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NONDEGENERATE INVARIANT SYMMETRIC BILINEAR FORMS ON SIMPLE LIE SUPERALGEBRAS IN CHARACTERISTIC 2

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Abstract. As is well-known, the dimension of the space of non-degenerate invariant symmetric bilinear forms (NISes) on any simple finite-dimensional Lie algebra or Lie superalgebra is equal to at most 1 if the characteristic of the ground field is distinct from 2.

We prove that in characteristic 2, the superdimension of the space of NISes can be equal to 0, or 1, or $0|1$, or $1|1$. This superdimension is equal to $1|1$ if and only if the Lie superalgebra is a queerification (defined in arXiv:1407.1695) of a simple restricted Lie algebra with a NIS (for examples of such Lie algebras, although mainly in characteristic distinct from 2, see arXiv:1806.05505).

We give examples of NISes on deformations with both even and odd parameter of several simple finite-dimensional Lie superalgebras in characteristic 2.

1. Main theorem

This paper is a sequel to [BKLS], where non-degenerate invariant symmetric bilinear forms (NISes) on known simple Lie algebras and Lie superalgebras (finite-dimensional and $\mathbb{Z}$-graded of polynomial growth) are listed if the characteristic p of the ground field $\mathbb{K}$ is distinct from 2, and for $p = 2$ there are given occasional examples . Here we mainly consider the algebraically closed field $\mathbb{K}$ for $p = 2$. For numerous applications of Lie (super)algebras with a NIS, see [DSB].

1.1. **On the contents of this paper.** In the background we give a little more information than is strictly needed to understand the Main Theorem 1.3 — the most interesting result of the paper. Partly, this information is not as well-known, as it deserves; it places the strictly needed facts in proper surroundings.

1.2. **Generalities.** The ground field is algebraically closed of characteristic p (mainly, $p = 2$ or 0), unless otherwise specified. We consider only finite-dimensional spaces; for infinite-dimensional examples, see [BGLLS] and [KS, KT].

In this note, all (super)commutative (super)algebras are supposed to be associative with 1; their morphisms should send 1 to 1, and the morphisms of supercommutative superalgebras should preserve parity.

From the super K_0-functor point of view, see [Mi], the *superdimension* of a given superspace V is $\operatorname{sdim} V := \dim V_{\bar 0} + \varepsilon \dim V_{\bar 1}$, where $\varepsilon^2 = 1$; usually one writes $\operatorname{sdim} V = \dim V_{\bar 0} \mid \dim V_{\bar 1}$, so $\varepsilon = 0|1$. We write $\dim V := \dim V_{\bar 0} + \dim V_{\bar 1}$.

A non-degenerate invariant symmetric bilinear form on a Lie (super)algebra will be briefly called NIS; just invariant and symmetric one will be briefly called IS. Speaking about "the space of non-degenerate forms" we exercise the usual abuse of the language: we are speaking about the space spanned by all non-degenerate forms, but not all forms in this space have to be non-degenerate. For example, the zero form is never non-degenerate (and in the case where

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"the space of non-degenerate forms" is of superdimension $1|1$, it contains a 1-dimensional inhomogeneous subspace of degenerate forms.)

In § 2, we recall basics on Lie superalgebras, especially for $p = 2$. We also give the definitions in terms of the functor of points needed to take into account odd parameters of deformations.

Our main result is a newly discovered fact in characteristic 2: the description of the possible superdimension of the space of NISes on a given simple Lie superalgebra.

For completeness, we consider Theorem 1.3 for any characteristic since we do not know if it was ever published in full generality, although well-known as a folklore. If $p \neq 2$, Theorem 1.3 seems to be "just a direct generalization" of a well-known fact about Lie algebras, *unless we realize that we might encounter* ***inhomogeneous*** *objects.* For $p = 2$, the situation is even more delicate; its complete description is a new result.

1.3. Theorem. 1) *If $p \neq 2$, any NIS on a simple finite-dimensional Lie superalgebra is homogeneous with respect to parity and the dimension of the space of NISes is ≤ 1.*

More precisely, the superdimension of the space of NISes is either 0, *or* 1 (*in this case, all NISes are even*), *or* ε (*in this case, all NISes are odd*).

2) *If $p = 2$, the superdimension of the space of NISes on a simple finite-dimensional Lie superalgebra is equal to either* 0, *or* 1, *or* ε, *or* $1|1$.

This superdimension is equal to $1|1$ *if and only if the Lie superalgebra is a queerification* (see Subsection 2.5) *of a simple restricted* (in the classical sense, see Subsection 2.3; for other restrictednesses, see [BLLS]) *Lie algebra with a NIS* (for examples of such Lie algebras, see [BKLS]).

1.4. Comment. The proof of Theorem 1.3 in case $p = 2$ actually proves the following Theorem 1.4.1, which implies Theorem 1.3 for $p = 2$ in its turn.

1.4.1. Theorem. *Let $p = 2$.*

If a simple finite-dimensional Lie superalgebra $\mathfrak{g}$ has a NIS, then

- *the dimension of the space of even IS-forms on $\mathfrak{g}$ is ≤ 1;*
- *the dimension of the space of odd IS-forms on $\mathfrak{g}$ is ≤ 1.*

(That is, the space of all IS-forms on $\mathfrak{g}$ is a superspace whose superdimension is equal to either 0, or 1, or ε, or $1|1$ since the even and odd components of an IS-form are also IS-forms).

- *Any homogeneous IS-form on any simple finite-dimensional Lie superalgebra $\mathfrak{g}$ is either* 0 *or non-degenerate.*

Proof. **Beginning of the proof of Theorem 1.3**. Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra over an algebraically closed field $\mathbb{K}$ of characteristic p. Let us first prove several lemmas which allow us to restrict ourselves to homogenous NIS forms.

1.5. Lemma. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra over a field of characteristics $p \neq 2$. Then, any homogenous IS on $\mathfrak{g}$ is either* 0 *or non-degenerate.*

Proof. Let ω be a homogenous IS-form on $\mathfrak{g}$. Then, $\operatorname{Ker}\omega$ is a subsuperspace of $\mathfrak{g}$ invariant with respect to $\operatorname{ad}_{\mathfrak{g}}$, i.e., it is an ideal. Since $\mathfrak{g}$ is simple, either $\operatorname{Ker}\omega = 0$, in which case ω is non-degenerate, or $\operatorname{Ker}\omega = \mathfrak{g}$, in which case $\omega = 0$. □

The statement of Lemma 1.5 is not true in characteristic 2. The problem is that even though $\operatorname{Ker}\omega$ is invariant with respect to $\operatorname{ad}_{\mathfrak{g}}$, it may be not an ideal, since it may be not closed under squaring. Before we formulate a similar statement for $p = 2$, let us prove two more lemmas, where we assume $p = 2$. (The lemmas are true for $p \neq 2$ as well, but they are trivial in that case.)

1.6. Lemma. *Let $p = 2$. Let $\mathfrak{g}$ be a simple Lie superalgebra with a NIS $(-,-)$. Then, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.*

1.6.1. Comment. One might think "since $[\mathfrak{g},\mathfrak{g}]$ is an ideal in $\mathfrak{g}$ which is supposed to be simple, we have nothing to prove". But for Lie **super**algebras in characteristic 2 there is a difference between the first derived algebra and the commutant $[\mathfrak{g},\mathfrak{g}] := \mathrm{Span}([x,y] \mid x,y \in \mathfrak{g})$, which is not necessarily closed with respect to squaring, and hence may be not an ideal. Recall that the *derived Lie superalgebras* of $\mathfrak{g}$ are defined to be (for $i \geq 0$)

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = \begin{cases} [\mathfrak{g}^{(i)},\mathfrak{g}^{(i)}] & \text{for } p \neq 2,\\ [\mathfrak{g}^{(i)},\mathfrak{g}^{(i)}] + \mathrm{Span}\{g^2 \mid g \in \mathfrak{g}^{(i)}_{\bar{1}}\} & \text{for } p = 2.\end{cases} \tag{1}$$

Proof. Suppose $[\mathfrak{g},\mathfrak{g}] \neq \mathfrak{g}$. Then, the orthogonal complement to $[\mathfrak{g},\mathfrak{g}]$ with respect to $(-,-)$ contains a non-zero element u. Since $\mathfrak{g}$ is simple, it has zero center, and hence there exists an $x \in \mathfrak{g}$ such that $[u,x] \neq 0$. Since the form $(-,-)$ is non-degenerate, there exists a $y \in \mathfrak{g}$ such that $([u,x],y) \neq 0$. But then $(u,[x,y]) = ([u,x],y) \neq 0$, contradicting the fact that $u \in [\mathfrak{g},\mathfrak{g}]^\perp$. □

The next statement is a direct corollary of Lemma 1.6.

1.6.2. Corollary. *Let $p = 2$. Let $\mathfrak{g}$ be a simple Lie superalgebra and $\boldsymbol{F}(\mathfrak{g})$ its desuperization, i.e., $\boldsymbol{F}$ is the functor that forgets squaring and parity. Let $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, then $\mathfrak{g}$ has a NIS if and only if $\boldsymbol{F}(\mathfrak{g})$ has a NIS.*

1.7. Lemma. *Let $p = 2$. Let $\mathfrak{g}$ be a simple Lie superalgebra such that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, and $S \subseteq \mathfrak{g}$ its subsuperspace such that $[S,\mathfrak{g}] \subseteq S$. Then, either $S = 0$ or $S = \mathfrak{g}$.*

Proof. Let $S \neq 0$, let $\overline{S}$ be the completion of S with respect to squaring. Since $\overline{S}$ is a subsuperspace of $\mathfrak{g}$, is closed under squaring, and $[\overline{S},\mathfrak{g}] = [S,\mathfrak{g}] \subseteq S$, it follows that $\overline{S}$ is an ideal, and hence $\overline{S} = \mathfrak{g}$. But then $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] = [\overline{S},\mathfrak{g}] \subseteq S$, and therefore $S = \mathfrak{g}$. □

Now we can formulate the statement we will use instead of Lemma 1.5 when $p = 2$:

1.8. Lemma (Analog of Lemma 1.5). *Let $p = 2$. Let $\mathfrak{g}$ be a simple Lie superalgebra with a NIS. Then, any homogenous IS on $\mathfrak{g}$ is either zero or non-degenerate.*

Proof. The proof is analogous to the proof of Lemma 1.5, but we use Lemmas 1.6 and 1.7 to show that $\mathrm{Ker}\,\omega$ is either 0 or $\mathfrak{g}$. □

Completion of the proof of Theorem 1.3. Now we can restrict ourselves to homogenous ω_1 and ω_2, since for any inhomogenous IS, its even and odd components are invariant and symmetric, and hence non-degenerate by Lemmas 1.5 and 1.8.

Fix a basis in $\mathfrak{g}$, and let B_1 and B_2 be Gram matrices of **non-degenerate** homogenous invariant symmetric forms ω_1 and ω_2; consider the 1-parameter family of invariant symmetric forms ω_λ with Gram matrices $B_\lambda = B_1 + \lambda B_2$. Consider B_λ just as a matrix, not supermatrix, and calculate its determinant; it is a polynomial of λ. Since $\mathbb{K}$ is algebraically closed, there exists a $\lambda_0 \in \mathbb{K}$ such that $\det B_{\lambda_0} = 0$. Then, the form ω_{λ_0} is degenerate. If the forms ω_1 and ω_2 are of the same parity, then $\omega_{\lambda_0} = 0$ by Lemmas 1.5 and 1.8. This means that $\omega_1 = -\lambda_0\omega_2$, i.e., any two homogeneous NISes of the same parity are proportional to one another. So, if ω_1 and ω_2 are even, then the superdimension of the space of even NISes is equal to 1; if ω_1 and ω_2 are odd, then the superdimension of the space of odd NISes is equal to ε.

Let now forms ω_1 and ω_2 be of different parity. Then, $\operatorname{Ker}\omega_{\lambda_0}$ is a non-trivial $\operatorname{ad}_{\mathfrak g}$-invariant subspace, but it is not a subsuperspace. Consider two subspaces

$$\begin{aligned}V &:= \operatorname{Ker}\omega_{\lambda_0}\cap\mathfrak g_{\bar 0}\oplus\operatorname{Ker}\omega_{\lambda_0}\cap\mathfrak g_{\bar 1},\\ W &:= \operatorname{pr}_{\bar 0}(\operatorname{Ker}\omega_{\lambda_0})\oplus\operatorname{pr}_{\bar 1}(\operatorname{Ker}\omega_{\lambda_0}),\end{aligned}$$

where $\operatorname{pr}_{\bar 0}$ and $\operatorname{pr}_{\bar 1}$ are projections to $\mathfrak g_{\bar 0}$ and $\mathfrak g_{\bar 1}$, respectively. Since both V and W are $\operatorname{ad}_{\mathfrak g}$-invariant sub**super**spaces, they are either 0 or $\mathfrak g$ by Lemmas 1.6 and 1.7. Since ω_{λ_0} is non-zero, $V\neq\mathfrak g$, so $V=0$; since ω_{λ_0} is degenerate, $W\neq 0$, so $W=\mathfrak g$. Hence, there exists an odd isomorphism f between linear superspaces $\mathfrak g_{\bar 0}$ and $\mathfrak g_{\bar 1}$ such that

$$\operatorname{Ker}\omega_{\lambda_0}=\operatorname{Span}\{x+f(x)\mid x\in\mathfrak g_{\bar 0}\}.$$

Since $\operatorname{Ker}\omega_{\lambda_0}$ is $\operatorname{ad}_{\mathfrak g}$-invariant, we see that for all $a,b\in\mathfrak g_0$:

$$\begin{aligned}&[a,b+f(b)]=[a,b]+[a,f(b)]\Longrightarrow[a,f(b)]=f([a,b]);\\ &[a+f(a),b]=[a,b]+[f(a),b]\Longrightarrow[f(a),b]=f([a,b]);\\ &[f(a),b+f(b)]=[f(a),b]+[f(a),f(b)]\\ &\;=[f(a),f(b)]+f([a,b])\Longrightarrow f([a,b])=f([f(a),f(b)]).\end{aligned}\tag{2}$$

The bottom line in (2) implies that

$$[f(a),f(b)]=[a,b]\text{ for all }a,b\in\mathfrak g_{\bar 0}.\tag{3}$$

Up to this moment our reasoning did not depend on p.

Now, let $p\neq 2$. Notice that the left-hand side of the equality (3) is symmetric while the right-hand side is anti-symmetric. Hence,

$$[f(a),f(b)]=[a,b]=0\text{ and }[f(a),b]=f([a,b])=0\text{ for all }a,b\in\mathfrak g_{\bar 0},$$

and so $\mathfrak g$ is commutative, i.e., has zero bracket. This contradicts the simplicity of $\mathfrak g$, and hence there can not exist two NISes of different parity on a simple Lie superalgebra over an algebraically closed field of characteristic $p\neq 2$. There can not exist an inhomogeneous NIS in this situation either, as was mentioned above. This completes the proof of the theorem if $p\neq 2$.

If $p=2$, no such conclusion follows from equality (3); it only tells us that $\mathfrak g$ is a queerification of $\mathfrak g_{\bar 0}$ and $\mathfrak g_{\bar 0}$ is restricted. Besides, the restriction of the even of the two forms ω_i to $\mathfrak g_{\bar 0}$ is a NIS.

Conversely, let $\mathfrak g$ be a queerification of a simple restricted Lie algebra $\mathfrak g_{\bar 0}$ with a NIS ω. Then, $\mathfrak g$ can be represented in the form $\mathfrak g=\mathfrak g_{\bar 0}\otimes\mathcal A$, where $\mathcal A$ is an associative and commutative, but **not** supercommutative,[1] superalgebra spanned by an even element 1 (unit) and an **odd** one a, subject to the relation $a^2=1$, and the natural bracket (for squaring, see Subsection 1.8.1):

$$[x\otimes\varphi,y\otimes\psi]=[x,y]\otimes\varphi\psi\text{ for any }x,y\in\mathfrak g_{\bar 0},\ \varphi,\psi\in\mathcal A.$$

Determine two bilinear forms on $\mathfrak g:=\mathfrak g_{\bar 0}\otimes\mathcal A$:

$$\omega_i(x\otimes\varphi,y\otimes\psi)=\omega(x,y)f_i(\varphi\psi)\text{ for }i=1,2,$$

where $f_1(\alpha\cdot 1+\beta\cdot a)=\alpha$ and $f_2(\alpha\cdot 1+\beta\cdot a)=\beta$ for any $\alpha,\beta\in\mathbb K$. It is clear that both these forms are non-degenerate, invariant, and symmetric; ω_1 is even and ω_2 is odd. □

[1] Recall that in any *supercommutative* superalgebra $\mathcal A$, we have $a^2=0$ for any $a\in\mathcal A_{\bar 1}$.

1.8.1. **Under what conditions on $\boldsymbol{\mathcal{A}}$ is $\mathfrak{g}_{\bar 0}\otimes\boldsymbol{\mathcal{A}}$ a Lie superalgebra?.** (A more general, but non-super, setting is discussed in [Z].) It is well-known that for any Lie algebra L and any commutative associative algebra A, one can introduce a Lie algebra structure on $L\otimes A$ by setting

$$[l_1\otimes a_1, l_2\otimes a_2] := [l_1, l_2]\otimes a_1a_2 \text{ for any } l_i\in L \text{ and } a_i\in A.$$

The commutativity of A is required for this bracket to be antisymmetric.

Analogously, if $p\neq 2$, L is any Lie superalgebra, and A is any supercommutative associative superalgebra, then one can introduce a Lie superalgebra structure on $L\otimes A$ by setting

$$[l_1\otimes a_1, l_2\otimes a_2] := (-1)^{p(l_2)p(a_1)}[l_1, l_2]\otimes a_1a_2.$$

Again, the supercommutativity of A is required for this bracket to be super antisymmetric.

If $p=2$, then the bracket on the Lie superalgebra is again just antisymmetric, so one could assume that the above definition would work even if A is just commutative, not supercommutative (note that if $p=2$, then a supercommutative superalgebra is commutative as well). But if $p=2$, one has to define the squaring on $(L\otimes A)_{\bar 1}$ separately from the bracket. For an arbitrary Lie superalgebra L, this can be done only if A is a supercommutative associative superalgebra, and the definition is as follows:

$(l\otimes a)^2=0$ for $l\in L_{\bar 0}$, $a\in A_{\bar 1}$;

$(l\otimes a)^2=l^2\otimes a^2$ for $l\in L_{\bar 1}$, $a\in A_{\bar 0}$;

$$\left(\sum_{1\leq i\leq n} l_i\otimes a_i\right)^2 = \sum_{1\leq i\leq n}(l_i\otimes a_i)^2+\sum_{1\leq i<j\leq n}(-1)^{p(l_j)p(a_i)}[l_i, l_j]\otimes a_ia_j$$

for any homogenous $l_i\in L$ and $a_i\in A$ such that $p(l_i)+p(a_i)=\bar 1$ for all i. (The sign $(-1)^{p(l_j)p(a_i)}$ is not needed if $p=2$, but it is introduced here so that the definition would work for other characteristics as well.)

Observe that if A is just commutative, not supercommutative, it is impossible to define $(l\otimes a)^2$ for any l even and a odd. One can not set $(l\otimes a)^2:=l^2\otimes a^2$, because for any even l its square is not defined. If A is supercommutative, then for a odd, we set $a^2=0$, and everything is OK.

However, if $p=2$ and L is a Lie superalgebra with a $2|4$-structure (in particular, if L is a restricted Lie algebra with a 2-structure $l\longmapsto l^{[2]}$ for any $l\in L$), then one can introduce a Lie superalgebra structure on $L\otimes A$ even if A is a commutative associative superalgebra, as follows:

$(l\otimes a)^2=l^{[2]}\otimes a^2$ for any $l\in L_{\bar 0}$ and $a\in A_{\bar 1}$;

$(l\otimes a)^2=l^2\otimes a^2$ for any $l\in L_{\bar 1}$ and $a\in A_{\bar 0}$;

$$\left(\sum_{1\leq i\leq n} l_i\otimes a_i\right)^2 = \sum_{1\leq i\leq n}(l_i\otimes a_i)^2+\sum_{1\leq i<j\leq n}[l_i, l_j]\otimes a_ia_j$$

for any homogenous $l_i\in L$ and $a_i\in A$ such that $p(l_i)+p(a_i)=\bar 1$ for all i. (No signs here because $2|4$-structure exists only if $p=2$.)

1.9. **On *degenerate* invariant symmetric bilinear forms.** Let $p=2$ and let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra such that $[\mathfrak{g},\mathfrak{g}]\neq\mathfrak{g}$, i.e., there are elements which can not be obtained by bracketing, only by squaring, and let $k:=\operatorname{codim}[\mathfrak{g},\mathfrak{g}]$; for example, if $\mathfrak{g}=\mathfrak{osp}^{(1)}_{I\Pi}(1|2)$, then $k=2$. The dimension of the space of **degenerate** invariant symmetric bilinear forms on $\mathfrak{g}$ is $\geq\frac{1}{2}k(k+1)=\dim S^2(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$. These are forms whose kernel contains $[\mathfrak{g},\mathfrak{g}]$. Such forms $(-,-)$ are invariant because for them

$$([x,y],z)=(x,[y,z])=0 \text{ for any } x,y,z\in\mathfrak{g}.$$

2. Basics, naively

2.1. **Sign Rule, skew and anti.** The definitions of *Lie superalgebra* are the same for any $p \neq 2$ or 3: they are obtained from the definition of the Lie algebra using the Sign Rule "if something of parity p is moved past something of parity q, the sign $(-1)^{pq}$ accrues; formulas defined on homogeneous elements are extended to any elements via linearity".

In addition to the Sign Rule, note that *morphisms* of superalgebras are only even ones.

Observe that sometimes applying the Sign Rule requires some dexterity (we have to distinguish between two versions both of which turn in the nonsuper case into one, called either skew- or anti-commutativity, see [Gr])

$$\begin{array}{ll} ba = (-1)^{p(b)p(a)}ab & \text{(supercommutativity)} \\ ba = -(-1)^{p(b)p(a)}ab & \text{(superanticommutativity)} \\ ba = (-1)^{(p(b)+1)(p(a)+1)}ab & \text{(superskew-commutativity)} \\ ba = -(-1)^{(p(b)+1)(p(a)+1)}ab & \text{(superantiskew-commutativity)} \end{array}$$

The *skew* formulas are those that can be "straightened" by the change of parity of the space on which the structure is considered, whereas the prefix *anti* requires an overall minus sign regardless of parity. In what follows the symmetry of bilinear forms and commutativity of superalgebras are named according to the above definitions.

2.2. **Lie superalgebra, pre-Lie superalgebra, Leibniz superalgebra.**

For any p, a *Lie superalgebra* is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ such that the even part $\mathfrak{g}_{\bar 0}$ is a Lie algebra, the odd part $\mathfrak{g}_{\bar 1}$ is a $\mathfrak{g}_{\bar 0}$-module (made into the two-sided one by *anti*-symmetry, i.e., $[y, x] = -[x, y]$ for any $x \in \mathfrak{g}_{\bar 0}$ and $y \in \mathfrak{g}_{\bar 1}$) and on $\mathfrak{g}_{\bar 1}$, a *squaring* $x \mapsto x^2$ and the *bracket* are defined via a linear[2] map $s : S^2(\mathfrak{g}_{\bar 1}) \longrightarrow \mathfrak{g}_{\bar 0}$, where $S^2(\mathfrak{g}_{\bar 1})$ is the symmetric square of $\mathfrak{g}_{\bar 1}$, as follows:

$$x^2 := s(x \otimes x). \tag{4}$$

$$[x, y] := s(x \otimes y + y \otimes x) \text{ for any } x, y \in \mathfrak{g}_{\bar 1}. \tag{5}$$

The linearity of the $\mathfrak{g}_{\bar 0}$-valued function s implies that

$$(ax)^2 = a^2x^2 \text{ for any } x \in \mathfrak{g}_{\bar 1} \text{ and } a \in \mathbb{K}, \text{ and} \tag{6}$$

$$[x, y] = (x + y)^2 - x^2 - y^2 \text{ for any } x, y \in \mathfrak{g}_{\bar 1}, \tag{7}$$

$$\text{and the bracket on } \mathfrak{g}_{\bar 1} \text{ is a bilinear form with values in } \mathfrak{g}_{\bar 0}. \tag{8}$$

For $p \neq 2$ or 3, the Jacoby identity for three equal to each other odd elements is

$$3[x, [x, x]] = 0$$

---

[2]Squaring is a nonlinear map from $\mathfrak{g}_{\bar 1}$ to $\mathfrak{g}_{\bar 0}$, whereas s is a linear map from $S^2(\mathfrak{g}_{\bar 1})$ to $\mathfrak{g}_{\bar 0}$. From this linearity we deduce that

$$(ax)^2 = s((ax) \otimes (ax)) = s(a^2 \cdot x \otimes x) = a^2 s(x \otimes x) = a^2x^2$$

and

$$\begin{gathered}(x + y)^2 - x^2 - y^2 = s((x + y) \otimes (x + y)) - s(x \otimes x) - s(y \otimes y) \\ = s((x + y) \otimes (x + y) - x \otimes x - y \otimes y) = s(x \otimes y + y \otimes x) = [x, y].\end{gathered}$$

What is nice in this approach: before, we had to state conditions

"$(ax)^2 = a^2x^2$" and

"$(x + y)^2 - x^2 - y^2$ is a bilinear form"

separately; now, they both follow from the definition "$x^2 = s(x \otimes x)$, where s is a linear map on $S^2(\mathfrak{g}_{\bar 1})$". Both concepts "linear map" and "symmetric square" are simple and natural.

which is equivalent to

$$[x, [x, x]] = 0 \text{ for any } x \in \mathfrak{g}_{\bar{1}} \text{ if } p \neq 3. \tag{9}$$

<u>If $p = 3$</u>, to the antisymmetry and Jacoby identity amended by the Sign Rule, we have to add condition (9), separately. The superalgebra satisfying the antisymmetry and Jacoby identity, but not the condition (9) is called *pre-Lie superalgebra*; for examples, see [BBH].

<u>If $p = 2$</u>, the *antisymmetry* for $p = 2$ should be replaced by an equivalent for $p \neq 2$, but otherwise stronger *alternating* or *antisymmetry* condition

$$[x, x] = 0 \text{ for any } x \in \mathfrak{g}_{\bar{0}}.$$

The *Jacobi identity* involving odd elements takes the form of the following two conditions:

$$[x^2, y] = [x, [x, y]] \text{ for any } x \in \mathfrak{g}_{\bar{1}}, y \in \mathfrak{g}_{\bar{0}}, \tag{10}$$

$$[x^2, x] = 0 \text{ for any } x \in \mathfrak{g}_{\bar{1}}. \tag{11}$$

The superalgebra satisfying Jacoby identity, but just symmetry not the antisymmetry for its even elements is a special case of *Leibniz superalgebra.*

Over $\mathbb{Z}/2$, the condition (11) must (see Example 2.2.1) be replaced with a more general one:

$$[x^2, y] = [x, [x, y]] \text{ for any } x, y \in \mathfrak{g}_{\bar{1}}. \tag{12}$$

For any other ground field this more general condition is a corollary of condition (11).

2.2.1. **Example.** This example shows that over $\mathbb{Z}/2$, condition (12) is not a corollary of condition (11). Take a $2|3$-dimensional algebra $\mathfrak{g}$ with the even part spanned by elements A and B, the odd part spanned by elements X, Y and Z, and the algebraic structure given as follows:

- the even part is commutative;
- the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is given by the following multiplication table

	X	Y	Z
A	0	Z	0
B	Z	0	0

- the squaring on $\mathfrak{g}_{\bar{1}}$ is given by the formula

$$(aX + bY + cZ)^2 = a^2A + b^2B \text{ for all } a, b, c \in \mathbb{Z}/2.$$

This algebra $\mathfrak{g}$ satisfies (11) since

$$[(aX + bY + cZ)^2, aX + bY + cZ] = [a^2A + b^2B, aX + bY + cZ] = (a^2b + ab^2)Z,$$

and since $a, b \in \mathbb{Z}/2$, we have $a^2 = a$ and $b^2 = b$, so $a^2b + ab^2 = 0$.

But the algebra $\mathfrak{g}$ does not satisfy (12): $[X^2, Y] = Z \neq [X, [X, Y]] = 0$.

2.2.2. **Other requirements.** If one wants $\mathfrak{der}\, \mathfrak{g}$ to be a Lie superalgebra for any Lie superalgebra $\mathfrak{g}$, one has to add the condition

$$D(x^2) = [Dx, x] \text{ for any odd element } x \in \mathfrak{g} \text{ and any } D \in \mathfrak{der}\, \mathfrak{g}. \tag{13}$$

Condition (13) is a generalization of (10), (11) for $D = \mathrm{ad}_x$, where $x \in \mathfrak{g}$.

By an *ideal* of a Lie superalgebra one always means an *homogeneous* ideal; for $p = 2$, the ideal should be closed with respect to squaring.

The Lie superalgebra $\mathfrak{g}$ is said to be *simple* if $\dim \mathfrak{g} > 1$ and $\mathfrak{g}$ has no nontrivial (distinct from 0 and $\mathfrak{g}$) ideals.

For $p = 2$, the definition of the derived algebra of the Lie superalgebra $\mathfrak{g}$ changes, see eq. (1).

An even linear map $r:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ is said to be a *representation of the Lie superalgebra* $\mathfrak{g}$ and V is a $\mathfrak{g}$-*module* if

$$\begin{array}{l} r([x,y])=[r(x),r(y)]\quad \text{for any } x,y\in\mathfrak{g};\\ r(x^2)=(r(x))^2 \text{ for any } x\in\mathfrak{g}_{\bar 1}.\end{array}$$

2.3. **The $p|2p$-structure or restricted Lie superalgebra.** Let the ground field $\mathbb{K}$ be of characteristic $p>0$, and $\mathfrak{g}$ a Lie algebra. For every $x\in\mathfrak{g}$, the operator $(\mathrm{ad}_x)^p$ is a derivation of $\mathfrak{g}$. If this derivation is an inner one, then the Lie algebra $\mathfrak{g}$ is said to be *restricted* or *having a p-structure*. In other words, a *p-structure* is a map $[p]:\mathfrak{g}\longrightarrow\mathfrak{g},\ x\mapsto x^{[p]}$ such that

$$\begin{array}{l} [x^{[p]},y]=(\mathrm{ad}_x)^p(y)\quad \text{for any } x,y\in\mathfrak{g},\\ (ax)^{[p]}=a^px^{[p]}\quad \text{for any } a\in\mathbb{K},\ x\in\mathfrak{g},\\ (x+y)^{[p]}=x^{[p]}+y^{[p]}+\sum\limits_{1\leq i\leq p-1}s_i(x,y)\quad \text{for any } x,y\in\mathfrak{g},\end{array}$$

where $is_i(x,y)$ is the coefficient of λ^{i-1} in $(\mathrm{ad}_{\lambda x+y})^{p-1}(x)$.

For a Lie superalgebra $\mathfrak{g}$ in characteristic $p>0$, let the Lie algebra $\mathfrak{g}_{\bar 0}$ be restricted and

$$[x^{[p]},y]=(\mathrm{ad}_x)^p(y)\quad \text{for any } x\in\mathfrak{g}_{\bar 0},\ y\in\mathfrak{g}. \tag{14}$$

This gives rise to the map (recall that the bracket of odd elements is the polarization of the squaring $x\mapsto x^2$)

$$[2p]:\mathfrak{g}_{\bar 1}\to\mathfrak{g}_{\bar 0},\ x\mapsto (x^2)^{[p]},$$

satisfying the condition

$$[x^{[2p]},y]=(\mathrm{ad}_x)^{2p}(y)\quad \text{for any } x\in\mathfrak{g}_{\bar 1},\ y\in\mathfrak{g}. \tag{15}$$

The pair of maps $[p]$ and $[2p]$ is called a *p-structure* (or, sometimes, a *$p|2p$-structure*) on $\mathfrak{g}$, and $\mathfrak{g}$ is said to be *restricted*. It suffices to determine the $p|2p$-structure on any basis of $\mathfrak{g}$; on simple Lie superalgebras there is at most one $p|2p$-structure.

• If (15) is not satisfied, the p-structure on $\mathfrak{g}_{\bar 0}$ does not have to generate a $p|2p$-structure on $\mathfrak{g}$: even if the actions of $(\mathrm{ad}_x)^p$ and $\mathrm{ad}_{x^{[p]}}$ coincide on $\mathfrak{g}_{\bar 0}$, they do not have to coincide on the whole of $\mathfrak{g}$. The *restricted universal enveloping* $U^{[p]}(\mathfrak{g})$ is defined for Lie algebras $\mathfrak{g}$ as the quotient of the universal enveloping $U(\mathfrak{g})$ modulo the two-sided ideal $\mathfrak{i}$ generated by $g^{\otimes p}-g^{[p]}$ for any $g\in\mathfrak{g}$.

For the Lie superalgebra $\mathfrak{g}$ The *restricted universal enveloping* $U^{[p]}(\mathfrak{g})$ is the quotient of $U(\mathfrak{g})$ modulo the two-sided ideal $\mathfrak{i}$ generated by $g^{\otimes p}-g^{[p]}$ for any $g\in\mathfrak{g}_{\bar 0}$.

The seemingly needed further factorization modulo the two-sided ideal generated by the elements $g^{\otimes 2p}-g^{[2p]}$ for any $g\in\mathfrak{g}_{\bar 1}$ is not needed: these elements are in $\mathfrak{i}$ automatically, as is not difficult to show.

• If $p=2$, there are **other**, seemingly natural, versions of restrictedness, see [BLLS]; we will not consider them in this text.

2.4. **Linear (matrix) Lie superalgebras.** Certain basics of Linear Superalgebra are not well-known, no harm in reminding a bit more than is strictly necessary: (1) the Lie superalgebra of series $\mathfrak{q}$ is an analog of $\mathfrak{gl}$; (2) we consider non-degenerate symmetric bilinear forms, so it is natural to introduce the Lie superalgebras of series $\mathfrak{osp}$ (resp. $\mathfrak{pe}$) which preserve even (resp. odd) such forms.

The *general linear* Lie superalgebra of all supermatrices of size Par corresponding to linear operators in the superspace $V=V_{\bar 0}\oplus V_{\bar 1}$ over the ground field $\mathbb{K}$ is denoted by $\mathfrak{gl}(\mathrm{Par})$, where $\mathrm{Par}=(p_1,\dots,p_{|\mathrm{Par}|})$ is an ordered collection of parities of the basis vectors of V for which we take only vectors *homogeneous with respect to parity* and $|\operatorname{Par}| := \dim V$. Usually, for the *standard* (simplest from a certain point of view) format, $\mathfrak{gl}(\bar 0, \dots, \bar 0, \bar 1, \dots, \bar 1)$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar 0}|\dim V_{\bar 1})$. Any supermatrix in $\mathfrak{gl}(\operatorname{Par})$ can be uniquely expressed as the sum of its even and odd parts; in the standard format this is the following block expression; on non-zero summands the parity is defined:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad p\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right) = \bar 0,\ p\left(\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right) = \bar 1.$$

The *supertrace* is the map $\mathfrak{gl}(\operatorname{Par}) \longrightarrow \mathbb{K}$, $(X_{ij}) \longmapsto \sum (-1)^{p_i} X_{ii}$. Thus, in the standard format, $\operatorname{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{tr} A - \operatorname{tr} D$. Observe that for Lie superalgebra $\mathfrak{gl}_{\mathcal{C}}(p|q)$ over a supercommutative superalgebra $\mathcal{C}$, i.e., for supermatrices with elements in $\mathcal{C}$, we have

$$\begin{aligned} &\operatorname{str} X = \operatorname{tr} A - (-1)^{p(X)} \operatorname{tr} D \text{ for any } X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \\ &\text{where } p(X) = p(A_{ij}) = p(D_{kl}) = p(B_{il}) + \bar 1 = p(C_{kj}) + \bar 1, \end{aligned}$$

so on odd supermatrices with entries in $\mathcal{C}$ such that $\mathcal{C}_{\bar 1} \neq 0$, the supertrace coincides with the trace.

Since $\operatorname{str}[x, y] = 0$, the subsuperspace of supertraceless matrices constitutes a Lie subsuperalgebra called *special linear* and denoted $\mathfrak{sl}(\operatorname{Par})$.

There are, however, at least two super versions of $\mathfrak{gl}(n)$, not one; for reasons, see [Lsos, Ch1, Ch.7]. The other version — $\mathfrak{q}(n)$ — is called the *queer* Lie superalgebra and is defined as the one that preserves — if $p \neq 2$ — the complex structure given by an *odd* operator J, i.e., $\mathfrak{q}(n)$ is the centralizer $\mathrm{C}(J)$ of J:

$$\mathfrak{q}(n) = \mathrm{C}(J) = \{X \in \mathfrak{gl}(n|n) \mid [X, J] = 0\}, \text{ where } J^2 = -\operatorname{id}.$$

It is clear that by a change of basis we can reduce J to the form (shape) J_{2n} in the standard format, and then the elements of $\mathfrak{q}(n)$ take the form

$$\mathfrak{q}(n) = \left\{ (A, B) := \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \text{ where } A, B \in \mathfrak{gl}(n) \text{ and } J_{2n} := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}. \tag{16}$$

(Over any algebraically closed field $\mathbb{K}$, instead of J we can take any odd operator K such that $K^2 = a \operatorname{id}_{n|n}$, where $a \in \mathbb{K}^\times$; and the Lie superalgebras $\mathrm{C}(K)$ are isomorphic for distinct K; if $p = 2$, it is natural to select $K^2 = \operatorname{id}$.)

On $\mathfrak{q}(n)$, the *queertrace* is defined: $\operatorname{qtr}\colon (A, B) \longmapsto \operatorname{tr} B$. Denote by $\mathfrak{sq}(n)$ the Lie superalgebra of *queertraceless* matrices; set $\mathfrak{psq}(n) := \mathfrak{sq}(n)/\mathbb{K}1_{2n}$.

Clearly, $\mathfrak{gl}$ and $\mathfrak{q}$ correspond to the super version of Schur's lemma over an algebraically closed field: an irreducible module over a collection S of homogeneous operators can be *absolutely irreducible*, i.e., have no proper invariant subspaces, then the only operator commuting with S is a scalar (the $\mathfrak{gl}$ case), or can have in invariant subspace, which is not a sub**super**space, then the superdimension of the module is of the form $n|n$ and an odd operator K interchanges the homogeneous components of the module (the $\mathfrak{q}$ case).

2.4.1. **Supermatrices of operators.** To the linear map of superspaces $F : V \longrightarrow W$ there corresponds the dual map $F^* : W^* \longrightarrow V^*$ between the dual superspaces. In bases consisting of homogeneous vectors $v_i \in V$ of parity $p(v_i)$, and $w_j \in W$ of parity $p(w_j)$, the formula $F(v_j) = \sum_i w_i X_{ij}$ assigns to F the supermatrix X. In the dual bases, the *supertransposed* matrix X^{st} corresponds to F^*:

$$(X^{st})_{ij} = (-1)^{(p(v_i)+p(w_j))p(w_j)} X_{ji}.$$

In the standard supermatrix format we have

$$X := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto X^{st} = \begin{pmatrix} A^t & (-1)^{p(X)}C^t \\ -(-1)^{p(X)}B^t & D^t \end{pmatrix} = \begin{cases} \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix} & \text{if } p(X) = \bar{0}, \\ \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix} & \text{if } p(X) = \bar{1}. \end{cases}$$

2.4.2. **Supermatrices of bilinear forms.** Having selected a basis (by definition consisting of vectors homogeneous with respect to parity) of the superspace V, we define the Gram matrix $B = (B_{ij})$ of the bilinear form B^f on V by the formula

(17) $$B_{ij} = (-1)^{p(B)p(v_i)} B^f(v_i, v_j) \text{ for the basis vectors } v_i \in V.$$

This formula for the Gram matrix of B^f allows us to identify any bilinear form $B(V, W)$ with an operator, an element of $\mathrm{Hom}(V, W^*)$, see [Lsos, Ch.1].

Recall that the *upsetting* of bilinear forms $u\colon \mathrm{Bil}(V, W) \longrightarrow \mathrm{Bil}(W, V)$ is given by the formula

(18) $$u(B^f)(w, v) = (-1)^{p(v)p(w)} B^f(v, w) \text{ for any } v \in V \text{ and } w \in W.$$

Let now $W = V$, and $\mathrm{Bil}(V) := \mathrm{Bil}(V, V)$. The shape of the Gram matrix B^u of a homogeneous form $u(B^f)$ in the standard format of V is as follows

(19) $$B^u = \begin{pmatrix} R^t & (-1)^{p(B)}T^t \\ (-1)^{p(B)}S^t & -U^t \end{pmatrix} \text{ for } B = \begin{pmatrix} R & S \\ T & U \end{pmatrix},$$

The form B^f is said to be *symmetric* if $B^u = B$, and *antisymmetric* if $B^u = -B$. (Here we correct terminology of [BKLS]: there are no *super*symmetric bilinear forms.) In particular, the form on $\mathfrak{gl}(V)$ (resp. $\mathfrak{q}(V)$) given by

$$(X, Y) = \mathrm{str}(XY) \text{ for any } X, Y \in \mathfrak{gl}(V) \text{ (resp. } (X, Y) = \mathrm{qtr}(XY) \text{ for any } X, Y \in \mathfrak{q}(V))$$

is symmetric.

Clearly, the *upsetting* of Gram matrices of bilinear forms is *not* supertransposition.

Observe that **the passage from V to $\Pi(V)$ turns every symmetric form B on V into an antisymmetric one on $\Pi(V)$ and vice versa**.

Most popular normal shapes of the (Gram matrices of) the even non-degenerate symmetric form are the ones which in the standard format are as follows:

$$B_{ev}(m|2n) = \mathrm{diag}(1_m, J_{2n}) := \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix} \text{ or } \mathrm{diag}(A_m, J_{2n}) := \begin{pmatrix} A_m & 0 \\ 0 & J_{2n} \end{pmatrix},$$
$$\text{where } J_{2n} = \mathrm{antidiag}(1_n, -1_n) := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \text{ and } A_m = \mathrm{antidiag}(1, \dots, 1).$$

The Lie superalgebra $\mathfrak{aut}(B) \subset \mathfrak{gl}(\mathrm{Par})$ that preserves the Gram matrix B of the form $B^f \in \mathrm{Bil}(V)$ consists of the supermatrices $X \in \mathfrak{gl}(\mathrm{Par})$ such that

$$X^{st}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(\mathrm{Par}).$$

The usual notation for $\mathfrak{aut}(B_{ev}(m|2n))$ is $\mathfrak{osp}(m|2n)$; sometimes one writes more explicitly, $\mathfrak{osp}^{sy}(m|2n)$. Observe that the antisymmetric non-degenerate bilinear form is preserved by the "symplectico-orthogonal" Lie superalgebra $\mathfrak{osp}^{a}(m|2n)$ isomorphic to $\mathfrak{osp}^{sy}(m|2n)$.

A non-degenerate **symmetric** odd bilinear form $B_{odd}(n|n)$ can be reduced to a normal shape whose matrix in the standard format is J_{2n}, see (19), NOT $\Pi_{2n} := \text{antidiag}(1_n, 1_n)$ which is **anti**symmetric, see [Lsos], contrary to hasty expectations. The usual notation for $\mathfrak{aut}(B_{odd}(\text{Par}))$ is $\mathfrak{pe}(\text{Par})$. The passage from V to $\Pi(V)$ establishes an isomorphism $\mathfrak{pe}^{sy}(\text{Par}) \cong \mathfrak{pe}^{a}(\text{Par})$. These isomorphic Lie superalgebras are called, as A. Weil suggested, *periplectic*.

A large class of Lie superalgebras either simple, or relatives of simple, have a Cartan matrix. Neither periplectic superalgebras nor their simple relatives have Cartan matrices; this is not so for $p = 2$, see [BGL1] for a classification of all finite-dimensional modular Lie superalgebras with indecomposable Cartan matrix over any algebraically closed field.

2.5. **Queerification for $p = 2$ (from** [BLLS]**).** If $p = 2$, then we can queerify any restricted Lie algebra $\mathfrak{g}$ as follows. We set $\mathfrak{q}(\mathfrak{g})_{\bar 0} = \mathfrak{g}$ and $\mathfrak{q}(\mathfrak{g})_{\bar 1} = \Pi(\mathfrak{g})$; define the multiplication involving the odd elements as follows:

$$[x, \Pi(y)] = \Pi([x, y]); \quad (\Pi(x))^2 = x^{[2]} \quad \text{for any } x, y \in \mathfrak{g}. \tag{20}$$

Clearly, if $\mathfrak{g}$ is restricted and $\mathfrak{i} \subset \mathfrak{q}(\mathfrak{g})$ is an ideal, then $\mathfrak{i}_{\bar 0}$ and $\Pi(\mathfrak{i}_{\bar 1})$ are ideals in $\mathfrak{g}$. So, if $\mathfrak{g}$ is restricted and simple, then $\mathfrak{q}(\mathfrak{g})$ is a simple Lie superalgebra. (Note that $\mathfrak{g}$ has to be simple as a Lie algebra, not just as a *restricted* Lie algebra, i.e., $\mathfrak{g}$ is not allowed to have **any** ideals, not only restricted ones.)

As an aside remark, observe that (the generalization of) the queerification is one of the two procedures producing all simple Lie superalgebras, see [BLLS].

3. Basics, continued: the functor of points approach

3.1. **Morphisms of supervarieties.** Recall the definition of supermanifolds, see, e.g., [Del, Ch.1], [MaG]. Same as manifolds are glued from coordinate patches locally diffeomorphic to a ball in $\mathbb{R}^n$, supermanifolds are *ringed spaces*,[3] i.e., pairs $\mathcal{M} := (M, \mathcal{O}_\mathcal{M})$, where M is an m-dimensional manifold and $\mathcal{O}_\mathcal{M}$ is the structure sheaf of $\mathcal{M}$, locally isomorphic to $C^\infty(U) \otimes \Lambda^\bullet(n)$, where U is a domain in M, and $\Lambda^\bullet(n)$ is the Grassmann algebra with n anticommuting generators. Pairs $\mathcal{U} := (U, C^\infty(U) \otimes \Lambda^\bullet(n))$ are called *superdomains*. Morphisms of supermanifolds $\mathcal{M} \longrightarrow \mathcal{N}$ are pairs (φ, φ^*), where $\varphi : M \longrightarrow N$ is a diffeomorphism and $\varphi^* : \mathcal{O}_\mathcal{N} \longrightarrow \mathcal{O}_\mathcal{M}$ is a preserving the natural parity of Grassmann-valued sheaves of functions morphism of sheaves.

That was definition of a superdomain over $\mathbb{R}$. Over any ground field $\mathbb{K}$, we will later define affine superschemes (which play the role of superdomains in the algebraic setting).

Consider a superdomain $\mathcal{U}$ of superdimension $0|n$. Unlike superdomains of superdimension $a|b$ with $a \neq 0$, we can consider $\mathcal{U}$ over any ground field $\mathbb{K}$ and call it *superpoint*. The underlying domain (or variety) of $\mathcal{U}$ is a single point. Since $\mathcal{O}(\mathcal{U}) = \Lambda^\bullet(n)$, the superpoint $\mathcal{U}$ has a lot of nontrivial automorphisms, namely the group $\text{Aut}^{\bar 0}\, \Lambda^\bullet(n)$ of parity preserving automorphisms of $\Lambda^\bullet(n)$. All such automorphisms are of the form (here the ξ_i are generators of $\Lambda^\bullet(n)$)

$$\xi_j \;\mapsto\; \sum_r \varphi_j^r \xi_r + \sum_{s\geq 1} \sum_{j_1<\dots<j_{2s+1}} \varphi_j^{j_1\dots j_{2s+1}} \xi_{j_1} \dots \xi_{j_{2s+1}}, \tag{21}$$

where the matrix (φ_j^r) with elements in $\mathbb{K}$ is invertible. We see that such automorphisms constitute the algebraic group (or a Lie group if $\text{char}\, \mathbb{K} = 0$) whose Lie algebra consists of the even elements of the Lie superalgebra $\mathfrak{vect}(0|n) := \mathfrak{der}\, \Lambda^\bullet(n)$.

[3] To understand the definition of ringed spaces, read [MaAG].

What corresponds to the odd vector fields from $\mathfrak{der}\,\Lambda^{\boldsymbol{\cdot}}(n)$? We consider the answer in the following more general setting, the one involving both even and odd indeterminates, but only for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. In the absence of even indeterminates the displayed formulas below in this subsection are meaningful for any $\mathbb{K}$, i.e., for superpoints.

Let E be the trivial vector bundle over a domain U of dimension m with fiber V of dimension n; let $\Lambda^{\boldsymbol{\cdot}}(E)$ be the exterior algebra of E. To the bundle E, we assign the superdomain $\mathcal{U}=(U,C^\infty(\mathcal{U}))$, where $C^\infty(\mathcal{U})$ is the superalgebra of smooth sections of $\Lambda^{\boldsymbol{\cdot}}(E)$. Clearly, each automorphism of the pair $(U,\Lambda^{\boldsymbol{\cdot}}(E))$, i.e., of the vector bundle $\Lambda^{\boldsymbol{\cdot}}(E)$, induces an automorphism of the superdomain $\mathcal{U}$.

However, *not all* automorphisms of the superdomain $\mathcal{U}$ are obtained in this way. By definition, every *morphism of superdomains* $\varphi\colon\mathcal{U}\longrightarrow\mathcal{V}$ is in one-to-one correspondence with a homomorphism of the superalgebras of functions $\varphi^*\colon C^\infty(\mathcal{V})\longrightarrow C^\infty(\mathcal{U})$.

Every homomorphism φ^* is defined on the (topological[4]) generators of the superalgebra, in other words: coordinates. Consider the corresponding formulas

$$\begin{cases}\varphi^*(u_i) &= \varphi_i^0(u)+\boxed{\sum\limits_{r\geq1}\ \sum\limits_{i_1<\dots<i_{2r}}\varphi_i^{i_1\dots i_{2r}}(u)\xi_{i_1}\dots\xi_{i_{2r}}},\\ \varphi^*(\xi_j) &= \sum\limits_{r\geq0}\ \sum\limits_{j_1<\dots<j_{2r+1}}\psi_j^{j_1\dots j_{2r+1}}(u)\xi_{j_1}\dots\xi_{j_{2r+1}}.\end{cases}\tag{22}$$

A) The terms $\varphi^*(u_i)=\varphi_i^0(u)$ determine an endomorphism of the underlying domain U.

B) The linear terms $\varphi^*(\xi_j)=\sum_i\psi_j^i(u)\xi_i$ determine endomorphisms of the fiber V (over each point its own fiber, as the dependence on u shows).

C) The higher terms in ξ from the right-hand side of the expression of $\varphi^*(\xi_j)$ in (22) determine an endomorphism of the larger fiber — the Grassmann superalgebra $\Lambda^{\boldsymbol{\cdot}}(E)$.

The endomorphisms A)–C) existed in Differential Geometry, and no need to introduce a flashy prefix "super" was felt.

The difference between the vector bundle $\Lambda^{\boldsymbol{\cdot}}(E)$ and the superdomain $\mathcal{U}$ is most easily understood when the reader looks at the boxed terms in (22). These terms, meaningless in the conventional Differential Geometry, make sense in the new paradigm:

In the category of superdomains there are more morphisms than in the category of vector bundles: morphisms with non-vanishing boxed terms in (22) are exactly the additional ones.

However, even the boxed terms in eq. (22) is not all we get in the new setting: we still did not describe any of odd parameters of endomorphisms. **To account for the odd parameters, we have to consider the functor from the category of supercommutative superalgebras to the category of groups** $C\longmapsto\mathrm{Aut}_C^{\bar0}(C^\infty(\mathcal{U})\otimes C)$,i.e., the parity preserving[5] C-linear

[4]A *topological algebra* A over a topological field $\mathbb{K}$ is a topological vector space together with a bilinear multiplication $A\times A\longrightarrow A$, continuous in a certain sense, and such that A is an algebra over $\mathbb{K}$. Usually the continuity of the multiplication means that the multiplication is continuous as a map between topological spaces $A\times A\longrightarrow A$. A set S is a *generating set* of a topological algebra A if the smallest closed subalgebra of A containing S is A.

[5]It took a while to acknowledge the fact that there are automorphisms of the Grassmann algebra $\mathbb{C}[\xi]$, considered as an associative algebra, and more generally, of $\mathbb{C}[x,\xi]$, where $x=(x_1,\dots,x_m)$ and $\xi=(\xi_1,\dots,\xi_n)$, not preserving parity, see [LSe]. The meaning of such general automorphisms is unclear at the moment. D.L. conjectures that by analogy with supersymmetries that are wider than symmetries under groups or Lie algebras, these general automorphisms further widen supersymmetries. Recently, U. Iyer proved that *Volichenko algebras*,

automorphisms of the form

$$\begin{cases} \varphi^*(u_i) &= \varphi_i^0(u) + \boxed{\sum\limits_{r\geq 1}\ \sum\limits_{i_1<\dots<i_r} \varphi_i^{i_1\dots i_r}(u)\xi_{i_1}\dots\xi_{i_r}}, \\ \varphi^*(\xi_j) &= \sum\limits_{r\geq 0}\ \sum\limits_{j_1<\dots<j_{2r+1}} \psi_j^{j_1\dots j_{2r+1}}(u)\xi_{j_1}\dots\xi_{j_{2r+1}} \\ & \quad + \boxed{\psi_i^0(u) + \sum\limits_{r\geq 1}\ \sum\limits_{j_1<\dots<j_{2r}} \psi_j^{j_1\dots j_{2r}}(u)\xi_{j_1}\dots\xi_{j_{2r}}}, \end{cases} \tag{23}$$

where

$$\varphi_i^0(u), \varphi_i^{i_1\dots i_{2r}}(u),\ \psi_j^{j_1\dots j_{2r+1}}(u) \in C_{\bar 0} \text{ whereas } \psi_i^0(u), \psi_j^{j_1\dots j_{2r}}(u),\ \varphi_i^{i_1\dots i_{2r+1}}(u) \in C_{\bar 1}$$

for all r are parameters of the infinite-dimensional supergroup (infinitesimally: Lie superalgebra $\mathfrak{vect}(m|n)$) of automorphisms of $C^\infty(\mathcal{U})$ or, equivalently, of diffeomorphisms of $\mathcal{U}$..

3.2. **Supervarieties and superschemes (after [L1]).** Over the ground field $\mathbb{R}$ or $\mathbb{C}$, let E be a vector bundle over M with fiber V. Let $\Lambda^\bullet(V)$ be the Grassmann superalgebra of V and $U \subset M$ is an open domain.

A *supervariety* is a ringed space $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$, where M is a variety, and the sheaf $\mathcal{O}_\mathcal{M}$ is locally isomorphic to $\mathcal{O}_U \otimes \Lambda^\bullet(V)$, or its quotient. A supervariety isomorphic to the ringed space whose structure sheaf is the sheaf of sections of the vector bundle $\Lambda^\bullet(E)$ over M is called *split*.

Observe that every object in the category of smooth supervarieties (= supermanifolds) is split (for a 1-line proof, see [MaG]), and therefore there is a one–to–one correspondence between the set of objects in the category of vector bundles over manifolds M and the set of objects in the category of smooth supermanifolds. The latter category has, however, many more morphisms than the former, see eq. (23).

A purely algebraic version of the supermanifold over any field (or any commutative ring) of any characteristic is an *affine superscheme* $\operatorname{Spec} C$, where C is a supercommutative superalgebra or a superring. The affine superscheme is defined literally as the affine scheme: its points are prime ideals defined literally as in the commutative case, i.e., $\mathfrak{p} \subsetneq C$ is *prime*[6]

$$\text{if } a, b \in C \text{ and } ab \in \mathfrak{p}, \text{ then either } a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}. \tag{24}$$

The space of any affine scheme is endowed with Zariski topology and the structure sheaf, defined as in the commutative case, see [MaAG], whose 1968 edition was the source of inspiration for [L1].

NB: there is just one subtlety: the localization of the superalgebra (or superring) C at the prime ideal $\mathfrak{p}$ should be performed with respect to the multiplicative system $S_\mathfrak{p} := C \setminus \mathfrak{p}$ and it is not at all obvious if we should consider — in order to have well-defined fractions with

defined as the nonhomogeneous subalgebras of Lie superalgebras, play the role of Lie algebras for the groups of C-points of such general automorphisms, see [I].

[6]K. Coulembier pointed out to us that the so far conventional definitions in the non-commutative case are at variance with the common sense: at the moment, if (24) holds, $\mathfrak{p}$ is called (say, in Wikipedia) *completely prime* while it would be natural to retain the term *prime*, as is done in [L1] and by J. Bernstein, P. Deligne et al in [Del], since the definition is the same as in the commutative case despite the fact that supercommutative rings are not commutative, whereas the term *prime* is (so far) applied to any ideal $P \subsetneq R$ of the non-commutative ring R which for any two ideals A and B in R satisfies the following version of (24):

$$\text{if } AB \subset P, \text{ then either } A \subset P \text{ or } B \subset P.$$

non-homogeneous denominators — only left fractions $b^{-1}a$ or only right fractions ab^{-1} or the equivalence (equality) of fractions does not depend on the choice (left or right). This is the only non-trivial place in the transition from Grothendieck's schemes to superschemes.

3.3. **The functor of points represented by a supervariety or a superscheme.** Smooth manifolds can be described as sets of points with a topology. For *manifolds-with-boundary* (which, strictly speaking, are not manifolds, hence the suggestive notation "in one word") or over fields of characteristic $p > 0$, the set of points does not define the variety or scheme; the same is true for supervarieties and superschemes. To determine one of such objects $\mathcal{M}$, we consider it parametrized by a superscheme $\operatorname{Spec} C$. In other words, we consider collections $\operatorname{Hom}(\operatorname{Spec} C, \mathcal{M})$ of $\operatorname{Spec} C$-points of $\mathcal{M}$, usually called *C-points of $\mathcal{M}$*. If $\mathcal{M}$ can be recovered from its algebra of functions $\mathcal{F}$, as is the case, e.g., with affine (super)schemes, we can consider $\operatorname{Hom}(\mathcal{F}, C)$ instead of $\operatorname{Hom}(\operatorname{Spec} C, \mathcal{M})$.

3.3.1. **Linear supervariety$\longleftrightarrow$linear superspace.** First, recall that there is a one-to-one correspondence between linear (a.k.a. vector) superspaces V and the *linear supervarieties*

$$\mathcal{V} = (V_{\bar 0}, \mathcal{O}_{V_{\bar 0}} \otimes \Lambda^{\bullet}(V_{\bar 1}^*)).$$

The *morphisms* of linear superspaces constitute the space $\operatorname{Hom}(V, W) := (\underline{\operatorname{Hom}}(V, W))_{\bar 0}$, whereas the *supervariety of linear homomorphisms* $V \longrightarrow W$ is the linear supervariety corresponding to the superspace $\underline{\operatorname{Hom}}(V, W) := \operatorname{Hom}(V, W)$.

In various instances, e.g., dealing with actions of supergroups, it is more convenient for many[7] to consider, instead of vector superspace V and even the linear supermanifold $\mathcal{V}$, the functor $\mathsf{ScommSalgs}_{\mathbb{K}} \rightsquigarrow \mathsf{Mod}_{\mathbb{K}}$ from the category of supercommutative $\mathbb{K}$-superalgebras C to the category of C-modules represented by V and $\mathcal{V}$:

$$C \longmapsto \mathcal{V}(C) = V(C) := V \otimes C \text{ for any } C,$$

where "any" is understood inside a suitable category (e.g., finitely generated over $\mathbb{K}$).

3.3.2. **Lie superalgebras.** In the above terms, a *Lie superalgebra in the category of supervarieties* is a vector superspace $\mathfrak{g}$, or a linear supervariety (supermanifold) $\mathcal{G}$ corresponding to it, corepresenting the functor from the category of supercommutative $\mathbb{K}$-superalgebras C to the category of Lie superalgebras understood "naively".

In other words, considering **co**representing functor instead of a representing one, we replace $\mathfrak{g}$, or the linear supervariety corresponding to it, by the algebra $P(\mathfrak{g})$ of polynomial functions on $\mathfrak{g}$. (Even over $\mathbb{R}$ we have to replace the spaces by the algebras of polynomial functions on these spaces.)

Clearly, $P(\mathfrak{g})$ is a free supercommutative (and associative with 1) superalgebra generated by $\mathfrak{g}^*$, i.e., there is a natural isomorphism of functors

$$C \mapsto \operatorname{Hom}_{\mathsf{ScommSalgs}_{\mathbb{K}}}(P(\mathfrak{g}), C) \text{ and } C \mapsto \operatorname{Hom}_{\mathbb{K}\text{-}Vect}(\mathfrak{g}^*, C),$$

whereas the second functor is naturally isomorphic to $C \mapsto \operatorname{Forget}_{\mathbb{K}\text{-}Vect \to \mathsf{Sets}}(\mathfrak{g} \otimes C)$.

At least, so it works for finite-dimensional and purely even $\mathfrak{g}$.

To the Lie superalgebra homomorphisms (in particular, to representations) a morphism of the respective functors should correspond.

[7]This convenience is not just a matter of taste or experience and habit. More precisely, one has to work with either commuting diagrams, like in [MaAG, Section 1.15.4], or with matrix realizations, as one does when working with Lie group actions. The language of points allows one to *actually compute* something, like passing from the invariant language of operators (commutative diagrams) to matrices.

Clearly, if $\mathfrak{g}$ is a Lie superalgebra, then $\mathfrak{g}(C) := \mathfrak{g} \otimes C$ is also a Lie superalgebra for any C *functorially in* C. The last three words mean that

> for any morphism of supercommutative superalgebras $C \longrightarrow C'$, there exists a morphism of Lie superalgebras $\mathfrak{g} \otimes C \longrightarrow \mathfrak{g} \otimes C'$ so that a composition of morphisms of supercommutative superalgebras
>
> $$C \longrightarrow C' \longrightarrow C'' \tag{25}$$
>
> goes into the composition of Lie superalgebra morphisms
>
> $$\mathfrak{g}(C) \longrightarrow \mathfrak{g}(C') \longrightarrow \mathfrak{g}(C'');$$
>
> the identity map goes into the identity map, etc.

An *ideal* $\mathfrak{h} \subset \mathfrak{g}$ represents the collection of ideals $\mathfrak{h}(C) \subset \mathfrak{g}(C)$ for every C.

In the above terms the action of Lie or algebraic (over any field) supergroup $\mathcal{G}$ (which is a group in the category of supervariaties) in the superspace V is the action of $\mathcal{G}(C)$ in $V(C)$ for every C and these actions should be compatible with morphisms of supercommutative superalgebras $C \longrightarrow C'$ in the same sense as in (25).

3.3.3. **Non-split supervarieties.** The category C of analytic (over $\mathbb{C}$) and algebraic (over any field) supervarieties have non-split objects. In other words, in the categories of analytic and algebraic supervarieties with a fixed underlying variety, there are not only more morphisms than in the category of the vector bundles over the underlying variety, but there are even more objects. Observe that to every non-split object there corresponds its split version. The obstructions to splitness were first (and most lucidly) described in [Gre]; for examples, see [MaG, Va, Vi]. We will not consider such non-split horrors in this note for the following reasons, see [AD]:

$$\textbf{every } C^\infty \textbf{ supermanifold is locally split}; \tag{26}$$

over any contractible paracompact subset in $\mathbb{R}^n$ and over any affine supervariety, all vector bundles are trivial. Since all (finite-dimensional) linear supervarieties are affine, we do not have to bother about non-split supervarieties thinking about Lie superalgebras.

Observe that triviality of the vector bundles does not necessarily takes place for algebraic supervarieties and supeschemes, and also over other fields and non-affine schemes, see [AD].

However, we can consider any such category C; then any object $\mathfrak{g} \in \operatorname{Ob} \mathsf{C}$ of this category representing the functor $C \mapsto \mathfrak{g}(C) := \operatorname{Hom}_{\mathsf{C}}(\operatorname{Spec} C, \mathfrak{g})$, i.e., satisfying conditions (25), is said to be a *Lie superalgebra in the category* C.

3.3.4. **The "even rules principle", see** [Del, §§1.7, 1.8]**, does not work for** $\boldsymbol{p = 2}$**.** For $p = 2$, the functor

$$C \longmapsto \underline{\mathfrak{g}}(C) := (\mathfrak{g} \otimes C)_{\bar 0}$$

defines at best a $\mathbb{Z}/2$-graded Lie algebra. To define a Lie bracket in $(\mathfrak{g} \otimes C)_{\bar 0}$, it suffices to know the bracket in $\mathfrak{g}$ (and the multiplication in C, of course), whereas the squaring in $\mathfrak{g}$ is not needed. This leads to the following two problems:

First, the squaring can be non-uniquely recovered from the functor, moreover, different squarings corresponding to the same functor can determine non-isomorphic Lie superalgebras.

For example, consider a $1|1$-dimensional superspace spanned by an even element A and an odd X. On this space, consider two Lie superalgebra structures: in both of them the bracket is identically equal to 0, but in one of them, denote it $\mathfrak{g}_1$, we set $X^2 = 0$, i.e., this is a commutative Lie superalgebra, whereas in the other one, denote it $\mathfrak{g}_2$, we set $X^2 = A$. These Lie superalgebras are non-isomorphic, but the functors of points corresponding to them coincide (to any supercommutative superalgebra C the functors assign a commutative Lie algebra), because the brackets in the Lie superalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the same.

Second, To the functor of points there might correspond a $\mathbb{Z}/2$-graded Lie algebra, to which a superalgebra corresponds on which it is impossible to define any squaring. For example, consider $\mathfrak{osp}(1|2)$, spanned by H, $X_\pm$, $X_\pm^2$. Now take its first "derived algebra" ignoring squaring, i.e., take $\mathfrak{h} := \mathrm{Span}\{[x, y] \mid x, y \in \mathfrak{osp}(1|2)\}$; this $\mathfrak{h}$ is spanned by H and $X_\pm$. This "derived algebra" is closed with respect to the bracket, so it represents the functor $\mathsf{ScommSalgs} \rightsquigarrow \mathsf{LieAlgs}$ given by $C \longmapsto (\mathfrak{h} \otimes C)_{\bar 0}$. But it is impossible to define squaring on $\mathfrak{h}$: the square of X_+ should be an even element x such that ad_x sends X_- to X_+, but there is no such element x in $\mathfrak{h}$.

3.3.5. **The "even rules principle", see** [Del, §§1.7, 1.8]**, gives too much for $p = 3$.** Any *pre-Lie super algebra* represents a functor $\mathsf{ScommSalgs} \rightsquigarrow \mathsf{LieAlgs}_\mathbb{K}$ given by $C \longmapsto (\mathfrak{g} \otimes C)_{\bar 0}$. Whereas we'd like to have a functor represented by *Lie superalgebras*, not *pre-Lie superalgebras*.

3.4. **Deformations and deforms with odd parameters.** Which of the infinitesimal deformations can be extended to a global one is a separate much tougher question, usually solved *ad hoc*; for examples over fields of characteristics 3 and 2, see [BLW] and references therein. Deformations with odd parameters are always integrable. Let us give two graphic examples.

1) **Deformations of representations**. Consider a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. The tangent space of the moduli superspace of deformations of ρ is isomorphic to $H^1(\mathfrak{g}; V \otimes V^*)$. For example, if $\mathfrak{g}$ is the $0|n$-dimensional (i.e., purely odd) Lie superalgebra (with the only bracket possible: identically equal to zero), its only irreducible representations are the 1-dimensional trivial one, $\mathbb{1}$, and $\Pi(\mathbb{1})$. Clearly,

$$\mathbb{1} \otimes \mathbb{1}^* \simeq \Pi(\mathbb{1}) \otimes \Pi(\mathbb{1})^* \simeq \mathbb{1},$$

and, because the Lie superalgebra $\mathfrak{g}$ is commutative, the differential in the cochain complex is zero. Therefore

$$H^1(\mathfrak{g}; \mathbb{1}) = E^1(\mathfrak{g}^*) \simeq \Pi(\mathfrak{g}^*),$$

so there are $\dim\ \mathfrak{g}$ odd parameters of deformations of the trivial representation. If we consider $\mathfrak{g}$ "naively", all of these odd parameters will be lost.

2) **Deformations of the brackets**. Let C be a supercommutative superalgebra.

Recall, see [Ru], where the non-super case is considered, that a *deformation* of a Lie superalgebra $\mathfrak{g}$ over $\mathrm{Spec}\, C$, is a Lie algebra $\mathfrak{G}$ such that $\mathfrak{G} \simeq \mathfrak{g} \otimes C$, as spaces. The deformation is *trivial* if $\mathfrak{G} \simeq \mathfrak{g} \otimes C$, as Lie superalgebras over C, not just as C-modules, and *non-trivial* otherwise.

Generally, the *deforms* — the results of deformations — of a given Lie superalgebra $\mathfrak{g}$ over $\mathbb{K}$ are Lie superalgebras $\mathfrak{G} \otimes_I \mathbb{K}$, where I is any closed point in $\mathrm{Spec}\, C$.

In particular, consider a deformation with an odd parameter τ. This is a Lie superalgebra $\mathfrak{G}$ isomorphic to $\mathfrak{g} \otimes \mathbb{K}[\tau]$ as a **super space**; if, moreover, $\mathfrak{G} \simeq \mathfrak{g} \otimes \mathbb{K}[\tau]$ as a **Lie superalgebra**, i.e.,

$$[a \otimes f, b \otimes g] = (-1)^{p(f)p(b)}[a, b] \otimes fg \text{ for all } a, b \in \mathfrak{g} \text{ and } f, g \in \mathbb{K}[\tau],$$

then the deformation is considered *trivial* (and *non-trivial* otherwise). Observe that $\mathfrak{g} \otimes \tau$ is not an ideal of $\mathfrak{G}$: the ideal should be a free $\mathbb{K}[\tau]$-module.

Comment. In a sense, the people who ignore odd parameters of deformations have a point: we (rather they) consider classification of simple Lie superalgebras (or whatever other problem) over the ground field $\mathbb{K}$, right? However, the odd parameters of deformations are no less natural, actually, than the odd part of the Lie superalgebra itself. In order to see these parameters, we have to consider whatever we are deforming not over $\mathbb{K}$, but over $\mathbb{K}[\tau]$.

We do the same, actually, when τ is even and we consider formal deformations over $\mathbb{K}[[\tau]]$. If the formal series in τ converges in a domain D, we can evaluate τ for any $\tau \in D$ and — if $\dim \mathfrak{g} < \infty$ — consider copies $\mathfrak{g}_\tau$, where $\tau \in D$, of the same dimension as $\mathfrak{g}$. If the parameter is formal or odd, such an evaluation is only possible trivially: $\tau \mapsto 0$.

4. Examples

For examples (even classification in several cases) of deforms of known symmetric simple modular Lie superalgebras, see [BGL2], where the cocycles we consider below are given explicitly, in terms of a Chevalley basis. Here we consider one of the simplest examples of deformations with an odd parameter and several other examples.

4.1. Lemma. *Consider the Lie superalgebra* $\mathfrak{oo}^{(1)}_{III}(1|2)$ *and its deform with the help of the cocycle* c_{-2}*, and* $\mathfrak{oo}^{(1)}_{II}(1|2)$ *and its deforms with the help of the cocycles* c_1 *or* c_2*, see* [BGL2, §7.2 and §7.3]. *There is no NIS on any of these deformed Lie superalgebras* $\mathfrak{g} := \mathfrak{g}_{c_i}$.

Proof. The direct computations show that $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Hence, no NIS on $\mathfrak{g}$ due to Lemma 1.6. □

4.2. **On a map of cochains of $\mathbf{F}(\mathfrak{g})$ to cochains of $\mathfrak{g}$.** Let $\mathfrak{g}$ be a Lie superalgebra and $\mathbf{F}(\mathfrak{g})$ its desuperization, i.e., $\mathbf{F}$ is the functor that forgets squaring and parity. The choice of a basis in $\mathfrak{g}$ induces the corresponding choice of a basis in $\mathbf{F}(\mathfrak{g})$ (the opposite is not true since a basis vector in $\mathbf{F}(\mathfrak{g})$ might be inhomogeneous in $\mathfrak{g}$). This correspondence between bases defines a $\mathbb{K}$-linear map $i\colon \mathbf{F}(\mathfrak{g}) \to \mathfrak{g}$. If $p = 2$, then $E(V) \subsetneq S(V)$, where $E(V)$ is the exterior algebra and $S(V)$ is the symmetric algebra of the space V, and the map i induces an injective map $i_*\colon C^\bullet(\mathbf{F}(\mathfrak{g})) \longrightarrow C^\bullet(\mathfrak{g})$ between spaces of cochains. The map i_* does not necessary commute with the differential: as it was noted in [BGL2, § 7.11], not every cocycle of $\mathbf{F}(\mathfrak{g})$ defines a cocycle of $\mathfrak{g}$. Interestingly, sometimes, the map i_* allows us to express some of cocycles of $\mathfrak{g}$, representing cohomology classes, in terms of images of cocycles of $\mathbf{F}(\mathfrak{g})$ under i_* (plus perhaps terms defining a deformation of the squaring), see examples in Lemmas 4.3.1 and 4.3.2.

4.3. **Two lemmas on NISes on deforms.** For $n = 3$ and 4, the NISes on deforms of $\mathfrak{wk}(n;\alpha)$ can be directly translated to the corresponding superizations — deforms of $\mathfrak{bgl}(n;\alpha)$ — because the squaring is not involved at all in the invariance condition for the bilinear form. In what follows $\mathfrak{g}_c$ denotes a deform(ation) of $\mathfrak{g}$ with the help of a cocycle c.

4.3.1. Lemma. *For Lie superalgebra* $\mathfrak{g} = \mathfrak{bgl}(4;\alpha)$*, where* $\alpha \neq 0, 1$*, all deforms depend on even parameters; see* [BGL2]. *Choose a basis in* $\mathfrak{g}$ *as in* [BGL2].

These deforms preserve a NIS with the same Gram matrix as that of the NIS on $\mathfrak{g}$*, except for the deform* $\mathfrak{g}_{c_0}$ *of* $\mathfrak{bgl}(4;\alpha)$ *with cocycle* c_0 *in which case the Gram matrix* Γ_{c_0} *is a different one. The desuperization of* Γ_{c_0} *coincides with the Gram matrix, described in* [BKLS, Claim 3.3]*, of the corresponding deform of* $\mathfrak{wk}(4;\alpha)$ *with cocycle* c_0.

Proof. For a basis in $H^2(\mathbf{F}(\mathfrak{g}); \mathbf{F}(\mathfrak{g}))$ take (the classes of) $c_{\pm12}$, $c_{\pm10}$, $c^1_{\pm8}$, $c^2_{\pm8}$, $c^1_{\pm6}$, $c^2_{\pm6}$, $c^1_{\pm4}$, $c^2_{\pm4}$, $c_{\pm2}$, c_0; for their explicit expressions, see [BGL2, § 7.12]. Due to the symmetry of the root system, it suffices to consider only cocycles of non-positive degree. As it was noted in [BGL2, § 7.13], the corresponding basis elements (of non-positive degree) in $H^2(\mathfrak{g}; \mathfrak{g})$ are the following cocycles

$$\begin{aligned}
&\bar c_{-12} = i_*(c_{-12}), && \bar c_{-10} = i_*(c_{-10}),\\
&\bar c^1_{-8} = i_*(c^1_{-8}), && \bar c^2_{-8} = i_*(c^2_{-8}) + \alpha^2(1+\alpha)h_3\otimes(\hat x_{11})^{\wedge 2},\\
&\bar c^1_{-6} = i_*(c^1_{-6}) + (1+\alpha)h_3\otimes(\hat x_9)^{\wedge 2}, && \bar c^2_{-6} = i_*(c^2_{-6}) + \alpha^2(1+\alpha)(h_3+h_4)\otimes(\hat x_8)^{\wedge 2},\\
&\bar c^1_{-4} = i_*(c^1_{-4}) + (1+\alpha)(h_3+h_4)\otimes(\hat x_6)^{\wedge 2}, && \bar c^2_{-4} = i_*(c^2_{-4}) + (1+\alpha)(h_3+h_4)\otimes(\hat x_6)^{\wedge 2},\\
&\bar c_{-2} = i_*(c_{-2}) + (1+\alpha)h_4\otimes(\hat x_1)^{\wedge 2}, && \bar c_0 = i_*(c_0).
\end{aligned}$$

where x_i, h_i, y_i are elements of the basis described in [BGL1]. Hence, since the squaring is not involved in the invariance condition, a NIS on $\mathbf{F}(\mathfrak{g}_c)$ induces a NIS on $\mathfrak{g}_c$. □

4.3.2. Lemma. *For Lie superalgebra $\mathfrak{g} = \mathfrak{bgl}^{(1)}(3;\alpha)/\mathfrak{c}$, where $\alpha \neq 0, 1$, all deforms depend on even parameters; see* [BGL2]. *Choose a basis in $\mathfrak{g}$ as in* [BGL2].

These deforms preserve a NIS with the same Gram matrix as that of the NIS on $\mathfrak{g}$, except for the deform $\mathfrak{g}_{c_0}$ of $\mathfrak{bgl}^{(1)}(3;\alpha)/\mathfrak{c}$ with cocycle c_0 in which case the Gram matrix Γ_{c_0} is a different one. The desuperization of Γ_{c_0} coincides with the Gram matrix, described in [BKLS, Claim 3.4]*, of the corresponding deform of $\mathfrak{wk}^{(1)}(3;\alpha)/\mathfrak{c}$ with cocycle c_0.*

Proof. For the basis in $H^2(\mathbf{F}(\mathfrak{g});\mathbf{F}(\mathfrak{g}))$ take (the classes of) $c_{\pm6}$, $c^1_{\pm4}$, $c^2_{\pm4}$, $c_{\pm2}$, c_0; for their explicit expressions, see [BGL2, § 7.10]. Due to symmetry of the root system, it suffices to consider only cocycles of non-positive degree. The corresponding basis elements (of non-positive degree) in $H^2(\mathfrak{g};\mathfrak{g})$ are the following cocycles, see [BGL2, § 7.11]

$$\bar c_{-4} = i_*(c^2_4),\quad \bar c_{-2} = i_*(c_{-2}),\quad \bar c_0 = i_*(c_0).$$

Hence, since the squaring is not involved in the invariance condtion, a NIS on $\mathbf{F}(\mathfrak{g}_c)$ induces a NIS on $\mathfrak{g}_c$. □

4.4. Claim. *For $\mathfrak{g} = \mathfrak{q}(\mathfrak{sl}(3))$, consider its two deformations: $\mathfrak{g}_A(\tau)$ given by the odd cocyle A with parameter τ and $\mathfrak{g}_B(\varepsilon)$ with parameter ε given by the even cocycle B; for explicit expressions of these cocycles, see* [BGL2, Lemma 8.3].

The space of NISes on $\mathfrak{g}_A(\tau)$ is of rank $1|1$ over $\mathbb{K}[\tau]$.

The spaces of NISes on $\mathfrak{g}_B(\varepsilon)$ is $1|1$-dimensional if $\varepsilon \neq 1$, and 0-dimensional if $\varepsilon = 1$.

Proof. Although the proof is analytical, the statement is called Claim because the expressions of NISes are obtained with the aid of *SuperLie*.

Choose a basis in $\mathfrak{g}$ given by the Chevalley basis in $\mathfrak{sl}(3)$, namely, x_i, y_i, h_i, and Πx_i, Πy_i, Πh_i, also called *elements of the Chevalley basis*.

A) There are two NISes, $\omega_{\rm tr}$ and $\omega_{\rm qtr}$, on $\mathfrak{q}(\mathfrak{sl}(3))$ defined by the trace and queer trace, respectively. Then, a NIS ω on $\mathfrak{g}_A(\tau)$ is defined as follows

$$\omega = c_1\omega_{\rm tr} + c_2\omega_{\rm qtr} + \tau(c_3\omega_{\rm tr} + c_4\omega_{\rm qtr} + c_1B_1 + c_2B_2),\quad c_1, c_2, c_3, c_4 \in \mathbb{K},$$

where B_1 (resp. B_2) is an odd (resp. even) bilinear form for which

$$\begin{array}{l}
B_1(\Pi h_1, h_1) = B_1(\Pi h_1, h_2) = B_1(\Pi h_2, h_1) = B_1(\Pi h_2, h_2) = 1,\\
B_2(y_1, x_1) = B_2(y_3, x_3) = B_2(\Pi h_2, \Pi h_1) = B_2(\Pi y_2, \Pi x_2) = 1.
\end{array}$$

and zero on all other pairs of the Chevalley basis elements. Observe that $\mathfrak{g}_A(\tau)$ is a free module over $\mathbb{K}[\tau]$. An arbitrary element $t = a + b\tau \in \mathbb{K}[\tau]$ is defined by a pair of numbers $a, b \in \mathbb{K}$. Set $t_1 = c_1 + c_3\tau$ and $t_2 = c_2 + c_4\tau \in \Lambda[\tau]$. We have

$$\omega = t_1(\omega_{\rm tr} + \tau B_1) + t_2(\omega_{\rm qtr} + \tau B_2)$$

Therefore, the space of NISes on $\mathfrak{g}_A(\tau)$ is of rank $1|1$ over $\mathbb{K}[\tau]$.

B) Note that the bracket in $\mathfrak{g}_B(\varepsilon)$ is nonlinear with respect to ε. There are two NISes on $\mathfrak{g}_B(\varepsilon)$ when $\varepsilon \neq 1$:

1) An even NIS ω_e for which (and zero on all other pairs of the Chevalley basis elements)

$$\begin{array}{ll} \omega_e(h_1, h_2) = 1, & \omega_e(x_1, y_1) = 1 + \varepsilon^2, \\ \omega_e(x_2, y_2) = 1 + \varepsilon^2, & \omega_e(x_3, y_3) = 1 + \varepsilon, \\ \omega_e(\Pi h_1, \Pi h_2) = 1 + \varepsilon, & \omega_e(\Pi x_1, \Pi y_1) = 1 + \varepsilon^2, \\ \omega_e(\Pi x_2, \Pi y_2) = 1 + \varepsilon^2, & \omega_e(\Pi x_3, \Pi y_3) = 1. \end{array}$$

2) An odd NIS ω_o for which (and zero on all other pairs of the Chevalley basis elements)

$$\begin{array}{ll} \omega_o(h_1, \Pi h_2) = 1, & \omega_o(h_2, \Pi h_1) = 1 + \varepsilon, \\ \omega_o(x_1, \Pi y_1) = 1 + \varepsilon^2, & \omega_o(y_1, \Pi x_1) = 1 + \varepsilon, \\ \omega_o(x_2, \Pi y_2) = 1 + \varepsilon^2, & \omega_o(y_2, \Pi x_2) = 1 + \varepsilon, \\ \omega_o(x_3, \Pi y_3) = 1, & \omega_o(y_3, \Pi x_3) = 1. \end{array}$$

It is easy to see that ω_e is a deformation of the NIS defined by the trace, and ω_o is a deformation of the NIS defined by the queer trace.

There is no NIS on $\mathfrak{g}_B(\varepsilon)$ for $\varepsilon = 1$.

Thus, the space of NISes on $\mathfrak{g}_B(\varepsilon)$ is $(1|1)$-dimensional if $\varepsilon \neq 0$ and 0-dimensional if $\varepsilon = 1$. □

4.5. Lemma. *The Lie superalgebra* $\mathfrak{k}(1; n|1)$ *and its* $(n-2)$*-parametric family of even deforms described in* [KL, Theorem 6.2] *have no NIS.*

Proof. For these Lie superalgebras, we have $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, and hence no NIS due to Lemma 1.6. □

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