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Mini-Workshop: Kronecker, Plethysm, and Sylow Branching Coefficients and their Applications to Complexity Theory

Organized by
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Abstract. The Kronecker, plethysm and Sylow branching coefficients describe the decomposition of representations of symmetric groups obtained by tensor products and induction. Understanding these decompositions has been hailed as one of the definitive open problems in algebraic combinatorics and has profound and deep connections with representation theory, symplectic geometry, complexity theory, quantum information theory, and local-global conjectures in representation theory of finite groups. The overarching theme of the Mini-Workshop has been the use of hidden, richer representation theoretic structures to prove and disprove conjectures concerning these coefficients. These structures arise from the modular and local-global representation theory of symmetric groups, graded representation theory of Hecke and Cherednik algebras, and categorical Lie theory.

Mathematics Subject Classification (2010): 05E10, 20C30, 20C08.

Introduction by the Organizers

The mini-workshop *Kronecker, plethysm, and Sylow branching coefficients and their applications to complexity theory*, organised by Christine Bessenrodt (Leibniz University Hannover), Chris Bowman (University of Kent), and Eugenio Giannelli (University of Florence) was attended by 17 researchers from Europe and North America. The expertise of the participants ranged from algebraic combinatorics, group theory and representation theory to their connections to complexity theory.

Main objects of study. We let $\mathfrak{S}_n$ denote the symmetric group on n letters. Given λ a partition of n, we have a corresponding simple $\mathbb{C}\mathfrak{S}_n$-module $\mathbf{S}_\lambda^{\mathbb{C}}$ called a *Specht module.* For $n = a \times b$, we let $\mathfrak{S}_a \wr \mathfrak{S}_b \leq \mathfrak{S}_n$ denote a wreath product subgroup of $\mathfrak{S}_n$. For a prime, p, we let $P_n \leq \mathfrak{S}_n$ denote a Sylow p-subgroup of $\mathfrak{S}_n$. We define the *Kronecker, plethysm* and *Sylow branching coefficients* to be the decomposition multiplicities of the following representations

$$\mathbf{S}_\lambda^{\mathbb{C}} \otimes \mathbf{S}_\mu^{\mathbb{C}} \qquad \mathrm{ind}_{\mathfrak{S}_a \wr \mathfrak{S}_b}^{\mathfrak{S}_n}(\mathbf{S}_\lambda^{\mathbb{C}} \oslash \mathbf{S}_\mu^{\mathbb{C}}) \qquad \mathrm{ind}_{P_n}^{\mathfrak{S}_n}(U) \tag{1}$$

as a direct sum of simple $\mathbb{C}\mathfrak{S}_n$-modules (for U any simple $\mathbb{C}P_n$-module). These coefficients have been described as *"perhaps the most challenging, deep and mysterious objects in algebraic combinatorics"* and were identified by Richard Stanley on his list of the definitive open problems in algebraic combinatorics.

The week at Oberwolfach. The workshop brought together specialists with different but overlapping and well complementing backgrounds in ordinary and modular representation theory (Law, Morotti, Navarro, Olsson, Sutton, Vallejo, Tiep, Wildon), and algebraic combinatorics and its applications (Ikenmeyer, Orellana, Pak, Panova, Rosas, Zabrocki). The format of the workshop was designed to foster collaborations and to inspire new research: the Monday was spent entirely on introductory talks to each of the focus research areas and there was plenty of time throughout the week for collaborative work. This format was very successful in this regard and the highlights were announced by research teams at the end of the week. We detail the programme of the week.

- In their introductory lectures, both Rosa Orellana and Chris Bowman focussed on stability properties of Kronecker and plethysm coefficients, which have been conjectured to be simpler than their non-stable counterparts in a number of ways; both speakers highlighted a fifteen year old conjecture of Klyachko and Kirillov which states that stable Kronecker coefficients form a saturated semigroup. This led to a lively and ongoing debate (which lasted through the week) and a bet was placed by Igor Pak and Chris Bowman on the validity of this conjecture (the former staking it to be incorrect and the latter that it was correct). This bet was resolved by Greta Panova who described a counterexample to this conjecture on the Thursday of the workshop.
- Local-global conjectures are at the heart of modern representation theory of finite groups. Among these, the most important and most studied is certainly the McKay conjecture that dates back to 1972 and has been seminal for many important counting conjectures on characters. On these lines, one of the spectacular highlights of the meeting was Gabriel Navarro's Monday lecture, reporting on joint work with Pham Tiep. In his talk he presented a new and striking conjecture which proposes a direct relation between McKay correspondences and restriction of characters to Sylow p-subgroups. This conjecture provides a fundamental shift of perspective in local-global representation theory and promises to be the source of many new avenues of research. Navarro also announced several results, bringing

new deep information on fields of values of characters, and at the same time providing evidence towards the validity of his new conjecture.

We are pleased to mention that after a week of intense work and exchange of ideas with other participants (Navarro, Tiep, Law and Vallejo), Eugenio Giannelli managed to prove the new Navarro–Tiep conjecture for symmetric and alternating groups.

- Pham Tiep's talk started playfully with the question "Why do we care about tensor products?". His answer was compelling in its sheer breadth: detailing his recent work on the Aschbacher–Scott program and his verification of conjectures of John Thompson and Michael Larsen and their connections across representation theory, group theory, algebraic geometry and number theory.
- Igor Pak detailed new upper and lower bounds for Kronecker coefficients by using recent work on 3-dimensional contingency tables (generalising the notion of a 3-dimensional Young diagram or "pyramid"). Such bounds were utilised by Pak–Panova to understand the asymptotically largest Kronecker coefficients and by Ikenmeyer–Mulmuley–Walter to show that the problem of deciding positivity of Kronecker coefficients is NP-hard.
- Greta Panova and Christian Ikenmeyer told a story of the recent highs and lows in the Geometric Complexity Theory program. Greta Panova focussed on their recent *negative* results. The central idea of GCT is to provide *obstructions* which separate the arithmetic versions of P and NP. Classically, this means proving that the determinantal complexity of the permanent grows super-polynomially. The GCT machine translates this into proving strict inequalities between Kronecker and plethysm coefficients. Greta Panova discussed their recent seismic paper in which they proved that this problem is much more difficult than anticipated: *"occurrence obstructions"* (inequalities in which the Kronecker coefficient is zero and the plethysm coefficient is strictly positive) do not exist. In other words to prove $\mathsf{P} \neq \mathsf{NP}$, it is *not enough* to find pairs such that the Kronecker coefficient is zero and the plethysm coefficient is (a perhaps *unknown*) positive number. Rather, we require "multiplicity obstructions" involving precise information about the *exact values* of both (non-zero) coefficients.

  Christian Ikenmeyer, on the other hand, told us that this is not the end of the world. In an enlightening talk, he focussed on possible replacements for the determinant and permanent as the polynomials in the GCT program. Their recent work in the context of Chow varieties has shown that multiplicity obstructions do exist and that they are stronger than occurrence obstructions. Christian also ran through a "miniature GCT program" involving the elementary symmetric and power sum polynomials. This highlighted how each step of the program should work (given an input of a pair of polynomials) in order to separate out complexity classes.

- In their talks on Tuesday and Wednesday, Stacey Law and Carolina Vallejo presented new results on Sylow Branching Coefficients for symmetric and alternating groups. Vallejo also mentioned significant new results towards a complete proof of a long standing open problem proposed by Malle and Navarro in 2012. The completion of this project was at the centre of collaborative work between Giannelli, Law and Vallejo during the afternoons at Oberwolfach.
- The quest for Sylow Branching Coefficients started with the study of McKay bijections for symmetric groups at the prime $p = 2$. In this sense it is incredible and a bit frustrating that we are currently unable to decompose the permutation characters induced from Sylow 2-subgroups. Bessenrodt, Giannelli and Olsson discussed this problem at Oberwolfach (and also before this meeting). Despite some encouraging progress, a complete solution seems still far from reach.
- A current benchmark problem for Kronecker coefficients is the Saxl conjecture which claims that all Kronecker coefficients for the square of the staircase character are positive. The existence of such an S_n-character for all $n > 9$ was conjectured by Heide, Saxl, Tiep and Zalesski, but without providing candidates. On her Simons Visiting Professorship, Greta Panova visited Christine Bessenrodt at Hannover, and they began work on a vast generalisation of Saxl's conjecture. Based on computer calculations, they formulated a conjectural classification of all irreducible characters such that the Kronecker coefficients of their squares are all positive. This opens up Saxl's original conjecture (phrased only for triangular numbers) to all integers and hence promises the possibility of an inductive attack.
- In 2019, Bowman, De Visscher, and Enyang had provided a "lattice permutation condition" for calculating *stable* Kronecker coefficients, and Orellana, Zabrocki had provided a "semistandard condition" for calculating stable Kronecker coefficients. Both teams used tableaux-theoretic ideas arising in slightly different Schur, Weyl and Howe dualities involving (multi-set) partition algebras. During the week, Chris Bowman, Rosa Orellana, and Mike Zabrocki pored over many intricate calculations; they attempted to reconcile these approaches and began several pump-priming projects (including the search for a presentation of the multiset partition algebra, an action of this algebra on multiset tableaux, and implementation of algorithms in SAGE) with the eventual aim of solving the stable Kronecker problem in its entirety. This was also the subject of Orellana and Zabrocki's talks.
- New links of the Kronecker problem to modular representation theory appeared in talks by Louise Sutton on Wednesday and Lucia Morotti on Friday. In her talk, Louise Sutton presented new results on Kronecker coefficients with applications to Saxl's conjecture, and connected to decomposition numbers of symmetric groups and Hecke algebras. In the

situation of the modular Kronecker problem, even determining when a tensor product of representations for the symmetric and alternating groups is irreducible is difficult. Morotti spoke about this problem and discussed her impressive results completing the classification of irreducible tensor products for these groups as well as their double covers at characteristic $p \neq 2$.

Wider categorical, combinatorial, and character-theoretic perspectives. A number of themes relating to wider contexts arose repeatedly throughout the week. Bowman and Wildon suggested a number of starting points for the search of a "plethystic crystal", only to discover that Zabrocki and Orellana had been looking at this exact same problem in an ongoing collaboration hosted at the American Institute of Mathematics.

The modular decomposition number problem also arose again and again during the week, culminating on the final day when Bowman announced the proof of a vast generalisation of Riche–Williamson's work on p-Kazhdan–Lusztig polynomials and decomposition numbers of symmetric groups. In a similar vein, Mark Wildon proposed an entirely new avenue of research "What plethystic isomorphisms have modular analogues?". Unfortunately, this question was only raised on the final day of the workshop and so did not receive the attention it deserved.

Zeros of characters, character values and more generally the study of the field of values of irreducible characters intertwined at several points in the workshop. In one direction, while unique factorisation of Kronecker products was disproved by a counterexample produced by Bessenrodt, the tools used in this context led to a number of wider questions concerning the vanishing sets of irreducible characters. In another direction, such character theoretic questions also arose in Navarro and Tiep's new conjecture which was proposed at the workshop.

It was widely agreed that these topics deserve much further investigation at a larger workshop, and that Oberwolfach would make an ideal location for such a future event.

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Table of Contents

Abstracts

Character values and Sylow Subgroups

Gabriel Navarro
(joint work with Pham Huu Tiep)

1. Introduction. I think that, in a certain sense, this talk and my presence here closes a circle that was opened in December 2012. I met E. Giannelli in Florence and asked him about the decomposition into irreducible constituents of the permutation character $(1_P)^G$ in symmetric groups, where P is a Sylow p-subgroup of G. I received an e-mail a month later telling me that he knew how to do a few cases. Later, in 2018 ([2]), he determined, with Stacey Law, and for p odd, the constituents of the character $(1_P)^G$, what we are going to call $\mathrm{Irr}((1_P)^G)$, but not the multiplicities (yet). These multiplicities are now called, in S_n, **branching Sylow coefficients**, and it is the reason of my presence in the workshop.

Why my interest in this particular character?

Let me start by saying that one of the main problems in representation theory of finite groups, is to understand the global/local connections between a group and the local subgroups. Richard Brauer was a pioneer in this idea. Examples of that are that simple groups could be classified by using centralizers of involutions; Brauer's first main theorem; or Problem 12 of his famous list: what does the character table $X(G)$ of G, know about $P \in \mathrm{Syl}_p(G)$? In fact, let me raise a more general problem here: what does $X_p(G)$ know about P, where $X_p(G)$ is the submatrix of the character table that corresponds to p-elements? (For instance, if $p \neq 3, 5$, $X_p(G)$ determines if P is abelian. Is this true in general?)

About this particular character $(1_P)^G$:

• In 1998 ([5]), I wrote a paper "Fusion in the Character Table", proving that from the character table of G you can determine the character $(1_P)^G$ in p-solvable groups. Why the name fusion? It is easy to see that knowing $(1_P)^G$ is equivalent to knowing $|x^G \cap P|$ for every p-element $x \in G$. I don't think this is known even if P is abelian. Recall that an abelian p-group is determined by the multiset of the orders of its elements.

• In 2012 ([4]), I wrote a paper with Gunter Malle about the constituents of $(1_P)^G$. One of the most essential results in character theory degrees is the Itô-Michler Theorem asserting that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if P is normal and abelian. We have, if Brauer's Height Zero Conjecture is true, that p does not divide $\chi(1)$ for all characters in the principal p-block if and only if P is abelian. We were able to separate "normal" from "abelian", and the answer was $(1_P)^G$: $P \triangleleft G$ if and only if p does not divide the degrees of the characters in $\mathrm{Irr}((1_P)^G)$.

But overall, I was essentially interested in characters of p'-degree (not divisible by p) and the McKay conjecture.

2. McKay connections. After Brauer, the most important thing that has happened in my part of Representation Theory is the McKay conjecture. For colleagues not entirely familiar with this problem, the McKay conjecture from 1971 asserts that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$, where $\mathrm{Irr}_{p'}(G)$ is the set of irreducible complex characters of G of degree not divisible by p.

Looking for some explanations, one wonders if there are some special types of bijections: $^* : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$, and the first thing that comes to your mind is to restrict characters to $\mathbf{N}_G(P)$ and check if you see something. If χ is a character of G and H is a subgroup, then χ_H is the restriction. Spoiler: restriction to $\mathbf{N}_G(P)$ is not the answer in general. In $G = \mathsf{S}_n$, there are different p'-degree characters that have the same restriction to $\mathbf{N}_G(P)$. Perhaps the experts have an explanation for this: for $p = 5$ the only example that I know of is S_5, and the two characters of degree 4. For $p = 3$, I only know of two examples: in S_{10} and S_{19}. I have checked up to S_{25}. No examples for $p = 2$. In any case, even when there is a natural bijection (for example, S_n and $p = 2$), this approach is too naive.

Let us introduce some important notation. If K is a field (of characteristic zero, almost all the time $\mathbb{Q}$) and χ is a character then $K(\chi)$ is the smallest field containing the values of χ and K. There are two important numbers associated to χ: the degree $\chi(1)$ and the **conductor** $c(\chi)$, which is the smallest $f \geq 1$ such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_f$, where here $\mathbb{Q}_f$ is the f-th cyclotomic field. (That is, $c(\chi)$ is the conductor of the field $\mathbb{Q}(\chi)$.)

An uncontroversial assertion, perhaps, is that if there is a *canonical bijection* * : $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ then one should have $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*)$ for all $\chi \in \mathrm{Irr}_{p'}(G)$. But this is false in general. The following seems to be working, however ([6]).

Conjecture (Galois-McKay) *There is a bijection* $^* : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ *such that* $\mathbf{Q}_p(\chi) = \mathbf{Q}_p(\chi^*)$*, where* $\mathbf{Q}_p$ *is the field of* p*-adics.*

Here we are interested in the field of values of the p'-degree characters over $\mathbb{Q}$.

If $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}_n$, then it is elementary to show that there is a finite group G and $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi) = F$. However, we were realizing that $\mathcal{F}_2 = \{\mathbb{Q}(\chi)/\mathbb{Q} \,|\, \chi \in \mathrm{Irr}_{2'}(G),\, G \text{ is a finite group}\}$ are not all the abelian extensions! What are these fields? My interest was to find more global/local connections, perhaps even find a new refinement of McKay. We proved the following last year ([3]).

Theorem *Suppose that* $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{d})$*, where* $d \neq \pm 1$ *is a square-free integer. If* $\chi(1)$ *is odd, then* $d \equiv 1$ *mod 4.*

So for instance, there is no $\chi(1)$ odd such that $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{2}i)$, etc. Looks elementary. But it is not.

Our main result here (whose proof was completed at the Institute) is the following:

Theorem (N-Tiep). *$\mathcal{F}_2$ is the set of abelian extensions $F/\mathbb{Q}$ such that $\mathbb{Q}_{2^a} \subseteq F \subseteq \mathbb{Q}_n$, where $n = 2^a m$ is the conductor of F, and m is odd.*

There is an odd version of this result, which occurs if the quasi-simple groups satisfy a natural condition. All these results, and others (such as Gow's conjecture on real groups and the exponent of P/P' [7]) would be a consequence of the following.

Conjecture (Restriction to Sylow, N-Tiep, 2020) *There is a bijection*

$$^* : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$$

such that $\mathbb{Q}(\chi_P) = \mathbb{Q}((\chi^*)|_P)$ for $\chi \in \mathrm{Irr}_{p'}(G)$.

3. Close the circle. I come back to $(1_P)^G$. What is the subset of $\mathrm{Irr}_{p'}(G)$, if it exists, that corresponds via a McKay bijection * onto $\mathrm{Irr}(\mathbf{N}_G(P)/P)$? Giannelli has an answer for symmetric groups ([1]). For p-solvable groups, these are the p'-special characters of Gajendragadkar. What about groups of Lie type? These characters should be **strongly p-rational** (if $G \triangleleft \Gamma$ and Γ/G is a p'-group, then all $\mathrm{Irr}(\Gamma|\chi)$ are p-rational). In p-solvable groups, we also have that $\chi^0 \in \mathrm{IBr}(G)$ for these characters. Perhaps there is something about χ^0 in general.

References

[1] E. Giannelli, A canonical class of p'-degree characters of S_n, J. Algebra **491**(2017), 141–157.

[2] E. Giannelli, S. Law, On permutation characters and Sylow p-subgroups of S_n, J. Algebra **506** (2018), 409–428.

[3] I. M. Isaacs, M. W. Liebeck, G. Navarro, P. H. Tiep, Fields of values of odd-degree irreducible characters, Advances in Mathematics **354** (2019) 106757, 26pp.

[4] G. Malle, G. Navarro, Characterizing normal Sylow p-subgroups by character degrees. J. Algebra **370** (2012), 402–406.

[5] G. Navarro, Fusion in the character table. Proc. Amer. Math. Soc. **126** (1998), no. 1, 65–66,

[6] G. Navarro, The McKay conjecture and Galois automorphisms. Ann. of Math. (2) **160** (2004), no. 3, 1129–1140.

[7] G. Navarro, P.H. Tiep, Real Groups and Sylow 2-subgroups, Advances in Mathematics **299** (2016), 331-360.

Introduction to Geometric Complexity Theory

Christian Ikenmeyer

Geometric complexity theory studies specific orbit closure questions that have implications in computational complexity theory. In this short talk I highlighted different orbit closure containment questions besides Valiant's well-known determinant versus permanent question that can be studied with representation theoretic branching rules.

The starting point of geometric complexity theory [1] is an adapted result of Valiant [2] that can be phrased as follows. Let

$$\det_n := \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) \prod_{i=1}^n x_{i,\pi(i)}.$$

Fix $m \in \mathbb{N}$ and a homogeneous polynomial f of degree m. Then there exists $n \in \mathbb{N}$ such that

$$x_1^{n-m} f \in \overline{\mathrm{GL}_{n^2} \det_n}. \tag{1}$$

In the talk I proved this result via *algebraic branching programs*, which is a convenient way of depicting sparse iterated matrix multiplication. The smallest n that is required for this inclusion is called the *border determinantal complexity* $\underline{\mathsf{dc}}(f)$ of f. The *permanent* polynomial is defined as

$$\mathrm{per}_m := \sum_{\pi \in \mathfrak{S}_m} \prod_{i=1}^m x_{i,\pi(i)}.$$

Mulmuley and Sohoni's strengthening of Valiant's conjecture can be phrased as

the sequence of natural numbers $\underline{\mathsf{dc}}(\mathrm{per}_m)$ is not polynomially bounded,

i.e., there does not exist a univariate real polynomial q such that $\underline{\mathsf{dc}}(\mathrm{per}_m) \leq q(m)$.

The proof of Valiant's theorem naturally leads to other orbit closures that can be used instead of $\overline{\mathrm{GL}_{n^2} \det_n}$. For example, define the (degree d, width w) iterated matrix multiplication polynomial (homogeneous of degree d in $(d-2)w^2 + 2w$ many variables) as follows:

$$\mathrm{imm}_w^d := \sum_{i_1,\ldots,i_{d-1} \in \{1,\ldots,w\}} x_{1,i_1}^{(1)} x_{i_1,i_2}^{(2)} x_{i_2,i_3}^{(3)} \cdots x_{i_{d-1},1}^{(d)}$$

Then in the conjecture above we can replace $\overline{\mathrm{GL}_{n^2} \det_n}$ by

$$\overline{\mathrm{GL}_{(n-2)n^2+2n} \mathrm{imm}_n^n}$$

A paper by Ben-Or and Cleve [3] lets us simplify this orbit closure in a way that it still has a very similar impact on algebraic complexity theory:

$$\overline{\mathrm{GL}_{9(n-2)+6} \mathrm{imm}_3^n}$$

Very recently, [4] shows that we can simplify further:

$$\overline{\mathrm{GL}_{4(n-2)+4} \mathrm{imm}_2^n}$$

Moreover, the polynomial imm_2^n can be replaced by a homogeneous degree n polynomial in n variables, so that the group action is GL_n.

A different approach is homogenization, where the left hand side of equation (1) is also adjusted by removing the *padding* with x_1^{n-m}. In fact, for the purposes of algebraic complexity theory, instead of studying

$$x_{1,1}^{n-m} \mathrm{per}_m \overset{?}{\in} \overline{\mathrm{GL}_{(n-2)n^2+2n} \mathrm{imm}_n^n}$$

we can also study the cleaner version

$$\mathrm{per}_m \overset{?}{\in} \overline{\mathrm{GL}_{(m-2)n^2+2n} \mathrm{imm}_n^m}$$

In this cleaner version, the recent no-go results about occurrence obstructions [5, 6] do not hold, which is already an indication that we gain by studying the homogenized setting.

References

[1] K.D. Mulmuley and M. Sohoni. Geometric Complexity Theory. I. An approach to the P vs. NP and related problems. *SIAM J. Comput.*, 31(2):496–526, 2001.

[2] Leslie G. Valiant. Completeness classes in algebra. In *Conference Record of the Eleventh Annual ACM Symposium on Theory of Computing (Atlanta, Ga., 1979)*, pages 249–261. ACM, New York, 1979.

[3] Michael Ben-Or and Richard Cleve. Computing algebraic formulas using a constant number of registers. *Proceedings 20th Annual ACM Symposium on Theory of Computing 1988*, pages 554–257, 1988.

[4] Karl Bringmann, Christian Ikenmeyer, and Jeroen Zuiddam. On algebraic branching programs of small width. *J. ACM*, 65(5), August 2018.

[5] Christian Ikenmeyer and Greta Panova. Rectangular kronecker coefficients and plethysms in geometric complexity theory. *Advances in Mathematics*, 319:40 – 66, 2017. A conference version appeared in: Proceedings IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS 2016), 396–405.

[6] Peter Bürgisser, Christian Ikenmeyer, and Greta Panova. No occurrence obstructions in geometric complexity theory. *Journal of the American Mathematical Society*, 32:163–193, 2019. A conference version appeared in: Proceedings IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS 2016), 386–395.

An introduction to plethysm

Mark Wildon

(joint work with Melanie de Boeck and Rowena Paget)

Let Λ be the ring of symmetric functions and let $s_\lambda = \sum_{t \in \mathrm{SSYT}(\lambda)} x^t$ be the Schur function labelled by the partition λ, defined combinatorially as the generating function enumerating semistandard tableaux of shape λ. For example,

$$(1) \qquad s_{(2,1)}(x_1,x_2,x_3) = x^{\ytableaushort{11,2}} + x^{\ytableaushort{11,3}} + x^{\ytableaushort{12,2}} + x^{\ytableaushort{12,3}} + x^{\ytableaushort{13,2}} + x^{\ytableaushort{13,3}} + x^{\ytableaushort{22,3}} + x^{\ytableaushort{23,3}}$$
$$= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3.$$

Informally, the *plethysm* $f \circ g$ of $f, g \in \Lambda$ is defined by substituting the monomials in g for the variables in f. This definition is unambiguous and easy to work with when g is a sum of distinct monomials. We give an example using $s_{(2)}(x_1,x_2) = x_1^2 + x_1x_2 + x_2^2$ and $s_{(2)}(y_1,y_2,y_3) = y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_1y_3 + y_2y_3$. Substituting monomials we find

$$(2) \qquad (f \circ g)(x_1,x_2) = f(x_1^2, x_2^2, x_1x_2) = x_1^4 + x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3 + x_2^4.$$

Note that since f is a symmetric function, it does not matter how we order the monomials of g; for instance,

$$(3) \qquad f(x_1^2, x_2^2, x_1x_2) = f(x_1x_2, x_1^2, x_2^2)$$

Moreover, since g is symmetric, $f \circ g$ is symmetric. If g has a repeated monomial then it is substituted in f according to its multiplicity: for instance if $g = (x_1+x_2)^2$ then $s_{(2)} \circ g = s_{(2)}(x_1^2, x_2^2, x_1x_2, x_1x_2)$. As this may indicate, there are subtleties in extending the plethysm product to arbitrary g: see [9] for the general definition and an excellent introduction to plethysm.

A fundamental open problem in algebraic combinatorics is to find the coefficients $\langle s_\nu \circ s_\mu, s_\lambda \rangle$ in the decomposition of the plethysm $s_\nu \circ s_\mu$ as a linear combination of Schur functions. This problem can be attacked using representations of general linear and symmetric groups, invariant theory, and ideas from combinatorial enumeration, such as the cycle index and the plethystic semistandard tableaux defined below. In my talk I surveyed some of these connections and gave some of the more useful rules for computing plethysms. I ended with a summary of the state of the art on Foulkes' Conjecture.

A combinatorial model. Let $\mathrm{PSSYT}(\mu^\nu)$ be the set of semistandard ν-tableaux whose entries are themselves semistandard μ-tableaux. (This requires the semistandard μ-tableaux to be ordered in some way: as seen in (3), the choice of order is irrelevant.) We define the *weight* of a plethystic semistandard tableau to be the sum of the weights of its μ-tableau entries. The 'substitute monomials' rule implies that $s_\nu \circ s_\mu = \sum_{T \in \mathrm{PSSYT}(\mu^\nu)} x^T$. This definition appears in [7, Definition 3.1], where it is used to find the maximal constituent of $s_\nu \circ s_\mu$ in the reverse lexicographic order on partitions. To give a small example, $\begin{array}{|c|c|}\hline \boxed{1}\boxed{1} & \boxed{1}\boxed{2} \\ \hline \end{array}$ has weight $(3,1)$ and, using the same formalism as (1),

$$
\begin{aligned}
(s_{(2)} \circ s_{(2)})(x_1, x_2) &= x^{\begin{array}{|c|c|}\hline \boxed{1}\boxed{1} & \boxed{1}\boxed{1} \\ \hline \end{array}} + x^{\begin{array}{|c|c|}\hline \boxed{1}\boxed{1} & \boxed{1}\boxed{2} \\ \hline \end{array}} + x^{\begin{array}{|c|c|}\hline \boxed{1}\boxed{1} & \boxed{2}\boxed{2} \\ \hline \end{array}} \\
&\quad + x^{\begin{array}{|c|c|}\hline \boxed{1}\boxed{2} & \boxed{1}\boxed{2} \\ \hline \end{array}} + x^{\begin{array}{|c|c|}\hline \boxed{1}\boxed{2} & \boxed{2}\boxed{2} \\ \hline \end{array}} + x^{\begin{array}{|c|c|}\hline \boxed{2}\boxed{2} & \boxed{2}\boxed{2} \\ \hline \end{array}} \qquad (4) \\
&= x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4 \\
&= s_{(4)}(x_1, x_2) + s_{(2,2)}(x_1, x_2). \qquad (5)
\end{aligned}
$$

This agrees with (2). Working with further variables gives nothing new: in fact $s_{(2)} \circ s_{(2)} = s_{(4)} + s_{(2,2)}$.

General linear groups and invariant theory. Given $\lambda \in \mathrm{Par}(r)$, let ∇^λ denote the corresponding Schur functor: thus if V is a polynomial representation of $\mathrm{GL}_d(\mathbb{C})$ of degree s then $\nabla^\lambda(V)$ is a polynomial representation of degree rs. For example, $\nabla^{(r)}$ and $\nabla^{(1^r)}$ are the rth symmetric power and rth exterior power functors, respectively. Let Φ_W denote the formal character of a representation W; for instance, if E is the natural representation of $\mathrm{GL}_d(\mathbb{C})$ then

$$\Phi_{\nabla^\lambda(E)} = s_{(r)}(x_1, \ldots, x_d)$$

and correspondingly, $\nabla^\lambda(E)$ has a canonical basis of weight vectors indexed by $\mathrm{SSYT}(\lambda)$. The fundamental bridge between plethysm and Schur functors is the relation

$$\Phi_{\nabla^\nu\left(\nabla^\mu(E)\right)} = (s_\nu \circ s_\mu)(x_1, \ldots, x_d). \qquad (6)$$

For example, if $E = \langle e_1, e_2 \rangle$ then $\mathrm{Sym}^2 E = \langle e_1^2, e_1e_2, e_2^2 \rangle$ and

$$\mathrm{Sym}^2(\mathrm{Sym}^2 E) = \big\langle (e_1^2)(e_1^2), (e_1^2)(e_1e_2), (e_1^2)(e_2^2), (e_1e_2)(e_1e_2), (e_1e_2)(e_2^2), (e_2^2)(e_2^2) \big\rangle$$

where the basis vectors are ordered to correspond with (4). Using this we may verify (6) and see the decomposition in (4) algebraically: the 'multiply out' map $\mathrm{Sym}^2(\mathrm{Sym}^2 E) \to \mathrm{Sym}^4 E$ has kernel spanned by $(e_1^2)(e_2^2) - (e_1e_2)(e_1e_2)$, which is a highest weight vector in $\nabla^{(2,2)}E$. This is interpreted geometrically in a very instructive example in [6, §11.3]; in my talk I sketched a proof using the related invariant theory of $\mathrm{SL}_2(\mathbb{C})$ that

$$\mathrm{Sym}^2\mathrm{Sym}^n E \cong \sum_{0 \leq s \leq n/2} \nabla^{(2m-2s,2s)}E. \tag{7}$$

Symmetric groups. Now suppose that $E = \langle e_1, \ldots, e_d \rangle$ where $d \geq r$. Let λ be a partition of r. The polynomial representation $\nabla^\lambda(E)$ of $\mathrm{GL}_d(\mathbb{C})$ has a (1^r)-weight space, denoted $\nabla^\lambda(E)_{(1^r)}$, in which the diagonal matrix $\mathrm{diag}(\alpha_1, \ldots, \alpha_d)$ acts by multiplication by $x_1 \ldots x_d$. This weight space is invariant under the permutation matrices in $\mathrm{GL}_d(\mathbb{C})$ that permute $e_1, \ldots, e_d$ amongst themselves. The fundamental bridge between representations of general linear and symmetric groups is that $\nabla^\lambda(E)_{(1^r)} \cong S^\lambda$, where S^λ is the Specht module canonically labelled by λ.

To see how composition of polynomial representations is reflected in weight spaces, an example is helpful. Observe that $\mathrm{Sym}^r E_{(1^r)} = \langle e_1 e_2 \ldots e_r \rangle$ is the trivial module and, more generally,

$$\left(\mathrm{Sym}^m E\right)^{\otimes n}_{(1^{mn})} = \langle e_{i_1} \ldots e_{i_m} \otimes \cdots \otimes e_{j_1} \ldots e_{j_m} \rangle$$

where (in slightly informal notation), $\big(\{i_1, \ldots, i_m\}, \ldots, \{j_1, \ldots, j_m\}\big)$ is an ordered partition of $\{1, \ldots, r\}$. Hence the weight space is isomorphic to the permutation module of S_{mn} acting on the cosets of the Young subgroup $S_m \times \cdots \times S_m$ of S_{mn}. Suppose we replace $\otimes^n$ with the Schur functor Sym^n. The basis for the weight space then becomes

$$(e_{i_1 \ldots i_m}) \ldots (e_{j_1} \ldots e_{j_m}) \in \mathrm{Sym}^n \mathrm{Sym}^m E$$

where concatenation shows the product for Sym^n. The order of sets in the partition is now irrelevant, and so the weight space is isomorphic to the permutation module of S_{mn} acting on the cosets of the wreath product $S_m \wr S_n$ containing the Young subgroup $S_m \times \cdots \times S_m$ as its base group. This is the *Foulkes module* $H^{(m^n)}$.

More generally, one can show that

$$\nabla^\nu\big(\nabla^\mu(E)\big)_{(1^{nm})} \cong \Big(\widetilde{(S^\mu)}^{\otimes n} \otimes \mathrm{Inf}_{S_n}^{S_m \wr S_n} S^\nu\Big)\mathrm{Ind}_{S_m \wr S_n}^{S_{mn}}. \tag{8}$$

Here the tilde denotes that the action of $S_m \times \cdots \times S_m$ on $(S^\mu)^{\otimes n}$ is extended to a top group S_n in the wreath product $S_m \wr S_n$ by permuting factors; in the example above, the representation we induce is the trivial representation of $S_m \wr S_n$.

Rules for computing plethysm. Generalizing the result of Iijima [7] mentioned above, de Boeck, Paget and the author [4, Theorem 1.5] proved the following theorem.

Theorem 1. *The maximal constituents of $s_\nu \circ s_\mu$ are precisely the maximal weights of the plethystic semistandard tableaux of shape μ^ν.*

This strengthened an earlier result proved by Paget and Wildon in [11] using (8). Also in [4], the authors gave a simpler proof of a result originally due to Brion [1], strengthened with an explicit combinatorial bound on the stable multiplicity.

Theorem 2. *Let $\nu \in \mathbb{P}(n)$ and let μ be a partition. If $r \in \mathbf{N}$ then*

$$\langle s_\nu \circ s_{\mu+(1^r)}, s_{\lambda+(n^r)}\rangle \geq \langle s_\nu \circ s_\mu, s_\lambda\rangle$$

for all partitions λ. Moreover $\langle s_\nu \circ s_{\mu+N(1^r)}, s_{\lambda+N(n^r)}\rangle$ is constant for $N \geq n(\mu_1 + \cdots + \mu_{r-1}) + (n-1)\mu_r + \mu_{r+1} - (\lambda_1 + \cdots + \lambda_r)$.

Still in [4], the authors proved the following two theorems, generalizing results due to Newell, Conca and Varbaro [2], and Ikenmeyer [8, Theorem 4.3.4] respectively.

Theorem 3. *Let $\nu \in \mathbb{P}(n)$ and let μ be a partition. If r is at least the greatest part of μ then $\langle s_\nu \circ s_{(r)\sqcup\mu}, s_{(nr)\sqcup\lambda}\rangle = \langle s_\nu \circ s_\mu, s_\lambda\rangle$ for all partitions λ.*

Theorem 4. *Let μ be a partition. If $\langle s_{(n^\star)} \circ s_\mu, s_{\lambda^\star}\rangle \geq 1$ then $\langle s_{(n+n^\star)} \circ s_\mu, s_{\lambda+\lambda^\star}\rangle \geq \langle s_{(n)} \circ s_\mu, s_\lambda\rangle$.*

Many further results on plethysm are known and it will be clear that the selection above is biased to the author's work.

Foulkes' Conjecture. In the language of symmetric functions, Foulkes' Conjecture states that if $n \geq m$ then $\langle s_{(n)} \circ s_{(m)}, s_\lambda\rangle \geq \langle s_{(m)} \circ s_{(n)}, s_\lambda\rangle$ for all partitions λ of mn. Equivalently, using the symmetric group, $H^{(n^m)}$ is isomorphic to a submodule of $H^{(m^n)}$. Foulkes' Conjecture is proved only when $n \leq 5$ (see [3] for the case $n = 5$), when $m+n \leq 19$ (computationally in [5] for $m+n \leq 19$, extending [10]) and when n is very large compared to m (see [1]). The full decomposition of $s_{(n)} \circ s_{(m)}$ is known for all m only when $n = 2$, when we have (7) and $s_{(n)} \circ s_{(2)} = \sum_{\lambda \in \mathrm{Par}(n)} s_{2\lambda}$. Problem 9 in Stanley's influential survey article [12] is to find a combinatorial interpretation of the multiplicity $\langle s_{(n)} \circ s_{(m)}, s_\lambda\rangle$. Even a solution in the special case $s_{(n)} \circ s_{(3)}$ would be of considerable interest.

References

[1] Michel Brion, *Stable properties of plethysm: on two conjectures of Foulkes*, Manuscripta Math. **80** (1993), no. 4, 347–371.

[2] Winfried Bruns, Aldo Conca, and Matteo Varbaro, *Relations between the minors of a generic matrix*, Adv. Math. **244** (2013), 171–206.

[3] Man-Wai Cheung, Christian Ikenmeyer, and Sevak Mkrtchyan, *Symmetrizing tableaux and the 5th case of the Foulkes conjecture*, J. Symbolic Comput. **80** (2017), no. part 3, 833–843.

[4] Melanie de Boeck, Rowena Paget, and Mark Wildon, *Plethysms of symmetric functions and highest weight representations*, Submitted. ArXiv:1810.03448 (September 2018), 35 pages.

[5] Anton Evseev, Rowena Paget, and Mark Wildon, *Character deflations and a generalization of the Murnaghan-Nakayama rule*, J. Group Theory **17** (2014), no. 6, 1035–1070.

[6] William Fulton and Joe Harris, *Representation theory, a first course*, Graduate Texts in Mathematics (Readings in Mathematics), vol. 129, Springer, 1991.

[7] Kazuto Iijima, *The first term of plethysms*, arXiv:1108.4915 (2011), 11.

[8] Christian Ikenmeyer, *Geometric complexity theory, tensor rank, and Littlewood–Richardson coefficients*, Ph.D. thesis, Universität Paderborn, 2012.

[9] Nicholas A. Loehr and Jeffrey B. Remmel, *A computational and combinatorial exposé of plethystic calculus*, J. Algebraic Combin. **33** (2011), no. 2, 163–198.
[10] Jürgen Müller and Max Neunhöffer, *Some computations regarding Foulkes' conjecture*, Experiment. Math. **14** (2005), no. 3, 277–283.
[11] Rowena Paget and Mark Wildon, *Generalized Foulkes modules and maximal and minimal constituents of plethysms of Schur functions*, Proc. Lond. Math. Soc. (3) **118** (2019), no. 5, 1153–1187.
[12] Richard P. Stanley, *Positivity problems and conjectures in algebraic combinatorics*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 295–319.

Sylow branching coefficients for symmetric groups

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(joint work with E. Giannelli)

A central problem in character theory is to ask what the character table, or even just the first column of character degrees, can tell us about the structure of a finite group G. Immediately, we can recover properties such as the size of G, or the size of the abelianisation of G, from the number of linear (i.e. degree 1) characters. Of course, these are quite elementary properties. To know more sophisticated information, for instance, the local structure of G, we need to introduce a prime p.

A classical result on character degree is the Itô–Michler Theorem, which asserts that if P is a Sylow p-subgroup of G, then P is abelian and normal in G if and only if all irreducible characters of G have degree coprime to p. The topic of whether P is abelian was considered in Brauer's Problem 12 [1], or more famously in Brauer's Height Zero Conjecture, looking more generally at defect groups and irreducible characters of p-blocks of finite groups. Indeed, fundamental questions about the relationship between the representation theory of a finite group and that of its Sylow subgroups have driven much research in the area over the last several decades.

To highlight a number of recent results, we note that given a finite group G and a Sylow p-subgroup P, the permutation character $(\mathbb{1}_P)^G$ controls important structural properties. Here $\mathbb{1}_P$ denotes the trivial character of P.

Theorem (Malle–Navarro (2012) [5])**.** *Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$. Then $P \lhd G$ if and only if $p \nmid \chi(1)$ for all irreducible constituents χ of $(\mathbb{1}_P)^G$.*

At the other end of the spectrum, we have the following result which determines not when P is normal, but when P is self-normalising.

Theorem (Navarro–Tiep–Vallejo (2014) [8])**.** *Let G be a finite group and p an odd prime. Let $P \in \mathrm{Syl}_p(G)$. Then $P = N_G(P)$ if and only if $\mathbb{1}_G$ is the only irreducible constituent of $(\mathbb{1}_P)^G$ of degree coprime to p.*

However, not very much is known yet in general about what characters induced from a Sylow p-subgroup look like, or what information we can learn from them. This is the case even for symmetric groups, which is our main focus.

Understanding this induced character $(\mathbb{1}_P)^G$, for instance, is equivalent by Frobenius Reciprocity to understanding when $\mathbb{1}_P$ is a constituent of restrictions of characters from G to $P \in \mathrm{Syl}_p(G)$. More generally, our main object of investigation is the restriction of irreducible characters of S_n to P_n and their decomposition into irreducible constituents.

Definition. *Let p be a prime, $n \in \mathbb{N}$ and $P_n \in \mathrm{Syl}_p(S_n)$. For $\chi \in \mathrm{Irr}(S_n)$, let*

$$\chi_{P_n} = \sum_{\phi \in \mathrm{Irr}(P_n)} Z^\chi_\phi \phi,$$

*where each **Sylow branching coefficient** $Z^\chi_\phi \in \mathbb{N}_0$ is the multiplicity of ϕ as an irreducible constituent of the restriction χ_{P_n}. In other words, $Z^\chi_\phi = [\chi_{P_n}, \phi] = [\chi, \phi^{S_n}]$.*

Of course, we can define this for general groups G, but the rich theory of symmetric groups allows us to ask, and sometimes answer, questions for S_n which might otherwise be out of reach for arbitrary groups.

A first question for Sylow branching coefficients is that of *positivity*: when is $Z^\chi_\phi > 0$?

To fix some notation, it is well-known that the set $\mathrm{Irr}(S_n)$ of irreducible characters of S_n is in bijection with $\mathcal{P}(n)$, the set of partitions of n; we simply denote the character corresponding to a partition λ by χ^λ, or even by λ itself when clear from context. We begin with the trivial character of P_n.

Theorem (Giannelli–L. [2]). *Let $n \in \mathbb{N}$ and p be an odd prime. Let $P_n \in \mathrm{Syl}_p(S_n)$ and $\chi^\lambda \in \mathrm{Irr}(S_n)$. Then $Z^\lambda_{\mathbb{1}_{P_n}} = 0$ if and only if $n = p^k$ and $\lambda \in \{(n-1,1), (2, 1^{n-2})\}$, with 8 exceptions when $p = 3$ and $n \leq 10$.*

This allows us to give a description of $\mathrm{Irr}(\mathcal{H})$, where $\mathcal{H}$ is the Hecke algebra corresponding to $(\mathbb{1}_{P_n})^{S_n}$. In particular, understanding the dimensions of these irreducibles is equivalent to knowing exactly the values of the Sylow branching coefficients when $\phi = \mathbb{1}_{P_n}$, which leads to the following:

Question. *Is there a combinatorial description of the map*

$$Z_{\mathbb{1}_{P_n}} : \mathcal{P}(n) \to \mathbb{N}_0, \quad \lambda \mapsto Z^\lambda_{\mathbb{1}_{P_n}}?$$

Our results on when $Z_{\mathbb{1}_{P_n}} > 0$ were also recently applied to the representation theory of simple groups by Malle and Zalesski in [6], as part of their study of so-called Sylow p-regular characters and Steinberg-like characters.

Moving on to general irreducible characters ϕ, we define

$$\Omega(\phi) := \{\chi \in \mathrm{Irr}(S_n) \mid Z^\chi_\phi > 0\}.$$

Note the above shows that $\Omega(\mathbb{1}_{P_n})$ is almost all of $\mathrm{Irr}(S_n)$. In fact, this is the case for all $\phi \in \mathrm{Lin}(P_n)$, the linear characters of P_n. For p an odd prime and $\Omega_n := \cap_{\phi\in\mathrm{Lin}(P_n)}\Omega(\phi)$, we have shown in [3] that $\lim_{n\to\infty} \frac{|\Omega_n|}{|\mathrm{Irr}(S_n)|} = 1$. This follows from explicit bounds on the sets $\Omega(\phi)$. Letting $\mathcal{B}_n(t) = \{\chi^\lambda \in \mathrm{Irr}(S_n) \mid \lambda_1, l(\lambda) \le t\}$, we have shown the following (note the case of $p = 3$ is in a separate preprint).

Theorem (Giannelli–L. [3]). *Let p be an odd prime and $n \in \mathbb{N}$. Let $P_n \in \mathrm{Syl}_p(S_n)$ and $\phi \in \mathrm{Lin}(P_n)$. We determine exactly*

$$m(\phi) := \max\{t \mid \mathcal{B}_n(t) \subseteq \Omega(\phi) \quad \text{and} \quad M(\phi) := \min\{t \mid \Omega(t) \subseteq \mathcal{B}_n(t)\}.$$

We remark that $m(\phi)$ is always large (to give a rough estimate, it is always greater than $\frac{n}{2}$, easily giving the earlier limit), and the difference between $M(\phi)$ and $m(\phi)$ is very small. The main ideas of our proofs is to restrict and induct between various wreath product subgroups of S_n and apply Mackey theory, along with the Littlewood–Richardson rule and knowledge of certain plethysm coefficients.

In the course of this work, we also prove the following, a symmetric group analogue of a result of Navarro [7].

Theorem (Giannelli–L.–Long [4]). *Let p be any prime and $n \in \mathbb{N}$. Let $P_n \in \mathrm{Syl}_p(S_n)$ and $N = N_{S_n}(P_n)$. Let ϕ, $\psi \in \mathrm{Lin}(P_n)$. Then the inductions ϕ^{S_n} and ψ^{S_n} are equal if and only if ϕ and ψ are N–conjugate.*

A next main challenge is to compute the multiplicities Z^χ_ϕ themselves, that is, to give combinatorial formulas or interpretations for the values of the Sylow branching coefficients. During the course of this mini-workshop, we have done so for $Z^\lambda_{\mathbb{1}_{P_n}}$ for some special shapes of partitions λ, including hooks and certain two-row partitions; more general λ and also what happens when $p = 2$ are work in progress.

References

[1] R. Brauer, Representations of finite groups, in *Lectures on Modern Mathematics*, vol. I, ed. by T. Saaty (Wiley, 1963).

[2] E. Giannelli and S. Law, On permutation characters and Sylow p-subgroups of $\mathfrak{S}_n$, *J. Algebra* **506** (2018), 409–428.

[3] E. Giannelli and S. Law, Sylow branching coefficients for symmetric groups, arxiv:1909.09446.

[4] E. Giannelli, S. Law and J. Long, Linear characters of Sylow subgroups of symmetric groups, preprint.

[5] G. Malle and G. Navarro, Characterizing normal Sylow p-subgroups by character degrees, *J. Algebra* **370** (2012), 402–406.

[6] G. Malle and A. Zalesski, Steinberg-like characters for finite simple groups, *J. Group Theory* **23** (2020), 25–78.

[7] G. Navarro, Linear characters of Sylow subgroups, *J. Algebra* **269** (2003), 589–598.

[8] G. Navarro, P. H. Tiep, and C. Vallejo, McKay natural correspondences on characters, *Algebra and Number Theory* **8** (2014), 1839–1856.

Introduction to Geometric Complexity Theory. Kronecker and plethysm coefficients in GCT

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(joint work with C. Ikenmeyer and P. Bürgisser)

In his landmark paper from 1979, Valiant defined algebraic complexity classes for computing polynomials in formal variables. Later these classes were denoted by VP and VNP, and represented the algebraic analogues of the original P and NP classes. The flagship problem in arithmetic complexity theory is to show that $\mathsf{VP} \neq \mathsf{VNP}$ and is closely related to $\mathsf{P} \neq \mathsf{NP}$, see below. As with P vs NP, the general strategy is to identify complete problems for VNP, i.e. complete polynomials, and show they do not belong to VP. Valiant identified such VNP-complete polynomials, most notably the permanent of a $n \times n$ variable matrix. At the same time he showed that the determinant polynomial is VP–universal, i.e. every polynomial from VP can be computed as a polynomially sized determinant of a matrix whose entries are affine linear forms in the original variables. This sets the general strategy of distinguishing VP from VNP by showing that the permanent is not a determinant of some poly-sized matrix.

Originally, the Geometric Complexity Theory (GCT) program of Mulmuley and Sohoni aims to prove the algebraic version of the "P vs NP" problem (VP_s vs VNP) by distinguishing permanents from determinants in an algebro-geometric and representation theoretic setting following the foundational work of Valiant. In particular, the aim is to show that the permanent of a $m \times m$ matrix per_m cannot be expressed as $n \times n$ determinant $\det_n$ of a matrix with affine linear entries in the variables x_{ij} when n is polynomial in m. By the universality of the determinant for VP, this would imply that $\mathrm{per}_m \notin \mathsf{VP}$, and since per_m is VNP-complete, that $\mathsf{VP} \neq \mathsf{VNP}$. More generally, the GCT approach can be used to obtain lower bounds, and applied to other problems like the complexity of Matrix Multiplication, general tensor rank questions etc.

In order to distinguish complexity classes we rely on universal polynomial representatives. For every polynomial $p(x_1, \ldots, x_n)$ in any number of variables there exists some n such that $p = \det(A)$, where A is an $N \times N$ matrix whose entries are affine linear forms in the variables $\mathbf{x} = (x_1, \ldots, x_n)$, i.e. $B\mathbf{x}^T + v$ for some $B \in Mat_{n\times n}(\mathbb{C})$ and $v \in \mathbb{C}^n$. The *determinantal complexity* $\mathsf{dc}(p)$ is the minimal such N. E.g. if $p = x_1^2 + x_1x_2 + x_2$ then $\mathsf{dc}(p) = 2$ since $p = \det \begin{bmatrix} x_1 & x_1 + 1 \\ -x_2 & x_1 \end{bmatrix}$ and 2 is minimal possible since $\mathsf{dc}(p) \geq \deg(p) = 2$. Valiant's universality theorem gives that $\mathsf{VP} = \{f : \mathsf{dc}(f) \leq \mathit{poly}(n)\}$. Since per_n is VNP-complete, then $\mathsf{VP} \neq \mathsf{VNP}$ is equivalent to

Conjecture 1 (Valiant). $\mathsf{dc}(\mathrm{per}_m)$ *grows superpolynomially.*

It is only known that $\mathsf{dc}(\mathrm{per}_m) \leq 2^m - 1$, and $\mathsf{dc}(\mathrm{per}_m) \geq \frac{m^2}{2}$.

To approach this conjecture, GCT considers the algebraic varieties arising as the collection of all possible affine linear forms of the variables $[X_{ij}]$. To formalize this algebraically, such a set can be obtained under the orbit of action of the general linear group $\mathrm{GL}_{n^2}(\mathbb{C})$ over the vectors of n^2 variables $\mathbf{X} = (X_{ij})_{i,j=1}^n$, taking the determinant of the newly formed matrices, and taking the closure of these polynomials to account for all possible affine linear forms. The resulting set (algebraic variety) is $\overline{\mathrm{GL}_{n^2}} \circ \det_n$, and if a polynomial $p \neq \det_n(B\mathbf{X} + v)$ then $p \notin \overline{\mathrm{GL}_{n^2}} \circ \det_n$. Hence we have

Proposition 2 (Lower bounds via geometry, Mulmuley–Sohoni)**.** *If $p \notin \overline{\mathrm{GL}_{n^2} \det_n}$, then $\mathsf{dc}(p) > n$.*

And since per_m is VNP-complete the following Conjecture implies $\mathsf{VP} \neq \mathsf{VNP}$.

Conjecture 3 (GCT: Mulmuley and Sohoni)**.** *The maximal n for which per_m is not in $\overline{\mathrm{GL}_{n^2} \det_n}$ (which is $\leq \mathsf{dc}(\mathrm{per}_m)$) grows superpolynomially.*

One can now apply the GL action on the LHS on the padded permanent $\mathrm{per}_m^n := (X_{1,1})^{n-m}\mathrm{per}_m$ and exploit this group action on both sides. We have the correspondence

$$\mathrm{per}_m^n \in \overline{GL_{n^2}\mathrm{det}_n} \iff \overline{\mathrm{GL}_{n^2}\mathrm{per}_m^n} \subseteq \overline{GL_{n^2}\mathrm{det}_n} \tag{1}$$

To distinguish the permanent from the determinant, it is algebraically more convenient to consider the coordinate rings of these orbit closures. The group action carries over turning these rings into group modules (representations), which can be decomposed into irreducible components with multiplicities denoted by δ for the determinant and γ for the permanent:

$$\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \quad \text{and} \quad \mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}}.$$

In order to show the inclusion in (1) does not hold it is enough to find one component λ which violates it.

Definition 4 (Representation theoretic obstruction)**.** *If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a* **representation theoretic obstruction**. *Its existence shows $\overline{\mathrm{GL}_{n^2}\mathrm{per}_m^n} \not\subseteq \overline{\mathrm{GL}_{n^2} \det_n}$ and so $\mathsf{dc}(\mathrm{per}_m) > n$. If $n > poly(m)$ then $\mathsf{VP} \neq \mathsf{VNP}$.*

Conjecture 5 (GCT: Mulmuley–Sohoni)**.** *There exist representation theoretic obstructions that show superpolynomial lower bounds on $\mathsf{dc}(\mathrm{per}_m)$.*

If also $\delta_{\lambda,d,n} = 0$, then λ is an **occurrence obstruction**.

Conjecture 6 (Mulmuley–Sohoni)**.** *There exist occurrence obstructions for $n > poly(m)$, i.e. showing superpolynomial lower bounds on $\mathsf{dc}(\mathrm{per}_m)$ and thus $\mathsf{VP} \neq \mathsf{VNP}$.*

The connections with algebraic combinatorics arise by considering the multiplicities of the representations above. The rectangular Kronecker coefficients $g(\lambda, d \times n, d \times n)$ appear as the upper bounds for the multiplicity of λ in the coordinate ring of $\det_n$, and the plethysm coefficients $a_\lambda(d[n]) = \mathrm{mult}_\lambda Sym^d Sym^n V$

are upper bounds for the multiplicity of λ in the coordinate ring of the orbit closure of the padded permanent $X_{1,1}^{n-m}\text{perm}_m$, as well as any homogenous polynomial. In particular, we have that $\delta_{\lambda,d,n} \leq \text{sk}(\lambda, n^d) \leq g(\lambda, n^d, n^d)$ and $a_\lambda(d[n]) := \text{mult}_\lambda Sym^d(Sym^n(V)) \geq \gamma_{\lambda,d,n,m}$. With Ikenmeyer we showed that using Kronecker coefficients as bounds cannot resolve this conjecture, and this was greatly generalized, using the combinatorics of Young tableaux and similar objects, leading to

Theorem 7 (Bürgisser–Ikenmeyer–Panova)**.** *Let n, d, m be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[\overline{\text{GL}_{n^2} X_{11}^{n-m} per_m}]$, then λ also occurs in $\mathbb{C}[\overline{\text{GL}_{n^2} \cdot \det_n}]$. In particular, Conjecture 5 is false, there are no "occurrence obstructions".*

Other models that can lead to $\mathsf{VP} \neq \mathsf{VNP}$ replace the determinant with other universal polynomials, e.g. the Iterated Matrix Multiplication (IMM) and the trace of the matrix power as a special case. Define the trace of the power polynomial as $\mathsf{Pow}_n^m := \text{tr}(X^m)$, where $X = (X_{i,j})$, let $\text{pc}(\text{per}_m)$ be the smallest n such that $per_m = \text{tr}(A^m)$, where A is an $n \times n$ matrix with entries $A_{i,j}$ *homogeneous* linear forms. Noam Nisan proves that $\text{pc}(\text{per}_m)$ and $\mathsf{dc}(\text{per}_m)$ are polynomially equivalent.

Conjecture 8 (VP vs VNP equivalent)**.** *The sequence* $\text{pc}(\text{per}_m)$ *grows superpolynomially. If* $\overline{\text{GL}_{n^2}\text{per}_m} \not\subseteq \overline{\text{GL}_{n^2} Pow_n^m}$*, then* $\text{pc}(\text{per}_m) > n$.

Theorem 9 (Gesmundo–Ikenmeyer–Panova)**.** *For $n, m \geq 3$ we have*

$$\mathbb{C}[\text{GL}_{n^2}\mathsf{Pow}_n^m]_d = \mathbb{C}[\text{GL}_{n^2}]_d^{\mathcal{S}} = \bigoplus_\lambda V_\lambda^{\oplus sm(\lambda,n)},$$

where the sum is over all $\lambda \vdash md$ and $sm(\lambda, n) := \sum_{\mu \vdash dm, \ell(\mu) \leq n} sk(\lambda, \mu)$ is a sum of symmetric Kronecker coefficients.

Unfortunately, in this case the *occurrence* obstructions also do not exist.

Theorem 10 (Gesmundo–Ikenmeyer–Panova)**.** *Let $m \geq 10$ and $n \geq m + 2$. For every $\lambda \vdash dm$ that satisfies $q_\lambda(d[m]) > 0$ we have $sm(\lambda, n) > 0$.*

Despite the lack of occurrence obstructions (for det vs per), there is still hope that multiplicity obstructions for lower bounds can be found. With Dörfler and Ikenmeyer, we considered a toy problem of distinguishing the polynomials of the forms $f = \ell_1' \ldots \ell_n'$ and power sum $p = \ell_1^n + \cdots + \ell_k^n$, where ℓ_i' and ℓ_j are linear forms in the variables $X_1, \ldots, X_m$ via the GCT approach. Namely, replacing det and per, to show that not all p can be expressed in the form of f one can consider their corresponding coordinate rings decomposed as GL_m-irreducible modules.

Theorem 11 (Dörfler–Ikenmeyer–Panova)**.** *Let $m \geq 3$, $n \geq 2$, $k = d = n + 1$, $\lambda = (n^2 - 2, n, 2)$. We have*

$$\text{mult}_\lambda(\mathbb{C}[\overline{\text{GL}_m \circ (\ell_1 \cdots \ell_n)}]_d) < \text{mult}_\lambda(\mathbb{C}[\overline{\text{GL}_m \circ (\ell_1^n + \cdots + \ell_k^n)}]_d),$$

i.e., λ is a multiplicity obstruction that shows

$$\overline{\{\ell_1^n + \cdots + \ell_k^n\}} \not\subseteq \overline{\{\ell_1 \cdots \ell_n | \ \ell_i - \text{linear forms}\}}.$$

Moreover, here there are no occurrence obstructions, i.e., all partitions λ appearing in the power sum ring also appear with positive multiplicity in $\mathbb{C}[\overline{\mathrm{GL}_m \circ (\ell_1 \cdots \ell_n)}]$.

Merging the Kronecker and modular decomposition number problems

Louise Sutton

(joint work with C. Bessenrodt and C. Bowman)

Let $\mathfrak{S}_n$ be the symmetric group on n letters. Recently Heide, Saxl, Tiep and Zalesski [3] provided insight into the Kronecker positivity problem. It was conjectured that for any $n \neq 2, 4, 9$ there is an irreducible $\mathbb{C}\mathfrak{S}_n$-module in which the Kronecker coefficients in the decomposition of its tensor square are always positive. Saxl suggested in 2012 that the irreducible $\mathbb{C}\mathfrak{S}_n$-module $D^{\mathbb{C}}(\rho)$ indexed by a *staircase partition* $\rho = (k, k-1, k-2, \dots, 2, 1)$ is such a candidate.

Saxl's Conjecture. Let $\rho = (k, k-1, k-2, \dots, 2, 1) \vdash n$ be a staircase partition. Then

$$D^{\mathbb{C}}(\rho) \otimes D^{\mathbb{C}}(\rho) = \bigoplus_{\lambda} g(\rho, \rho, \lambda) D^{\mathbb{C}}(\lambda)$$

with multiplicity $g(\rho, \rho, \lambda) \neq 0$ for all partitions λ of n.

The Kronecker coefficients $g(\rho, \rho, \lambda)$ are known to be positive when λ is a hook or two-part partition with n sufficiently large [5], when n is arbitrary and λ is a hook partition [4, 1] or a double-hook partition [1], and for any λ comparable to ρ in dominance order in [4]. In joint work with Bessenrodt and Bowman [2], we determine Kronecker positivity for large new families of partitions by connecting the Kronecker positivity problem to the 2-modular representation theory of $\mathfrak{S}_n$.

We now observe that the Kronecker multiplicities in Saxl's conjecture are intimately related to the 2-modular representation theory of $\mathfrak{S}_n$. Let $\mathbb{F}$ be a field of characteristic 2. Then the Specht module $S^{\mathbb{F}}(\rho)$ indexed by the staircase partition $\rho = (k, k-1, \dots, 2, 1)$ is both projective and simple. Thus its tensor square decomposes into indecomposable projective modules as follows

$$D^{\mathbb{F}}(\rho) \otimes D^{\mathbb{F}}(\rho) = \bigoplus_{\nu \text{ 2-regular}} G(\rho, \rho, \nu) P^{\mathbb{F}}(\nu),$$

where $G(\rho, \rho, \nu)$ is the corresponding multiplicity of the projective module $P^{\mathbb{F}}(\nu)$. Comparing the above decompositions, we can write the associated coefficients in terms of the other:

$$(\dagger) \qquad g(\rho, \rho, \lambda) = \sum_{\nu} G(\rho, \rho, \nu) d_{\lambda,\nu},$$

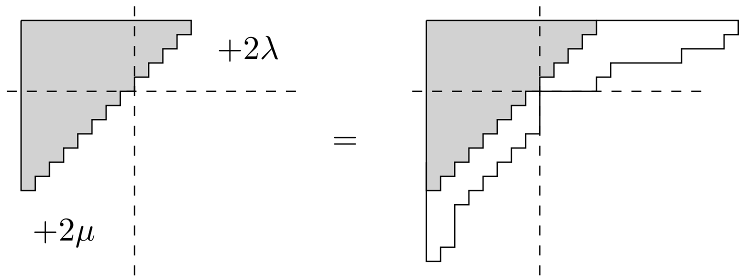
where $d_{\lambda,\nu}$ is the composition multiplicity $[S^{\mathbb{F}}(\lambda) : D^{\mathbb{F}}(\nu)]$. Thus to determine Kronecker positivity we study a family of Specht modules $S(\lambda)^{\mathbb{F}}$ whose corresponding decomposition numbers $d^{\mathbb{F}}_{\lambda,\nu}$ are non-zero, and moreover the multiplicities

$G(\rho, \rho, \nu)$ are also non-zero. Thus, by applying Ikenmeyer's main result from [4], we obtain a new family of partitions that satisfy Kronecker positivity in Saxl's conjecture.

Theorem. Let $n = k(k+1)/2$, $\rho = (k, k-1, \dots, 2, 1)$ and $\lambda \vdash n$ such that the complex irreducible $\mathfrak{S}_n$-character χ^λ is of height 0. Then $g(\rho, \rho, \lambda) > 0$. In particular, all χ^λ of odd degree are constituents of the Saxl square.

One now observes that the symmetric group algebra is obtained from the Iwahori–Hecke algebra of type A, denoted $\mathcal{H}_{q,\mathbb{F}}(\mathfrak{S}_n)$, upon fixing the parameters $e = \operatorname{char} \mathbb{F}$, where q is a cyclotomic eth root of unity. Since decomposition numbers in characteristic zero provide a lower bound to decomposition numbers in positive characteristic, so that $d^{\mathbb{C}}_{\lambda,\nu} \leq d^{\mathbb{F}}_{\lambda,\nu}$, it suffices to study the representation theory of these Specht modules as $\mathcal{H}_{-1,\mathbb{C}}(\mathfrak{S}_n)$-modules (that is, with $e = 2$).

We study the decompositions of Specht modules indexed by 2-*separated partitions*. Given a staircase partition ρ, we obtain a 2-separated partition, denoted ρ^λ_μ, from ρ by adding 2 copies of a partition λ to the right of ρ and 2 copies of a partition to the bottom of ρ such that λ and μ do not touch (except perhaps diagonally), as pictured below.



We determine that Specht modules indexed by 2-separated partitions are semisimple, with the following decomposition.

Theorem. Let $e = 2$ and let ρ^λ_μ denote a 2-separated partition of n. Then

$$S^{\mathbb{C}}_{-1}(\rho^\lambda_\mu) = \bigoplus_\nu c(\nu^T, \lambda^T, \mu) D^{\mathbb{C}}_{-1}(\rho^\nu_\varnothing)$$

where $c(\nu^T, \lambda^T, \mu)$ is the corresponding Littlewood–Richardson coefficient.

Using this result and Ikenmeyer's result [4], together with the observation †, we determine that a large new family of 2-separated partitions satisfies Kronecker positivity in Saxl's conjecture.

Theorem. Let $w = k(k+1)/2$, $n = w(2w+1)$, $\rho(2w) = (2w, 2w-1, \dots, 1)$, $\tau = \rho(2w-1) = (2w-1, 2w-2, \dots, 1)$ and $\rho(k) = (k, k-1, \dots, 1)$. Then for λ, μ any pair such that $c(\rho(k), \lambda, \mu^T) > 0$ we have that

$$g(\rho(2w), \rho(2w), \tau^\lambda_\mu) \geq c(\rho(k), \lambda, \mu^T) > 0.$$

Moreover, this result provides us with a new infinite family of partitions whose Kronecker coefficients are unbounded.

References

[1] C. Bessenrodt, *Critical classes, Kronecker products of spin characters, and the Saxl conjecture*, Algebr. Comb. **1** (2018), no. 3, 353–369.

[2] C. Bessenrodt, C. Bowman and L. Sutton, *Kronecker positivity and* 2*-modular representation theory*, arXiv:1903.07717.

[3] G. Heide, J. Saxl, P. Tiep, and A. E. Zalesski, *Conjugacy action, induced representations and the Steinberg square for simple groups of Lie type*, Proc. Lond. Math. Soc. (3) **106** (2013), no. 4, 908–930.

[4] C. Ikenmeyer, *The Saxl conjecture and the dominance order*, Discrete Math. **338** (2015), no. 11, 1970–1975.

[5] I. Pak, G. Panova, and E. Vallejo, *Kronecker products, characters, partitions, and the tensor square conjectures*, Adv. Math. **288** (2016), 702–731.

The multiset partition algebra and Kronecker product

Mike Zabrocki

(joint work with Rosa Orellana)

The actions of GL_n and $\mathbb{C}S_k$ on the k-fold tensor of an n dimensional vector space centralize each other. This algebra action is known as Schur–Weyl duality and, as a consequence of this centralizer property, facts about the representation theory of one algebra can be deduced from the other and, if we denote the irreducible representations of an algebra A by W_A^λ, then

$$V_n^{\otimes k} \cong \bigoplus_\lambda W^\lambda_{\mathrm{GL}_n} \otimes W^\lambda_{\mathbb{C}S_k}.$$

The theory of centralizer algebras has been used to study the representation theory of other pairs of algebras acting on other spaces. Another well known duality is that of the action of GL_n and GL_k acting on the polynomial ring in the set of variables $X_{n\times k} := \{x_{ij}$ with $1 \leq i \leq n$ and $1 \leq j \leq k\}$, then it is known that the actions of GL_n acting on the first index of the variable and GL_k acting on the second index centralize each other and this polynomial ring decomposes as

$$\mathbb{C}[X_{n\times k}]_{deg\ r} \cong \bigoplus_\lambda W^\lambda_{\mathrm{GL}_n} \otimes W^\lambda_{\mathrm{GL}_k}.$$

Denote the multiplicity of an irreducible in a module by $\left\langle W_A^\lambda, - \right\rangle$, then in particular, Schur–Weyl duality relates the 'branching rule' for the restriction from S_k to S_{k-1} and tensor by V_n as

$$\left\langle W^\mu_{S_{k-1}}, Res\downarrow^{S_k}_{S_{k-1}} W^\lambda_{S_k} \right\rangle = \left\langle W^\lambda_{\mathrm{GL}_n}, W^\mu_{\mathrm{GL}_n} \otimes V_n \right\rangle$$

and this happens if and only if the partition μ differs from the partition λ by 1 in a single row. The second duality relates the branching from GL_k to GL_{k-1} to tensoring by a polynomial ring, showing that

$$\left\langle W^{\mu}_{\mathrm{GL}_{k-1}}, Res \downarrow^{\mathrm{GL}_k}_{\mathrm{GL}_{k-1}} W^{\lambda}_{\mathrm{GL}_{k-1}} \right\rangle = \left\langle W^{\lambda}_{\mathrm{GL}_n}, W^{\mu}_{\mathrm{GL}_n} \otimes \mathbb{C}[X_{n\times 1}]_{deg\ |\lambda|-|\mu|} \right\rangle .$$

This multiplicity is 1 if and only if the partitions are interleaved $\mu_1 \le \lambda_1 \le \mu_2 \le \lambda_2 \le \cdots$. These two rules are known as Pieri rules and are used as the base step for developing the Littlewood–Richardson rule for the common multiplicities

$$\begin{aligned} c^{\nu}_{\lambda\mu} &= \left\langle W^{\nu}_{\mathrm{GL}_n}, W^{\lambda}_{\mathrm{GL}_n} \otimes W^{\mu}_{\mathrm{GL}_n} \right\rangle \\ &= \left\langle W^{\lambda}_{\mathrm{GL}_k} \otimes W^{\mu}_{\mathrm{GL}_\ell}, Res \downarrow^{\mathrm{GL}_{k+\ell}}_{\mathrm{GL}_k \times \mathrm{GL}_\ell} W^{\nu}_{\mathrm{GL}_{k+\ell}} \right\rangle \\ &= \left\langle W^{\lambda}_{S_k} \otimes W^{\mu}_{S_\ell}, Res \downarrow^{S_{k+\ell}}_{S_k \times S_\ell} W^{\nu}_{S_{k+\ell}} \right\rangle . \end{aligned}$$

In the 1990s, Martin introduced a diagram algebra $P_k(n)$, known as the partition algebra, which is isomorphic to the centralizer algebra of the symmetric group S_n acting on the tensor space $V_n^{\otimes k}$ where $S_n \subseteq \mathrm{GL}_n$ as permutation matrices. In analogy to Schur–Weyl duality we have the decomposition

$$V_n^{\otimes k} \cong \bigoplus_{\lambda} W^{\lambda}_{S_n} \otimes W^{\lambda}_{P_k(n)} .$$

In this talk I presented the analogous algebra $M\!P_{r,k}(n)$, an algebra with a basis indexed by multiset partitions [4], which is the centralizer algebra when S_n acts on the polynomial ring, thus giving us the other analogous decomposition

$$\mathbb{C}[X_{n\times k}]_{deg\ r} \cong \bigoplus_{\lambda} W^{\lambda}_{S_n} \otimes W^{\lambda}_{M\!P_{r,k}(n)}.$$

Besides the beauty of the algebra itself, the reason we are interested in this centralizer algebra is that we would like to give a combinatorial interpretation to the common multiplicities

$$\begin{aligned} g_{\lambda\mu\nu} &= \left\langle W^{\nu}_{S_n}, W^{\lambda}_{S_n} \otimes W^{\mu}_{S_n} \right\rangle \\ &= \left\langle W^{\lambda}_{M\!P_{d,k}(n)} \otimes W^{\mu}_{M\!P_{r-d,\ell}(n)}, Res \downarrow^{M\!P_{r,k+\ell}(n)}_{M\!P_{d,k}(n) \times M\!P_{r-d,\ell}(n)} W^{\nu}_{M\!P_{r,k+\ell}(n)} \right\rangle \\ &= \left\langle W^{\lambda}_{P_k(n)} \otimes W^{\mu}_{P_\ell(n)}, Res \downarrow^{P_{k+\ell}(n)}_{P_k(n) \times P_\ell(n)} W^{\nu}_{P_{k+\ell}(n)} \right\rangle \end{aligned}$$

known as Kronecker coefficients.

The presentation also included an idea for an approach to completing this goal by developing a combinatorial interpretation [3] for the repeated Pieri rule (with the restriction that n was sufficiently large) which we developed using a basis for the symmetric functions [1, 2]. The repeated Pieri rule was stated in terms of set valued tableaux and multiset tableaux that are analogous to standard tableaux and semi-standard tableaux. The hope is that we can generalize the combinatorics that was used to arrive at the Littlewood–Richardson rule.

References

[1] R. Orellana, M. Zabrocki, *Characters of the symmetric group as symmetric functions*, `arXiv:1605.06672`.

[2] R. Orellana, M. Zabrocki, *The Hopf structure of symmetric group characters as symmetric functions*, `arXiv:1901.00378`.

[3] R. Orellana, M. Zabrocki, *Products of symmetric group characters*, J. Combin. Theory Ser. A 165 (2019), 299–324. `arXiv:1709.08098`.

[4] R. Orellana, M. Zabrocki, *Howe duality of the symmetric group and a multiset partition algebra*, preprint.

Restriction of characters and Sylow subgroups

Carolina Vallejo Rodríguez

(joint work with E. Giannelli and S. Law)

Let G be a finite group and p be a fixed prime, the McKay conjecture asserts that the number of irreducible complex characters (simple $\mathbb{C}G$-modules) of G of degree (dimension) coprime to p can be computed locally, namely in the normalizer of a Sylow p-subgroup P of G. This astonishing conjecture lies at the core of the global-local principles in Group Representation Theory and has been generalized in many different ways taking into account congruences of character degrees (Isaacs and Navarro), values over the field of p-adic numbers (Navarro), Schur indices (Turull), Brauer p-blocks (Alperin), derived equivalences (Broué), values of restrictions to Sylow subgroups (this latter generalization due to Navarro and Tiep has been announced in the MFO mini-workshop 2009a), etc. In the self-normalizing case $P = N_G(P)$ we are looking for ways of relating irreducible characters of degree coprime to p of G and linear (that is one-dimensional) characters of P. If p is odd, then there is an extremely nice way of relating these two sets [7], so that algebraic properties are preserved. More precisely, there is a bijection for the McKay conjecture $\chi \mapsto \chi^*$ such that the correspondent linear character χ^* of P appears in the restriction to P of the given character χ of degree coprime to p of G (and conversely χ appears in the induction $(\chi^*)^G$ by Frobenius reciprocity). For $p = 2$ this nice situation does not happen in general outside solvable groups [6] or symmetric groups acting on 2^k letters [1]. Nevertheless a beautiful correspondence does exist for symmetric groups and $p = 2$, which can be described both by combinatorial [2] and purely character theoretical means [4]. For these particular cases, either p odd and P self-normalizing or $p = 2$ and $G = \mathfrak{S}_{2^k}$, the decomposition of the permutation character $(1_P)^G$ obtained by inducing the trivial representation from the Sylow 2-subgroup P up to G is sufficiently understood. In the case of symmetric groups, $p = 2$ and more complicated 2-adic expansions of n, the decomposition of this permutation character is still unknown and results on the direction of understanding $(1_P)^G$ are hard to achieve (see [3]).

On another line, building on the existence of nice correspondences for the McKay conjecture it is possible to show that for odd primes, the property of having a self-normalizing Sylow p-subgroup depends only on the number of representations of degree coprime to p appearing in the decomposition of the permutation character $(1_P)^G$ (see [7]). At the opposite side of the spectrum, the normality of P in G can also be characterized in terms of the decomposition of $(1_P)^G$ for any prime by [5]. More precisely, P is normal in G if, and only if, every irreducible constituent of $(1_P)^G$ has degree coprime to p. Recall that the Itô–Michler theorem asserts that P is normal and abelian in G if, and only if, every irreducible character of G has degree coprime to p. Somehow, the result of Malle and Navarro shows that in order to detect the normality of P in G (hence isolating this property from the commutativity) it is enough to look at irreducible constituents of the permutation character $(1_P)^G$ and use the same criterion (namely, impose that every character has degree coprime to p). The authors of [5] go one step further and propose the following.

Conjecture (Malle, Navarro). Let G be a finite group, p a prime and P a Sylow p-subgroup of G. The subgroup P is normal in G if, and only if, every irreducible constituent of $(1_P)^G$ appearing with multiplicity coprime to p has degree coprime to p.

This conjecture tells us that in order to characterize the normality of P in the manner of Itô–Michler, a smaller subset of irreducible characters of G suffices, namely those constituents of $(1_P)^G$ appearing with multiplicity coprime to p. In joint work with E. Giannelli and S. Law, and following the approach of Malle and Navarro, we show that proving the above conjecture reduces to a problem on Sylow branching coefficients for symmetric (and alternating) groups at the prime $p \in \{2,3\}$. Let P be a Sylow p-subgroup $\mathfrak{S}_n$ where p is a fixed but arbitrary prime. Recall that given ϕ and irreducible character of P and a partition λ of n, the Sylow branching coefficient Z^λ_ϕ of λ with respect to ϕ is the multiplicity of χ^λ, the character labelled by the partition λ, as an irreducible constituent of $(\phi)^G$. That is, $Z^\lambda_\phi = [\chi^\lambda, (\phi)^G] = [(\chi^\lambda)_P, \phi]$. Ultimately, for $p \in \{2,3\}$, we must work to find a partition λ of n with the following properties:

- λ is not self-adjoint.
- The degree $\chi^\lambda(1)$ is divisible by p.
- If $p = 3$, then p does not divide the multiplicity of χ^λ in $(1_P)^G$. In other words, the Sylow branching coefficient $Z^\lambda_{1_P}$ is coprime to p.
- If $p = 2$, then the sum of the multiplicity of χ^λ in $(1_P)^G$ and in $(\mathrm{sgn}|_P)^G$ is odd, where $\mathrm{sgn} = \chi^{(1^n)}$. In other words, $Z^\lambda_{1_P} + Z^\lambda_{\mathrm{sgn}|P}$ is odd.

During the MFO mini-workshop 2009a we found partitions λ as above whenever $n = p^k$, and we continue working on more complicated p-adic expansions.

References

[1] E. Giannelli, *Characters of odd degree of symmetric groups*, J. London Math. Soc. **96** (2017), 1–14.

[2] E. Giannelli, A. Kleshchev, G. Navarro, and P. H. Tiep, *Restriction of odd degree characters and natural correspondences*, Int. Math. Res. Notices **22** (2017), 6089–6118.

[3] E. Giannelli, and S. Law, *On permutation characters and Sylow p-subgroups of* $\mathfrak{S}_n$, Journal of Algebra **506** (2018) 409–428.
[4] I. M. Isaacs, G. Navarro, J. B. Olsson, and P. H. Tiep, *Character restrictions and multiplicities in symmetric groups*, J. Algebra **478** (2017) 271–282.
[5] G. Malle and G. Navarro, *Characterizing normal Sylow p-subgroups by character degrees*, J. Algebra **370** (2012), 402–406.
[6] G. Navarro, *Linear characters of Sylow subgroups*, J. Algebra **269** (2003), 589–598.
[7] G. Navarro, P. H. Tiep, and C. Vallejo, *McKay natural correspondences of characters*, Algebra Number Theory **8** (2014), 1839–1856.

Bounds on Kronecker coefficients via contingency tables

Igor Pak
(joint work with Greta Panova)

We present general upper bounds for Kronecker coefficients by using recent work on 3-dimensional *contingency tables*. The *Kronecker coefficients* $g(\lambda, \mu, \nu)$ are defined as structure constants in products of S_n-characters:

$$\chi^\mu \cdot \chi^\nu = \sum_{\lambda \vdash n} g(\lambda, \mu, \nu) \chi^\lambda \,,$$

where $\lambda, \mu, \nu \vdash n$.

Theorem 1. *Let* $\lambda, \mu, \nu \vdash n$ *such that* $\ell(\lambda) = \ell$, $\ell(\mu) = m$, *and* $\ell(\nu) = r$. *Then:*

$$g(\lambda, \mu, \nu) \;\le\; \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r}$$

In particular, when $\ell m r \le n$, this gives $g(\lambda, \mu, \nu) \le 4^n$. The bound in the theorem is often much sharper than the *dimension bound* $g(\lambda, \mu, \nu) \le \min\{f^\lambda, f^\mu, f^\nu\}$, which is the only known general upper bound for Kronecker coefficients. For example, when $n = \ell^3$, $\lambda = \mu = \nu = (\ell^2, \ldots, \ell^2)$, ℓ times, the dimension bound gives only

$$g(\lambda, \mu, \nu) \le f^\lambda = e^{\frac{1}{3} n \log n + O(n)}.$$

Our tool is the following general upper bound. Let $\lambda, \mu, \nu \vdash n$. Denote by $\mathrm{T}(\lambda, \mu, \nu)$ the number of 3-dimensional $\ell(\lambda) \times \ell(\mu) \times \ell(\nu)$ contingency tables with 2-dimensional sums orthogonal to three coordinates are given by λ, μ and ν, respectively.

Lemma 2. *Let* $\mathrm{T}(\lambda, \mu, \nu)$ *be the number of* 3*-dimensional contingency tables with marginals* $\lambda, \mu, \nu \vdash n$. *Then* $g(\lambda, \mu, \nu) \le \mathrm{T}(\lambda, \mu, \nu)$.

The theorem follows from the lemma and the analysis of known upper bounds on $\mathrm{T}(\lambda, \mu, \nu)$ in some special cases plus majorization technology.

We also compare our upper bound with the upper and lower bounds coming from counting *binary contingency tables*. Let us single out the following curious asymptotic inequality:

Theorem 3. *Let $\mathcal{L}_n = \{\lambda \vdash n, \lambda = \lambda'\}$. We have:*

$$\sum_{\lambda \in \mathcal{L}_n} g(\lambda, \lambda, \lambda) \geq e^{cn^{2/3}} \quad \text{for some} \;\; c > 0.$$

The proof is based on the following general lower bound. Let $\lambda, \mu, \nu \vdash n$. A 3-dimensional binary (0/1) contingency table $X = (x_{ijk}) \in \mathcal{B}(\lambda, \mu, \nu)$ is called a *pyramid* if whenever $x_{ijk} = 1$, we also have $x_{pqr} = 1$ for all $p \leq i$, $q \leq j$, $r \leq k$. Denote by $\mathrm{Pyr}(\lambda, \mu, \nu)$ the number of pyramids with margins λ, μ, ν.

Lemma 4 (Vallejo, Ikenmeyer–Mulmuley–Walter). *We have: $g(\lambda, \mu, \nu) \geq \mathrm{Pyr}(\lambda', \mu', \nu')$.*

The theorem used the lemma and the asymptotic analysis of pyramids (plane partitions).

Multiset Tableaux and the Kronecker Product

Rosa Orellana
(joint work with Mike Zabrocki)

The *classical Schur–Weyl duality* is a fundamental theorem in representation theory that relates the representation theory of the symmetric, S_k, and general linear, $GL_n(\mathbb{C}) = GL_n$, groups. Let $V = \mathbb{C}^n$. Then GL_n acts diagonally on the k-tensor power $V^{\otimes k}$, i.e.,

$$A \cdot (v_1 \otimes \cdots \otimes v_k) = Av_1 \otimes \cdots \otimes Av_k, \quad \text{for all } A \in GL_n, \tag{1}$$

and the symmetric group S_k acts by permuting tensor coordinates, i.e.,

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \quad \text{for all } \sigma \in S_k. \tag{2}$$

A key observation is that these two actions commute; thus $V^{\otimes k}$ is a $GL_n \times S_k$-bimodule that has a multiplicity-free decomposition as follows,

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} V^\lambda \otimes S^\lambda, \tag{3}$$

where V^λ is a polynomial irreducible GL_n-representation and S^λ is an irreducible S_k-representation and λ ranges over partitions of k (denoted $\lambda \vdash k$). An important and beautiful consequence of this duality is the following: When we compute the character of $V^{\otimes k}$ at an element $(A, \sigma) \in GL_n \times S_k$, using Equation (3), where A has eigenvalues $x_1, x_2, \ldots, x_n$ and σ has cycle type μ, we get the Frobenius identity:

$$p_\mu(x_1, x_2, \ldots, x_n) = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda(x_1, x_2, \ldots, x_n) \tag{4}$$

where p_μ is the power sum symmetric function, s_λ is the Schur function and $\chi^\lambda(\mu)$ is the irreducible character of S_k evaluated at the conjugacy class indexed by μ.

A key observation is that S_n is a subgroup of GL_n realized as the permutation matrices. We can then restrict the diagonal action, see (1), of GL_n on $V^{\otimes k} = (\mathbb{C}^n)^{\otimes k}$ to just the permutation matrices, i.e.,

$$\sigma \cdot (v_1 \otimes \cdots v_k) = \sigma v_1 \otimes \cdots \otimes \sigma v_k, \qquad \text{for all } \sigma \in S_n.$$

We can then ask: What commutes with this action of S_n on $V^{\otimes k}$? The answer was given by Jones [2], it is the partition algebra $P_k(n)$. The partition algebra has a beautiful realization as a diagram algebra as introduced by Martin [3]. In fact, Jones showed that $V^{\otimes k}$ decomposes as a $P_k(n) \times S_n$-bimodule as follows:

$$V^{\otimes k} = \bigoplus_{\lambda} L_k(\lambda) \otimes S^{\lambda}, \tag{5}$$

where $L_k(\lambda)$ is a representation of $P_k(n)$ and λ is a partition with $\lambda_2 + \lambda_3 + \cdots \leq k$.

If we compute the character of $V^{\otimes k}$ in Equation (5), on the left-handside we get $p_\mu[\Xi_\alpha]$ evaluated at the eigenvalues of permutation matrices, i.e., Ξ_α are the eigenvalues of a permutation matrix of cycle type α. In [1], Halverson assigned to every element of the partition algebra a partition type. The character at an element (d_μ, σ) in $P_k(n) \times S_n$ where σ has eigenvalues Ξ_α was given by Zabrocki and I in [4], where we introduced a new non-homogeneous basis of symmetric functions $\tilde{s}_\lambda(x_1, \ldots, x_n)$ such that

$$p_\mu(\Xi_\alpha) = \sum_{\lambda} \chi^{\lambda}_{P_k(n)}(d_\mu) \tilde{s}_\lambda(\Xi_\alpha).$$

The new basis of symmetric functions $\{\tilde{s}_\lambda\}$ has the stable (reduced) Kronecker coefficients as structure coefficients,

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_{\nu} \bar{g}(\lambda, \mu, \nu) \tilde{s}_\nu.$$

A well-know open problem is to find a combinatorial interpretation of these coefficients in the spirt of the Littlewood–Richardson rule. In his talk, Zabrocki presented two strategies for finding a combinatorial interpretation. Both of these strategies depend on first finding descriptions of products of the form

$$\tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_k} \tilde{s}_\lambda$$

where λ is a partition and μ_i's are positive integers. Notice that these products contain as a special case the Pieri rules. In my talk, I introduced *multiset tableaux*, which are the objects that govern the Schur–Weyl duality between the symmetric group and the partition algebra. Recently, Zabrocki and I in [5] described combinatorial descriptions for several products of stable Kronecker coefficients. In particular, we have shown that the coefficients occurring in these types of products are enumerated by multiset tableaux that satisfy a lattice condition. The main result described in my talk was a combinatorial interpretation for the coefficients $r(\mu, \lambda, \gamma)$ in the product

$$\tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_k} \tilde{s}_\lambda = \sum_{\gamma} r(\mu, \lambda, \gamma) \tilde{s}_\gamma.$$

The details of these results can be found in [5].

References

[1] T. Halverson. Characters of Partition Algebras. *Journal of Algebra* 238,502–533 (2001).
[2] V.F.R. Jones The Potts model and the symmetric group. In: *Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras (Kyuzeso, 1993)* World Sci. Publishing, River Edge, NJ, 1994, 259–267.
[3] P.P. Martin. Temperley-Lieb algebras for non-planar statistical mechanics–the Partition algebra construction. *J. Knot Theory Ramifications* 3 (1994), 51–82.
[4] R. Orellana, M. Zabrocki, *Characters of the symmetric group as symmetric functions*, `arXiv:1605.06672`.
[5] R. Orellana, M. Zabrocki, *Products of symmetric group characters*, J. Combin. Theory Ser. A 165 (2019), 299–324. `arXiv:1709.08098`.

Schur generating functions and the asymptotics of structural constants from combinatorial representation theory

Mercedes Rosas
(joint work with Emmanuel Briand)

Many families of structural constants coming from the representation theory of the general linear groups, including the Littlewood–Richardson, Kronecker, plethysm, Heisenberg, and reduced Kronecker coefficients share the property that they can be defined by *Schur generating series*. Let

$$\sigma[X] = \prod_{x\in X} \frac{1}{1-x} = \sum_{n\geq 0} h_n[X]$$

be the generating series for the homogeneous symmetric functions. The following series are the Schur generating functions for the the Littlewood–Richardson, Kronecker coefficients and plethysm coefficients (with fixed μ), respectively:

$$\sigma[XZ+YZ] = \sum_{\lambda,\mu,\nu} c^{\lambda}_{\mu,\nu} s_\mu[X] s_\nu[Y] s_\lambda[Z]$$

$$\sigma[XYZ] = \prod_{x_i,y_j,z_k} \frac{1}{1-x_i y_j z_k} = \sum_{\lambda} g_{\mu,\nu,\lambda} s_\mu[X] s_\nu[Y] s_\lambda[Z]$$

$$\sigma[X s_\mu[Y]] = \sum_{\lambda} p_{\mu,\nu,\lambda} s_\mu[X] s_\nu[Y] s_\lambda[Z].$$

Let $F(X_1, X_2, \ldots, X_m)$ be a symmetric function in m alphabets X_1, X_2, $\ldots$, X_m, with no constant term. The Schur generating series corresponding to the symmetric function F is defined as

$$\sigma[F(X_1, X_2, \ldots, X_m)] = \sum_{\omega} m_F(\omega) s_{\omega_1}[X_1] s_{\omega_2}[X_2] \cdots s_{\omega_m}[X_m], \tag{1}$$

where we are summing over all sequences of partitions $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ such that $\ell(\omega_i) \leq |X_i|$.

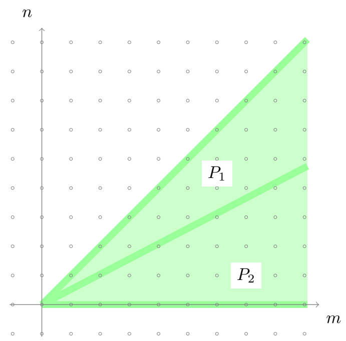


Figure 1. The chamber complex of a piecewise quasipolynomial.

What do multiplicity functions look like? A function f defined on $\mathbb{Z}^r$ is *piecewise quasipolynomial* if

i) The convex cone $C(f)$ in $\mathbb{R}^m$ generated by the support of f is a rational polyhedral convex cone.
ii) There exists a complex of rational convex polyhedral cones subdividing $C(f)$, and for each of its (closed) chambers σ, a quasipolynomial p_σ (defined on the linear span of $C(f)$) coinciding with f on $\mathbb{Z}^m \cap \sigma$. This complex is called the *chamber complex of f*. See Figure 1.

In the case where F is Schur positive, a major theorem of Meinrenkein and Sjamaar [5] implies that multiplicity functions m_F appearing in Eq. (1) are piecewise quasipolynomials, see also [1, 2]. In particular, this result implies that for fixed ω, the function $m_F(k\omega)$ is a quasipolynomial on k.

Let ω^0 be an arbitrary tuple of partitions, with no restriction on their lengths. We are interested in the study of the coefficients $m_F(\omega^0 + k\omega)$ where ω is a fixed tuple of partitions, and k varies.

A well-studied example of this phenomenon is given by the *reduced Kronecker coefficients*. Murnaghan [6] discovered that for any triple of partitions λ, μ and ν, the sequence $g_{\lambda+(k),\mu+(k),\nu+(k)}$ stabilizes as k grows. The stable limits of these sequences are the reduced Kronecker coefficients appearing in so many of the talks of this mini-workshop. Our methods allow us to easily recover Murnaghan's result, along with a formula for the stable limits (a Schur symmetric function for the reduced Kronecker coefficients), and bounds on k for the stabilization to occur.

A way of describing Murnaghan's result is saying that the rate of growth of the sequences $(g_{\lambda+(k),\mu+(k),\nu+(k)})_{k\geq 0}$ is proportional to the rate of growth of the simpler sequence $(g_{(k),(k),(k)})_{\geq 0}$, regardless of the initial shapes λ, μ and ν. On the other hand, the constant of proportionality depends on the triple of partitions obtained after deleting the first parts of λ, μ and ν. (These are the partitions indexing the corresponding reduced Kronecker coefficient.)

Our main result greatly generalizes Murnaghan's result. Under mild hypotheses on the function m_F, we find large domains where $m_F(\omega_0 + k\omega)$ behaves asymptotically like $m_F(k\omega)$, up to a constant that is described by means of a new and

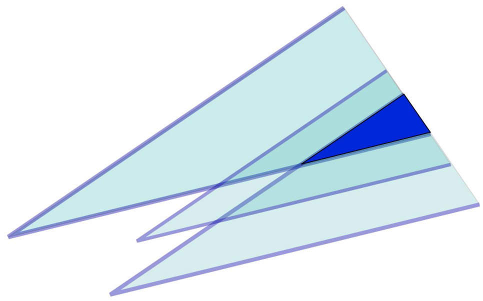


Figure 2. An affine combination of cones. Under mild hypothesis, as we start increasing k on $m_F(\omega_0 + k\omega)$ we hit the dark region, that contains a translate of the original cone.

easily computable Schur generating series. For instance, when m_F is constant on a cone, we show that $\omega \mapsto m_F(\omega_0 + \omega)$ is constant on some translate of this cone.

These general results can be applied to particular cases: in addition to Murnaghan stability for Kronecker coefficients and Brion's formula for the reduced Kronecker coefficients, we also recover Weintraub's known stability results for some families of plethysm coefficients. We get new stability or asymptotic results for Heisenberg coefficients extending some recent work of Li Ying [4], and new stability results for the reduced Kronecker coefficients.

Allowing the presence of the plethystic minus into our alphabets, we deduce from this approach a new kind of stability for Kronecker coefficients, that we call "hook stability". These results have been obtained in collaboration with A. Rattan and E. Briand, see [3] for related results.

Finally, we study what happens when we iterate our constructions allowing us to obtain information regarding the asymptotic growth of the sequences as we increase the sizes of multiple rows/hooks.

Our main tool is the use of Vertex Operators, as developed by J.-Y. Thibon in [7, 8]. Fix ω^0 as above. We associate to $m_F(\omega_0 + \omega)$ an ordinary generating series $\Phi_F^{\omega^0}$. Then, we show that $\Phi_F^{\omega^0}$ always factors as the product of the generating series corresponding to the particular case of $m_F(\omega)$ (that is, when ω^0 is a sequence of empty partitions), multiplied by a Laurent polynomial. This Laurent polynomial determines the affine shifts of the cones of $m_F(\omega_0 + \omega)$. It only depends on the "tails" of the partitions.

References

[1] V. Baldoni, M. Vergne, and M. Walter. Computation of dilated Kronecker coefficients. *Journal of Symbolic Computation*, 84:113 – 146, 2018.

[2] Velleda Baldoni and Michèle Vergne. Multiplicity of compact group representations and applications to Kronecker coefficients. *ArXiv e-prints*, June 2015.

[3] Emmanuel Briand, Amarpreet Rattan and Mercedes Rosas. On the growth of the Kronecker coefficients. . *ArXiv e-prints*, 1607.02887 (2016)

[4] Li Ying. Stability of the Heisenberg products on symmetric functions . *ArXiv e-prints*, 1701.03203 (2017)

[5] Eckhard Meinrenken and Reyer Sjamaar. Singular reduction and quantization. *Topology*, 38(4):699–762, 1999.
[6] Francis D. Murnaghan. The Characters of the Symmetric Group. *Amer. J. Math.*, 59(4):739–753, 1937.
[7] Thomas Scharf and Jean-Yves Thibon. A Hopf-algebra approach to inner plethysm *Adv. Math.*, 104(1):30–58, 1994.
[8] Thomas Scharf, Jean-Yves Thibon, and Brian G. Wybourne. Reduced notation, inner plethysms and the symmetric group *J. Phys. A,* 26(24):7461–7478,1993.

Plethystic tableaux and applications

Chris Bowman
(joint work with R. Paget and C. Bessenrodt)

Schur functions are important examples of symmetric functions and are indexed partitions. Given 2 Schur functions, there are 3 distinct ways of 'multiplying' them together to obtain a new symmetric function. These products are the *Kronecker* the *plethysm* and the *Littlewood–Richardson* products.

The focus of this talk is on the plethysm product. Plethystic tableaux provide a new combinatorial gadget with which to try and understand these products. Examples of these "tableaux of tableaux" are pictured in the figure below. We use these plethystic tableaux to advance our understanding of the plethysm product by analogy with the other (better understood) products.

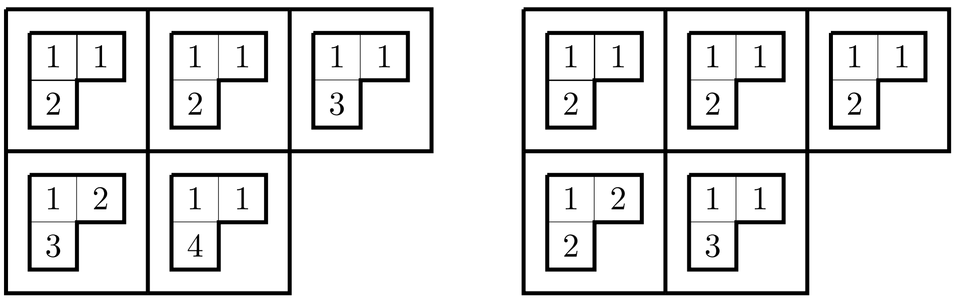


Figure 1. Examples of plethystic tableaux

(+) The first and most fundamental problem is to find a positive combinatorial interpretation for the decomposition multiplicities in these products. This problem seems out of reach at the moment and so we set benchmarks for our understanding: Saxl's tensor square conjecture is one such benchmark. In this talk we focus on the homogeneous and multiplicity-free Kronecker and plethysm products.

Multiplicity-free representations are of interest due to their manyfold applications in algebraic combinatorics. For example, the multiplicity-freeness of Littlewood–Richardson products of $\mathfrak{S}_n$-characters/Schur functions labelled by pairs of rectangles was used to count self-complementary plane partitions [5] and to provide explicit bijections related to quantised Littlewood–Richardson coefficients [7, 4, 8].

Multiplicity-free products are also of interest as their endomorphism algebras are commutative. Howe surveys the invariant-theoretic importance of multiplicity-free decompositions in [2].

Stembridge generalised this multiplicity-free question to tensor products of arbitrary algebraic groups in [6]. However, we wish to remain firmly rooted in the land of symmetric functions and algebraic combinatorics — therefore we will instead focus on the multiplicity-freeness of Kronecker and plethysm products. The following result was originally conjectured by Bessenrodt in 1999 and was solved by Bessenrodt–Bowman almost 20 years later.

Theorem. The Kronecker product $\chi^\lambda \otimes \chi^\mu$ is multiplicity-free if and only if the partitions λ, μ satisfy one of the following conditions (up to conjugation):

- One of the partitions is (n), and the other one is arbitrary;
- one of the partitions is $(n-1,1)$, and the other one is a fat hook (here, a fat hook is a partition with at most two different parts, i.e. it is of the form (a^b, c^d), $a \geq c$);
- $n = 2k+1$ and $\lambda = (k+1,k) = \mu$, or $n = 2k$ and $\lambda = (k,k) = \mu$;
- $n = 2k$, one of the partitions is (k,k), and the other one is $(k+1, k-1), (n-3,3)$ or a hook;
- one of the partitions is a rectangle, and the other one is one of $(n-2,2), (n-2,1^2)$;
- the partition pair is one of the pairs $((3^3),(6,3))$, $((3^3),(5,4))$, and $((4^3), (6^2))$.

Finally, multiplicity-free plethysm products of $\mathfrak{S}_n$-characters were recently classified in [1], thus completing the picture for the last remaining product of symmetric functions.

Theorem. The plethysm product $s_\nu \circ s_\mu$ is multiplicity-free if and only if one of the following holds:

- either ν or μ is the partition (1) and the other is arbitrary;
- $\nu \vdash 2$ and μ is (a^b), $(a+1, a^{b-1})$, $(a^b, 1)$, $(a^{b-1}, a-1)$ or a hook;
- $\mu \vdash 2$ and ν is linear or ν belongs to a small list of exceptions $\nu \in \{(4,1),(3,1),(2,1^a),(2^2),(3^2),(2^2,1) \mid 1 \leq a \leq 6\}$;
- ν and μ belong to a finite list of small rank exceptional products. In particular ν and μ are both linear and $|\nu| + |\mu| \leq 8$ and $(\nu,\mu) \notin \{((5),(3)),((4),(4)),((4),(1^4))\}$; or $\nu = (1^2)$ and $\mu \in \{(4,2),(2^2,1^2)\}$; or $\nu = (1^3)$ and $\mu \in \{(6),(1^6),(2^2)\}$; or $\nu = (2,1)$ and $\mu \in \{(3),(1^3)\}$.

($\div$) A fundamental question one can ask of any (representation theoretic) product is: *"does it factorise uniquely?"*. For the Littlewood–Richardson product, this question was answered by Rajan [3] in 2004 (perhaps later than one would expect!).

Theorem. Given partitions λ, μ and α, β we have that

$$\chi^\lambda * \chi^\mu = \chi^\alpha * \chi^\beta$$

if and only if $\alpha = \lambda$ and $\mu = \beta$ (up to reordering of the partitions).

In fact, Rajan proved that tensor products factorise uniquely for any simple Lie algebra. This might lead an optimist to ask whether the same is true for Kronecker and plethysm products. This is *almost true* for the plethysm product, modulo a few trivial cases:

Theorem. Let μ, ν, π, ρ be arbitrary partitions. If $\chi^{\nu} \circ \chi^{\mu} = \chi^{\rho} \circ \chi^{\pi}$ then either $\nu = \rho$ and $\mu = \pi$; or we are in one of five exceptional small rank cases,

$$\chi^{(2,1^2)} \circ \chi^{(1)} = \chi^{(1^2)} \circ \chi^{(1^2)}, \qquad \chi^{(3,1)} \circ \chi^{(1)} = \chi^{(1^2)} \circ \chi^{(2)},$$
$$\chi^{(2,1^2)} \circ \chi^{(2)} = \chi^{(1^2)} \circ \chi^{(3,1)}, \qquad \chi^{(2,1^2)} \circ \chi^{(1^2)} = \chi^{(1^2)} \circ \chi^{(2,1^2)},$$
$$\chi^{\nu} \circ \chi^{(1)} = \chi^{(1)} \circ \chi^{\nu}.$$

However, things appear to be much murkier for the Kronecker product. Here the first counter example to unique factorisability (modulo conjugation) is not until $n = 16$. In fact, this counterexample shows that the Grothendick ring of $\mathbb{C}\mathfrak{S}_n$ is not even an integral domain:

Example. Kronecker products do not factorise uniquely. We have that

$$\chi^{(8,3^2,1^2)} \otimes \chi^{(6,4,2^2,1^2)} = \chi^{(5^2,4,1^2)} \otimes \chi^{(6,4,2^2,1^2)}$$

This counterexample was found by Christine Bessenrodt using a computer. The key point here is that $(6, 4, 2^2, 1^2)$ labels a self-dual partition which takes many zero values. It seems highly likely that many such examples will appear for larger ranks.

However, we should not dismay. There is still hope, in the form of the stable products. We believe the following theorem is very striking indeed and that it provides significant evidence for our belief that stable products are more natural and elementary than their non-stable counterparts.

Theorem. Stable Kronecker and plethysm products factorise uniquely.

References

[1] C. Bessenrodt, C. Bowman, and R. Paget, *The classification of multiplicity-free plethysms of Schur functions*, preprint.

[2] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proc., vol. 8, Bar-Ilan Univ., Ramat Gan, 1995, pp. 1–182.

[3] C. S. Rajan, *Unique decomposition of tensor products of irreducible representations of simple algebraic groups*, Ann. of Math. (2) **160** (2004), no. 2, 683–704.

[4] M. Shimozono, *Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties*, J. Algebraic Combin. **15** (2002), no. 2, 151–187.

[5] R. P. Stanley, *Symmetries of plane partitions*, J. Combin. Theory Ser. A **43** (1986), no. 1, 103–113.

[6] J. R. Stembridge, *Multiplicity-free products and restrictions of Weyl characters*, Represent. Theory **7** (2003), 404–439.

[7] A. Schilling and S. Warnaar, *Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials*, Comm. Math. Phys. **202** (1999), no. 2, 359–401.

[8] M. Shimozono and Dennis E. White, *A color-to-spin domino Schensted algorithm*, Electron. J. Combin. **8** (2001), no. 1, Research Paper 21, 50.

Tensor products and moments

Pham Huu Tiep

Why do we care about **tensor products**? Perhaps we have already seen enough answers to this question, from many talks at the workshop. But I will try to discuss this question from my own perspective.

1. Aschbacher–Scott program and tensor products. It is important to understand the **subgroup structure** of G, where G is a (simple) group, finite or algebraic. This subject has a rich history, dating back at least to Galois' 1832 letter to Chevalier. It has become a much more active, and useful, area since the CFSG.

First and foremost, one would like to focus on understanding maximal subgroups of finite or algebraic groups G. In fact, in most problems, the **Aschbacher–O'Nan–Scott theorem** allows one to concentrate on the case where G is *almost quasi-simple.* The results of Liebeck–Praeger–Saxl and Liebeck–Seitz then allow one to assume furthermore that G is a **classical** group $Cl(V) = \mathrm{GL}(V)$, $\mathrm{Sp}(V)$, $\mathrm{GO}(V)$, and $\mathrm{GU}(V)$.

Theorem 1 (**Aschbacher** [1])**.** *Let $\mathcal{G} = Cl(V)$ be a classical group over an algebraically closed field $\mathbb{F}$, and let $M < \mathcal{G}$ be a maximal Zariski closed subgroup of $\mathcal{G}$. Then*

$$M \in \bigcup_{i=1}^{4} \mathcal{C}_i \cup \mathcal{S},$$

where $\mathcal{C}_i$, $1 \leq i \leq 4$, are collections of certain "natural" subgroups of $\mathcal{G}$, and $\mathcal{S}$ consists of the subgroups of the form $M = N_{\mathcal{G}}(S)$ for some quasisimple closed subgroup $S < \mathcal{G}$ such that the restriction $V \downarrow_S$ is irreducible.

In particular, $\mathcal{C}_4$ consists of stabilizers of tensor decompositions $V = V_1 \otimes V_2$, or $V = V_1 \otimes V_2 \otimes \ldots \otimes V_m$ with $V_1 \cong V_2 \cong \ldots \cong V_m$.

What about the **converse**? Given the hypothesis of Aschbacher's Theorem 1, assume now that $M \in \cup_{i=1}^4 \mathcal{C}_i \cup \mathcal{S}$. When can one say that M is indeed a maximal subgroup of $\mathcal{G}$? If $M \in \cup_{i=1}^4 \mathcal{C}_i$, then the maximality of M has been determined by Kleidman–Liebeck [5]. So we may assume that $M = N_{\mathcal{G}}(S) \in \mathcal{S}$. Suppose in addition that M is **not** maximal. Then $M < N < \mathcal{G}$, where N is a maximal subgroup of G, and we can again apply Aschbacher's Theorem 1 to N: $N \in \cup_{i=1}^4 \mathcal{C}_i \cup \mathcal{S}$. Assume furthermore that $N \in \mathcal{C}_4$. The tensor-induced subcase is treated by Magaard–Tiep. So we may assume for the quasisimple subgroup $S < \mathrm{GL}(V)$ that the S-module $V \downarrow_S = A \otimes B$ is irreducible and tensor decomposable (by passing to the universal cover of S). Thus we arrive at the question:

Problem 2. *When can the tensor product $A \otimes B$ of two S-representations be irreducible?*

The case where S is a quasisimple group of Lie type in defining characteristic is handled by a classical result of Seitz and Steinberg, whereas the case where

S is a (covering group) of S_n or A_n is mostly resolved by work of Bessenrodt, Bessenrodt–Kleshchev, Kleshchev–Tiep, and Morotti.

The main remaining case, where S is a finite quasisimple group of Lie type over a field of characteristic different from $\mathrm{char}(\mathbb{F})$, is the subject of the following theorem:

Theorem 3. [6], [7], [9] *Suppose S is a finite quasisimple group defined over $\mathbb{F}_q$ with q coprime to $p = \mathrm{char}(F)$, and suppose that the $\mathbb{F}S$-module V is irreducible and tensor decomposable. Then one of the following holds.*

(i) $q \leq 3$, *but* $G \not\cong \mathrm{SL}_n(q)$.
(ii) $G = \mathrm{Sp}_{2n}(5)$.
(iii) $2|q$, $G = F_4(q)$ *or* ${}^2F_4(q)$, *and p divides* $|G|$.

2. Symmetric powers. Of course, there remains also the case where $N \in \mathcal{S}$, that is, an N-module V is irreducible over a proper (quasisimple) subgroup S. Thus we arrive at the **Irreducible Restriction Problem**, a very difficult and important problem. Work of Magaard–Röhrle–Testerman [8] essentially reduces one configuration of the Irreducible Restriction Problem to

Problem 4 (Kollár–Larsen problem on symmetric powers). *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $V = \mathbb{F}^d$ with $d \geq 5$. Which Zariski closed subgroups of $\mathcal{G} = GL(V)$ act irreducibly on some symmetric power $\mathrm{Sym}^k(V)$ of V for some $k \geq 4$?*

Theorem 5. [4] *Assume a Zariski closed subgroup H of $\mathcal{G} := \mathrm{GL}(V)$ acts irreducibly on $\mathrm{Sym}^k(V)$ for some $k \geq 4$. Then $L \lhd H \leq N_{\mathcal{G}}(L)$, and one of:*

(i) $L \in \{\mathrm{SL}(V), \mathrm{Sp}(V)\}$;
(ii) $\mathrm{char}(\mathbb{F}) = p$, $L = \mathrm{SL}_d(q)$, $\mathrm{SU}_d(q)$, *or* $\mathrm{Sp}_d(q)$, $q = p^a$ *and* $d = \dim(V)$;
(iii) $k = 4, 5$, *and* $(\dim(V), L) = (6, 2J_2)$, $(12, 2G_2(4))$, $(12, 6Suz)$;
(iv) $k = 4, 5$, $p = 5, 7$, *and* $L =$ *Monster.*

As shown by Balaji–Kollár [2], this result has nice implications on holonomy groups and stability of vector bundles.

3. Semi-invariants and the α-invariant. For a Kähler manifold X and a compact subgroup $G \leq \mathrm{Aut}(X)$, Tian [11] defined an invariant $\alpha_G(X)$. In particular, Tian showed that a Fano variety X admits a G-invariant Kähler-Einstein metric if $\alpha_G(X) > \frac{\dim(X)}{\dim(X)+1}$.

Consider the case a finite $G < GL_{n+1}(\mathbb{C})$ acts on the projective space $\mathbb{P}^n$. Then Tian's invariant $\alpha_G(\mathbb{P}^n)$ is known to algebraic geometers as the *log-canonical threshold* $\mathrm{lct}(\mathbb{P}^n, G)$.

Theorem 6 (Thompson [10]**).** *Suppose that $G < GL_{n+1}(\mathbb{C})$ is any finite group. Then $\alpha_G(\mathbb{P}^n) \leq 4(n+1)$.*

In fact, a much stronger bound should hold asymptotically:

Conjecture 7 (Thompson [10]**).** *There exists a constant $C > 0$ such that $\alpha_G(\mathbb{P}^n) \leq C$ for all n and all finite subgroups $G < \mathrm{GL}_{n+1}(\mathbb{C})$.*

How did it happen that this strong upper bound on $\alpha_G(\mathbb{P}^n)$ was proved by John Thompson six years before the invariant was defined?

Recall that $G < GL(V)$ is said to have *a semi-invariant of degree k on V* if $\mathrm{Sym}^k(V)$ contains a one-dimensional G-submodule. Now the connection between $\alpha_G(\mathbb{P}^n)$ and semi-invariants of $G < GL_{n+1}(\mathbb{C})$ was observed by Cheltsov–Shramov to be as follows

$$\alpha_G(\mathbb{P}^n) \leq \frac{\min\{k \mid G \text{ has a semi-invariant of degree } k \text{ on } \mathbb{C}^{n+1}\}}{n+1}.$$

Theorem 8. [12] *Thompson's conjecture holds, with* $C = 1184036$.

Corollary 9. *Let $G \leq \mathrm{GL}(V)$ be a finite group for $V = \mathbb{C}^n$. Then G has a nonzero polynomial invariant, of degree at most* $\min(1184036 \cdot \dim(V) \cdot \exp(G/G'), |G|)$.

4. Moments. Let $V = \mathbb{C}^d$ and let $X \leq \mathrm{GL}(V)$ be a Zariski closed subgroup. Then Katz defined the $2k$**-moment** of X on V to be

$$M_{2k}(X,V) = \int_X |\mathrm{tr}(g)|^{2k} d\mu = \dim(V^{\otimes k} \otimes (V^*)^{\otimes k})^X.$$

For instance, if $\dim V > 2k$ then

$$M_{2k}(X,V) = \begin{cases} k!, & X = \mathrm{SL}(V), \\ (2k-1)!!, & X = \mathrm{Sp}(V), \mathrm{GO}(V). \end{cases}$$

Conjecture 10 (**Larsen**). *Let $V = \mathbb{C}^d$ with $d \geq 5$, $G = \mathrm{GL}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{GO}(V)$. Let $X \leq G$ be a Zariski closed subgroup, with X° being reductive. Assume $M_8(X,V) = M_8(G,V)$. Then X is big, i.e. $X \geq [G,G]$.*

This conjecture of Larsen is motivated by work of Deligne and Katz on the monodromy groups of Lefschetz pencils of hypersurface sections on a smooth projective complex variety, and more recent work of Katz on the monodromy groups of families of character sums over finite fields. The latter include *Kloosterman sums*

$$\sum_{x \in \mathbb{F}_p^\times} \psi\left(ax + \frac{b}{x}\right), \; a, b \in \mathbb{F}_p^\times, \; \psi \in \mathrm{Hom}((\mathbb{F}_p, +), \mathbb{C}^\times),$$

as well as

$$\sum_{x \in \mathbb{F}_p} \chi(x^3 + ax + b), \; a, b \in \mathbb{F}_p, \; \chi(\cdot) = \left(\frac{\cdot}{p}\right) \text{ – the Legendre symbol}$$

which (suitably adjusted) counts the $\mathbb{F}_p$-points of the elliptic curve $y^2 = x^3 + ax + b$.

Weil's celebrated proof of **the Riemann hypothesis** for curves over function fields implies that (the modulus of) the above sum is bounded by $2\sqrt{p}$. Deligne and Katz showed that the behavior of these sums over $\mathbb{F}_{q^k}$ when $k \to \infty$ is controlled by the monodromy group X which, in certain cases, should be determined by its moments. Hence, the truth of Larsen's Conjecture 10 has nice consequences on the distribution of eigenvalues and the trace of Frobenius element and L-functions of elliptic curves.

Theorem 11. [3] *Larsen's conjecture is true, aside from exactly one exception:* $V = \mathbb{C}^6,\ G = \mathrm{Sp}(V),\ X = 2J_2,$ *for which* $M_{2k}(G, V) = M_{2k}(X, V)$ *when* $2 \leq k \leq 5$.

Note that $M_{12}(2J_2, \mathbb{C}^6) = 10660 > 9449 = M_{12}(\mathrm{Sp}_6(\mathbb{C}), \mathbb{C}^6)$.

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References

[1] M. Aschbacher, On the maximal subgroups of the finite classical groups, *Invent. Math.* **76** (1984), 469–514.
[2] V. Balaji and J. Kollár, Holonomy groups and stable vector bundles, *Publ. Math. RIMS* **44** (2008), 183–211.
[3] R. M. Guralnick and Pham Huu Tiep, Decompositions of small tensor powers and Larsen's conjecture, *Represent. Theory* **9** (2005), 138–208.
[4] R. M. Guralnick and Pham Huu Tiep, Symmetric powers and a conjecture of Kollár and Larsen, *Invent. Math.* **174** (2008), 505–554.
[5] P. B. Kleidman and M. W. Liebeck, '*The Subgroup Structure of the Finite Classical Groups*', London Math. Soc. Lecture Note Ser. no. **129**, Cambridge University Press, 1990.
[6] A.S. Kleshchev and Pham Huu Tiep, On restrictions of modular spin representations of symmetric and alternating groups, *Trans. Amer. Math. Soc.* **356** (2004), 1971–1999.
[7] K. Magaard and Pham Huu Tiep, Irreducible tensor products of representations of finite quasi-simple groups of Lie type, in: '*Modular Representation Theory of Finite Groups*', M. J. Collins, B. J. Parshall, L. L. Scott, eds., Walter de Gruyter, Berlin et al, 2001, pp. 239–262.
[8] K. Magaard, G. Röhrle, and D. Testerman, On the irreducibility of symmetrizations of cross-characteristic representations of finite classical groups, *J. Pure Appl. Algebra* **217** (2013), 1427–1446.
[9] G. Navarro and Pham Huu Tiep, On irreducible products of characters, (in preparation).
[10] J. G. Thompson, Invariants of finite groups, *J. Algebra* **69** (1981), 143–145.
[11] G. Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, *Invent. Math.* **89** (1987), 225–246.
[12] Pham Huu Tiep, The alpha-invariant and Thompson's conjecture, *Forum of Math. Pi* **4** (2016), e5, 28 pages.

Implementing geometric complexity theory: On the separation of orbit closures via symmetries

Christian Ikenmeyer

(joint work with Umangathan Kandasamy)

The idea of using the symmetries of the determinant $\det_n$ and the permanent per_m to separate algebraic complexity classes was pioneered by Mulmuley and Sohoni in 2001 [12]. This approach is based on the observation that $\det_n$ and per_m are both characterized by their respective symmetry groups.

A *homogeneous projection* of a homogeneous polynomial is its evaluation at a point whose coordinates are given by homogeneous linear polynomials. The set of all homogeneous projections of $\det_n$ to polynomials in the variables $x_1, \ldots, x_N$ can then be written as $\{\det_n(\ell_1,\dots,\ell_{n^2}) \mid \ell_i$ is a homogeneous linear polynomial in $x_1,\dots,x_N\}$. The polynomial function $(x_{1,1},\dots,x_{n,n}) \mapsto \det_n(\ell_1,\dots,\ell_{n^2})$ equals the composition $\det_n \circ A$, where A is the linear map $(x_{1,1},\dots,x_{n,n}) \mapsto (\ell_1,\dots,\ell_{n^2})$. As it is common in representation theory, we write $A{\cdot}\det_n$ or just $A\det_n$ for $\det_n \circ A$. The *endomorphism orbit* $\mathrm{End}_{n^2}\det_n$ is defined as $\{A\det_n \mid A \in \mathbb{C}^{n^2\times n^2}\}$. For fixed m, n with $m < n$, define the *padded permanent* $\mathrm{per}_{m,n} := (x_{n,n})^{n-m}\cdot \mathrm{per}_m$. Let $\mathsf{dc}'(\mathrm{per}_{m,n})$ denote the smallest n such that $\mathrm{per}_{m,n} \in \mathrm{End}_{n^2}\det_n$. Valiant's $VBP \neq VNP$ conjecture is equivalent to the conjecture that $\mathsf{dc}'(\mathrm{per}_m)$ grows superpolynomially.

It turns out that if we restrict $\mathrm{End}_{n^2}\det_n$ to only the points $A\det_n$ for which A is invertible, we get the much simpler *group orbit* $\mathrm{GL}_{n^2}\det_n := \{g\det_n \mid g \in \mathrm{GL}_{n^2}\} \subseteq \mathrm{End}_{n^2}\det_n$.

Let $H_{\det_n} \subseteq \mathrm{GL}_{n^2}$ denote the symmetry group of $\det_n$. From the viewpoint of algebraic geometry, the set $\mathrm{GL}_{n^2}\det_n$ is an affine variety ([7, Sec 4.2], [11, Cor., p. 206]) and a homogeneous space that is isomorphic to the quotient $\mathrm{GL}_{n^2}/H_{\det_n}$. It is crucial to note here that the group $H_{\det_n}$ does not carry any information about the fact that we study a space of polynomials! In this way, we study the orbit $\mathrm{GL}_{n^2}\det_n$ independently of its embedding into the space of polynomials. This gives a particularly beautiful description of its coordinate ring via invariant theory:

$$\mathbb{C}[\mathrm{GL}_{n^2}\det_n] = \mathbb{C}[\mathrm{GL}_{n^2}]^{H_{\det_n}},$$

where $\mathbb{C}[\mathrm{GL}_{n^2}]^{H_{\det_n}}$ is the set of regular $H_{\det_n}$-invariant functions on the variety GL_{n^2} (see [7, (5.2.6)]. The coordinate ring of GL_{n^2} has classically been studied: It is a $\mathrm{GL}_{n^2}\times\mathrm{GL}_{n^2}$-representation whose representation theoretic decomposition is multiplicity free:

$$\mathbb{C}[\mathrm{GL}_{n^2}] = \bigoplus_{\lambda}\{\lambda\}\otimes\{\lambda^*\}, \tag{1}$$

where λ runs over all nonincreasing lists of n^2 integers and $\{\lambda\}$ denotes the irreducible rational representation of GL_{n^2} corresponding to λ. Eq. (1) is known as the algebraic Peter-Weyl theorem. It implies that the multiplicity of λ^* in $\mathbb{C}[\mathrm{GL}_{n^2}\det_n]$ equals the dimension of the $H_{\det_n}$-invariant space $\{\lambda\}^{H_{\det_n}}$. This coefficient $\dim\{\lambda\}^{H_{\det_n}}$ is known to be the symmetric rectangular Kronecker coefficient.

If we replace $\det_n$ by other polynomials, we get an analogous theory that is often equally beautiful. The power sum polynomial and the product $x_1\cdots x_D$ are of particular interest in this talk. The corresponding coefficients are not Kronecker coefficients, but plethysm coefficients and related coefficients that appear in algebraic combinatorics.

As we just have seen, group orbits have several desirable properties and can be understood directly via symmetry groups and algebraic combinatorics. But endomorphism orbits do not behave that nicely. In general, $\mathrm{End}_{n^2}\det_n$ is not a variety. In order to enable the study of $\mathrm{End}_{n^2}\det_n$ with methods from algebraic geometry, we go to the closure (Euclidean closure and Zariski closure coincide here by general principles, see [14, §2.C]): $\overline{\mathrm{End}_{n^2}\mathrm{det}_n}$, which coincides with the group orbit closure $\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}$, see e.g. [1, Sec. 3.5]. Hence we have a chain of inclusions $\mathrm{GL}_{n^2}\mathrm{det}_n \subseteq \mathrm{End}_{n^2}\mathrm{det}_n \subseteq \overline{\mathrm{GL}_{n^2}\mathrm{det}_n}$. The border determinantal complexity $\underline{\mathsf{dc}}(\mathrm{per}_m)$ is defined as the smallest n such that $\mathrm{per}_{m,n} \in \overline{\mathrm{GL}_{n^2}\mathrm{det}_n}$. Mulmuley and Sohoni's conjecture (closely related to Bürgisser's conjecture) [2, hypothesis (7)]) is that $\underline{\mathsf{dc}}(\mathrm{per}_m)$ grows superpolynomially. Since $\underline{\mathsf{dc}}(\mathrm{per}_m) \leq \mathsf{dc}'(\mathrm{per}_m)$, this would imply Valiant's conjecture. Mulmuley and Sohoni's conjecture can be attacked by representation theoretic multiplicities as follows. If we assume for the sake of contradiction that $\overline{\mathrm{GL}_{n^2}\mathrm{per}_{m,n}} \subseteq \overline{\mathrm{GL}_{n^2}\mathrm{det}_n}$, then by Schur's lemma the multiplicities satisfy $\mathrm{mult}_\lambda\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{per}_{m,n}}] \leq \mathrm{mult}_\lambda\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}]$. Thus, if there exists (λ, d) that satisfies

$$\mathrm{mult}_\lambda\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{per}_{m,n}}]_d > \mathrm{mult}_\lambda\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}]_d, \tag{2}$$

then $\underline{\mathsf{dc}}(m) > n$. Such a pair (λ, d) is called a *multiplicity obstruction*.

The algebraic geometry of $\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}$ and the representation theory of its coordinate ring are rather difficult to understand, see e.g. [10, 3]. But the close relationship between orbit and orbit closure gives hope that results can be transferred from the orbit to the closure. Indeed, $\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}] \subseteq \mathbb{C}[\mathrm{GL}_{n^2}\mathrm{det}_n]$ is a subalgebra, and hence we have $\mathrm{mult}_\lambda\mathbb{C}[\mathrm{GL}_{n^2}\mathrm{det}_n] \geq \mathrm{mult}_\lambda\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}]$. Getting lower bounds on multiplicities in $\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}]$ seems much harder. But as a first step towards lower bounds on $\mathrm{mult}_\lambda\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}]$, [6] proved that $\mathrm{GL}_{n^2}\mathrm{det}_n$ is open in its closure and that the ring $\mathbb{C}[\mathrm{GL}_{n^2}\mathrm{det}_n]$ is a localization of $\mathbb{C}[\overline{\mathrm{GL}_{n^2}\mathrm{det}_n}]$.

In this talk we present how to tighten the results from [6] in the case of the power sum polynomial. For $m \geq D$ let $p := x_1^D + x_2^D + \cdots + x_m^D$ and let $q := x_1x_2\cdots x_D$. Let $G := \mathrm{GL}_m$. For $m = D$ we separate the two families of orbit closures $\overline{Gp} \not\subseteq \overline{Gq}$ of polynomials p and q using multiplicity obstructions λ, i.e., $\mathrm{mult}_\lambda\mathbb{C}[\overline{Gp}] > \mathrm{mult}_\lambda\mathbb{C}[\overline{Gq}]$. Our key contribution is a method of proof that for the first time implements closely the strategy in [12, 13]: Both the lower bound on $\mathrm{mult}_\lambda\mathbb{C}[\overline{Gp}]$ and the upper bound on $\mathrm{mult}_\lambda\mathbb{C}[\overline{Gq}]$ are obtained directly from the symmetry groups of p and q and the dimension of the spaces of H_p- and H_q-invariants in irreducible GL_m-representations. This is the result of our tightening of the relationship between $\mathrm{mult}_\lambda\mathbb{C}[Gp]$ and $\mathrm{mult}_\lambda\mathbb{C}[\overline{Gp}]$.

Before our paper, all existence proofs of multiplicity obstructions $\overline{Gp} \not\subseteq \overline{Gq}$ for any p and q required to explicitly construct (with multilinear algebra) copies of irreducible representations in $\mathrm{mult}_\lambda\mathbb{C}[\overline{Gp}]$. These papers only took into account the symmetry group of q instead of both symmetry groups, see [4, 5, 9, 8].

References

[1] Markus Bläser and Christian Ikenmeyer. Introduction to geometric complexity theory. lecture notes, summer 2017 at Saarland University.

[2] Peter Bürgisser. The complexity of factors of multivariate polynomials. In *42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001)*, pages 378–385. IEEE Computer Soc., Los Alamitos, CA, 2001.

[3] Peter Bürgisser, Jesko Hüttenhain, and Christian Ikenmeyer. Permanent versus determinant: not via saturations. *Proceesings of the American Mathematical Society*, 145(3):1247–1258, 2017.
[4] Peter Bürgisser and Christian Ikenmeyer. Geometric complexity theory and tensor rank. *Proceedings 43rd Annual ACM Symposium on Theory of Computing 2011*, pages 509–518, 2011.
[5] Peter Bürgisser and Christian Ikenmeyer. Explicit lower bounds via geometric complexity theory. *Proceedings 45th Annual ACM Symposium on Theory of Computing 2013*, pages 141–150, 2013.
[6] Peter Bürgisser and Christian Ikenmeyer. Fundamental invariants of orbit closures. *Journal of Algebra*, 477(Supplement C):390 – 434, 2017.
[7] Peter Bürgisser, J.M. Landsberg, Laurent Manivel, and Jerzy Weyman. An overview of mathematical issues arising in the Geometric complexity theory approach to VP v.s. VNP. *SIAM J. Comput.*, 40(4):1179–1209, 2011.
[8] Julian Dörfler, Christian Ikenmeyer, and Greta Panova. On geometric complexity theory: Multiplicity obstructions are stronger than occurrence obstructions. accepted for SIAGA.
[9] Fulvio Gesmundo, Christian Ikenmeyer, and Greta Panova. Geometric complexity theory and matrix powering. *Differential Geometry and its Applications*, 2017.
[10] Shrawan Kumar. A study of the representations supported by the orbit closure of the determinant. *Compositio Mathematica*, 151:292–312, 2 2015.
[11] Yozô Matsushima. Espaces homogènes de Stein des groupes de Lie complexes. *Nagoya Math. J*, 16:205–218, 1960.
[12] K.D. Mulmuley and M. Sohoni. Geometric Complexity Theory. I. An approach to the P vs. NP and related problems. *SIAM J. Comput.*, 31(2):496–526 (electronic), 2001.
[13] K.D. Mulmuley and M. Sohoni. Geometric Complexity Theory. II. Towards explicit obstructions for embeddings among class varieties. *SIAM J. Comput.*, 38(3):1175–1206, 2008.
[14] D. Mumford. *Algebraic geometry. I: Complex projective varieties.* Classics in mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1976 edition in Grundlehren der mathematischen Wissenschaften, vol. 221.

Irreducible tensor products for symmetric and related groups

Lucia Morotti

Given a group G and a field F, an interesting question is to study for which irreducible FG-representations V and W the tensor product $V \otimes W$ is also irreducible. This is always the case if V or W is 1-dimensional, so the question could be restated as:

Given a group G and V and W irreducible FG-representations, both of which are not 1-dimensional, when is $V \otimes W$ irreducible?

Such irreducible tensor products are called non-trivial. Their classification is relevant to the Aschbacher–Scott classification of maximal subgroups of finite classical groups.

For example for symmetric groups in characteristic 0, it was first shown by Zisser in [13] that no non-trivial irreducible tensor products exist. In the same paper however it was shown that non-trivial irreducible tensor products for alternating groups exist and such products were characterised. In works of Bessenrodt [1] and Bessenrodt and Kleshchev [2, 5] (almost) homogeneous tensor products of symmetric, alternating groups and double covers of symmetric groups in characteristic 0 have been studied, thus solving the question for any of these groups in characteristic 0.

In positive characteristic p, however the situation is more complicated. It was conjectured by Gow and Kleshchev in [7] that non-trivial irreducible tensor products for symmetric groups exist only for $p = 2$. Parts of the conjecture, including the odd characteristic case, have been proved shortly after the conjecture was formulated by Bessenrodt and Kleshchev [3] and Graham and James [6]. Similarly, non-trivial tensor products for alternating groups in large characteristic (that is $p \geq 7$) have been studied by Bessenrodt and Kleshchev in [4]. For covering groups some results were obtained by Kleshchev and Tiep in [8]. In the last few years I have been able to finish studying non-trivial irreducible tensor products for these groups, apart for some still open cases for alternating groups in characteristic 2, see [9, 10, 11, 12].

In order to state results we need to define notation for the irreducible representations of these groups. The irreducible representations of S_n are given by the modules D^λ, while those of A_n by the modules E^λ or $E^\lambda_\pm$, with λ a p-regular partition of n. When considering restrictions from S_n to A_n, it is well known that $D^\lambda{\downarrow_{A_n}} \cong E^\lambda$ or $E^\lambda_+ \oplus E^\lambda_-$. These representations can also be viewed as irreducible representations of the double covers $\widetilde{S}_n$ and $\widetilde{A}_n$ respectively. The other irreducible representations of $\widetilde{S}_n$ and $\widetilde{A}_n$ are the spin irreducible representations, which only exist if $p \neq 2$. In characteristic 0 it is well known that (pairs) of spin representations are labeled by partitions in distinct parts. In odd characteristic labelings of (pairs) of spin representations have been found by Brundan and Kleshchev, using certain subsets of partitions. We will write $D(\lambda, 0)$ or $D(\lambda, \pm)$ (resp. $E(\lambda, 0)$ or $E(\lambda, \pm)$) for the spin irreducible representation(s) of $\widetilde{S}_n$ (resp. $\widetilde{A}_n$) labeled by λ. As it can be expected and is known in characteristic 0, $D(\lambda, 0){\downarrow_{\widetilde{A}_n}} \cong E(\lambda, +) \oplus E(\lambda, -)$, while $D(\lambda, \pm){\downarrow_{\widetilde{A}_n}} \cong E(\lambda, 0)$.

A special family of representations that plays a major role in the classification of irreducible tensor products are basic spin representations. A representation V is basic spin if V is any composition factor of the reduction modulo p of a spin representation in characteristic 0 labeled by the partition (n). Note that if $p \neq 2$ then basic spin representations are spin representations, while if $p = 2$ basic spin representations are also representations of S_n or A_n.

The following two theorems can be obtained from the aforementioned papers:

Theorem. *Let V and W be irreducible representations of $F\widetilde{S}_n$, both of which are not 1-dimensional. Then, up to exchange of V and W and tensoring with the sign representation, $V \otimes W$ is irreducible if and only if:*

- *(i)* $V \cong D^{(n-1,1)}$, $W \cong D(\lambda, \pm)$ *with* $E(\lambda, 0) \downarrow_{\widetilde{A}_{n-1}}$ *irreducible and* $n \not\equiv 0 \bmod p$,
- *(ii)* V *is basic spin,* $W \cong D^{(n-k,k)}$ *with* $D^{(n-k,k)} \downarrow_{\widetilde{S}_{n-1}}$ *irreducible and* $n \not\equiv 0, \pm 2 \bmod p$ *is even if* $p \neq 2$ *or* $n \equiv 2 \bmod 4$ *and* k *is odd if* $p = 2$,
- *(iii)* $V \cong D((6), \pm)$, $W \cong D((3, 2, 1), \pm)$ *and* $p \neq 2, 3$ *or* 5.

Theorem. *Let V and W be irreducible representations of $F\widetilde{A}_n$, both of which are not 1-dimensional. If $V \otimes W$ is irreducible then one of the following holds up to exchange of V and W:*

(i) $V \cong E^{(n-1,1)}$, $W \cong E^\lambda_\pm$ *with* $D^\lambda \downarrow_{\widetilde{S}_{n-1}}$ *irreducible and* $n \not\equiv 0 \bmod p$,

(ii) $V \cong E^{(n-1,1)}$, $W \cong E(\lambda, \pm)$ *with* $D(\lambda, 0) \downarrow_{\widetilde{S}_{n-1}}$ *irreducible and* $n \not\equiv 0 \bmod p$,

(iii) $p \neq 2$, *V is basic spin,* $W \cong E^{(n-k,k)}$ *with* $D^{(n-k,k)} \downarrow_{\widetilde{S}_{n-1}}$ *irreducible and* $n \not\equiv 0, \pm 2 \bmod p$ *is odd,*

(iv) $p = 2$, *V is basic spin and at least one between V and W does not extend to S_n,*

(v) $n \in \{5, 6, 9\}$.

In cases (i), (ii) and (iii) $V \otimes W$ is always irreducible.

In each of the above cases, if $V \otimes W$ is known to be irreducible, formulas for $V \otimes W$ are also known. For $n \leq 9$ it is easy to check which tensor products are irreducible, since decomposition matrices for such n are known. So the only still open case is case (iv) of the second theorem.

References

[1] C. Bessenrodt, On mixed products of complex characters of the double covers of the symmetric groups, *Pacific J. Math.* **199** (2001), 257–268.

[2] C. Bessenrodt, A.S. Kleshchev, On Kronecker products of complex representations of the symmetric and alternating groups, *Pacific J. Math.* **190** (1999), 201–223.

[3] C. Bessenrodt, A.S. Kleshchev, On tensor products of modular representations of symmetric groups, *Bull. London Math. Soc.* **32** (2000), 292–296.

[4] C. Bessenrodt, A.S. Kleshchev, Irreducible tensor products over alternating groups, *J. Algebra* **228** (2000), 536–550.

[5] C. Bessenrodt, A.S. Kleshchev, On Kronecker products of spin characters of the double covers of the symmetric groups, *Pacific J. Math.* **198** (2001), 295–305.

[6] J. Graham, G. James, On a conjecture of Gow and Kleshchev concerning tensor products, *J. Algebra* **227** (2000), 767–782.

[7] R. Gow, A.S. Kleshchev, Connections between the representations of the symmetric group and the symplectic group in characteristic 2, *J. Algebra* **221** (1999), 60–89.

[8] A.S. Kleshchev, P.H. Tiep, On restrictions of modular spin representations of symmetric and alternating groups, *Trans. Amer. Math. Soc.* **356** (2004), 1971–1999.

[9] L. Morotti, Irreducible tensor products for symmetric groups in characteristic 2, *Proc. Lond. Math. Soc. (3)* **116** (2018), 1553–1598.

[10] L. Morotti, Irreducible tensor products for alternating groups in characteristic 5, *Algebr. Represent. Theory* (to appear).

[11] L. Morotti, Irreducible tensor products for alternating groups in characteristic 2 and 3, preprint.

[12] L. Morotti, Irreducible tensor products of representations of covering groups of symmetric and alternating groups, preprint.

[13] I. Zisser, Irreducible products of characters in A_n, *Israel J. Math.* **84** (1993), 147–151.

Modular plethysms for $\mathbf{SL_2}(F)$

Mark Wildon
(joint work with Eoghan McDowell and Rowena Paget)

Let E be a two-dimensional complex vector space. The finite-dimensional irreducible polynomial representations of $\mathrm{SL}_2(\mathbb{C})$ are, up to isomorphism, the symmetric powers $\mathrm{Sym}^\ell E$ for $\ell \in \mathbf{N}_0$. Working in invariant theory, Hermite discovered the isomorphism

$$\mathrm{Sym}^r\mathrm{Sym}^\ell E \cong \mathrm{Sym}^\ell\mathrm{Sym}^r E. \tag{1}$$

This is one of many *plethystic isomorphisms* of $\mathrm{SL}_2(\mathbb{C})$-representations. Another important example is the Wronskian isomorphism $\mathrm{Sym}^r\mathrm{Sym}^\ell E \cong \bigwedge^r \mathrm{Sym}^{\ell+r-1}E$ (see for instance [1]). More generally, let ∇^λ denote the Schur functor canonically labelled by the partition λ. We ask: *when is there an* $\mathrm{SL}_2(\mathbb{C})$*-isomorphism* $\nabla^\lambda\mathrm{Sym}^\ell E \cong \nabla^\mu\mathrm{Sym}^m E$? In my talk I surveyed some of the answers to this question and then considered the modular analogue in which $\mathbb{C}$ is replaced with an infinite field of prime characteristic.

The first part is on joint work with Rowena Paget [6]; the second is on work in progress with my Ph.D. student Eoghan McDowell.

Part 1: Complex plethystic isomorphisms. Let s_λ denote the Schur function canonically labelled by the partition λ. By the bridge between representation theory and symmetric functions seen in my introductory talk, there is a plethystic isomorphism $\nabla^\lambda\mathrm{Sym}^\ell E \cong \nabla^\mu\mathrm{Sym}^m E$ if and only if $(s_\lambda \circ s_{(\ell)})(x^{-1}, x) = (s_\mu \circ s_\mu)(x^{-1}, x)$. (It is correct to specialize the variables x_1, x_2 so that they satisfy $x_1x_2 = 1$ because this relation is satisfied by the eigenvalues of every matrix in $\mathrm{SL}_2(\mathbb{C})$.) Substituting $x = q^2$ one obtains (iii) in the theorem below; this is the combinatorial statement that the generating functions enumerating $\mathrm{SSYT}_{\{1,\ldots,\ell\}}(\lambda)$ and $\mathrm{SSYT}_{\{1,\ldots,m\}}(\mu)$ by the sum of the contents of each tableau are equal, up to a power of q.

Theorem 1. *The following are equivalent:*
- (i) $\nabla^\lambda\mathrm{Sym}^\ell E \cong \nabla^\mu\mathrm{Sym}^m E$*;*
- (ii) $(s_\lambda \circ s_{(\ell)})(x^{-1}, x) = (s_\mu \circ s_{(m)})(x^{-1}, x)$*;*
- (iii) $s_\lambda(1, q, \ldots, q^\ell) = s_\mu(1, q, \ldots, q^m)$ *up to a (known) power of* q*;*
- (iv) $C(\lambda) + \ell + 1/H(\lambda) = C(\mu) + m + 1/H(\mu)$.

In (iv), $C(\lambda) = \{j - i : (i, j) \in [\lambda]\}$ is the multiset of contents of λ, $H(\lambda) = \{h_{(i,j)} : (i, j) \in [\lambda]\}$ is the multiset of hook lengths, and $/$ denotes the difference of multisets, *allowing negative multiplicities.* (This is clarified in the example following Theorem 2 below.) The equivalence of (iii) and (iv) is proved using a unique factorization property of the quantum integers $[m]_q = (q^m - 1)/(q - 1) = 1 + \cdots + q^{m-1}$, and Stanley's *Hook Content Formula* [7, Theorem 7.12.2], namely that

$$s_\lambda(1, q, \ldots, q^\ell) = q^B \frac{\prod_{(i,j)\in[\lambda]}[j - i + \ell + 1]_q}{\prod_{(i,j)\in[\lambda]}[h_{(i,j)}]_q}$$

where q^B is a (known) power of q. For example, Hermite reciprocity (1) follows from (iv), since $\{1+\ell,\dots,r+\ell\}/\{1,\dots,r\} = \{1+r,\dots,\ell+r\}/\{1,\dots,\ell\}$. The Wronskian isomorphism may be established still more easily, because in this case the difference multisets on either side of (iv) are equal even before cancellation.

The following theorem is a typical example of a plethystic isomorphism. It was first proved by King [5, §4.2]. A stronger version including a converse is proved using the equivalence of (i) and (iii) in Theorem 1.5 of [6].

Theorem 2. *Let λ be a partition contained in a box with $\ell+1$ rows and a columns. Let $\lambda^\bullet$ be its complement in this box. Then*

$$\nabla^\lambda \mathrm{Sym}^\ell E \cong \nabla^{\lambda^\bullet} \mathrm{Sym}^\ell E.$$

As a corollary of (iv) in Theorem 1 we obtain the following appealing result.

Corollary 3. *Let λ be a partition contained in a box with $\ell+1$ rows and a columns. Let $\lambda^\bullet$ be its complement in this box. There is an equality of multisets*

$$\big(C(\lambda)+\ell+1\big) \cup H(\lambda^\bullet) = \big(C(\lambda^\bullet)+\ell+1\big) \cup H(\lambda).$$

For example, if $\lambda = (4,3,3,1)$ and the box has 4 rows and 5 columns then $\lambda^\bullet = (4,2,2,1)$ and the equality in Corollary 3 may be checked using the bold numbers in the tableaux below.

$C(\lambda)+4$

$\mathbf{4}_0$	$\mathbf{5}_1$	$\mathbf{6}_2$	$\mathbf{7}_3$	$\mathbf{1}_0$
$\mathbf{3}_0$	$\mathbf{4}_1$	$\mathbf{5}_2$	$\mathbf{1}_0$	$\mathbf{3}_1$
$\mathbf{2}_0$	$\mathbf{3}_1$	$\mathbf{4}_2$	$\mathbf{2}_0$	$\mathbf{4}_1$
$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{2}_1$	$\mathbf{5}_2$	$\mathbf{7}_3$

$H(\lambda^\bullet)$

$H(\lambda)$

$\mathbf{7}_3$	$\mathbf{5}_2$	$\mathbf{4}_1$	$\mathbf{1}_0$	$\mathbf{1}_0$
$\mathbf{5}_2$	$\mathbf{3}_1$	$\mathbf{2}_0$	$\mathbf{3}_1$	$\mathbf{2}_0$
$\mathbf{4}_2$	$\mathbf{2}_1$	$\mathbf{1}_0$	$\mathbf{4}_1$	$\mathbf{3}_0$
$\mathbf{1}_0$	$\mathbf{7}_3$	$\mathbf{6}_2$	$\mathbf{5}_1$	$\mathbf{4}_0$

$C(\lambda^\bullet)+4$

The author is grateful to Christine Bessenrodt for observing that Corollary 3 holds in a stronger version also considering arm-lengths, as indicated above by subscripts. This was proved by Bessenrodt [3] by an ingenious application of [2, Theorem 3.2]. A longer inductive proof can be given by adapting the proof of Corollary 3 in [8]. Finding a representation theoretic interpretation of this stronger result was suggested at the workshop as an open problem.

In [6], many further plethystic isomorphisms, and obstructions to such isomorphisms, are proved. In particular, in [6, Theorem 1.4] we extend another result of King [5, §4] to give a complete classification of all isomorphisms between $\nabla^\lambda \mathrm{Sym}^\ell E$ and $\nabla^{\lambda'} \mathrm{Sym}^m E$, where λ' is the conjugate partition to λ. In [6, §10] we give a complete classification of all isomorphisms $\nabla^\lambda \mathrm{Sym}^\ell E \cong \nabla^\mu \mathrm{Sym}^m E$ in which λ and μ are (separately) either hook partitions, two-row partitions, or two-column partitions. One curious family we obtain is $\nabla^{(3\ell-3,2\ell-1)} \mathrm{Sym}^\ell E \cong \nabla^{(\ell+1,1^{\ell-2})} \mathrm{Sym}^{3\ell-4} E$ for all $\ell \geq 2$. The author suggests finding a geometric or invariant theory interpretation of this isomorphism as an open problem.

Part 2: Modular plethysms. Let F be an infinite field of prime characteristic p and let E be the natural representation of $\mathrm{SL}_2(F)$. It is now important to distinguish the two versions of the symmetric power. Given a polynomial representation V of $\mathrm{SL}_2(F)$, let $\mathrm{Sym}_r V = (V^{\otimes r})^{S_r}$ be the invariant submodule under the place permutation action of S_r on $V^{\otimes r}$ and let

$$\mathrm{Sym}^r V = V^{\otimes r} \big/ \langle v^{(1)} \otimes \cdots \otimes v^{(r)} \cdot \sigma - v^{(1)} \otimes \cdots \otimes v^{(r)} \rangle$$

be the module of coinvariants. For example, the matrices giving the action of

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(F)$$

on $\mathrm{Sym}^2 E$ and $\mathrm{Sym}_2 E$ in a basis e_1, e_2 of E are

$$\begin{matrix} e_1^2 & e_2^2 & e_1 e_2 \end{matrix} \qquad \begin{matrix} e_1 \otimes e_1 & e_2 \otimes e_2 & e_1 \otimes e_2 + e_2 \otimes e_1 \end{matrix}$$

$$\begin{pmatrix} \alpha^2 & \beta^2 & \alpha\beta \\ \gamma^2 & \delta^2 & \gamma\delta \\ 2\alpha\gamma & 2\beta\delta & \alpha\delta + \beta\gamma \end{pmatrix} \qquad \begin{pmatrix} \alpha^2 & \beta^2 & 2\alpha\beta \\ \gamma^2 & \delta^2 & 2\gamma\delta \\ \alpha\gamma & \beta\delta & \alpha\delta + \beta\gamma \end{pmatrix}$$

respectively. (Here, as usual e_1^2 is the image of $e_1 \otimes e_1$ in the quotient module defined above.) Observe that if $p = 2$ then $\mathrm{Sym}^2 E$ has a 2-dimensional invariant submodule $\langle e_1^2, e_2^2 \rangle$, whereas $\mathrm{Sym}_2 E$ has this 2-dimensional module only as a quotient. More generally, it is known that $\mathrm{Sym}^r E \cong (\mathrm{Sym}_r E)^\circ$ where $\circ$ denotes contravariant duality, defined on a representation $\rho : \mathrm{SL}(E) \to \mathrm{GL}(V)$ by $\rho^\circ(g) = \rho(g^{\mathrm{t}})^{\mathrm{t}}$ (see [4, §2.7 and p44 Example 1]).

The distinction between the two versions of the symmetric power is critical in the following modular generalization of the Wronskian isomorphism.

Theorem 4. *For all r, $\ell \in \mathbf{N}$, there is an $\mathrm{SL}_2(F)$-isomorphism*

$$\mathrm{Sym}_r \mathrm{Sym}^\ell E \cong \bigwedge^r \mathrm{Sym}^{r+\ell-1} E.$$

We prove this isomorphism by an explicit construction: it is non-obvious and slightly subtle to prove $\mathrm{SL}_2(F)$-equivariance. We also generalize Theorem 2.

Theorem 5. *Let λ be a partition contained in a box with $\ell+1$ rows and a columns. Let $\lambda^\bullet$ be its complement in this box. Then*

$$\nabla^\lambda \mathrm{Sym}^\ell E \cong \nabla^{\lambda^\bullet} \mathrm{Sym}_\ell E.$$

One important idea in the proof is that if V is a polynomial representation of $\mathrm{SL}_2(F)$ of dimension d then $\bigwedge^r V \cong \bigwedge^{d-r} V^\star \cong \bigwedge^{d-r} V^\circ$.

It follows from the theorem of King on conjugation of partitions mentioned above that there is an $\mathrm{SL}_2(\mathbb{C})$-isomorphism $\nabla^{(a+1,1^b)} \mathrm{Sym}^\ell E \cong \nabla^{(b+1,1^a)} \mathrm{Sym}^{\ell+a-b} E$ for all $a, b \in \mathbf{N}$ and $\ell \geq b$. The final result below shows that this does not extend to the modular case.

Theorem 6. *There exist infinitely many pairs (a, b) such that, provided e is sufficiently large, the eight representations of $\mathrm{SL}_2(F)$ obtained from $\nabla^{(a+1,1^b)} \mathrm{Sym}^{p^e+b}$ by*

(i) *Replacing* ∇ *with its contravariant dual functor* ∇°*;*
(ii) *Replacing* $(a+1, 1^b)$ *with* $(b+1, 1^a)$ *and* $p^e + b$ *with* $p^e + a$*;*
(iii) *Replacing* $\mathrm{Sym}^\ell E$ *with* $\mathrm{Sym}_\ell E$
are all non-isomorphic.

Determining which of the other plethystic isomorphisms in [6] have modular generalizations appears to be a fruitful topic for further research.

References

[1] Abdelmalek Abdesselam and Jaydeep Chipalkatti, *On the Wronskian combinants of binary forms*, J. Pure Appl. Algebra **210** (2007), no. 1, 43–61.
[2] Christine Bessenrodt, *On hooks of Young diagrams*, Ann. Comb. **2** (1998), no. 2, 103–110.
[3] Christine Bessenrodt, personal communication, April 2019.
[4] James A. Green, *Polynomial representations of* GL_n, Lecture Notes in Mathematics, vol. 830, Springer-Verlag, Berlin, 1980.
[5] Ronald C. King, *Young tableaux, Schur functions and* SU(2) *plethysms*, J. Phys. A **18** (1985), no. 13, 2429–2440.
[6] Rowena Paget and Mark Wildon, *Plethysms of symmetric functions and representations of* $\mathrm{SL}_2(\mathbf{C})$, arXiv:1907.07616 (July 2019), 51 pages.
[7] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
[8] Mark Wildon, *A corollary of Stanley's Hook Content Formula*, arXiv:1904.08904 (April 2019), 8 pages.

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