

Random matrix theory: Dyson Brownian motion

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The theory of random matrices was introduced by John Wishart (1898–1956) in 1928. The theory was then developed within the field of nuclear physics from 1955 by Eugene Paul Wigner (1902–1995) and later by Freeman John Dyson, who were both concerned with the statistical description of heavy atoms and their electromagnetic properties. In this snapshot, we show how mathematical properties can have unexpected links to physical phenomena. In particular, we show that the eigenvalues of some particular random matrices can mimic the electrostatic repulsion of the particles in a gas.

1 Mathematical background

In order to define and talk about random matrices and their properties, we must first give a crash-course in linear algebra.

1.1 Linear algebra

Let us first recall that an $(n \times m)$ -*matrix* is a table of real numbers with n rows and m columns. We use the notation $\mathbb{R}^{n \times m}$ to refer to the collection of

all $n \times m$ matrices with real-number entries. For example, if we let

$$M = \begin{bmatrix} 0 & 2 \\ 10 & 2.5 \end{bmatrix}, \text{ and } N = \begin{bmatrix} -1 & 2 & 0 \\ 4.75 & 0 & \sqrt{6} \end{bmatrix},$$

then M is a 2×2 matrix, while N is a 2×3 matrix. If the number of columns is 1, we have a *vector* in \mathbb{R}^m .

We will only need matrices with the same number of rows and columns, also called *square matrices*, we denote this number by d and call it the *dimension* of the matrix. For example, the matrix M above is of dimension 2. If two matrices M and N have the same dimension d then it is possible to define their *sum* and *product* in the following way:

$$\begin{aligned} M &= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, & N &= \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}, \\ M + N &= \begin{bmatrix} m_{11} + n_{11} & m_{12} + n_{12} \\ m_{21} + n_{21} & m_{22} + n_{22} \end{bmatrix}, \\ MN &= \begin{bmatrix} m_{11}n_{11} + m_{12}n_{21} & m_{11}n_{12} + m_{12}n_{22} \\ m_{21}n_{11} + m_{22}n_{21} & m_{21}n_{12} + m_{22}n_{22} \end{bmatrix}, \end{aligned}$$

The example above is for dimension 2; the definition can be extended in a natural way to all dimensions $d > 0$. In fact, the product can be defined for any $(n \times m)$ matrix M and $(m \times k)$ matrix N , and the product MN will be a matrix of dimension $(n \times k)$. We now give a list of the other definitions and properties of matrices that we will need.

- The *identity matrix* I_d is the square matrix of dimension d with 1s on the diagonal and 0s everywhere else. For example, in dimension 3 it is given by

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- A square matrix M of dimension d is said to be *invertible* if there exists a matrix M^{-1} with the property that

$$MM^{-1} = M^{-1}M = I_d.$$

- The *transpose* M^t of a matrix M is the matrix obtained by switching the rows and columns of M :

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}, \quad M^t = \begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}.$$

Notice that the entries on the diagonal are unchanged by the transposition operation.

- If a matrix M has the property that $M = M^t$, we say that M is *symmetric*.
- If instead the matrix M has the property that $M^t = M^{-1}$, that is, that $MM^t = I_d$, we say that M is *orthogonal*. Here is an example of an orthogonal matrix:

$$U = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix},$$

$$U^t U = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- If there exists a non-zero vector \underline{x} in \mathbb{R}^n such that

$$M\underline{x} = \lambda\underline{x}$$

for some scalar λ , then λ is called an *eigenvalue* for M with *eigenvector* \underline{x} . The set of all eigenvalues of a matrix M is called the *spectrum* of M and denoted by $\text{Spec}(M)$. For a square matrix of dimension d , there are d eigenvalues (although they may not all be different). The eigenvalues of a matrix are extremely important for many applications in physics and engineering where we have a matrix that comes from a system of equations.^[1] We shall shortly see their importance for an application in atomic physics.

We are interested in studying the properties a symmetric matrix can retain when an “orthogonal transformation” is applied. That is, given a symmetric matrix M and an orthogonal matrix U , what do M and $U^t M U$ have in common? It turns out that the eigenvalues of M and $U^t M U$ are equal. We say that the eigenvalues are *invariant under orthogonal transformation*. Furthermore, a symmetric matrix M can always be decomposed in the form $M = U D U^t$ where D is a diagonal matrix whose entries are the eigenvalues of M and U is an orthogonal matrix. For example:

$$M = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}, \quad U = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}, \quad U^t M U = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix},$$

In this example, the spectrum $\text{Spec}(M) = \{1, 5\}$. This process of writing a symmetric matrix in terms of the diagonal matrix of its eigenvalues is referred to as *spectral decomposition*.

1.2 Probability theory

Now we need to consider some concepts from probability theory, which will give us the “random” part of the theory of random matrices. We will extensively

[1] We refer to <http://mathworld.wolfram.com/Eigenvalue.html> for more information on eigenvalues and how to calculate them.

use the concept of a “random variable”. The best way to picture this notion is to consider a six-sided die. Every time we roll a die, we generate an integer number between 1 and 6 and each number is equally likely to appear, that is, the probability is $p = \frac{1}{6}$ for each face of the die. Figure 1 shows the simulation of 1000 dice rolls and the frequency of outcome for each number.

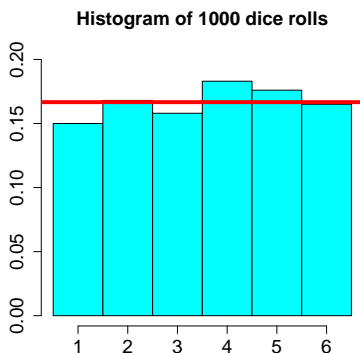


Figure 1: Simulation (light blue) and distribution (red) of a dice roll.

More generally, a *random variable* is a variable whose possible values are the outcome of a random phenomenon. This phenomenon can be genuinely random, as in the rolling of a die example, or randomness that is the result of incomplete knowledge of a system. Random variables can be either *discrete* if they can take only finitely many values (as in the die example), or *continuous* if they take infinitely many values. The *probability distribution* of a discrete random variable is a list of all its possible values together with the respective probabilities of obtaining them. For a continuous random variable, the probability distribution is instead given by the integral of a function (called the “probability density”). If the values of a random variable are all equally likely, we say it follows a *uniform distribution*. This is the case for a random variable that models rolling a die, or, for a continuous example, a random number generator on an interval (a, b) . Obviously not all random variables are uniformly distributed; the most common example of one that is not is the “Gaussian” random variable, which behaves according to the Gaussian distribution function, denoted by $\mathcal{N}(\mu, \sigma^2)$ and pictured in Figure 2. Here μ is the *mean value*, that is, the value with the highest probability of outcome, and σ^2 is the *variance*, which is a measure of how the values spread around the mean. A special case is the variable $\mathcal{N}(0, 1)$, called the *standard Gaussian*. It can be shown that any other Gaussian distribution can

be obtained through the relation $\mu + \sigma\mathcal{N}(0, 1)$. A Gaussian variable $\mathcal{N}(\mu, \sigma^2)$ takes values in an interval $[a, b]$ with probability

$$p_{\mu, \sigma^2}([a, b]) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Figure 2 shows the simulation of two different Gaussian variables and the frequencies of the realizations. We see how a smaller variance makes the observations less spread out around the mean. We say two random variables are

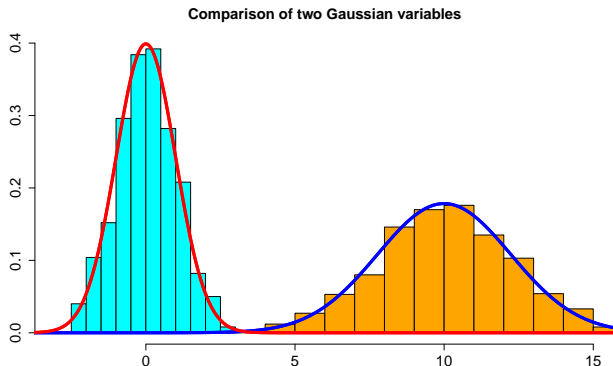


Figure 2: Simulations of a $\mathcal{N}(0, 1)$ variable and a $\mathcal{N}(10, 5)$ (light blue and orange respectively) and their theoretical distributions (red and blue respectively).

independent if knowledge about one of them doesn't carry information about the other. For example, if we roll two ordinary six-sided dice together, the score on one die is independent of the score on the other. We say two random variables have the same distribution if they assign the same probability to the same outcomes. For instance, the dice from the previous example have the same distribution.

Sometimes it is useful to study a sequence of random variables as a mathematical object in its own right; we refer to this sequence as a *random process*. Consider this simple example: Start from 0 and flip a coin. If the coin shows heads add +1, and if the coin shows tails, add -1. Repeat this process as many times as you like. We call this particular example of a process a *random walk*, as it can be imagined as an object “walking” on the integers, forward when adding 1 and backwards when adding -1. We can describe it more precisely

with the following recurrence relations:

$$W_0 = 0,$$

$$W_k = W_{k-1} + C_{k-1},$$

where $k > 0$ is the integer number of coin tosses, W_k is the position at time k and C_{k-1} is the coin toss at step $k - 1$ which yields either $+1$ or -1 . What does W_k look like? And what happens if we repeat the same construction multiple times? Figure 3 shows the simulation of a single trajectory and a sample of multiple realizations.

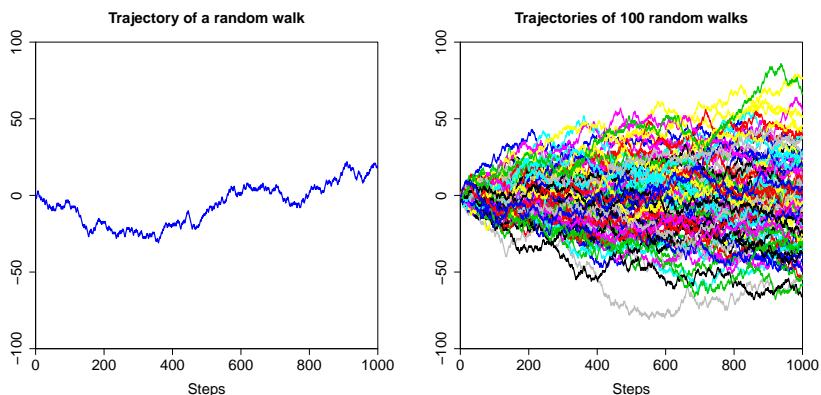


Figure 3: Simulations of random walks with 1000 steps.

It is also common to define random processes with continuous recurrence relations. One important example is “Brownian motion”, which can be defined as follows: We begin the process with the value 0 and for every increment of time dt we add the value given by one outcome of the Gaussian variable $\mathcal{N}(0, dt)$. We say the Brownian motion has “independent Gaussian increments” and we write

$$B_0 = 0,$$

$$B_{t+dt} = B_t + \sqrt{dt} \cdot \mathcal{N}(0, 1).$$

The Brownian motion described by the system of equations above is depicted in Figure 4, which shows a simulation of a single trajectory and a sample of multiple realizations. This process is used to model a great variety of natural phenomena, from the motion of tiny particles in a fluid to the motion of massive bodies in space responding to gravitational forces from surrounding stars.

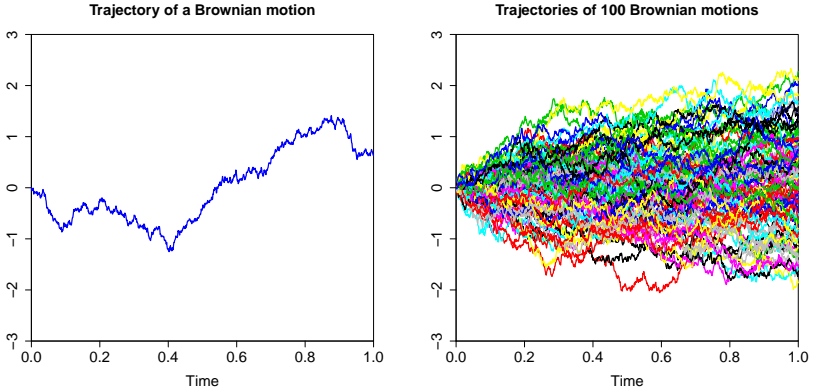


Figure 4: Simulations of Brownian motions.

The simulations of the random walk and of Brownian motion look surprisingly similar. This is not a coincidence and the relationship between the two can be explained. Let us fix an integer n and set $dt = \frac{1}{n}$. Then consider the Brownian motion $B_{\frac{k}{n}}$ and the rescaled random walk $V_{\frac{k}{n}}$ defined as follows:

$$B_0 = 0, \quad B_{\frac{k}{n}} = B_{\frac{k-1}{n}} + \frac{1}{\sqrt{n}} \cdot \mathcal{N}(0, 1),$$

$$W_0 = 0, \quad V_{\frac{k}{n}} = \frac{1}{\sqrt{n}} \cdot W_k,$$

where now $k \leq n$. If we let n increase to infinity, then these two processes will have the same distribution. Figure 5 illustrates this phenomenon.

1.3 Gaussian ensemble and symmetric Brownian motion

We are now prepared to introduce our main topic. A *random matrix* is defined to be a matrix whose entries are random variables, and a *matrix process* is defined to be a sequence of random matrices. Let us also define the *Gaussian Orthogonal Ensemble* (GOE), which is the set of all random matrices H which are symmetric (recall that this means they have the property $H^t = H$), and which satisfy

$$H_{ij} \stackrel{\text{ind}}{\sim} \begin{cases} \mathcal{N}(0, 1) & \text{if } i \neq j, \\ \sqrt{2} \cdot \mathcal{N}(0, 1) & \text{if } i = j. \end{cases}$$

In other words, the GOE is the set of all symmetric random matrices whose entries are independent standard Gaussian variables, and rescaled Gaussian

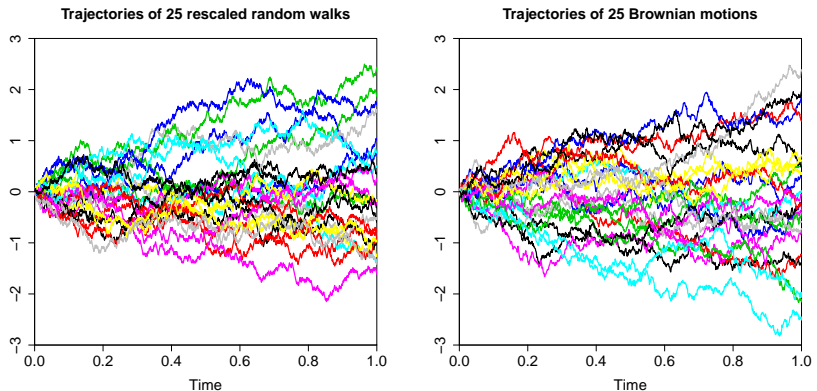


Figure 5: Rescaled random walks and Brownian motions where $n = 1000$.

variable along the diagonal. If H belongs to the GOE then the eigenvalues of H , which will be random variables themselves. Wishart [3] gave an explicit form for the distribution of these eigenvalues, which was an important step forward in the theory of random matrices. Wishart's work was subsequently developed by Wigner [2], who showed that when the dimension d grows to infinity, the distribution of each of the eigenvalues is given by the *semi-circular law*, which assigns probability

$$p_{sc}([a, b]) = \frac{2}{\pi} \int_a^b \sqrt{1 - x^2} dx$$

to an interval $[a, b] \subset [-1, 1]$; see Figure 6 to see where the name of this distribution comes from. In particular, this means that all the eigenvalues have the same distribution. Dyson [1] generalises the GOE in the same way as the Brownian motion generalises a Gaussian variable. We can define the *symmetric Brownian motion* to be the matrix process S_t whose entries are the independent Brownian motions

$$(S_t)_{ij} = \begin{cases} \frac{1}{\sqrt{d}} B_t^{ij} & \text{if } i \neq j, \\ \frac{\sqrt{2}}{\sqrt{d}} B_t^{ii} & \text{if } i = j, \end{cases}$$

rescaled inversely with the square root of the dimension d of the matrices B_t that form the process. Note also the rescaling by $\sqrt{2}$ along the diagonal, as in the GOE case shown above. We denote by $\Lambda_t = \text{Spec}(S_t)$ the process of eigenvalues of S_t (where we recall that each eigenvalue of a random matrix

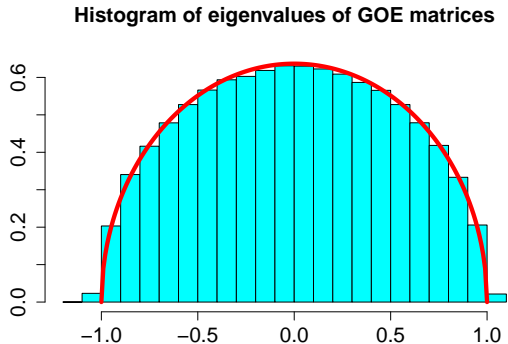


Figure 6: Simulation of eigenvalues from a GOE of dimension $d = 50$ (light blue) and semi-circular distribution (red).

is itself a random variable). Figure 7 shows how these eigenvalues evolve in time. The fascinating aspect of the dynamics of Λ_t is not only the fact that the eigenvalues do not cross each other, but they also repel each other when they are too close (like equal poles of two magnets). For example, let us have a closer look at the two upper curves in Figure 7 (the ones drawn in red and black). If we zoom into the graph, we see that the red curve always remains above the black curve, that is, they never intersect. Furthermore, when at some point the curves get very close, shortly after that they start to move away from each other again. If we inspect any other pair of neighbouring curves we will observe the same behaviour. Dyson [1] gave a rigorous mathematical description of these dynamics using the following model

$$P_0^{(i)} = 0,$$

$$P_{t+dt}^{(i)} = P_t^{(i)} + \sqrt{\frac{2dt}{d}} \cdot \mathcal{N}(0, 1) + \frac{1}{d} \sum_{j=1}^d \frac{1}{P_t^{(i)} - P_t^{(j)}}.$$

The process P_t describes a set of particles which follow the trajectory of independent Brownian motions under the effect of mutual repulsion. We call this model a *Dyson Brownian motion* or *Dyson Gas*. Figure 8 shows a simulation of the system. The particles are characterized by numbers on the real line, thus the gas formed of the ensemble of these particles is confined to a one dimensional space (the line). Notice the striking resemblance between Figures 7 and 8.

Dyson proved that the process Λ_t of the eigenvalues of S_t and the process P_t given by the Dyson gas have the same distribution. This is remarkable because

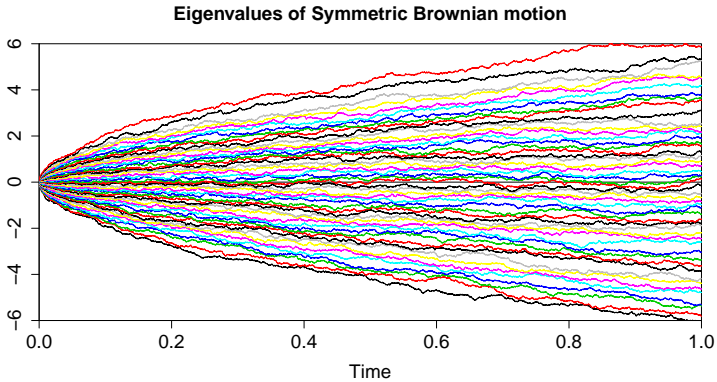


Figure 7: Eigenvalues of the symmetric Brownian motion with $d = 50$.

the symmetric Brownian motion S_t is a process of random matrices whose entries are independent Brownian motions, there is no interaction going on, the only constraint is the symmetry of the matrices. One might think the eigenvalues should follow independent Brownian motions themselves, maybe non-intersecting Brownian motions^[2] as in Figure 9, but that is not the case. The non-intersecting Brownian motions do not spread as much as the eigenvalues of the symmetric Brownian motion, both around zero and from each Brownian motion to the other (see Figure 9 and compare to Figure 7). What is then causing this unexpected behaviour? It is the symmetry structure, the emergence of the repulsion between the eigenvalues is an intrinsic consequence of the invariance under orthogonal transformations.

2 Wigner postulate and Dyson interpretation

The original motivation for the development of random matrix theory was atomic physics. In the 1950s, physicists were trying to develop a model which would accurately predict the energy spectrum of “heavy atoms”.

Atoms are constituted by a nucleus and one or several electrons, which are particles much smaller and lighter than the nucleus. The nucleus is the core

^[2] *Non-intersecting Brownian motions* are a system of N independent Brownian motions $B_1(t), \dots, B_N(t)$ indexed on a interval $[0, T]$ with the additional condition $B_1(t) < \dots < B_N(t)$ for all t in $[0, T]$. Non-intersecting Brownian motions can also be called *ordered Brownian motions* because the non-intersecting condition requires these Brownian motions to keep the same order for all t in $[0, T]$.

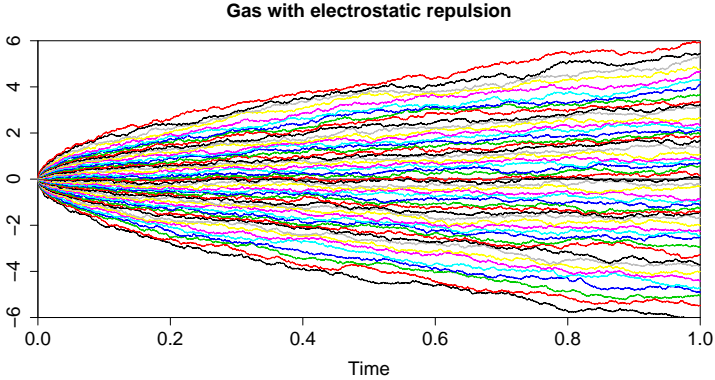


Figure 8: A simulation of the Dyson gas with $d = 50$ particles.

of the atom, it is stable and barely moves when the atom is at rest. The electrons surround the nucleus, and they constitute the most reactive part of the atom. For example, when particles of light (*photons*) hit an atom, these can be absorbed by it; when this happens, the energy of some of the atom's electrons increases by the exact amount of energy carried by the photons. The rules of quantum physics – which describe the behaviour of microscopic components of matter such as atoms – give only a few specific possible values for the energy levels of atoms. We say that the atomic energy levels are *discrete*. If the photon hits the atom with an energy that is not precisely the difference between two energy levels of the atom, then nothing actually happens. But if the energy of the photon is exactly this difference of energy, then it is absorbed by the atom and one electron of the atom will pick this difference in energy, causing the atom to reach a new energy state. The set of the possible energy levels of an atom is called its *spectrum*. The spectrum of the Hydrogen atom – the only atom with a single electron – can be computed analytically. However, for the other atoms (the *heavy atoms*), only approximate values can be obtained for their energy spectrum, and the more electrons that an atom has, the more difficult is this calculation, and the more imprecise are the values obtained.

Computing these spectra meant finding the eigenvalues of enormously large matrices (called “Hamiltonian matrices”), which was computationally impossible, only the lowest energy levels could be modelled with any accuracy in this way. Given the intractability of this problem, researchers then began to consider the statistical distribution of the energy levels. Wigner had the idea that the Hamiltonian matrices could be treated as arrays of random numbers with some symmetry properties, since the individual entries in the matrices seemed to

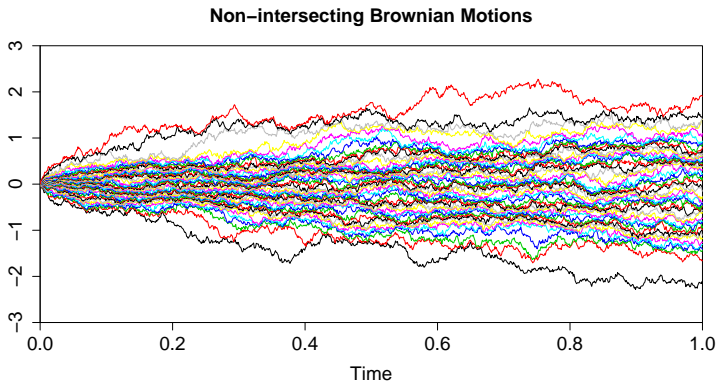


Figure 9: Simulation of $d = 50$ non-intersecting Brownian motions.

have no obvious correlations. As we have seen, simply imposing symmetry structure on random matrices is enough to produce the behaviour that is seen experimentally.

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References

- [1] F. J. Dyson, *A Brownian motion model for the eigenvalues of a random matrix*, *Journal of Mathematical Physics* **3**, **6** (1962), 1191–1198.
- [2] E. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions*, *Annals of Mathematics* **62**, **3** (1955), 548–564.
- [3] J. Wishart, *The generalised product moment distribution in samples from a normal multivariate population*, *Biometrika* **20A**, **1–2** (1928), 32–52.

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