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Convex Geometry and its Applications

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Abstract. The geometry of convex domains in Euclidean space plays a central role in several branches of mathematics: functional and harmonic analysis, the theory of PDE, linear programming and, increasingly, in the study of algorithms in computer science. The purpose of this meeting was to bring together researchers from the analytic, geometric and probabilistic groups who have contributed to these developments.

Mathematics Subject Classification (2010): 52A (68Q25, 60D05).

Introduction by the Organisers

The meeting *Convex Geometry and its Applications*, organized by Franck Barthe, Martin Henk and Monika Ludwig, was held from December 9 to December 15, 2018. It was attended by 53 participants working in all areas of convex geometry. Of these 20% were female and more than one third were younger participants. There were 12 plenary lectures of one hour's duration and 18 shorter lectures. They illustrated the diversity of research activities in the field, from theoretical aspects to applications. Among the main topics, we can list the study of geometric inequalities (including Brunn-Minkowski theory, isoperimetric inequalities), classification of valuations, stochastic geometry, high dimensional convex geometry and its probabilistic approaches, including random matrices and net arguments, combinatorial geometry, algorithmic problems but also applications to tomography, quantum information theory or stereology.

Some highlights of the program were as follows. In the opening lecture, Bo′az Klartag presented striking connections between two major open questions in convex geometry: the slicing problem (a.k.a. isotropy constant problem) and the

Mahler conjecture on the volume of the dual body. In the first part of his lecture, he disproved a conjecture, related to the slicing problem, about the trace of the product of the covariances of a body and of its dual. It was expected that the Euclidean ball would maximize this quantity, and observed that the simplex gives the very same value. Klartag described the construction of a body with a much bigger trace of product. The focus of the second part was on explaining the coincidence of the values for the ball and the simplex, in terms of projective transformations and homogeneous cones. Developing these novel ideas yields the following surprising fact: if simplices have maximal isotropy constant then they minimize the Mahler volume product.

In her talk, Eva Vedel Jensen presented new results on rotational integral geometry for Minkowski tensors. These tensors were introduced by McMullen and characterized by Alesker when they have polynomial behavior with respect to translations. Here the questions come from applications in local stereology and translation are not allowed. Still explicit Crofton formulas were obtained (in joint work with Anne Marie Svane) and their application in the stereological analysis of particle processes was demonstrated. The talk showed the strong connections between research in the geometric theory of valuations and applications.

Emanuel Milman gave an impressive talk on his recent solution, joint with Joe Neeman, of the Gaussian double-bubble and multi-bubble conjectures. The classical Gaussian isoperimetric inequality states that the optimal way to decompose $\mathbb{R}^n$ into two sets of prescribed Gaussian measure, so that the (Gaussian) area of their interface is minimal, is by using two complementing half-planes. A natural generalization is to decompose $\mathbb{R}^n$ into $q \geq 3$ sets of prescribed Gaussian measure. It was conjectured that when $q \leq n+1$, the configuration whose interface has minimal (Gaussian) area is given by the Voronoi cells of q equidistant points. Milman and Neeman prove this conjecture for $q = 3$ (the "double-bubble conjecture) and also for all $3 < q \leq n+1$ (the "multi-bubble conjectures).

In her talk, Alina Stancu presented her very recent results on a new centro-affine curvature flow depending on an origin-symmetric reference body. She showed that starting with an origin-symmetric convex body, the flow converges to a convex body with the same centro-affine curvature as the reference body. She announced that this result can be used to establish geometric inequalities and ultimately the logarithmic Minkowski inequality for two orgin-symmetric convex bodies in general dimensions (and thus the logarithmic Brunn-Minkowski inequality), thereby proving conjectures of Böröczky, Lutwak, Yang, and Zhang from 2012.

Also in the shorter talks, remarkable new results were presented. Yair Shenfeld (joint work with Ramon van Handel) presented a surprising new proof of the Alexandrov-Fenchel inequality via the Bochner method in Riemannian Geometry. In her inspiring lecture, Sophie Huiberts (joint work with Daniel Dadush) presented within the smoothed analysis framework of Spielman and Teng a polynomial time simplex algorithm whose expected running time improves on former results. The improvement is based on a better bound on the expected number of edges of the projection of a polyhedron onto a two-dimensional plane. Fabian

Mussnig presented his new classification of valuations on coercive, convex functions and a new functional corresponding to polar volume on this space. In his talk, Boaz Slomka (joint work with Han Huang, Tomasz Tkocz and Beatrice-Helen Vritsiou) showed a sub-exponential improvement on the best known bound in the Hadwiger covering problem. The key ingredient is a lower bound on the volume of the largest symmetric set contained in a convex body, which improves (sub-exponentially) a previous bound by V. Milman and Pajor and which is based on thin-shell estimates. Dmitry Ryabogin presented a recent solution (with M. Angeles Alfonseca, Fedor Nazarov, and Vlad Yaskin) of a local version of the fifth Busemann-Petty problem from 1956. They give an affirmative answer to the classical problem for bodies sufficiently close to the Euclidean ball in the Banach-Mazur distance.

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Workshop: Convex Geometry and its Applications

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Abstracts

Isotropic constants and Mahler volumes

Bo'az Klartag

Below is an edited version of informal notes prepared for my lecture at Oberwolfach. Please refer to [10] for a rigorous mathematical discussion and for precise references related to these notes.

The following question is known as Bourgain's slicing problem [3, 4]: Let $K \subseteq \mathbb{R}^n$ be convex with $Vol_n(K) = 1$. Does there always exist a hyperplane $H \subseteq \mathbb{R}^n$ with

$$Vol_{n-1}(K \cap H) \geq \frac{1}{100}?$$

Perhaps with some other universal constant $c > 0$ in place of $1/100$?

This is not merely a curious riddle. In fact it shows up in the study of almost any question pertaining to volume distribution in high dimension under convexity assumptions. We know that it suffices to look at hyperplane sections through the barycenter, according to Makai and Martini [13]. We may furthermore reduce matters to the centrally-symmetric case [6].

Hensley [5] proved the following theorem: Let $K \subseteq \mathbb{R}^n$ be a convex body of volume one. Assume that $K = -K$ (it suffices to require that the barycenter of K lie at the origin). Then for any unit vector $\theta \in \mathbb{R}^n$,

$$c \leq Vol_{n-1}(K \cap \theta^{\perp}) \cdot \sqrt{\int_K \langle x, \theta \rangle^2 dx} \leq C,$$

where $c, C > 0$ are universal constants and $\theta^{\perp} = \{x \in \mathbb{R}^n \, ; \, \langle x, \theta \rangle = 0\}$ is the hyperplane orthogonal to θ.

It follows from Hensley's theorem that the slicing problem may be reformulated as a question on the relation between the *covariance matrix* and the *volume* (or entropy) of convex sets. This entropic point of view was emphasized by K. Ball. The covariance matrix $Cov(K) = (Cov_{ij})_{i,j=1,\dots,n}$ is given by

$$Cov_{ij} = \int_K x_i x_j \frac{dx}{|K|} - \int_K x_i \frac{dx}{|K|} \cdot \int_K x_j \frac{dx}{|K|},$$

where $|K| = Vol_n(K)$. In Bourgain's notation, the isotropic constant is defined as

$$L_K = \frac{\det^{\frac{1}{2n}} Cov(K)}{|K|^{1/n}}.$$

The isotropic constant is affinely-invariant. The slicing problem is equivalent to the question of whether $L_K < C$ for some universal constant $C > 0$, for any convex body K in any dimension. It is known that $L_K \geq L_{B^n} \geq c$, where B^n is the Euclidean unit ball centered at the origin in $\mathbb{R}^n$.

Are there any relations between isotropic constants and duality? The polar body to $K \subseteq \mathbb{R}^n$ is

$$K^\circ = \{x \in \mathbb{R}^n \,;\, \forall y \in K,\ \langle x, y \rangle \leq 1\}.$$

Note that

$$L_K \cdot L_{K^\circ} = [\det Cov(K) \cdot \det Cov(K^\circ)]^{\frac{1}{2n}} \cdot (|K| \cdot |K^\circ|)^{-1/n}.$$

According to the Bourgain-Milman and Santaló inequalities,

$$c \leq n(|K||K^\circ|)^{1/n} \leq C$$

whenever the barycenter of K or of K° lies at the origin. We thus learn that Bourgain's slicing problem is equivalent to the question of whether the following inequality holds:

$$\det\left(Cov(K)Cov(K^\circ)\right) \leq \left(\frac{C}{n}\right)^{2n}. \tag{1}$$

An idea which appears in the unpublished Ph.D. dissertations by Ball '86 and by Giannopoulos '93 is to consider the trace of the matrix in (1). Perhaps the trace is easier to analyze than the determinant. Given a convex body $K \subseteq \mathbb{R}^n$ with barycenter at zero we set

$$\phi(K) = Tr[Cov(K)Cov(K^\circ)].$$

According to the arithmetic/geometric means inequality,

$$L_K^2 L_{K^\circ}^2 \leq Cn\phi(K).$$

The quantity $\phi(K)$ has the following probabilistic interpretation: Let X be a random vector, distributed uniformly in K. Let Y be an independent random vector, distributed uniformly in K°. Then $\phi(K) = \mathbb{E}\langle X, Y\rangle^2$. We thus see that $0 \leq \phi(K) \leq 1$ when $K = -K$.

In the case where $K \subseteq \mathbb{R}^n$ has the symmetries of the cube (i.e., it is the unit ball of a "1-symmetric norm"), we know quite a lot about the distribution of the random variable $\langle X, Y\rangle$ in high dimensions. In this case, the random variable $\langle X, Y\rangle$ is approximately a Gaussian random variable of mean zero and variance bounded by C/n. This follows from the results of [9].

The central limit theorem for convex sets [7, 8] states that for any convex body $K \subseteq \mathbb{R}^n$, there exists $0 \neq \theta \in \mathbb{R}^n$ such that $\langle X, \theta\rangle$ is approximately a standard Gaussian, in the sense that the total variation distance to the Gaussian distribution does not exceed C/n^α where $C, \alpha > 0$ are universal constants. With X and Y as above, one may wonder whether $\langle X, Y\rangle$ is approximately Gaussian in high dimensions. This would imply that $\phi(K)$ is much smaller than one.

An amusing fact is that $\phi(K)$ attains the same value $n/(n+2)^2$ when K is either a Euclidean ball B^n or a simplex Δ^n, see [1]. Here Δ^n stands for any n-dimensional simplex whose barycenter lies at the origin. It was conjectured by

Kuperberg [11], following earlier unpublished work by Ball and by Giannopoulos, that for any centrally-symmetric, convex body $K \subseteq \mathbb{R}^n$,

$$\phi(K) \leq \frac{C}{n}$$

for a universal constant $C > 0$. In fact, it was conjectured more precisely that

$$\phi(K) \leq n/(n+2)^2.$$

Supporting evidence for this conjecture includes the fact, proven by Kuperberg, that the Euclidean ball is a local maximizer of $\phi(K)$ among C^2-smooth perturbations, and also the result by Alonso–Gutiérrez [1] which verifies the conjecture in the particular case where $K = B_p^n = \{x \in \mathbb{R}^n\, ;\, \sum_i |x_i|^p \leq 1\}$ for some $p \geq 1$.

Balls and simplices are extremals for a few well-known functionals in convexity. Nevertheless, we find that there exists a counter-example to Kuperberg's conjecture. Namely, we exhibit a centrally-symmetric convex set $K \subseteq \mathbb{R}^n$ with

$$\phi(K) \geq c$$

where $c > 0$ is a universal constant. In fact, our convex set is unconditional, i.e., for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$(x_1, \ldots, x_n) \in K \qquad \Longleftrightarrow \qquad (|x_1|, \ldots, |x_n|) \in K.$$

Thus there are convex bodies in high dimension for which the random variable $\langle X, Y \rangle$ is far from Gaussian. Our counter-example is essentially a one-dimensional perturbation of the cross-polytope. Its construction exploits the instability of volume under duality in high dimensions. Specifically, we use the fact that for $K_1 = B_1^n \cap \sqrt{3/n} B_2^n$ and $K_2 = B_1^n$, we have $K_1 \subseteq K_2 \subseteq \mathbb{R}^n$ and

(1) $|K_1| \geq \frac{1}{3} \cdot |K_2|$

(2) $|K_1^\circ \cap (1+c)K_2^\circ| \geq \frac{1}{6} |(1+c)K_2^\circ|$ for a universal constant $c > 0$.

Let us now explain the "coincidence" mentioned earlier, that $\phi(K)$ attains the same value $n/(n+2)^2$ when K is a Euclidean ball and when K is a simplex. The reason behind this phenomenon is that both the Euclidean ball and the simplex are hyperplane sections of homogeneous cones.

An open, convex cone $V \subseteq \mathbb{R}^{n+1}$ with apex at 0 is *homogeneous* if for any two points $x, y \in V$ there exists a linear map $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $A(V) = V$ and $Ax = y$. Examples for homogeneous cones include the positive orthant $\mathbb{R}_+^n$, the Lorentz cone and the cone of positive-definite symmetric $n \times n$ matrices.

We shall consider certain canonical constructions in convex cones. Such constructions necessarily respect the symmetries of the cone, when such symmetries exist. Roughly speaking, the quantity $\phi(K) - n/(n+2)^2$ is something like the "Laplacian" of a function $s : V \to \mathbb{R}$ which has the symmetries of V. The function s is constant when the cone is homogeneous, and hence $\phi(K) = n/(n+2)^2$ for all of the hyperplane sections of a homogeneous cone. These canonical constructions are described in detail in [10], and they involve the *Mahler volume* of a convex body $K \subseteq \mathbb{R}^n$, defined as

$$\bar{s}(K) = |K| \cdot \inf_{x \in K} |(K-x)^\circ|.$$

The Mahler conjecture [12] from 1939 suggests that $\bar{s}(K) \geq (n+1)^{n+1}/(n!)^2$ for any convex body $K \subseteq \mathbb{R}^n$, with equality for the simplex Δ^n.

We prove that any local minimizer $K \subset \mathbb{R}^n$ of the functional $K \mapsto \bar{s}(K)$ satisfies

$$L_K \cdot L_{K^\circ} \cdot \bar{s}(K)^{1/n} \geq \frac{1}{n+2}. \tag{2}$$

There is equality in (2) in the case where K is a ball or a simplex. It follows that any global minimizer K of the Mahler volume satisfies $L_K \geq L_{\Delta^n}$ or $L_{K^\circ} \geq L_{\Delta^n}$.

The strong slicing conjecture suggests that $L_K \leq L_{\Delta^n}$ for any convex body $K \subseteq \mathbb{R}^n$. We conclude that the strong slicing conjecture implies Mahler's conjecture. We remark that it was shown by Rademacher [14] that the simplex is the only local maximizer of the isotropic constant L_K in the class of simplicial polytopes.

It is known that the isotropic constant may become bounded after a small perturbation. Our last theorem in this lecture states that the perturbation can be always made projective. That is, for any convex body $K \subseteq \mathbb{R}^n$ with barycenter at the origin and $0 < \varepsilon < 1$, there exists a convex set $T \subseteq \mathbb{R}^n$ with three properties:

(1) $(1-\varepsilon)K \subseteq T \subseteq (1+\varepsilon)K$.
(2) $T^\circ = K^\circ - y$ for some point y in the interior of K°.
(3) $L_T \leq C/\sqrt{\varepsilon}$ where $C > 0$ is a universal constant.

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The Alexandrov-Fenchel inequality via the Bochner method

Yair Shenfeld
(joint work with Ramon van Handel)

One of the deepest theorems in the theory of Convex Bodies is the Alexandrov-Fenchel Inequality [1] which states that the coefficients of the volume polynomial satisfy hyperbolic inequalities. If $K_1, \dots, K_m \subset \mathbb{R}^n$ are convex bodies (m being a positive integer), then it is a result of Minkowski [5] that the function

$$(t_1, \dots, t_m) \mapsto \mathrm{Vol}(t_1K_1 + \cdots + t_mK_m)$$

is a homogeneous polynomial of degree n. The coefficients of this polynomial $V(K_{i_1}, \dots, K_{i_n})$ are called *mixed volumes* and they carry important geometric information about the bodies $K_1, \dots, K_m$ and the relations between them. The Alexandrov-Fenchel inequality reads

$$V(K_1, K_2, K_3, \dots, K_n)^2 \geq V(K_1, K_1, K_3, \dots, K_n)V(K_2, K_2, K_3, \dots, K_n) \quad (1)$$

for any convex bodies $K_1, \dots, K_n \subset \mathbb{R}^n$. In this talk we provide a new proof [6] of (1) which is considerably simpler than all other known proofs of the inequality, and in addition sheds a new light on related inequalities. Our method is spectral in nature (an approach which goes back to Hilbert [4]) and it starts with an integral representation formula for mixed volumes of smooth convex bodies [2], p. 64:

$$V(K_1, \dots, K_n) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_1 D(D^2h_2, \dots, D^2h_n) d\mathcal{H}^{n-1}. \quad (2)$$

Here, $h_i : \mathbb{R}^n \to \mathbb{R}$ is the support function of K_i:

$$h_i(u) = \sup_{x \in K_i} \langle u, x\rangle$$

and D^2h_i is the restriction of the Hessian of h_i to the tangent spaces of the unit sphere $\mathbb{S}^{n-1}$. The term $D(D^2h_2, \dots, D^2h_n)$ is called *mixed discriminant* and these quantities arise as the coefficients of the homogeneous polynomial

$$(t_1, \dots, t_m) \mapsto \det(t_1M_1 + \cdots + t_mM_m)$$

for $(n-1) \times (n-1)$ matrices $M_1, \dots, M_m$ (e.g. D^2h_i). It is a result due to Alexandrov [1] (and also a consequence of our method) that under some assumptions on the matrices, mixed discriminants satisfy hyperbolic inequalities analogous to the Alexandrov-Fenchel inequalities. Namely,

$$(3)\quad D(M_1, M_2, M_3, \ldots, M_{n-1})^2 \geq D(M_1, M_1, M_3, \ldots, M_{n-1}) D(M_2, M_2, M_3, \ldots, M_{n-1}).$$

The way in which we use (3) to prove (1) is inspired by the Bochner method in Riemannian Geometry [3]. Specifically, by using the representation (2) we define an appropriate differential operator $\mathcal{L}$ on $C^2(\mathbb{S}^{n-1})$ and a measure μ so we can write (1) as

$$(4)\qquad \left(\int_{\mathbb{S}^{n-1}} h_1(\mathcal{L}h_2)d\mu\right)^2 \geq \int_{\mathbb{S}^{n-1}} h_1(\mathcal{L}h_1)d\mu \int_{\mathbb{S}^{n-1}} h_2(\mathcal{L}h_2)d\mu.$$

The normalization of $\mathcal{L}$ is such that its maximal eigenvalue is 1 with eigenfunction h_3, and standard elliptic regularity theory implies that this eigenvalue is simple. Now, an equivalent way of stating (4), is saying that $\mathcal{L}$ has at most one eigenfunction with positive eigenvalue. (The proof of this equivalence is similar to the proof of the Cauchy-Schwarz inequality.) Hence the hyperbolic inequality (4) would follow if we show that the rest of the eigenvalues of $\mathcal{L}$ are non-positive. The Bochner method deduces this result from the inequality

$$(5)\qquad \int_{\mathbb{S}^{n-1}} (\mathcal{L}f)^2 d\mu \geq \int_{\mathbb{S}^{n-1}} f(\mathcal{L}f)d\mu,$$

for any nice enough function f on $\mathbb{S}^{n-1}$, by plugging in eigenfunctions. To obtain the inequality (5) one uses *pointwise* information on $(\mathcal{L}f)^2$. Our key observation is that this method is suitable for our purposes since the inequality (3) is exactly a lower bound on $(\mathcal{L}f)^2$ which upon integration with respect to μ gives (5). This establishes (1) for smooth convex bodies and a standard approximation argument then completes the proof the Alexandrov-Fenchel inequality.

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A friendly smoothed analysis of the simplex method

Sophie Huiberts
(joint work with Daniel Dadush)

One key application of convex geometry is in optimization. Convex optimization problems are tractably solvable by various algorithms, both in theory and in practice. However, some algorithms perform much better in practice than theoretical worst-case results would suggest. The bad instances don't seem to occur in practice. One area where this happens is in solving linear programs (LP's) using the simplex method. In a linear program, we aim to maximize a linear function over a feasible set given by linear inequalities, and a solution consist of either an optimal point or an infinite feasible ray which certifies that the objective value is unbounded.

$$\max c^\mathsf{T} x$$
$$\text{subject to } Ax \leq b.$$

Roughly, the simplex method solves such problems by first finding any vertex of the feasible set, and then repeatedly moving (*pivoting*) to neighboring vertices until the optimal solution has been found. Different variants of the simplex method differ in how the initial vertex is found and by which rule a neighboring vertex is chosen.

In practice, the simplex method takes a number of pivot steps that is roughly linear in $d+n$, where d is the number of variables and n is the number of constraints. The theoretical worst-case performance is at least sub-exponential in d for all pivot rules that have been analyzed, though the existence of a polynomial-time pivot rule remains an open problem.

What property of real-world LP's makes them so easy to solve? Is there a geometric quantity that we can use to explain the good performance of the simplex method? Average-case analyses have been done in the past, but the chosen probability distribution might not resemble the real-world distribution in important ways.

The smoothed analysis framework of Spielman and Teng [4] aims to show that difficult instances are unlikely to occur by considering the expected running time under a small perturbation of the input data, and under this regime, Spielman and Teng managed to prove a polynomial running time bound for a specific simplex method. Their results have been improved in various ways by other authors since. We improve over previous running time bounds in all parameter regimes, with a substantially simpler and more general proof.

Theorem 1. *There is a self-dual simplex method such that, if a linear program* $\max c^\mathsf{T} x$ st $Ax \leq b$, *in d variables with n inequalities, has its constraint vectors* (a_i, b_i) *distributed with* $\|\mathbb{E}[(a_i, b_i)]\| \leq 1$ *and independent Gaussian noise of variance* σ^2 *on every entry of* (A, b), *the algorithm solves the program in expected time* $O(d^2\sqrt{\log n}\ \sigma^{-2} + d^5 \log^{3/2} n \log d)$.

Underlying this running time bound is a geometric statement about the expected number of edges of the projection of a polyhedron onto a two-dimensional plane. Our main contribution was proving a better bound on this quantity.

Theorem 2. *Let $W \subset \mathbb{R}^d$ be a fixed two-dimensional subspace, $n \geq d \geq 3$ and let $a_1, \ldots, a_n \in \mathbb{R}^d$, be independent Gaussian random vectors with variance σ^2 and centers of norm at most 1. We write A for the matrix with $a_1, \ldots, a_n$ as its rows. For $P := \{x : Ax \leq 1\}$, the number of edges of the projection polygon $\pi_W(P)$ of P onto W is bounded by*

$$\mathbb{E}[|\text{edges}(\pi_W(P))|] \leq \mathcal{D}_g(n, d, \sigma),$$

where the function $\mathcal{D}_g(d, n, \sigma)$ is defined as

$$\mathcal{D}_g(d, n, \sigma) := O(d^2 \sqrt{\log n}\ \sigma^{-2} + d^{2.5} \log n\ \sigma^{-1} + d^{2.5} \log^{1.5} n).$$

Open Questions.

(1) For d fixed, $\mathbb{E}[A] = 0$, and $n \to \infty$, Borgwardt [1] proved a tight bound of $\Theta(d^{1.5}\sqrt{\ln n})$ on the expected number of edges of $\pi_W(P)$. Can the smoothed upper bound be improved to match it for $\sigma \to \infty$? Can we prove any lower bound dependent on σ?

(2) Real-world LP's are sparse while smoothed LP's are dense. Can we give smoothed complexity bounds when only a random ϵ-fraction of entries of (A, b) get perturbed?

(3) Can anything meaningful be said about the smoothed complexity of other pivot rules? We do not expect such probability calculations to be easy for e.g., Dantzig's rule, as even determining if a constraint ever enters the basis is PSPACE-complete.[3]

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The smallest singular value of heavy-tailed not necessarily i.i.d. random matrices via random rounding

Galyna V. Livshyts

We are concerned with the small ball behavior of the smallest singular value of random matrices. Often, establishing such results involves, in some capacity, a discretization of the unit sphere. This requires bounds on the norm of the matrix, and the latter bounds require strong assumptions on the distribution of the entries, such as bounded fourth moments (for a weak estimate), sub-gaussian tails (for a strong estimate), and structural assumptions such as mean zero and variance one. Recently, Rebrova and Tikhomirov [1] developed a discretization procedure which does not rely on strong tail assumptions for the entries. However, their argument still required the structural assumptions of mean zero, variance one i.i.d. entries. In this talk, we discuss an efficient discretization of the unit sphere, which works with exponentially high probability, does not require any such structural assumptions, and, furthermore, does not require independence of the rows of the matrix. We show the existence of nets near the sphere, which compare values of any (deterministic) random matrix on the sphere and on the net via *a refinement of the Hilbert-Schmidt norm.* Such refinement is a form of averaging, and enjoys strong large deviation properties.

As a consequence we show, in particular, that the smallest singular value $\sigma_n(A)$ of an $N \times n$ random matrix A with i.i.d. mean zero variance one entries enjoys the following small ball estimate, for any $\epsilon > 0$:

$$P\left(\sigma_n(A) < \epsilon(\sqrt{N+1} - \sqrt{n})\right) \leq (C\epsilon)^{N-n+1} (\log 1/\epsilon)^{N-n+2} + e^{-cN},$$

which matches (up to a logarithmic error), for *heavy-tailed matrices with arbitrary aspect ratio*, the corresponding sub-gaussian behavior (as per work of Rudelson and Vershynin [3]). Allowing dependent rows in the discretization part is essential for this result.

Furthermore, in the case of the square $n \times n$ matrix A with independent entries having concentration function separated from 1, and such that $\mathbb{E}||A||_{HS}^2 \leq cn^2$, one has

$$P\left(\sigma_n(A) < \frac{\epsilon}{\sqrt{n}}\right) \leq C\epsilon + e^{-cn},$$

for any $\epsilon > \frac{c}{\sqrt{n}}$. Under the additional assumption of i.i.d. rows, this estimate is valid for all $\epsilon > 0$. In addition, we show that for an i.i.d. random matrix A, it suffices to assume, for an arbitrary $p > 0$, that $(\mathbb{E}|Ae_i|^p)^{\frac{1}{p}} \leq C\sqrt{n}$, to conclude the strong small ball property of $\sigma_n(A)$. Our estimates generalize the previous results of Rudelson and Vershynin [2], which required the sub-gaussian mean zero variance one assumptions. This condition is rather restrictive, however it shows that some random matrices with certain pathological *very heavy-tailed* entries (whose distribution depends on n), still enjoy the nice sub-gaussian behavior of the smallest singular value.

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Theorems of Carathédory, Helly, and Tverberg without dimension

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(joint work with Karim A. Adiprasito, Nabil Mustafa, Tamás Terpai)

Carathéodory's classical theorem [4] from 1907 says that every point in the convex hull of a point set $P \subset \mathbb{R}^d$ is in the convex hull of a subset $Q \subset P$ with at most $d+1$ points. One cannot require here that $|Q| \le r$ for some fixed $r \le d$ because for instance when P is finite, the union of the convex hull of all r-element subsets of P has measure zero while $\operatorname{conv} P$ may have positive measure. Instead one may to try to find, given $a \in \operatorname{conv} P$, a subset $Q \subset P$ with $|Q| \le r$ so that a is *close* to $\operatorname{conv} Q$. This is the content of the following theorem:

Theorem 1. *Let P be a set of n points in $\mathbb{R}^d$, $r \in [n]$ and $a \in \operatorname{conv} P$. Then there exists a subset Q of P with $|Q| = r$ such that $d(a, \operatorname{conv} Q) < \frac{\operatorname{diam} P}{\sqrt{2r}}$.*

In the statement of the theorem the dimension d has disappeared. So one can think of the n-element point set P as a set in $\mathbb{R}^n$ (or $\mathbb{R}^{n-1}$) with $a \in \operatorname{conv} P$. The conclusion is that for every $r < n$ the set P has a subset Q of size r whose convex hull is close to a. That is why we like to call the result "no-dimension Carathéodory theorem". The appearance of the factor $\operatorname{diam} P$ is quite natural here. The dependence on r is best possible: when $d = n-1$ and P is the set of vertices of a regular $(n-1)$-dimensional simplex whose centre is a, then for every $Q \subset P$ with $|Q| = r$,

$$d(a, \operatorname{conv} Q) = \sqrt{\frac{1}{2r} - \frac{1}{2n}} \operatorname{diam} P,$$

which is asymptotically the same as the upper bound in Theorem 1 in the no dimension setting.

The coloured version of Carathéodory's theorem [3] states that if $a \in \bigcap_1^{d+1} \operatorname{conv} P_i$, where $P_i \subset \mathbb{R}^d$, then there is a transversal $T = \{p_1, \dots, p_{d+1}\}$ such that $a \in \operatorname{conv} T$. Here a *transversal* of the set system $P_1, \dots, P_{d+1}$ is a set $T = \{p_1, \dots, p_{d+1}\}$ such that $p_i \in P_i$ for all $i \in [d+1]$. We extend this to the no-dimension case as follows.

Theorem 2. *Let $P_1, \dots, P_r$ be $r \ge 2$ point sets in $\mathbb{R}^d$ such that $a \in \bigcap_1^r \operatorname{conv} P_i$. Define $D = \max_{i \in [r]} \operatorname{diam} P_i$. Then there exists a transversal T such that*

$$d(a, \operatorname{conv} T) < \frac{D}{\sqrt{2r}}.$$

The proof is an averaging argument that can be turned into a randomized algorithm that finds the transversal T in question; the method of conditional probabilities also gives a deterministic algorithm. We give another, also algorithmic, proof is which based on the Frank-Wolfe procedure [5].

There are earlier results of the same type by Starr, Cassels, Maurey, Carl, Carl and Pajor, Bárány and Füredi, Barman. Further details of the history can be found in the full version of the paper [1].

Several results in combinatorial convexity have similar no-dimension versions. Our first example is Helly's theorem.

Theorem 3. *Assume $K_1, \dots, K_n$ are convex sets in $\mathbb{R}^d$ and $k \in [n]$. For $J \subset [n]$ define $K(J) = \bigcap_{j \in J} K_j$. If the Euclidean unit ball $B(b,1)$ centered at $b \in \mathbb{R}^d$ intersects $K(J)$ for every $J \subset [n]$ with $|J| = k$, then there is point $q \in \mathbb{R}^d$ such that*

$$d(q, K_i) \leq \frac{1}{\sqrt{k}} \text{ for all } i \in [n].$$

The precise bound in this theorem is

$$d(q, K_i) \leq \sqrt{\frac{n-k}{k(n-1)}} = \sqrt{\frac{1}{k} - \frac{1}{n-1} + \frac{1}{k(n-1)}}. \tag{1}$$

The proof is based on a geometric inequality about simplices that seems to be new.

Theorem 4. *Let Δ be a (non-degenerate) simplex on n vertices with inradius r and let $k \in [n]$. Then any ball intersecting the affine span of each $(k-1)$-dimensional face of Δ has radius at least $\lambda_n r$ where $\lambda_n = \sqrt{\frac{(n-1)(n-k)}{k}}$ is the optimal ratio for the regular simplex.*

Theorem 3 extends to the colourful version of Helly's theorem, due to Lovász (see [3]), and to the fractional Helly theorem of Katchalski and Liu [6]. Their proofs are based on a more general result. To state it some preparation is needed. We write $B(a, \rho)$ for the Euclidean ball centred at $a \in \mathbb{R}^d$ of radius ρ. Suppose $\mathcal{F}_1, \dots, \mathcal{F}_k$ are finite and non-empty families of convex sets in $\mathbb{R}^d$, $\mathcal{F}_i$ can be thought of as a collection of convex sets of colour i. A *transversal* $\mathcal{T}$ of the system $\mathcal{F}_1, \dots, \mathcal{F}_k$ is just $\mathcal{T} = \{K_1, \dots, K_k\}$ where $K_i \in \mathcal{F}_i$ for all $i \in [k]$. We define $K(\mathcal{T}) = \bigcap_1^k K_i$. Given $\rho_i > 0$ for all $i \in [k]$, set $\rho = \sqrt{\rho_1^2 + \ldots + \rho_k^2}$.

Theorem 5. *Assume that, under the above conditions, for every $p \in \mathbb{R}^d$ there are at least m_i sets $K \in \mathcal{F}_i$ with $B(p, \rho_i) \bigcap K = \emptyset$ for all $i \in [k]$. Then for every $q \in \mathbb{R}^d$ there are at least $\prod_1^k m_i$ transversals $\mathcal{T}$ such that*

$$d(q, K(\mathcal{T})) > \rho,$$

with the convention that $d(q, \emptyset) = \infty$.

The no-dimensional version of Tverberg's famous theorem [7] is the following.

Theorem 6. *Given a set P of n points in $\mathbb{R}^d$ and an integer $2 \leq k \leq n$, there exists a point $q \in \mathbb{R}^d$ and a partition of P into k sets $P_1, \dots, P_k$ such that*

$$d\left(q, \operatorname{conv} P_i\right) \leq (2 + \sqrt{2}) \cdot \sqrt{\frac{k}{n}} \operatorname{diam} P \text{ for every } \in [k].$$

Actually this result is a corollary to the more general coloured Tverberg theorem (cf. [8]), no-dimension version. We assume that the sets $C_1, \dots, C_r \subset \mathbb{R}^d$ (considered as colours) are disjoint and each has size k. Set $P = \bigcup_1^r C_j$.

Theorem 7. *Under the above conditions there is a point* $q \in \mathbb{R}^d$ *and a partition* $P_1, \dots, P_k$ *of* P *such that* $|P_i \cap C_j| = 1$ *for every* $i \in [k]$ *and every* $j \in [r]$ *satisfying*

$$d(q, \operatorname{conv} P_i) \leq (1+\sqrt{2})\frac{\operatorname{diam} P}{\sqrt{r}} \text{ for every } i \in [k].$$

The bounds given in Theorems 6 and 7 are best possible apart from the constants. We remark further that several results in combinatorial convexity have no-dimension versions, for instance the center point theorem, the first selection lemma, the weak ε-net theorem, and also the (p,q) theorem of Alon and Kleitman [2].

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Dar's conjecture and the log-Brunn-Minkowski inequality

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(joint work with Gangsong Leng)

Let $\mathcal{K}^n$ be the class of convex bodies (compact, convex sets with non-empty interiors) in Euclidean n-space $\mathbb{R}^n$, and let $\mathcal{K}^n_o$ be the class of members of $\mathcal{K}^n$ containing o (the origin) in their interiors. The classical Brunn-Minkowski inequality states that

$$|K+L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}, \tag{1}$$

with equality if and only if K and L are homothetic. Here $K, L \in \mathcal{K}^n$, $|\cdot|$ denotes the n-dimensional Lebesgue measure, and $K+L$ denotes the Minkowski sum of K and L:

$$K + L = \{x + y : x \in K \text{ and } y \in L\}.$$

In his survey article, Gardner [5] summarized the history of the Brunn-Minkowski inequality and some applications in many other fields.

In 1999, Dar [3] conjectured that

$$|K+L|^{\frac{1}{n}} \geq M(K,L)^{\frac{1}{n}} + \frac{|K|^{\frac{1}{n}}|L|^{\frac{1}{n}}}{M(K,L)^{\frac{1}{n}}}, \tag{2}$$

for convex bodies K and L. Here $M(K,L)$ is defined by

$$M(K,L) = \max_{x\in\mathbb{R}^n} |K \cap (x+L)|.$$

Dar [3] showed that (2) implies (1) for convex bodies. He also proved (2) in some special cases.

1. Relationship with the stability of the B-M inequality

Dar's conjecture has a close relationship with the stability of the Brunn-Minkowski inequality. The stability estimates are actually strong forms of the Brunn-Minkowski inequality in special circumstances. Original works about this issue are due to Diskant, Groemer, and Schneider.

Figalli, Maggi, and Pratelli [4] tackled the stability problem for convex bodies with a more natural distance, i.e., "relative asymmetry" (which has a close relationship with the functional $M(K,L)$), by using mass transportation approach. Using the same distance as in [4], Segal [6] improved the constants that appeared in the stability versions in these inequalities for convex bodies. He also showed in [6, Page 391] that Dar's conjecture (2) will lead to a stronger stability version of Brunn-Minkowski inequality for convex bodies.

In 2012, Campi, Gardner, and Gronchi [2, Page 1208] pointed out that Dar's conjecture "seems to be open even for planar o-symmetric bodies". Besides, the equality condition of (2) is also unknown.

2. The solution of Dar's conjecture in dimension two

We proved that the inequality (2) holds for all planar convex bodies, and we also give the equality condition.

Theorem 1. *Let K, L be planar convex bodies. Then, we have*

$$|K+L|^{\frac{1}{2}} \geq M(K,L)^{\frac{1}{2}} + \frac{|K|^{\frac{1}{2}}|L|^{\frac{1}{2}}}{M(K,L)^{\frac{1}{2}}}. \tag{3}$$

Equality holds if and only if one of the following conditions holds:

(i) K and L are parallelograms with parallel sides, and $|K| = |L|$;
(ii) K and L are homothetic.

In our proof of Theorem 1, the definition of "dilation position" (the definition is in next section) plays a key role. It enables us to further study the other stronger version of (1), i.e., the log-Brunn-Minkowski inequality (see [1, 7] for details).

3. Log-Brunn-Minkowski inequality for non-symmetric convex bodies

The log-Brunn-Minkowski inequality for planar o-symmetric (symmetry with respect to the origin) convex bodies was established by Böröczky, Lutwak, Yang, and Zhang [1]. After that, they proposed the following problem for non-symmetric convex bodies.

Problem. *Let $K, L \in \mathcal{K}^2$. Is there a "good" position of the origin o, such that K and an "appropriate" translate of L satisfy the log-Brunn-Minkowski inequality?*

We gave a weak answer to this problem. Before this, we give the definition of the so-called dilation position.

Let $K, L \in \mathcal{K}^n$. We say K and L are at a *dilation position*, if $o \in K \cap L$, and

$$r(K,L)L \subset K \subset R(K,L)L. \tag{4}$$

Here $r(K,L)$ and $R(K,L)$ are *relative inradius* and *relative outradius* of K with respect to L, i.e.,

$$r(K,L) = \max\{t > 0 : x + tL \subset K \text{ and } x \in \mathbb{R}^n\},$$

$$R(K,L) = \min\{t > 0 : K \subset x + tL \text{ and } x \in \mathbb{R}^n\}.$$

It is clear that

$$r(K,L) = 1/R(L,K). \tag{5}$$

By the definition, it is clear that two o-symmetric convex bodies are always at a dilation position.

When K and L are at a dilation position, o may be in $\partial K \cap \partial L$. Therefore, we should extend the definition of "geometric Minkowski combination" slightly. Let $K, L \in \mathcal{K}^n$ with $o \in K \cap L$. The *geometric Minkowski combination* of K and L is defined as follows:

$$(1-\lambda)\cdot K +_o \lambda \cdot L := \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}\}, \tag{6}$$

for $\lambda \in (0,1)$; $(1-\lambda)\cdot K +_o \lambda \cdot L := K$ for $\lambda = 0$; and $(1-\lambda)\cdot K +_o \lambda \cdot L := L$ for $\lambda = 1$.

We can prove that $(1-\lambda)\cdot K +_o \lambda \cdot L$ defined by (6) is always a convex body, as long as K and L are at a dilation position. The followings are the general *log-Brunn-Minkowski inequality* and the general *log-Minkowski inequality* for planar convex bodies.

Theorem 2. *Let $K, L \in \mathcal{K}^2$ with $o \in K \cap L$. If K and L are at a dilation position, then for all real $\lambda \in [0,1]$,*

$$|(1-\lambda)\cdot K +_o \lambda \cdot L| \geq |K|^{1-\lambda}|L|^{\lambda}. \tag{7}$$

When $\lambda \in (0,1)$, equality in the inequality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.

Theorem 3. *Let $K, L \in \mathcal{K}^2$ with $o \in K \cap L$. If K and L are at a dilation position, then*

$$\int_{S^1} \log \frac{h_L}{h_K} dV_K \geq \frac{|K|}{2} \log \frac{|L|}{|K|}. \tag{8}$$

Equality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.

Here V_K denotes the cone-volume measure. It can be seen from (4) that $\{h_K = 0\} = \{h_L = 0\}$. The integral in (8) should be understood to be taken on S^1 except the set $\{h_K = 0\}$, which is of measure 0, with respect to the measure V_K.

It can be easily seen from the fact $(1-\lambda) \cdot K +_o \lambda \cdot L \subset (1-\lambda)K + \lambda L$ that (7) implies the classical Brunn-Minkowski inequality (1) for all planar convex bodies.

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On a characterization of (dual) mixed volumes

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(joint work with David Alonso-Gutiérrez, Martin Henk)

Let $\mathcal{K}^n$ denote the set of convex bodies (compact and convex sets) in $\mathbb{R}^n$. Given $K_1, \ldots, K_m \in \mathcal{K}^n$, $\lambda_1, \ldots, \lambda_m \geq 0$, the volume of $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is given by

$$\mathrm{vol}(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \mathrm{V}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $\mathrm{V}(K_{i_1}, \ldots, K_{i_n}) \geq 0$ are the *mixed volumes* of $K_1, \ldots, K_m$, and are symmetric in the indices. Therefore, there are $N_{n,m} = \binom{n+m-1}{n}$ mixed volumes associated to the m sets. In the particular case of two convex bodies $K, L \in \mathcal{K}^n$, the mixed volumes $\mathrm{V}\big(K[n-i], L[i]\big) = \mathrm{W}_i(K, L)$ are called *relative quermassintegrals* of K with respect to L, $i = 0, \ldots, n$.

Relating the mixed volumes we find the well-known Aleksandrov-Fenchel inequalities: if $\mathcal{C}_{n-r}$ is any $(n-r)$-tuple of the m convex bodies $K_1, \ldots, K_m$, then

$$\mathrm{V}(K_i, K_j, \mathcal{C}_{n-2})^2 \geq \mathrm{V}(K_i, K_i, \mathcal{C}_{n-2}) \mathrm{V}(K_j, K_j, \mathcal{C}_{n-2}), \tag{1}$$

as well as the more general ones

(2) $\mathrm{V}\big(K_i[r{-}1], K_j, \mathcal{C}_{n-r}\big)\mathrm{V}\big(K_i, K_j[r{-}1], \mathcal{C}_{n-r}\big) \geq \mathrm{V}\big(K_i[r], \mathcal{C}_{n-r}\big)\mathrm{V}\big(K_j[r], \mathcal{C}_{n-r}\big).$

In [4] Shephard got a family of determinantal inequalities for mixed volumes: if

$$M = \begin{pmatrix} \mathrm{V}(K_1, K_1, \mathcal{C}_{n-2}) & \mathrm{V}(K_1, K_2, \mathcal{C}_{n-2}) & \dots & \mathrm{V}(K_1, K_n, \mathcal{C}_{n-2}) \\ \mathrm{V}(K_1, K_2, \mathcal{C}_{n-2}) & \mathrm{V}(K_2, K_2, \mathcal{C}_{n-2}) & \dots & \mathrm{V}(K_2, K_n, \mathcal{C}_{n-2}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{V}(K_1, K_n, \mathcal{C}_{n-2}) & \mathrm{V}(K_2, K_n, \mathcal{C}_{n-2}) & \dots & \mathrm{V}(K_n, K_n, \mathcal{C}_{n-2}) \end{pmatrix},$$

every leading s-minor of M is ≥ 0 or ≤ 0 according to whether s is odd or even.

Then, Shephard posed the question whether the known inequalities relating the mixed volumes are enough in order to characterize them, in the following sense: given $N_{n,m}$ non-negative real numbers satisfying the inequalities, do there exist n convex bodies whose mixed volumes are the given numbers? He solved this question when two convex bodies come into play:

Theorem 1 ([4])**.** *Any given set of* $n+1$ *non-negative real numbers* $\mathrm{W}_0, \dots, \mathrm{W}_n$ *satisfying the inequalities* (2)*, i.e.,* $\mathrm{W}_i\mathrm{W}_j \geq \mathrm{W}_{i-1}\mathrm{W}_{j+1}$*,* $1 \leq i \leq j \leq n-1$*, arises as the set of relative quermassintegrals of two convex bodies.*

He provided a beautiful constructive proof when all $\mathrm{W}_i > 0$, whereas the general case was obtained by a rather non-constructive topological argument. In [3] we reproved Shephard's result with two slight advantages: i) we extended the construction of the two convex bodies to the non-negative case, i.e., $\mathrm{W}_i \geq 0$; ii) we reduced the number of involved inequalities to (1), namely, $\mathrm{W}_i^2 \geq \mathrm{W}_{i-1}\mathrm{W}_{i+1}$.

In [4] Shephard also proved that for $m = n+2$ sets, the previously mentioned families of inequalities are not enough in order to characterize the mixed volumes. The case $n = 2$ and $m = 3$ has also a positive answer: Heine [2] showed that (1), together with the determinantal inequality, characterize the mixed volumes of 3 planar convex bodies. For $3 \leq m \leq n+1$ (and $n \geq 3$) the problem is still open.

Recently we have considered the corresponding Shephard problem within the *dual Brunn-Minkowski theory*, i.e., to look for necessary and sufficient conditions for a set of positive real numbers to be the dual quermassintegrals of two star bodies in $\mathbb{R}^n$. In order to define the star bodies, we call a non-empty set $S \subseteq \mathbb{R}^n$ *starshaped* (with respect to the origin) if the segment $[0,x] \subseteq S$ for all $x \in S$. A compact starshaped set K whose *radial function* $\rho_K(u) = \max\{\rho \geq 0 : \rho u \in K\}$, $u \in \mathbb{S}^{n-1}$, is positive and continuous, is called a *star body.* We write $\mathcal{S}_0^n$ to denote the set of all star bodies in $\mathbb{R}^n$. Moreover, the Minkowski sum $+$ is replaced in this setting by the so-called *radial addition* $\widetilde{+}$, namely: for $x, y \in \mathbb{R}^n$,

$$x \widetilde{+} y = \begin{cases} x+y, & \text{if } x, y \text{ are linearly dependent,} \\ 0, & \text{otherwise.} \end{cases}$$

Then the volume of the radial sum $K \widetilde{+} \lambda L$, for $K, L \in \mathcal{S}_0^n$ and $\lambda \geq 0$, is also expressed as a polynomial, whose coefficients $\widetilde{\mathrm{W}}_i(K, L)$ are the (relative) *dual quermassintegrals* of K and L. Dual quermassintegrals have an integral representation in terms of the radial functions of K and L,

$$\widetilde{\mathrm{W}}_i(K,L)=\frac{1}{n}\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n-i}\rho_L(u)^i\,\mathrm{d}\sigma(u)$$

(here σ is the usual spherical Lebesgue measure), which allows us to define dual quermassintegrals for any real index $i\in\mathbb{R}$. In this setting we have proved the following result:

Theorem 2 ([1]). *For $R=(r_0=0,r_1,\dots,r_m)$, with $r_j\in\mathbb{R}$, $1\le j\le m$, and $0<a<b$, we write $C^R_{a,b}=\mathrm{pos}\big\{(1,t^{r_1},\dots,t^{r_m})^\intercal : t\in[a,b]\big\}\subset\mathbb{R}^{m+1}$. Let $\omega_i\in\mathbb{R}_{>0}$, $0\le i\le m$, be positive real numbers and let $n\ge 2$. Then there exist $K,L\in\mathcal{S}^n_0$ such that*

$$\widetilde{W}_{r_i}(K,L)=\omega_i,\quad 0\le i\le m,$$

if and only if

(1) either there exist $0<a<b$ such that

$$(\omega_0,\omega_1,\dots,\omega_m)^\intercal\in\mathrm{int}C^R_{a,b},$$

(2) or $\omega_i=\lambda^{r_i}\omega_0$ for some $\lambda>0$ and $1\le i\le m$ (in this case $L=\lambda K$).

The characterization of dual quermassintegrals is related to the famous *moment problem* via Riesz's theorem, which is also a key tool in the proof of our theorem.

Moreover, based on this relation, new determinantal inequalities among the dual quermassintegrals can be derived:

Corollary 1 ([1]). *Let $K,L\in\mathcal{S}^n_0$ and let $m\in\mathbb{N}$, $m\ge 1$. For pairwise distinct numbers $r_1,\dots,r_m\in\mathbb{R}$, let $A_m\in\mathbb{R}^{m\times m}$ be the Hankel matrix*

$$A_m=\left(\widetilde{\mathrm{W}}_{r_i+r_j}(K,L)\right)_{1\le i,j\le m}.$$

Then $\det A_m\ge 0$, with equality if and only if $K=\lambda L$ for some $\lambda>0$.

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On a conjecture about the tensor product of cones

Guillaume Aubrun
(joint work with Ludovico Lami, Carlos Palazuelos)

We work in a finite-dimensional real vector space V and consider a cone $\mathcal{C} \subset V$ (all cones are implicitly convex, closed, salient and generating). The dual cone $\mathcal{C}^* \subset V^*$ is defined as $\mathcal{C}^* = \{\phi \in V^* : \phi(x) \geqslant 0 \ \forall x \in \mathcal{C}\}$.

Given two such cones $\mathcal{C}_1 \subset V_1$, $\mathcal{C}_2 \subset V_2$, there are two canonical ways to define the tensor product:

(1) The *minimal tensor product*

$$\mathcal{C}_1 \otimes_{\min} \mathcal{C}_2 = \operatorname{conv}\{x_1 \otimes x_2 : x_i \in \mathcal{C}_i\}.$$

(2) The *maximal tensor product*

$$\mathcal{C}_1 \otimes_{\max} \mathcal{C}_2 = (\mathcal{C}_1^* \otimes_{\min} \mathcal{C}_2^*)^* .$$

By analogy with the terminology used in C^*-algebras, we say that $(\mathcal{C}_1, \mathcal{C}_2)$ is a nuclear pair if $\mathcal{C}_1 \otimes_{\min} \mathcal{C}_2 = \mathcal{C}_1 \otimes_{\max} \mathcal{C}_2$ (the inclusion $\subset$ always holds). We point a reformulation in terms of operators: the pair $(\mathcal{C}_1, \mathcal{C}_2)$ is nuclear if and only if every linear map $\Phi : V_1^* \to V_2$ which is positive (i.e such that $\Phi(\mathcal{C}_1^*) \subset \mathcal{C}_2$) is a sum of rank 1 positive maps.

We say that a cone $\mathcal{C}$ is classical if it has a basis which is a simplex. It is easy to see that a pair of cones is nuclear whenever one of them is classical. We conjecture that the converse holds.

Conjecture 1. *For any cones $\mathcal{C}_1$, $\mathcal{C}_2$,*

$$(\mathcal{C}_1, \mathcal{C}_2) \text{ is nuclear} \iff \mathcal{C}_1 \text{ or } \mathcal{C}_2 \text{ is classical.}$$

Our conjecture has a strong motivation from physics. The fact that the pair (PSD, PSD) is not nuclear (PSD is the cone of positive semi-definite matrices) is related to the phenomenon of quantum entanglement. In the language of "generalized probabilistic theories", our conjecture means that entanglement exists between any two non-classical theories.

The only related work is a 1969 paper by Namioka–Phelps [1] where it is proved that

$$(\mathcal{C}_\square, \mathcal{C}) \text{ is nuclear} \iff \mathcal{C} \text{ is classical,}$$

where $\mathcal{C}_\square \subset \mathbf{R}^3$ is the cone with a square as a basis.

We give support to our conjecture by proving its validity in the following particular cases.

Theorem 1. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two 3-dimensional cones. Then*

$$(\mathcal{C}_1, \mathcal{C}_2) \text{ is nuclear} \iff \mathcal{C}_1 \text{ or } \mathcal{C}_2 \text{ is classical.}$$

Theorem 2. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two polyhedral cones. Then*

$$(\mathcal{C}_1, \mathcal{C}_2) \text{ is nuclear} \iff \mathcal{C}_1 \text{ or } \mathcal{C}_2 \text{ is classical.}$$

Theorem 3. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be cones with a centrally symmetric basis and* $\dim \mathcal{C}_i \geqslant 3$. *Then* $(\mathcal{C}_1, \mathcal{C}_2)$ *is not nuclear.*

The ingredient in the proof of Theorem 1 is a non-symmetric version of Auerbach's lemma for planar convex bodies. The proof of Theorem 2 is by induction on the dimension, with Theorem 1 as the base case; it uses the known fact that simplices are the only polytopes which are both simple and simplicial. The proof of Theorem 3 connects with the projective and injective tensor products of Banach spaces, for which analagous questions were studied in [2].

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On Rogers-Shephard type inequalities for general measures

Jesús Yepes Nicolás

(joint work with David Alonso-Gutiérrez, María A. Hernández Cifre, Michael Roysdon, Artem Zvavitch)

The classical Rogers-Shephard inequality in $\mathbb{R}^n$, originally proven in [4], states that

$$\mathrm{vol}(K-K) \leq \binom{2n}{n} \mathrm{vol}(K),$$

for any convex body $K \subset \mathbb{R}^n$, with equality if and only if K is a simplex. Here $K - K = \{x - y : x \in K, y \in K\}$ denotes the so-called difference set whereas by a convex body we mean a compact convex set with non-empty interior.

In [5], in addition to $K-K$, Rogers and Shephard considered two other centrally symmetric convex bodies associated with K. The first one is

$$CK = \big\{(x,\theta) \in \mathbb{R}^{n+1} : x \in (1-\theta)K + \theta(-K),\, \theta \in [0,1]\big\},$$

whose volume is given by

$$\mathrm{vol}_{n+1}(CK) = \int_0^1 \mathrm{vol}\big((1-\theta)K + \theta(-K)\big)\,\mathrm{d}\theta.$$

The second one is just $\mathrm{conv}\big(K \cup (-K)\big)$. The relation of the volumes of CK and $\mathrm{conv}\big(K \cup (-K)\big)$ to the volume of K was proved in [5], obtaining on one hand that for any convex body $K \subset \mathbb{R}^n$ containing the origin

$$\int_0^1 \mathrm{vol}\big((1-\theta)K + \theta(-K)\big)\,\mathrm{d}\theta \leq \frac{2^n}{n+1}\,\mathrm{vol}(K), \tag{1}$$

with equality if and only if K is a simplex. On the other hand,

$$\mathrm{vol}\Big(\mathrm{conv}\big(K \cup (-K)\big)\Big) \leq 2^n\,\mathrm{vol}(K), \tag{2}$$

with equality if and only if K is a simplex with the origin as a vertex.

In this talk we discuss whether one may obtain some Rogers-Shephard type inequalities for convex bodies, in the spirit of the above results, when dealing with measures on the Euclidean space associated to general densities. To this aim, we point out that certain assumptions must be imposed to the densities as well as some reformulation on the 'structure' of the inequalities has to be taken into account. In relation to the latter, we observe that one cannot expect to obtain

$$\mu(K-K) \leq \binom{2n}{n}\mu(K)$$

without having certain control on the 'position' of the body K, as it may be shown by considering the standard Gaussian measure and taking as K a unit ball with center x, with $|x|$ large enough. To solve this issue, given a measure μ on $\mathbb{R}^n$, we define its *translated-average* $\overline{\mu}$ as

$$\overline{\mu}(K) = \frac{1}{\mathrm{vol}(K)}\int_K \mu(-y+K)\,\mathrm{d}y.$$

Then, with this notion, our first main result reads as follows.

Theorem 1 ([2]). *Let $K \subset \mathbb{R}^n$ be a convex body. Let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ is radially decreasing. Then*

$$\mu(K-K) \leq \binom{2n}{n}\min\bigl\{\overline{\mu}(K), \overline{\mu}(-K)\bigr\}. \tag{3}$$

Moreover, if ϕ is continuous at the origin then equality holds in (3) if and only if μ is a constant multiple of the Lebesgue measure on $K-K$ and K is a simplex.

Regarding analogs of both (1) and (2) in the setting of measures with radially decreasing density we have the following result:

Theorem 2 ([2]). *Let $K \subset \mathbb{R}^n$ be a convex body containing the origin. Let μ be a measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(x) = \phi(x)\,\mathrm{d}x$, where $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ is radially decreasing. Then*

$$\int_0^1 \mu\bigl((1-\theta)K + \theta(-K)\bigr)\,\mathrm{d}\theta \leq \frac{2^n}{n+1}\sup_{\substack{y\in K\\ \theta\in(0,1]}}\frac{\mu\bigl((1-\theta)y-\theta K\bigr)}{\theta^n} \tag{4}$$

and

$$\mu\Bigl(\mathrm{conv}\bigl(K\cup(-K)\bigr)\Bigr) \leq 2^n\sup_{\substack{y\in K\\ \theta\in(0,1]}}\frac{\mu\bigl((1-\theta)y-\theta K\bigr)}{\theta^n}. \tag{5}$$

Moreover, if ϕ is continuous at the origin then equality holds in (4) if and only if μ is a constant multiple of the Lebesgue measure on $\mathrm{conv}\bigl(K\cup(-K)\bigr)$ and K is a simplex, and equality holds in (5) if and only if μ is a constant multiple of the Lebesgue measure on $\mathrm{conv}\bigl(K\cup(-K)\bigr)$ and K is a simplex with the origin as a vertex.

We also prove some Rogers-Shephard type inequalities involving both the projection $P_H K$ (onto a plane H) and the maximal measure section (through translates of the orthogonal complement of H) of a convex body $K \subset \mathbb{R}^n$, namely,

$$\mu_{n-k}\big(P_H K\big)\mu_k\big(K \cap H^\perp\big) \leq \binom{n}{k}\mu_n(K),$$

underlying which is the main difference with respect to the classical setting: the necessity of assuming that $P_H K \subset K$.

Finally, in a similar way, we show that certain functional versions of classical Rogers-Shephard type inequalities may be also derived as consequences of our approach. This generalizes some previous results from [1, 3].

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Volume, polar volume and Euler characteristic for convex functions

Fabian Mussnig

A map Z defined on the subset $\mathcal{S}$ of a lattice is called a *valuation* if

$$\mathrm{Z}(u) + \mathrm{Z}(v) = \mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v)$$

whenever $u, v, u \vee v, u \wedge v$ are in $\mathcal{S}$. Valuations defined on the set of convex bodies (compact convex sets) in $\mathbb{R}^n$ have been studied since Dehn's solution of Hilbert's Third Problem in 1901. In this case, $\vee$ and $\wedge$ denote union and intersection, respectively, and the first classification of valuations on convex bodies was obtained by Blaschke in the 1930s, thus characterizing the Euler characteristic and the n-dimensional volume.

In recent years, valuations on function spaces have been introduced and studied. Here, $u \vee v$ denotes the pointwise maximum of u and v and $u \wedge v$ denotes the pointwise minimum of two functions $u, v \in \mathcal{S}$, where $\mathcal{S}$ is a space of real-valued functions on $\mathbb{R}^n$. Together with Andrea Colesanti and Monika Ludwig the following analogue of Blaschke's result was established on the space

$$\begin{aligned}\mathrm{Conv}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to (-\infty, +\infty] \,:\, &u \text{ is convex, l.s.c.},\\ &\lim\nolimits_{|x|\to+\infty} u(x) = +\infty, u \not\equiv +\infty\}.\end{aligned}$$

Theorem 1 ([2]). *For $n \geq 2$, a map* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to [0, \infty)$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation if and only if there exist continuous functions* $\zeta_0, \zeta_1 : \mathbb{R} \to [0, \infty)$ *where* $\int_0^\infty t^{n-1} \zeta_1(t)\,\mathrm{d}t < \infty$ *such that*

$$\mathrm{Z}(u) = \zeta_0(\min_{x \in \mathbb{R}^n} u(x)) + \int_{\mathrm{dom}\, u} \zeta_1\big(u(x)\big)\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Here, continuity of Z is understood with respect to epi-convergence (also called Γ-convergence), which coincides with pointwise convergence if the functions are convex and finite. Furthermore, Z is called *translation invariant* if $\mathrm{Z}(u \circ \tau^{-1}) = \mathrm{Z}(u)$ for every translation τ on $\mathbb{R}^n$ and $\mathrm{SL}(n)$ *invariant* if $\mathrm{Z}(u \circ \phi^{-1}) = \mathrm{Z}(u)$ for every $\phi \in \mathrm{SL}(n)$.

More recently, this result could be improved by restricting to the space

$$\mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) = \{u \in \mathrm{Conv}(\mathbb{R}^n) \,:\, u(x) < +\infty \quad \forall x \in \mathbb{R}^n\}.$$

There, $u^*(x) = \sup_{y \in \mathbb{R}^n} \big(x \cdot y - u(y)\big)$ denotes the *convex conjugate* or *Legendre transform* of the function u.

Theorem 2 ([4]). *For $n \geq 2$, a map* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to [0, \infty)$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation if and only if there exist continuous functions* $\zeta_0, \zeta_1, \zeta_2 : \mathbb{R} \to [0, \infty)$ *where* $\int_0^\infty t^{n-1} \zeta_1(t)\,\mathrm{d}t < \infty$ *and* $\zeta_2(t) = 0$ *for all* $t \geq T$ *with some* $T \in \mathbb{R}$ *such that*

$$\mathrm{Z}(u) = \zeta_0(\min_{x \in \mathbb{R}^n} u(x)) + \int_{\mathbb{R}^n} \zeta_1\big(u(x)\big)\,\mathrm{d}x + \int_{\mathrm{dom}\, u^*} \zeta_2\big(\nabla u^*(x) \cdot x - u^*(x)\big)\,\mathrm{d}x \tag{1}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$.

The proof of this result is based on a characterization of the Euler characteristic, volume and volume of the polar body as continuous, $\mathrm{SL}(n)$ invariant valuations on the space of convex bodies that contain the origin their interiors by Haberl and Parapatits [3].

Remark 1. For a function $u \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n)$, the new term in (1) can be rewritten as

$$\int_{\mathbb{R}^n} \zeta_2(u(x)) \det(\mathrm{D}^2 u(x))\,\mathrm{d}x$$

where $\mathrm{D}^2 u(x)$ is the Hessian matrix of u and $\det(\mathrm{D}^2 u(x))$ denotes its determinant. This is also a special case of the so-called *Hessian valuations* that were introduced in [1].

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The logarithmic Minkowski problem, the logarithmic Brunn-Minkowski conjecture and relatives

Károly J. Böröczky

For any convex body K (compact, convex, non-empty interior) in $\mathbb{R}^n$, we write S_K to denote the surface area measure on S^{n-1} (see Schneider [8]). If ∂K is C^2_+ and $\kappa(u)$ stands for the Gaussian curvature at the boundary point of ∂K with exterior unit normal $u \in S^{n-1}$, then

$$dS_K(u) = \kappa^{-1}(u)\, du$$

with respect to the Lebesgue measure du on S^{n-1}, and we have

$$\det(\nabla^2 h + h\, I) = \kappa^{-1}$$

where $h(u) = h_K(u) = \max\{\langle u, x\rangle : x \in K\}$ is the support function. For an admissible Borel measure μ on S^{n-1}, the classical Minkowski problem asks for a convex body K satisfying $\mu = S_K$. Uniqueness of the solution up to translation follows *via* the Brunn-Minkowski inequality

$$V((1-\lambda)K + \lambda\, C)^{\frac{1}{n}} \geq (1-\lambda)V(K)^{\frac{1}{n}} + \lambda\, V(C)^{\frac{1}{n}},$$

for $\lambda \in [0, 1]$, which can be written in the dimension invariant form

$$V((1-\lambda)K + \lambda\, C) \geq V(K)^{1-\lambda} V(C)^{\lambda}. \tag{1}$$

Various argument is known to prove the Brunn-Minkowski inequality. The first clean optimal transportation proof is due to Gromov, which was developed into a sharp stability result by Figalli, Maggi, Pratelli.

The L_p-Brunn-Minkowski theory was initiated by Lutwak [7] where $p = 1$ is the classical case. For a finite Borel measure μ on S^{n-1} and $p \in \mathbb{R}$, the corresponding L_p-Minkowski problem asks for a convex body K with $o \in K$ such that $d\mu = h_K^{1-p} dS_K$; or in other words, the corresponding Monge-Ampère equation is

$$h_K^{1-p} \det(\nabla^2 h + h\, I) = f.$$

Here $dV_K = h_K dS_K$ is the cone-volume measure of Gromov, Milman in the $p = 0$ case. Böröczky, Lutwak, Yang, Zhang [1] charactherized even cone-volume measures solving the even logarithmic Minkowski problem, and conjectured that $V_K = V_C$ for o-symmetric convex bodies K, C with C^2_+ boundary implies $K = C$.

For $p \geq 0$, $\lambda \in [0, 1]$ and convex bodies K, C with $o \in \operatorname{int} K, \operatorname{int} C$, the L_p-linear combination is defined by

$$\begin{aligned} (1-\lambda)K +_p \lambda\, C &= \{x \in \mathbb{R}^n : \langle x, u\rangle^p \leq (1-\lambda)h_K(u)^p + \lambda\, h_C(u)^p\} \text{ if } p > 0, \\ (1-\lambda)K +_0 \lambda\, C &= \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u)^{1-\lambda} h_C(u)^{\lambda}\} \end{aligned}$$

where the L_1 addition is just Minkowski addition. If $p \geq 1$ and K, C are any convex bodies with $o \in \operatorname{int} K, \operatorname{int} C$, then the inequality

$$V((1-\lambda)K +_p \lambda\, C) \geq V(K)^{1-\lambda} V(C)^{\lambda} \tag{2}$$

is a direct consequence of the Brunn-Minkowski inequality (1).

According to the L_p-Brunn-Minkowski conjecture due to Böröczky, Lutwak, Yang, Zhang [1], (2) holds for any $p \in [0,1)$ and o-symmetric convex bodies K, C (it does not hold say for shifted cubes so needs some assumption on a center). It is known that the uniqueness of the solution of the even L_p-Minkowski problem for absolutely continuous measures would yield the L_p-Brunn-Minkowski conjecture for $p \in [0,1)$.

The $p = 0$ case, which has been verified say in the plane and for unconditional bodies in $\mathbb{R}^n$, is the celebrated so-called logarithmic Brunn-Minkowski conjecture. It yields (2) for $p \in [0,1)$ and o-symmetric convex bodies K, C not only for Lebesgue measure but for any even log-concave measure according to Saraglou, and also yields the Gardner-Zvavitch conjecture for Gaussian measure (see ([5]).

The last months saw exiting new developments related to the logarithmic Brunn-Minkowski conjecture. Kolesnikov, Milman [6] considered the renormalized Hilbert operator, and proved the existence of $p_n \in (0,1)$, $p_n > 1 - \frac{c}{n^{3/2}}$ such that (2) and the uniqueness of the smooth even L_p-Minkowski problem holds in some C^2 neighbourhood of any o-symmetric convex body M with C^2_+ boundary. Extending this result *via* Schauder theory in PDE, Chen, Huang, Li, Liu [3] verified (2) for $p \in (p_n, 1)$ and any o-symmetric convex bodies K, C by establishing he uniqueness of the solution of the even smooth L_p-Minkowski problem. Using also some estimates in [6], Kolesnikov, Livshyts [5] proved a weaker form

$$\gamma((1-\lambda)K + \lambda\, C)^{\frac{1}{2n}} \geq (1-\lambda)\gamma(K)^{\frac{1}{2n}} + \lambda\, \gamma(C)^{\frac{1}{2n}}$$

of the Gardner-Zvavitch conjecture where γ is the standard Gaussian measure and K and C are o-symmetric convex bodies.

For the logarithmic Brunn-Minkowski conjecture itself, Kolesnikov [4] provided the optimal tranportation set up. In addition, again extending some result in [6], [3] proved (2) if K is C^2-close to the unit ball and both K and C are o-symmetric.

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Rotational Crofton formulae for Minkowski tensors

Eva B. Vedel Jensen
(joint work with Anne Marie Svane)

Motivated by applications in local stereology, we consider rotational Crofton formulae for Minkowski tensors. First, we discuss the following special case

$$\int_{G(n,q)} V_j(K \cap L)\, \nu_q(\mathrm{d}L), \quad K \in \mathcal{K}^n, \tag{1}$$

$q = 1, \dots, n-1$, $j = 0, \dots, q$, where $G(n,q)$ is the set of q-dimensional linear subspaces of $\mathbb{R}^n$, V_j is the intrinsic volume of order j and ν_q is the unique rotation invariant probability measure on $G(n,q)$. For $j = q$, (1) can be expressed as a simple integral with respect to Lebesgue measure in $\mathbb{R}^n$, while for $j < q$ it can be shown that (1) is an integral over the normal bundle $\operatorname{nor} K$ with respect to the $(n-1)$th support measure of K and with an integrand not only depending on $x \in \operatorname{bd} K$ and an outer unit normal of K at x, but also on the principal directions.

These results have recently been generalized to the case of Minkowski tensors ([2]). More specifically, integrals of the form (1) with V_j replaced by (the intrinsically defined) Minkowski tensor $\Phi_{jL}^{r,s}$ of order j and rank r, s have been studied in [2]. Again, the case $j = q$ has a simple solution. For $j = q - 1$, an explicit expression can be derived, involving hypergeometric functions. We also discuss briefly the 'opposite problem' of expressing a Minkowski tensor $\Phi_j^{r,s}$, defined in $\mathbb{R}^n$, as a rotational integral. This problem has partly been solved in [1]. Finally, it is shown how the results can be used in the stereological analysis of particle processes.

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Improved bounds for Hadwiger's covering problem via thin-shell estimates

Boaz A. Slomka
(joint work with Han Huang, Tomasz Tkocz, Beatrice-Helen Vritsiou)

A central problem in discrete geometry, known as Hadwiger's covering problem, asks what is the smallest natural number $N(n)$ such that every convex body in $\mathbb{R}^n$ can be covered by a union of the interiors of at most $N(n)$ of its translates. It is conjectured that $N(n) \le 2^n$ where 2^n translates are needed only for parallelotpes.

This problem was posed by Hadwiger [6] for $n \ge 3$ but was already considered and settled for $n = 2$ a few years earlier by Levi [7]. An equivalent formulation, in which the interior of the convex body is replaced by smaller homothetic copies of it, was independently posed by Gohberg and Markus [4]. For a comprehensive survey of this problem and most of the progress made so far towards its solution see e.g. [2, 3, 9].

Despite continuous efforts, the best general upper bound known for this number remains as it was more than sixty years ago, of the order of $\binom{2n}{n} n \ln n$ (which is a consequence of Rogers' estimate [10] and the Rogers-Shephard inequaity [11]).

In this talk, I will present a new result in which we improve this bound by a sub-exponential factor. That is, we prove a bound of the order of $\binom{2n}{n} e^{-c\sqrt{n}}$ for some universal constant $c > 0$.

Our approach combines ideas from [1] with tools from asymptotic geometric analysis. One of the key steps is proving a new lower bound for $\Delta_{KB}(K)$, the Kövner-Besicovitch measure of symmetry for a convex body $K \subset \mathbb{R}^n$. Namely, we prove that for some universal constant $c > 0$ and every convex body $K \subset \mathbb{R}^n$,

$$\Delta_{KB}(K) := \max_{x \in \mathbb{R}^n} \frac{\mathrm{Vol}((x - K) \cap (K - x))}{\mathrm{Vol}(K)} \geq 2^{-n} e^{c\sqrt{n}}.$$

The proof of this lower bound involves using the property of an isotropic log-concave measure to concentrate in a thin shell, and in particular a quantitative form of it by Guédon and E. Milman [5]. In fact, we are able to establish the same lower bound for $\mathrm{Vol}(K \cap (-K))/\mathrm{Vol}(K)$ when the barycenter of K is at the origin, thus improving a previous result of V. Milman and Pajor [8]. This is done by combining the above mentioned thin-shell estimates with the notion of entropy.

Using the same ideas, we establish an exponentially better bound for $N(n)$ when restricting our attention to convex bodies that are ψ_2. By a slightly different approach, an exponential improvement is established also for classes of convex bodies with positive modulus of convexity.

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On the maximal perimeter of hyperplane sections of the cube

Alexander Koldobsky
(joint work with Hermann König)

A well-known result of Ball [1] states that the hyperplane section of the n-dimensional unit cube $B_\infty^n = [-\frac{1}{2}, \frac{1}{2}]^n$ perpendicular to $a_{max} := \frac{1}{\sqrt{2}}(1, 1, 0, \dots, 0)$ has the maximal $(n-1)$-dimensional volume among all hyperplane sections, i.e. for any $a = (a_1, ..., a_n) \in S^{n-1} \subset \mathbb{R}^n$

$$\mathrm{vol}_{n-1}(B_\infty^n \cap a^\perp) \le \mathrm{vol}_{n-1}(B_\infty^n \cap a_{max}^\perp) = \sqrt{2},$$

where $a^\perp$ is the central hyperplane orthogonal to a.

Pełczyński [3] asked whether the same hyperplane section is also maximal for intersections with the boundary of the n-cube, i.e. whether for all $a \in S^{n-1} \subset \mathbb{R}^n$

$$\mathrm{vol}_{n-2}(\partial B_\infty^n \cap a^\perp) \le \mathrm{vol}_{n-2}(\partial B_\infty^n \cap a_{max}^\perp) = 2((n-2)\sqrt{2} + 1).$$

He proved it for $n = 3$ when $\mathrm{vol}_1(\partial B_\infty^3 \cap a^\perp)$ is the perimeter of the quadrangle or hexagon of intersection. We answer Pełczyński's question affirmatively for all $n \ge 3$.

Ball used his result to prove that the answer to the Busemann-Petty problem is negative in dimensions 10 and higher. The Busemann-Petty problem asks the following question. Suppose that origin-symmetric convex bodies K, L in $\mathbb{R}^n$ satisfy $\mathrm{vol}_{n-1}(K \cap a^\perp) \le \mathrm{vol}_{n-1}(L \cap a^\perp)$ for all $a \in S^{n-1}$. Does it follow that the n-dimensional volume of K is smaller than that of L, i.e. $\mathrm{vol}_n K \le \mathrm{vol}_n L$? The problem was solved as the result of work of many mathematicians, and the answer is affirmative for $n \le 4$, and it is negative for $n \ge 5$. We refer to the monograph [2] for details.

Ball's result was one of the steps of the solution. He showed that the answer is negative when $n \ge 10$, K is the unit cube and L is the Euclidean ball in $\mathbb{R}^n$ whose radius is chosen so that the $(n-1)$-dimensional volume of central hyperplane sections is equal to $\sqrt{2}$.

We consider the following analogue of the Busemann-Petty problem for the surface area. Suppose that origin-symmetric convex bodies K, L in $\mathbb{R}^n$ satisfy

$$\mathrm{vol}_{n-2}(\partial K \cap a^\perp) \le \mathrm{vol}_{n-2}(\partial L \cap a^\perp)$$

for all $a \in S^{n-1}$, i.e. the surface area (perimeter) of every central hyperplane section of K is smaller than the same for L. Does it follow that the surface area of K is smaller than that of L, i.e.

$$\mathrm{vol}_{n-1}(\partial K) \le \mathrm{vol}_{n-1}(\partial L)?$$

We prove that the answer is negative for $n \geq 14$, with K being the unit cube and L the Euclidean ball whose radius is chosen so that the perimeter of every central hyperplane section is $2((n-2)\sqrt{2}+1)$.

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Reciprocals and flowers in convexity

Vitali Milman

(joint work with Emanuel Milman, Liran Rotem)

We study new classes of convex bodies and star bodies with unusual properties. First we define the class of reciprocal bodies, which may be viewed as convex bodies of the form “$1/K$”. The map $K \mapsto K'$ sending a body to its reciprocal is a duality on the class of reciprocal bodies, and we study its properties.

To connect this new map with the classic polarity we use another construction, associating to each convex body K a star body which we call its flower and denote by $K^{\clubsuit}$. Let B_x denote the Euclidean ball with center $\frac{x}{2}$ and radius $\frac{|x|}{2}$, or equivalently with the interval $[0, x]$ as a diameter. Then the flower $K^{\clubsuit}$ is defined by

$$K^{\clubsuit} = \bigcup_{x \in K} B_x.$$

The mapping $K \mapsto K^{\clubsuit}$ is a bijection between the class $\mathcal{K}_0^n$ of convex bodies and the class $\mathcal{F}^n$ of flowers. Even though flowers are in general not convex, their study is very useful to the study of convex geometry. For example, we show that the polarity map $\circ : \mathcal{K}_0^n \to \mathcal{K}_0^n$ decomposes into two separate bijections: First our flower map $\clubsuit : \mathcal{K}_0^n \to \mathcal{F}^n$, followed by a slight modification Φ of the spherical inversion which maps $\mathcal{F}^n$ back to $\mathcal{K}_0^n$. Each of these maps has its own properties, which combine to create the various properties of the polarity map.

We study the various relations between the four maps $\prime$, $\circ$, $\clubsuit$ and Φ and use these relations to derive some of their properties. For example, we show that a convex body K is a reciprocal body if and only if its flower $K^{\clubsuit}$ is convex.

We show that the class $\mathcal{F}^n$ has a very rich structure, and is closed under many operations, including the Minkowski addition. This structure has corollaries for the other maps which we study. For example, we show that if K and T are reciprocal bodies so is their “harmonic sum” $(K^\circ + T^\circ)^\circ$. We also show that the volume $\left|\left(\sum_i \lambda_i K_i\right)^{\clubsuit}\right|$ is a homogeneous polynomial in the λ_i's, whose coefficients can be called “$\clubsuit$-type mixed volumes”. These mixed volumes satisfy natural geometric inequalities, such as an elliptic Alexandrov-Fenchel inequality. More geometric inequalities are also derived.

Gaussian concentration and convexity

Petros Valettas
(joint work with Grigoris Paouris, Konstantin Tikhomirov)

The concentration of measure phenomenon is considered by now an indispensable tool for the study of high-dimensional structures. For context let us recall the classical concentration inequality in Gauss' space, proved independently in [1, 10].

Theorem 1 (Borell, 1975, Sudakov, Tsirel'son, 1974)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an L-Lipschitz map. Then, for every $t > 0$*

$$(1)\qquad \max\{\mathbb{P}(f(G) - \mathbb{E}f(G) \geq tL), \mathbb{P}(f(G) - \mathbb{E}f(G) \leq -tL)\} \leq \exp(-t^2/2),$$

where G is the standard Gaussian vector in $\mathbb{R}^n$.

Nowadays, this fundamental inequality is customary addressed as consequence of the Gaussian isoperimetric principle. For f being a norm, it is known (see [5, Corollary 3.2]) that the estimate is optimal (up to universal constants) in the large deviation regime, i.e. for $t \geq \mathbb{E}f(G)/L$. For all its qualities the concentration (of norms) in terms of the Lipschitz constant has a drawback: it does not yield bounds of optimal order in many key situations. However, one-sided improvements can always be obtained in the presence of convexity. More precisely, for f *convex* (not necessarily Lipschitz) the Gaussian isoperimetry implies (see [9, §5.2] for a proof)

$$(2)\qquad \mathbb{P}\left(f(G) - \mathbb{E}f(G) \leq -t\sqrt{\mathbb{E}\|\nabla f(G)\|_2^2}\right) \leq \exp(-t^2/2), \quad t > 0.$$

Clearly, the latter improves the lower tail in (1), since $\mathbb{E}\|\nabla f(G)\|_2^2 \leq L^2$. Further refinements can be achieved, if we additionally exploit the convexity properties of the Gaussian distribution. Namely, using the Gaussian analogue of Brunn-Minkowski inequality due to Ehrhard [3], we obtain in [8] the following (see also [11] for an alternative proof which yields its sharp form):

Theorem 2 (Paouris, Valettas, 2016)**.** *Let f be a convex function in $\mathbb{R}^n$. Then,*

$$\mathbb{P}\left(f(G) - \mathbb{E}f(G) \leq -t\sqrt{\mathrm{Var}[f(G)]}\right) \leq \exp(-t^2/(4\pi)), \quad t > 0.$$

Main features of this inequality are: (a) it demonstrates a new type of concentration, which is explained by convexity rather than isoperimetry as opposed to the previous cases, and (b) it improves upon (2), in view of the classical Poincaré inequality $\mathrm{Var}[f(G)] \leq \mathbb{E}\|\nabla f(G)\|_2^2$, thus exploiting the *superconcentration* phenomenon (following Chatterjee [2]) whenever occurs. Further, the, intuitively clear, fact that a convex function of a Gaussian vector exhibits skew behavior was rigorously established in [11], again as application of Gaussian convexity.

Proposition 1 (Valettas, 2017)**.** *Let f be a convex function in $\mathbb{R}^n$. Then,*

$$\mathbb{P}(f(G) \leq \mathrm{med}(f(G)) - t) \leq \mathbb{P}(f(G) > \mathrm{med}(f(G)) + t), \quad t > 0.$$

The (improved) distributional inequalities discussed so far, provide strong evidence that the concentration for convex functions (in particular for norms) below the mean is more drastic. This behavior had only been confirmed in concrete problems-examples using ad-hoc methods. A comprehensive approach, which also highlights the underlying principles for this previously unexplored phenomenon, is treated in a recent joint work with G. Paouris and K. Tikhomirov [7]. A sample of findings from this work can be summarized in the following:

Theorem 3 (Paouris, Tikhomirov, Valettas, 2018)**.** *Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$.*

(1) If $\|\cdot\|$ is 1-unconditional with $\mathbb{E}|\partial_i\|G\|| = \mathbb{E}|\partial_j\|G\||$ for $i,j = 1,\ldots,n$, then for any $\delta \in (0,1/2)$ one has

$$\mathbb{P}(\|G\| \leq c\delta\mathbb{E}\|G\|) \leq \exp(-cn^{1-c\delta^2}), \quad \mathbb{P}(\|G\| \leq (1-\delta)\mathbb{E}\|G\|) \leq \exp(-n^{c\delta}).$$

(2) For the general case, there exists a linear map T such that for $\delta \in (0,1/2)$

$$\mathbb{P}(\|TG\| \leq c\delta\mathbb{E}\|TG\|) \leq \exp(-cn^{\frac{1}{4}-c\delta^2}),$$
$$\mathbb{P}(\|TG\| \leq (1-\delta)\mathbb{E}\|TG\|) \leq \exp(-n^{c\delta}),$$

where $c > 0$ is a universal constant.

The link between lower-deviation estimates and Dvoretzky-type results is well known, see [4, 6]. For example, it follows from Theorem 3 (and Alexandrov's inequality) that any symmetric convex body K in $\mathbb{R}^n$ admits a linear image $\tilde{K}$ such that all the (normalized) quermassintegrals of $\tilde{K}$ up to polynomial order are comparable. More precisely, for $k \leq cn^{1/5}$ one has

$$W_{[k]}(\tilde{K}) := \left(\frac{1}{|B_2^k|}\int_{G_{n,k}} |P_F\tilde{K}|\, d\nu_{n,k}(F)\right)^{1/k} \geq cW_{[1]}(\tilde{K}).$$

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Angular curvature measures

Thomas Wannerer

The curvature measures Federer introduced in his seminal work [8] on the curvature of non-smooth subsets of $\mathbb{R}^n$ take on a particularly simple form in the special case of convex polytopes:

$$\Phi_k(P,U) = \sum_F \gamma(F,P)\,\mathrm{vol}_k(F\cap U),$$

where $P\subset\mathbb{R}^n$ is a polytope, $0\le k\le n$ is an integer, $U\subset\mathbb{R}^n$ is a Borel subset, the sum extends over all k-faces of P, and $\gamma(F,P)$ is the external angle of P at the face F. Given any function f on the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$, one may consider the weighted sums

$$\Phi(P,U) = \sum_F f(\overline{F})\gamma(F,P)\,\mathrm{vol}_k(F\cap U), \tag{1}$$

where the sum is over all k-faces of P and $\overline{F}$ is the translate of the affine hull of F containing the origin. The obvious question arises whether such expressions can be extended to curvature measures of more general subsets of $\mathbb{R}^n$; any linear combination of such curvature measures is called angular.

From a different perspective, angular curvature measures arise naturally in the theory of valuations on Riemannian manifolds. Over the last decade it has become clear that the theory of valuations on convex bodies, a classical line of research in convex geometry, admits a natural continuation in the setting of general smooth manifolds. According to the pioneering work of Alesker [1, 2, 3, 4, 5], to each smooth manifold M is associated the commutative filtered algebra of smooth valuations $\mathcal{V}(M)$ on M, with the Euler characteristic χ as multiplicative identity. Loosely speaking, smooth valuations on M are finitely additive set functions satisfying a smoothness condition and the Alesker product of valuations reflects the operation of intersection of subsets of M. It was soon realized that this new structure can be used to solve classical problems in integral geometry: the description of explicit kinematic formulas in complex space forms – a problem first taken up by Blaschke and his school in the 1930s with special cases solved by Santaló, Gray, Shifrin, and others – had to wait until 2014 when it was finally found by Bernig, Fu, and Solanes [6] using the new tools from valuation theory introduced by Alesker.

Smooth valuations may be localized, albeit non-uniquely. The resulting space of smooth curvature measures on M, denoted by $\mathcal{C}(M)$, is naturally a module over

$\mathcal{V}(M)$ with respect to the Alesker product. Bernig, Fu, and Solanes [6] observed that a Riemannian metric on M induces a canonical isomorphism

$$\tau\colon \mathcal{C}(M) \to \Gamma(\mathrm{Curv}(TM))$$

between smooth curvature measures on M and smooth sections of the bundle of translations-invariant smooth curvature measures on the tangent spaces of M.

Following [6] this allows us to make the following definition: a curvature measure on M is called angular if $\tau_p\Phi$ is angular for every point $p \in M$. Let $\mathcal{A}(M)$ denote the space of angular curvature measures on M. It may seem surprising, but there are natural curvature measures, e.g., arising in hermitian integral geometry [6], that are not angular.

The intrinsic volumes of a convex body play a fundamental role in convex geometry. Their extension to Riemannian manifolds are the Lipschitz-Killing valuations. Their existence is non-trivial: Alesker first observed that it follows from a classical theorem of H. Weyl on the volume of tubes in combination with the Nash embedding theorem; an alternative approach in the spirit of Chern's intrinsic proof of the Gauss-Bonnet theorem can be found in [9]. The closely related Lipschitz-Killing curvatures of M have remarkable properties; they arise for example in the asymptotic expansion of the trace of the heat kernel and they converge under approximations of a Riemannian manifold by a piecewise linear one. The conjectures presented in the next paragraph shed new light on the geometric meaning of the Lipschitz-Killing valuations.

The Lipschitz-Killing valuations form a finite-dimensional subalgebra $\mathcal{LK}(M)$ of $\mathcal{V}(M)$. Motivated by their results on the integral geometry of complex space forms, Bernig, Fu, and Solanes [6] formulated the following

Angularity conjecture. *Let M be a Riemannian manifold. Then $\mathcal{A}(M)$ is invariant under the action of the Lipschitz-Killing algebra,*

$$\mathcal{LK}(M) \cdot \mathcal{A}(M) \subset \mathcal{A}(M)$$

Here $\cdot$ denotes the Alesker product. We call a valuation μ angular if $\mu \cdot \mathcal{A}(M) \subset \mathcal{A}(M)$. The angularity conjecture states that the Lipschitz-Killing valuations are angular. Bernig, Fu, and Solanes [6] conjecture that this property even characterizes the Lipschitz-Killing valuations:

Conjecture 1. *The algebra of angular valuations on M equals $\mathcal{LK}(M)$.*

In the presence of additional invariance assumptions the angularity conjecture is known to be true in the following special cases: translation-invariant curvature measures on $\mathbb{R}^n$ and isometry-invariant curvature measures in complex projective space $\mathbb{C}P^n$. Both results are contained in [6]. Also for Conjecture 1 the integral geometry of complex space forms provides evidence [7].

The main result presented in the talk is

Theorem 1 ([10])**.** *The angularity conjecture is true.*

The proof of Theorem 1 relies on a complete classification of translation-invariant angular curvature measures on $\mathbb{R}^n$, Theorem 2 below. Clearly, every angular curvature measure (1) is even in the sense that $\Phi(-P,-U)=\Phi(P,U)$. It is not difficult to see that for $k=n-1$ this is the only restriction and thus (1) extends for every smooth function f to a smooth curvature measure. Choosing f to be constant yields the well-known curvature measures introduced by Federer [8]. Further examples of angular curvature measures of degree $k<n-1$ are harder to come by. A whole family of examples are the constant coefficient curvature measures introduced by Bernig, Fu, and Solanes [6].

Let $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)$ denote the oriented Grassmannian, the manifold of oriented k-dimensional linear subspaces of $\mathbb{R}^n$. We call a function on $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)$ even if it is invariant under change of orientation. Note that even functions on the oriented Grassmannian $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)$ correspond bijectively to functions on $\mathrm{Gr}_k(\mathbb{R}^n)$. The oriented Grassmannian smoothly embeds into the exterior power $\bigwedge^k\mathbb{R}^n$ as $E\mapsto\vec{E}$, where $\vec{E}=e_1\wedge\cdots\wedge e_k$ for some positively oriented orthonormal basis of E. This map is called the Plücker embedding.

Theorem 2 ([10])**.** *Let $0\leq k<n-1$ be an integer and f be a function on $\mathrm{Gr}_k(\mathbb{R}^n)$. Then (1) extends to a translation-invariant smooth curvature measure on $\mathbb{R}^n$ if and only if f is the restriction of a 2-homogeneous polynomial on $\bigwedge^k\mathbb{R}^n$ to the image of the Plücker embedding. Consequently, the space of translation-invariant angular curvature measures of degree k has dimension*

$$\frac{1}{n-k+1}\binom{n}{k}\binom{n+1}{k+1}$$

and coincides with the space of constant coefficient curvature measures.

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Applications of Grünbaum-type inequalities

Vlad Yaskin

(joint work with Matthew Stephen)

Let K be a convex body in $\mathbb{R}^n$. The centroid of K is the point

$$\frac{1}{\mathrm{vol}_n(K)} \int_K x \,\mathrm{d}x \in \mathrm{int}(K).$$

Makai and Martini [4] conjectured the following: for integers $1 \leq k < n$, any convex body $K \subset \mathbb{R}^n$ with centroid at the origin, and any k-dimensional subspace $E \in G(n,k)$,

$$\mathrm{vol}_k(K \cap E) \geq \left(\frac{k+1}{n+1}\right)^k \max_{x \in K} \mathrm{vol}_k\big((K-x) \cap E\big). \tag{1}$$

They were able to prove (1) for $k = 1,\ n-1$. Shortly thereafter, Fradelizi [1] proved the conjecture for all k, including sharpness and a complete characterization of the equality conditions.

We generalize (1) to intrinsic and dual volumes. Recall that intrinsic volumes arise as the coefficients in the Steiner formula. For a convex and compact set $L \subset \mathbb{R}^n$ and the n-dimensional Euclidean ball B_2^n with unit radius, Steiner's formula expands the volume of the Minkowski sum $L + tB_2^n$ into a polynomial of t:

$$\mathrm{vol}_n\big(L + tB_2^n\big) = \sum_{i=0}^{n} \kappa_{n-i} V_i(L) t^{n-i} \qquad \forall\ t \geq 0.$$

The coefficient $V_i(L)$ is the ith intrinsic volume of L, and κ_{n-i} denotes the $(n-i)$-dimensional volume of B_2^{n-i}. We prove the following:

Theorem 1. *Consider integers $1 \leq i \leq k < n$. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin, and let $E \in G(n,k)$. Then*

$$V_i(K \cap E) \geq \left(\frac{i+1}{n+1}\right)^i \max_{x \in K} V_i\big((K-x) \cap E\big). \tag{2}$$

The constant in this inequality is the best possible.

The radial sum of a star body $L \subset \mathbb{R}^n$ with the ball tB_2^n of radius $t > 0$ is the star body $L \widetilde{+} tB_2^n$ whose radial function is equal to $\rho_L(\xi) + t$ for all $\xi \in S^{n-1}$, where ρ_L is the radial function of L. The dual Steiner formula expands the volume of $L \widetilde{+} tB_2^n$ into a polynomial of t:

$$\mathrm{vol}_n\big(L \widetilde{+} tB_2^n\big) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{V}_i(L) t^{n-i} \qquad \forall\ t \geq 0.$$

The coefficient $\widetilde{V}_i(L)$ is the ith dual volume of L. We prove the following:

Theorem 2. *Consider integers $1 \le i \le k < n$. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin, and let $E \in G(n,k)$. Then*

$$\widetilde{V}_i\big(K \cap E\big) \ge \left(\frac{i+1}{n+1}\right)^i \max_{x \in K} \widetilde{V}_i\big((K-x) \cap E\big), \tag{3}$$

where the dual volumes are taken within the k-dimensional subspace E. The constant in this inequality is the best possible.

One of the main ingredients in the proofs of Theorem 1 and Theorem 2 is "Grünbaum's inequality for sections", due to Myroshnychenko, Stephen, and Zhang [6], which says the following: for integers $1 \le k \le n$, a convex body $K \subset \mathbb{R}^n$ with centroid at the origin, and $E \in G(n,k)$,

$$\mathrm{vol}_k(K \cap E \cap \xi^+) \ge \left(\frac{k}{n+1}\right)^k \mathrm{vol}_k(K \cap E) \quad \text{for all} \quad \xi \in S^{n-1} \cap E. \tag{4}$$

The latter inequality is part of a series of results dedicated to Grünbaum-type inequalities. The reader is referred to [3], [2], [7], [5] for previous results on this topic, including the original paper of Grünbaum. In particular, it is worth mentioning "Grünbaum's inequality for projections",

$$\mathrm{vol}_k\big((K|E) \cap \xi^+\big) \ge \left(\frac{k}{n+1}\right)^k \mathrm{vol}_k(K|E) \quad \text{for all} \quad \xi \in S^{n-1} \cap E, \tag{5}$$

which was proved in [7].

We prove an analogue of (4) and (5) for dual volumes.

Theorem 3. *Consider integers $1 \le i \le k \le n$. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin, and let $E \in G(n,k)$. Then*

$$\widetilde{V}_i(K \cap E \cap \xi^+) \ge \left(\frac{i}{n+1}\right)^i \widetilde{V}_i(K \cap E) \tag{6}$$

$$\text{and} \qquad \widetilde{V}_i\big((K|E) \cap \xi^+\big) \ge \left(\frac{i}{n+1}\right)^i \widetilde{V}_i(K|E) \tag{7}$$

for all $\xi \in S^{n-1} \cap E$, where the dual volumes are taken within the k-dimensional subspace E. The constant in each inequality is the best possible.

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The convex hull of random points on the boundary of a simple polytope

Elisabeth Werner
(joint work with Matthias Reitzner, Carsten Schütt)

The convex hull of N independent random points chosen on the boundary of a simple polytope in $\mathbb{R}^n$ is investigated. Asymptotic formulas for the expected number of vertices and facets, and for the expectation of the volume difference are derived. This is the first successful attempt of investigations which lead to rigorous results for random polytopes which are neither simple nor simplicial. The results contrast existing results when points are chosen in the interior of a convex set.

Choosing random points from the interior of a convex set always produces a simplicial polytope with probability one. Yet often applications in computational geometry, the analysis of the average complexity of algorithms and optimization necessarily deal with non simplicial polytopes and the question became important if there are analogous results for random polytopes without this very specific combinatorial structure. In this paper we are discussing the case that the points are chosen from the boundary of a simple polytope P. This produces random polytopes which are neither simple nor simplicial and thus our results are a huge step in taking into account the first point mentioned above. The applications in computational geometry, the analysis of the average complexity of algorithms and optimization need formulas for the combinatorial structure of the involved random polytopes and thus the question on the number of facets f_{n-1} and vertices f_0 and the expected volume V_n are of interest. This is the content of our theorem.

Theorem 1 ([1]). *Choose N uniform random points on the boundary of a simple polytope P. Let P_N be the convex hull of the N randomly chosen points. For the expected number of vertices and facets of the random polytope P_N, we have*

$$\mathbb{E}f_0(P_N) \quad = \quad c_{n,0} f_0(P)(\ln N)^{n-2}(1+o(1)),$$

and

$$\mathbb{E}f_{n-1}(P_N) \quad = \quad c_{n,n-1} f_0(P)(\ln N)^{n-2}(1+o(1)),$$

with some $c_{n,0} > 0$ and $c_{n,n-1} > 0$. For the expected volume difference between P and the random polytope P_N we have

$$\mathbb{E}(V_n(P) - V_n(P_N)) = c_{n,P} N^{-\frac{n}{n-1}}(1+o(1))$$

with some $c_{n,P} > 0$.

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The Gaussian double-bubble and multi-bubble conjectures

Emanuel Milman
(joint work with Joe Neeman)

The classical Gaussian isoperimetric inequality, established in the 70s independently by Sudakov-Tsirelson and Borell, states that the optimal way to decompose $\mathbb{R}^n$ into two sets of prescribed Gaussian measure, so that the (Gaussian) area of their interface is minimal, is by using two complementing half-planes. This is the Gaussian analogue of the classical Euclidean isoperimetric inequality, and is therefore referred to as the "single-bubble case.

A natural generalization is to decompose $\mathbb{R}^n$ into $q \geq 3$ sets of prescribed Gaussian measure. It is conjectured that when $q \leq n+1$, the configuration whose interface has minimal (Gaussian) area is given by the Voronoi cells of q equidistant points. For example, for $q = 3$ (the "double-bubble conjecture) in the plane ($n = 2$), the interface is conjectured to be a "tripod or "Y - three rays meeting at a single point in 120 degree angles. For $q = 4$ (the "triple-bubble conjecture) in $\mathbb{R}^3$, the interface is conjectured to be a tetrahedral cone.

We confirm the Gaussian double-bubble and, more generally, multi-bubble conjectures for all $3 \leq q \leq n+1$. The double-bubble case $q = 3$ is simpler, and we will explain why. None of the numerous methods discovered over the years for establishing the classical $q = 2$ case seem amenable to the $q \geq 3$ cases, and our method consists of establishing a matrix-valued partial differential inequality satisfied by the isoperimetric profile. To treat $q > 3$, we first prove that locally minimal ("stable") configurations must have flat interfaces, and thus convex polyhedral cells. Uniqueness of minimizers up to null-sets is also established.

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Local and general inequalities for projections

Silouanos Brazitikos

The core of the talk was to study local and functional forms of the Loomis-Whitney inequality, which compares the volume $|K|$ of a convex body K in $\mathbb{R}^n$ with the geometric mean of the volumes $|P_i(K)|$ of its orthogonal projections onto $e_i^\perp$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$.

One local analogue is for example that if K is a convex body in $\mathbb{R}^n$ and $u, v \in S^{n-1}$. If $P_{u,v}(K) = P_{\mathrm{span}\{u,v\}^\perp}(K)$, then

$$|P_u(K)|\,|P_v(K)| \geq \frac{n}{2(n-1)}|u \wedge v|\,|K|\,|P_{u,v}(K)|,$$

where $u \wedge v$ is the area of the parallelogram formed by u, v.

We then discuss general versions, for more than two projections and even for mixed volumes, and ask for functional analogues, as well as, some consequences of all these inequalities.

A sausage body is a unique solution to the reverse isoperimetric problem

Kateryna Tatarko

(joint work with Roman Chernov, Kostiantyn Drach)

The classical isoperimetric problem asks which domain, among all domains with a fixed surface area, has maximal volume. The question has a long and beautiful history, and has been generalized to a variety of different settings (see [5, 9]). In particular, among bodies in $\mathbb{R}^{n+1}$ with a fixed surface area, the isoperimetric inequality asserts that the Euclidean ball has the largest possible volume.

On the other hand, one can state the reverse isoperimetric problem: under which conditions can one minimize the volume among all domains of a given constraint. In order to avoid trivial solutions, one must consider a family of sets with additional conditions imposed on it. For example, one natural constraint to consider is convexity or strict convexity. The answer to the reverse isoperimetric problem in the family of affine equivalence classes of convex bodies was given by K. Ball in his celebrated results [1, 2]. He showed that among all convex bodies in $\mathbb{R}^{n+1}$ (modulo affine transformations), the standard simplex has the smallest volume for a given surface area. The necessity condition in the equality case was settled by Barthe [3].

A different approach is to consider some curvature constraints for the boundary (see, for example, [4, 7, 8]). We study the class of λ-concave bodies in $\mathbb{R}^{n+1}$; that is, convex bodies with the property that each of their boundary points supports a tangent ball of radius $\frac{1}{\lambda}$ that lies locally (around the boundary point) inside the body. If the boundary ∂K of a convex body K is at least C^2-smooth, then K is λ-concave if and only if the principal curvatures $k_i(p)$ for all $i \in \{1, \ldots, n\}$ are non-negative and uniformly bounded above by λ, i.e. $0 \leq k_i(p) \leq \lambda$ for every i and $p \in \partial K$.

For λ-concave bodies we completely solve the reverse isoperimetric problem in any dimension. We deduce this result by proving a more general reverse Bonnesen-style family of quersmassintegral inequalities for λ-concave bodies.

Recall that for a convex body $K \subset \mathbb{R}^{n+1}$ quermassintegrals $W_i(K)$ arise as coefficients in the polynomial expansion

$$\mathrm{Vol}_{n+1}(K+tB) = \sum_{i=0}^{n+1} \binom{n+1}{i} W_i(K) t^i$$

known as the Steiner formula; here B is the unit Euclidean ball in $\mathbb{R}^{n+1}$ and '+' stands for the Minkowski addition. In particular, $W_0(K) = \mathrm{Vol}_{n+1}(K)$, $W_1(K) = \mathrm{Vol}_n(\partial K)/(n+1)$ and $W_{n+1}(K) = \omega_{n+1}$, where ω_{n+1} is the volume of the Euclidean unit ball in $\mathbb{R}^{n+1}$.

Definition 1 (λ-sausage body)**.** A λ-*sausage body* in $\mathbb{R}^{n+1}$ is the convex hull of two balls of radius $1/\lambda$.

We are now ready to state the main results.

Theorem 1 (Reverse quermassintegrals inequality for λ-concave bodies)**.** *Let $K \subset \mathbb{R}^{n+1}$ be a convex body. If K is λ-concave, then*

$$(k-j)\frac{W_i(K)}{\lambda^i} + (i-k)\frac{W_j(K)}{\lambda^j} + (j-i)\frac{W_k(K)}{\lambda^k} \geq 0$$

for every triple (i,j,k) with $0 \leq i < j < k \leq n+1$. Moreover, equality holds if and only if K is a λ-sausage body.

Taking the triple $(i,j,k) = (0,1,n+1)$, we immediately obtain the following result

Theorem 2 (Reverse isoperimetric inequality for λ-concave bodies)**.** *Let $K \subset \mathbb{R}^{n+1}$ be a convex body. If K is λ-concave (for some $\lambda > 0$), then*

$$\mathrm{Vol}_{n+1}(K) \geq \frac{\mathrm{Vol}_n(\partial K)}{n\lambda} - \frac{\omega_{n+1}}{n\lambda^{n+1}}, \tag{1}$$

where ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$. Moreover, equality holds if and only if K is a λ-sausage body.

For all proofs and more details we refer to [6].

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A PDE approach to geometric inequalities

Alina Stancu

The idea of using curvature flows to derive geometric inequalities goes back to mid-nineties, shortly after these flows were introduced. A somewhat classical example is that of Andrews' use of asymptotic behavior of the flow of convex hypersurfaces along their affine normal to obtain a new proof of the affine isoperimetric inequality for smooth, strictly convex bodies in $\mathbb{R}^n$. By showing that each normalized, smooth, strictly convex hypersurface evolving with speed determined by their affine normal converges in the $\mathcal{C}^\infty$-norm to an ellipsoid, and that, for any initial hypersurface the affine isoperimetric ratio is increasing during the flow, the inequality follows, see Theorem 7.1, [1]. The same paper provides with similar reasoning a new proof of the Blaschke-Santaló inequality for smooth, strictly convex bodies in $\mathbb{R}^n$, see Theorem 7.3, [1].

In what follows, we will exploit similar phenomena for a centro-affine invariant flow. We will fix a smooth, centrally symmetric, strictly convex reference body $\tilde{C}$ in $\mathbb{R}^n$ and define a flow on the boundary of any other smooth, centrally symmetric, strictly convex body C which is weighted in the direction of its affine normal by a power of the centro-affine normal curvature of $\tilde{C}$. As each normalized, smooth, strictly convex hypersurface evolving by this flow converges in the $\mathcal{C}^\infty$-norm to a convex hypersurface with same centro-affine curvature as that of $\tilde{C}$, we use the monotonicity of a centro-affine functional to conclude a geometric inequality for which further corollaries are investigated.

To state the results, we first start with a list of notations. All convex bodies, denoted by C or other variants, will contain the origin in their interior and will be parameterized by their support function as a function on the unit sphere, $h_C : \mathbb{S}^{n-1} \to [0, \infty)$, $h(u) = \max_{x \in C} x \cdot u$, where the latter is usual scalar product in $\mathbb{R}^n$. Let $\kappa_C(u)$ be the Gauss curvature at the boundary point of unit normal u and position vector $x(u)$, thus overall a function on the unit sphere $\kappa_C : \mathbb{S}^{n-1} \to [0, \infty)$, as well as the centro-affine curvature $\kappa_{C,0} : \mathbb{S}^{n-1} \to [0, \infty)$ defined pointwise simply as the ratio $\kappa_{C,0}(u) = \frac{\kappa_C(u)}{h_C^{n+1}(u)}$. Further, we may drop the index C identifying the convex body, as well as omit the point u when there is no risk of confusion. The affine normal vector at u is denoted by $\mathcal{N}(u)$ and is considered to point outward. Finally, by $dv_C = \frac{1}{n} \frac{h_C(u)}{\kappa_C(u)} \, d\mu_{\mathbb{S}^{n-1}}$ we denote the cone-volume measure of C, and by $d\bar{v}_C$, the normalized cone-volume measure $d\bar{v}_C = \frac{1}{|C|} \, dv_C$, with $|C| = \int_{\mathbb{S}^{n-1}} dv_C$ being the usual Lebesgue measure of C as a subset of $\mathbb{R}^n$, referred to as the volume of C, and $d\mu_{\mathbb{S}^{n-1}}$ is the usual surface area measure of the unit sphere. For detailed references on these notions, the reader is directed to [3], [4], [5].

We will now define the centro-affine flow (CAF), first as a Cauchy problem for the position vector of family of hypersurfaces bounding a family of convex bodies $\{C(t) \mid t \geq 0\}$, then, in its equivalent scalar, normalized form, as a Cauchy problem for the support functions of the bodies $C(t)$ normalized to have constant volume:

$$\begin{cases} \dfrac{\partial x(u,t)}{\partial t} = -\kappa_{\tilde{C},0}^{-\frac{1}{n+1}}(u)\,\mathcal{N}(u,t), \\ u(x,0) = x_C(u), \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{\partial h(u,t)}{\partial t} = -\kappa_{\tilde{C},0}^{-\frac{1}{n+1}}(u)\,\kappa^{\frac{1}{n+1}}(u,t) + h(u,t)\,I(t), \\ h(x,0) = h_C(u), \end{cases}$$

where $I(t) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \left(\frac{\kappa_{\tilde{C},0}(u)}{\kappa_{C(t),0}(u)}\right)^{-\frac{1}{n+1}} d\bar{v}_{C(t)}$. It is precisely this valuation I on convex bodies, a particular form of the general upper semi-continuous $\mathrm{SL}(n)$-invariant valuations characterized by Ludwig and Reitzner in [5], that is monotone increasing along the flow for any initial convex body C.

Note that if $\tilde{C}$ has constant centro-affine curvature, hence $\tilde{C}$ is an ellipsoid, then $I(t)$ becomes the affine surface area of $C(t)$ and the flow is the flow along the affine normal defined by Andrews. The asymptotic behavior of the former flow is summarized by the following result.

Theorem 1. *Let $\tilde{C}$ be a fixed origin-symmetric, smooth convex body with strictly positive Gauss curvature. Then, up to an* $\mathrm{SL}(n)$*-transformation, any origin-symmetric, smooth convex body C with strictly positive Gauss curvature converges in $\mathcal{C}^\infty$-norm under the normalized CAF to an origin-symmetric, smooth convex body with same centro-affine curvature function on $\mathbb{S}^{n-1}$ as that of the convex body $\tilde{C}$.*

Several consequences follow among which the most important is stated below in a flow-independent form:

Proposition 1. *Let C and $\tilde{C}$ be two origin-symmetric smooth convex bodies with strictly positive Gauss curvature. Then*

$$\int_{\mathbb{S}^{n-1}} \ln \frac{\kappa_{C,0}(u)}{\kappa_{\tilde{C},0}(u)}\, d\bar{v}_C(u) \leq 2 \ln \frac{|\tilde{C}|}{|C|}, \tag{1}$$

with equality if and only if C and $\tilde{C}$ have the same centro-affine curvature function on $\mathbb{S}^{n-1}$.

This is further used toward the uniqueness of solutions to the logarithmic Minkowski problem and, respectively, to the logarithmic Minkowski inequality. Due to the antisymmetric form of (1), we obtain the following corollary:

Corollary 1. *Assume that C and $\tilde{C}$ are two origin-symmetric, strictly convex, smooth convex bodies with the same cone-volume measure. Then*

$$\int_{\mathbb{S}^{n-1}} \ln \frac{\kappa_{C,0}(u)}{\kappa_{\tilde{C},0}(u)}\, d\bar{v}_C(u) = 0,$$

thus C and $\tilde{C}$ have the same centro-affine curvature everywhere.

I would like to thank Christos Saroglou who pointed out that this implies right away the uniqueness of smooth solutions to the logarithmic Minkowski problem. Moreover, the formulation of the logarithmic Minkowski problem as an optimization problem, implies the logarithmic Minkowski inequality for centrally symmetric, smooth, strictly convex bodies with equality if and only if the two convex bodies are homothetic to each other.

Furthermore, as the subset of smooth and strictly convex bodies is dense in the set of all convex bodies, [6], the logarithmic Minkowski inequality, and thus the equivalent logarithmic Brunn-Minkowski inequality [2], follows for all centrally symmetric convex bodies:

Theorem 2 (The logarithmic Minkowski inequality)**.** *Let C and $\tilde{C}$ be two centrally symmetric convex bodies in $\mathbb{R}^n$, then*

$$\int_{\mathbb{S}^{n-1}} \ln \frac{h_{\tilde{C}}(u)}{h_C(u)}\, d\bar{v}_C(u) \geq \frac{1}{n} \ln \frac{|\tilde{C}|}{|C|}.$$

The asymptotic behavior of the flow is currently being written and will be available at a later time. It seems probable in fact that Proposition 1 can be obtained bypassing the technicalities of the flow's asymptotic behavior.

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Stars of empty simplices

Matthias Reitzner
(joint work with Daniel Temesvari)

Let $\xi \subset \mathbb{R}^d$ be a finite point set in general position. We call a $(d+1)$-tuple $\{x_1, \dots, x_{d+1}\} \in \binom{\xi}{d+1}$ an *empty simplex* if the simplex which is the convex hull of these points satisfies $[x_1, \dots, x_{d+1}]^o \cap \xi = \emptyset$. For given $\{x_1, \dots, x_k\} \in \binom{\xi}{k}$ we define the k-degree $\deg_k(x_1, \dots, x_k; \xi)$ as the number of $(d-k+1)$-tuples $\{x_{k+1}, \dots, x_{d+1}\} \in \xi \setminus \{x_1, \dots, x_k\}$ such that $\{x_1, \dots, x_{d+1}\}$ is an empty simplex.

$$\deg_k(x_1, \dots, x_k; \xi) = \sum_{\{x_{k+1}, \dots, x_{d+1}\} \in \binom{\xi \setminus \{x_1, \dots, x_k\}}{d-k+1}} \mathbb{1}([x_1, \dots, x_{d+1}]^o \cap \xi = \emptyset).$$

The union of these $\deg_k(x_1, \dots, x_k; \xi)$ empty simplices is what we call a 'star of empty simplices'. The k-degree $\deg_k(\xi)$ of the point set ξ is defined as the degree of the maximal star, i.e.,

$$\deg_k(\xi) = \max_{\{x_1, \dots, x_k\} \in \binom{\xi}{k}} \deg_k(x_1, \dots, x_k; \xi). \tag{1}$$

The quantity $\deg_d(\xi)$ was introduced by Erdős [3] in the planar case. He posed the question whether the $\deg_2(\xi)$ goes to infinity as the number of points in ξ goes to infinity. Even in the planar case the question is still open.

In [2] Bárány, Marckert and Reitzner turned their attention to a random point set $\xi_n \subset \mathbb{R}^2$ consisting of n iid uniformly chosen points from a convex body $K \subset \mathbb{R}^2$. Here the expected number of empty triangles is known by a work of Valtr [4] and is asymptotically $\leq 2n^2$. Since the number of pairs of points is $\binom{n}{2}$ we see that the degree of a typical pair of points is

$$\mathbb{E}\deg_2(x_1, x_2; \xi_n) \approx 12.$$

For general dimensions $d \geq 3$ a result by Bárány and Füredi [1] states that the expected number of empty simplices in a uniform random point set is $\leq c(d)n^d$. Because there are $\binom{n}{d}$ simplices of dimension $(d-1)$ this shows that the typical degree again is constant,

$$1 \leq \mathbb{E}\deg_d(x_1, \dots, x_d; \xi_n) \leq c(d).$$

On the other hand it is clear that

$$1 \leq \deg_d(\xi_n) \leq n,$$

and the stochastic version of the problem of Erdös asks whether

$$\deg_d(x_n) \to \infty$$

as $n \to \infty$. Bárány, Marckert and Reitzner [2] showed that for sufficiently large n the assertion holds true in expectation, i.e., that

$$\mathbb{E}\deg_d(\xi_n) \geq c(d)(\ln n)^{-1}\, n.$$

Observe that this lower bound is surprisingly close to the trivial upper bound, up to a logarithmic factor.

In our work presented at the Oberwolfach workshop we are able to remove the logarithmic factor completely and determine the asymptotic order with a significantly simpler proof as in [2]. Thus the expected degree of a uniform random point set is surprisingly large: there is a star of empty simplices where the number of spikes is at least a constant proportion of *all* random points.

There is a constant $c(d,K) > 0$ *such that*

$$c(d,K)n \le \mathbb{E} deg_d x_n \le n.$$

The more case of $\deg_k(\xi_n)$, $k = 1,\dots,d-1$ turn out to be more involved. Here, one easily sees that asymptotically n^{d-k} is a lower bound, and n^{d-k+1} is a trivial upper bound on $\deg_k(\xi_n)$. In contrast to the case $k = d$ where the *upper* bound gives the correct order, we are showing that for the case $k = 1$ the *lower* bound gives indeed the correct asymptotic behavior.

There are constants $c(d), c(d,K)$ *such that*

$$c(d)n^{d-1} \le \mathbb{E} deg_1 \xi_n \le c(d,K)n^{d-1}.$$

The cases $k = 2,\dots,d-1$ get computationally much more involved and intricate and we have not been able to prove these.

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Brascamp–Lieb inequalities for even functions

Liran Rotem

(joint work with Dario Cordero-Erausquin)

Let γ denote the standard Gaussian measure on $\mathbb{R}^n$. The classical *Gaussian Poincaré inequality* states that for every smooth enough function $f : \mathbb{R}^n \to \mathbb{R}$ with $\int f \mathrm{d}\gamma = 0$ one has

$$\int f^2 \mathrm{d}\gamma \le \int |\nabla f|^2 \mathrm{d}\gamma,$$

where $|\cdot|$ denotes the Euclidean norm. This inequality is sharp, and equality holds if f is a linear function. In fact the same inequality holds if γ is replaced with any measure μ which is log-concave with respect to γ, which means that $\frac{\mathrm{d}\mu}{\mathrm{d}\gamma} = e^{-V}$ for some convex function $V : \mathbb{R}^n \to \mathbb{R}$.

Cordero-Erausquin, Fradelizi and Maurey proved in [2] that if μ and f are as before but f is also assumed to be **even**, then we actually have the stronger inequality

$$\int f^2 \mathrm{d}\mu \le \frac{1}{2} \int |\nabla f|^2 \mathrm{d}\mu.$$

They used this stronger inequality to prove the (B)-conjecture for the Gaussian measure: If $K \subseteq \mathbb{R}^n$ is a symmetric convex body than the map $t \mapsto \gamma(e^t K)$ is log-concave.

The Brascamp-Lieb inequality ([1]) generalizes the Poincaré inequality. It states that if $\frac{\mathrm{d}\mu}{dx} = e^{-V}$ for a smooth enough convex function V, then

$$\int f^2 \mathrm{d}\mu \le \int \left(\nabla^2 V\right)^{-1} \nabla f \cdot \nabla f \mathrm{d}\mu$$

for every function $f : \mathbb{R}^n \to \mathbb{R}$ with $\int f \mathrm{d}\mu = 0$. This inequality is related to possible extensions of the (B)-conjecture from Gaussian measures to other measures. However, just like in the Poincaré case, the classical Brascamp-Lieb inequality is not strong enough, and one needs a stronger inequality under the additional assumption that f is even. More precisely, we have the following relation:

Theorem 1. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a p-homogeneous, even, smooth convex function, and let μ be the measure with density $\frac{\mathrm{d}\mu}{\mathrm{d}x} = e^{-V}$. Assume*

$$\int f^2 \mathrm{d}\nu \le \frac{p-1}{p} \int \left(\nabla^2 V\right)^{-1} \nabla f \cdot \nabla f \mathrm{d}\nu$$

for all even measures ν which are log-concave with respect to μ and for all even smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ with $\int f \mathrm{d}\nu = 0$. Then for every symmetric convex body $K \subseteq \mathbb{R}^n$ the map $t \mapsto \mu(e^t K)$ is log-concave.

In our work we concentrate on the simplest p-homogeneous convex function, which is $V_p(x) = \frac{|x|^p}{p}$. Our theorem then reads as follows:

Theorem 2. *Fix $p \ge 2$ and define $V_p(x) = \frac{|x|^p}{p}$. Let μ_p be the measure with density $\frac{\mathrm{d}\mu_p}{\mathrm{d}x} = e^{-V_p}$. Then for every smooth enough function $f : \mathbb{R}^n \to \mathbb{R}$ with $\int f \mathrm{d}\mu_p = 0$ one has*

$$\int f^2 \mathrm{d}\mu_p \le \frac{p-1}{p} \int \left(\nabla^2 V_p\right)^{-1} \nabla f \cdot \nabla f \mathrm{d}\mu_p.$$

Note that when $p = 2$ one recovers the Gaussian Poincaré inequality for even functions. The constant $\frac{p-1}{p}$ is sharp for all p.

The proof of the theorem begins in a similar way to the proof from [2]. However, the proof also requires a new ingredient: a new weighted Poincaré inequality for the measures μ_p which holds for *odd* functions. The exact statement is as follows:

Theorem 3. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be an odd smooth function. Then*

$$\int |x|^{p-2} h^2 \mathrm{d}\mu_p \le \int |\nabla h|^2 \mathrm{d}\mu_p$$

with equality if and only if h is linear.

We believe this inequality may be of independent interest.

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Constant parts of a function via isotropicity of its sections

Christos Saroglou
(joint work with Andreas Halilaj, Ioannis Purnaras)

A function $f : S^{n-1} \to \mathbb{R}$ is called isotropic if the map

$$S^{n-1} \ni u \mapsto \int_{S^{n-1}} \langle x, u\rangle^2 dx$$

is constant. The following was proved in [2].

Theorem A. *Let $f : S^{n-1} \to \mathbb{R}$ be a measurable, bounded a.e. and even function, $n \geq 3$. If for almost every $u \in S^{n-1}$ the restriction $f|_{S^{n-1}\cap u^\perp}$ of f to $S^{n-1} \cap u^\perp$ is isotropic (i.e. the restriction of f to almost every equator is isotropic), then f is almost everywhere equal to a constant.*

Theorem A was used to confirm a conjecture of Ryabogin stating that if all central sections of a centrally symmetric star body K have the symmetries of the cube, then K has to be a Euclidean ball. This result can be viewed in connection with classical characterizations of the Euclidean space such as those established in [1] or [4]. Notice of course that any even function (or centrally symmetric set) that has the symmetries of the cube, is necessarily isotropic.

One can also ask about a local version of the previously stated result. More specifically, if for some $U \subseteq S^{n-1}$ and for all $u \in U$, $K \cap u^\perp$ has the symmetries of the cube, is it true that $K \cap U^\perp = B \cap U^\perp$ for some Euclidean ball B centered at the origin? Here $U^\perp$ stands for the union of all great subspheres of S^{n-1} which are perpendicular to some vector from U.

To deal with such a question, one naturally needs a local version of Theorem A. This was explicitly asked in [2]. Our main result states as follows:

Theorem 1. *Let U be an open subset of the sphere S^{n-1}, $n \geq 3$ and $f : U^\perp \to \mathbb{R}$ be an even bounded measurable function. If $f|_{S^{n-1}\cap u^\perp}$ is isotropic for almost all $u \in U$, then f is almost everywhere equal to a constant on $U^\perp$.*

The proof uses the following observation: Assume that f is strictly positive and smooth and set $Z(f)$ to be the zonoid whose support function is given by

$$h_{Z(f)}(u) = \int_{S^{n-1}} |\langle x, u\rangle| f(x) dx, \qquad u \in S^{n-1}.$$

Then the contact point of $Z(f)$ with its supporting hyperplane with outer unit normal vector $u \in U$ is umbilical (i.e. the principal curvatures at this point are all equal) if and only if $f|_{S^{n-1}\cap u^\perp}$ is isotropic. One then has to make use of a classical result in Differential Geometry, stating that if all points of a smooth (enough) hypersurface M in $\mathbb{R}^n$ are umbilical, then M is contained in a Euclidean sphere. A little more work is required to remove the regularity assumptions.

Let us state another application of Theorem 1. First let us recall the following theorem due to Ryabogin [3]:

Theorem B (Ryabogin). *Let $f, g : S^2 \to \mathbb{R}$ be two continuous functions, such that for any $u \in S^2$, there exists an orthogonal map $T_u : u^\perp \to u^\perp$, such that $f(x) = g(T_u x)$, for all $x \in S^2 \cap u^\perp$. Then, $f(x) = g(x)$, for all $x \in S^2$ or $f(x) = g(-x)$, for all $x \in S^2$.*

The core of the proof of Theorem B is probably the following fact: If $f : S^2 \to \mathbb{R}$ is a continuous function and U is an open subset of S^2, such that for all $u \in U$, there exists an orthogonal map $T_u : u^\perp \to u^\perp$, different than $\pm Id$, with the property $f(x) = f(T_u x)$, for all $x \in S^2 \cap u^\perp$, then f is constant on $U^\perp$. Notice that a function on S^1 that has a non-trivial symmetry (i.e. different than $\pm Id$) is always isotropic, therefore the previous fact follows immediately from Theorem 1.

It should be noted that Theorem B turned out to be false for $n \geq 4$ (see [5]). However, it is hoped that it could be true under some extra assumptions (for instance, as proposed in [5], not to allow T_u to be idempotent) and Theorem 1 (or variants of it) seems to be a good starting point towards this direction.

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Flag area measures

Andreas Bernig

(joint work with Judit Abardia-Evéquoz, Susanna Dann)

Flag area measures are a generalization of the classical area measures associated to compact convex bodies. The latter are valuations whose values are measures on the unit sphere, the former are valuations whose values are measures on a certain partial flag manifold. Hinderer-Hug-Weil [5] used a Steiner formula approach to introduce a class of flag area measures. See [4, 6, 7] for a more detailed study of flag area measures.

Let

$$F^\perp(n,p) := \{(v,E) \in S^{n-1} \times \mathrm{Gr}_p(V) : v \perp E\},$$

which is a partial flag manifold.

Theorem 1 (Hinderer-Hug-Weil, [5]). *Let $0 \leq p \leq n-1$, $0 \leq k \leq n-p-1$. There is a unique weakly continuous flag area measure $S_k^{(p)}$ on convex bodies such that for a polytope P*

$$S_k^{(p)}(P,\beta) = \binom{n-p-1}{k}^{-1} \frac{\omega_{n-p}}{\omega_n} \times$$

$$\times \sum_{F \in \mathcal{F}_k(P)} \mathrm{vol}_k(F) \int_{\mathrm{n}(P,F)} \int_{\mathrm{Gr}_{p+1}(v)} \mathbf{1}_{(v, E\cap v^\perp)\in\beta} \cos^2(E^\perp, F) dE\, dv.$$

Here $\mathrm{Gr}_{p+1}(v)$ denotes the Grassmannian of all $(p+1)$-planes containing v, endowed with an invariant probability measure dE; $\beta \subset F^\perp(n,p)$ is a Borel subset and $\cos^2(E^\perp, F)$ denotes the squared cosine between the subspaces $E^\perp$ and F. In particular, $S_k^{(p)}$ is translation-invariant and $\mathrm{O}(n)$-equivariant.

The relative position of two linear subspaces is measured in terms of Jordan angles [3, 9]. If $\theta_1, \ldots, \theta_m, m := \min(k, n-k-1, p, n-p-1)$ are the Jordan angles between $E^\perp$ and F, then $\cos^2(E^\perp, F) = \cos^2\theta_1 \cdot \ldots \cdot \cos^2\theta_m$. More generally, let us denote by $\sigma_i(E^\perp, F)$ the i-th elementary symmetric polynomial in $\cos^2\theta_1, \ldots, \cos^2\theta_m$.

Our main theorem is the following.

Theorem 2. *For every $0 \leq p, k \leq n-1, 0 \leq i \leq m := \min\{k, n-k-1, p, n-p-1\}$, there exists a unique weakly continuous flag area measure on convex bodies such that for a polytope $P \subset V$ and $\beta \subset F^\perp(n,p)$,*

$$S_k^{(p),i}(P,\beta) = c_{n,k,p,i} \times \tag{1}$$

$$\times \sum_{F \in \mathcal{F}_k(P)} \mathrm{vol}_k(F) \int_{\mathrm{n}(P,F)} \int_{\mathrm{Gr}_{p+1}(v)} \mathbf{1}_{(v, E\cap v^\perp)\in\beta} \sigma_i(E^\perp, F) dE\, dv, \tag{2}$$

where

$$c_{n,k,p,i} := \binom{n-1}{k}^{-1} \binom{m}{i}^{-1} \binom{|k-(n-1-p)|+m}{i}^{-1} \binom{n-1}{i}.$$

The proof uses a new definition of *smooth flag area measures* which is based on differential forms and the conormal cycle. The constant $c_{n,k,p,i}$ is chosen in such a way that the push-forward under the projection $F^\perp(n,p) \to S^{n-1}, (v,E) \mapsto v$ yields the area measure S_k. For this, we need a theorem by James [8] on the distribution on Jordan angles as well as the computation of certain Selberg type integrals [2, 10] due to Aomoto [1].

In addition, we show that the above flag area measures form a basis of the space of smooth, translation-invariant and $\mathrm{O}(n)$-equivariant flag area measures.

If n is odd, there is another smooth flag area measure which is translation-invariant and $\mathrm{SO}(n)$-equivariant, but not $\mathrm{O}(n)$-equivariant.

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Random polytopes obtained by matrices with heavy tailed entries

Olivier Guédon

(joint work with Alexander E. Litvak, Kateryna Tatarko)

In this talk, I have presented recent results from [4] and I refer to this paper for more detailed explanations. We consider rectangular $N \times n$ matrices $\Gamma = \{\xi_{ij}\}_{\substack{1\le i\le N\\ 1\le j\le n}}$, with $N \geq n$, where the entries are real-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We will mainly assume that:

$$\text{(1)} \qquad \begin{cases} \forall i, j, \xi_{ij} \text{ are independent, symmetric and } \mathbb{E}\xi_{ij}^2 = 1, \\ \text{in each row, the entries are identically distributed.} \end{cases}$$

We are interested in geometric parameters of the random polytope generated by Γ, that is, the absolute convex hull of rows of Γ. In other words, the random polytope under cosideration is $\Gamma^* B_1^N$, where B_1^N is the N-dimensional octahedron. Such random polytopes have been extensively studied in the literature, especially in the Gaussian case and in the Bernoulli case. The Gaussian random polytopes in the case when N is proportional to n have many applications in the Asymptotic Geometric Analysis (see e.g. the survey [5]). The Bernoulli case corresponds to 0/1 random polytopes. Their geometric parameters have been studied in [1, 3]. In the compress sensing theory, it is shown in [2] that an $n \times N$ matrix A satisfies not only the ℓ_1-quotient property but is also robust to noise-blind ℓ_1-minimization if

$$\text{(2)} \qquad AB_1^N \supset b^{-1} \left(B_\infty^n \cap \sqrt{\ln(N/n)}\, B_2^n \right).$$

The main purpose of this note is to prove such an inclusion with weaker assumptions on the distribution of the entries than in [2].

Theorem 1. *Let* $\Gamma = \{\xi_{ij}\}_{\substack{1\le i\le N\\ 1\le j\le n}}$, *with* $N \geq n$ *satisfying* (1). *Let* $K_N = \Gamma^* B_1^N$. *Assume that there exists* $u, v \in (0,1)$ *such that*

$$\forall i,j \quad \sup_{\lambda\in\mathbb{R}} \mathbb{P}\big\{|\xi_{ij}-\lambda| \le u\big\} \le v.$$

Let $\beta \in (0,1)$. *There are two positive constants* $M = M(u,v,\beta)$ *and* $C(u,v,\beta)$ *which depend only on* u, v, β, *such that for every* $N \geq Mn$, *one has*

$$\mathbb{P}\left(K_N \supset C(u,v,\beta)\left(B_\infty^n \cap \sqrt{\ln(N/n)}\, B_2^n\right)\right) \ge 1 - 4\exp\left(-cn^\beta N^{1-\beta}\right),$$

where c *is an absolute positive constant.*

Our proof follows the scheme of [3] with a very delicate change – in [3] there was an assumption that the operator norm of Γ is bounded by $C\sqrt{N}$ with high probability. However it is known that such a bound does not hold in general unless fourth moments are bounded. To avoid using the norm of Γ, we use ideas appearing in [6], where the authors constructed a certain deterministic ϵ-net (in ℓ_2-metric) $\mathcal{N}$ such that $A\mathcal{N}$ is a good net for AB_2^n for most realizations of a square random matrix A. We extend their construction in three directions. First, we work with rectangular random matrices, not only square matrices. Second, we need a net for the image of a given convex body (not only for the image of the unit Euclidean ball). Finally, instead of approximation in the Euclidean norm, we use approximation in the following norm

$$\|a\|_{k,2} = \Big(\sum_{i=1}^{k} (a_i^*)^2\Big)^{1/2}, \tag{3}$$

where $1 \le k \le N$ and $a_1^* \ge a_2^* \ge \ldots \ge a_m^*$ is the decreasing rearrangement of the sequence of numbers $|a_1|, \ldots, |a_m|$. This norm appears naturally and plays a crucial role in our proof of inclusion (2). This approach allows also to recover sharp estimates for the smallest singular value of tall matrices, see [4] for the details.

Theorem 2. *Let* $n \in [N]$, $0 \le \delta \le 1$, $0 < \epsilon \le 1$. *Let* $k \in [N]$ *such that* $k\ln(eN/k) \ge n$. *Let* T *be a non-empty subset of* $\mathbb{R}^n$ *and denote* $M := N(T, \epsilon B_\infty^n)$. *There exists a set* $\mathcal{N} \subset T$ *and a collection of parallelepipeds* $\mathcal{P}$ *in* $\mathbb{R}^n$ *such that*

$$\max\{|\mathcal{N}|, |\mathcal{P}|\} \le M\, F(\delta, n, N) e^{\delta N}.$$

Moreover, for any random matrix Γ *satisfying assumption* (1), *with probability at least* $1 - e^{-k\ln(eN/k)} - e^{-\delta N/4}$, *we have*

$$\begin{cases} \forall x \in T\ \exists y \in \mathcal{N} \quad \textit{such that} \quad \|\Gamma(x-y)\|_{k,2} \le C\epsilon\sqrt{\dfrac{kn}{\delta}\ln\left(\dfrac{eN}{k}\right)} \\ \forall x \in T\ \exists P \in \mathcal{P} \quad \textit{such that} \quad x \in P \quad \textit{and} \quad \Gamma P \subset \Gamma x + C\epsilon\sqrt{\dfrac{kn}{\delta}\ln\left(\dfrac{eN}{k}\right)}\ \mathbf{B}_{k,2} \end{cases}$$

where C *is a positive absolute constant.*

We believe that the new approximation in $\|\cdot\|_{k,2}$ norms will find other applications in the theory.

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On a local version of the fifth Busemann-petty problem

Dmitry Ryabogin

(joint work with M. Angeles Alfonseca, Fedor Nazarov, Vlad Yaskin)

In 1956, Busemann and Petty [2] posed a series of questions about symmetric convex bodies, of which only the first one has been solved ([5]; see also [6] for the history of the solution of the first problem). Their fifth problem asks the following.

Problem 5. *If for an origin symmetric convex body $K \subset \mathbb{R}^n$, $n \geq 3$, we have*

$$\forall \theta \in S^{n-1} \quad h_K(\theta) vol_{n-1}(K \cap \theta^{\perp}) = c, \tag{1}$$

where the constant c is independent of θ, must K be an ellipsoid?

Here $S^{n-1} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sqrt{x_1^2 + \dots + x_n^2} = 1\}$ is the unit sphere in $\mathbb{R}^n$, $\theta^{\perp} = \{x \in \mathbb{R}^n : x \cdot \theta = 0\}$ is the hyperplane passing through the origin and orthogonal to the unit direction $\theta \in S^{n-1}$, $h_K(\theta) = \max_{\{x \in K\}} x \cdot \theta$ is the support function of a convex body $K \subset \mathbb{R}^n$, and $x \cdot \theta = x_1\theta_1 + \dots + x_n\theta_n$ is the usual inner product in $\mathbb{R}^n$.

Problem 5 is related to the notion of normality in a Minkowski space. A Minkowski space is a finite dimensional real vector space V with a norm $\|\cdot\|_K$; the unit ball K corresponding to the given norm is an origin symmetric convex body with non empty interior. Normality between vectors in a Minkowski space is defined as follows. A vector x is normal to a vector y (denoted by $x \dashv y$) if $\|x\|_K \leq \|x + ty\|_K$ for every $t \in \mathbb{R}$. In general, normality between vectors is not a symmetric relation. In dimension 2, the Minkowski spaces with symmetric normality are precisely those with unit circles for which the triangles given by (1) have constant areas. It was shown by Radon that there are 2 dimensional non-Euclidean norms with this property, and the boundary of the corresponding convex body is known as a Radon curve [3]. Blaschke and Birkhoff established that for $n \geq 3$, the only Minkowski spaces where normality between vectors is symmetric are the Euclidean ones.

In dimension $n \geq 3$, a different concept of normality, the normality between lines and hyperplanes was defined by Busemann [1] (note that both concepts coincide for two dimensional spaces). Busemann showed that symmetry of this relation is equivalent to the fact that the volume of the cone with base $K \cap \theta^\perp$ and height $h_K(\theta)$ is independent of $\theta \in S^{n-1}$, which is what equation (1) states. Therefore, an affirmative answer to Problem 5 would mean that the only Minkowski spaces where normality between lines and hyperplanes is a symmetric relation are the Euclidean ones.

The Euclidean ball clearly satisfies (1). If a body K satisfies (1), then so does TK for any linear transformation $T \in \mathrm{GL}(n)$ (with constant $c \cdot \det T$), and hence (1) is satisfied by ellipsoids.

Let $\mathcal{S}_n$ be the set of equivalence classes of convex bodies in $\mathbb{R}^n$, where two bodies are equivalent if one can be obtained from the other by a linear transformation. On $\mathcal{S}_n$ we consider the Banach-Mazur distance

$$d_{BM}(K, L) = \inf\left\{\frac{b}{a} : \exists T \in \mathrm{GL}(n) \text{ such that } aK \subseteq TL \subseteq bK\right\}.$$

Our main result is

Theorem 1. *Let $n \geq 3$. If a symmetric convex body $K \in \mathbb{R}^n$ satisfies (1) and is sufficiently close to the Euclidean ball in the Banach-Mazur metric, then K must be an ellipsoid.*

We remark that in dimension 2, there are convex bodies satisfying (1) that are not ellipsoids (the bodies bounded by a Radon curve, mentioned earlier), but, nevertheless, they can be arbitrarily close to a unit disc. To see this, let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. The body can be defined as B_p^2 in the first and third quadrant, and as B_q^2 in the second and fourth quadrants, where

$$B_p^n = \Big\{x \in \mathbb{R}^n : \sum_{j=1}^n |x_j|^p \leq 1\Big\}.$$

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Conic support measures

Rolf Schneider

Recent applications (see [2], e.g.) have led to new interest in spherical intrinsic volumes and their integral geometry (see [3] for local extensions), conveniently translated into the conic situation. This was our motivation for a detailed study of conic support measures.

For a closed convex cone C in $\mathbb{R}^d$, let C° be its polar cone, and let Π_C denote the nearest-point map of C. Let $\widehat{\mathcal{B}}(\mathbb{R}^d \times \mathbb{R}^d)$ denote the σ-algebra of all Borel sets η in $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $(\lambda x, \mu y) \in \eta$ for $(x,y) \in \eta$ and $\lambda, \mu > 0$. First considering a polyhedral cone C, let $k \in \{0,\dots,d\}$ and define the k-skeleton $\mathrm{skel}_k C$ of C as the union of the relative interiors of all k-faces of C. Denoting by $\mathbf{g}$ a standard Gaussian random vector in $\mathbb{R}^d$, the kth conic support measure of C can be defined by

$$\Omega_k(C,\eta) = \mathbb{P}(\Pi_C(\mathbf{g}) \in \mathrm{skel}_k C,\ (\Pi_C(\mathbf{g}), \Pi_{C^\circ}(\mathbf{g})) \in \eta)$$

for $\eta \in \widehat{\mathcal{B}}(\mathbb{R}^d \times \mathbb{R}^d)$, where $\mathbb{P}$ denotes probability. The total measure $v_k(C) = \Omega_k(C, \mathbb{R}^d \times \mathbb{R}^d)$ is the kth conic intrinsic volume of C.

Our first result is an extension of the 'Master Steiner formula' of [4]. For a measurable function $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ define

$$\varphi_f(C,\eta) = \mathbb{E}\left[f(\|\Pi_C(\mathbf{g})\|^2, \|\Pi_{C^\circ}(\mathbf{g})\|^2) \cdot \mathbf{1}_\eta(\Pi_C(\mathbf{g}), \Pi_{C^\circ}(\mathbf{g}))\right]$$

for $\eta \in \widehat{\mathcal{B}}(\mathbb{R}^d \times \mathbb{R}^d)$, where $\mathbb{E}$ denotes expectation. If $\varphi_f(C,\cdot)$ is finite, the result says that

$$\varphi_f(C,\eta) = \sum_{k=0}^{d} I_k(f) \cdot \Omega_k(C,\eta), \tag{1}$$

where the coefficients are given by $I_k(f) = \varphi_f(L_k, \mathbb{R}^d \times \mathbb{R}^d)$ with an arbitrary k-dimensional subspace L_k of $\mathbb{R}^d$. For polyhedral cones, the proof uses the Moreau decomposition, as in [4], but the extension to general convex cones requires additional arguments.

By specialization, one gets a local Steiner formula for the Gaussian measure $\mu_\lambda(C,\eta)$ of the local parallel set

$$M^a_\lambda(C,\eta) = \{x \in \mathbb{R}^d : 0 < d_a(x,C) \le \lambda,\ (\Pi_C(x), \Pi_{C^\circ}(x)) \in \eta\},$$

where d_a denotes the angular distance. Since $\mu_\lambda(C,\cdot)$ depends weakly continuously on C, one can use the local Steiner formula to extend the conic support measures, and then also formula (1), to general closed convex cones.

The weak continuity of the conic support measures, which was obtained on the way, is considerably strengthened by our second result. For finite measures μ, ν on $\widehat{\mathcal{B}}(\mathbb{R}^d \times \mathbb{R}^d)$, we define

$$d_{bL}(\mu,\nu) = \sup\left\{\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f_h\,d\mu - \int_{\mathbb{R}^d\times\mathbb{R}^d} f_h\,d\nu\right|\right\},$$

where the supremum is taken over all (degree 0) homogeneous extensions f_h of real functions $f : \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ (where $\mathbb{S}^{d-1}$ denotes the unit sphere of $\mathbb{R}^d$) satisfying

$$\sup_{a\neq b} \frac{|f(a) - f(b)|}{\|a - b\|} \leq 1, \quad \sup_{a} |f(a)| \leq 1.$$

Then d_{bL} is a metric which metrizes the weak convergence of finite measures on $\widehat{\mathcal{B}}(\mathbb{R}^d \times \mathbb{R}^d)$. Our result is the Hölder type inequality

$$\delta_{bL}(\Omega_k(C, \cdot), \Omega_k(D, \cdot)) \leq c\delta_a(C, D)^{1/2}$$

for closed convex cones C, D, with a constant c depending only on the dimension; here δ_a denotes the angular Hausdorff metric induced by the angular distance, and it is assumed that $\delta_a(C, D) \leq 1$.

Proofs of the preceding are in [6], and the following can be found in [5].

In [1], a new approach to the kinematic integral-geometric formula for conic intrinsic volumes was developed, and we verified that the method can also be used locally. The kth conic curvature measure of the closed convex cone C is defined by $\Phi_k(C, A) = \Omega_k(C, A \times \mathbb{R}^d)$ for conic Borel sets $A \subset \mathbb{R}^d$. For closed convex cones C, D and conic Borel sets A, B in $\mathbb{R}^d$, the formula

$$\int_{\mathrm{SO}_d} \Phi_k(C \cap \vartheta D, A \cap \vartheta B)\, \nu(d\vartheta) = \sum_{i=k}^{d} \Phi_i(C, A)\Phi_{d+k-i}(D, B)$$

holds for $k = 1, \dots, d$, where SO_d is the rotation group of $\mathbb{R}^d$ and ν is now the normalized Haar measure on SO_d. This is well known (the spherical case was treated in [3] in a different way), but the new approach allows some extensions in the case of lower-dimensional cones, where an additional function of the generalized sine function between the affine hulls of the cones may appear in the integrand.

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