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Mini-Workshop: Algebraic Tools for Solving the Yang–Baxter Equation

Organized by
Eric Jespers, Brussels
Victoria Lebed, Caen
Wolfgang Rump, Stuttgart
Leandro Vendramin, Buenos Aires

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Abstract. The workshop was focused on three facets of the interplay between set-theoretic solutions to the Yang–Baxter equation and classical algebraic structures (groups, monoids, algebras, lattices, racks etc.): structures used to construct new solutions; structures as invariants of solutions; and YBE as a source of structures with interesting properties.

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Introduction by the Organizers

A fruitful approach to fundamental problems in mathematical physics is to identify and study their underlying algebraic structures. One such problem is to systematically construct and classify solutions to the Yang–Baxter equation (YBE). During the last decade, a whole new layer of algebraic structures appeared in (or were adapted to) the study of the YBE. Braces, bijective group 1-cocycles, Hopf–Galois extensions and self-distributive structures are just a few examples. These structures brought a whole new spectrum of mathematics into the study of the YBE: group theory, ring theory, non-commutative geometry, computational and cohomological algebra, low-dimensional topology.

The purpose of this meeting was to bring together experts in the algebraic approach to the set-theoretic YBE and related fields. Discussions were channelled into four directions:

Associative invariants of solutions: Given a set X, a map $r\colon X \times X \to X \times X$ is said to satisfy the *Yang–Baxter equation (YBE)* if one has

$$(r \times \mathrm{Id}) \circ (\mathrm{Id} \times r) \circ (r \times \mathrm{Id}) = (\mathrm{Id} \times r) \circ (r \times \mathrm{Id}) \circ (\mathrm{Id} \times r)$$

on X^3. To any solution

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

of the YBE on the set X, one associates its *structure monoid*

$$M(X, r) = \langle x \in X \mid xy = \lambda_x(y)\rho_y(x), \text{ for all } x, y \in X\rangle;$$

its *structure group* $G(X, r)$ with an analogous presentation; its *structure algebra* $kM(X, r)$ over a field k; its *permutation group*

$$\mathcal{G}(X, r) = \langle \lambda_x \mid x \in X\rangle,$$

which is a subgroup of the symmetric group $\mathbb{S}_X$; its *derived monoid*

$$A(X, r) = \langle x \in X \mid x\lambda_x(y) = \lambda_x(y)\lambda_{\lambda_x(y)}(\rho_y(x)), \text{ for all } x, y \in X\rangle$$

and its *derived group* with an analogous presentation, which are also the structure monoid and group of the rack solution (X, r') with

$$r'(x, y) = (y, \lambda_y\rho_{\lambda_x^{-1}(y)}(x)).$$

For the last constructions we need all λ_x to be bijective; such solutions are called *left non-degenerate.*

Besides being important invariants of solutions, the above groups and algebras enjoy interesting properties (Bieberbach, Garside, Koszul, Artin–Schelter regular, left orderable etc.), provided that the solution is from a suitable class. This has already been used for constructing counterexamples to group-theoretic conjectures. The talks of Jespers, Kubat, Okniński, Chouraqui, and Lebed highlighted recent advances in this direction and presented multiple open problems.

In particular, Lebed presented Ryder's conjectural characterisation of structure groups of a particular type of solutions. The next day Eisermann found a counterexample, and together they started collaboration on an amended version of this conjecture, and its group-cohomological implications.

Braces and classification of solutions: A (left) *skew brace* is the data of two group operations $+$ and $\circ$ on a set B, compatible in the sense of

$$a \circ (b + c) = a \circ b - a + a \circ c. \tag{1}$$

It is called a (left) *brace* if the operation $+$ is commutative. The classification of certain types of solutions boils down to the classification of (skew) braces, and the latter are now actively studied using group- and ring-theoretic tools. It is not surprising that together with people from the YBE community (Vendramin, Gateva–Ivanova, Cedó, van Antwerpen), in this section we included talks by Byott (a specialist in Hopf–Galois theory) and Ballester–Bolinches (a group theorist). They presented spectacular applications of their domains to the YBE.

Meta-structures and connections: In this section Brzeziński presented his work on trusses, a very general structure which interpolates between braces and rings; Dietzel explained the work of Rump and himself on l-algebras, which generalise structure groups and connect them to lattice theory; Catino generalised skew braces to semi-braces, and showed how to construct solutions out of them; Verwimp explained how skew lattices give rise to YBE solutions; and Wiertel presented a complete classification of symmetric group actions obtained from linear unitary involutive solutions.

Yang–Baxter cohomology: Cohomology groups of YBE solutions can be seen as solution invariants, related to structure groups (in a highly non-trivial way!). In another vein, they yield linear YBE solutions as deformations of set-theoretic ones. They are also used in knot theory and Hopf algebra classification, and are thus interesting on their own right. Eisermann and Lebed gave an overview of this direction, and presented their recent results.

A particular accent in the talks was made on open questions and conjectures, which were then discussed among participants. This was especially useful for young researchers, well represented in the mini-workshop. At the end of the week, all participants agreed that the meeting had been extremely productive, allowing them to keep up with the numerous recent developments in the field, to initiate collaborations, and to discuss new approaches to open problems in the area.

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Mini-Workshop: Algebraic Tools for Solving the Yang–Baxter Equation

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Abstracts

Associative structures associated to set-theoretic solutions of the Yang–Baxter equation

Eric Jespers

(joint work with Ferran Cedó and Charlotte Verwimp)

Let (X, r) be a set-theoretic solution of the Yang–Baxter equation. Gateva-Ivanova and Majid in [3] showed that the study of such solutions is determined by solutions (M, r_M), where $M = M(X, r)$ is the structure monoid of (X, r), and r_M restricts to r on X^2. For left non-degenerate solutions, it has been shown [4] that $M(X, r)$ is a regular submonoid of $A(X, r) \rtimes \mathcal{G}(X, r)$, where we use notations from the Introduction. The elements of $A = A(X, r)$ are normal, i.e. $aA = Aa$ for all $a \in A$. It is this "richer structure" that has been exploited by several authors to obtain information on the structure monoid $M(X, r)$ and the structure algebra $kM(X, r)$ over a field k.

In this talk we report on some recent investigations, by Cedó, Jespers and Verwimp [2], i.e. set-theoretic solutions that are not necessarily left non-degenerate nor bijective. We prove that there is a unique 1-cocycle $\pi\colon M(X, r) \to A(X, r)$, with respect to the natural left action $\lambda'\colon M(X, r) \to \mathrm{End}(A(X, r))$, such that $\pi(x) = x$, and a unique 1-cocycle $\pi'\colon M(X, r) \to A'(X, r)$, with respect to natural right action $\rho'\colon M(X, r) \to \mathrm{End}(A'(X, r))$ such that $\pi'(x) = x$. Here A' is the right derived monoid

$$A'(X, r) = \langle x \in X \mid \rho_y(x)y = \rho_{\rho_y(x)}(\lambda_x(y))\rho_y(x), \text{ for all } x, y \in X\rangle.$$

It is determined when these mappings are injective, surjective, respectively bijective. One then obtains a monoid homomorphism $M(X, r) \to A(X, r) \rtimes \langle \lambda_x \mid x \in X\rangle$. The mapping π is injective when all λ_x are injective, i.e. when (X, r) is a left non-degenerate solution. It is surjective if and only if each λ_x is surjective. Hence, it is bijective if (X, r) is a left non-degenerate solution (a result earlier proven in [4]). In the latter case, we determine the left cancellative congruence η on $M(X, r)$ and show that (X, r) induces a set-theoretic solution on $M(X, r)/\eta$. Hence one obtains that left non-degenerate set-theoretic solutions are linked with semi-trusses as introduced by Brzeziński [1]. Conversely it can be shown that left cancellative semi-tusses correspond to left non-degenerate solutions of the YBE. Similar results are obtained for right non-degenerate set-theoretic solutions.

Problems of interest are:

Problem 1. *Determine precisely when the 1-cocycle π is injective.*

Problem 2. *Determine the algebraic structure of $kM(X, r)$ when π is injective.*

Problem 3. *Determine the algebraic structure of $kM(X, r)$ for arbitrary solutions.*

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Structure algebras of solutions of the Yang–Baxter equation

Łukasz Kubat
(joint work with Eric Jespers and Arne Van Antwerpen)

For a finite involutive non-degenerate solution (X, r) of the Yang–Baxter equation, it is known that the structure monoid $M = M(X, r)$ is a monoid of I-type, and the structure algebra kM over a field k shares many properties with commutative polynomial algebras. In particular, it is a Noetherian PI-domain, a maximal order in its division ring of fractions, and also an Auslander-regular, Cohen–Macaulay, Koszul algebra satisfying

$$\operatorname{clKdim} kM = \operatorname{GKdim} kM = |X|$$

(see e.g. [1, 4, 10, 9] and also [3, 5, 11, 12]). In this talk I am going to focus on certain ring-theoretical properties of structure algebras of an arbitrary finite bijective (one-sided) non-degenerate solution (X, r) of the YBE. Although the structure of both the monoid M and the algebra kM is much more complicated than in the involutive case, it is still possible to show the following result:

Theorem 1 ([8, Theorem 3])**.** *There exists a finitely generated commutative normal submonoid $T \subseteq M$ and a finite subset $F \subseteq M$ such that $M = \bigcup_{f\in F} Tf$. In particular, kM is a module-finite normal extension of the commutative affine subalgebra kT. Hence kM is a Noetherian PI-algebra. Moreover,*

$$\operatorname{clKdim} kM = \operatorname{GKdim} kM = \operatorname{rk} M \leqslant |X|,$$

and the equality holds if and only if the solution (X, r) is involutive.

Moreover, the classical Krull dimension of the algebra kM is expressible (and easily computable) in a purely combinatorial way (see [6, Theorems 3.5 and 3.8]). It is also possible to characterize, in ring-theoretical or homological terms of kM, when (X, r) is an involutive solution or when M is a cancellative monoid (a recent question of Gateva-Ivanova, see [2, Conjecture 3.20]).

Theorem 2 ([7, Theorem 4.6])**.** *The following conditions are equivalent:*

(1) *(X, r) is an involutive solution.*
(2) *M is a cancellative monoid.*
(3) $\operatorname{rk} M = |X|$.

(4) *kM is a prime algebra.*
(5) *kM is a domain.*
(6) $\operatorname{clKdim} kM = |X|$.
(7) $\operatorname{GKdim} kM = |X|$.
(8) *kM has finite global dimension.*
(9) *kM is an Auslander–Gorenstein algebra.*
(10) *kM is an Auslander-regular algebra.*

The above mentioned results allow one to control the prime spectrum of kM and to describe the Jacobson and prime radicals of kM as well (see [6, Proposition 5.6 and Theorem 6.5]). Finally, these results lead also to a matrix-type representations of prime images of the algebra kM.

Theorem 3 ([6, Theorem 7.1]). *If P is a prime ideal of kM, then there exists an ideal I of kM contained in P, a finitely generated abelian-by-finite group G which is the group of quotients of a cancellative subsemigroup of M, and a prime ideal Q of kM such that $kM/I \subseteq M_n(kG)$ and $kM/P \subseteq M_n(kG/Q)$ for some $n \geqslant 1$. Moreover, the algebra $M_n(kG)$ is a localization of kM/I. In particular, $\mathrm{Q}_{cl}(kM/P) \cong \mathrm{Q}_{cl}(M_n(kG/Q))$. If, furthermore, kM is semiprime then there exist finitely many finitely generated abelian-by-finite groups, say $G_1, \dots, G_s$, each being the group of quotients of a cancellative subsemigroup of M, such that kM embeds into $M_{n_1}(kG_1) \times \cdots \times M_{n_s}(kG_s)$ for some $n_1, \dots, n_s \geqslant 1$.*

Note also that results obtained in [6, 8, 7] serve as a motivation to posing the following questions, which may be also considered as possible directions of further research concerning ring-theoretical aspects of the Yang–Baxter equation.

Problem 4. *Determine the structure of prime images kM/P of kM. When is kM/P a domain? When is kM/P a maximal order in its quotient ring? When does kM/P have some good homological properties? When is kM/P Koszul or Auslander-regular or Cohen–Macaulay? How do these properties affect the solution (X, r)?*

Problem 5. *When is kM semiprime? How does the semiprimeness of kM affect the solution (X, r)?*

Problem 6. *When does kM/P come from a solution of the YBE related to (X, r)? In such a case, try to investigate the nature of this (probably simpler) solution. How do properties of such solutions for various P reflect the properties of (X, r)?*

Problem 7. *Try to find new (and understand better already known) matrix representations of $K[M]$ in both semiprime and non-semiprime case.*

Problem 8. *Is it possible to find an algebraic invariant determining the order of the map r?*

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Structure algebras and the minimality condition

Jan Okniński
(joint work with Ferran Cedo and Eric Jespers)

Given a finite non-degenerate (bijective) set-theoretic solution (X, r) of the Yang–Baxter equation and a field k, the structure algebra $kM(X, r)$ is a graded algebra: $kM = \oplus_{m\geq 0} A_m$, where A_m is the linear span of all the elements $x_1 \cdots x_m$, for $x_1, \ldots, x_m \in X$. Moreover, for every $m \geq 2$, X^m receives a natural action of the group generated by $r_{1,2}, \ldots, r_{m-1,m}$, whose orbits are in a one to one correspondence with the elements of degree m in the structure monoid $M(X, r)$. Here $r_{i,i+1} \colon X^m \to X^m$ is the map that acts as r on components $i, i+1$ and as the identity on the remaining components. So, this yields an action of the braid group B_m, and one also gets an action of the symmetric group $\mathbb{S}_m$ in the case of involutive solutions (X, r), i.e. when $r^2 = \mathrm{Id}$.

The latter case (a finite involutive non-degenerate solution (X, r)) is an important particular case, that was first studied in [2, 3]. Namely, one of the known results asserts that the maximal possible value of $\dim(A_2)$ corresponds to involutive solutions and implies several deep and important properties of kM. In this case, if X is finite of cardinality n, then there are n singleton orbits and $\binom{n}{2}$ orbits of cardinality 2 in X^2 under the action of the group $\langle r\rangle$, so that $\dim A_2 = n + \binom{n}{2}$. Moreover, the structure algebra kM shares many properties with the polynomial algebra $k[x_1, \dots, x_n]$. In particular, it is a domain, it satisfies a polynomial identity and it is a PBW algebra of Gelfand–Kirillov dimension $\mathrm{GK}(kM) = n$. It also shares various nice homological properties with $K[x_1, \dots, x_n]$. The structure monoid $M(X, r)$ is embedded in the structure group $G(X, r)$, which is isomorphic to $\mathrm{gr}((x, \lambda_x) \mid x \in X) \subseteq \mathbb{Z}^n \rtimes \mathbb{S}_X$, where $\mathbb{S}_X$ acts naturally on the free abelian group $\mathbb{Z}^n$ of rank n. Also, in this case, the structure group $G(X, r)$ is solvable and a Bieberbach group of dimension n, i.e. a torsion-free group with a free abelian subgroup of rank n of finite index.

Following recent ideas of Gateva-Ivanova [4], we focus on the minimal possible values of the dimension of A_2, and their impact on the growth function and the structure of the algebra kM. We determine lower bounds and classify solutions (X, r) for which these bounds are attained in the general case and also in the square-free case, i.e. when $r(x, x) = (x, x)$ for every $x \in X$. This is done in terms of the so called derived solution, introduced by Soloviev [6] and closely related with racks and quandles. Namely, a recent result in [5] allows to reduce the problem to the derived solution (X, s), which is of the form $s(x, y) = (\sigma_x(y), x)$, for $x, y \in X$, where $\sigma_x \in \mathbb{S}_X$ for every $x \in X$.

We list some of the main results, obtained in [1]. Recall that a quadratic set is a pair (X, r) with $r\colon X^2 \to X^2$. It is called braided if r satisfies the Yang–Baxter equation, see [4]. First, we show that there exists a finite non-degenerate quadratic set (X, r) such that $\dim(A_m) = 1$, for all $m > 1$; so, in particular, $\mathrm{GK}(kM) = 1$.

If furthermore (X, r) is a braided set, then we prove that $\dim(A_2) \geq \frac{|X|}{2}$. Moreover, we describe explicitly all cases when the lower bound $\frac{|X|}{2}$ or $\frac{|X|+1}{2}$ (depending on whether $|X|$ is even or odd) is reached, in terms of the derived solution (X, s). As a consequence, it follows in these cases that the solution (X, r) is indecomposable.

For square-free non-degenerate quadratic sets (X, r) it is easy to see that $|X| + 1$ is the smallest possible value of $\dim(A_2)$. We present a construction with $\dim(A_m) = |X| + 1$ for all $m > 1$; so that $\mathrm{GK}(kM) = 1$ in this case as well.

Let (X, r) be a finite square-free non-degenerate braided set. We prove that $\dim(A_2) \geq 2|X| - 1$. One of the main cases that arises in this context is when (X, s) is the braided set associated to the so called dihedral quandle. It can be defined as $(\mathbb{Z}/(n), s)$, where $n \geq 3$ and $s(k, t) = (2k - t, k)$, for all $k, t \in \mathbb{Z}/(n)$. We discuss this case in detail, as it turns out to be crucial in the context of the impact of the size of A_2 on the algebra kM. Then our main result shows that the above lower bound is achieved if and only if the derived solution (X, s) is of one of the following types:

(1) $|X|$ is an odd prime and (X, s) is the braided set associated to the dihedral quandle; moreover, $\mathrm{GK}(A) = 1$ in this case,
(2) $|X| = 2$ and (X, s) is the trivial braided set, that is $s(x, y) = (y, x)$ for all $x, y \in X$; and $\mathrm{GK}(A) = 2$ in this case,
(3) $X = \{1, 2, 3\}$ and $\sigma_1 = \sigma_2 = \mathrm{id}$, $\sigma_3 = (1, 2)$; and $\mathrm{GK}(A) = 3$ in this case.

In particular, our results answer a number of questions raised in [4].

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On the left-orderability of structure groups of set-theoretic solutions of the Yang–Baxter equation

Fabienne Chouraqui

A group G is *left-orderable* if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication, that is $g \prec h$ implies $fg \prec fh$ for all f, g, h in G. If a group G is left-orderable, then it satisfies *the unique product property*, that is for any finite subsets $A, B \subseteq G$, there exists at least one element $x \in AB$ that can be uniquely written as $x = ab$, with $a \in A$ and $b \in B$. We call a strict total ordering that is invariant under left multiplication a *left order*. The *positive cone of* a left order $\prec$ is defined by $P = \{g \in G \mid 1 \prec g\}$ and it satisfies:

(1) P is a semigroup, that is $P \cdot P \subseteq P$;
(2) G is partitioned by P, that is $G = P \cup P^{-1} \cup \{1\}$ and $P \cap P^{-1} = \emptyset$.

Conversely, if there exists a subset P of G that satisfies (1) and (2), then P determines a unique left order $\prec$ defined by $g \prec h$ if and only if $g^{-1}h \in P$.

A subgroup N of a left-orderable group G is called *convex* (with respect to $\prec$), if for any $x, y, z \in G$ such that $x, z \in N$ and $x \prec y \prec z$, we have $y \in N$. A left order $\prec$ is *Conradian* if for any strictly positive elements $a, b \in G$, there is $n \in \mathbb{N}$ such that $b \prec ab^n$. A left-orderable group G is called *Conradian* if it admits a Conradian left order. A left-orderable group G is Conradian if and only if G is locally indicable [4, 1, 14]. Conradian left-orderable groups share many of the properties of the bi-orderable groups.

Definition 1. [13, Definition 3.2], [12, Definition 3.1] *A left order $\prec$ in a countable group G is* recurrent (for every cyclic subgroup) *if for every $g \in G$ and every finite sequence $h_1 \prec h_2 \prec \ldots \prec h_r$ with $h_i \in G$, there exists $n_i \to \infty$ such that $\forall i$, $h_1 g^{n_i} \prec h_2 g^{n_i} \prec \ldots \prec h_r g^{n_i}$.*

A recurrent left order is Conradian [13]. The set of all left orders of a group G is denoted by $LO(G)$ and it is a topological space (compact and totally disconnected with respect to the topology induced by the Tychonoff topology on the power set of G) [15]. The set $LO(G)$ cannot be countably infinite [11]. A left order $\prec$ is *finitely determined* if there is a finite subset $\{g_1, g_2, ..., g_k\}$ of G such that $\prec$ is the unique left-invariant ordering of G satisfying $1 \prec g_i$ for $1 \leq i \leq k$. A finitely determined left order $\prec$ is also called *isolated*, since $\prec$ is finitely determined if and only if it is not a limit point of $LO(G)$. If the positive cone of $\prec$ is a finitely generated semigroup, then $\prec$ is isolated.

If the order $\prec$ is also invariant under right multiplication, then G is said to be *bi-orderable*. The braid group B_n with $n \geq 3$ strands is left-orderable but not bi-orderable [6], and if $n \geq 5$ none of these orders is Conradian [14]. In [7], the question whether every Garside group is left-orderable is raised (Question 3.3, p.292, also in [8]). It is a very natural question as the Garside groups extend the braid groups in many respects and it motivated our research in the context of the structure group of a non-degenerate symmetric set-theoretical solution of the Yang–Baxter equation. This group is a Garside group that satisfies many interesting properties [2, 3, 5, 9]. In this note, we show this group is not bi-orderable and we find the question whether it is left-orderable has a wide range of answers. We now state our main results for the structure group $G(X, r)$ of a non-degenerate symmetric (non-trivial) set-theoretical solution of the YBE.

Theorem 2. *The group $G(X, r)$ has generalised torsion elements, and is thus not bi-orderable.*

Theorem 3. *Assume that (X, r) is a retractable solution and $|X| \geq 3$. Then*

(1) *$G(X, r)$ has a recurrent left order.*
(2) *The space of left orders of $G(X, r)$ is homeomorphic to the Cantor set.*
(3) *All the left orders of $G(X, r)$ are Conradian.*

Note that under the assumptions of Theorem 3, $G(X, r)$ is locally indicable (each non-trivial finitely generated subgroup has a quotient isomorphic to $\mathbb{Z}$), as the existence of a recurrent left order implies local indicability [13]. In contrast, for $n \geq 5$, the braid group B_n is not locally indicable [7, p. 287] and hence B_n has no recurrent left order like most of the left-orderable groups. E. Jespers and J. Okninski prove that the structure group of a retractable solution is poly-(infinite)cyclic [10, p. 223], which implies locally indicable.

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Structure groups of racks and quandles

Victoria Lebed
(joint work with A. Mortier and L. Vendramin)

This talk was an overview of what is known on the structure groups $G = G(X, \lhd)$ of YBE solutions coming from finite racks and quandles $(X, \lhd)$. After an introduction to self-distributive structures, I presented, following [6], a dichotomy for such a group G:

(1) either G is boring (free abelian),
(2) or G is interesting (non-abelian, with torsion, non-left-orderable, but not too wild either: it is virtually free abelian in a very explicit way).

I then explained the recent classification of quandles with boring $G(X, \lhd)$, done in [1] for the groups $G = \mathbb{Z}$ and $\mathbb{Z}^2$, and in [4] for the general case $\mathbb{Z}^k$. Further, I showed that these quandles have no torsion in the cohomology group H^2, which is bad news for applications, since it is this torsion that is most useful in practice.

I also explicitly described the structure groups of a wide class of quandles called *abelian quandles*, defined by the additional axiom

$$(a \lhd b) \lhd c = (a \lhd c) \lhd b.$$

Their H^2 can have torsion. These torsion groups T^2 are related to the structure groups in a way reminiscent of the connected quandle case [2, 3]. Apart from several examples treated in [4], a precise description of these T^2 is missing.

Problem 1. *Compute the torsion part of the second cohomology groups of finite abelian quandles.*

To each left non-degenerate YBE solution (X, r) one can associate a self-distributive operation $\lhd$ on X (called the *derived operation*), which is a rack operation if r is bijective [5]. Its structure group $G(X, \lhd)$ is often referred to as the *derived group* of (X, r). The groups $G(X, r)$ and $G(X, \lhd)$ are related by a bijective 1-cocycle, particularly useful when its target $G(X, \lhd)$ is free abelian. This has been extensively exploited for involutive solutions, for which the operation $\lhd$ is trivial, and hence $G(X, \lhd)$ free abelian. An answer to the following question would then allow one to extend properties of structure groups of involutive solutions to a wider class of solutions:

Problem 2. *What property of a YBE solution corresponds to its derived rack having an abelian structure group?*

In the same vein,

Problem 3. *What property of a YBE solution corresponds to its derived rack being abelian?*

In the end I recalled the conjugation quandles, which yield a right adjoint Conj to the structure group functor StrGr. Given a group Γ, these functors allow one to construct a central extension

$$0 \longrightarrow K \longrightarrow \mathrm{StrGr}(\mathrm{Conj}(\Gamma)) \xrightarrow{\pi} \Gamma \longrightarrow 0.$$

If Γ is the structure group of a rack, then the natural projection π has an explicit section. In her PhD thesis, Ryder conjectured that it is an if-and-only-if statement: the above short exact sequence splits iff Γ is the structure group of a rack [7]. After the talk, Eisermann found an explicit counterexample. However, it might be possible to amend the conjecture:

Problem 4. *Does the splitting of the above sequence coupled with some extra condition characterise the structure groups of racks?*

More generally,

Problem 5. *Are there any group-theoretic characterisations of the structure groups of racks?*

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Some problems on skew braces and the Yang–Baxter equation

Leandro Vendramin

The first problem I would like to mention comes from the theory of Hopf–Galois extensions and it was first formulated by Byott [4].

Problem 1 (Byott)**.** *Let A be a finite skew left brace with solvable additive group. Is the multiplicative group of A solvable?*

To state the second problem I will first recall the following question: Which finite solvable groups are IYB groups? Recall that IYB-groups are those groups that are multiplicative groups of skew left braces of abelian type. The problem was solved by Bachiller in [1], following the ideas of Rump [12]. However, I would like to understand better the ideas behind this proof.

Problem 2. *Which is the minimal size of a finite solvable group that is a non-IYB-group?*

The following two similar problems appear in [5]:

Problem 3 (Cedó–Jespers–Okniński)**.** *Is every finite nilpotent group of nilpotency class two the multiplicative group of a (two-sided) skew brace of abelian type?*

Problem 4 (Cedó–Jespers–Okniński)**.** *Which finite nilpotent groups are multiplicative groups of (two-sided) skew braces of abelian type?*

Problem 4 is interesting even in the particular case of nilpotency class ≤ 3.

Problem 5 (Rump)**.** *Is there an example of a non-IYB finite group where all the Sylow subgroups are IYB groups?*

Now I mention some problems related to involutive multipermutation solutions. For a finite non-degenerate involutive solution (X, r), the following properties are equivalent:

(1) (X, r) is a multipermutation solution.
(2) The structure group $G(X, r)$ is left orderable
(3) The structure group $G(X, r)$ is diffuse.

The implication 1) $\implies$ 2) was proved by Jespers and Okniński [9] and independently by Chouraqui [6]. The implications 2) $\implies$ 1) and 3) $\implies$ 1) were proved in [2] and in [10], respectively. All these results answer a question of Gateva–Ivanova [8]. The following problem appears in [10].

Problem 6. *Let (X, r) be a finite involutive non-degenerate solution to the Yang–Baxter equation. When does $G(X, r)$ have the unique product property?*

Benson proved in [3] that groups that contain a finite index subgroup isomorphic to $\mathbb{Z}^n$ have rational growth series. The structure group of a finite solutions has a finite index subgroup isomorphic to $\mathbb{Z}^n$, see [7, 11, 13].

Problem 7. *Let (X, r) be a finite non-degenerate invertible solution. Compute explicitly the growth series of the structure group $G(X, r)$.*

Finally, let me discuss a problem related to multipermutation solutions. It seems that "almost all" non-degenerate involutive solutions to the Yang–Baxter equation of size ≤ 8 are multipermutation solutions:

$$\frac{\text{number of multipermutation solutions}}{\text{number of solutions}} = \frac{36115}{38698} > 0.93.$$

For these calculation one needs the list of small solutions computed of [7].

Problem 8. *Is it true that*

$$\frac{\textit{number of multipermutation solutions of size } n}{\textit{number of solutions of size } n} \underset{n\to\infty}{\longrightarrow} 1?$$

This question makes sense for non-involutive solutions as well.

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A combinatorial approach to noninvolutive set-theoretic solutions of the Yang–Baxter equation

Tatiana Gateva-Ivanova

Let X be a nonempty set (possibly infinite), and let $r\colon X\times X\to X\times X$ be a bijective map. Such a pair (X,r) is referred to as a *quadratic set* [2]. It is called

- *square-free* if $r(x,x)=(x,x)$ for all $x\in X$; square-free, and involutive;
- a *braided set* if r is a YBE solution;
- a *symmetric set* if r is an involutive YBE solution.

We propose to study a braided set in terms of the properties of its structure monoid $M(X,r)$, its structure algebra $A=kM(X,r)$, and its Koszul dual, $A^!$. More generally, we continue our systematic study of nondegenerate quadratic sets (X,r) and the associated algebraic objects. Next we investigate the class of square-free solutions (X,r). It contains the special class of self-distributive solutions (quandles). We make a detailed characterization in terms of various algebraic and combinatorial properties, each of which shows the contrast between involutive and noninvolutive square-free solutions. We introduce and study a class of finite square-free solutions (X,r) of order $n\geq 3$ which satisfy the *minimality condition* **M**, that is $\dim_k A_2=2n-1$. Examples are some simple racks of prime order p. Finally, we discuss general extensions of solutions and introduce the notion of a *generalized strong twisted union* $Z=X\natural^*Y$ of solutions (X,r_X), and (Y,r_Y), where the map r has high, explicitly prescribed order.

Theorem 1 ([3])**.** *Let (X,r) be a square-free non-degenerate quadratic set of finite order $|X|=n$. Let $A=kM(X,r)$ be its associated quadratic algebra.*

(I) *If (X,r) is 2-cancellative then the following inequalities hold:*

$$2n-1\leq \dim_k A_2=n+q\leq \binom{n+1}{2},$$

where the upper bound is exact for all $n\geq 3$, and the lower bound is exact whenever $n=p>2$ is a prime number.

(II) *The following conditions are equivalent:*

(1) *The Hilbert series of A is*

$$H_A(z)=\frac{1}{(1-z)^n}.$$

(2) *A is a PBW algebra with a set of PBW generators $X=\{x_1,\cdots,x_n\}$ and with polynomial growth.*

(3) *A is a PBW algebra with a set of PBW generators $X=\{x_1,\cdots,x_n\}$ and with finite global dimension,* $\operatorname{gl}\dim A<\infty$*.*

(4) *A is a PBW Artin–Schelter regular algebra.*

(5) *There exists an enumeration $\{x_1,\cdots,x_n\}$ of X such that the set*

$$\mathcal{N}=\{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\mid \alpha_i\geq 0 \text{ for } 1\leq i\leq n\}$$

is a k-basis of A.

(6)

$$\dim_k A_2 = \binom{n+1}{2}, \quad \textit{and} \quad \dim_k A_3 = \binom{n+2}{3}.$$

(7) *A is a binomial skew polynomial ring in the sense of* [1].

(8) (X, r) *is a symmetric set.*

In this case A is a Noetherian domain. Moreover, A is Koszul, and

$$\operatorname{GKdim} A = n = \operatorname{gl}\dim A,$$

where GKdim A *is the Gelfand–Kirillov dimension of A.*

For the definitions of a Koszul algebra, a PBW algebra and the Koszul dual algebra $A^!$, see [5].

Corollary 2 ([3])**.** *(Characterization of* noninvolutive *square-free braided sets.) Let* (X, r) *be a square-free non-degenerate braided set of order* $|X| = n$*, suppose* $r^2 \neq \mathrm{Id}_{X\times X}$*. Then the following conditions hold:*

(1) *The algebra A is not Koszul.*

(2) *The set of quadratic relations* $xy = \lambda_x(y)\rho_y(x)$, $x, y \in X$ *is not a Gröbner basis with respect to any enumeration of X.*

(3) *A is not a binomial skew polynomial ring, with respect to any enumeration of X.*

(4) $2n - 1 \leq \dim_k A_2 \leq \binom{n+1}{2} - 1$.

(5) $\operatorname{GKdim} A < n$.

(6) $\dim_k A_3 < \binom{n+2}{3}$.

(7) *The Koszul dual* $A^!$ *satisfies* $0 \leq \dim_k A^!_3 < \binom{n}{3}$*, and* $A^!_3 = 0$ *whenever* $\dim_k A_2 = 2n - 1$.

(8) *There exist* $x, y \in X$*, such that* $x \neq y$*, and* $x^p = y^p$ *holds in the group* $G(X, r)$.

(9) *There exist* $a, b, x, y \in X$*, such that* $x \neq y, x \neq a, y \neq b$*, and the equality* $axx = byy$ *holds in* $M(X, r)$*. Moreover, the structure monoid* $M(X, r)$ *is not cancellative.*

(10) *The algebra A is not a domain.*

Theorem 3 ([3])**.** *Suppose* (Z, r) *is a non-degenerate 2-cancellative braided set (possibly infinite), which splits as a generalized strong twisted union* $Z = X\natural^* Y$ *of its r-invariant subsets X and Y. Let* (X, r_1) *and* (Y, r_2) *be the induced sub-solutions. Let* $S = M(X, r_1)$, $T = M(Y, r_2)$, $U = M(Z, r)$ *be the associated monoids,and* (S, r_S), (T, r_T), (U, r_U) *be the corresponding braided monoids, see* [4]*. Let* (G_Z, r_{G_Z}), (G_X, r_{G_X}), (G_Y, r_{G_Y}) *be the associated braided groups. Then the following conditions hold:*

(1) *The braided monoid* (U, r_U) *has a canonical structure of a generalized strong twisted union:*

$$(U, r_U) = (S, r_S)\natural^*(T, r_T),$$

extending the ground actions of the strong twisted union $Z = X\natural^* Y$.

(2) *Suppose (Z, r) is injective (i.e. Z is embedded in G_Z). Then (X, r_1) and (Y, r_2) are also injective, and the braided group (G_Z, r_{G_Z}) has a canonical structure of a strong twisted union*

$$(G_Z, r_{G_Z}) = (G_X, r_{G_X})\natural^*(G_Y, r_{G_Y}).$$

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Finite simple left braces: constructions and problems

Ferran Cedó

(joint work with David Bachiller, Eric Jespers, and Jan Okniński)

Recall the following definitions for a left brace $(B, +, \cdot)$:

- B comes with an action $\lambda\colon (B, \cdot) \to \mathrm{Aut}(B, +)$ defined by $\lambda(a) = \lambda_a$ and $\lambda_a(b) = a \cdot b - a$, for $a, b \in B$;
- B is called *trivial* if $ab = a + b$ for all $a, b \in B$;
- a *left ideal* of B is a subgroup L of $(B, +)$ such that $\lambda_a(b) \in L$ for all $b \in L$ and all $a \in B$;
- an *ideal* of B is a normal subgroup I of $(B, \cdot)$ such that $\lambda_a(b) \in I$ for all $b \in I$ and all $a \in B$;
- B is *simple* if $\{0\}$ and B are distinct, and are the only ideals of B.

Theorem 1 (Etingof, Schedler, Soloviev, [6])**.** *The multiplicative group of a finite left brace is solvable.*

Theorem 2 (Rump, [7])**.** *Every finite simple left brace with nilpotent multiplicative group is a trivial brace of prime cardinality.*

Theorem 3 (Bachiller, [1])**.** *Let p_1, p_2 be two prime numbers such that $p_2 \mid p_1 - 1$. Then, for every positive integer k, there exists a finite simple left brace with additive group isomorphic to*

$$\mathbb{Z}/(p_1) \times (\mathbb{Z}/(p_2))^{k(p_1-1)+1}.$$

1. Construction of left braces

We say that a left brace B is a matched product of left ideals $L_1, L_2, \dots, L_n$ if the additive group of B is the direct sum of these left ideals.

Note that if B is a finite left brace, then every Sylow subgroup of its additive group is a left ideal and thus B is the matched product of these left ideals.

Hegedűs left braces.
Let p be a prime. Hegedűs constructed examples of nonabelian regular subgroups of the affine group $\mathrm{AGL}(n,p)$ for $p > 2$ and $n > 3$ or $p = 2$ and $n \geq 3$,n odd. Catino and Rizzo presented these examples as left braces.

Theorem 4 (Bachiller, Cedó, Jespers, Okniński, [2])**.** *Let s be an integer greater than 1 and let $p_1, p_2, \dots, p_s$ be different prime numbers. Then there exists a simple left brace B with additive group*

$$(\mathbb{Z}/(p_1))^{p_1(p_2-1)+1} \times \cdots \times (\mathbb{Z}/(p_{s-1}))^{p_{s-1}(p_s-1)+1} \times (\mathbb{Z}/(p_s))^{p_s(p_1-1)+1}.$$

Furthermore, B is the matched product of the left ideals corresponding to the Sylow subgroups of $(B,+)$, and every such left ideal is a Hegedűs left brace.

Asymmetric product of left braces.
Catino, Colazzo and Stefanelli in [4] introduced the asymmetric product of two left braces.

Theorem 5 (Bachiller, Cedó, Jespers, Okniński, [3])**.** *Let $A = \mathbb{Z}/(l_1) \times \cdots \times \mathbb{Z}/(l_s)$ be a finite abelian group and let p be a prime such that $p \mid q-1$ for every prime divisor q of the order of A. Then there exists a finite simple left brace B with additive group*

$$A \times (\mathbb{Z}/(p))^{l_1+\cdots+l_s-s+1}$$

Furthermore, B is an asymmetric product

$$((\mathbb{Z}/(p))^{l_1+\cdots+l_s-s} \rtimes A) \rtimes_\circ \mathbb{Z}/(p)$$

of a semidirect product of two trivial braces $(\mathbb{Z}/(p))^{l_1+\cdots+l_s-s} \rtimes A$ by the trivial brace $\mathbb{Z}/(p)$.

Using a technical construction of asymmetric product of two trivial braces we have proved the following result.

Theorem 6 (Cedó, Jespers, Okniński, [5])**.** *Take an integer $n > 1$ and distinct primes $p_1, p_2, \dots, p_n$. There exist positive integers $l_1, l_2, \dots, l_n$, only depending on $p_1, p_2, \dots, p_n$, such that for each n-tuple of integers $m_1 \geq l_1$, $m_2 \geq l_2, \dots, m_n \geq l_n$ there exists a simple left brace of order $p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}$ that has a metabelian multiplicative group with abelian Sylow subgroups.*

2. Questions and references

Problem 7. *Describe the structure of all left braces of order p^n, for a prime p. And describe the group* $\mathrm{Aut}(B,+,\cdot)$ *of automorphisms for all such left braces.*

Problem 8. *Determine for which prime numbers p, q and positive integers α, β, there exists a simple left brace of cardinality $p^\alpha q^\beta$.*

Problem 9. *Let B be a non-trivial finite simple left brace that has a metabelian multiplicative group with abelian Sylow subgroups. Is B the asymmetric product of two trivial left braces?*

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Factorizations of skew left braces and multipermutation solutions to the Yang–Baxter equation

Arne Van Antwerpen
(joint work with E. Jespers, Ł. Kubat and L. Vendramin)

To study bijective non-degenerate set-theoretic solutions of the Yang–Baxter equation, Rump [4] and Guarnieri and Vendramin [2] introduced (skew) left braces. An important notion for tying properties of YBE solutions and skew left braces is right nilpotency. Cedo, Smoktunowicz and Vendramin [1] showed that an involutive non-degenerate solution is multipermutation if and only if $G(X,r)$ is a right nilpotent skew left brace. In this talk, we study factorizations of skew left braces. We call a left ideal L of B a strong left ideal, if $(L,+)$ is a normal subgroup of $(B,+)$. The skew left brace B is said to have a factorization, if there exist strong left ideals L, N of B such that $L+N=B$.

Note that a natural example is given by examining a decomposition of a bijective non-degenerate solution (X,r). If $X = Y \cup Z$, where Y and Z are subsolutions such that $r(Y\times X) = X \times Y$ and $r(X \times Y) = Y \times X$, write $< Y >$ and $< Z >$ for the additive subgroup generated by Y and Z respectively in the skew left brace $G(X,r)$. Then, $G(X,r)$ is factorized by $< Y >$ and $< Z >$.

We will discuss in this talk that a skew left brace B, which is factorized by two strong left ideals, which are trivial as skew left braces, is right nilpotent. In particular, this can be applied to show that some solutions are multipermutation. Furthermore, we will discuss that these conditions can be relaxed slightly, but that the relaxed conditions are necessary.

A continuation of this research will allow to further investigate the effect of a decomposition on the solution. Further questions that might be interesting are:

Problem 1. *Is the sum of a trivial skew left brace and a right nilpotent skew left brace again right nilpotent?*

Problem 2. *Can the Kegel–Wielandt result be adapted to skew left braces? For this a novel approach will be needed, as a naive adaptation does not work.*

Problem 3. *Can a classical result of Huppert, i.e. the finite sum of finite cyclic groups is supersolvable, be adapted to skew left braces? In particular, what are both notions for skew left braces? What is their impact on a solution?*

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Hopf–Galois structures and skew braces of squarefree order

Nigel Byott

As observed by Bachiller [3], there is a connection between braces and Hopf–Galois theory. This extends to skew braces. I will outline the relationship between the problem of counting finite skew braces (up to isomorphism) with given additive group A and given multiplicative group M, and the problem of counting Hopf–Galois structures of a given type A on a Galois extension of fields with given Galois group M. Both problems involve finding the regular subgroups in the holomorph $\mathrm{Hol}(A)$ of A which are isomorphic to M. The number of Hopf–Galois structures is obtained by multiplying the number of such regular subgroups by the factor $|\operatorname{Aut}(M)|/|\operatorname{Aut}(A)|$, whereas the number of skew braces is the number of $\operatorname{Aut}(A)$-orbits of such regular subgroups, and in general these orbits will not all have the same size.

As an illustration of how to solve these interrelated counting problems, I will then describe joint work [1, 2] with my former PhD student Ali Alabdali (University of Mosul, Iraq) in which we treat Hopf–Galois structures on Galois extensions of arbitrary squarefree degree, and skew braces of arbitrary squarefree order. A group of squarefree order n has the form

$$G = G(d,e,k) = \langle \sigma, \tau : \sigma^e = 1 = \tau^d, \tau\sigma\tau^{-1} = \sigma^k \rangle$$

where $de = n$ and where k has order d in $\mathbb{Z}_e^\times$. Given such a group, we factorise e as $e = gz$ where $z = \gcd(k-1, e)$ is the order of the centre of G. We fix two groups $G = G(d,e,k)$ and $\Gamma = G(\epsilon, \delta, \kappa)$ of squarefree order n. Let g, z be the quantities defined above for G, and let γ, ζ be the corresponding quantities for Γ. Also let $w = \varphi(\gcd(d,\delta))$ where φ is Euler's totient function, and let $\omega(g)$ be the number of prime factors of g. We then have the following enumeration result for skew braces:

Theorem 1. *The number of skew braces (up to isomorphism) with additive group G and multiplicative group Γ is*

$$\begin{cases} 2^{\omega(g)} w & \text{if } \gamma \mid e, \\ 0 & \text{if } \gamma \nmid e. \end{cases}$$

To prove this, we must first find all regular subgroups of $\mathrm{Hol}(G)$ isomorphic to Γ, from which we can easily deduce the number of Hopf–Galois structures. These regular subgroups fall into w families, in which κ is replaced by certain elements $\kappa_1, \ldots, \kappa_w$ of $\mathbb{Z}_\epsilon^\times$. For each prime q dividing e, let r_q be the order of k in $\mathbb{F}_q^\times$, and let ρ_q be defined similarly for Γ. We define sets of primes

$$S_h = \{q \mid \gcd(g,\gamma) \ : \ r_q = \rho_q > 2 \text{ and } \kappa_h \equiv k^{\pm 1} \pmod q\}$$

for $1 \le h \le w$, and

$$T = \{q \mid \gcd(g,\gamma) \ : \ r_q = \rho_q = 2\}.$$

Theorem 2. *The number of Hopf–Galois structures of type G on a Galois extension of fields with Galois group Γ is*

$$\begin{cases} \dfrac{2^{\omega(g)}\varphi(d)\gamma}{w} \left(\displaystyle\prod_{q \in T} \frac{1}{q} \right) \displaystyle\sum_{h=1}^{w} \prod_{q \in S_h} \frac{q+1}{2q} & \text{if } \gamma \mid e, \\ 0 & \text{if } \gamma \nmid e. \end{cases}$$

Having determined the regular subgroups, we then count their orbits under the action of $\mathrm{Aut}(G)$ to complete the proof of Theorem 1.

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Finite groups versus finite left braces

Adolfo Ballester-Bolinches
(joint work with Ramón Esteban-Romero, Neus Fuster-Corral
and Hangyang Meng)

Some results related to several aspects of the theory of finite left braces and their applications in the context of the Yang–Baxter equation are presented. Our approach is based on the abstract theory of finite groups.

Given an involutive non-degenerate set-theoretic solution (X, r) of the Yang–Baxter equation, its structure group $G(X, r)$ and permutation group $\mathcal{G}(X, r)$ admit structures of left braces as a consequence of a result of [3] based on earlier results of [8]. We can modify the Cayley graph of the permutation group with respect to its natural generating system to obtain the Cayley graph of an abelian group $(\mathcal{G}(X, r), +)$ in such a way $(\mathcal{G}(X, r), +, \circ)$ becomes a left brace, and we use these Cayley graphs to obtain a brace structure for $G(X, r)$. Both structures coincide with the ones of [3]. As consequences of this description, some results about the structure and the permutation groups become more transparent, we can recover a description of a finite Coxeter-like quotient for the structure group, and we prove that the quotient of $\mathcal{G}(X, r)$ by its socle is isomorphic to the permutation group of its retraction. These results appear in [6] and [7].

The semidirect product G associated to the natural action of the multiplicative group C of a skew left brace on its additive group K gives a triple factorisation $G = KC = KD = CD$, in which $\sigma\colon C \to K$ denotes the identity, which corresponds to a 1-cocycle or derivation between these groups with respect to the natural action, that makes $D = \{\sigma(c)c \mid c \in C\}$ to be a subgroup of G with $K \cap D = C \cap D = 1$. This approach was presented by Sysak in [10] and opens the door to the use of techniques of group theory to study braces. Recall that $a * b$ denotes $-a + ab - b$ for a, $b \in B$ and for two subsets X, Y of B, $X * Y$ denotes the subgroup of $(B, +)$ generated by $\{x * y \mid x \in X, y \in Y\}$, $L_1(X, Y) = Y$ and $L_n(X, Y) = X * L_{n-1}(X, Y)$. For example, given a set π of primes, we say that a skew left brace B is left π-nilpotent if $L_n(B, B_\pi) = 0$ for some n, where B_π is the Hall π-subgroup of the additive group of B. For a brace B with nilpotent additive group, and a set of primes π, we can prove that B is left π-nilpotent if, and only if, the multiplicative group is π-nilpotent. The proof depends of some computations with commutators in the semidirect product. This result forms part of [4] and extends some results published in [9] and [1].

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What can we learn from trusses?

Tomasz Brzeziński
(joint work with Bernard Rybołowicz)

The talk begins with recalling the definition of a heap [5], [1], i.e. a set with a ternary operation that satisfies the associative law and the Mal'cev identities, and modelled on the operation $[a, b, c] = a - b + c$ defined on a group. Next we explain the connection between heaps and groups, stressing not only similarities but also differences (e.g. there is an empty heap, there is no empty group). We the proceed to define *trusses*, introduced in [2], as systems with a binary semigroup operation distributing over a ternary (abelian) heap operation and explain how trusses are used to build bridges between braces and rings. In particular, given a ring R or a brace B, there are associated trusses $T(R)$ or $T(B)$.

We explain the notions of *paragons* which in trusses play the same role as ideals in rings, and modules over trusses [3]. We proceed to make some (possibly surprising) observations about rings and braces that arise from thinking about them as trusses of special kinds. In particular we show that every point of a quotient ring or, equivalently, every congruence class in a ring is a paragon in the truss corresponding to this ring. We characterise fully rings in which units form a paragon in the corresponding truss and the quotient truss is associated to the ring $\mathbb{Z}_2$. These are precisely rings R in which either $r \in R$ or $1 - r$ is a unit. Finally, we describe a method of extending a truss to a truss by a one-sided module. If the truss is associated to a brace, this gives a truss associated to brace as well, irrespective of the nature of the extending module. This construction and its one-sided version might lead to new examples of braces. The main new results presented here will appear in [4].

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Desarguesian Garside groups

Carsten Dietzel

A result of Chouraqui says that each structure group $G(X,r)$ associated to a non-degenerate cycle set X is a Garside group whose Garside monoid is generated by X. Rump characterised the groups arising in this way as right l-groups with a duality and showed that they form a subclass of the broader class of geometric Garside groups. These can also be shown to stem from a cyclic structure. We demonstrate how to construct a geometric Garside group for each Desarguesian finite geometry and give an explicit description of the underlying cyclic structures. At the end of the talk, we address related results and open questions regarding more general geometries.

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Generalized semi-braces and set-theoretic solutions of the Yang–Baxter equation

Francesco Catino

(joint work with Ilaria Colazzo and Paola Stefanelli)

Recall that a semigroup $(S,\circ)$ is *completely regular* if any element a of S there exists a (unique) element a^- of S such that

$$a = a\circ a^-\circ a,\quad a^- = a^-\circ a\circ a^-,\quad a\circ a^- = a^-\circ a. \tag{1}$$

Conditions (1) imply that $a^0 := a\circ a^- = a^-\circ a$ is an idempotent element of $(S,\circ)$. Let S be a set with two operations $+$ and $\circ$ such that $(S,+)$ is a semigroup (not necessarily commutative) and $(S,\circ)$ is a completely regular semigroup. We say that $(S,+,\circ)$ is a *generalized (left) semi-brace* if

$$a\circ(b+c) = a\circ b + a\circ\left(a^- + c\right),$$

for all $a, b, c \in S$. Here we assume that the multiplication $\circ$ has higher precedence than the addition $+$.

In addition to left semi-braces [1, 5] examples of generalized left semi-braces may be obtained by any completely regular semigroup. Indeed, if $(S, \circ)$ is an arbitrary completely regular semigroup and $(S, +)$ is a right zero semigroup (or a left zero semigroup), then $(S, +, \circ)$ is a generalized left semi-brace.

Among the several ways by obtaining generalized left semi-braces, we point out the following construction. Let Y be a (lower) semilattice. For each $\alpha \in Y$, let S_α be a generalized left semi-brace and assume that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair α, β of elements of Y such that $\alpha \geq \beta$, let $\phi_{\alpha,\beta} \colon S_\alpha \to S_\beta$ be a homomorphism such tha $\phi_{\alpha,\alpha}$ is the identical automorphism of S_α, for every $\alpha \in Y$, and $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$, for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$. Then, $S = \bigcup\{S_\alpha \mid \alpha \in Y\}$ with addition and multiplication defined by the rule that, for each $a \in S_\alpha$ and $b \in S_\beta$,

$$a + b = \phi_{\alpha,\alpha\beta}(a) + \phi_{\beta,\alpha\beta}(b), \qquad a \circ b = \phi_{\alpha,\alpha\beta}(a) \circ \phi\beta, \alpha\beta(b).$$

is a generalized left semi-brace. If each $S_\alpha (\alpha \in Y)$ is a left semi-brace with $(S, +)$ left cancellative, we show that map $r : S \times S \longrightarrow S \times S$ defined by

$$r(a, b) = (\ a \circ (a^- + b),\ (a^- + b)^- \circ b\ ) \tag{2}$$

is a set-theoretic solution of the Yang–Baxter equation. We obtain this by Theorem 9 of [1] and the following novel construction of set-theoretic solutions of the Yang–Baxter equation.

Let Y be a (lower) semilattice. For each $\alpha \in Y$, let r_α be a solution on a set X_α, and assume that $X_\alpha \cap X_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair α, β of elements of Y such that $\alpha \geq \beta$, let $\phi_{\alpha,\beta} \colon X_\alpha \to X_\beta$ be a map such that $\phi_{\alpha,\alpha}$ is the identity map of S_α, for every $\alpha \in Y$, and $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$, for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$ and $(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta}) r_\alpha = r_\beta(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta})$ for all $\alpha, \beta \in Y$.

If $X = \bigcup\{X_\alpha \mid \alpha \in Y\}$, then the map $r \colon S \times S \longrightarrow S \times S$ defined by

$$r(a, b) = r_{\alpha\beta}(\phi_{\alpha,\alpha\beta}(a),\ \phi_{\beta,\alpha\beta}(b)),$$

for each $a \in S_\alpha$ and $b \in S_\beta$, is a set-theoretic solution of the Yang–Baxter equation.

Finally, we remark that if $(S, +, \circ)$ is a generalized left semi-brace with $(S, +)$ right zero semigroup, the map r as in (2) in a solution if and only if $(S, \circ)$ is a right cryptogroup, that is $(a \circ b)^0 = (a^0 \circ b)^0$, for all $a, b \in S$. So we get new idempotent solutions that are added to those obtained in [6, 7, 3].

Some questions arise naturally:

Question 1. *Given a generalized radical ring S* [2, 4]*, is the map r defined as in (2) a solution?*

Question 2. *What are the generalized left semi-braces S for which the map r defined as in (2) is a solution?*

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Skew lattices and set-theoretic solutions of the Yang–Baxter equation

Charlotte Verwimp
(joint work with Karin Cvetko-Vah)

Finding all set-theoretic YBE solutions is a fundamental open problem. Recently introduced algebraic structures, called braces and cycle sets, are related to special classes of solutions. In an attempt to describe more general solutions, skew- and semi-braces were defined. Still, an algebraic structure describing all solutions is unknown. Hence, we begin this talk with stating the following problem:

Problem 1. *Find algebraic structures that provide (degenerate) set-theoretic solutions of the Yang–Baxter equation.*

An answer to this question comes when looking at skew-lattices, an algebraic structure that up until recently had not been related to the YBE. Solutions that are obtained using skew lattices are degenerate in general, and thus different from solutions obtained from braces, racks and other known structures. We recall basic information on skew lattices and state some results from [1], regarding YBE solutions obtained using skew lattices. The main result is the following.

Theorem 2. *Let* $(S, \wedge, \vee)$ *be a skew lattice. Then the map defined by* $r(x, y) = ((x \wedge y) \vee x, y)$ *is an idempotent set-theoretic solution of the Yang–Baxter equation.*

It turns out that all obtained solutions (S, r) are either idempotent (i.e. $r^2 = r$) or cubic (i.e. $r^3 = r$). The following question is then natural to ask:

Problem 3. *Can we describe more set-theoretic solutions of the Yang–Baxter equation using skew lattices and what kind of solutions do we obtain?*

The fact that skew lattices give solutions motivates the second part of this talk, which is on constructions of skew lattices. We recall a construction from [1] and give some new results and ideas towards the following problem:

Problem 4. *Find constructions of skew lattices. Do the constructed skew lattices inherit properties of structures that we started from?*

We end the talk with some other open problems:

Problem 5. *Are the solutions coming from skew lattices still solutions for other non-commutative lattices? Do non-commutative lattices give new solutions?*

Problem 6. *Study the algebraic structures, like the structure group, associated to set-theoretic solutions that are obtained using skew lattices.*

Problem 7. *Study the relation between skew lattices and other algebraic structures that provide set-theoretic solutions of the Yang–Baxter equation.*

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Solutions of the Yang–Baxter equation and representation theory of the infinite symmetric group

Magdalena Wiertel

One of the approaches to solving the Yang–Baxter equation is to understand the structure of the set of solutions modulo an appropriate equivalence relation. During the talk I will follow the recent paper [1] in which this idea is developed.

It is well known that any involutive R–matrix R, treated as element of End $(V \otimes V)$ for some finitely dimensional space V, generates unitary representations $\rho_R^{(n)}$ of the symmetric group S_n, $n \in \mathbb{N}$, given on the transpositions τ_k, $k = 1, \dots, n-1$ (natural generating set of S_n) as

$$\rho_R^{(n)}(\tau_k) = \mathrm{id}_V^{\otimes k-1} \otimes R \otimes \mathrm{id}_V^{n-k-1} \in \mathrm{End}\ V^{\otimes n}.$$

We will say that two involutive R–matrices are equivalent if and only if for every $n \in \mathbb{N}$ the corresponding representations $\rho_R^{(n)}$ and $\rho_S^{(n)}$ are equivalent.

It turns out that classes of this equivalence relation are in one to one correspondence with pairs of Young diagrams. What is more, involutive R–matrices on the space of dimension d correspond to pairs of Young diagrams with d boxes in total.

The results use the fact that the S_n–representations $\rho_R^{(n)}$ define a representation of the infinite symmetric group S_∞. Theorems from the representation theory of the group S_∞ and some operator–algebraic techniques are essential in the presented approach.

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Yang–Baxter cohomology: Diagonal deformations and knot invariants

Victoria Lebed

Set-theoretic solutions to the YBE are used for colouring braids, knots, knotted surfaces etc. Counting such colourings, one obtains algebraic invariants for topological objects. Following Carter et al. [2], these invariants can be refined by counting colourings with Boltzmann-type weights. These weights are constructed using maps satisfying certain conditions. A convenient way of expressing this is to say that such a map should be a cocycle for an explicitly defined cochain complex. The same complex appears if one wants to encode diagonal deformations of set-theoretic YBE solutions, as it was done in the Hopf algebra classification program of Andruskiewitsch et al. [1]. This complex defines the *braided cohomology* $H^*(X, r)$ of a solution (X, r).

I explained or briefly mentioned several conceptual interpretations of this cohomology theory:

(1) in terms of a topological space (an explicit CW-complex) [7]; its fundamental group happens to be precisely the structure group $G(X, r)$!
(2) using a graphical calculus based on braids and knotted trivalent graphs [9];
(3) involving the quantum shuffle coproduct [9];
(4) in terms of an explicit d.g. bialgebra associated to a solution [3].

The last approach suggests that braided cohomology groups are more than just abelian groups: $H^*(X, r)$ carries a graded commutative associative product, which for suitable coefficients refines to a Zinbiel structure.

Computation-wise, I mentioned

(1) the formula for the Betti numbers (the ranks of the free parts of $H^k(X, r)$), computed for solutions obtained from self-distributive structures [5];
(2) a splitting of $H^*(X, r)$ into two parts for a wide class of solutions (e.g., those coming from quandles as opposed to racks) [13, 14, 12].

Problem 1. *Compute the Betti numbers for arbitrary solutions.*

Computer-aided calculations for small solutions suggest a rather intricate behaviour of these Betti numbers even for involutive solutions.

I concluded with two types of relations between the braided cohomology and the structure groups and algebras of solutions:

(1) an explicit map, the quantum symmetriser $\mathcal{QS}$, relating $H^*(X, r)$ to the Hochschild cohomology $HH^*(kM(X, r))$;
(2) connections between $G(X, r)$ and $H^2(X, r)$, interpretated as solution invariants [5, 4, 8, 12, 11].

Problem 2. *Is the quantum symmetriser $\mathcal{QS}$ bijective for general solutions?*

The answer is known to be positive for involutive and idempotent solutions [6, 10]. In particular, the idempotent case led to results on the cohomology of factorisable groups and plactic monoids.

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Yang–Baxter cohomology: General deformations and rigidity

Michael Eisermann

In his study of quantum groups, Drinfel′d suggested to consider set-theoretic solutions (X, r) of the Yang–Baxter equation as a discrete analogon of linear solutions. Given any commutative ring A, the pair (X, r) can be linearized to the free A–module $V = AX$ with basis X and the A–linear map $c\colon V \otimes V \to V \otimes V$ extending r. This transfers the set-theoretic solution (X, r) in the category of sets to an A–linear solution (V, c) in the category of A–modules.

In order to study deformations of c, we use the ring $A = K[\![h]\!]$ of formal power series over some field $K < A$, whence $A = K \oplus \mathfrak{m}$ with the maximal ideal $\mathfrak{m} = hA$. Our aim is to study deformations of c within the space of Yang–Baxter operators

$$\mathrm{YB}(V) = \left\{ \tilde{c}\colon V \otimes V \to V \otimes V \;\middle|\; \tilde{c}_1\tilde{c}_2\tilde{c}_1 = \tilde{c}_2\tilde{c}_1\tilde{c}_2 \right\}$$

such that $\tilde{c} \equiv c \bmod \mathfrak{m}$. Infinitesimal deformations, modulo $\mathfrak{m}^2$, are classified by the Yang–Baxter cohomology, as laid out in [1, 2]. In this talk I presented the general setting and some (homotopy) retraction theorems, which organize the investigation and greatly simplify calculations. These results establish a relationship between the classical deformation theory following Gerstenhaber and the cohomology theory of racks and quandles, all of which have numerous applications in knot theory.

— (∗) —

As a parenthesis, I presented a counter-example to a conjecture of Hayley Ryder [3]. This open question was brought to our attention by Lebed in her talk on the first day of the workshop, and its solution during this week manifests that the exchange of ideas was immediately fruitful.

We study an adjoint pair of functors: First, the functor Conj: `Groups` $\to$ `Quandles` sending each group $(G, \cdot)$ to its conjugation quandle $(G, \lhd)$, where $a \lhd b = b^{-1} \cdot a \cdot b$. Second, the functor Grp: `Quandles` $\to$ `Groups` sending each Quandle $(Q, \lhd)$ to its structure group given by the presentation

$$\mathrm{Grp}(Q, \lhd) = \left\langle\, x_a \colon a \in Q \;\middle|\; x_a \cdot x_b = x_b \cdot x_{a \lhd b} \colon a, b \in Q \,\right\rangle.$$

This begs the natural question: Given a group $(G, \cdot)$, when is it (isomorphic to) the structure group $\mathrm{Grp}(Q, \lhd)$ of some quandle? We have a simple necessary criterion: The group $\tilde{G} = \mathrm{Grp}(\mathrm{Conj}(G))$ always yields a central extension

$$E \,:\, 1 \longrightarrow K \xhookrightarrow[\text{central}]{} \tilde{G} \underset{p}{\overset{s}{\rightleftarrows}} G \longrightarrow 1$$

via $p \colon \tilde{G} \to G \colon x_g \mapsto g$ for each $g \in G$. If $G = \mathrm{Grp}(Q)$, then E splits via $s \colon G \to \tilde{G} \colon a = x_q \mapsto x_a$ for each $q \in Q$. Ryder conjectured the converse: If E splits, then G is the structure group of some quandle Q, thus $G \cong \mathrm{Grp}(Q)$.

This turns out to be false: The easiest counter-example is provided by any non-trivial superperfect group, that is, a group $G \neq \{1\}$ satisfying $H_1(G;\mathbb{Z}) = H_2(G;\mathbb{Z}) = 0$. By the universal coefficient theorem, this implies $H^2(G, K) = 0$ for any abelian group K. Therefore *every* central extension of G splits, and so does E in particular. (In the setting of perfect groups, central extension behave like coverings, and any superperfect group G is its own universal central extension.) If we had $G \cong \mathrm{Grp}(Q)$ for some n–component quandle Q, then $H_1(G;\mathbb{Z}) \cong G_{\mathrm{ab}} \cong \mathbb{Z}^n$, whence $n = 0$ and $Q = \emptyset$, which contradicts $G \neq \{1\}$.

The smallest example of such a group G is the binary icosahedral group $\tilde{A}_5 \cong \mathrm{SL}_2\,\mathbb{F}_5$, of order 120. Historically, this group and its homology $H_1(\tilde{A}_5;\mathbb{Z}) = H_2(\tilde{A}_5;\mathbb{Z}) = 0$ played a crucial rôle in Poincaré's construction of his famous homology 3–sphere $M = \mathbb{S}^3/\tilde{A}_5$: This is a closed 3–dimensional manifold, and we know $M \not\cong \mathbb{S}^3$ by virtue of its fundamental group $\pi_1(M, *) \cong \tilde{A}_5$. Nevertheless, it satisfies $H_n(M) \cong H_n(\mathbb{S}^3)$ for all $n \in \mathbb{N}$, disproving Poincaré's initial conjecture that homology detects the sphere among all closed 3–manifolds, as it does among all closed 2–manifolds (surfaces).

Following Ryder's conjecture, it remains an interesting open question to characterize and further analyze those groups $G \cong \mathrm{Grp}(Q)$ that arise as structure groups of quandles. This will be worked out in an article in preparation.

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Reporters: Victoria Lebed, Leandro Vendramin

Participants

Prof. Dr. Adolfo Ballester-Bolinches
Departament d'Algebra
Universitat de València
Carrer del Dr. Moliner, 50
46100 Burjassot (València)
SPAIN

Prof. Dr. Tomasz Brzezinski
Department of Mathematics
Swansea University
Fabian Way
Swansea SA1 8EN
UNITED KINGDOM

Prof. Dr. Nigel P. Byott
Department of Mathematics
University of Exeter
Harrison Building
Exeter EX4 4QE
UNITED KINGDOM

Prof. Dr. Francesco Catino
Dipartimento di Matematica e Fisica
"Ennio de Giorgi"
Università del Salento
Via Per-Arnesano
73100 Lecce (LE)
ITALY

Prof. Dr. Ferran Cedó
Departament de Matematiques
Universitat Autònoma de Barcelona
Campus Universitari
08193 Bellaterra (Barcelona)
SPAIN

Prof. Dr. Fabienne Chouraqui
Department of Mathematics and Physics
University of Haifa at Oranim
Kiryat Tiv'on 36006
ISRAEL

Prof. Dr. Carsten Dietzel
Fachbereich Mathematik
Institut für Algebra und Zahlentheorie
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Prof. Dr. Michael Eisermann
Institut für Geometrie und Topologie
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Prof. Dr. Tatiana Gateva-Ivanova
Department of Mathematics and Science
American University in Bulgaria
Balkanski Academic Center, Rm. 322
1 Georgi Izmirliev Square
Blagoevgrad 2700
BULGARIA

Prof. Dr. Eric F. Jespers
Department of Mathematics
Vrije Universiteit Brussel
Pleinlaan 2
1050 Bruxelles
BELGIUM

Dr. Łukasz Kubat
Department of Mathematics
Vrije Universiteit Brussel
Pleinlaan 2
1050 Bruxelles
BELGIUM

Prof. Dr. Victoria Lebed
Laboratoire de Mathématiques Nicolas Oresme
UMR 6139 CNRS
P.O. Box 5186
14032 Caen Cedex
FRANCE

Prof. Dr. Leandro Vendramin
IMAS - Departamento de Matemática - FCEN
Universidad de Buenos Aires
Pabellón I, Ciudad Universitaria
Buenos Aires C 1428 EGA
ARGENTINA

Prof. Dr. Jan Okninski
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warszawa
POLAND

Prof. Dr. Charlotte Verwimp
Department of Mathematics
Vrije Universiteit
Pleinlaan 2
1050 Bruxelles
BELGIUM

Prof. Dr. Arne van Antwerpen
Department of Mathematics
Vrije Universiteit
Pleinlaan 2
1050 Bruxelles
BELGIUM

Prof. Dr. Magdalena Wiertel
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warszawa
POLAND