \nabla\nabla p.$$

The following complementary slackness conditions also hold: an admissible pair (u_*, φ_*) is optimal only if

$$\lim_{\delta \to 0} \int_\Omega \left\langle (\operatorname{cof} \nabla\nabla\varphi_*)_\delta, \frac{1}{2}\nabla p \otimes \nabla p\, dx - e(u_*) \right\rangle = 0.$$

Combining the previous results we deduce a method for proving the existence of stable directions.

Corollary 2. *Let φ_* solve the dual problem from Theorem 2 and assume there exists an open set U for which $\varphi_* \in C^2(U)$. Any sequence of almost minimizers must satisfy the improved convergence*

$$\langle \operatorname{cof} \nabla\nabla\varphi_*, \nabla w_{b,k} \otimes \nabla w_{b,k} \rangle \to 0 \quad \textit{strongly in } L^1_{loc}(U).$$

To demonstrate how this last result can be applied to deduce the existence of stable directions, we consider it in the case that φ_* satisfies

$$\operatorname{cof} \nabla\nabla\varphi_* = \lambda \hat{n} \otimes \hat{n}$$

for some continuous $\hat{n} : U \to S^1$ and $\lambda : U \to (0, \infty)$. Such φ_* naturally arise when we consider the application of our results to the setting of [1, 2]. In such a case, the result of Corollary 2 implies that any sequence of almost minimizers must satisfy the improved convergence

$$\hat{n} \cdot \nabla w_{b,k} \to 0 \quad \text{strongly in } L^2_{loc}(U).$$

The situation is neatly summarized by saying that almost minimizers are stable to asymptotic out-of-plane perturbations parallel to $\hat{n}$ on U.

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Translating solutions of mean curvature flow

Brian White

A **translator** with velocity $\mathbf{v}$ is a hypersurface M in $\mathbf{R}^{n+1}$ such that

$$t \mapsto M + t\,\mathbf{v}$$

is a mean curvature flow, i.e., such that normal component of the velocity at each point is equal to the mean curvature at that point:

$$\overrightarrow{H} = \mathbf{v}^{\perp}.$$

By rotating and scaling, we can make the velocity equal to $-\mathbf{e}_{n+1}$; unless otherwise specified, I will assume that the velocity has been so normalized.

If a translator M (with velocity $-\mathbf{e}_{n+1}$) is the graph of function $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}$, we will say that M is a **translating graph**; in that case, we also refer to the function u as a translator, and we say that u is complete if its graph is a complete submanifold of $\mathbf{R}^{n+1}$. Thus $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ is a translator if and only if it solves the translator equation (the nonparametric form of $\overrightarrow{H} = -\mathbf{e}_{n+1}^{\perp}$):

$$D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) = -\frac{1}{\sqrt{1+|Du|^2}}.$$

An example is the **grim reaper curve**:

$$\{(x,y) : y = \log(\cos x)),\ x \in (-\pi/2, \pi, 2)\}.$$

Translators are interesting for a number of reasons:

(1) They provide simple examples of mean curvature flows.
(2) They provide possible models for singularity formation in mean curvature flow. For example, consider a figure 8 curve $M(t)$ in the plane moving by (mean) curvature flow. It will develop a singularity at some finite time T. Let $p(t)$ be the point of maximum curvature $\kappa(t)$. Then $\kappa(t)(M(t) - p(t))$ converges smoothly as $t \to T$ to the grim reaper curve (modulo a rotation of $\mathbf{R}^2$).

(3) They are interesting as examples in minimal surface theory. Ilmanen observed that M is a translator with velocity $\mathbf{v}$ if and only if M is a minimal surface (i.e., critical point of the area functional) with respect to the Riemannian metric

$$g_{ij}(x) = (e^{-x\cdot\mathbf{v}})^{2/n}\delta_{ij}.$$

(4) Ilmanen's elliptic regularization [6] scheme lets one get general mean curvature flows as limits of translators.

In this talk, I described recent joint work with David Hoffman, Tom Ilmanen, and Francisco Martín: (1) a classification of all the complete translating graphs in $\mathbf{R}^3$, and (2) new families of examples of non-graphical translators in $\mathbf{R}^3$.

Before stating the classification theorem, I recall the known examples of translating graphs in $\mathbf{R}^3$. First, the Cartesian product of the grim reaper curve with $\mathbf{R}$ is a translator:

$$\begin{aligned} &\mathcal{G} : \mathbf{R} \times (-\pi/2, \pi/2) \to \mathbf{R}, \\ &\mathcal{G}(x, y) = \log(\cos y). \end{aligned}$$

It is called the **grim reaper surface**.

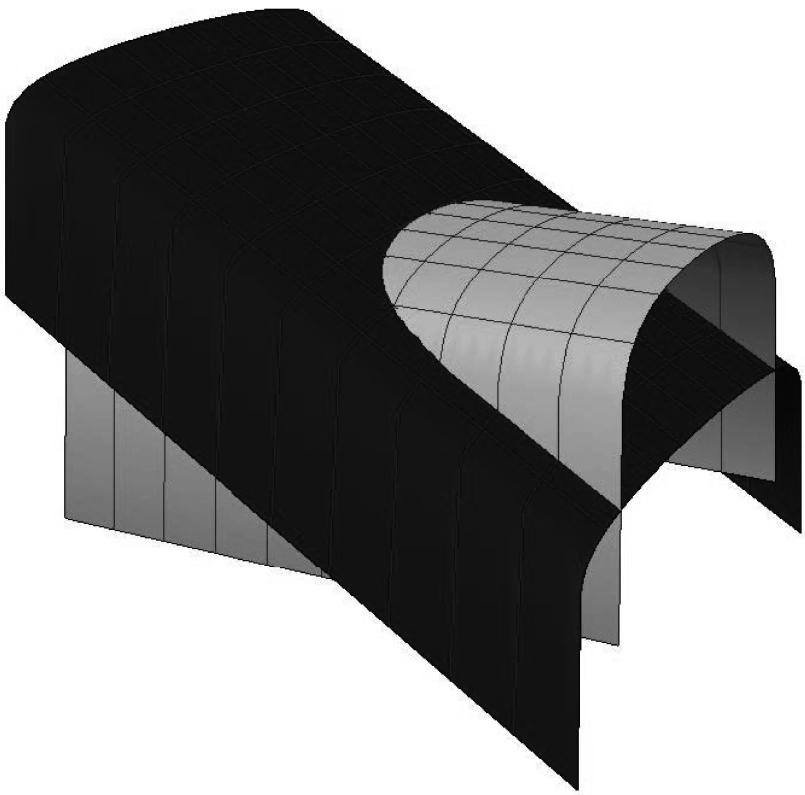

FIGURE 1. The grim reaper surface in $\mathbb{R}^3$, and that surface tilted by angle $\theta = -\pi/4$ and dilated by $1/\cos(\pi/4)$. (Picture by F. Martin.)

Second, if we rotate the grim reaper surface by an angle $\theta \in (0, \pi/2)$ about the y-axis and dilate by $1/\cos\theta$, the resulting surface is again a translator, given by

$$\mathcal{G}_\theta : \mathbf{R} \times (-b, b) \to \mathbf{R},$$
$$\mathcal{G}_\theta(x,y) = \frac{\log(\cos(y\cos\theta))}{\cos^2\theta} + x\tan\theta,$$

where $b = \pi/(2\cos\theta)$. Note that as θ goes from 0 to $\pi/2$, the width $2b$ of the strip goes from π to ∞. These examples are called **tilted grim reaper surfaces**.

Every translator $\mathbf{R}^3$ with zero Gauss curvature is (up to translations and up to rotations about a vertical axis) a grim reaper surface, a tilted grim reaper surface, or a vertical plane.

In [3], J. Clutterbuck, O. Schnürer and F. Schulze (see also [1]) proved for each $n \geq 2$ that there is a unique (up to vertical translation) entire, rotationally invariant function $u : \mathbf{R}^n \to \mathbf{R}$ whose graph is a translator. It is called the **bowl soliton**.

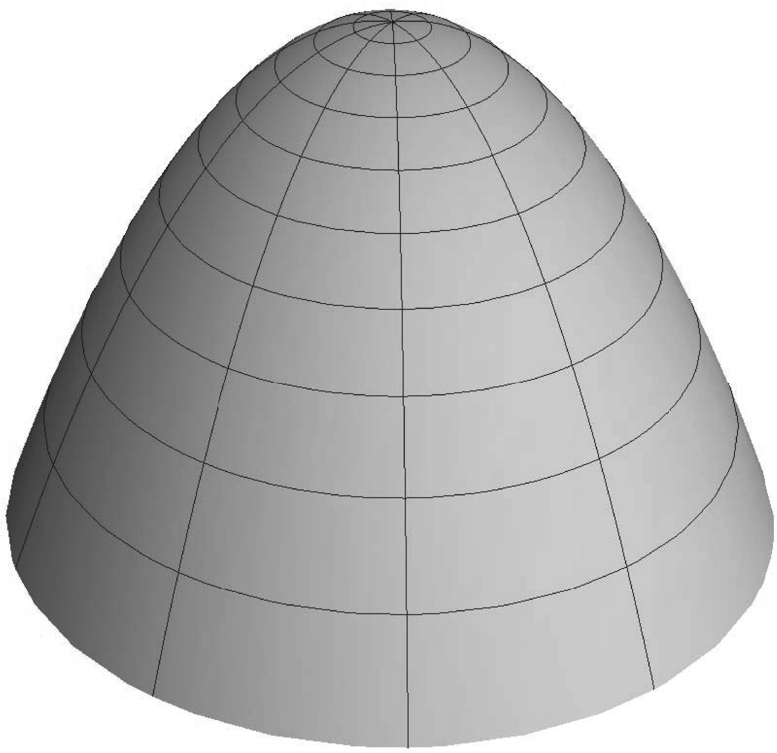

FIGURE 2. The bowl soliton. As one moves down, the slope tends to infinity, and thus the end is asymptotically cylindrical. (Picture by F. Martín.)

In addition to the examples described above, many years ago, Ilmanen (in unpublished work) proved that for each $0 < k < 1/2$, there is a translator u :

$\Omega \to \mathbf{R}$ with the following properties: $u(x,y) \equiv u(-x,y) \equiv u(x,-y)$, u attains its maximum at $(0,0) \in \Omega$, and

$$D^2u(0,0) = \begin{bmatrix} -k & 0 \\ 0 & -(1-k) \end{bmatrix}.$$

The domain Ω is either a strip $\mathbf{R} \times (-b\mathbf{R}$ or $\mathbf{R}^2$. He referred to these examples as **Δ-wings**. As $k \to 0$, he showed that the examples converge to the grim reaper surface. Uniqueness (for a given k) was not known. It was also not known which strips $\mathbf{R} \times (-b, b)$ occur as domains of such examples.

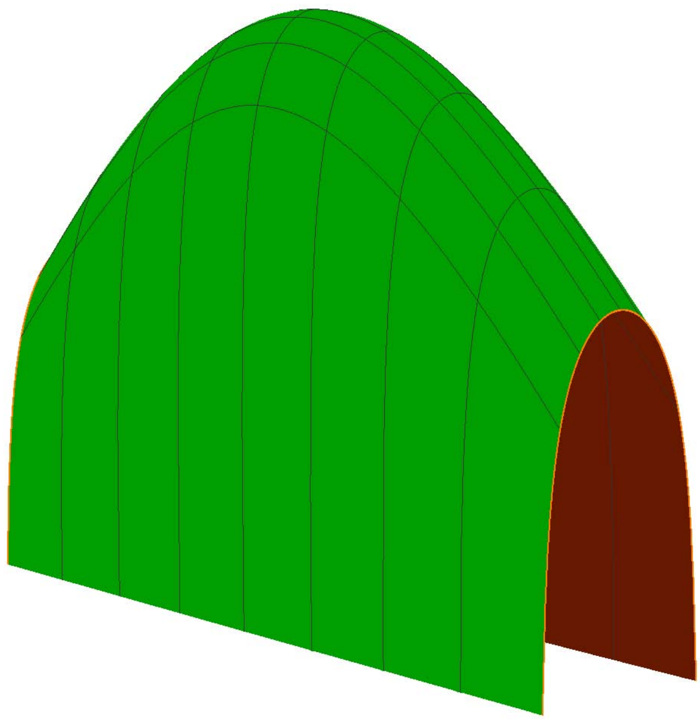

FIGURE 3. The Δ-wing of width $\sqrt{2}\,\pi$. As $y \to \pm\infty$, this Δ-wing is asymptotic to the tilted grim reapers $\mathcal{G}_{-\frac{\pi}{4}}$ and $\mathcal{G}_{\frac{\pi}{4}}$, respectively. (Picture by F. Martín.)

Our new classification theorem is the following:

Theorem 1. [5] *For every $b > \pi/2$, there is (up to translation) a unique complete, strictly convex translator $u^b : \mathbf{R} \times \mathbf{R}, b) \to \mathbf{R}$. Up to isometries of $\mathbf{R}^2$, the only other complete translating graphs in $\mathbf{R}^3$ are the grim reaper surface, the tilted grim reaper surfaces, and the bowl soliton.*

I remark that Bourni, Langford, and Tinaglia [2] have recently given a different proof of part of this theorem: they proved existence (but not uniqueness) of strictly convex translating graphs defined over strips.

The proof of Theorem 1 uses some important earlier work. In particular, Spruck and Xiao recently proved the very powerful theorem that every translating graph in $\mathbf{R}^3$ is convex [8, Theorem 1.1]. Thus it sufficed to classify convex examples. L. Shahriyari [7] proved that if $u : \Omega \subset \mathbf{R}^2 \to \mathbf{R}$ is a complete translator, then Ω is (up to rigid motion) one of the following: the plane $\mathbf{R}^2$, a halfplane, or a strip $\mathbf{R} \times (-b, b)$ with $b \geq \pi/2$. Also, in [9], X. J. Wang proved that the only entire convex translating graph is the bowl soliton, and that there are no complete translating graphs defined over halfplanes. Thus by the Spruck-Xiao Convexity Theorem, the bowl soliton is the only complete translating graph defined over a plane or halfplane.

In the lecture, I also described a new family of examples of complete, non-graphical translating annuli. Clutterbuck, Schnürrer, and Schulze found all rotationally invariant examples [3]. In particular, there is a one-parameter family of such examples, parametrized by the necksize. Each example has two ends, and each end is asymptotic to a bowl soliton. Thus the example looks like two bowl solitons, one above the other, joined by a neck; see Figure .

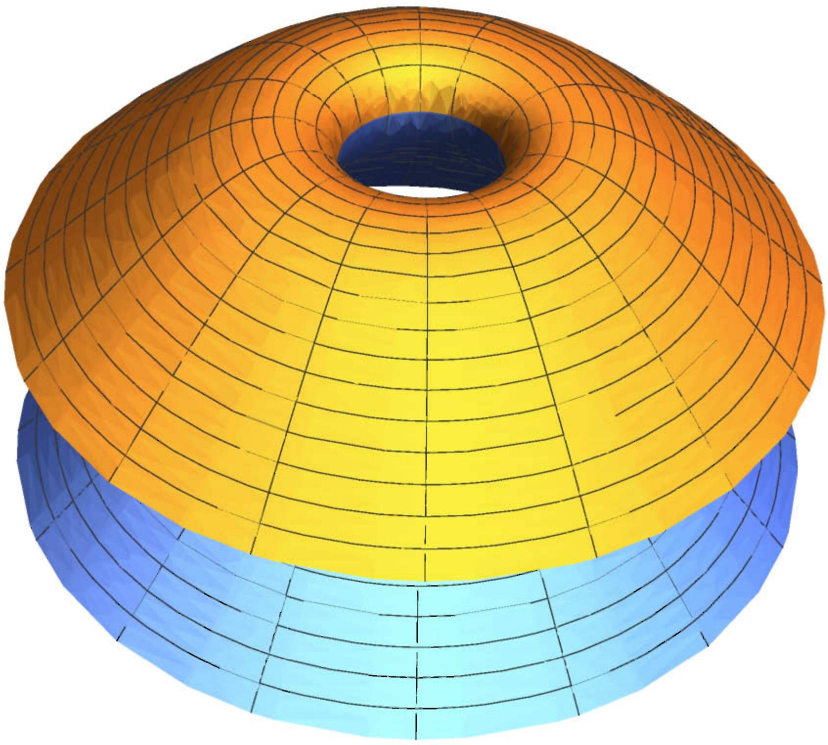

Figure 4. A rotationally invariant translating annulus. Each of the two ends is asymptotic to a bowl soliton. (Picture by F. Martín.)

Hoffman, Ilmanen, Marten, and I have shown that those rotationally invariant examples are part of a larger, two-parameter family of complete, translating annuli [4]. One parameter is the necksize, which we define to be the shortest (Euclidean) length of a homotopically nontrivial curve in the annulus. The necksize takes all values in $(0, \infty)$. The other parameter is an "inner width" w, which takes all values in $[\pi, \infty]$. When the inner width is infinite, the examples are the rotationally invariant examples analyzed by Clutterbuck et. al.

For every finite width w, an example M with neck size n and inner width w has the following property: as $z \to -\infty$, M is asymptotic to four parallel planes, and w is the distance between the inner pair. That is, by making a translation and a rotation, we can assume that as $z \to -\infty$, M is asymptotic to the planes $y = c$, $y = w/2$, $y = -w/2$, and $y = -c$. Here $w/2 \leq c < w/2 + \pi$.

Note: we prescribe the inner width w and the necksize n. For each $(w, n) \in [\pi/2, \infty] \times (0, \infty)$, there is at least one example M with inner width w and neck size n. We would guess that there is exactly one example, but we do not know how to prove that.

What about c? For a long time, we were puzzled by it. We had some heuristic arguments that it should be $w/2$, but other heuristic arguments that it should be strictly greater than $w/2$. In fact, both arguments are right. It turns out (for each finite inner width w) that when the necksize is very small, $c = w/2$. However, as the necksize tends to infinity, c tends to $w/2 + \pi$.

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The inverse problem of finding a phase field energy for 2D branched transport and other functionals of normal currents

Benedikt Wirth

Branched transport. Denote by $\mathcal{P}(\mathbb{R}^2)$ the set of probability measures on $\mathbb{R}^2$ and by $\mathcal{M}(\mathbb{R}^2; \mathbb{R}^2)$ the set of vector-valued Radon measures on $\mathbb{R}^2$. Most optimal transportation problems on $\mathbb{R}^2$ can be formulated as the task of finding the optimal way to move all material from a given source $\mu_0 \in \mathcal{P}(\mathbb{R}^2)$ to a given sink $\mu_1 \in \mathcal{P}(\mathbb{R}^2)$. For a particular subclass of optimal transport problems the cost can be expressed based on the material flux $\sigma \in \mathcal{M}(\mathbb{R}^2;\mathbb{R}^2)$, which necessarily satisfies the mass conservation law

$$\mathrm{div}\sigma = \mu_0 - \mu_- \tag{1}$$

in a distributional sense. For instance, in Wasserstein-1-transport the cost to be minimized is the total variation $|\sigma|(\mathbb{R}^2)$ of σ among all material fluxes σ satisfying (1) [3, §4.2]. In so-called *branched transport* and related models the cost depends concavely on the transported mass and is given by

$$E^\tau[\sigma] = \int_S \tau(|w|)\,\mathrm{d}\mathcal{H}^1 + \tau'(0)|\sigma^\perp|(\mathbb{R}^2) \qquad \text{for} \qquad \sigma = w\theta\mathcal{H}^1\llcorner S + \sigma^\perp\,,$$

where S is a 1-rectifiable set with approximate tangent $\theta : S \to \mathbb{S}^1$, $w : S \to \mathbb{R}$ an $\mathcal{H}^1\llcorner S$-measurable function, $\sigma^\perp$ the union of a Lebesgue-continuous and a Cantor part, and where $\tau : [0,\infty) \to [0,\infty)$ is an *admissible transport cost*, that is, $\tau(0) = 0$ and τ is nondecreasing, concave, and continuous. The quantity $\tau(w)$ here has the interpretation of the cost for transporting mass w along a unit distance, and a consequence of its concavity is that optimal (that is, cost-minimizing) mass fluxes σ typically are concentrated on ramified 1-rectifiable transportation networks S.

Phase field approximation. One approach to compute optimal mass fluxes σ numerically is via a phase field approach. In more detail, we wish to replace E^τ by an approximating functional E^τ_ε whose minimizer $u : \mathbb{R}^2 \to \mathbb{R}^2$ shall be a smooth approximation of the optimal σ in the sense that all lines of the network S are thickened to a width of roughly $\varepsilon > 0$. To this end we make the ansatz of a Modica–Mortola or Allen–Cahn functional

$$E^\tau_\varepsilon[u] = \int_{\mathbb{R}}^2 \frac{\varepsilon^\alpha}{2}|\nabla u|^2 + \frac{1}{\varepsilon^\beta}c(\varepsilon^\gamma|u|)\,\mathrm{d}x$$

in which the phase field potential $c : [0,\infty) \to [0,\infty)$ has to be chosen appropriately so as to yield an approximation of E^τ (the real powers α, β, γ will be fixed further below). In [2] the authors made the ansatz $c(s) = s^\zeta$ for some $\zeta > 0$ and showed this to be the appropriate choice for $\tau(w)$ a concave power of w (for alternative Ambrosio–Tortorelli type phase field models see [1]). Such a potential c can be interpreted as a double well with one well at 0 and another one at infinity so that the functional-minimizing u will preferably be either zero or have a large magnitude, resulting in a flux concentration within small regions (those regions will be the thickened S).

For simplicity consider the situation in which the minimizer u of E^τ_ε (among all phase field functions satisfying a smoothed version of the mass conservation (1)) describes a vertical flux of mass w. Without loss of generality we may assume $u(x_1, x_2) = m(x_1)(0,1)^T$ for some function $m : \mathbb{R} \to [0,\infty)$ symmetrically decreasing about $x_1 = 0$ (otherwise the cost could be reduced without changing the vertical flux by altering u to this form) so that we obtain

$$w = \int_{\mathbb{R}} u(x_1,x_2)\cdot\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}x_1 = \int_{\mathbb{R}} m(x_1)\,\mathrm{d}x_1 = \int_{\mathbb{R}} \psi(y)\,\mathrm{d}y\,, \tag{2}$$

where we performed the change of variables $y = x_1/\varepsilon$ and introduced the rescaled flux magnitude $\psi(y) = m(\varepsilon y)\varepsilon$. For E_ε^τ to be an approximation of E^τ we would like the vertical phase field cost per unit length (that is, a horizontal slice of E_ε^τ) to equal $\tau(w)$,

$$\begin{aligned}(3)\quad \tau(w) &= \int_{\mathbb{R}} \frac{\varepsilon^\alpha}{2}|\nabla u|^2 + \frac{1}{\varepsilon^\beta}c(\varepsilon^\gamma|u|)\,\mathrm{d}x_1\\ &= \int_{\mathbb{R}} \frac{\varepsilon^\alpha}{2}|m'|^2 + \frac{1}{\varepsilon^\beta}c(\varepsilon^\gamma|m|)\,\mathrm{d}x_1 = \int_{\mathbb{R}} \frac{\varepsilon^{\alpha-3}}{2}|\psi'|^2 + \varepsilon^{1-\beta}c(\varepsilon^{\gamma-1}|\psi|)\,\mathrm{d}y\,.\end{aligned}$$

Thus, the natural parameter choice is $\alpha = 3, \beta = \gamma = 1$, and the phase field potential c is to be determined from

$$(4)\qquad \tau(w) = \inf\{F^c[\psi]\,|\,(2)\}\quad\text{for}\quad F^c[\psi] = \int_{\mathbb{R}} \frac{1}{2}|\psi'|^2 + c(|\psi|)\,\mathrm{d}y\,.$$

During the remainder of this note we answer the question whether such a function c exists and how it can be calculated.

Formula and examples. It is more convenient to work with the mass-specific potential $z(s) = c(s)/s$ rather than with c itself. A heuristic calculation now yields a formula for z (or rather for $g(s) = (z^{-1}(s))^{3/2}$, assuming z to be invertible): The optimality conditions for the optimal ψ_w in (4) read

$$0 = -\psi_w'' + c'(\psi_w) + \lambda\,,\qquad 0 = \int_{\mathbb{R}} \psi_w\,\mathrm{d}y - w$$

for a Lagrange multiplier $\lambda \in \mathbb{R}$. Differentiating $\tau(w) = F^c[\psi_w]$ we now obtain

$$\begin{aligned}\tau'(w) = \partial_\psi F^c[\psi_w](\partial_w\psi_w) &= \int_{\mathbb{R}} \psi_w'\partial_w\psi_w' + c'(\psi_w)\partial_w\psi_w\,\mathrm{d}y\\ &= \int_{\mathbb{R}} (-\psi_w'' + c'(\psi_w))\,\partial_w\psi_w\,\mathrm{d}y = -\lambda\partial_w\int_{\mathbb{R}}\psi_w\,\mathrm{d}y = -\lambda\,.\end{aligned}$$

Thus, testing the optimality conditions with ψ_w' and integrating we obtain

$$0 = -|\psi_w'|^2 + 2(c(\psi_w) - \tau'(w)\psi_w)$$

(where the integration constant is zero due to $\lim_{|y|\to\infty}\psi_w(y) = \lim_{|y|\to\infty}\psi_w'(y) = 0$), which implies $|\psi_w'| = \sqrt{2(z(\psi_w) - \tau'(w))\psi_w}$ as well as $z(\psi_w(0)) = \tau'(w)$. Thus,

$$\begin{aligned}\tau(w) - \tau'(w)w &= F^c[\psi_w] - \tau'(w)\int_{\mathbb{R}}\psi_w\,\mathrm{d}y = \int_{\mathbb{R}} \frac{|\psi_w'|^2}{2} + \frac{2(z(\psi_w) - \tau'(w))\psi_w}{2}\,\mathrm{d}y\\ &= 2\int_0^\infty |\psi_w'|\sqrt{2(z(\psi_w)-\tau'(w))\psi_w}\,\mathrm{d}y = 2\int_0^{\psi_w(0)} \sqrt{2(z(\phi) - \tau'(w))\phi}\,\mathrm{d}\phi\\ &= \int_0^{z^{-1}(\tau'(w))} \sqrt{2(z(\phi)-\tau'(w))\phi}\,\mathrm{d}\phi = -\frac{4}{3}\int_0^{z^{-1}(\tau'(w))} \phi^{\frac{3}{2}}\frac{z'(\phi)}{\sqrt{2(z(\phi)-\tau'(w))}}\,\mathrm{d}\phi\\ &= -\frac{4}{3}\int_\infty^{\tau'(w)} \left[z^{-1}(s)\right]^{\frac{3}{2}}\frac{1}{\sqrt{2(s-\tau'(w))}}\,\mathrm{d}s = [g * r](\tau'(w))\quad\text{for}\quad r(s) = -\frac{2}{3\sqrt{-s}}\,.\end{aligned}$$

With $w = (\tau')^{-1}(t)$ this yields

$$[-\tau(-\cdot)]^*(t) = [g * r](t)\,. \tag{5}$$

This formula (which will be made rigorous below) can be used to derive c for examples such as the following:

- $\tau(w) = w^\zeta$ is induced by $c(\phi) = \text{const.}\phi^{(4\zeta-2)/(\zeta+1)}$,
- $\tau(w) = \min\{aw, bw+d\}$ is induced by $c(\phi) = \max\left\{a\phi - \frac{2\pi^2}{9}\left(\frac{d}{a-b}\right)^2\phi^4, b\phi\right\}$.

Existence and further properties. For piecewise constant nonincreasing mass-specific phase field potential z one can explicitly solve the optimisation in (4) for ψ. It turns out that this leads to a one-to-one relation between piecewise constant nonincreasing potentials z and piecewise affine admissible costs τ. One even obtains an explicit formula to derive a solution z of (4) for any piecewise affine admissible cost τ. This formula can be used to prove an existence result for general admissible τ via approximation:

Theorem 1 (Existence)**.** *Problem* (4) *has a solution* $c : [0,\infty) \to [0,\infty)$ *for any given admissible* τ*. Furthermore,* c *can be chosen such that* $z(s) = c(s)/s$ *is lower semi-continuous and nonincreasing.*

With that c one could now prove the Γ-convergence $E_\varepsilon^\tau \to E^\tau$ along the lines of [2, 1]. In general, c is nonunique. Note that our phase field ansatz approximates exactly the branched transport problems with admissible τ:

Theorem 2 (Obtainable τ)**.** *Let* $c : [0,\infty) \to [0,\infty)$ *be Borel measurable with* $c(0) = 0$ *and define* $\tau : [0,\infty) \to [0,\infty)$ *via* (4)*. Then* τ *is admissible.*

Finally, the heuristic formula (5) can be made rigorous:

Theorem 3 (Necessary and sufficient conditions)**.** *Let* $c(s) = z(s)s$ *solve* (4)*, then* (5) *holds for all* $t \in \{\tau'(w) \mid \tau$ *differentiable at* $w\}$*. Vice versa, if* (5) *holds for all* $t > 0$*, then* $c(s) = z(s)s$ *solves* (4)*.*

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Stochastic homogenisation of free-discontinuity problems

Caterina Ida Zeppieri
(joint work with Filippo Cagnetti, Gianni Dal Maso, Lucia Scardia)

In the Calculus of Variations the terminology *free-discontinuity problems* usually refers to those minimisation problems involving competing volume and surface functionals. A prototipical free-discontinuity problem consists in minimising an integral functional of the form

$$E(u) = \int_A f(x, \nabla u)\, dx + \int_{S_u \cap A} g(x, [u], \nu_u)\, d\mathcal{H}^{n-1}, \tag{1}$$

depending on both a possibly discontinuous function $u\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ and on its unknown discontinuity set S_u. Functionals of type (1) are referred to as *free-discontinuity functionals* and are ubiquitous in Material Sciences, Mathematical Physics, and Computer Vision. Indeed free-discontinuity energies account for crack growth and crack initiation in the theory of brittle fracture, interface formation between different phases of Cahn-Hilliard fluids, surface tension between small drops of liquid crystals, and are employed for pattern recognition in computer vision to detect surfaces corresponding to sudden changes in images (such as edges of objects or shadows). The natural functional space where to minimise (1) is that of De Giorgi and Ambrosio's space of *special functions of bounded variation* $SBV(A, \mathbb{R}^m)$. Here ∇u denotes the approximate differential of u, $[u]$ stands for the difference $u^+ - u^-$ between the approximate limits of u on both sides of the discontinuity set S_u, and ν_u denotes the (generalised) normal to S_u.

In the typical applications one deals with families of functionals of type (1); *i.e.,* functionals depending on some small positive parameter ε, whose nature depends on the specific problem under examination, and tries to establish some emergent properties in the limit as ε tends to zero. Further, in many relevant applications (such as *e.g.*, in the study of brittle composite materials) the integrands f and g may also vary according to some (spatial) periodicity or, more in general, to some random law. One then considers sequences of functionals of the form

$$E_\varepsilon(\omega)(u) = \int_A f\left(\omega, \frac{x}{\varepsilon}, \nabla u\right)\, dx + \int_{S_u \cap A} g\left(\omega, \frac{x}{\varepsilon}, [u], \nu_u\right)\, d\mathcal{H}^{n-1}, \tag{2}$$

where the random parameter ω belongs to the sample space Ω of a probability space $(\Omega, \mathcal{T}, P)$ and denotes an instance of the medium. The integrands f and g are random fields and only their statistical specification is known. When f and g do not depend on ω and are periodic in the spatial variable, the limit behaviour of E_ε can be determined appealing to the classical homogenisation theory [3]. The latter asserts that, under standard growth and coercivity conditions (and mild regularity assumptions) on f and g the functionals E_ε behave macroscopically like a homogeneous free-discontinuity functional. Further, in the homogenisation process there is no interaction between volume and surface energies.

In this talk we extend the classical deterministic *periodic* homogenisation result as above to the stochastic *stationary* setting. We assume that the realisations of the random variables f and g satisfy the following assumptions: Fix five constants $p, c_1, \dots, c_4$, with $1 < p < +\infty$, $0 < c_1 \leq c_2 < +\infty$, and $0 < c_3 \leq c_4 < +\infty$, and two nondecreasing continuous functions σ_1, $\sigma_2\colon [0,+\infty) \to [0,+\infty)$ such that $\sigma_1(0) = \sigma_2(0) = 0$; then

The volume integrand $f\colon \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0,+\infty)$ satisfies:

($f1$) (measurability) f is Borel measurable on $\mathbb{R}^n \times \mathbb{R}^{m\times n}$;

($f2$) (continuity in ξ) for every $x \in \mathbb{R}^n$ we have

$$|f(x,\xi_1) - f(x,\xi_2)| \leq \sigma_1(|\xi_1 - \xi_2|)\big(1 + f(x,\xi_1) + f(x,\xi_2)\big)$$

for every ξ_1, $\xi_2 \in \mathbb{R}^{m\times n}$;

($f3$) (bounds) for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m\times n}$

$$c_1|\xi|^p \leq f(x,\xi) \leq c_2(1 + |\xi|^p).$$

The surface integrand $g\colon \mathbb{R}^n \times \mathbb{R}^m_0 \times \mathbb{S}^{n-1} \to [0,+\infty)$ satisfies:

($g1$) (measurability) g is Borel measurable on $\mathbb{R}^n \times \mathbb{R}^m_0 \times \mathbb{S}^{n-1}$;

($g2$) (continuity in ζ) for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$ we have

$$|g(x,\zeta_2,\nu) - g(x,\zeta_1,\nu)| \leq \sigma_2(|\zeta_1 - \zeta_2|)\big(g(x,\zeta_1,\nu) + g(x,\zeta_2,\nu)\big)$$

for every ζ_1, $\zeta_2 \in \mathbb{R}^m_0$;

($g3$) (bounds) for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m_0$, and $\nu \in \mathbb{S}^{n-1}$

$$c_3(1 + |\zeta|) \leq g(x,\zeta,\nu) \leq c_4(1 + |\zeta|);$$

($g4$) (symmetry) for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m_0$, and $\nu \in \mathbb{S}^{n-1}$

$$g(x,\zeta,\nu) = g(x,-\zeta,-\nu).$$

The random environment is described by a group of P-preserving transformations $(\tau_z)_{z\in\mathbb{Z}^n}$ defined on the probability space $(\Omega, \mathcal{T}, P)$. In the random setting the analogue of periodicity is *periodicity in law* which can be quantified in terms of $(\tau_z)_{z\in\mathbb{Z}^n}$ by requiring that f and g are *stationary*, that is for every $z \in \mathbb{Z}^n$ and P-almost surely

$$f(\omega, x+z, \xi) = f(\tau_z\omega, x, \xi) \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^{m\times n}, \tag{3}$$

$$g(\omega, x+z, \zeta, \nu) = g(\tau_z\omega, x, \zeta, \nu) \quad \forall (x,\zeta,\nu) \in \mathbb{R}^n \times \mathbb{R}^m_0 \times S^{n-1}. \tag{4}$$

The two conditions as above ensure that the statistical properties of the medium are invariant under translations and will allow us to reconstruct, in a suitable statistical sense, the overall limit behaviour of E_ε by the knowledge of its local behaviour on a sequence of increasingly larger “fundamental cells”.

To determine the homogenised limit of E_ε we first regard ω as a fixed parameter and study the limit behaviour of the deterministic (and in general non periodic) functionals $E_\varepsilon(\omega)$. The convergence of functionals of type $E_\varepsilon(\omega)$ has been studied in [5] where, among other things, the homogenisation of free-discontinuity functionals without periodicity assumptions has been addressed. Specifically, [5, Theorem 3.8] provides us with a sufficient condition for the Γ-convergence of the family $\big(E_\varepsilon(\omega)\big)_\varepsilon$ towards a homogeneous free-discontinuity functional of the form

$$E_{\rm hom}(\omega)(u) := \int_A f_{\rm hom}\left(\omega, \nabla u\right)\, dx + \int_{S_u\cap A} g_{\rm hom}\left(\omega, [u], \nu_u\right)\, d\mathcal{H}^{n-1},$$

for suitable Borel functions $f_{\rm hom}(\omega,\cdot)$ and $g_{\rm hom}(\omega,\cdot,\cdot)$. The aforementioned sufficient condition amounts to the existence and independence of x of the two following limits

$$\lim_{r\to 0^+}\frac{1}{r^n}\inf\int_{Q_r(rx)} f(\omega, y, \nabla u(y))dy =: f_{\rm hom}(\omega,\xi), \tag{5}$$

and

$$\lim_{r\to 0^+}\frac{1}{r^{n-1}}\inf\int_{S_u\cap Q_r^\nu(rx)} g(\omega, y, [u](y), \nu_u(y))d\mathcal{H}^{n-1}(y) =: g_{\rm hom}(\omega,\zeta,\nu) \tag{6}$$

where the infimum in (5) is taken among functions in $W^{1,p}(Q_r(rx),\mathbb{R}^m)$ satisfying $u(y)=\xi y$ near $\partial Q_r(rx)$, while the infimum in (6) is taken among all functions u in $SBV(Q_r^\nu(rx),\mathbb{R}^m)$ satisfying $\nabla u = 0$ $\mathcal{L}^n$-a.e. in $Q_r^\nu(rx)$ and

$$u(y) = u_{rx,\zeta,\nu}(y) := \begin{cases}\zeta & \text{if } (y-rx)\cdot\nu \geq 0\\ 0 & \text{if } (y-rx)\cdot\nu < 0\end{cases} \qquad \text{near } \partial Q_r^\nu(rx).$$

Then in a second (stochastic) step we show that the sufficient condition as above is fulfilled *almost surely*; *i.e.,* the two limits in (5) and (6) exist for P-a.e. $\omega\in\Omega$. As for the case of stochastic homogenisation of volume functionals [6] this step heavily relies on the stationarity assumption on f and g. Specifically, the almost sure existence of the limit in (5) follows as in the classical result of Dal Maso and Modica [6] by first proving that, for every fixed $\xi\in\mathbb{R}^{m\times n}$, the map

$$(\omega, A)\mapsto \inf\left\{\int_A f(\omega, y, \nabla u(y))dy\colon u\in W^{1,p}(A,\mathbb{R}^m),\ u(y)=\xi y \text{ near } \partial A\right\}$$

defines a *subadditive stochastic process* on $\Omega\times\mathcal{I}_n$ (where $\mathcal{I}_n$ denotes the class of n-dimensional intervals) and then invoking the pointwise subadditive Ergodic Theorem of Ackcoglou and Krengel [1]. Though the proof of the existence of the limit in (6) follows a similar strategy as for the volume case, the analysis of surface random functionals is particularly delicate and requires some additional care. Indeed, two main differences between volume and surface energies are immediately apparent from (6). Namely, the latter shows a "mismatch" between the surface scaling r^{n-1} and the minimisation problem

$$\inf\left\{\int_{S_u\cap Q_r^\nu(rx)} g(\omega, y, [u], \nu_u)d\mathcal{H}^{n-1}\colon u\in SBV(Q_r^\nu(rx),\mathbb{R}^m),\right. \tag{7}$$
$$\left.\nabla u = 0, \text{ a.e. in } Q_r^\nu(rx),\ u = u_{rx,\zeta,\nu} \text{ on } \partial Q_r^\nu(rx)\right\}$$

which is defined on the n-dimensional cube $Q^\nu_r(rx)$. Moreover, the explicit dependence of the boundary datum $u_{rx,\zeta,\nu}$ on the spatial variable x results into an additional difficulty in the proof of (6). Then, similarly as in [2], we first set $x = 0$ in (7) and then provide a systematic way to associate to (7) a map defined on $\Omega \times \mathcal{I}_{n-1}$ ($\mathcal{I}_{n-1}$ being the class $(n-1)$-dimensional intervals). This map then turns out to be the sought for $(n-1)$-dimensional subadditive stochastic process, the main difficulty being here the proof of the measurability of the process. As a final step we show that the choice $x = 0$ is not "special"; *i.e.,* that the limit in (6) actually defines a homogeneous random surface-integrand $g_{\rm hom}$. This is done appealing to the Birkhoff's Ergodic Theorem in the spirit of [4], where a similar issue is solved by proving the translation invariance of a first passage percolation formula. Finally, if f and g are ergodic (*i.e.,* they satisfy (3) and (4) for $(\tau_z)_{z\in\mathbb{Z}^n}$ ergodic) the homogenisation becomes effective and the functional $E_{\rm hom}$ is deterministic.

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