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Mini-Workshop: Positional Games

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Abstract. This mini-workshop focused on Positional Games and related fields. Positional Games Theory is a branch of Combinatorics whose main aim is to systematically develop an extensive mathematical basis for a variety of two-player games of perfect information and without chance moves, usually played on discrete objects. These include popular recreational games such as Tic-Tac-Toe and Hex as well as purely abstract games played on graphs and hypergraphs. Though a close relative of the classical Game Theory of von Neumann and of Nim-like games, popularized by Conway and others, Positional Games are quite different and are more of a combinatorial nature. The subject is strongly related to several other branches of Combinatorics like Ramsey Theory, Extremal Graph and Set Theory, and the Probabilistic Method. It has also proven to be instrumental in deriving central results in Theoretical Computer Science, in particular in derandomization and algorithmization of important probabilistic tools. Despite being a relatively young topic, there are already three textbooks dedicated to Positional Games as well as one invited talk at the International Congress of Mathematicians. During this mini-workshop, several new exciting developments in the field were presented and discussed. We have also made some progress towards solving various open problems in Positional Games Theory and related areas.

Mathematics Subject Classification (2010): 91A24 (Positional games), 05C57 (Games on graphs), 91A43 (Games involving graphs), 91A46 ()Combinatorial games), 05C65 (Hypergraphs), 05C80 (Random graphs), 05D40 (Probabilistic methods), 05D10 (Ramsey theory).

Introduction by the Organisers

Positional games involve two players who alternately occupy the free elements of a given set V, which we call the *board* of the game. The focus of their attention is a given family $\mathcal{F} = \{E_1, \dots, E_k\} \subseteq 2^V$ of subsets of V, usually referred to as the *winning sets*. In the general version there are two additional parameters – positive integers p and q, where the first player claims p free board elements per turn and the second player responds by claiming q free elements (in the basic version we have $p = q = 1$, the so-called unbiased game). It remains to specify who wins the game, each such specification leading to a standard type of positional games.

There are several standard types of Positional Games. The most frequently played is probably the so-called *strong game*, where both players compete to be the first to claim a winning set. Both Tic-Tac-Toe and 5-in-a-row are of this type. Tools such as strategy stealing and Ramsey-type statements are of utmost value here. Strong games are well-known to be hard to analyze. Nevertheless, various interesting results on strong games were obtained recently. A close relative is the *Maker-Breaker game*, where the first player, called Maker, wins if he fully claims a winning set by the end of the game, while the second player, called Breaker, aims to prevent Maker from fulfilling his goal. For example, Hex can be cast into this framework. In *Avoider-Enforcer games*, Avoider *loses* if he claims a winning set, or, in other words, in order to win he has to avoid claiming a winning set to the end of the game. Much of the ground work on Avoider-Enforcer games was layed down by the organizers in three papers. Exciting progress on some of the questions that were raised in those papers was achieved recently. In recent years, various other types of positional games have attracted growing attention. For example, in *Waiter-Client* and *Client-Waiter* games, the first player, called Waiter, offers the second player, called Client, $p + q$ board elements. Client then chooses p of these elements which he claims and the remaining q elements are claimed by Waiter. Client wins (respectively, loses) the Client-Waiter (respectively, Waiter-Client) game if he fully claims a winning set by the end of the game.

Typical general results in Positional Games include winning criteria for one of the players, in some cases also supplying an efficient winning strategy for that player. The proofs utilize an array of various combinatorial arguments (Ramsey Theory, Extremal Graph and Set Theory, etc.); sometimes – perhaps somewhat surprisingly – probabilistic strategies are used to analyze completely deterministic games of perfect information. This connection was first indicated by Paul Erdős. Subsequently, it was discussed in detail and masterfully implemented by József Beck. Recent developments in the field have affirmed the crucial role of probabilistic arguments in positional games.

The mini-workshop on Positional games was attended by 17 people, arriving from various geographic regions (namely, England, Germany, Israel, the Netherlands, Poland, Serbia, Switzerland, and the United States). The participants had different backgrounds in Positional Games Theory (though all of them had a considerable level of familiarity with the field) and different levels of research experience ranging from M. Sc. students to Full Professors.

In the first few days, most participants gave research talks (13 in total) which presented many interesting new developments in Positional Games and related fields (in particular, in Ramsey Theory and Random Graphs). The talks also included many open problems which indicated new research directions to be explored.

In the evening of the first day of the mini-workshop we held an open problems session in which participants offered many "good" open problems (some were good problems in the sense that solving them is likely to have an impact on the field, others were good in the sense that it seemed plausible one could make some progress towards solving them in a week, some were perhaps good in both ways). After this session, the participants were divided into four groups. Starting on the second day, these groups have engaged in focused open problem solving activities. This continued until the end of the week and will hopefully continue (in one way or another) for a much longer period. We have tried to form the groups in a way which will foster new and lasting collaborations between researchers with different levels of experience. All groups have reported some progress during the week and we expect several publications to result from this mini-workshop. All in all, we believe the mini-workshop was a great success.

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Workshop: Mini-Workshop: Positional Games

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Abstracts

When playing randomly is optimal

Tibor Szabó
(joint work with Christopher Kusch, Juanjo Rué, Christoph Spiegel)

The concept of biased Maker-Breaker games, introduced by Chvtal and Erdős, is a central topic in the field of positional games, with deep connections to the theory of random structures. For any given hypergraph $\mathcal{H}$ the main questions is to determine the smallest bias $q(\mathcal{H})$ that allows Breaker to force that Maker ends up with an independent set of $\mathcal{H}$. Here we prove matching general winning criteria for Maker and Breaker when the game hypergraph satisfies a couple of natural container-type regularity conditions about the degree of subsets of its vertices. This will enable us to derive a hypergraph generalization of the $\mathcal{H}$-building games, studied for graphs by Bednarska and Łuczak. Furthermore, we investigate the biased version of generalizations of the van der Waerden games introduced by Beck. We refer to these generalizations as Rado games and determine their threshold bias up to constant factors by applying our general criteria. We find it quite remarkable that a purely game theoretic deterministic approach provides the right order of magnitude for such a wide variety of hypergraphs, when the generalizations to hypergraphs in the analogous setup of sparse random discrete structures are usually quite challenging.

Lengths of strong Ramsey games

Alexey Pokrovskiy
(joint work with Dan Hefetz, Christopher Kusch, Lothar Narins, Clément Requilé, Amir Sarid)

For an arbitrary graph G and $n \in \mathbb{N}$, the strong Ramsey game $SR(G, n)$ on n vertices with target graph G is defined as follows:

- The board consists of the edges of K_n.
- The winning sets consist of copies of the graph G
- The players alternate choosing edges. The first player to occupy a winning set wins.

Strategy stealing implies that the second player can never win $SR(G, n)$ for any G and n. Ramsey's theorem implies that for sufficiently large finite n, there are no final drawing positions in the strong Ramsey game. Thus the first player always has a wining strategy—however we have no idea what this strategy is for most target graphs G.

Define the length of the strong Ramsey game, denoted $L(G, n)$, to be the number of moves that $SR(G, n)$ lasts under optimal play by both players. Formally, when studying $L(G, n)$ player 1 plays a winning strategy for $SR(G, n)$ which guarantees him a win in as few moves as possible, while player 2 plays a losing strategy which keeps the game going for as long as possible.

The following is the central conjecture in strong Ramsey games

Conjecture 1 (Beck)**.** *For every t, there is a constant c_t such that $L(K_t, n) \leq c_t$.*

This conjecture is known to hold for $t \leq 4$. For $t \geq 5$, this conjecture is extremely difficult since the only known method for proving the finiteness $L(K_t, n)$ is to find an explicit wining strategy for the first player. Known explicit strategies in strong games tend to be complicated with no hope of being generalised to higher t. Even the $t = 5$ case of the above conjecture is a separate problem of Beck.

More generally, one can ask whether $L(G, n) \leq c_t$ for every graph G. The main result I presented at Oberwolfach was a construction of hypergraphs for which this is not the case.

Theorem 2 (Hefetz, Kusch, Narins, Pokrovskiy, Requilé, Sarid, [1])**.** *There exists a 5-uniform hypergraph $\mathcal{H}$ such that $L(\mathcal{H}, n) \geq \Omega(n)$.*

I also proposed the following problem for the workshop.

Problem 3. *Show that $L(K_5, n) \leq o(n^2)$.*

At first glance the above problem looks as difficult as Beck's Conjecture since it's not clear how one can find bounds on $L(K_5, n)$ without giving explicit strategies for the players. However occasionally techniques from extremal combinatorics can be used to give bounds on $L(G, n)$ without working out explicit strategies:

Proposition 4. *For any bipartite graph G we have $L(G, n) \leq o(n^2)$.*

Proof. By the Kovari-Sos-Turán Theorem $ex(G, n) \leq o(n^2)$. After $ex(G, n)$ moves, both players must have a copy of G in their graph and so the game must have ended before this point. □

Of course K_5 is non-bipartite, so $ex(K_5, n)$ only gives a quadratic bound on $L(K_5, n)$. However sometimes by combining extremal and game-theoretic arguments it is possible to give a subquadratic upper bound on $L(G, n)$ even for non-bipartite G.

Proposition 5. *Let G be a nonbipartite graph with an edge e such that $G - e$ is bipartite. Then $L(G, n) \leq o(n^2)$.*

Proof. By the Kovari-Sos-Turán Theorem, after $o(n^2)$ moves, player 1 will have a copy of $K_{|G|,|G|}$. Let X and Y be the parts of $K_{|G|,|G|}$. If player 2 controls all the edges in X and Y then he has a copy of G (and so the game would have ended by now). Otherwise, *since player 1 is playing optimally*, on his next move player 1 should claim a non-edge in X or Y in order to win the game. □

Since K_5 is reasonably close to a complete biparite graph, I think there's a chance that $L(K_5, n) \leq o(n^2)$ could be proved by a more complicated version of the arguments in the above propositions. Natural intermediate steps towards this would be to first prove that $L(G, n) \leq o(n^2)$ for $G = K_{m,m} + e + f$ and then $G = K_{m,m} + e + f + h$.

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Strong Ramsey Games in Unbounded Time

Marius Tiba

(joint work with Stefan David, Ivailo Hartarsky)

For two graphs B and H the strong Ramsey game $\mathcal{R}(B, H)$ on the board B and with target H is played as follows. Two players alternately claim edges of B. The first player to build a copy of H wins. If none of the players win, the game is declared a draw. A notorious open question of Beck [1, 2] asks whether the first player has a winning strategy in $\mathcal{R}(K_n, K_k)$ in bounded time as $n \to \infty$. Surprisingly, in a recent paper [3] Hefetz, Kusch, Narins, Pokrovskiy, Requilé and Sarid constructed a 5-uniform hypergraph $\mathcal{H}$, for which they prove that the first player does not have a winning strategy in $\mathcal{R}(K_n^{(5)}, \mathcal{H})$ in bounded time. They naturally ask whether the same result holds for graphs. In this paper we make further progress in decreasing the rank.

In our first main result, we construct a graph G (in fact $G = K_6 \setminus K_4$) and prove that the first player does not have a winning strategy in $\mathcal{R}(K_n \sqcup K_n, G)$ in bounded time. As an application of this result we deduce our second main result in which we construct the 4-uniform hypergraph G' (in fact G' is obtained from G by adding two new vertices and extending each edge to include them) and prove that the first player does not have a winning strategy in $\mathcal{R}(K_n^{(4)}, G')$ in bounded time. This improves the result in the paper above.

By compactness, an equivalent formulation of our first result is that the game $\mathcal{R}(K_\omega \sqcup K_\omega, G)$ is a draw. Another reason for interest in the board $K_\omega \sqcup K_\omega$ is a folklore result that the disjoint union of two finite positional games both of which are first player wins is also a first player win. An amusing corollary of our first result is that at least one of the following two natural statements is false: (1) for every graph H, $\mathcal{R}(K_\omega, H)$ is a first player win; (2) for every graph H if $\mathcal{R}(K_\omega, H)$ is a first player win, then $\mathcal{R}(K_\omega \sqcup K_\omega, H)$ is also a first player win. Surprisingly we cannot decide between the two.

Inspired by the idea in [3], we view the graph $G = K_6 \setminus K_4$ composed of a core $C = K_5 \setminus K_3$ and a pair of connected edges P, and we construct a drawing strategy for the second player in $\mathcal{R}(K_\omega \sqcup K_\omega, G)$ as follows. The strategy is divided into 3 stages. In the first stage, the second player builds the core C in K_ω^2 assuming that the first player played her first edge in K_ω^1. Though the second player cannot promise to construct the core directly and can be delayed by the first player, the second player can promise that in K_ω^2 he is always ahead of the first player. At the end of this stage, the first player could have a threat in K_ω^1. In the second stage, the second player responds to a possibly infinite sequence of consecutive threats in K_ω^1. However, the first player cannot force a win in K_ω^1 by repeating threats and eventually she must play somewhere else. In the third stage, the second player makes a infinite sequence of threats in K_ω^2 using the core C and playing one of the edges of a free pair P that, together with the core C, forms G. Moreover the second player makes sure that the responses of the first player to his threats cannot be used by the first player in a threat of her own.

By increasing the rank, we show that $\mathcal{R}(K_\omega^{(4)}, G')$ is a draw as follows. The idea is to cover the board $K_\omega^{(4)}$ by a family of almost disjoint copies of the board $K_\omega^{(2)}$ and to use a slightly simplified version of the strategy above that makes use of the extra freedom provided by the higher rank. To do this note that for a fixed pair of vertices $\{A, B\}$ we can identify the set of hyperedges of the board $K_\omega^{(4)}$ that contain $\{A, B\}$ with a copy of the board K_ω, and hence we can obtain a cover $K_\omega^{(4)} := \cup_{A,B} K_\omega^{A,B}$. Moreover, note that any copy of G' in the board $K_\omega^{(4)}$ is contained in some board $K_\omega^{A,B}$, and for any two disjoint pairs $\{A, B\}$ and $\{C, D\}$ the boards $K_\omega^{A,B}$ and $K_\omega^{C,D}$ intersect only in one hyperedge $\{A, B, C, D\}$.

One natural question that arises after this analysis is whether there exist a graph H such that $\mathcal{R}(K_\omega, H)$ is a first player win, but $\mathcal{R}(K_\omega \sqcup K_\omega, H)$ is a draw.

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Positional games on the vertex set of random graphs

Gal Kronenberg

(joint work with Adva Mond and Alon Naor)

In a (graph) **Maker-Breaker game** there are two players, claiming edges (or vertices) of some graph. In the $(1 : b)$ Maker-Breaker game, in each round Maker claims one element from the board, while Breaker claims b elements. For the random graph version of the H-game, the goal is to determine who is the typical winner in a game in which Maker is trying to build a predetermined graph H from his elements, and Breaker is trying to prevent Maker from achieving this goal. Stojaković and Szabó [3] were the first to study games played on the **edge** set of a random graph, and since then the subject has become quite popular. In particular, the study of Maker-Breaker H-games in this setting was continued by Müller and Stojaković [1], who found the threshold probability for the unbiased K_k-game where $k \geq 4$, by giving a lower bound on the threshold probability matching the upper bound given in [3]. For the K_3-game they provided a *hitting time* result, thus achieving a better understanding of this game, whose threshold probability was already determined in [3]. In [2], Nenadov, Steger, and Stojaković solved the unbiased Maker-Breaker H-game for almost all H, showing that the threshold probability is determined by the maximum 2-density of the graph H. In a joint work with Mond and Naor, we study Maker-Breaker games, as well as other positional games, where the board of the game is the **vertex** set of $G(n,p)$ (as opposed to the previously prevalent setting of games played on the edges of $G(n,p)$) and the winning sets are all spanning subgraphs containing a fixed graph H. It is important to note that for a fixed graph H, the *vertex H-game* is bias monotone: if Maker wins the game with bias $(1:b)$, he also wins this game with bias $(1:b')$, for every $b' \leq b$. In other words, claiming more vertices cannot harm Maker. Furthermore, given a graph H and an integer $b \geq 1$, "being Maker's win in the $(1:b)$ vertex H-game" is also a monotone increasing graph property. Thus we can study the threshold function for this game, namely the function $p^* = p^*(n,b,H)$ that satisfies

$$\lim_{n\to\infty} \Pr\left[G \sim G(n,p) \text{ is Maker's win in the } (1:b) \text{ vertex } H\text{-game}\right]$$
$$= \begin{cases} 1 p = \omega(p^*), \\ 0 p = o(p^*) \end{cases}$$

We give an exact solution (that is, find the threshold probability) for the case that the target graph is a clique or a cycle, and show the analogy between the vertex version and the edge version of the game. In particular, we show that, similarly to the edge version of the game, there is a strong connection between the threshold probability for these games and the one for the vertex-Ramsey property (that is, the property that every r-vertex-coloring of $G(n,p)$ spans a monochromatic copy of H). We also show that the cases where H is a triangle or a forest sometimes have a different behavior. Among other things, we prove the following. Let $d_1(H) = \frac{|E(H)|}{|V(H)|-1}$ be the 1-*density* of H and let $m_1(H) = \max\{d_1(H') \mid H' \subseteq H,\ |V(H')| \geq 2\}$ be its *maximum 1-density*.

Theorem 1. *Let k, b be constant integers such that $k \geq 4,\ b \geq 1$ or $k = 3,\ b \geq 2$. Let H be a graph for which there exists $H' \subseteq H$ such that $d_1(H') = m_1(H)$ and $H' = K_k$ or $H' = C_k$. Consider the $(1:b)$ Maker-Breaker H-game played on $V(G)$ where $G \sim G(n,p)$. Then there are constants $c := c(b,H)$ and $C := C(b,H)$ such that*

$$\lim_{n\to\infty} \Pr\left[G \textit{ is Maker's win in the } (1:b)\ H\textit{-game on } V(G)\right]$$
$$= \begin{cases} 1 p \geq Cn^{-1/m_1(H)}, \\ 0 p \leq cn^{-1/m_1(H)}. \end{cases}$$

Note that the case $H' = K_3$ and $b = 1$ was excluded from Theorem 1.It turns out, that in this case Maker wins at the game in the exact same moment that the special graph appears.

Let $\mathcal{G}_{DD}$ and $\mathcal{G}_{2DD}$ be the graph properties of containing one or two vertex disjoint copies of the graph DD as a subgraph, respectively. Note that when fixing a graph H and the bias $b \geq 1$, the graph property of "being Maker's win in the $(1 : b)$ H-game" is a monotone increasing graph property. We show the following.

Theorem 2. *Let* $\mathcal{M}^F_{K_3}$ *and* $\mathcal{M}^S_{K_3}$ *be the graph properties of being Maker's win in the* $(1 : 1)$ *triangle game played on the vertex set of the graph, where Maker moves first or second, respectively. For a random graph process* $\tilde{G}$*, w.h.p.* $\tau(\tilde{G}, \mathcal{M}^F_{K_3}) = \tau(\tilde{G}, \mathcal{G}_{DD})$ *and* $\tau(\tilde{G}, \mathcal{M}^S_{K_3}) = \tau(\tilde{G}, \mathcal{G}_{2DD})$*.*

We also study other classical positional games under this setting such as Avoider-Enforcer games, Waiter-Client and Client-Waiter games.

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The threshold bias of the clique-factor game

Rajko Nenadov

Two milestones in the extremal and probabilistic combinatorics are a result by Hajnal and Szemerédi, which determines the best possible minimum degree condition which enforces a K_r-factor, and a result by Johansson, Kahn, and Vu which determines the threshold for the appearance of a K_r-factor in the random graph $G(n, p)$. Our goal is to establish a similar result in the realm of Maker-Breaker games.

Let $r > 3$ be an integer and consider the following game on the complete graph K_n such that $r|n$: Two players, Maker and Breaker, alternately claim previously unclaimed edges of K_n such that in each turn Maker claims one and Breaker claims b edges. Maker wins if her graph contains a K_r-factor, that is a collection of n/r vertex-disjoint copies of K_r, and Breaker wins otherwise. In other words, we consider a b-biased K_r-factor Maker-Breaker game. We show that the threshold bias for this game is of order $n^{2/(r+2)}$. This makes a step towards determining the threshold bias for making bounded-degree spanning graphs and extends a result of Allen et al. who resolved the case $r = 3$ and $r = 4$ up to the logarithmic factor.

The random strategy in Maker-Breaker graph minor games

Ander Lamaison

We consider a biased Maker-Breaker game played on the edge set of K_n. The bias is $(1 : b)$, that is, in every round Maker claims one edge and Breaker claims b edges. We say that a function $f(n)$ is a threshold bias, and denote it by $b^*_{CC} \sim f$ if Maker wins for $b = o(f(n))$ and Breaker wins for $b = \omega(f(n))$. Similarly $f(n)$ is a sharp threshold, and denote it by $b^*_{CC} \approx f$, if Maker wins for $b = (1 + o(1))f(n)$ and Breaker wins for $b = (1 - o(1))f(n)$. This assumes that both players use their best strategy possible.

One could ask what would happen if one or both players are limited in the kind of strategies that they can use. A particularly interesting case takes place when one or both players is forced to play uniformly at random on every move. We are interested in the case where Maker plays randomly and Breaker follows the best counterstrategy available. We call the players Clever- or Random- depending on the strategy used. As in the previous case, we denote by $b^*_{RC} \sim f$ or $b^*_{RC} \approx f$ whenever the winner of the game is determined with high probability whenever the bias differs from f by a factor of at least $\omega(1)$ or $1 + o(1)$, respectively.

We are interested in finding games for which the two biases defined previously are close to each other, with $b^*_{CC} \sim b^*_{RC}$ or even $b^*_{CC} \approx b^*_{RC}$. This implies that the random strategy is in some sense close to optimal. This phenomenon was proved to hold for the H-subgraph game by Bednarska and Łuczak, in the weaker sense of $b^*_{CC} \sim b^*_{RC}$, and has since been observed to hold in other settings.

Here we turn our attention to the H-minor game: Maker wins by claiming the edges of a graph which contains H as a minor. For every choice of H, we determine the sharp threshold b^*_{CC}, and show that it is matched in some cases by b^*_{CC}. Here $\tau(G)$ denotes the maximum number of edges in a component of G:

Theorem 1. *In the H-minor game:*

- $b^*_{CC} \approx \frac{n^2}{2e(H)-2} \approx b^*_{RC}$ *if* $\tau(H) = 1$,
- $b^*_{CC} \approx 2n \approx b^*_{RC}$ *if* $\tau(H) = 2$,
- $b^*_{CC} \approx n \sim b^*_{RC}$, *if H is a forest with* $\tau(H) \geq 3$,
- $b^*_{CC} \approx \frac{n}{2} \approx b^*_{RC}$, *if H is not a forest.*

The natural question would be whether $b^*_{RC} \approx n$ if H is a forest with $\tau(G) \geq 3$. This is not always the case, with the path on eleven vertices being an explicit counterexample:

Theorem 2. *Let H be the path on eleven vertices. In the H-minor game with $b = 0.99n$, CleverBreaker wins against RandomMaker with high probability.*

We also study the related notion of topological minors. In the H-subdivision game, Maker wins by claiming the edges of a subdivision of H. This game is equivalent to the H-minor game for $\Delta(H) \leq 2$, so we focus on the case $\Delta(G) \geq 3$. We observe the optimality of the random strategy in the weak sense:

Theorem 3. *Let H be a graph with $\Delta(H) \geq 3$. CleverBreaker wins against CleverMaker for bias $b = \frac{n}{\Delta(H)-1}$, while RandomMaker wins with high probability against CleverBreaker for bias $b = \frac{(1+o(1))n}{2(\Delta(H)-1)}$. In particular, $b^*_{CC} \sim b^*_{RC}$.*

Making Hamiltonian cycles on small graphs

Miloš Stojaković
(joint work with Nikola Trkulja)

A positional game is a hypergraph $(X, \mathcal{F})$, where X is a finite set representing the board of the game, and $\mathcal{F} \subseteq 2^X$ is a family of sets that we call *winning sets*. In Maker-Breaker positional games, the players are called Maker and Breaker. In the course of the game, Maker and Breaker alternately claim unclaimed elements of the board X, one element at a time, until all of the elements are claimed. Maker wins the game if he claims all elements of a winning set, while Breaker wins if he claims an element in every winning set. Each game can be observed in two variants, depending on which player is to play first. One of the main questions in the theory of positional games is the existence of a *winning strategy* for one of the players, when both are playing optimally. The player that has a strategy to win the game is referred to as the winner of the game.

The positional games we intend to study are played on graphs, and in particular, on the edge set of a complete graph K_n. Our prime interest lies with Hamiltonicity game $\mathcal{HAM}_n$, where the winning sets are the edge sets of all Hamiltonian cycles in K_n. The game was first introduced by Chvátal and Erdős in [1], and since then it has been one of the most studied positional games on graphs, see [3] for details. It was shown in [1] that Maker has a winning strategy for all sufficiently large n. Papaioannou [5] later proved that Maker wins the game for all $n \geq 600$, and at the same time conjectured that the smallest n for which Maker can win is 8. Hefetz and Stich [4] further improved the upper bound by showing that Maker wins for all $n \geq 29$. We note that these statements hold under the assumption that Maker is the first to play.

We determine the outcome of Hamiltonicity game for every value of n, and for each of the players starting the game. In particular, this resolves the mentioned long-standing conjecture of Papaioannou in the affirmative. As the trivial cases $n \leq 3$ can be handled directly, from now on we assume $n \geq 4$.

Theorem 1. *In the Maker-Breaker $\mathcal{HAM}_n$ game on $E(K_n)$, Maker, as first or second player, wins if and only if $n \geq 8$.*

Next, we look at two games where Maker's goal is to claim a Hamiltonian path. In Hamiltonian Path game $\mathcal{HP}_n$, first introduced in [5], the winning sets are the edge sets of all Hamiltonian paths in K_n. We are able to show the following.

Theorem 2. *In the Maker-Breaker Hamiltonian Path game $\mathcal{HP}_n$ on $E(K_n)$, Maker, as first or second player, wins if and only if $n \geq 5$.*

This theorem is a strengthening of Papaioannou's result from [5] where he proved that Maker, as first player, can win $\mathcal{HP}_n$ if and only if $n \geq 5$.

In Fixed Hamiltonian Path game $\mathcal{FHP}_n$, the goal of Maker is to claim a Hamiltonian path between two fixed (predetermined) vertices, $u, v \in V(K_n)$. Even though in the literature this game does not draw as much interest as $\mathcal{HAM}_n$ and $\mathcal{HP}_n$, it has appeared as an auxiliary game when studying some other games on graphs.

Theorem 3. *In the Maker-Breaker Fixed Hamiltonian Path game $\mathcal{FHP}_n$ on $E(K_n)$, Maker, as first player, wins if and only if $n \geq 7$, and as second player he wins if and only if $n \geq 8$.*

Note that in the game $\mathcal{FHP}_n$ the edge between the fixed vertices u and v actually does not participate in any winning set, so right away we obtain the same result for the game played on $E(K_n) \setminus \{(u, v)\}$.

Let us note that even though answering the question of who wins a game on a small board (with n fixed) is a finite problem, it still may have a greater scientific impact for several reasons. First of all, the approaches we use to resolve standard positional games when n is large often do not apply for small n, and in that case in order to determine the outcome we need to develop new methods. As we saw, resolving the "small cases" is straightforward for some games, but not for all.

Also, standard positional games, like Hamiltonicity, Hamiltonian Path, Connectivity, etc. are often used as auxiliary games when studying other positional games on graphs. Sometimes these auxiliary games have boards of fixed size, and knowing their outcome is essential for completing the analysis. An example of that can be found in [2], where one of the problems tackled was to estimate the smallest number of edges $\hat{m}(n)$ a graph on n vertices can have, knowing that Maker as first player can win the Maker-Breaker Hamiltonicity game played on its edges. It was proved in [2] that $2.5n \leq \hat{m}(n) \leq 21n$, for all $n \geq 1600$. We show how to apply our results about Hamiltonicity game in Theorem 1 and Fixed Hamiltonian Path game in Theorem 3 to improve this upper bound, eventually obtaining the following.

Theorem 4. *For $n \geq 336$, we have $\hat{m}(n) \leq 4n$.*

Positional games are combinatorial games (sequential two player games with perfect information and no randomness involved), and it is well known that we can find out which of the players has a strategy to win by simply traversing the whole game tree of the game. But this fact alone is of limited practical use, knowing that already for games on relatively small boards the game trees are way too big (they are exponentially large in the size of the game board) to be completely traversed by a computer. In particular, if the board of the game is $E(K_n)$, its size is $\binom{n}{2}$, so there are $\binom{n}{2}!$ different game plays.

Often there is no need to search through the whole game tree, as some moves are "analogue" to the others, we may arrive to the same game position more than once, and on top of that some game positions are "similar" to the others. We devise a sophisticated set of algorithms that formalizes and exploits these "similarities" as part of the optimization of the brute force search algorithm. This enables us to write a computer program that efficiently calculates the outcome of all three mentioned games for enough initial values of n to inductively determine the outcome of the game for every n.

When implementing the algorithms we aimed at doing it in a generic way to make our code easily adaptable for other positional games on graphs. Even though some algorithms are tailored to fit the particularities of the games we analyzed, most of them are generally applicable for determining the outcome of positional games on graphs with the help of a computer.

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Waiter-Client Maximum Degree Game

Michael Krivelevich
(joint work with Nadav Trumer)

The Waiter-Client maximum degree with bias q is played by Waiter and Client on the edges of the complete graph K_n on n vertices. In each round of the game, Waiter offers to Client $q+1$ previously unoffered edges, Client claims one of them, and the rest goes to Waiter. If at most q edges are left unclaimed on the board, Waiter claims all of them. Client's goal is to minimize the maximum degree in the graph of his edges by the end of the game.

We determine the asymptotic value of Client's maximum degree for all values of the bias $q(n)$ outside of the critical region $q(n) = \Theta(n/\log n)$. Also, for the very important unbiased case $q = 1$, we show that, assuming the perfect play of both players, Client's maximum degree D satisfies: $D = n/2 + \Theta(\sqrt{n \log n})$. The obtained results comply very well with the probabilistic intuition, derived from observing the typical maximum degree of the random graph $G(n, m)$ with the corresponding value of of $m(n)$: $m = \frac{n^2}{2(q+1)}$.

Monadic second order logic and random graphs from minor closed classes

Tobias Müller
(joint work with Peter Heinig, Marc Noy, Anusch Taraz)

A classical result of Glebskii et al. 1969 and independently Fagin 1976 states that in the Erdos-Renyi model with edge-probability $p = 1/2$ every graph property that can be expressed as a sentence in first order logic holds with probability tending to either zero or one.

A class of graphs is minor closed, if it is closed under the operations of removing edges and of ”contracting” edges. (An example of a minor closed class of graphs is the class of all graphs that have a crossing-free drawing on some fixed surface S.) I will discuss a recent work, joint with P. Heinig, M. Noy and A.Taraz, on analogues of the classical result of Glebskii et al./Fagin for random graphs from a minor closed class (i.e. we sample a graph uniformly at random from all graphs on n vertices from some minor closed class), where we consider monadic second order logic sentences. The proofs build on the major progress that was made in recent years in the study of these random graph models.

The odd-cycle game

Adva Mond
(joint work with Jan Corsten, Alexey Pokrovskiy, Christoph Spiegel, Tibor Szabó)

In the $(1 : b)$ *Maker-Breaker game* $(X, \mathcal{F})$, the two players, called Maker and Breaker, claiming previously unclaimed elements of the board. In each round Maker claims one element and Breaker claims b. Maker wins if at some point of the game she claimed all elements of some winning set $F \in \mathcal{F}$ and Breaker wins otherwise (there are no draws). Here assume that Maker starts the game.

It is well-known that Maker-Breaker games are *bias monotone*, i.e. if Breaker wins the $(1 : b)$ game $\mathcal{F}$ then he also wins the $(1 : b')$ game $\mathcal{F}$ for every $b' \geq b$. Therefore, we define the *threshold bias* to be the maximal value of b for which Maker wins the $(1 : b)$ Maker-Breaker game $\mathcal{F}$ and denote it by $b_{mb}(\mathcal{F})$. Determining the threshold bias for various natural games is one of the central problems in the study of Maker-Breaker games.

Considering the *cycle game*, the game where Maker's goal is to occupy the edges of a cycle of any length, denoted by $\mathcal{C}_n$, Bednarska and Pikhurko proved in [1] that $b_{mb}(\mathcal{C}_n) = \lceil n/2 \rceil - 1$. Furthermore, Krivelevich [3] proved that Maker can always build a linearly-long cycle when $b \leq (1/2 - o(1))n$. In [2] Bednarska and Pikhurko discussed the *even-cycle game* and the *odd-cycle game*, denoted by $\mathcal{EC}_n$ and $\mathcal{OC}_n$ respectively. In these games Maker's goal is to occupy the edges of some cycle of an odd/even length. Since building a cycle of a certain parity is more difficult for Maker than building just any cycle, we have $b_{mb}(\mathcal{EC}_n), b_{mb}(\mathcal{OC}_n) \leq b_{mb}(\mathcal{C}_n) = \lceil n/2 \rceil - 1$. The authors of [2] proved that asymptotically the threshold bias of the even cycle game behaves the same as the one of the cycle game, i.e. $b_{mb}(\mathcal{EC}_n) = (1/2 + o(1))n$. Furthermore, they also proved that $b_{mb}(\mathcal{OC}_n) \geq (1 - 1/\sqrt{2} - o(1))n$ (approximately $0.2929n$). However, no upper bound separating the parameters $b_{mb}(\mathcal{OC}_n)$ and $b_{mb}(\mathcal{C}_n)$ is known yet. Hence, in [2] the authors asked the following question

Question 1. *Do we have* $b_{mb}(\mathcal{OC}_n) = (1/2 + o(1))n$*?*

We give the following small improvement for the lower bound.

Theorem 2. *The threshold bias for the Maker-Breaker odd-cycle game satisfies*

$$b_{mb}(\mathcal{OC}_n) \geq \left(\frac{4-\sqrt{6}}{5} + o(1)\right) n \geq 0.3101n$$

Wanting to get closer to answering 1, we note that the best known strategies of Maker are playing connected, i.e. claiming only edges incident to any vertex she has visited so far. Both Maker's strategies in the proof of 2 and in Bednarska's and Pikhurko's proof in [2] are connected. It is therefore natural to try to answer 1 under the additional assumption that Maker follows connected rules. If indeed playing connected is optimal for Maker then investigating this variant of the game will lead to an answer to 1. Hence we introduce this new variant of the odd-cycle Maker-Breaker game in which Maker is allowed to claim only edges that keep her graph connected. We show that, under these connected rules, the answer to 1 is no. Given that, in order to give a negative answer to 1 in general, one could show that Maker's optimal strategy is playing disconnected.

The $(1 : b)$ *connected Maker-Breaker game* $(E(K_n), \mathcal{F})$ is exactly as the $(1 : b)$ Maker-Breaker game $(E(K_n), \mathcal{F})$ except for one additional restriction. Maker wins the game if she claims a full winning set while keeping her graph having exactly one non-trivial connected component during the game up to this point. Alternatively, Breaker wins the game if one of the following happens: either the game ends with Maker's graph not containing any winning set, or Maker's graph has at least two non-trivial connected components at some point while her graph still does not contain any full winning set. This means that in every round Maker must claim only an edge that keeps her graph connected, or otherwise she loses. It is not hard to see that connected Maker-Breaker games are also bias-monotone. Thus, for every connected Maker-Breaker game $(E(K_n), \mathcal{F})$, there also exists a threshold bias $b^c_{mb}(\mathcal{F})$, so that Breaker wins the $(1 : b)$ game $(X, \mathcal{F})$ if and only if $b \geq b^c_{mb}(\mathcal{F})$. As connected games are a restriction for Maker, we have $b^c_{mb}(\mathcal{OC}_n) \leq b_{mb}(\mathcal{OC}_n) \leq \lceil n/2 \rceil - 1$ for every $n \in \mathbb{N}$.

We give the following improvement that shows that if $b_{mb}(\mathcal{OC}_n) = (1/2 - o(1))n$, it means that the threshold biases for the connected odd-cycle game and the unrestricted one differ, and hence every optimal strategy of Maker in the unrestricted game must not be connected during the whole game.

Theorem 3. *Breaker wins the* $(1 : b)$ *Maker-Breaker game under connected rules for every large enough* n *and every* $b \geq 0.498n$*, i.e.* $b^c_{mb}(\mathcal{OC}_n) \leq 0.498n$ *for every large enough* n*.*

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A semi-random graph game

Dan Hefetz
(joint work with Omri Ben-Eliezer, Gal Kronenberg, Olaf Parczyk, Clara Shikhelman, Miloš Stojaković)

In this talk we introduce and analyze a general semi-random multigraph process, which is motivated by games. It arises from an interplay between a sequence of random choices on the one hand, and a strategy of our choice (that may also involve randomness) on the other. The process is defined as follows. We start with an empty graph on the vertex set $[n]$. In each round, *Builder* is offered a vertex v, chosen uniformly at random with replacement from the set $[n]$, independently of all previous choices. Builder then irrevocably chooses an additional vertex u and adds the edge uv to his (multi)graph, with the possibility of creating multiple edges and loops.

The (possibly randomized) algorithm that Builder uses in order to add edges throughout this process is called the *strategy* of Builder. As a special case, we also show how the process can be used to approximate (using suitable strategies) some well-known random graph models such as the Erdős-Renyi random graph model (see [1]), the random multigraph model (see [2]), the k-out model (see [3]), and the min-degree process (see [4]).

Given a positive integer n and a monotone increasing graph property $\mathcal{P}$, we consider the one-player game in which Builder's goal is to build a multigraph with vertex set $[n]$ satisfying $\mathcal{P}$ as quickly as possible; we denote this game by $(\mathcal{P}, n)$. The general problem discussed in this paper is to determine the typical number of rounds Builder needs in order to construct such a multigraph under optimal play. We mostly focus on the *online version* of the game, where in each round Builder is presented with the next random vertex only after he chose a vertex in the previous round and added the corresponding edge to his graph, but also consider the *offline version*, in which Builder is given the entire sequence of random vertex choices before the game starts.

For both the online and offline versions of the game, we prove lower and upper bounds (which, in most cases, are fairly close) on the typical length of play (assuming Builder follows an optimal strategy) for the following graph properties: admitting a copy of any fixed graph, having minimum degree k, being k-connected, admitting a perfect matching, and admitting a Hamilton cycle.

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On the singularity probability of random symmetric matrices

Asaf Ferber

Let M_n denote a random symmetric $n \times n$ matrix whose diagonal and upper diagonal entries are independent and identically distributed Bernoulli random variables (which take values 1 and -1 with probability $1/2$ each). It is widely conjectured that M_n is singular with probability at most $(2+o(1))^{-n}$. On the other hand, the best known upper bound on the singularity probability of M_n, due to Vershynin (2011), is 2^{-n^c}, for some unspecified small constant $c > 0$. This improves on a polynomial singularity bound due to Costello, Tao, and Vu (2005), and a bound of Nguyen (2011) showing that the singularity probability decays faster than any polynomial. In this talk, improving on all previous results, we show that the probability of singularity of M_n is at most $2^{-n^{1/4}\sqrt{\log n}/1000}$ for all sufficiently large n. The proof utilizes and extends a novel combinatorial approach to discrete random matrix theory, which has been recently introduced by myself together with Jain, Luh and Samotij.

On minimal Ramsey graphs and Ramsey equivalence for multiple colours

Dennis Clemens

(joint work with Anita Liebenau, Damian Reding)

For an integer $q \geq 2$, a graph G is called q-Ramsey for a graph H if every q-colouring of the edges of G contains a monochromatic copy of H. If G is q-Ramsey for H, yet no proper subgraph of G has this property then G is called q-Ramsey-minimal for H.

Burr, Erdős, and Lovász [5] initiated the study of properties of graphs that are 2-Ramsey-minimal for K_k, where as usual K_k denotes the complete graph on k vertices. Their seminal paper raised numerous questions on minimal Ramsey graphs that were addressed by various mathematicians in subsequent years [3, 4, 7, 10, 12].

Denote with $\mathcal{M}_q(H)$ the sets of all graphs being q-Ramsey-minimal for H. We are mainly interested in the interplay between $\mathcal{M}_q(H)$ and $\mathcal{M}_r(H')$ when $q \neq r$ or $H \ncong H'$. Clearly, every graph G that is a q-Ramsey-minimal graph for some graph H is r-Ramsey for H, for all $2 \leq r \leq q$, and thus contains an r-Ramsey-minimal graph as a subgraph. Our first contribution complements this observation in the sense that every r-Ramsey-minimal graph F can be obtained this way from a q-Ramsey-minimal graph G, as long as H satisfies some connectivity conditions.

Theorem 1. *Let H be a 3-connected graph or $H \cong K_3$ and let $q > r \geq 2$ be integers. Then for every $F \in \mathcal{M}_r(H)$ there are infinitely many graphs $G \in \mathcal{M}_q(H)$ such that F is an induced subgraph of G.*

The result is proved by generalising an argument of Burr, Nešetřil and Rödl [7] and Burr, Faudree and Schelp [6], giving the following more general theorem.

Theorem 2. *Let H be a 3-connected graph or $H \cong K_3$, let $q \geq 2$ be an integer and let F be a graph which is not q-Ramsey for H. Then there are infinitely many graphs $G \in \mathcal{M}_q(H)$ such that F is an induced subgraph of G.*

For all such H considered in these two theorems, we further obtain the following consequences.

- For every $q \geq 3$, there are q-Ramsey-minimal graphs for H of arbitrarily large maximum degree, genus, and chromatic number.
- The collection $\{\mathcal{M}_q(H) : H \text{ is 3-connected or } K_3\}$ forms an antichain, where $\mathcal{M}_q(H)$ denotes the set of all graphs that are q-Ramsey-minimal for H.

We also address the question which pairs of graphs satisfy $\mathcal{M}_q(H_1) = \mathcal{M}_q(H_2)$, a question that has been introduced by Szabó, Zumstein and Zürcher [13] in the case when $q = 2$. Let H_1 and H_2 be called q-equivalent if this relation holds.

From results of Nešetřil and Rödl [11] as well as Fox, Grinshpun, Liebenau, Person and Szabó [9] it follows that every graph being 2-equivalent to K_k needs to be a disjoint union $K_k + H$ with $\omega(H) < k$, while Bloom and Liebenau [2] also proved that $K_k \sim_2 K_k + K_{k-1}$ for every $k \geq 4$. We extend these results to more than two colours and also show that $K_3 \sim_q K_3 + K_2$ for every $q \geq 3$ [8].

Apart from that, we do not have a deep understanding of which connected graphs H satisfy $K_k \sim_2 K_k + H$. In particular, the following problem seems to be challenging.

Problem 3. *(see* [8],[9]*) Determine*

- *the largest $t = t(k)$ such that $K_k \sim_2 K_k + K_{1,t}$.*
- *the largest $t = t(k)$ such that there is a tree T on t vertices satisfying $K_k \sim_2 K_k + T$.*

Indeed, looking at all the results that are known so far, it is not even clear whether there exist two non-isomorphic connected graphs being equivalent.

Question 4. *(see* [8],[9]*) For given $q \geq 2$, are there two non-isomorphic connected graphs H and H' that are q-equivalent?*

From our research [8] it follows that in case the answer is yes, then at least one of the graphs H and H' cannot be 3-connected.

Theorem 5. *Let H and H' be non-isomorphic graphs that are either 3-connected or isomorphic to K_3. Then $\mathcal{M}_q(H) \neq \mathcal{M}_q(H')$ for all $q \geq 2$.*

Moreover, we wonder how the concepts of 2-minimality and q-minimality are related to each other. While in general q-equivalence for some $q \geq 3$ does not necessarily imply 2-equivalence, we show that two graphs H and H' are q-equivalent for every even q if they are 2-equivalent [8]. However, the following question is still unsolved.

Question 6. *(see* [8]*) Is it true that any two 2-equivalent graphs H and H' are also 3-equivalent?*

Note that Axenovich, Rollin and Ueckerdt [1] could show that 2-equivalence implies q-equivalence for every q, when restricted to graphs H and H' satisfying $H \subseteq H'$. Also note that if the answer to this last question is yes, then we can show that 2-equivalence implies q-equivalence in general [8].

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