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SPLITTING NECKLACES, WITH CONSTRAINTS

DUŠKO JOJIĆ, GAIANE PANINA, AND RADE ŽIVALJEVIĆ

Abstract. We prove several versions of *Alon's necklace-splitting theorem*, subject to additional constraints, as illustrated by the following results.

(1) The "almost equicardinal necklace-splitting theorem" claims that, without increasing the number of cuts, one guarantees the existence of a fair splitting such that each thief is allocated (approximately) one and the same number of pieces of the necklace, provided the number of thieves $r = p^\nu$ is a prime power.

(2) The "binary splitting theorem" claims that if $r = 2^d$ and the thieves are associated with the vertices of a d-cube then, without increasing the number of cuts, one can guarantee the existence of a fair splitting such that adjacent pieces are allocated to thieves that share an edge of the cube.

This result provides a positive answer to the "binary splitting necklace conjecture" of Asada at al. (Conjecture 2.11 in [5]) in the case $r = 2^d$.

1. Introduction

The *Splitting Necklace Theorem* of Noga Alon [1, 2] is one of the best known early results of topological combinatorics where the methods of algebraic topology were applied with great success. The name of the theorem stems from the interpretation of the interval $[0, 1]$ as an open (unclasped), continuous necklace where n probability measures μ_i describe the distribution of "precious gemstones" of n different type. The result, together with its discrete version [1, 3], solves the problem of finding the minimum number of the cuts of the necklace needed for a fair distribution of pieces among r persons (r "thieves" who stole the necklace).

Theorem 1.1. ([1]) *Let $\mu_1, \mu_2, \dots, \mu_n$ be a collection of n continuous probability measures on $[0, 1]$. Let $r \geq 2$ and $N := (r-1)n$. Then there exists a partition of $[0, 1]$ by N cut points into $N + 1$ intervals $I_1, I_2, \dots, I_{N+1}$ and a function $f : \{1, 2, \dots, N + 1\} \to \{1, \dots, r\}$ such that for each μ_i and each $j \in \{1, 2, \dots, r\}$,*

$$\sum_{f(p)=j} \mu_i(I_p) = 1/r\,. \qquad (*)$$

The "*fair splitting condition*" $(*)$ illustrates the requirement that each thief should be treated fairly and receive an equal net value of the necklace, as evaluated by each of the measures μ_i. Theorem 1.1 is optimal, as far as the

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number of cuts is concerned, which means that for a generic choice of measures a fair partition with less than $(r-1)n$ cuts is not possible. However, it is an interesting question if the *necklace-splitting theorem* can be refined by adding extra conditions (constraints) on how the pieces of the necklace are distributed among the thieves.

1.1. **Splitting necklaces with additional constraints.** Additional constraints in the necklace splitting problem (and in the general Tverberg problem) were originally introduced and studied in [21]. The emphasis in this and in a subsequent paper [10] was on finding good lower bounds on the number of distinct fair splittings of a generic necklace.

The more recent paper [5] links the necklace splitting problem with other fair division problems and emphasizes the importance of the so called "binary splitting of necklaces" for studying the equipartitions of mass distributions by hyperplanes.

Our central new results are necklace splitting theorems with constraints of the following two types:

(1) *"Almost equicardinal splitting"* (Theorem 4.3, Corollary 4.4). Assuming that r is a prime power we show (Theorem 4.3) that, with the same number of cuts $N := (r-1)n$ as in the original Alon's theorem, it is always possible to fairly divide the necklace such that each of the thieves is given either t or $t+1$ pieces where $t := \lfloor (N+1)/r \rfloor$. In the special case when $N+1$ is divisible by r we obtain as a corollary the result that there exists an *equicardinal* fair splitting of the necklace when each thief is given exactly the same number $t := (N+1)/r$ of pieces.

An interesting feature of the proof is that we initially use a larger number of cuts. Eventually we get rid of superfluous cuts and end up with the desired number $N := (r-1)n$. Unlike the original Alon's theorem we need the condition that $r = p^\nu$ is a prime power and it remains an interesting open problem if this condition can be relaxed.

(2) *"Binary splitting"* (Theorem 5.1, Conjecture 5.2). Suppose that $r = 2^d$ and assume that thieves are positioned at the vertices of the d-dimensional cube. A *binary necklace splitting* is a fair splitting with $N = (r-1)n$ cuts with the additional constraint that adjacent (possibly degenerate) pieces of the necklace are allocated to thieves whose vertices share an edge.

A binary necklace splitting theorem is proven in [5] for $d = 2$, that is for the case of 4 thieves. The idea of the proof was to embed the necklace into the Veronese (moment) curve, and apply an equipartition result by two hyperplanes, which turns out to be a binary splitting.

We prove the existence of a binary splitting (Theorem 5.1) for any $r = 2^d$ and with the same number of cuts $N := (r-1)n$, by applying a more direct combinatorial/topological argument.

1.2. Fair splitting of a discrete necklace.

The following theorem is referred to as the discrete necklace-splitting theorem.

Theorem 1.2. ([1]) *Every unclasped necklace with n types of beads and ra_i beads of type $i \in [n]$ has a fair splitting among r thieves with at most $(r-1)n$ cuts.*

Theorem 1.2 is a consequence of Theorem 1.1 by an elementary combinatorial argument, see [1, p. 249] [3, Lemma 7] or [5, Lemma 2.3]. By a similar argument each continuous necklace-splitting theorem with additional constraints, mentioned in Section 1.1, has an obvious discrete version.

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2. Preliminaries and main definitions

2.1. Partition/allocation of a necklace.

A partition of a necklace $[0,1]$ into $m = N+1$ parts is described by a sequence of cut points

$$0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_N \leq x_m = 1\,.$$

(Here and in the sequel, $m = N+1$.)

The associated, possibly degenerate intervals $I_j := [x_{j-1}, x_j]\,(j = 1,\ldots,m)$ are distributed among the thieves by an *allocation function* $f : [m] \to [r]$. The pair (x, f), where $x = (x_1, x_2, \ldots, x_N)$ is the sequence of cuts is called a *partition/allocation* of a necklace.

2.2. Fair and (k,s)-equicardinal partitions/allocations.

(1) A *partition/allocation* (x, f) of a necklace is *fair* if each measure is evenly distribute among the thieves, i.e. if for each measure μ_j and each thief $i \in [r]$,

$$\mu_j\Big(\bigcup_{\nu \in f^{-1}(i)} I_\nu\Big) = \frac{1}{r}.$$

(2) A partition/allocation (x, f) is (k,s)-*equicardinal* if
(i) each thief gets no more than $k+1$ parts (intervals), and (ii) the number of thieves receiving exactly $k+1$ parts is not greater than s.

Note that for an equicardinal fair division it is not important where we allocate the degenerate (one-point) segments. Actually, in our setting for almost equicardinal necklace-splitting, we prefer (Section 3) not to allocate them at all. However, we will use degenerate segments in a binary splitting.

2.3. **Collectively unavoidable complexes.** Collectively unavoidable r-tuples of complexes are introduced in [13]. They were originally studied as a common generalization of pairs of Alexander dual complexes, Tverberg unavoidable complexes of [7] and r-unavoidable complexes from [12].

Definition 2.1. An ordered r-tuple $\mathcal{K} = \langle K_1, \dots, K_r\rangle$ of subcomplexes of $2^{[m]}$ is *collectively r-unavoidable* if for each ordered collection $(A_1, \dots, A_r)$ of pair-wise disjoint sets in $[m]$ there exists i such that $A_i \in K_i$.

2.4. **Balanced simplicial complexes.**

Definition 2.2. We say that a simplicial complex $K \subseteq 2^{[m]}$ is *(m,k)-balanced* if it is positioned between two consecutive skeleta of the simplex on m vertices,

$$\binom{[m]}{\leqslant k} \subseteq K \subseteq \binom{[m]}{\leqslant k+1}. \tag{1}$$

2.5. **Borsuk-Ulam theorem for fixed point free actions.**

Theorem 2.3. (Volovikov [20]) *Let p be a prime number and $G = (\mathbb{Z}_p)^k$ an elementary abelian p-group. Suppose that X and Y are fixed-point free G-spaces such that $\widetilde{H}^i(X, \mathbb{Z}_p) \cong 0$ for all $i \leq n$ and Y is an n-dimensional cohomology sphere over $\mathbb{Z}_p$. Then there does not exist a G-equivariant map $f : X \to Y$.*

2.6. **Connectivity of symmetrized deleted joins.**

Definition 2.4. The *deleted join* [17, Section 6] of a family $\mathcal{K} = \langle K_i\rangle_{i=1}^r = \langle K_1, \dots, K_r\rangle$ of subcomplexes of $2^{[m]}$ is the complex $\mathcal{K}^*_\Delta = K_1 *_\Delta \cdots *_\Delta K_r \subseteq (2^{[m]})^{*r}$ where $A = A_1 \sqcup \cdots \sqcup A_r \in \mathcal{K}^*_\Delta$ if and only if A_j are pairwise disjoint and $A_i \in K_i$ for each $i = 1, \dots, r$. In the case $K_1 = \cdots = K_r = K$ this reduces to the definition of r-fold deleted join K^{*r}_Δ, see [17].

The *symmetrized deleted join* [16] of $\mathcal{K}$ is defined as

$$SymmDelJoin(\mathcal{K}) := \bigcup_{\pi \in S_r} K_{\pi(1)} *_\Delta \cdots *_\Delta K_{\pi(r)} \subseteq (2^{[m]})^{*r}_\Delta\,,$$

where the union is over the set of all permutations of r elements and $(2^{[m]})^{*r}_\Delta \cong [r]^{*m}$ is the r-fold deleted join of a simplex with m vertices.

An element $A_1 \sqcup \cdots \sqcup A_r \in (2^{[m]})^{*r}_\Delta$ is from here on recorded as $(A_1, A_2, \dots, A_r; B)$ where B is the complement of $\cup_{i=1}^r A_i$, so in particular $A_1 \sqcup \cdots \sqcup A_r \sqcup B = [m]$ is a partition of $[m]$ such that $A_i \neq \emptyset$ for some $i \in [r]$.

Lemma 2.5. *The dimension of the simplex can be read of from $|B|$ as*

$$\dim(A_1, \dots, A_r; B) = m - |B| - 1.$$

The following theorem is one of the two main results from [14].

Theorem 2.6. *Suppose that $\mathcal{K} = \langle K_i\rangle_{i=1}^r = \langle K_1, \dots, K_r\rangle$ is a* collectively r-unavoidable *family of subcomplexes of $2^{[m]}$. Moreover, we assume that there* *exists $k \geq 1$ such that K_i is (m,k)-balanced for each $i = 1, \dots, r$. Then the associated symmetrized deleted join*

$$SymmDelJoin(\mathcal{K}) = SymmDelJoin(K_1, \dots, K_r)$$

is $(m-r-1)$-connected.

The following theorem [16, Theorem 3.3] was originally proved by a direct shelling argument. As demonstrated in [14] it can be also deduced from Theorem 2.6.

Theorem 2.7. *Let $r, d \geq 2$ and assume that $rt + s = (r-1)d$ where r and s are the unique integers such that $t \geq 1$ and $0 \leq s < r$. Let $N = (r-1)(d+2)$ and $m = N+1$. Then the symmetric deleted join $SymmDelJoin(K_1, \dots, K_r)$ of the following skeleta of the simplex $\Delta^N = 2^{[N+1]}$,*

$$K_1 = \dots = K_s = \binom{[N+1]}{\leqslant t+2}, \quad K_{s+1} = \dots = K_r = \binom{[N+1]}{\leqslant t+1}. \tag{2}$$

is $(m-r-1)$-connected.

3. New configuration spaces for constrained splittings

Perhaps the main novelty in our approach and the central new idea, emphasizing the role of *collectively unavoidable complexes*, is the construction and application of *modified (refined)* configuration spaces for splitting necklaces.

We begin by recalling a "deleted join" version of the *configuration space/test map scheme* [25], applied to the problem of splitting necklaces, as described in [21] (see also [17] for a more detailed exposition).

3.1. **Primary configuration space.** The configuration space of all sequences $0 = x_0 \leq x_1 \leq \dots \leq x_N \leq x_m = 1\,(m = N+1)$ is an N-dimensional simplex Δ^N, where the numbers $\lambda_j := x_j - x_{j-1}\,(j = 1, \dots, m)$ play the role of barycentric coordinates. For a fixed allocation function $f : [m] \to [r]$, the set of all partitions/allocations (x, f) is also coordinatized as a simplex $C_f \cong \Delta^N$. The primary configuration space, associated to the necklace-splitting problem, is obtained by gluing together N-dimensional simplices C_f, one for each function $f : [m] \to [r]$. Note that the common face of C_{f_1} and C_{f_2} is the set of all pairs (x, f_1) $(\sim (x, f_2))$ such that $I_j = [x_{j-1}, x_j]$ is degenerate if $f_1(j) \neq f_2(j)$.

The simplicial complex obtained by this construction turns out to be (the geometric realization of) the deleted join $(\Delta^N)^{*r}_\Delta \cong [r]^{*m}$. Indeed, a simplex $\tau = (A_1, A_2, \dots, A_r; B) \in (\Delta^N)^{*r}_\Delta$ is described as a partition $A_1 \sqcup A_2 \sqcup \dots \sqcup A_r \sqcup B = [m]$, and a partition/allocation (x, f) is in (the geometric realization of) τ if and only if $B = \{j \in [m] \mid I_j = [x_{j-1}, x_j] \text{ is degenerate}\}$ and $A_i = f^{-1}(i) \setminus B$ is the set of all non-degenerate intervals allocated to $i \in [r]$.

In other words, (x, f) is in the common face $\tau = (A_1, A_2, \dots, A_r; B)$ of C_{f_1} and C_{f_2} iff $B = \{j \in [m] \mid f_1(j) \neq f_2(j)\}$ and for each $i \in [r]$, $A_i = f_1^{-1}(i) \setminus B = f_2^{-1}(i) \setminus B$.

3.2. **The test map for detecting fair splittings.** Let $\mu = (\mu_1, \dots, \mu_n)$ be the vector valued measure associated to the collection of measures $\{\mu_j\}_{j=1}^n$. If $(x, f) \in (A_1, \dots, A_r; B) \in [r]^{*m}$ is a partition/allocation of the necklace let

$$\phi_i(x, f) := \mu(\bigcup_{j\in A_i} I_j) = \sum_{j\in A_i} \mu(I_j) \in \mathbb{R}^n$$

be the total μ-measure of all intervals $I_j = [x_{j-1}, x_j]$, allocated to the thief $i \in [r]$. If $\phi(x, f) := (\phi_1(x, f), \dots, \phi_n(x, f)) \in (\mathbb{R}^n)^r$ then (x, f) is a fair splitting if and only if $\phi(x, f) \in D$, where $D := \{(v, \dots, v) \mid v \in \mathbb{R}^n\} \subset (\mathbb{R}^n)^r$ is the diagonal subspace.
Summarizing, $(x, f) \in (\Delta^N)^{*r}_{\Delta}$ is a fair splitting of the necklace $([0, 1]; \{\mu_j\}_{j=1}^n)$ if and only if (x, f) is a zero of the composition map

$$\widehat{\phi} : (\Delta^N)^{*r}_{\Delta} \longrightarrow (\mathbb{R}^n)^r/D\,. \tag{3}$$

3.3. **The group of symmetries.** The final ingredient in applications of the configuration space/test map scheme is a group G of symmetries [25], characteristic for the problem. In the chosen scheme it is the p-toral group $G = (\mathbb{Z}_p)^\nu$, where p is a prime and $r = p^\nu$. The group G acts freely on the deleted join $(\Delta^N)^{*r}_{\Delta}$ and without fixed points on the sphere $S((\mathbb{R}^n)^r/D) \subset (\mathbb{R}^n)^r/D$.

Moreover, the map (3) is clearly G-equivariant.

3.4. **New (refined) configuration spaces.** For each constraint an adequate refined configuration space should be carefully designed. In principle the constraint dictates the choice of an appropriate configuration space, as a subspace of $(\Delta^N)^{*r}_{\Delta}$. However this choice may not be unique and even the initial choice of the parameter N may depend on the constraint.

Refined configuration spaces for the almost equicardinal splitting problem. In order to derive Alon's necklace-splitting theorem (Theorem 1.1) it is natural to choose N, the dimension of the primary configuration space $(\Delta^N)^{*r}_{\Delta}$, to be equal to the expected number of cuts, $N = (r - 1)n$.

Our basic new idea is to allow (initially) a larger number of cuts, but to force some of these cut points to coincide, by an appropriate choice of the configuration space. This is achieved by choosing a G-invariant, $(r - 1)n$-dimensional subcomplex K of the primary configuration space $(\Delta^N)^{*r}_{\Delta}$, where N is (typically) larger than the number $(r - 1)n$ of essential cut points.

Our first choice for a *refined configuration space* $K \subseteq (\Delta^N)^{*r}_{\Delta}$ is the symmetrized deleted join $SymmDelJoin(\mathcal{K})$ of a family $\mathcal{K} = \{K_i\}_{i=1}^r$ of collectively unavoidable subcomplexes of $2^{[m]}$ where $m = N+1 = (r-1)(n+1)+1$.

Refined configuration spaces for the binary splitting problem. For the binary splitting theorem we choose the usual parameter $N = (r - 1)n$, as in Alon's original theorem. Recall that maximal simplices (facets) of the primary configuration space $(\Delta^N)^{*r}_{\Delta} \cong [r]^{*(N+1)}$ can be interpreted as the graphs $\Gamma(f) \subset [N + 1] \times [r]$ of functions $f : [N + 1] \to [r]$.

Assuming that $r = 2^d$ thieves are positioned on the vertices of a d-dimensional cube, we consider a subcomplex $K \subset (\Delta^N)^{*r}_{\Delta}$ that includes the graphs of functions corresponding to binary splitting of the necklace. More explicitly $\Gamma(f) \in K$ if and only if for each $i \in [N]$ either $f(i) = f(i+1)$ or $\{f(i), f(i+1)\}$ is an edge of the d-cube.

4. Almost equicardinal necklace-splitting theorem

We begin with a very simple example of a necklace where all fair partitions/allocations are easily described.

Example 4.1. Assume that the measures μ_j $(j = 1, \dots, n)$ are supported by pairwise disjoint subintervals of $[0, 1]$. In this case we need at least $(r-1)n$ cuts which dissect the necklace into $(r-1)n+1$ parts. We observe that for this choice of measures there always exists a (k, s)-equicardinal, fair partition/allocation of measures to r thieves where k is the quotient and s the corresponding remainder, on division of $(r-1)n+1$ by r.

The choice of measures in Example 4.1 is rather special and it is natural to ask if such a partition is always possible.

Problem 4.2. For a given collection $\{\mu_j\}_{j=1}^n$ of continues measures on $[0, 1]$ and r thieves, is it always possible to find a fair, (k, s)-equicardinal partition/allocation of the necklace where k and s are chosen as in Example 4.1?

The following extension of the classical necklace theorem of Alon provides an affirmative answer to Problem 4.2.

Theorem 4.3. (Almost equicardinal necklace-splitting theorem) *For given positive integers r and n, where $r = p^\nu$ is a power of a prime, let $k = k(r, n)$ and $s = s(r, n)$ be the unique non-negative integers such that $(r-1)n+1 = kr+s$ and $0 \leq s < r$. Then for any choice of n continuous, probability measures on $[0, 1]$ there exists a fair partition/allocation of the associated necklace with $(r-1)n$ cuts which is also (k, s)-equicardinal in the sense that:*
(1) each thief gets no more than $k+1$ parts (intervals);
(2) the number of thieves receiving exactly $k+1$ parts is not greater than s.

Proof. As emphasized in Section 3.4, the basic idea of the proof is to initially allow a larger number of cuts, and then to force some of these cuts to be superfluous by an appropriate choice of the configuration space.

Our choice for a *refined configuration space* is the symmetric deleted join $K := SymmDelJoin(K_1, \dots, K_r)$ of the family $\mathcal{K} = \langle K_i \rangle_{i=1}^r$,

$$K_1 = \cdots = K_s = \binom{[N+1]}{\leqslant k+1}, \quad K_{s+1} = \cdots = K_r = \binom{[N+1]}{\leqslant k} \tag{4}$$

of subcomplexes of the simplex $\Delta^N = 2^{[N+1]}$, where $N = (r-1)(n+1)$, and

$$m = N+1 = (r-1)(n+1)+1 = r(k+1)+s-1\,. \tag{5}$$

By substituting $k = t+1$ and $n = d+1$ in Theorem 2.7 we observe that the complex K is $(m-r-1)$-connected. By construction (Section 3) a partition/allocation $(x,f) \in K$ corresponds to a fair division if and only if $\widehat{\phi}(x,f) = 0$, where $\widehat{\phi}$ is the test map described in the equation (3). If a fair division (x,f) does not exist there arises a G-equivariant map

$$\widehat{\phi} : K \longrightarrow S(\mathbb{R}^{nr}/D) \overset{G}{\simeq} S^{(r-1)n-1}$$

where $G = (\mathbb{Z}_p)^r$ and $S(V)$ is a G-invariant sphere in a G-vector space V. Since by (5)

$$m - r - 1 = [(r-1)(n+1)+1] - r - 1 = (r-1)n - 1$$

this contradicts Volovikov's theorem (Theorem 2.3).

Suppose that $(x,f) \in (A_1,\dots,A_r;B)$. Then, with a possible reindexing of thieves, $(x,f) \in \tau = (A_1,\dots,A_r;B)$ where $|A_i| \leq k+1$ for $i = 1,\dots,s$ and $|A_j| \leq k$ for $j = s+1,\dots,r$. From here it immediately follows that (x,f) describes a (k,s) balanced partition/allocation of the necklace. □

Corollary 4.4. (Equicardinal necklace-splitting theorem) *In the special case $s = 0$, or equivalently if $(r-1)n+1$ is divisible by r, Theorem 4.3 guarantees the existence of a fair partition/allocation which is* equicardinal *in the sense that each thief is allocated exactly the same number of pieces of the necklace. Here we tacitly assume that the necklace is generic, i.e. that all $(r-1)n$ cuts are needed.*

Splitting necklaces and collectively unavoidable complexes. Collectively unavoidable complexes $\mathcal{K} = \{K_i\}_{i=1}^r$, introduced in [13], include as a special case pairs of Alexander dual complexes [17] (in the case $r = 2$) and *unavoidable complexes* [7, 12] (in the case $K_1 = \cdots = K_r$). As shown in [14], they are a very useful tool for proving theorems of Van Kampen-Flores type. Here we demonstrate that they also provide a natural environment for necklace-splitting theorems with constraints.

Theorem 4.3 turns out to be a very special case of the following theorem where the constrains on the partition/allocation are ruled by a collectively unavoidable r-tuple of complexes.

As in Theorem 4.3, we assume that $r = p^\nu$ is a power of a prime number and $m = N+1 = (r-1)(n+1)+1$. Moreover, $k = k(r,n)$ and $s = s(r,n)$ are the unique non-negative integers such that $(r-1)n+1 = kr+s$ and $0 \leq s < r$.

Theorem 4.5. *Let $\mathcal{K} = \langle K_i\rangle_{i=1}^r = \langle K_1,\dots,K_r\rangle$ be a sequence of subcomplexes of $2^{[m]}$ such that:*

(1) *each complex K_i is (m,k)-balanced, and*
(2) *the sequence $\mathcal{K}$ is collectively unavoidable.*

Choose a collection $\{\mu_i\}_{i=1}^n$ of n continuous, probability measures on $[0,1]$. Then for any company $\mathcal{C}$ of r thieves there exists a fair partition/allocation

$(x,f)\in SymmDelJoin(\mathcal{K})$ of the associated necklace with at most $n(r-1)$ cuts. More explicitly, there exists a $(r-1)n$-dimensional simplex

$$(A_1,\dots,A_r;B)\in SymmDelJoin(\mathcal{K})$$

and a partition/allocation $(x,f)\in(A_1,\dots,A_r;B)$ which is fair for $\mathcal{C}$, with a suitable choice of a bijection $\mathcal{C}\leftrightarrow[r]$.

Proof. The proof is similar to the proof of Theorem 4.3, with an additional intermediate step allowing us to control the number of essential cut points.

As expected we use Theorem 2.6, instead of Theorem 2.7, which claims that, under the conditions of the theorem, the complex $K := SymmDelJoin(\mathcal{K})$ is $(m-r-1)$-connected. However, we refine the configuration space even more by selecting the $(m-r)$-dimensional skeleton $K^{(m-r)}$ of K as the domain for our test map $\widehat{\phi}$. The complex $K^{(m-r)}$ is also $(m-r-1)$-connected and the condition $\dim(K^{(m-r)})=(r-1)n$ guarantees that the number of superfluous cuts (indexed by B) is at least $r-1$. □

5. Binary necklace splitting

Recall (see [5] or Section 1.1) that if $r=2^d$ thieves are positioned at the vertices of the d-dimensional cube then a *binary necklace splitting* is a fair splitting with the additional constraint that adjacent (possibly degenerate) pieces of the necklace are allocated to thieves whose vertices share an edge.

Theorem 5.1. (Binary necklace-splitting theorem) *Given a necklace with n kinds of beads and $r=2^d$ thieves, there always exists a binary necklace splitting with $n(r-1)$ cuts.*

Note that the authors of [5] originally introduced a slightly more general binary necklace splitting where r thieves are placed at the vertices of a cube of dimension $\lceil\log_2 r\rceil$ (allowing some vertices of the cube to remain unoccupied).

Theorem 5.1 provides an affirmative answer to the following conjecture (Conjecture 2.11 in [5]) in the case when r is a power of two.

Conjecture 5.2. Given a necklace with n kinds of beads and $r\geq 4$ thieves, there exists a binary necklace splitting of size $n(r-1)$.

Both versions of the binary necklace splitting are special cases of the graph-constrained or G-constrained necklace splitting, where $G=(V,E)$ is a connected graph on the set $V\cong[r]$ of thieves.

Definition 5.3. Let $G=(V,E)$ be a connected graph where $V\cong[r]$ is the set of thieves. A necklace splitting is G-constrained if the corresponding *partition/allocation* (x,f), where $f:[m]\to[r]$ is the allocation function (Section 2.1), has the property that for each $i=1,\dots,m-1$ either $f(i)=f(i+1)$ or $\{f(i),f(i+1)\}\in E$ is an edge of the graph G. A function f satisfying this condition will be referred to as a G-constrained allocation function.

Note that if for some $f(i) = f(i+1)$ the cut-point x_i is superfluous and can be removed from the necklace splitting.

The following G-constrained simplicial subcomplex $K_G^m \subseteq (\Delta^N)^{*r}_\Delta \cong [r]^{*m}$ of the primary configuration space $[r]^{*m}$ is a natural choice for a configuration space suitable for studying the G-constrained splittings of a necklace.

Definition 5.4. Let $G = (V, E) = ([r], E)$ be a connected graph. The G-constrained complex $K_G^m \subset (\Delta^N)^{*r}_\Delta \cong [r]^{*m}$ (G-complex for short) is defined as the union of all simplices C_f (Section 3.1) where $f : [m] \to [r]$ is a G-constrained allocation function (Definition 5.3).

The primary configuration space $[r]^{*m}$ can be interpreted as the order complex of a poset Π on the set $V \times [m] \cong [r] \times [m]$ where $(x, i) \preccurlyeq (y, j)$ in Π if and only if $i \leq j$.

Similarly let Π_G^m be a subposet of Π, defined on the same set of vertices $V \times [m]$, where in Π_G^m

$$(x, i) \preccurlyeq (y, j) \Leftrightarrow i \leqslant j \text{ and } dist(x, y) \leq j - i.$$

(The distance function $dist(x, y)$ is the graph-theoretic distance, i.e. the smallest number of edges in a path connecting the vertices $x, y \in V$.)

It is clear from the construction that $K_G^m \cong \Delta(\Pi_G^m)$ is the order complex of the poset Π_G^m.

Remark 5.5. By construction K_G^m is always a subcomplex of the standard (primary) configuration space $(\Delta^N)^{*r}_\Delta$ (where $m = N+1$) and $K_G^m = (\Delta^N)^{*r}_\Delta$ if G is the complete graph K_r. If $r = 2^d$ and C^d is the vertex-edge graph of the d-dimensional cube, then $K_{C^d}^m$ is a proper configuration space for the binary necklace-splitting problem.

As expected, in the course of the proof of Theorem 5.1 the main step is the proof that the complex $K_{C^d}^{N+1}$ is $N-1$ connected. For an inductive proof of this fact we need the following definition.

Definition 5.6. For a given graph $G = (V, E)$, where $V = \{v_i\}_{i=1}^r$, let $Prism(G) = (V', E')$ to be a new graph with $V' = \{v_i^{(1)}\}_{i=1}^r \bigcup \{v_i^{(2)}\}_{i=1}^r$, as the set of vertices. The vertices $v_i^{(1)}$ and $v_j^{(1)}$ (respectively, $v_i^{(2)}$ and $v_j^{(2)}$) share an edge in $Prism(G)$ if and only if $\{v_i, v_j\} \in E$. Moreover, the copies of the same vertex $v_i^{(1)}$ and $v_i^{(2)}$ always share an edge in $Prism(G)$.

Note that by definition $C^{d+1} = Prism(C^d) = Prism^d(\text{one-vertex graph})$.

Proposition 5.7. *Suppose that $G = (V, E)$ is a connected graph. If K_G^m is $(m-2)$-connected for all $m \geq 2$, then $K_{Prism(G)}^m$ is also $(m-2)$-connected for all $m \geq 2$.*

Proof. The proof is by induction on m. Let $\Pi_1 = \Pi_{Prism(G)}^m \setminus \{(v_i^{(1)}, m)\}_{i=1}^r$ and $\Pi_2 = \Pi_{Prism(G)}^m \setminus \{(v_i^{(2)}, m)\}_{i=1}^r$ be two subposets of $\Pi_{Prism(G)}^m$. By definition $\Pi_1 \cup \Pi_2 = \Pi_{Prism(G)}^m$ and $\Pi_1 \cap \Pi_2 = \Pi_{Prism(G)}^{m-1}$.

If $\Delta_1 = \Delta(\Pi_1)$ and $\Delta_2 = \Delta(\Pi_2)$ are the associated order complexes then $K^m_{Prism(G)} = \Delta_1 \cup \Delta_2$ and $K^{m-1}_{Prism(G)} = \Delta_1 \cap \Delta_2$.

Let us show that the complex $\Delta_1 \cong \Delta_2$ has the same homotopy type as the complex $K^m_G = \Delta(\Pi^m_G)$.

Let $e_1 : \Pi^m_G \to \Pi_1$ be the inclusion map which maps (v_i, j) to $(v_i^{(2)}, j)$ and let $\rho_1 : \Pi_1 \to \Pi^m_G$ be a monotone map of posets defined by the formula:

$$\rho_1(v^{(i)}, j) = \begin{cases} (v^{(2)}, j), & \text{if } i = 2; \\ (v^{(2)}, j+1), & \text{if } i = 1. \end{cases}$$

These maps satisfy the relations:

(1) $id = \rho_1 \circ e_1 : \Pi^m_G \to \Pi^m_G$, and
(2) $e_1 \circ \rho_1(x) \succcurlyeq x \quad \forall x \in \Pi_1$.

By the homotopy property of monotone maps, see Quillen [18, Section 1.3] (or Theorem 12 from [26]), we conclude that both e_1 and ρ_1 induce homotopy equivalences of the order complexes Δ_1 and K^m_G. Similarly, we have a homotopy equivalence $\Delta_2 \simeq K^m_G$.

By the inductive assumption, $\Delta_1 \cap \Delta_2 = K^{m-1}_{Prism(G)}$ is $(m-3)$-connected and since both Δ_1 and Δ_2 are $(m-2)$-connected by the *Gluing Lemma* (see [6, Lemma 10.3]) the complex $K^m_{Prism(G)}$ is also $(m-2)$-connected. □

Corollary 5.8. *The complex $K^{(N+1)}_{C^d}$ is $N-1$ connected.*

In light of the discrete-to-continuous reduction, described in Section 1.2, Theorem 5.1 is a consequence of the following result.

Theorem 5.9. (Continuous binary necklace-splitting theorem) *If the number of thieves is $r = 2^d$, then for each continuous necklace, with n continuous probability measures $\mu_1, \dots, \mu_n$ on $[0,1]$ representing the distribution of n kinds of beads, there exists a binary necklace splitting of with $n(r-1)$ cuts.*

Proof. The proof is similar to the proof of Theorem 4.3 with Corollary 5.8 playing the role of Theorem 2.7. □

5.1. Binary necklaces plitting and equipartitions by hyperplanes.

The *Grünbaum-Hadwiger-Ramos hyperplane mass partition problem* [19, 24, 8, 9, 22] is the question of finding the smallest dimension $d = \Delta(j,k)$ such that for every collection of j masses (measurable sets, measures) in $\mathbb{R}^d$ there exist k affine hyperplanes that cut each of the j masses into 2^k equal pieces.

Asada et al. in [5] obtained (the continuous version of) Theorem 5.1 in the case $r = 4$ by embedding the necklace ($= [0,1]$) into the moment curve $\{(t, t^2, \dots, t^D) \mid -\infty \leq t \leq +\infty\} \subset \mathbb{R}^D$ and using a necklace splitting arising from an equipartition of the necklace by two hyperplanes in $\mathbb{R}^D$.

These authors correctly observed that their approach would allow them to deduce the general case $r = 2^d$ of Theorem 5.1 from Ramos' conjecture [19] which says that each collection of n continuous measures in $\mathbb{R}^D$ admits an equipartition by d hyperplanes, provided $n(2^d - 1) = dD$.

Moreover, they claim (at the end of Section 2) that a partial converse is true, i.e. that Theorem 5.1 is strong enough to establish Ramos' conjecture for measures concentrated on the moment curve.

This is unfortunately not the case since there exist binary necklace splittings which do not arise from equipartitions by hyperplanes, as illustrated by Example 5.10. The reason is that hyperplane splittings have an additional property of being "balanced", due to the fact that each hyperplane contributes the same number of cuts.

Example 5.10. Let $r = 4$ and $n = 2$ and suppose that the thieves A, B, C, D are positioned in a cyclic order on the vertices of a square. By the necklace-splitting theorem of Alon a continuous necklace with two types of beads (two measures μ_1, μ_2) there exists a fair division with $n(r-1) = 6$ cuts. Assume that μ_1 and μ_2 are, as in Example 4.1, uniform probability measures on two disjoint intervals I and J. Suppose that this fair division arises from an equipartition by two planes H_1 and H_2 in $\mathbb{R}^3$. The interval I is subdivided into subintervals I_1, I_2, I_3, I_4 (by cut points $x_1 < x_2 < x_3$), similarly J is subdivided into J_1, J_2, J_3, J_4 by cut points $y_1 < y_2 < y_3$. By taking into account that each plane has at most three points in common with the moment curve, we observe that $\{x_1, x_3, y_2\} \subset H_1$ and $\{x_2, y_1, y_3\} \subset H_2$ (or vice versa). Assume that the intervals I_1, I_2, I_3, I_4 are in this order allocated to thieves A, B, C, D.

From here we deduce that A and B (respectively C and D) are on different sides of the hyperplane H_1. Similarly A and D (respectively B and C) are on different sides of the hyperplane H_2. The rest of the allocation is uniquely defined and reads as follows, $J_1 \mapsto D, J_2 \mapsto A, J_3 \mapsto B, J_4 \mapsto C$.

In turn this shows that the binary necklace splitting (Figure 5.1)

$$I_1 \mapsto A,\ I_2 \mapsto B,\ I_3 \mapsto C,\ I_4 \mapsto D \quad J_1 \mapsto D, J_2 \mapsto C, J_3 \mapsto B, J_4 \mapsto A$$

cannot be obtained from an equipartition by two hyperplanes.

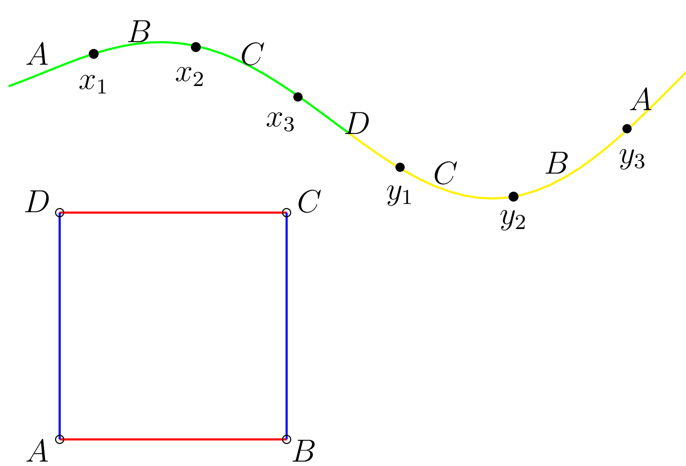

FIGURE 1. A non-balanced binary splitting.

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