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Tropical Geometry: new directions

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Abstract. The workshop *Tropical Geometry: New Directions* was devoted to a wide discussion and exchange of ideas between the leading experts representing various points of view on the subject, notably, to new phenomena that have opened themselves in the course of the last 4 years. This includes, in particular, refined enumerative geometry (using positive integer q-numbers instead of positive integer numbers), unexpected appearance of tropical curves in scaling limits of Abelian sandpile models, as well as a significant progress in more traditional areas of tropical research, such as tropical moduli spaces, tropical homology and tropical correspondence theorems.

Mathematics Subject Classification (2010): Primary 14-xx (algebraic geometry), Secondary 81Txx (quantum field theory, related classical field theories), 53Dxx (symplectic geometry, contact ge- ometry), 37-xx (dynamical systems and ergodic theory).

Introduction by the Organizers

The workshop *Tropical Geometry: New Directions*, organized by Ilia Itenberg (Paris), Hannah Markwig (Tübingen), Grigory Mikhalkin (Genève), and Eugenii Shustin (Tel Aviv), was held April 28th–May 5th, 2019.

The workshop was well attended by 50 participants from around the world. The program of the workshop consisted of 19 one-hour talks given by leading experts in the subject as well as 4 quarter an hour talks delivered by perspective young researchers. In addition, three informal discussions on open problems and on questions related to the main topics of the workshop were run during this week. Extended abstracts of the talks and reports on the discussions follow these introductory notes.

Tropical geometry has appeared as an ultimately successful approach to classical enumerative geometry, and up to date this direction remains very active and promising, though nowadays various other fields of mathematics and physics use methods and ideas born in tropical geometry and, on the other hand, suggest new challenging problems and research directions. The main goal of the workshop was to discuss and elaborate new directions in tropical geometry that have opened themselves in the course of the last years as well as new developments in related areas of mathematics which potentially may be linked to tropical geometry. We shortly comment on new and traditional trends in tropical geometry and on how they were reflected in the talks and discussions during the workshop.

Enumerative geometry is a traditional area of applications of tropical geometry, and it is still one of leading research directions revealing new challenging problems and perspective developments. It was closely addressed in the talks by H. Ruddat, R. Cavalieri, and L. Goettsche. In the talk by H. Ruddat the famous 2875 straight lines on a quintic threefold appeared on the other side of the mirror in the form of Lagrangian submanifolds diffeomorphic to 3-spheres and other graph-manifolds so that the total size of their first homology groups add up to 2875. This came as an illustration of the correspondence between tropical curves and Lagrangian submanifolds also presented in the talk by Diego Matessi. The tropical computation of psi-classes in moduli spaces of stable rational and elliptic marked curves as well as possible extension of these methods to higher genera were discussed in the talk by R. Cavalieri. The talk of L. Goettsche provided a detailed account of the refined count of algebraic and tropical curves. It contained both, several fascinating results in this topic and some conjectural relations to the geometry of moduli spaces of stable marked curves and their stable maps.

New connections to integrable systems, unexpected appearance of tropical curves in scaling limits of Abelian sandpile models, and the dynamics of tropical-like formations in more complicated models was the subject of the informal discussion led by N. Kalinin, M. Shkolnikov, and A. Sportiello. Exciting results linking these physical models with tropical geometry and further deep questions on the behavior of more general systems make this research direction very promising.

The core of most important applications of tropical geometry to the algebraic and symplectic geometry are correspondence relations between “classical” geometric objects and their tropical analogues. Various forms of such correspondences were reflected in several talks delivered during the workshop. In particular, I. Tyomkin presented a tropicalization and lifting procedures in the framework of Berkovich geometry, P. Bousseau showed how to compute Euler characteristics, Betti numbers and (intersection) Hodge numbers of moduli spaces of semistable vector bundles on the projective plane using relations to tropical geometry, D. Matessi considered Lagrangian fibrations of toric and Calabi-Yau varieties and analyzed the convergence of families of Lagrangian submanifolds to certain tropical limits. At last, I. Zharkov led a discussion on new enhanced tropical varieties that combine features of phase tropical varieties, complex amoebas and coamoebas, which potentially should reflect more geometry of their algebraic counterparts and might be more convenient in applications.

The classical-tropical correspondence relations yield important tasks to study geometry of tropical varieties towards new actual and potential applications to algebraic, symplectic and polyhedral geometry. Such problems were touched in the talks by K. Adiprasito, J. Rau, K. Shaw, M. Ulirsch, and D. Maclagan. A very interesting result was presented by K. Adiprasito, who addressed the problem of counting faces in polyhedral complexes from the point of view of a polyhedral version of the Lefschetz hyperplane section theorem largely influenced by tropical geometry. In turn, the talks by J. Rau and K. Shaw demonstrated applications of the tropical homology techniques for obtaining tropical analogues of Lefschetz theorems in classical algebraic geometry. In the talk by M. Ulirsch it was explained how to enhance the usual tropicalization of Riemann surfaces in order to catch the mapping class groups. D. Maclagan presented newly discovered connectivity properties of tropical varieties and their possible applications to polyhedral geometry.

Short communications of PhD students and postdocs Y. Ren, A. Gross, M. Hahn, and Y. Yamamoto demonstrated that a new generation of researchers actively works in various directions of tropical geometry, from computational aspects to delicate intrinsic geometry of tropical varieties in view of promising applications to classical geometry.

An integral part of the Oberwolfach workshops in tropical geometry has been formed by presentations of topics which are not elaborated by tropical geometry yet, but may designate perspectives for the future research. This time more than a third part of the program was scheduled for such presentations given in talks by E. Brugallé, X. Chen, P. Georgieva, T. Nishinou, A. Degtyarev, S. Finashin, O. Viro, and M. Temkin, and in an informal discussion led by V. Fock. The talks by E. Brugallé, X. Chen, and P. Georgieva were devoted the open Gromov-Witten/Welschinger theory, which is very natural application area for tropical geometry techniques as well as an important source of new ideas. The talks by A. Degtyarev, S. Finashin, and O. Viro exhibited new results and raised new questions in topology in real algebraic geometry, which has been an equally important source of new developments in tropical geometry. T. Nishinou presented new results in deformation theory, which seem to be quite interesting for establishing new correspondences between algebraic and tropical curves. The talk by M. Temkin was devoted to a new functorial algorithm of resolution of singularities, whose combinatorial nature definitely points to a potentially existing tropical version. An informal presentation by V. Fock brought together challenging relations between deformations of real singularities, cluster algebras and stable solutions of certain singular differential operators. None of this subjects has been well understood in the framework of tropical geometry, which makes the tasks to link this stuff with tropical geometry even more attractive.

We hope that the very intensive and substantial exchange of a broad spectrum of ideas during the workshop will stimulate the further research in the variety of discussed problems, which still are far from being completely settled.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Renzo Cavalieri in the "Simons Visiting Professors" program at the MFO.

Workshop: Tropical Geometry: new directions

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Abstracts

Generic intersection theory

Karim Adiprasito

The hard Lefschetz theorem, in almost all cases that we know, is connected to rigid algebro-geometric properties. Most often, it comes with a notion of an ample class, which not only induces the Lefschetz theorem but the induced bilinear form satisfies the Hodge-Riemann relations as well, which give us finer information about its signature (see for instance Voisin, CUP 2002).

Even in the few cases that we have the hard Lefschetz without the Hodge-Riemann relations, they are often at least conjecturally present in some form, as for instance in the case of Grothendieck's standard conjectures and Deligne's proof of the hard Lefschetz standard conjecture. This connection is deep and while we understand Lefschetz theorems even for singular varieties, to this day, we have no way to understand the Lefschetz theorem without such a rigid atmosphere for it to live in.

My goal and result in [A] is to provide a different criterion for varieties to satisfy the hard Lefschetz theorem that goes beyond positivity, and abandons the Hodge-Riemann relations entirely (but not the associated bilinear form); instead of finding Lefschetz elements in the ample cone of a variety, we give general position criteria for an element in the first cohomology group to be Lefschetz. The price I pay for this achievement is that the variety itself has to be sufficiently "generic".

For the current results I therefore turn to toric varieties, which allow for a sensible notion of genericity without sacrificing all properties of the variety, most importantly, without changing its Betti vector. Specifically, I consider varieties with a fixed equivariant cohomology ring, and allow variation over the Artinian reduction, i.e., the variation over the torus action. The main result can be summarized as follows:

Theorem 1. [A] *Consider a PL $(d-1)$-sphere Σ, and the associated graded commutative face ring $\mathbb{R}[\Sigma]$ (see Stanley, Birkhäuser Prog. in Math. 1996). Then there exists an open dense subset of the Artinian reductions $\mathcal{R}$ of $\mathbb{R}[\Sigma]$ and an open dense subset $\mathcal{L} \subset A^1(\Sigma)$, where $A(\Sigma) \in \mathcal{R}$, such that for every $k \leq \frac{d}{2}$, we have:*

(1) Generic Lefschetz theorem: *For every $A(\Sigma) \in \mathcal{R}$ and every $\ell \in \mathcal{L}$, we have an isomorphism*

$$A^k(\Sigma) \xrightarrow{\cdot \ell^{d-2k}} A^{d-k}(\Sigma).$$

(2) Hall-Laman relations: *The* Hodge-Riemann bilinear form

$$\begin{array}{rccccc} \mathrm{Q}_{\ell,k}: & A^k(\Sigma) & \times & A^k(\Sigma) & \longrightarrow & A^d(\Sigma) \cong \mathbb{R} \\ & a & & b & \longmapsto & \deg(ab\ell^{d-2k}) \end{array}$$

is nondegenerate when restricted to any squarefree monomial ideal in $A(\Sigma)$, as well as the annihilator of any squarefree monomial ideal.

The Lefschetz theorem is therefore as announced valid for generic Artinian reductions. In particular, the more algebrao-geometric reader may consult the following Corollary for easier visualization.

Corollary 1. *Consider $\mathfrak{F}$ a complete simplicial fan in $\mathbb{R}^d$. Then, after perturbing the rays of $\mathfrak{F}$ to a suitable rational fan $\mathfrak{F}'$, the Chow ring of the toric variety $X_{\mathfrak{F}'}$ satisfies the hard Lefschetz theorem with respect to a generic degree one element, while the equivariant Chow ring remains unchanged from $X_{\mathfrak{F}}$ to $X_{\mathfrak{F}'}$.*

These results have a myriad of consequences, among them:

(1) *g-conjecture, McMullen Isr. J. Math. 1971:* It proves that the f-vector, i.e. the number of vertices, edges, two-dimensional faces etc. of a simplicial sphere is also the f-vector of some simplicial polytope.
(2) *Grünbaum conjecture, J. Comb. Theor. 1970:* It generalizes a result of Déscartes: If Δ is a simplicial complex of dimension d that allows a PL embedding into $\mathbb{R}^{2d}$ then

$$f_d(\Delta) \ \leq \ (d+2) f_{d-1}(\Delta)$$

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Lefschetz $(1,1)$-theorem in tropical geometry

Johannes Rau
(joint work with Philipp Jell, Kristin Shaw)

The classical Lefschetz $(1,1)$-theorem gives a description of the cohomology classes of complex projective varieties which arise as Chern classes of complex line bundles. The theorem asserts that these classes are exactly the integral $(1,1)$-classes. It implies the Hodge conjecture (over $\mathbb{Z}$) for the degree 2 cohomology classes of a complex projective variety. In my talk, I will discuss analogous results for rational polyhedral and tropical spaces. These results are based on tropical homology groups and the tropical eigenwave operators ϕ, $\hat{\phi}$ introduced in [MZ14] and [IKMZ16].

In our setting, a rational polyhedral space is a topological space whose local models are of the form $Y \times \mathbb{T}^s$, with $Y \subset \mathbb{R}^n$ a rational polyhedral fan and $\mathbb{T} = [\infty, +\infty)$ the tropical affine line. (For technical reasons, we mostly assume the existence of a global polyhedral stratification of X). A tropical space (or sometimes tropical cycle) is a rational polyhedral space for which in addition each local fan Y satisfies the balancing condition. Finally, a tropical manifold is a tropical space if all local fans Y are degree one fans (equivalently, matroidal fans).

The tropical situation can be summarized in the following diagram:

$$\begin{array}{ccccccc}
\mathrm{CaDiv}(X) & \xrightarrow{\pi} & \mathrm{Pic}(X) & \xrightarrow{c_1} & \mathrm{H}^{1,1}(X,\mathbb{Z}) & \xrightarrow{\hat{\phi}} & \mathrm{H}^{0,2}(X,\mathbb{R}) \\
\Big\downarrow{\scriptstyle \mathrm{div}} & & & & \Big\downarrow{\scriptstyle \cap[X]} & & \Big\downarrow{\scriptstyle \cap[X]} \\
\mathrm{Z}_{n-1}(X) & & \xrightarrow{\quad\mathrm{cyc}\quad} & & \mathrm{H}^{\mathrm{BM}}_{n-1,n-1}(X,\mathbb{Z}) & \xrightarrow{\phi} & \mathrm{H}^{\mathrm{BM}}_{n,n-2}(X,\mathbb{R})
\end{array}$$

Note that the row maps exists for any rational polyhedral space, while the top-bottom maps are only defined in the case when X is a tropical space. Our main results, in decreasing level of generality, are as follows.

Theorem 1. *If X is a rational polyhedral space, then any $\alpha \in \mathrm{H}^{1,1}(X;\mathbb{Z})$ with $\hat{\phi}(\alpha) = 0$ is the first Chern class of a tropical line bundle,*

$$\mathrm{im}(c_1) = \ker(\hat{\phi}).$$

Theorem 2. *If X is a tropical space, then for any $\alpha \in \mathrm{H}^{1,1}(X;\mathbb{Z})$ with $\hat{\phi}(\alpha) = 0$, the class $\alpha \cap [X] \in \mathrm{H}^{\mathrm{BM}}_{n-1,n-1}(X,\mathbb{Z})$ is the fundamental class of a codimension one tropical cycle in X,*

$$\ker \hat{\phi} \cap [X] \subset \mathrm{im}(\mathrm{cyc}).$$

Theorem 3. *If X is a tropical manifold X, then any $\alpha \in \mathrm{H}^{\mathrm{BM}}_{n-1,n-1}(X,\mathbb{Z})$ with $\hat{\phi}(\alpha) = 0$ is the fundamental class of a codimension one tropical cycle in X,*

$$\mathrm{im}(\mathrm{cyc}) = \ker(\phi).$$

The tropical (co)homology groups (and their Borel-Moore variants) are denoted by $\mathrm{H}^{p,q}(X,Q)$ and $\mathrm{H}^{\mathrm{BM}}_{p,q}(X,Q)$. The maps ϕ and $\hat{\phi}$ are called tropical eigenwave operators. They have a purely combinatorial definition, but are related to Hodge-theoretic monodromy operators in the case of tropicalisations. By $\mathrm{Pic}(X)$ and $\mathrm{CaDiv}(X)$ we denote the groups of line bundles and line bundles with (rational) sections, respectively. The Chern class map c_1 is induced by the exact sequence of sheaves

$$0 \to \underline{\mathbb{R}} \to \mathrm{Aff}_{\mathbb{Z}} \to T^*X \to 0 \tag{1}$$

from constant functions to affine $\mathbb{Z}$-linear functions to covectors. The map cyc associates to a tropical subspace Y its fundamental class $\mathrm{cyc}(Y) = [Y]$, and $\cap[X]$ denotes the cap product with the fundamental class of X. Finally, the tropical divisor construction div associates to a rational section of a line bundle a tropical cycle of codimension one. For details, see [MZ14, IKMZ16, AR10, MR]. I will briefly present the necessary definitions and deduce the main theorems from the following key statements about the above diagram.

Theorem 4.

(1) *Up to sign, the tropical eigenwave $\hat{\phi}$ coincides with the coboundary map of (1).*
(2) *Every tropical line bundle admits a rational section. Equivalently, the map π is surjective.*

(3) *If X is a tropical space, then the diagram commutes.*
(4) *If X is a tropical manifold, then Poincaré duality over $\mathbb{Z}$ holds (i.e., the maps $\cap[X] : \mathrm{H}^{p,q}(X,\mathbb{Z}) \to \mathrm{H}^{\mathrm{BM}}_{n-p,n-q}(X,\mathbb{Z})$ are isomorphisms).*

Finally, we briefly discuss two applications/examples.

Theorem 5. *Let $X = \mathbb{R}^n/\Lambda$ be a tropical torus and $\alpha = \sum a_{ij} e_i^* e_j^* \in \mathrm{H}^{1,1}(X,\mathbb{Z}) = (\mathbb{Z}^n)^* \otimes \Lambda^*$. Then α can be represented by a tropical cycle if and only if (a_{ij}) is symmetric. If, in addition, (a_{ij}) is positive definite, then α can be represented by an effective cycle, namely the associated tropical theta divisor.*

Theorem 6. *For every $1 \leq \rho \leq 19$ there exists a smooth tropical quartic surface with Picard rank ρ. Moreover, such surfaces can be chosen to have the same combinatorial type.*

Open questions in this context are: Can the positivitiy criterion in Theorem 5 be inverted? Are there tropical counterparts to the Nakai–Moishezon/ Kleiman criteria? What are criteria for combinatorial types of hypersurfaces in order to admit a large/small range of Picard ranks?

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Towards a theory of tropical ψ classes in genus one.

Renzo Cavalieri

(joint work with Andreas Gross, Hannah Markwig)

The class ψ_i is a degree one Chow class on the moduli space of curves defined as the first Chern class of the i-th tautological (or cotangent) line bundle. These classes control the non-transversal intersections of boundary strata, but besides being important classes for the geometry of the moduli spaces of curves, they were brought to the spotlight by Witten's conjecture [5] (now Kontsevich's theorem), which predicts that a generating function for top intersections of ψ classes is a τ-function for the KdV hierarchy. A somewhat simplistic interpretation of such statement is that the intersection theory of ψ classes is highly combinatorial and related to the boundary stratification of the moduli spaces of curves. It is therefore unsurprising that we would like to have a parallel theory of tropical ψ classes, together with some correspondence theorems.

Such a theory of tropical ψ classes exists in genus zero. It was initially envisioned and described by Mikhalkin in [4], and further developed in [3]. It is simultaneously a purely combinatorial theory, and a mirror of the classical theory via the "operational"/toric intersection theory point of view. By this we mean

$$\prod (\psi_i^{trop})^{k_i} = \sum \left(\prod \psi_i^{k_i} \cdot \Delta_\tau \right) \tau, \tag{1}$$

where τ ranges among all cones of $M_{0,n}^{trop}$ and Δ_τ denotes the stratum in the algebraic moduli space which is dual to the cone τ.

What makes everything work in genus 0 is that $M_{0,n}^{trop}$ is naturally embedded in a real vector space as a balanced fan, and one can use tools of toric/tropical intersection theory [2]. For higher genus, such an embedding is not possible. At a very basic level, $M_{1,1}^{trop}$ consists of a single ray, which as such may not possibly become a balanced fan. This talk describes ongoing joint work with Andreas Gross and Hannah Markwig that aims at developing and computing a theory of ψ classes in genus 1.

We wish to exploit the fact that moduli space of tropical curves have been given a stack structure ([1]) that makes them into fine moduli spaces. In algebraic geometry, any geometric concept on a stack may be defined by looking at all of its pull-backs from representable maps from schemes to the stack. The technical heart of this project consists in accurately identifying the appropriate notion of representable morphism of tropical objects. In our specific case, we are concerned with maps:

$$f : C \to M_{1,n}^{trop},$$

where C belongs to a class $\mathcal{C}$ of well-behaved tropical objects, and f should be representable in the sense that the family X_f should also belong to the class $\mathcal{C}$.

As a **minimal requirement** we would like to make sure that our definitions include **realizable** families of tropical stable maps to toric targets, where the base of the family may need to be given a balanced structure as additional information.We also want the following forgetful morphisms to be representable:

$$M_{1,n}^{trop}(\mathbb{R}^N, \Delta) \to M_{1,n+|\Delta|}^{trop}$$

and

$$M_{1,n+1}^{trop} \to M_{1,n}^{trop}.$$

Then tropical ψ classes may be defined as follows: for any diagram where f is a "good" map,

$$\begin{array}{ccc} \Gamma & & \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle \pi} & & \\ C & \xrightarrow{\;f\;} & M_{1,n}^{trop} \end{array}$$

define

$$f^*\psi := -\pi_\star(Im(s)^2), \tag{2}$$

where s denotes the section corresponding to the i-th mark.

This prescription will define ψ on the stack. Further, after showing compatibility with fiber products (and therefore that the class is well-defined) it should be sufficient to describe $f^*\psi$

- for one good atlas (i.e. f surjective), OR
- for enough curves mapping into the stack,

to completely determine ψ.

Once we have defined tropical ψ classes this way, similarly we wish to define intersection cycles by:

$$f^\star\psi^I := (f^\star\psi)^I$$

(the RHS of this equation lives on C where we know how to do intersection theory).

In order to compute the theory, we envision a two step approach: first we compute the degree of the class ψ_1 on $M_{1,1}^{trop}$ to be $\frac{1}{24}[v]$. We choose as an atlas for $M_{1,1}^{trop}$ a one-dimensional family of realizable tropical stable maps of degree three to the tropical projective plane, passing through 8 fixed general points:

$$[\gamma_f] = \prod_{i=1}^{8} ev_i^*(P_i)_{|[\mathcal{R}M_{1,8}^{trop}(\mathbb{TP}^2,3)]} \in A_1(M_{1,8}^{trop}(\mathbb{TP}^2,3)). \tag{3}$$

(here we are denoting by $\mathcal{R}M_{1,8}^{trop}(\mathbb{TP}^2,3)]$ the representable locus in the moduli space of tropical stable maps, which is thought somehow like a virtual class).

The above cycle gives rise to a map

$$f : T \to M_{1,1}^{trop}, \tag{4}$$

where T is a trivalent tree endowed with the standard planar balanced local structure. We studied in detail the map f arising when we choose the points P_i to be in horizontally stretched position, and computed the degree of the map to be 12 and the self intersection of a section to be -1, thus recovering via a purely tropical intersection theoretic computation the degree of ψ_1 to be $1/24$ (It is $1/12$ the class of a point on the ray of $M_{1,1}^{trop}$, which carries a $B\mu_2$ structure.)

To compute the theory for more points and in higher codimension, we do not want to rely on explicit computations, but rather on the development of tatutological relations. In fact it would suffice to establish the tropical version of:

pull-back: the pull-back relation

$$\psi_i = \pi_{n+1}^*(\psi_i) + S_i,$$

where S_i is a suitable version of a cycle arising from the i-th section.

string:

$$\pi_{n+1,*}\prod_{i=1}^{n}\psi_i^{k_i} = \sum_{j=1}^{n}\psi_j^{k_j-1}\prod_{i\neq j}\psi_i^{k_i}.$$

dilation:

$$\pi_{n+1,*}\left(\psi_{n+1}\prod_{i=1}^{n}\psi_i^{k_i}\right) = n\prod_{i=1}^{n}\psi_i^{k_i}.$$

The resulting theory should be an operational theory which is very similar to the genus zero theory. The operational class ψ_i should be expressed as a tropical cycle as the sum of all cones where the i-th points lies on a four-valent, genus zero vertex.

Once the program is completed, we expect to recover a theory which corresponds to the classical one in the sense of equation (1). The main contribution of this work is to make such a theory arise from purely tropical intersection theoretic constructions, as opposed to just being a definition.

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On the invariance of Welschinger invariants

Erwan Brugallé

A *real symplectic manifold* $X_{\mathbb{R}} = (X, \omega_X, \tau_X)$ is a symplectic manifold (X, ω_X) equipped with an anti-symplectic involution τ_X. The *real part* of (X, ω_X, τ_X), denoted by $\mathbb{R}X$, is by definition the fixed point set of τ_X. An almost complex structure J on X is called *τ_X-compatible* if it is tamed by ω, and if τ_X is J-anti-holomorphic. In what follows, the manifold $X_{\mathbb{R}}$ will always be compact of dimension 4 with a non-empty real part, and rational. We denote by $H_2^{-\tau_X}(X;\mathbb{Z})$ the space of τ_X-anti-invariant classes. A non-singular projective real algebraic variety is always implicitly assumed to be equipped with some Kähler form which turns it into a real symplectic manifold. All algebraic surfaces considered here are assumed to be projective and non-singular.

Let $X_{\mathbb{R}} = (X, \omega_X, \tau_X)$ be a real rational compact symplectic manifold of dimension 4, and denote by $L_1, \cdots, L_k$ the connected components of $\mathbb{R}X$. Choose a class $d \in H_2(X;\mathbb{Z})$, and a vector $\rho = (r_1, \cdots, r_k) \in \mathbb{Z}^k_{\geq 0}$ such that

$$c_1(X)\cdot d - 1 - \sum_{i=1}^{k} r_i = 2s \in 2\mathbb{Z}_{\geq 0}.$$

Choose a configuration $\underline{x}$ made of r_i points in L_i for $i = 1, \cdots k$, and s pairs of τ_X-conjugated points in $X \setminus \mathbb{R}X$. Given a τ_X-compatible almost complex structure J, we denote by $\mathcal{C}_{X_\mathbb{R}}(d, \underline{x}, J)$ the set of real rational J-holomorphic curves in X realizing the class d, and passing through $\underline{x}$. Then we define the integer

$$W_{X_\mathbb{R},\rho}(d; s) = \sum_{C \in \mathcal{C}_{X_\mathbb{R}}(d,\underline{x},J)} (-1)^{m(C)},$$

where $m(C)$ is the number of nodes of C in $\mathbb{R}X$ with two τ_X-conjugated branches. For a generic choice of J, the set $\mathcal{C}_{X_\mathbb{R}}(d, \underline{x}, J)$ is finite, and $W_{X_\mathbb{R},\rho}(d; s)$ depends neither on $\underline{x}$, J, nor on the deformation class of $X_\mathbb{R}$ (see [Wel05, Wel15]). We call these numbers the *Welschinger invariants of* $X_\mathbb{R}$. Our main result, Theorem 1 below, is that when $X_\mathbb{R}$ is a real rational algebraic surface, Welschinger invariants eventually only depends on s and some homological data of $X_\mathbb{R}$.

Two real rational algebraic surfaces $X_{1,\mathbb{R}}$ and $X_{2,\mathbb{R}}$ are said to be *homologically equivalent* if both are obtained, up to deformation, as a real blow-up $\pi_i : X_{i,\mathbb{R}} \to X_{0,\mathbb{R}}$ of a real minimal algebraic surface $X_{0,\mathbb{R}}$ at p distinct real points and q distinct pairs of τ_{X_0}-conjugated points. We emphasize that the distributions of the p real points among connected components of $\mathbb{R}X_0$ may not coincide for π_1 and π_2. Note nevertheless that

$$\chi(\mathbb{R}X_1) = \chi(\mathbb{R}X_2) = \chi(\mathbb{R}X_0) - p.$$

Furthermore, the two maps π_1 and π_2 provide an identification of the groups $H_2(X_1; \mathbb{Z})$ and $H_2(X_2; \mathbb{Z})$ commuting with both intersection forms and action of the anti-symplectic involutions. We denote by $[X_\mathbb{R}]$ the homological equivalence class of a real rational algebraic surface $X_\mathbb{R}$.

Theorem 1. *If $X_\mathbb{R}$ is a real rational algebraic surface, then $W_{X_\mathbb{R},\rho}(d; s)$ does not depend on ρ, nor on a particular representative of $[X_\mathbb{R}]$.*

As a consequence of Theorem 1, we simply denote by $W_{[X_\mathbb{R}]}(d; s)$ the invariant $W_{X_\mathbb{R},\rho}(d; s)$.

Remark 1. *Loosely speaking, Theorem 1 states that $W_{X_\mathbb{R},\rho}(d; s)$ only depends on s and the lattice $H_2(X; \mathbb{Z})$ equipped with the intersection form and the action of $\tau_{X,*}$. It may be interesting to work this out more rigorously. It may also be interesting to study generalizations of Theorem 1 to modified Welschinger invariants introduced in* [IKS13]*, as well as to to higher genus Welschinger invariants introduced in* [Shu14]*, or to the higher dimensional invariants recently defined in* [Geo16, GZ15].

Theorem 1 easily implies next corollary, which generalizes [BP15, Theorem 1.1(1)] in the case $F = [\mathbb{R}X_\mathbb{R} \setminus L]$.

Corollary 1. *Let $X_\mathbb{R}$ be a compact real rational algebraic surface with a disconnected real part. Suppose that $X_\mathbb{R}$ is a real blow-up of another real rational algebraic surface in at least two real points, and denote by E_1 and E_2 the corresponding exceptional divisors. Then for any $d \in H_2(X; \mathbb{Z})$ such that both $d \cdot [E_1]$ and $d \cdot [E_2]$ are odd, one has $W_{[X_\mathbb{R}]}(d; s) = 0$.*

Combining Theorem 1 with [Wel07, Theorem 1.1], we obtain the following.

Theorem 2. *Let $X_\mathbb{R}$ be a compact real rational algebraic surface with a disconnected real part, and assume that $c_1(X)\cdot d - 1 - 2s > 0$. Then one has*

$$(-1)^{\frac{d^2-c_1(X)\cdot d+2}{2}} \cdot W_{[X_\mathbb{R}]}(d;s) \geq 0.$$

Furthermore, the invariant $W_{[X_\mathbb{R}]}(d;s)$ is sharp in the following sense: there exists a compact real rational algebraic surface $Y_\mathbb{R}$ in $[X_\mathbb{R}]$, a real configuration $\underline{x}$ of $c_1(X)\cdot d-1$ points in Y with $|\underline{x}\cap\mathbb{R}Y| = c_1(Y)\cdot d-2s$, and a generic τ_Y-compatible almost complex structure J on Y such that

$$Card(\mathcal{C}_{Y_\mathbb{R}}(d,\underline{x},J)) = |W_{[X_\mathbb{R}]}(d;s)|.$$

Remark 2. *A configuration $\underline{x}$ and a τ_Y-compatible almost complex structure J as in Theorem 2 may not exist for any representative $Y_\mathbb{R}$ of $[X_\mathbb{R}]$, even up to deformation, see* [Bru15, Remark 6.13]*.*

One of the main ingredients in our proof of Theorem 1 is a formula relating Welschinger invariants of two real symplectic 4-manifolds differing by a so-called *surgery along a real Lagrangian sphere*. This latter formula partially generalizes both [IKS15, Corollary 4.2] and [Bru18, Theorem 1.1, Remark 1.3]. We point out that our proof is an easy adaptation of the proof of [IKS15, Corollary 4.2], using [BP15, Theorem 2.5(1)]. It just required to believe in the correctness of the statement to prove it.

In its turn, this formula is obtained thanks to a real version of a (very simple instance) of the symplectic sum formula. It turns out that the same strategy provides a formula similar to Theorem 1 for relative Gromov-Witten invariants of symplectic 4-manifolds. This observation suggests a possible connection of our work to tropical refined invariants defined in [BG16, GS16]. In particular, we provide an alternative explanation for the specializations in $q = \pm 1$ of the tropical refined descendant invariants from [GS16]. We also show that a refined version of a conjecture by Itenberg, Kharlamov and Shustin [IKS04, Conjecture 6] holds, although it was known to be wrong in the non-refined case.

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Clusters and singularities

Vladimir Fock

This informal discussion brought together challenging relations between deformations of real singularities, cluster algebras and stable solutions of certain singular differential operators, coming from deformations of plane curve singularities. These relations shed a new light on the conjectural equivalence between the topological classifications of plane curve singularities and mutational equivalence of quivers recently pointed by S. Fomin et al.

From algebraic curves to tropical curves to Lagrangian submanifolds

Helge Ruddat
(joint work with Travis Mandel, Cheuk Yu Mak)

Abstract

A quintic threefold is the hypersurface in complex projective 4-space cut out by a homogeneous polynomial equation of degree 5. It is the most famous Calabi-Yau manifold and the source of many interesting discoveries in pure mathematics and mathematical physics. A general quintic threefold contains 2875 straight lines. Katz studied where these lines move when one deforms the quintic into the union of the coordinate hyperplanes of projective 4-space. From here, for each line contained in a hyperplane, we may map it under the component-wise logarithm map $(\mathcal{C}\setminus\{0\})^3 \to \mathbb{R}^3$ and find that the spine of its image is what is called a tropical line, a piecewise linear graph. Each of these tropical lines can in turn be used to construct a Waldhausen graph manifold fibering in 2-tori over the tropical curve. These graph manifolds embed as Lagrangian submanifolds in another Calabi-Yau manifold, the mirror dual of the quintic! They have many interesting properties.

1. Tropical lines in 3-space and their multiplicity

Consider a tropical tropical line γ in $\mathbb{R}^3$ with unbounded legs in the directions of the rays of the fan for $\mathbb{P}^3$, i.e. the standard basis vector directions e_1, e_2, e_3 and $e_0 := -e_1 - e_2 - e_3$. These lines move in a four-dimensional parameter space: translations give an $\mathbb{R}^3$ and, additionally, we can scale the length of the compact edge in γ. Let $A_i \subset \mathbb{R}^3$ be an affine two-plane in general position with the property that A_i contains a translate of $\mathbb{R}e_i$. Requiring for γ that the unbounded leg with direction e_i be contained in A_i uniquely fixes the parameters, so there is a unique tropical line γ with this property. We say γ is *rigid*. We assume that the tangent space of A_i is integrally generated, that is, if we identify $\mathbb{R}^3 = N \otimes_{\mathbb{Z}} \mathbb{R}$ for $N = \mathbb{Z}^3$ and set $M = \mathrm{Hom}(N, \mathbb{Z})$, then A_i is a translate of $m_i^\perp$ for some $m_i \in M$ and $i = 0, ..., 4$.

1.1. **Siebert-Nishinou multiplicity of γ.** From the correspondence theory of tropical curves with log Gromov-Witten invariants [3] (pioneered in [7, 8]), one associates the Siebert-Nishinou multiplicity to the line γ as follows. We consider the homomorphism of $\mathbb{Z}$-modules

$$\Phi : N \oplus N \to N/\mathbb{Z}u_e \oplus N/m_0^\perp \oplus ... \oplus N/m_4^\perp$$

where $u_e \in N$ is a primitive generator of the tangent space to the compact edge e of γ. If the vertices v_1, v_2 of e have the unbounded legs spanned by e_0, e_1 and e_2, e_3 attached, respectively, as in the above figure, then Φ is given by

$$\Phi : (a, b) \mapsto (a - b, a, a, b, b).$$

The *Siebert-Nishinou multiplicity* $\mathrm{mult}(\gamma)$ of γ is defined by

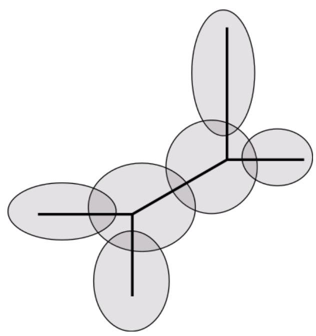
$$\mathrm{mult}(\gamma) = |\,\mathrm{coker}(\Phi)|$$

and is easily seen to depend on the choice of m_i.

1.2. **New interpretation for** $\mathrm{mult}(\gamma)$ **via the first homology of a 3-manifold.** The Strominger-Yau-Zaslow conjecture suggests that a *mirror symmetry pair* $X, \check{X}$ arises as fibrations $X \to B \leftarrow \check{X}$ over an affine base space B. An algebraic curve C in X projects to an amoeba in B whose spine is a tropical curve γ in the base B from which the mirror dual object to $\check{C}$ shall be constructed, a Lagrangian $L_\gamma \subset \check{X}$. Given a tropical line $\gamma \subset \mathbb{R}^3$ as above, we now propose a topological model for L_γ based on the construction in [9]. Let $\check{X}$ denote the quotient $T^*_B/\check{\Lambda}$ of the cotangent bundle T^*_B by the local system $\check{\Lambda}$ of integral cotangent vectors, so the stalks of $\check{\Lambda}$ are isomorphic to $\mathbb{Z}^3$. In this particular situation, the cotangent bundle is trivial, so we may identify

$$\check{X} = T^*_B/\check{\Lambda} = \underbrace{\mathbb{R}^3}_{B} \times \mathbb{R}^3/\mathbb{Z}^3.$$

To each edge e of γ, we consider the co-normal plane $e^\perp \subset \mathbb{R}^3$, i.e. the set of cotangent vectors that evaluate to zero on tangent vectors to e. The *co-normal torus* is the quotient 2-torus $T_e := e^\perp/(e^\perp \cap \check{\Lambda})$ and we obtain a Lagrangian 3-fold $L_e := e \times T_e \subset \check{X}$ where we view $\check{X}$ as a symplectic manifold with symplectic form inherited from the standard symplectic form of the cotangent bundle. The goal is to glue the disjoint union of all L_e to obtain a closed 3-manifold L_γ (a priori no longer as embedded in $\check{X}$). For a vertex $v \in \gamma$, we want to connect the three L_e for e the edges that meet v. Indeed, as a connecting patch, we may glue in the three-fold $H \times S^1$ for H a 3-punctured sphere. This can be done entirely inside the fiber F of $\check{X} \to B$ over the point v because the sum of the three tori T_e for e an edge meeting v is zero in $H_2(Z, \mathbb{Z})$ by the balancing condition of γ at v, hence indeed a filling $H \times S^1$ at v exists, topologically. Finally, we want to cap off the remaining open endings (for the unbounded edges of γ) by solid tori. A solid torus is the 3-manifold $D \times S^1$ where D denotes a two-dimensional disk. The boundary of a solid torus is $T_e \cong (S^1)^2$ and to say how this boundary gets filled by the solid torus is to say which (of the many) homology classes of circles in T_e becomes the meridian of the solid torus, i.e. the boundary of D. Any choice produces a closed 3-manifold but the resulting topological type is sensitive to the choice, so we want to be more explicit here. This is where the incidence planes A_i come into play. Note that $H_1(T_{e_i}, \mathbb{Z}) = e_i{}^\perp \cap \check{\Lambda}$, so m_i gives a non-trivial element and hence determines a homology class of a circle in T_{e_i}. We choose the class given by m_i to become the meridian of the to-be-glued-in solid torus to obtain a three-manifold L_γ unique up to diffeomorphism. Consider the cover of γ by open sets U_α displayed on the right.



Let $\pi : L_\gamma \to \gamma$ be the projection (adding vertices at infinity to the unbounded edges of γ to make this work). From a Čech complex consideration for this cover, one deduces

$$H^2(L_\gamma) = \operatorname{coker}\left(\bigoplus_{\alpha} H^1\big(\pi^{-1}(U_\alpha)\big) \to \bigoplus_{\alpha<\beta} H^1\big(\pi^{-1}(U_\alpha \cap U_\beta)\big)\right) = \operatorname{coker}(\Phi)$$

where Φ is the map given §1.1, so using the universal coefficient theorem, we conclude the following result

Theorem 1 (R.-Mak [2])**.**

$$\operatorname{mult}(\gamma) = |H_1(L_\gamma, \mathbb{Z})_{\mathrm{tor}}|$$

Here, the decoration tor refers to the torsion subgroup which can be omitted in this particular example because L_γ is a homology sphere, so H_1 consists of torsion only (no free part). For a Lagrangian homology sphere L, Joyce called the quantity $|H_1(L, \mathbb{Z})|$ the *weight* $w(L)$ of the Lagrangian L, conjecturing this to be the correct multiplicity when counting special Lagrangian spherical objects.

1.3. **Realizing L_γ as a Lagrangian submanifold.** For suitable choices of A_i, we want to embed L_γ as a Lagrangian submanifold in the mirror quintic threefold $\check{X}$, a symplectic 6-manifold. Similar ideas have been followed in [6, 5]. To see how, we first go back to the mirror dual side: to curves C in the quintic X; more precisely *lines* on the quintic. Recall that γ above is also a line and we want to view it as the tropicalization of a complex line on the quintic. Sheldon Katz [1] studied lines on the quintic threefold by considering the degeneration

$$tf_5(z_0, ..., z_4) = z_0 z_1 z_2 z_3 z_4$$

where f_5 is the equation of a general quintic hypersurface and t is the pencil parameter. For $t = 0$, the quintic degenerates to the union of coordinate hyperplanes X_0. The general quintic contains 2875 lines and Katz determined this number as follows. First note that X_0 contains ten plane quintic curves Z_{ij}, namely as the intersection of $\{f_5 = 0\}$ with each of the ten coordinate 2-planes $z_i = z_j = 0$. The degenerate quintic X_0 obviously contains infinitely many lines but only some of them deform into the nearby fibers when making t non-zero. The condition for a line $\ell \subset X_0$ to deform to $t \neq 0$ is that it meets four of the ten plane quintics Z_{ij} but is disjoint from any coordinate $\mathbb{P}^1$ given by $z_i = z_j = z_k = 0$ for i, j, k pairwise distinct. So if ℓ is contained in the component $\mathbb{P}^3 \cong \{z_0 = 0\}$ of X_0 then it needs to meet the four plane quintics contained in this component and not meet the coordinate lines inside. By a simple inclusion-exclusion argument combined with the knowledge that $2 \cdot 5^4$ lines meet four quintics in $\mathbb{P}^4$, one comes up with the number of 575 lines in each $\mathbb{P}^3$-component of X_0 that deforms to $t \neq 0$. This yields a total of $5 \cdot 575 = 2875$ lines as claimed above.

Tropicalizing the situation means looking for tropical lines in $\mathbb{R}^3$ so that its four legs lie on four tropical quintics (at tropical infinity). Assuming the tropical quintics to be generically perturbed, all tropical lines can be expected to meet only edges but not vertices of the quintics. Indeed, after removing garbage lines that are non-rigid, one finds a count of 575 tropical lines and most interestingly, this is a count with multiplicities and lines of multiplicity 1 *and* 2 both occur. This also means that the actual count of lines is slightly smaller than 575 since each multiplicity two line counts as two obviously. The multiplicity of each line γ is computed by the methods introduced in the previous sections, that is we have incidence conditions $A_0, ..., A_3$ where A_i is the unique affine two-plane spanned by the corresponding leg e_i of γ and the edge of the tropical quintic that is met by this leg. The lines were found with a computer search. Taking into account the tropicalizations of all five components of X_0, the total tropical count with multiplicity is

$$2875 = 2695 + 90 \cdot 2,$$

i.e. 2695 lines of multiplicity one and 90 lines of multiplicity two. In terms of the three-manifold of §1.2, the multiplicity one lines γ give $L_\gamma \cong S^3$ and the multiplicity two lines γ give $L_\gamma \cong \mathbb{RP}^3$. A tropical line is *admissible* if it meets only compact edges of the plane quintics. This holds for a bit more than half of the found tropical lines (precisely for 45 multiplicity two lines).

Theorem 2 (R.-Mak [2]). *For each admissible tropical line γ meeting four tropical quintics as described above, there is a Lagrangian L_γ of the diffeomorphic type given in* §1.2 *embedded in the mirror quintic $\check{X}$. If two such tropical lines are disjoint, so are the associated Lagrangians.*

Interestingly, many tropical lines meet one another; the incidence matrix is full rank; each tropical line meets at least one other. However, still more than 350 disjoint Lagrangians can be obtained by the theorem. All these Lagrangians are *spherical* in the sense of having the Betti numbers of a 3-sphere and these can be used to make Dehn-twists along them. Furthermore, one can show they are all homologous. We hence found an Abelian subgroup of rank > 350 in the symplectic automorphism group of the mirror quintic $\check{X}$.

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Tropical Lagrangian sumbanifolds inside toric and Calabi-Yau varieties

Diego Matessi

In this talk we describe a construction of Lagrangian submanifolds which lift tropical subvarieties in the base of a Lagrangian fibration on a symplectic manifold. We start with the cotangent bundle of a real torus $T = V/\Lambda$, where V is a real n-dimensional vector space and $\Lambda \cong \mathbb{Z}^n$ an n-dimensional sublattice. The cotangent bundle is $T^*T = V^* \times V/\Lambda$. The projection map $\pi : T^*T \to V^*$ is a Lagrangian torus fibration with respect to the canonical symplectic form on T^*T.

Let us now consider a d dimensional tropical subvariety Γ of V^*. To a first approximation we can lift Γ to a piecewise linear (PL) Lagrangian lift $\hat{\Gamma}_{PL} \subset T^*T$ of Γ. The projection map π restricts to a map $\hat{\Gamma}_{PL} \to \Gamma$. The shape of the fibres F_b of this map over a point $b \in \Gamma$ depend on the dimension of the smallest stratum containing b. If b lies only in the top dimensional stratum of Γ, i.e. it is a smooth point of Γ, then F_b is just the conormal bundle

$$F_b = \frac{(T_b\Gamma)^\perp}{(T_b\Gamma)^\perp \cap \Lambda}.$$

Notice that F_b is isomorphic to an $(n-d)$-dimensional torus. When b lies in smaller dimensional strata, the fibres F_b are certain simplicial chains whose boundaries match the fibers of adjacent larger strata. We call such fibres Lagrangian coamebas and they exist thanks to the balancing conditions of Γ. Under certain conditions, e.g. if Γ is smooth, these coamebas can be chosen so that $\hat{\Gamma}_{PL}$ is a topological submanifold of T^*T. Moreover $\hat{\Gamma}_{PL}$ is Lagrangian at smooth points. In [1, 2] we proved the following

Theorem 1. *If the torus T is 2 or 3 dimensional and Γ is a smooth tropical hypersurface of V^*, then there exists a one parameter family L_t of smooth Lagrangian submanifolds of T^*T such that*

a) *L_t is homeomorphic to $\hat{\Gamma}_{PL}$;*
b) *L_t converges to $\hat{\Gamma}_{PL}$ in the Hausdorff topology as t goes to zero.*

Mikhalkin [3] also proved a similar result for tropical curves in any codimension and Mak and Ruddat [4] have a similar construction of Lagrangian submanifolds lifting tropical curves in the mirror of the quintic Calabi-Yau.

There are various interesting examples of tropical Lagrangians inside toric varieties. When a tropical hypersurface is considered inside the moment polytope of a toric variety, one needs to check what happens to the Lagrangian lift as the tropical variety hits the boundary of the polytope. Does it compactify nicely to a smooth Lagrangian submanifold without boundary? In general this is rare, for instance, in two dimensional toric varieties this happens only when the tropical curve hits a corner along its bisectrice or an edge along a multiplicity two direction (see [3]). In the case of a tropical surface inside a three dimensional polytope, a delicate technical issue it to check what happens in the case of what we may call a "trisectrice" hitting a corner. A "trisectrice" is the tropical hypersurface inside $(\mathbb{R}_{\geq 0})^3$ traced out by the union of the three positive coordinate axes as we translate it along the ray with direction $(1, 1, 1)$. We believe that the Lagrangian lift of the trisectrice compactifies smoothly inside $\mathbb{C}^3$ and we will write up a proof in forthcoming work. Using this construction we can produce a monotone tropical Lagrangian in $\mathbb{P}^3$ diffeomorphic to $S^1 \times S^2$.

One of the main motivations for our construction comes from mirror symmetry. Indeed the Strominger-Yau-Zaslow conjecture claims that a pair X and $\check{X}$ of mirror Calabi-Yau manifolds should admit dual (special)-Lagrangian fibrations $f : X \to B$ and $\check{f} : \check{X} \to B$ over the same base B. If $\Delta \subset B$ is the discriminant locus of f, the set $B_0 = B - \Delta$ has an integral affine structure, therefore it makes sense to speak about tropical subvarieties of B_0. It is expected that tropical subvarieties of B_0 can be lifted (some times) to complex subvarieties of $\check{X}$ and to Lagrangian subvarieties of X. Indeed it is expected that this correspondence can be refined so to give an equivalence between the Fukaya category of X and the derived category of coherent sheaves on $\check{X}$, as predicted by the homological mirror symmetry conjecture. A precise prediction of this correspondence in some examples was described in [5] and recent work of J. Hicks [6] confirms this idea to an even deeper level. Indeed Hicks proves that there is a Lagrangian cobordism between a Lagrangian lift of a tropical hypersurface and a pair of Lagrangian sections. The sections are mirror to line bundles and the Lagrangian cobordim gives a cone in the Fukaya category which is mirror to a short exact sequence of the type $0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$, where D is the divisor lifting the tropical hypersurface.

Currently we are working on a construction of tropical Lagrangian spheres on a symplectic manifold (X, ω) with X diffeomorphic to a quintic in $\mathbb{P}^4$ and with a Lagrangian fibration $f : X \to B$ constructed in work of Gross [7] and Castano-Bernard, Matessi [8]. These spheres lift tropical hypersurfaces of B with boundary on the discriminant locus Δ and we expect to be able to construct enough Lagrangian spheres to generate $H_3(X, \mathbb{Z})$, which is 204 dimensional. The base B of the fibration can be identified with the boundary of the 4-simplex P in $\mathbb{R}^4$ with corners the points $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, $(-1, -1, -1, 4)$. This simplex is a reflexive polytope and the fan Σ constructed over a maximal unimodal subdivision of ∂P gives a four dimensional toric variety Y_Σ. The mirror $\check{X}$ of the quintic X is an anticanonical section of Y_Σ.

The discriminant locus Δ is contained in the 2-skeleton $(\partial P)^{[2]}$. Inside each two dimensional face of ∂P, Δ looks like a tropical quintic curve in the moment polytope of $\mathbb{P}^2$. Each edge of Δ which goes out to the 1-skeleton $(\partial P)^{[1]}$ ends at a 3-valent vertex of Δ. The tropical hypersurfaces which we consider are the connected components of $(\partial P)^{[2]} - \Delta$. They are in one to one correspondence with the integral points of $(\partial P)^{[2]}$ and we call them of type I, II or III depending on whether the integral point is respectively in the interior of a two face, of an edge or is a vertex of P. We claim that each of these tropical hypersurfaces lifts to a Lagrangian sphere. An easy count gives 60 tropical hypersurfaces of type I, 40 of type II and 5 of type III. This gives at least 105 Lagrangian spheres, plus we have the zero section of the Lagrangian fibration which gives 106. Not enough to generate $H_3(X,\mathbb{Z})$! On the other hand, each tropical hypersurface can be lifted to a Lagrangian submanifold in many different ways which are described by a similar combinatorics as the classification of line bundles on a toric variety (see [5]). We expect to be able to obtain enough different lifts so to generate all of $H_3(X,\mathbb{Z})$.

We believe that it is possible to predict the mirror sheaves to each Lagrangian sphere. Indeed each integral point $p \in (\partial P)^{[2]}$ corresponds to a toric divisor D_p in Y_Σ and hence to a divisor $W_p = \check{X} \cap D_p$ in $\check{X}$. Each Lagrangian lift of the tropical hypersurface corresponding to p should be mirror to a sheaf supported on W_p. We think there are a finite number of sheaves of this type which, together with the structure sheaf $\mathcal{O}_{\check{X}}$, split generate the derived category of $\check{X}$.

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Lefschetz section theorems for tropical hypersurfaces

Kristin Shaw

(joint work with Charles Arnal and Arthur Renaudineau)

Tropical homology is a homology theory with non-constant coefficients for polyhedral spaces. Under suitable conditions, the dimensions of the $\mathbb{Q}$-tropical homology groups of the tropical limit of a family of complex projective varieties are equal to the corresponding Hodge numbers of a generic member of the family [2]. In this talk, I explain the proof that the *integral* tropical homology groups of a non-singular tropical hypersurface in a toric variety satisfy a version of the Lefschetz hyperplane section theorem. Our main goal for proving a tropical version of the Lefschetz section theorem is establish conditions under which the integral tropical homology groups of hypersurfaces are torsion free. The motivation behind proving the torsion freeness of the integral tropical homology groups is to establish equality of the dimensions of the $\mathbb{Z}_2$-tropical homology groups and the Hodge numbers of complex hypersurfaces of toric varieties. By [5], the dimensions of the $\mathbb{Z}_2$-tropical homology groups bound Betti numbers of real algebraic hypersurfaces near the tropical limit.

The (p, q)-th tropical homology group of a polyhedral complex Z is denoted $H_q(Z; \mathcal{F}_p^Z)$ and the Borel-Moore homology group is denoted $H_q^{BM}(Z; \mathcal{F}_p^Z)$. When a polyhedral space Z is compact then $H_q(Z; \mathcal{F}_p^Z) = H_q^{BM}(Z; \mathcal{F}_p^Z)$. Here $\mathcal{F}_p^Z$ is the p-th multi-tangent cosheaf on the polyhedral space Z. For a non-singular tropical hypersurface X in a tropical toric variety Y to satisfy the Lefschetz section theorems we require two main assumptions. Firstly, the hypersurface must be *combinatorially ample* in Y. in short this condition implies that each connected component of the complement $Y \backslash X$ has the tropical homology of $\mathbb{R}^r \times \mathbb{T}^s$ for some r and s, where $\mathbb{T} = [-\infty, \infty)$. We also require that the pair (Y, X) be a *cellular pair*. This means that the polyhedral complex obtained by considering the subdivision of Y induced by X satisfies that its one point compactification is a regular CW complex. This condition is required in order to use the description of tropical homology as cellular cosheaf homology. The tropical Lefschetz section theorem for hypersurfaces is the following.

Theorem 1. *Let X be a non-singular and combinatorially ample tropical hypersurface of an $n+1$ dimensional non-singular tropical toric variety Y. Then there are maps*

$$i\colon H_q^{BM}(X; \mathcal{F}_p^X) \to H_q^{BM}(Y; \mathcal{F}_p^Y)$$

which are isomorphisms when $p + q < n$ and surjections when $p + q = n$.

If moreover (Y, X) is a cellular pair, then there are maps

$$i\colon H_q(X; \mathcal{F}_p^X) \to H_q(Y; \mathcal{F}_p^Y)$$

which are isomorphisms when $p + q < n$ and surjections when $p + q = n$.

The proof of this theorem is established by considering the following exact sequences of cosheaves:

$$0 \to \mathcal{F}_p^Y|_X \to \mathcal{F}_p^Y \to \mathcal{Q}_p \to 0 \qquad \text{and} \qquad 0 \to \mathcal{F}_p^X \to \mathcal{F}_p^Y|_X \to \mathcal{N}_p \to 0.$$

Once we establish the vanishing of the homology of the cosheaves $\mathcal{Q}_p$ and $\mathcal{N}_p$ up to the appropriate degrees, the main theorem follows from the long exact sequence in homology.

The assumption that the tropical hypersurface be combinatorially ample is necessary for Theorem 1 to hold, just as ampleness of the divisor in question is necessary for the classical Lefschetz section theorem to hold. Consider the standard tropical hyperplane $X \subset \mathbb{R}^{n+1}$. We can compactify X in the tropical toric variety Y corresponding to the blow up of $\mathbb{P}^{n+1}$ in a toric invariant fixed point. It can be computed that $\operatorname{rank} H_1(X, \mathcal{F}_1^X) = 1$ whereas $\operatorname{rank} H_1(Y, \mathcal{F}_1^Y) = 2$. Therefore, the map $H_1(X, \mathcal{F}_1^X) \to H_1(Y, \mathcal{F}_1^Y)$ is not an isomorphism.

Tropical homology with real or rational coefficients is the homology of the cosheaf of real vector spaces $\mathcal{F}_p \otimes \mathbb{R}$ or $\mathcal{F}_p \otimes \mathbb{Q}$, respectively. We wish to remark that a variant of Theorem 1 holds in the case of tropical homology with real coefficients for a singular tropical hypersurface X in a tropical toric variety Y.

To conclude torsion freeness of tropical homology in the case when X is compact we combine Theorem 1 with the integral version of tropical Poincaré duality from [3].

Corollary 1. *The integral tropical homology groups of a non-singular tropical hypersurface of a non-singular compact tropical toric variety are torsion free.*

The following theorem establishes a relation between the Euler characteristics of the chain complexes of tropical chains and the χ_y-characteristics using the tropical description of the motivic nearby fibre from [4].

Theorem 2. *Let X be an n-dimensional non-singular tropical hypersurface in a non-singular tropical toric variety Y. Let $X_{\mathbb{C}}$ be a complex hypersurface torically non-degenerate in the complex toric variety $Y_{\mathbb{C}}$ such that X and $X_{\mathbb{C}}$ have the same Newton polytope. Then*

$$(-1)^p \chi(C_\bullet^{BM}(X; \mathcal{F}_p^X)) = \sum_{q=0}^{n} e_c^{p,q}(X_{\mathbb{C}}),$$

and thus

$$\chi_y(X_{\mathbb{C}}) = \sum_{p=0}^{n} (-1)^p \chi(C_\bullet^{BM}(X; \mathcal{F}_p^X)) y^p.$$

As corollaries of Theorems 1 and 2, we can relate the ranks of the tropical homology groups to the Hodge numbers of complex hypersurfaces in the compact case, and to the Hodge-Deligne numbers for hypersurfaces in the torus for tropical hypersurfaces in $\mathbb{R}^{n+1}$ [1].

Corollary 2. *Let X be a non-singular and combinatorially ample compact tropical hypersurface in a non-singular compact toric variety Y and assume that X has Newton polytope Δ. Let $X_{\mathbb{C}}$ be a torically non-degenerate complex hypersurface in the compact toric variety $Y_{\mathbb{C}}$ also with Newton polytope Δ. Then for all p and q we have*

$$\dim H^{p,q}(X_{\mathbb{C}}) = \operatorname{rank} H_q(X; \mathcal{F}_p^X).$$

Corollary 3. *Let X be a non-singular tropical hypersurface in $\mathbb{R}^{n+1}$ and assume that X has Newton polytope Δ. Let $X_{\mathbb{C}}$ be a torically non-degenerate complex hypersurface in $(\mathbb{C}^*)^{n+1}$ also with Newton polytope Δ. Then*

$$\operatorname{rank} H_{n-p}^{BM}(X; \mathcal{F}_p) = \sum_{q=0}^{n-p} h^{p,q}(H_c^n(X_{\mathbb{C}})).$$

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What is the fundamental group of a tropical curve?

Martin Ulirsch
(joint work with Yoav Len, Mattia Talpo, and Dmitry Zakharov)

An (abstract) tropical curve Γ is a finite metric graph $\Gamma = (G = (V, E, L), |.|: E \to \mathbb{R}_{>0})$ (multi-edges, loops, and legs, denoted by L, are allowed) together with an integer vertex weight $h: V \to \mathbb{Z}$. Suppose further that G is connected. The topological fundamental group $\pi_1(G)$ of the underlying graph G is a finitely generated free group that classifies all topological covers. One might suspect that this is all there is. We argue, however, that there are at least three other candidates that answer the question in the title.

The crucial technical ingredient to make sense of these answers is the theory of graphs of groups, originally due to Bass [Bas93] and Serre [Ser77]. A *graph of groups* $\mathbb{G}$ consists of the following data:

- a finite graph G;
- groups G_v for every vertex $v \in V(G)$, G_e for every finite edge $e \in E(G)$, and G_l for every leg $l \in L(G)$, as well as
- injective homomorphisms $i_{e,v}: G_e \to G_v$ and $i_{e,v'}: G_e \to G_{v'}$ for every edge e connecting v to v' and an injective homomorphism $i_l: G_l \to G_v$ for every leg l emanating from v.

This datum formalises the original heuristic that graphs of groups are a graph-theoretic analogue of orbifolds. We may define the *fundamental group* $\pi_1(\mathbb{G})$ as the amalgamated product of $\pi_1(G)$ (generated by the edge classes $[e]$ in a complement of a spanning tree) with all the G_v (for $v \in V(G)$) subject to the relations $[e]i_{e,v'}(g_e)[e]^{-1} = i_{e,v'}(g_e)$ for all edges $e = [v, v']$ and $g_e \in G_e$.

We associate to a tropical curve Γ a graph of groups $\mathbb{G}(\Gamma)$ and and think of the fundamental group of $\mathbb{G}(\Gamma)$ as the fundamental group of Γ. There are at least three ways of doing so:

(1) Endow every vertex $v \in V(G)$ with the free group $F_{h(v)}$ and every edge and leg with the trivial group. This approach turns out be very useful when trying to give a modular interpretation of tropical Teichmüller space, as introduced in [CMV13].
(2) Endow every edge with a copy of $\mathbb{Z}$ and every vertex with the group $\mathbb{Z}^{\mathrm{val}(v)}$. With this approach one finds a fundamental group that classifies unramified finite harmonic morphisms to the metric graph $(G, |.|)$ (joint work in progress with Y. Len and D. Zakharov).
(3) Endow every edge with a copy of $\mathbb{Z}$ and every vertex with the fundamental group of val(v)-pointed Riemann surface of genus $h(v)$. This fundamental group classifies realizations of admissible covers of tropical curves in the

sense of [CMR16] (joint work in progress with M. Talpo). This idea is already present in the work of Ekedahl [Eke95] and Saïdi [Saï97].

The last approach allows for several applications: For once, we can use this to reprove the classical correspondence between algebraic and tropical Hurwitz numbers as in [BBM11, CJM10]. It also allows us to find a logarithmic/tropical reinterpretation of the compactification of the moduli space of curves with Teichmüller level structures constructed in [ACV03].

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Tropical geometry appearing in sandpile model

Nikita Kalinin, Mikhail Shkolnikov, Andrea Sportiello

The discussion concerned the relation between tropical geometry and *sandpile models*. The Abelian Sandpile [8] is a cellular automaton, popular because of its connections with combinatorics, number theory, and statistical mechanics. In its most basic realisation, it is defined by an underlying graph, and an initial configuration (that relaxes through an 'avalanche'). As common in lattice models of statistical mechanics, we are interested in the asymptotic properties as the lattice mesh goes to zero, so in particular we will have a family of graphs and initial configurations. Here is one of its versions, in which the graph is a portion of the square grid: we consider a convex domain $\Omega \subset \mathbb{R}^2$, and let the graph be $\Gamma_h = \Omega \cap h\mathbb{Z}^2$ where h, the lattice spacing, will tend to zero in the limit. Choose a finite subset P in Ω, and set the initial configuration at mesh parameter h as equal to three on every vertex of Γ_h, except for the points which best approximate P on the graph, where it is valued four. We think of these numbers as the numbers of grains at the corresponding vertices. Then, perform a relaxation: while it is possible, choose a vertex with at least four grains and redistribute four grains from it to its neighbours in $h\mathbb{Z}^2$. Grains falling outside of Γ_h disappear.

It was noted long ago [7] that processes in this fashion produce outcome configurations with peculiar regular patterns. In particular, it is frequent to observe linear defects in otherwise locally biperiodic configurations, these defects meet in (generically) trivalent vertices, which satisfy remarkable conservation laws [9, 2], implied by circuitation arguments on the function that records the number of topplings of the avalanche (which, in these cases, is piece-wise linear). A striking fact is that these rules are analogous to the ones which describe tropical curves in the plane in terms of their elementary linear portions, the sole modification being due to the intrinsic discreteness of the underlying space. The special initial configurations described above are taylored as to isolate the features of these defect lines, from other features, such as the emergence of two-dimensional fractals filled with biperiodic patterns, which are present in the Sandpile Model, but, as far as we presently know, have no counterpart in the tropical context.

In particular, Kalinin and Shkolnikov showed in [5, 4, 3] that, for these graphs and initial configurations, the result of the relaxation is a state which is equal to three everywhere except a set C_h of vertices of Γ_h which are close to a tropical curve $C \subset \Omega$ passing through P, and C_h tends to C as $h \to 0$. The curves resulting from this procedure have a number of marked points, $|P|$, equal to the genus of the closure $\overline{C}$ of the curve.

Caracciolo, Paoletti and Sportiello [10] enlarge the spectrum of the analogy with tropical curves, by introducing 'anti-toppling' operators (which consist of removing one grain, and then perform an inverse avalanche). A toppling at a give site, followed by the anti-toppling at the same site, in a sandpile configuration appearing as the discrete counterpart of a tropical curve, produces a modification of the curve analogous to the 'breathe' operation in Knutson–Tao honeycombs (see e.g. [11]). As emerging from observations of Mikhalkin, the significance of the anti-toppling operations in the tropical context is that they allow to explore the space of moduli of tropical curves in which the number of marked points is smaller than the genus of $\overline{C}$. Furthermore, the use of anti-toppling operators makes feasible the study of the sandpile/tropical curve analogy also for curves on the torus, where, in absence of anti-topplings, the sandpile avalanches would become infinite, and the outcome configurations would not be defined. We performed a preliminary study of the steady-state probability distribution for genus-2 curves with no marked points, on a generic torus, under the Markov dynamics of random breathe operations induced by the uniform measure on the torus. The results suggested a complex situation, in which the relation between the steady-state probability distribution and the measure induced by the natural parametrisation of the curves is still to be clarified.

On top of these conversations, M. Shkolnikov also presented his recent results [1] on the extended sandpile group: namely, the sandpile group can be naturally embedded to a real torus, and in this framework rescaling of the domain and a continuous flow make sense. Numerical explorations of special "harmonic" directions

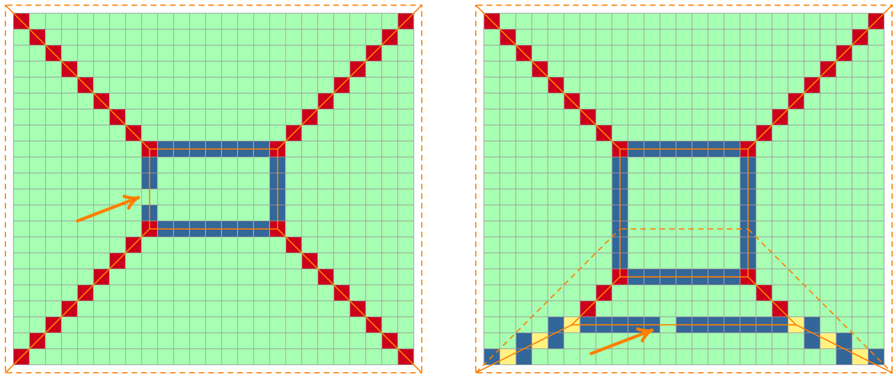

FIGURE 1. On the left: the outcome configuration, when Ω is a rectangle and P consists of a single point, in the position of te yellow arrow. On the right: the configuration resulting from an anti-toppling at the previous position, followed by the addition of an extra grain of sand in the position of the new arrow. Note that the genus of the curve has increased. The dotted trapezoid illustrates an instance of the breathe operation.

in this flow see again the emergence of regular fractal structures (which are present, for example, not only in the identity configuration z_{id}, but also in configurations which are of small order in the sandpile group, say z such that $z \oplus \cdots \oplus z = z_{\text{id}}$ for a number $\Theta(1)$ of summands). In particular, one can prove that there is a monomorphism between the sandpile group of the squares if a certain divisibility condition is satisfied, see [6].

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Relative Orientability, Lifted Cobordisms, and WDVV-Type Relations for Real Gromov-Witten Invariants

Xujia Chen

The Gromov-Witten (or GW) invariants of a symplectic manifold (X, ω) are counts of (pseudo-) holomorphic curves arising from classical enumerative geometry, symplectic topology, and string theory. These invariants enumerate J-holomorphic curves, for an ω-tame almost complex structure J on X, that represent a fixed element of $H_2(X)$ and pass through submanifolds $H_1, \ldots, H_l \subset X$. The resulting numbers do not depend on J or the choices of H_i in their homology classes $[H_i] \in H_*(X)$.

WDVV equations of string theory relate genus 0 GW-invariants representing different elements of $H_2(X)$; they are equivalent to the associativity of the quantum product on $H^*(X)$. These equations determine all counts of rational curves in many important smooth projective varieties. Kontsevich's recursion, proved by Ruan-Tian in the early 90s, explicitly demonstrates this phenomenon in the case of the complex projective plane $\mathbb{P}^2$.

A real symplectic manifold (X, ω, ϕ) is a symplectic manifold together with an anti-symplectic involution ϕ. We denote by $\mathcal{J}_\omega^\phi$ the space of ω-tame almost complex structures J on X such that $\phi^* J = -J$. The fixed locus X^ϕ is then a Lagrangian submanifold of (X, ω) which is totally real with respect to any $J \in \mathcal{J}_\omega^\phi$. A curve $C \subset X$ is called real if $\phi(C) = C$.

Welschinger [8, 9] defined invariant signed counts of real genus 0 J-holomorphic curves in real symplectic manifolds (X, ω, ϕ) of dimensions 4 and 6. Solomon interpreted Welschinger's invatiants as holomorphic disk counts in [6] and proposed WDVV-type relations for them in [7]. He also suggested an adaption of Ruan-Tian's homotopy style approach for proving his relations.

My recent work [2] established Solomon's relations for Welschinger's invariants of real symplectic 4-folds and led to their analogues for real symplectic 6-folds with some symmetry in [4]. As indicated in [3], these relations determine all counts of real rational curves in many important real projective varieties. They also recover the predictions of Alcolado [1] for extended Frobenius manifold structures for $\mathbb{P}^2$ and $\mathbb{P}^3$ and establish their existence for other real symplectic 4- and 6-folds. The proofs of these relations, outlined below, are based on lifting homology relations via the usual forgetful morphisms *together with* suitably chosen cobordisms; this makes it possible to determine the wall-crossing effects coming from the obstructions to the relative orientability of the relevant morphisms.

For $B \in H_2(X)$, let $\mathfrak{M}_{k,l}(B; J)$ be the moduli space of irreducible degree B real rational J-holomorphic curves in X with k real marked points and l conjugate pairs of marked points and $\overline{\mathfrak{M}}_{k,l}(B; J)$ be its stable map compactification. The latter is a stratified space with $\mathfrak{M}_{k,l}(B; J)$ as the top-dimensional stratum. The codimension 1 strata consist of curves with two real components. The domain and target of the natural evaluation morphism

$$\mathrm{ev} : \overline{\mathfrak{M}}_{k,l}(B; J) \longrightarrow (X^\phi)^k \times X^l$$

may not be orientable, but this morphism becomes relatively orientable (i.e. the pull-back of the first Stiefel-Whitney class w_1 of the target is the w_1 of the domain) after removing certain codimension 1 strata from the domain. We call these codimension 1 strata bad strata.

Let $\overline{\mathcal{M}}_{k',l'}$ be the Deligne-Mumford moduli space of k' real marked points and l' conjugate pairs of marked points on $\mathbb{P}^1$ with the standard real structure (i.e. $z \to \overline{z}$). For $k \geq k'$ and $l \geq l'$, let

$$\mathfrak{f}: \overline{\mathfrak{M}}_{k,l}(B;J) \longrightarrow \overline{\mathcal{M}}_{k',l'}$$

be the natural forgetful morphism (forgetting all data except for the first k' real and l' conjugate pairs of marked points). We choose a bordered hypersurface Υ in $\overline{\mathcal{M}}_{k',l'}$, with (k',l') being $(1,2)$ or $(0,3)$, so that $\partial\Upsilon$ consists of certain curves with three components and a conjugate pair of nodes.

Let $\mathbf{C} \subset X_{k,l}$ be a generic constraint so that the intersection

$$\overline{\mathfrak{M}}_{k,l}(B;J) \xrightarrow{\mathrm{ev}\times\mathfrak{f}_{k',l'}} X_{k,l} \times \overline{\mathcal{M}}_{k',l'} \supset \mathbf{C} \times \Upsilon$$

is transverse and

$$\dim \overline{\mathfrak{M}}_{k,l}(B;J) + \dim\big(\mathbf{C}\times\Upsilon\big) = \dim\big(X_{k,l}\times\overline{\mathcal{M}}_{k',l'}\big) + 1\,.$$

The intersection numbers then satisfy

$$(*) \qquad \overline{\mathfrak{M}}_{k,l}(B;J)\cdot(\mathbf{C}\times\partial\Upsilon) = \pm\, 2\,(\text{bad strata})\cdot(\mathbf{C}\times\Upsilon);$$

see also Figure 1 in [2]. The right-hand side of $(*)$ is the wall-crossing correction to lifting the homology relation on $\overline{\mathcal{M}}_{k',l'}$ determined by $\partial\Upsilon$ via the forgetful morphism $\mathfrak{f}$. This correction arises from crossing the bad strata of $\overline{\mathfrak{M}}_{k,l}(B,J)$, i.e. the strata that obstruct the orientability of the evaluation morphism. The identity $(*)$ and the splitting formulas, which express the counts of reducible curves appearing in $(*)$ in terms of counts of their irreducible components, yield the desired WDVV relations.

In order to obtain a splitting formula for the counts of two-component curves on the right-hand side of $(*)$, the bounding hypersurface Υ needs to be chosen subject to certain topological conditions. If $\dim X = 4$, the counts of the three-component curves on the left-hand side of $(*)$ reduce to counts of irreducible curves just as in [5]. If $\dim X = 6$, a splitting formula for the curve counts on the left-hand side of $(*)$ is obtained in the presence of a finite group G of automorphisms of (X,ω,ϕ) satisfying some conditions; see Definition 1.1 in [4]. If (X,ω,ϕ) is $\mathbb{P}^3$ with the Fubini-Study symplectic form and its standard conjugation, G can be taken to be the group generated by a reflection about a real hyperplane. Mikhalkin used such a reflection in 2003 to note that Welschinger's invariants of $\mathbb{P}^3$ in even degrees vanish. A similar vanishing phenomenon underpins the splitting formula for the right-hand side of $(*)$ in [4].

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The local real Gromov-Witten theory of curves

Penka Georgieva

In the talk we present the results of a joint work with E. Ionel on the real Gromov-Witten theory of local 3-folds over real curves. We show that this gives rise to a 2-dimensional Klein TQFT defined on an extension of the category of unorientable surfaces. We use this structure to completely solve the theory by providing a closed formula for the local RGW invariants in terms of representation theoretic data, extending earlier results of Bryan and Pandharipande. As a consequence we obtain the local version of the real Gopakumar-Vafa formula that expresses the connected real Gromov-Witten invariants in terms of integer invariants. In the case of the resolved conifold the partition function of the RGW invariants agrees with that of the SO/Sp Chern-Simons theory.

Computing zero-dimensional tropical varieties

Yue Ren

This talks touches upon some recent algorithmic and computational developments regarding tropicalizations of affine varieties. In particular, we will discuss the challenges of computing zero-dimensional tropical varieties. The motivation is twofold:

(1) Zero-dimensional tropical varieties play a central role in the computation of general tropical varieties [5].
(2) The recent resurgence of activity in the study of tropical lines on cubic surfaces, tropical bitangents to plane quartic curves, tropical tritangents to space sextic curves, etc, all of which are zero-dimensional tropical varieties.

We will show how zero-dimensional tropical varieties can be efficiently computed using projections, similar to Hept-Theobald's algorithm for general tropical varieties and Chan's specialised algorithm for tropical curves. Analysing this idea, we provide a brief argument for why the complexity of computing tropical varieties is dominated by the Groebner walk, and how tropical algebraic computations over p-adics can be orders of magnitudes faster by exploiting modular techniques [2]. All algorithms have been implemented in Singular [3].

We conclude the talk with a light demo of 3D printing tropical curves, tropical surfaces and combinations theoreof using the newest release of polymake [4] and blender [1].

The first part of the talk is joint work with Paul Görlach (MPI MiS Leipzig), Leon Zhang (UC Berkeley), the second part of the talk is joint work with Ronald Kriemann (MPI MiS Leipzig) and Henryk Nagel (TU Berlin).

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Properties of the tropical cycle class map

Andreas Gross
(joint work with Farbod Shokrieh)

Introduced by Itenberg, Katzarkov, Mikhalkin, and Zharkov [1], tropical homology groups are a new tool to associate algebraic invariants to the spaces appearing in tropical geometry. They are generalizations of the singular homology groups of these spaces that takes into account the piecewise linear structures appearing in tropical geometry. Eventually, one would like the theory of tropical homology groups to be similarly well-developed as the theory of singular homology groups in the sense that it satisfies certain functoriality properties and identities. In our work, we take a big step in this direction by relating the tropical homology groups to invariants coming from sheaf theory. More precisely, we exhibit the tropical homology groups as certain Ext-groups of the sheaves of tropical forms and the dualizing complex. The desired functoriality properties then follow directly from the functoriality properties of tropical forms and the dualizing complex. For example, we immediately obtain a proper push-forward and tropical cross products for tropical homology classes. We also obtain a very general definition of the tropical cycle class map which associates a tropical homology class to every tropical cycle. As we work in a sheaf-theoretic setup that is local by nature, our definition does not involve any global choices like that of polyhedral structures of triangulations. Most of the identities we want to prove about the tropical cycle class map reduce to local computations in our setup. For example, we show that the tropical cycle class map respects cross products, proper push-forwards, and intersections with tropical Cartier divisors. Unfortunately, not all identities are immediately reduced to local computations. Notably, this fails when one wants to prove that the cycle class of an intersection product is the intersection of the cycle classes. The reason for this failure is that the intersection of the cycle classes is of a global nature, even though the tropical intersection product is local. We are thus forced to take a different approach in proving this, and we strongly believe it is possible via a cocycle class map that associates a tropical cohomology class to every tropical cocycle. The first problem we encountered here is that the notion of tropical cocycles has not been known to be dual to the notion of tropical cycles on tropical manifolds. This is a local statement and we proved it by using the isomorphism between tropical cocycles on a tropical linear space respecting a given polyhedral structure and the Chow rings of the associated toric variety. We then showed that the Chow ring (with integer coefficients) of a toric variety whose fan is supported on a tropical linear space is a Poincaré duality ring, a statement which we believe is of independent interest. Poincaré duality for these Chow rings then implies the duality between cycles and cocycles on tropical manifolds.

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Tropical Jucys Covers and applications

Marvin Anas Hahn

(joint work with D. Lewanski; F. Leid, J.W. van Ittersum)

Hurwitz numbers enumerate branched morphisms between Riemann surfaces with fixed genera and fixed ramification data. These topological invariants play a significant role in several areas of mathematics, such as algebraic geometry, algebraic topology, representation theory, Gromov-Witten theory and many more. There are several interesting cases of Hurwitz numbers, obtained by specifying a certain kind of ramification data. In this talk, we discuss two of the most important cases: Double Hurwitz numbers and simple Hurwitz numbers for elliptic base curves. In order to define double Hurwitz numbers, we consider a non-negative integer g and partitions μ,ν of a positive integer d. We say branched degree d morphisms $f : S \to \mathbb{P}^1$ is of type (g,μ,ν), if S is of genus g, f ramifies with profile μ over 0, ν over ∞ and $(2,1,\dots,1)$ over $b = 2g-2+\ell(\mu)+\ell(\nu)$ fixed points on $\mathbb{P}^1$. Furthermore, we call two covers $f : S \to \mathbb{P}^1$, $f' : S' \to \mathbb{P}^1$ equivalent, if there exists an isomorphism $g : S \to S'$, such that $f = f' \circ g$. We then define the double Hurwitz numbers as $H_g(\mu,\nu) = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|}$, where we sum over all equivalence classes of branched morphisms of type (g,μ,ν).

It turns out that double Hurwitz numbers admit a piecewise polynomial structure. To be more precise, we define the parameter space of all tuples of partitions (μ,ν) of the same positive integer and with fixed lengths m, n as

$$W_{m,n} = \{(\mu,\nu) \in \mathbb{N}^m \times \mathbb{N}^n \mid \sum \mu_i = \sum \nu_j\}. \tag{1}$$

Furthermore, for fixed $I \subset \{1,\dots,m\}$, $J \subset \{1,\dots,n\}$, we consider the hyperplane $H_{I,J}$ in W cut out by $\sum_{i\in I} \mu_i - \sum_{j \in J} \nu_j = 0$. The complement of the hyperplane arrangement $(H_{I,J})_{I,J}$ for all possible choices I, J yields finitely many open chambers C_i, such that $W \backslash \{(H_{I,J})_{I,J}\} = \bigsqcup C_i$. It was proved in [8] that the function $H_g : C_i \to \mathbb{Q}$, $(\mu,\nu) \mapsto H_g(\mu,\nu)$ is polynomial, i.e. there exists a polynomial $P_i \in \mathbb{Q}[x_1,\dots,x_m,y_1,\dots,y_n]$, such that $P_i(\mu,\nu) = H_g(\mu,\nu)$ for all $(\mu,\nu) \in C_i$. Furthermore, for adjacent chambers C_i, C_j, it was proved in [17, 3, 16] that $P_i - P_j$ may be expressed in terms of Hurwitz numbers with smaller input data. We want to highlight the approach taken in [3], which starts from a tropical expressions of double Hurwitz numbers and proceeds with an involved combinatorial analysis of the space of possible weightings of the corresponding tropical covers.

We further note that there is a beautiful description of double Hurwitz numbers, which is essentially due to Hurwitz [15]. Namely, we have that $H_g(\mu,\nu)$ is equal to $\frac{1}{d!}$ times the numbers of tuples $(\sigma_1,\sigma_2,\tau_1,\dots,\tau_b)$ of elements in the symmetric group S_d, such that

- $\sigma_2 = \tau_b \cdots \tau_1 \sigma_1$;
- the cycle type of σ_1 (resp. σ_2) is μ (resp. ν) and the τ_i are transpositions
- the group generated by $\sigma_1,\sigma_2,\tau_1,\dots,\tau_b$ is a transitive subgroup of S_d.

We now turn our attention to so-called simple Hurwitz numbers for elliptic base curves. The term simple stems from the fact that one calls branch points with profile $(2,1,\dots,1)$ simple branch points and that we will only allow such ramification. More precisely, we fix positive integers $g, d \geq 1$, an elliptic curve E and call a branched degree d morphism $f : S \to E$ elliptic of type (g,d) if S is of genus g and f ramifies with profile $(2,1,\dots,1)$ over $2g-2$ fixed points in E. With the same notion of equivalence of covers as before, we define $N_{g,d} = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|}$, where we sum over all elliptic branched morphisms of type (g,d). A remarkable result due to Dijkgraaf [4] states that for $g \geq 2$ the generating function

$$\sum_d N_{g,d} q^d \tag{2}$$

is a quasimodular form, i.e. it may be expressed as polynomials in Eisenstein series. In particular, Dijkgraaf's result implies that for fixed g there exists d_g, such that the numbers $N_{g,d}$ with $d \leq d_g$ determine all numbers $N_{g,d}$. There are now many different proofs for this result (and various generalisations). We want to highlight the following result (conjectured in [2] and proved in [9]), which is a refinement of

(2): The numbers $N_{g,d}$ may be naturally written as finite sums of contributions $N^{\Sigma}_{g,d}$ of tropical covers with combinatorial type Σ, such that $\sum_{d>0} N^{\Sigma}_{g,d} q^d$ is a quasimodular form as well. In particular, this gives a new proof of Dijkgraaf's theorem.

Similar to double Hurwitz numbers, simple Hurwitz numbers for elliptic base curves admit a description in terms of factorisations in the symmetric group. More precisely, the invariant $N_{g,d}$ is equal to $\frac{1}{d!}$ times the number of tuples $(\tau_1, \ldots, \tau_{2g-2}, \alpha, \beta)$ with $\tau_i \in S_d$ transpositions and $\alpha, \beta \in S_d$ arbitrary, such that

- we have $\tau_b \cdots \tau_1 = \alpha\beta\alpha^{-1}\beta^{-1}$;
- the group generated by $\tau_1, \ldots, \tau_b, \alpha, \beta$ is a transitive subgroup of S_d.

In recent years several variants of Hurwitz numbers have appeared in the literature. One of the most inticing ones are the variants called monotone Hurwitz numbers. Surprisingly, they originate from the theory of random matrix theory where they appear as coefficients in the Maclaurin expansion of the Harish-Chandra–Itzykson–Zuber integral [7].

These enumerants are analogues of classical Hurwitz numbers where we obtain the monotone double Hurwitz numbers $\vec{H}_g(\mu, \nu)$ and the monotone Hurwitz numbers for elliptic base curves $\vec{N}_g(\mu, \nu)$ by considering the above enumerations of factorisations in the symmetric group with the additional condition that for $\tau_i = (r_i s_i)$ with $r_i < s_i$, we have $s_i \leq s_{i+1}$. A common theme in monotone Hurwitz theory is that monotone Hurwitz numbers share a lot of structural properties with their classical analogues, although the proofs may be quite different.

In this spirit, we want to study the polynomial behaviour of $\vec{H}_g(\mu, \nu)$ (see [6, 10, 11] for several advances to this problem) and the quasimodular behaviour of $\vec{N}_g(\mu, \nu)$. Motivated by the success of the tropical theory in studying the classical numbers, we start by deriving a tropical correspondence theorem for the monotone enumerants. While a tropical interpretation already appeared in [5, 10], it is not suitable for our purpose. Therefore, in a joint work with D. Lewanski [13], we start with a representation theoretic expression of monotone double Hurwitz numbers derived in [1] in terms of the fermionic Fock space. Via the Boson-Fermion correspondence, we obtain an expression in terms of the bosonic Fock space, which is well-known to yield an interpretation in the language of tropical covers. Surprisingly, the tropical covers involved may have less branch points than their classical analogues and are weighted by Gromov-Witten invariants. This points to an unknown non-trivial geometric connection. Furthermore, for partitons λ (resp. λ') of $2g-2+\ell(\mu)+\ell(\nu)$ (resp. $2g-2$), we find natural decompositions $\vec{H}_g(\mu, \nu) = \sum_\lambda \vec{H}^\lambda_g(\mu, \nu)$ and $\vec{N}_{g,d} = \sum_{\lambda'} \vec{N}^{\lambda'}_{g,d}$, such that

- the function $\vec{H}^\lambda_g : C_i \to \mathbb{Q}\ (\mu, \nu) \mapsto \vec{H}^\lambda_g(\mu, \nu)$ is given by a polynomial Q^λ_i. Moreover, for adjacent chambers the difference $Q^\lambda_i - Q^\lambda_j$ may be expressed in terms of the numbers H^λ_g with smaller input data (see [14])
- for $g \geq 2$, the generating series $\sum_d N^\lambda_{g,d} q^d$ is a quasimodular form (see [12]).

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Periods of tropical K3 hypersurfaces

Yuto Yamamoto

Let M be a free $\mathbb{Z}$-module of rank 3 and $N := \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice. We set $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}(M, \mathbb{R})$. Let $\Delta \subset M_{\mathbb{R}}$ be a smooth reflexive polytope of dimension 3, and $\check{\Delta} \subset N_{\mathbb{R}}$ be the polar polytope of Δ. Let further Σ be the normal fan to Δ. We consider a tropical Laurent polynomial

$$f(x) = \max_{n \in \check{\Delta} \cap N} \{a(n) + n_1 x_1 + n_2 x_2 + n_3 x_3\}.$$

Let $V(f) \subset \mathbb{R}^3$ be the tropical hypersurface defined by f. We construct a 2-sphere B equipped with an integral affine structure with singularities by contracting $V(f)$

in the way of Gross–Siebert program [1], [2]. Let $\iota\colon B_0 \hookrightarrow B$ denote the complement of singularities of B. Let further $\mathcal{T}_{\mathbb{Z}}$ be the local system on B_0 of integral tangent vectors. The cohomology group $H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}})$ has the cup product

$$\cup\colon H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}) \otimes H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}) \to H^2(B, \iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}}) \cong \mathbb{Z}$$

induced by the wedge product. Let Y be an anti-canonical hypersurface of the complex toric variety X_Σ associated with Σ, and

$$\mathrm{Pic}(Y)_{\mathrm{amb}} := \mathrm{Im}\left(\mathrm{Pic}(X_\Sigma) \hookrightarrow \mathrm{Pic}(Y)\right)$$

be the sublattice of $\mathrm{Pic}(Y)$ coming from the Picard group of the ambient space. Each element $n \in \left(\check{\Delta} \cap N\right) \setminus \{0\}$ is the primitive generator of a 1-dimensional cone in Σ. We write the toric divisor corresponding to this cone as $D_n \in \mathrm{Pic}(X_\Sigma)$.

Theorem 1. (1) *There is a primitive embedding*

$$\psi\colon \mathrm{Pic}(Y)_{\mathrm{amb}} \hookrightarrow H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}),$$

that preserves the pairing.

(2) *The radiance obstruction c_B of B is given by*

$$c_B = \sum_{n \in \left(\check{\Delta} \cap N\right) \setminus \{0\}} \{a(n) - a(0)\}\, \psi(D_n).$$

Let $K := \overline{\mathbb{C}\{t\}}$ be the convergent Puiseux series field, and $f = \sum_{n \in \check{\Delta} \cap N} k_n x^n \in K[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}]$ be a Laurent polynomial over K in three variables. For a sufficiently large $R \in \mathbb{R}^{>0}$, we set $f_R := f|_{t=1/R} \in \mathbb{C}\left[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}\right]$, and let V_R denote a minimal model of $\{f_R = 0\}$. Here V_R is mirror to Y. We consider the one-parameter family $\{V_R\}_R$ of complex K3 hypersurfaces. The period map of this family can be written as

$$\begin{aligned} \mathcal{P}\colon (R_0, \infty) &\to \{[\sigma] \in \mathbb{P}\left((U \oplus \mathrm{Pic}(Y)_{\mathrm{amb}}) \otimes \mathbb{C}\right) | (\sigma, \sigma) = 0, (\sigma, \overline{\sigma}) > 0\} \\ &\cong \{\sigma \in \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes \mathbb{C} | (\mathfrak{Re}\,\sigma, \mathfrak{Re}\,\sigma) > 0\}, \end{aligned}$$

where U denotes the hyperbolic plane. Let $V(\mathrm{trop}(f))$ be the tropical hypersurface defined by the tropicalization of f. We construct a 2-sphere B with an integral affine structure with singularities by contracting $V(\mathrm{trop}(f))$.

Corollary 1. *The leading term of the period map $\mathcal{P}(R)$ is given by*

$$\mathcal{P}(R) \sim \log R \cdot \psi^{-1}(c_B) \quad (R \to +\infty).$$

We can regard the element $\psi^{-1}(c_B) \in \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes_{\mathbb{Z}} \mathbb{R}$ as the "tropical period" of $B \cong V(\mathrm{trop}(f))$.

Corollary 2. *The element $\psi^{-1}(c_B) \in \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes_{\mathbb{Z}} \mathbb{R}$ satisfies*

$$(\psi^{-1}(c_B), \psi^{-1}(c_B)) > 0.$$

This inequality is the one which $\psi^{-1}(c_B)$ should satisfy in order to make the leading term of the period map satisfy Hodge–Riemann bilinear relation. Hence, it can be regarded as the tropical version of Hodge–Riemann bilinear relation.

There are several previous studies on the relationship between periods and tropical geometry. It is known that the valuation of the j-invariant of an elliptic curve over a non-archimedean valuation field coincides with the cycle length of the tropical elliptic curve obtained by tropicalization [4], [5]. The definition of periods for general tropical curves was given in [6]. It was also shown in [3] that the leading term of the period map of a degenerating family of Riemann surfaces is given by the period of the tropical curve obtained by tropicalization. Ruddat–Siebert computed periods of toric degenerations constructed from wall structures [7]. They calculated the integrations of holomorphic volume forms over cycles constructed from tropical 1-cycles on the intersection complex of the central fibers.

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Reduction and lifting of Berkovich curves with differentials

Ilya Tyomkin

(joint work with Michael Temkin)

In a recent paper [3], Bainbridge, Chen, Gendron, Grushevsky, and Möller studied what they called *Incidence compactification of strata of abelian differentials.* For a given pattern of zeroes (and poles) $\underline{\mu} \in \mathbb{N}^r$, they considered pairs $(C, \underline{p}; \omega)$ consisting of a smooth projective curve C with r marked points $\underline{p}$, and a (meromorphic) differential form ω up-to a multiplicative scalar, such that $div(\omega) = \sum \mu_i p_i$. The incidence compactification then is the closure of this locus in the projectivized Hodge bundle on $\overline{\mathcal{M}}_{g,r}$. The main result of [3] provides an explicit description of complex points of the incidence compactification in terms of level graphs (functions) and twisted differentials satisfying the usual compatibilities and a new striking condition introduced in [3] - the *global residue condition* for twisted differentials with respect to a level function.

The results of [3] have many important applications. In particular, Möller, Ulirsch, and Werner used [3] to provide a description of the liftable loci in the canonical systems on tropical curves [5]. More explicitly, given a tropical curve Γ and a divisor D in the canonical system on Γ, Möller, Ulirsch, and Werner provide a purely combinatorial necessary and sufficient condition for the pair (Γ, D) to be the tropicalization of a smooth curve X over a non-Archimedean field of zero equicharacterstic and an effective canonical divisor K on X.

In our work we studied meromorphic differential forms on *nice* k-analytic curves, i.e., quasi-smooth connected compact separated strictly k-analytic curves. One of our motivations was to find a Berkovich analytic proof of the main result of [3]. Starting with a nice curve X equipped with a non-zero meromrphic differential ω we describe a natural tropicalization datum associated to the pair. If (X, ω) is the analytification of an algebraic pair then the datum we associate to it almost coincides with the datum of [3] and [5], but in addition we associate a canonically defined residue function on the set of oriented edges of the skeleton Γ of $(X, div(\omega))$ with values in k. The residue function $\mathfrak{R}$ satisfies the very common in Berkovich geometry harmonicity condition: for any vertex x of Γ we have $\sum_{e \in Star(x)} \mathfrak{R}(e) = 0$. If (X, ω) is the analytification of an algebraic pair then the harmonicity condition of $\mathfrak{R}$ together with its compatibility with the residues of the associated twisted differential implies the global residue condition of [3].

Our main result is the lifting theorem asserting that given a tropical datum satisfying natural compatibility conditions and such that the residue function is harmonic, there exists a nice k-analytic curve X with a meromorphic differential ω, whose tropicalization coincides with the given datum. The proof of the theorem is based on the key lemma asserting that for any differential form $\omega_{\mathcal{A}}$ on an analytic annulus $\mathcal{A} = \mathcal{M}\{t, rt^{-1}\}$ that has neither zeroes nor poles, there exists a *good* analytic coordinate s on $\mathcal{A}$ such that $\omega_{\mathcal{A}} = ads^n + \mathfrak{R}\frac{ds}{s}$. The main conclusion from the key lemma is that a differential form on an annulus without zeroes and poles is determined by its norm and its residue uniquely up-to an orientation preserving automorphism. We shall emphasize that a similar lemma about the existence of good coordinates in the case of differential forms on small punctured complex discs was one of the ingredients also in the complex-analytic proof of the main theorem in [3]. Good coordinates allow us to patch local liftings along annuli similarly to the patchings of coverings of curves in the work of Amini, Baker, Brugallé, and Rabinoff [1, 2] in characteristic zero, and in the work of Brezner and Temkin [4] in positive characteristic. Also in the problem of patching of coverings of curves there were similar key lemmas providing explicit description of isomorphism classes of coverings of annuli, see e.g., [4, Thm. 4.3.8, Cor. 4.3.9]. To the best of our understanding, the patching technique we use in the Berkovich-analytic setting is a close analogue of the plumbing technique used in [3].

We shall also mention, that our tropical reduction datum contains one more ingredient. Namely, for any oriented edge e of Γ with head x and tail y, consider an open annulus whose skeleton is the edge e. Then the set of good coordinates on the annulus induces a canonical identification of the torsors of good formal coordinates for the reduction (C_x, ω_x) and (C_y, ω_y) at the points corresponding to e. This extra "stacky" piece of reduction is not needed in the proof of the lifting theorem, but as it is absolutely canonical, we expect it to be useful for other applications. The situation here is analogues to the tropical and stacky tropical reductions introduced in [6]. In [6], one could prove the lifting result for regular non-stacky tropical reductions, but for a correspondence theorem one had to work with the stacky reductions.

A version of this talk was given previously at Oberwolfach during the Workshop on Non-Archimedean Geometry and Applications in February 2019, [7].

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Connectivity of tropical varieties

Diane Maclagan
(joint work with Josephine Yu)

A key to the success of tropical geometry is that the tropicalization of a variety has combinatorial structure. The goal of this talk was to describe some extra structure (d-connectivity of the underlying complex) when the original variety is irreducible.

Throughout we assume that X is a d-dimensional irreducible subvariety of the algebraic torus, and $\operatorname{trop}(X) \subset \mathbb{R}^n$ is the non-archimedean amoeba of X: $\operatorname{trop}(X) = \operatorname{cl}(X(K'))$, where K'/K is a nontrivially valued algebraically closed field.

A key result in this version of tropical geometry is the Structure Theorem, which states that when X is irreducible of dimension d, the tropical variety $\operatorname{trop}(X)$ is the support of a pure d-dimensional polyhedral complex that is connected through codimension one. This connectivity means that one can walk from any facet of the complex to any other by passing only through codimension one faces (ridges), or equivalently that if every closed codimension-two face is removed, the space remains connected.

We note that the proof of this connectivity is significantly harder than the rest of the theorem. Over an arbitrary field, there are versions using the connectivity of the Berkovich space X^{an} [EKL06], or using tropical compactifications and deep connectivity results from algebraic geometry [CP12]. The hardest part is the connectivity of curves; one reduces to that case by slicing [BJSST07], [CP12].

The fact that tropical varieties are connected in this fashion has important computational implications. The main software, `gfan` [Jen], to compute tropical varieties makes crucial use of this fact, as it computes tropical varieties by (hyper)graph traversal. It is also an important part of the definition of an abstract embedded tropical variety as the support of a pure weighted balanced $\mathbb{R}$-rational polyhedral complex that is connected through codimension one.

We now generalize this condition.

Definition 1. *For a pure d-dimensional polyhedral complex, the facet-ridge hypergraph has vertices the facets (d-dimensional polyhedra) and hyperedges the ridges ($d-1$-dimensional polyhedra).*

Example 1. *When* $\operatorname{trop}(X)$ *is the standard tropical line in* $\mathbb{R}^2$*, the hypergraph has three vertices, labelled* $0, 1, 2$*, and one hyperedge* $\{0, 1, 2\}$ *corresponding to the origin.*

When $\operatorname{trop}(X)$ *is the standard tropical plane in* $\mathbb{R}^3$*, the hypergraph has six vertices, which we label* $\{01, 02, 03, 12, 13, 23\}$*, and four hyperedges:* $\{01, 02, 03\}$, $\{01, 12, 13\}$, $\{02, 12, 23\}$*, and* $\{03, 13, 23\}$*. The vertices correspond to the two-dimensional cones:* 12 *corresponds to* $\operatorname{pos}(\mathbf{e}_1, \mathbf{e}_2$*. The hyperedges correspond to the ridges, which are the rays of the fan, spanned by* $\mathbf{e}_0 = (-1, -1, -1), \mathbf{e}_1, \mathbf{e}_2$ *and* $\mathbf{e}_3$*.*

A hypergraph is connected if there is a path from any vertex to any other vertex where each step connects two vertices that are both in some hyperedge. It is d-*connected* if the hypergraph resulting from removing any $d-1$ vertices and all hyperedges containing them is still connected.

Theorem 1. *Let K be a field of characteristic* 0 *which is either algebraically closed, complete, or real closed with convex valuation ring. Let X be a d-dimensional irreducible subvariety of $(K^*)^n$. The tropicalization* $\operatorname{trop}(X)$ *is the support of a pure d-dimensional polyhedral complex Σ that is $(d-\ell)$-connected through codimension one, where ℓ is the dimension of the lineality space of Σ. In other words, the facet-ridge hypergraph of this complex is $(d-\ell)$-connected.*

Most requirements on the field come from the requirement that the tropicalization of a curve be connected, and are already present in the presentation given in [CP12]. This result can be considered a generalization of Balinski's theorem [Bal61] that the edge graph of a d-dimensional polytope is d-connected.

Example 2. *The standard tropical line is the tropicalization of a line, which is one-dimensional, so we expect the corresponding hypergraph to be* 1*-connected, which means connected. This is the case. Similarly, the standard tropical plane in* $\mathbb{R}^3$ *is the tropicalization of a plane, which is two-dimensional. The corresponding* *hypergraph remains connected when we remove any vertex and the two adjacent hyperedges, so is* 2*-connected.*

The idea of the proof is to slice to the curve case. This makes key use of a toric Bertini theorem by Fuchs, Mantova, and Zannier [FMZ18], with additions by Amoroso and Sombra [AS17] to prove a "tropical Bertini theorem": (under appropriate hypotheses) if Σ is the tropicalization of an irreducible variety, then set of hyperplanes H for which $\Sigma \cap H$ is the tropicalization of an irreducible variety is dense in the Grassmannian.

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Real algebraic curves with large finite number of real points

Alex Degtyarev

(joint work with Erwan Brugallé, Ilia Itenberg, Frédéric Mangolte)

We are interested in the maximal number $\delta(k)$ of real points of a real algebraic curve $C \subset \mathbb{P}^2$ of degree $d := 2k$ *with finite real part* $\mathbb{R}C$. (With the usual abuse of the language, we call C a *finite real curve*.) Asymptotically, we have

$$\frac{4}{3}k^2 \lesssim \delta(k) \lesssim \frac{3}{2}k^2.$$

For small degrees, the more precise upper and lower bounds are as follows:

k	1	2	3	4	5	6	7	8	9	10
$\delta(k) \leq$	1	4	10	19	31	46	64	85	109	136
$\delta(k) \geq$	1	4	10	19	30	45	59	78	98	123

Thus, the precise values are known only for $k \leq 4$. We also have a bound

$$\delta_g(k) \leq k^2 + g + 1$$

on the number of real points of a finite real curve $C \subset \mathbb{P}^2$ of degree $2k$ and a fixed genus g; this bound is sharp if $g \leq k - 3$. (Note that we do not assume the curves irreducible; therefore, the genus is defined via the relation $2-2g(C) = \chi(\tilde{C})$, where $\tilde{C}$ is the normalization of C.)

Most upper bounds are of purely topological nature (essentially, the Petrovsky–Comessatti inequality) and often are attained by pseudo-holomorphic curves. The examples for the lower bounds are mainly constructed using patchworking and squaring the coordinates (passing from a curve $f(x, y) = 0$ to $f(x, y^2) = 0$ or even $f(x^2, y^2) = 0$); the "elementary pieces" are curves in appropriate Hirzebruch surfaces constructed *via* the techniques of *dessins d'enfants*.

Some of our results extend to finite real curves in other surfaces, most notably rational ruled. For example, given a lattice polygon $\Delta \subset \mathbb{R}^2$, we construct a sequence of curves $C_k \subset \operatorname{Tor}(\Delta)$, $k \in \mathbb{Z}^+$, in the corresponding toric variety, with the Newton polygon $2k\Delta$ and such that

$$|\mathbb{R}C_k| \approx \frac{4}{3}k^2 \operatorname{Area}(\Delta).$$

Most of these results are published in [1].

Our bounds are closely related to Hilbert's 17-th problem. Denote by P_{2k} the cone of positive semi-definite ternary forms of degree $2k$, and let $\Sigma_{2k} \subset P_{2k}$ be the subcone of the forms representable as a sum of squares of forms of degree k. Given $p \in P_{2k}$, let

$$h_m(p) := \min\{h \mid pq \in \Sigma_{2h} \text{ for some } q \in P_{2h-2k}\},$$

and denote $h(k) := \max\{h_m(p) \mid p \in P_{2k}\}$. Then, clearly,

$$\delta(k) \leq h(k)^2.$$

The best known upper bound, due to Hilbert, is $h(k) \leq 2k - 2$, and our bounds on $\delta(k)$ imply the lower bound

$$h(k) \gtrsim \frac{2k}{\sqrt{3}}.$$

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Round dances of points on a curve

Oleg Viro

In Russian folklore there are dances called *khorovod's*. In a khorovod, dancers move along a closed curve. The word *khorovod* is translated to English as *round dance*. In this talk khorovods of points on a real algebraic curve are studied.

1. Problems

Let X be a non-singular projective real algebraic curve, $A = P_1 + \cdots + P_d$, $P_1, \ldots, P_d \in \mathbb{R}X$ be a simple real divisor. How can A move staying *simple real* and belonging to the same linear equivalence class?

Denote by $Emb(A, X)$ the space formed by such divisors. What is the fundamental group $\pi_1(Emb(A, X))$?

Each loop L in $Emb(A, X)$ is formed of d paths on $\mathbb{R}X$. Considered as singular 1-simplices, the paths form a *singular 1-cycle.* The homology class of this 1-cycle is called the *trace* of L, denoted by $\tau(L)$. It gives rise to a homomorphism $\tau : \pi_1(Emb(A, X)) \to H_1(X)$. *What can it be?*

2. Warm up problems and result

Say, let X be a plane projective M-curve of degree 3, $d = 9$ and A be the transverse intersection of X with other curve of degree 3. Is $\pi_1(Emb(X, A))$ trivial?

Sometimes yes, sometimes no, it depends on A. If A is contained in one component of $\mathbb{R}X$, then $\pi_1(Emb(A, X)) = 0$. Otherwise $\pi_1(Emb(A, X)) = \mathbb{Z}$.

Under a loop move, *trajectories* of points cover the components of $\mathbb{R}X$ the same number of times count according to a complex orientation of $\mathbb{R}X$. In other words, $\mathrm{Im}(\tau) = \mathbb{Z}$.

The first examples and observations which led to this picture on cubic curves belong to *Ayşegül Öztürkalan*, Abdullah Gül Üniversitesi, Turkey.

3. Main results

Recall: a non-singular real algebraic curve $\underline{X}$ is said to be of *type I* if $\mathbb{R}X$ bounds in $\mathbb{C}X$ and type II otherwise.

Let $A = P_1 + \cdots + P_d$, $P_1, \ldots, P_d \in \mathbb{R}X$ be a simple real divisor. Let $D(X, A)$ denote the linear equivalence class of divisors that contains A. Let $D_{\mathbb{R}}(X, A) \subset D(X, A)$ be formed by divisors contained in $\mathbb{R}X$.

Recall that $Emb(A, X)$ is the subset of $D(X, A)$ formed by simple real divisors.

Obviously, $\tau : \pi_1(Emb(A, X)) \to H_1(\mathbb{R}X)$ factors through $\tau : \pi_1(D_{\mathbb{R}}(X, A)) \to H_1(\mathbb{R}X)$.

Main Theorem. $\tau(\pi_1(D_{\mathbb{R}}(X, A))) = 0$*, unless* X *is of type I and* A *meets each connected component of* $\mathbb{R}X$*, when* $\tau(\pi_1(D_{\mathbb{R}}(X, A)))$ *is either trivial or infinite cyclic generated by the complex orientation.*

Lemma 1. *The inclusion homomorphism* $H_1(\mathbb{R}X) \to H_1(\mathbb{C}X)$ *maps* $\tau(\pi_1(D_{\mathbb{R}}(X, A))) \subset H_1(\mathbb{R}X)$ *to zero.*

Lemma 1 and Main Theorem above are corollaries of the following:

Lemma 2. *For any divisor* A *on a complex curve* X*,* $\tau(\pi_1(D(X, A))) \subset H_1(\mathbb{C}X)$ *is zero.*

Indeed, $D(X, A)$ is a complex projective space, and $\pi_1(D(X, A)) = 0$. □

Discussion: Phase tropical varieties

Ilia Zharkov

In addition to an affine complex hypersurface and its phase tropical counterpart we are going to introduce two new objects: *amplitude tropical* and *obertropical.* All four of these are homeomorphic to each other and can serve different geometric purposes.

Let $H \subset (\mathbb{C}^*)^n$ be a generic hypersurface with Newton polytope Q. We assume that Q is coherently triangulated and H is close to its tropical limit. The Viro's patchworking method will induce a pair-of-pants decomposition of H according to the triangulation, see [1]. If the triangulation is not unimodular then the pieces are abelian covers of the standard pairs-of-pants P. For the rest we will concentrate on one pair-of-pants as a building block for H.

We think of points in the pair-of pants as non-zero solutions of the homogeneous equation:

$$z_0 + z_1 + \cdots + z_n = 0,$$

and consider the map $(\mathbb{C}^*)^n \to (\mathbb{R}_+)^{n+1}/\mathbb{R}_+ \times (S^1)^{n+1}/S^1$ given by

$$(z_0, z_1, \ldots, z_n) \mapsto (|z_0|, |z_1|, \ldots, |z_n|) \times (\mathrm{Arg}(z_0), \mathrm{Arg}(z_1), \ldots, \mathrm{Arg}(z_n)).$$

Let P denote the closure of the image of the pair-of-pants in $\Delta \times T^n$, where Δ is the standard simplex. We further subdivide P as a regular CW complex by cyclicly ordering the variables such that their arguments $\theta_0, \theta_1, \ldots, \theta_n$ are ordered counter clock wise on the circle. We fix one such order, then we can assume that (after relabeling the variables) the vectors $z_0, z_1, \ldots, z_n$ form a convex polygon. Let $B \subset P$ be the closure of the part of the pair-of-pants with our fixed order. With G. Kerr [2] we showed that B is a closed ball of codimension 2.

To achieve complete symmetry between amoeba and coamoeba we can replace the arguments θ_i by the exterior angles α_i in the polygons and dehomogenize the simplex Δ by setting $\sum |z_i| = 2\pi$. Then the image of B in the product of two simplices $\Delta_1 \times \Delta_2$ lies in the product of two hypersimplices $O_1 \times O_2$ defined by

$$0 \le |z_i| \le \pi, \sum |z_i| = 2\pi \text{ and } 0 \le \alpha_i \le \pi, \sum \alpha_i = 2\pi,$$

respectively.

Each of the O_1 and O_2 have a nice skeleton in it. The skeleton $S \subset O_1$, also known as the tropical hyperplane is spanned by the baricenters of faces $I \subseteq \{0, 1, \ldots, n\}$ of Δ for the subsets I with at least two elements. The faces $F_{I,J}$ of S are indexed by pairs of subsets $I \subseteq J \subseteq \{0, 1, \ldots, n\}$. The most famous skeleton $\Sigma \subset O_2$ (known as the boundary of the permutahedron) has the same face structure but with a little bit off balance baricenters.

The part B_1 of the phase tropical pair-of-pants lies in the product $S \times O_2$ where the fiber of the face $F_{I,J}$ is given by there restriction of the reduced coamoeba $\mathrm{Arg}(\{\sum_{i \in I} z_i = 0\})$ to B. It has a nice combinatorial description as a polyhedral complex in terms of alcoves. We will omit the details. The point is that B_1 is also a codimension 2 ball in $O_1 \times O_2$.

The new, *amplitude* tropical object B_2 is the same as B_1 but with the roles of amoeba and coamoeba switched. It lies in $O_1 \times \Sigma$ and it is also a codimension 2 ball in $O_1 \times O_2$.

Finally, the *ober* tropical ball

$$B_3 = \bigcup F_I \times G_K$$

where the sets I of edges of the polygon and K of exterior angles (vertices) are *interlacing*. That means that not all vertices in K lie between two elements of I, that is there is a vertex-edge-vertex-edge combination in I, K. This is the ultimate tropical object, which is $(n-1)$-polyhedral in both amoeba and coamoeba directions.

One can also easily describe the obertropical pair-of-pants in its entirety not broken into balls. Indeed, the faces of the premutahedral skeleton are parameterized by the cyclicly ordered partitions σ of the set $\{0, 1, \ldots, n\}$. The condition for the face σ to lie over the tropical face F_I is that I has elements in at least two parts of σ.

All these four types of ball can be glued naturally to produce four homeomorphic objects, the first being the original complex hypersurface, the other three are polyhedral complexes of various degree of complexity.

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A dream desingularization algorithm

Michael Temkin

(joint work with Dan Abramovich, Jarosław Włodarczyk)

In a joint project with Abramovich and Włodarczyk, we construct a new desingularization algorithm in characteristic zero, which does not use history and exceptional divisors. It was well known for decades that such an algorithm does not exist in the usual setting, and the key of our success is in using orbifolds and non-representable weighted blow ups. Similar results were independently obtained by McQuillan and Marzo.

1. Classical embedded methods

1.1. **Reduction to principalization.** Until our works [1] and [2], all functorial (or canonical) desingularization methods followed the general framework of Hironaka and worked as follows. First, one locally embeds a singular variety X into a smooth variety $Y = Y_0$ and then only operates with sequences $Y_n \to \ldots Y_1 \to Y_0$ of blow ups whose centers $C_i \hookrightarrow Y_i$ are smooth. One takes $\mathcal{J}_0 \subset \mathcal{O}_{Y_0}$ to be the ideal of $X_0 = X$ and updates it via $\mathcal{J}_{i+1} = (\mathcal{J}_i \mathcal{O}_{Y_{i+1}})(\mathcal{I}_i \mathcal{O}_{Y_{i+1}})^{-d}$, where $\mathcal{I}_i$ is the ideal of C_i on Y_i and C_i lies in the locus where the order of $\mathcal{J}_i$ is at least d (or, equivalently, $\mathcal{J}_i \subseteq \mathcal{I}_i^d$).

The goal is to *principalize* $\mathcal{J}_0 \subseteq \mathcal{O}_Y$ by finding a sequence such that $\mathcal{J}_n = 1$. In this case, the last non-empty strict transform $X_m \hookrightarrow Y_m$ of X is smooth and hence resolves the singularities of X. This reduces the desingularization problem to the principalization problem.

1.2. **Dream algorithms.** In the above framework, one only needs to choose the centers C_i of the blow ups. The most natural approach is to look for an algorithm of the following type, that we call a *dream algorithm*: each time just take C_i to be the locus where the singularity is worst. Formally, this means that one should define an invariant $\mathrm{inv}_y(\mathcal{J})$ which measures the singularity of $\mathcal{J}$ at $y \in Y$ and accepts values in a totally ordered set, and each time one blows up the maximality locus of inv.

1.3. **Invariants and maximal contact.** Probably, the most natural attempt is to take a lexicographically ordered string $(d_1, \dots, d_l)$, where d_1 is the order of $\mathcal{J}$ at y, d_2 is a "secondary order" obtained by eliminating the first coordinate, etc. In the classical algorithms this is provided by the theory of maximal contact: one chooses x_1 to be a coordinate of "maximal contact" to $\mathcal{J}$, restricts a so called homogenized coefficient ideal $C(\mathcal{J})$ of order $d_1!$ onto $H_2 = V(x_1)$ obtaining an ideal $\mathcal{J}_2$, finds maximal contact x_2 to $\mathcal{J}_2$, etc., and defines $\mathrm{inv}_y(\mathcal{J}) = (d_1!, \mathrm{inv}_y(\mathcal{J}_2))$ recursively. Thus, $(d_1, d_2, \dots)$ is the string of normalized orders of $\mathcal{J}_i$.

1.4. **The counterexample: Whitney's umbrella.** Whitney's umbrella $V(x^2 - zy^2) \in \mathbb{A}^3$ is a classical example showing that dream algorithms do not exist in the usual setting. Indeed, the pinch point O at the origin is clearly the worst point. For example, the above invariant equals $(2, 3, 3)$ at O, while it equals $(2, 2)$ at the other points of the singular locus $S = V(x, y)$. Blowing up O produces the same singularity on the z-chart as $x' = x/z$ and $y' = y/z$ satisfy $x^2 - zy^2 = z^2(x'^2 - zy'^2)$ and the transform divides by z^2. In other simple examples, such as $V(x^2 + y^2 + z^m t^m)$, the natural invariant can even get worse.

1.5. **Classical solution.** Hironaka's solution of the above problem is to take the history into account. One worries that the exceptional divisor E is always snc and the invariant becomes of the form $(d_1, s_1, d_2, s_2, \dots)$, where s_i is the number of components of E through y. The s_i are not related to the nature of the current singularity, and are needed to encode the history. The Whitney umbrella, for example, is resolved by blowing up the pinch twice, and then (once enough "historical evidence" is accumulated) by blowing up the singular line.

2. Logarithmic algorithms

2.1. **Logarithmic smoothness.** In our work on resolution of morphisms, see [1, 2], it was important to replace varieties by log varieties and smoothness by log smoothness. In particular, instead of a smooth ambient variety Y we worked with a log smooth one, which étale locally looks as $\mathrm{Spec}(k[M][t_1, \dots, t_n])$ for a toric monoid M. The natural class of blow ups preserving log smoothness is wider: the centers are of the form $(m_1,\dots,m_r,t_1,\dots,t_l)$ for any monomials m_i.

2.2. **Kummer centers and blow ups.** To our surprise, the algorithm insisted to work with "fractional monomials" $m_i^{1/d}$ and, what we called, *Kummer centers* $\mathcal{I}=(m^{1/d},t)=(m_1^{1/d},\dots,m_r^{1/d},t_1,\dots,t_l)$. The technical solution was to work log étale locally: such $\mathcal{I}$ is an ideal for the Kummer log-étale topology on Y. Moreover, it is an honest ideal on the Kummer cover $Z=Y[m^{1/d}]$ of Y. The corresponding *Kummer blow up* along $\mathcal{I}$ is also defined log étale locally: one would like to define $Y'=\mathrm{Bl}_{\mathcal{I}}(Y)$ to be the quotient of $Z'=\mathrm{Bl}_{\mathcal{I}}(Z)$ by the Galois group G of Z/Y.

2.3. **Appearance of orbifolds.** The scheme theoretic quotient Y' is nothing but the normalized blow up of (m,t^d) or the weighted blow up of (m,t) with weights $(d,1)$. In general it is not log smooth, and we had to consider the finer orbifold quotient, which is log smooth. Thus, the Kummer blow up is defined as a non-representable modification $\mathcal{Y}'\to Y$ such that the pullback of $\mathcal{I}$ to $\mathcal{Y}'$ becomes a usual invertible ideal. It might be viewed as a refinement of Y', which is the course moduli space of $\mathcal{Y}'$.

Remark 1. *An important discovery of* [1] *and* [2] *was that there exist wider contexts, where principalization and embedded resolution can run. Using a larger supply of spaces and their modifications one can construct new algorithms. The use of generalized ideals and orbifolds seems almost inevitable.*

3. Weighted Hironaka and the dream algorithm

Kummer blow ups only use weighted blow ups of weights 1 and d, and the goal of the current project was to describe the natural algorithm in the context of arbitrary weighted blow ups. It turned out that this is a dream algorithm. Moreover, the coordinates defining the center are the classical iterative maximal order coordinates from §1.3. The only novelty is that one should blow them up with the weights they naturally come equipped with. Needless to say, this only becomes possible in the context of orbifolds.

3.1. ***h*-ideals and weighted blow ups.** The new algorithm operates with generalized ideals of the form $\mathcal{I}=(t_1^{1/w_1},\dots,t_n^{1/w_n})^l$. This time there is no log structure, so we view them as ideals for the h-topology, or simply ideals on fine enough alterations of Y. Any ideal on Y is invertible as an h-ideal since it becomes invertible on an appropriate modification. By the same reason, any finitely generated h-ideal is invertible. This reminds valuation rings, and not by accident – valuation rings are stalks of $\mathcal{O}_Y$ in the h-topology. In addition, different ideals may generate the same h ideal, in particular, $(t_1^{1/w_1},\dots,t_n^{1/w_n})^l=(t_1^{l/w_1},\dots,t_n^{l/w_n})$ as h-ideals. We define weighted blow ups along such h-ideals similarly to Kummer blow ups.

3.2. **Admissible centers.** By a $\mathcal{J}$-admissible center we mean an h-ideal $\mathcal{I}$ locally given by $(t_1^{d_1},\dots,t_n^{d_n})$ with $d_1\le d_2\le\ldots\le d_n$ and such that $\mathcal{J}\subseteq\mathcal{I}$.

3.3. **The dream algorithm.** The following theorem constructs a dream algorithm:

Theorem 1. *Let* $\mathcal{J} \subseteq \mathcal{O}_Y$ *be an ideal, then*

(i) There exists a unique $\mathcal{J}$*-admissible center* $\mathcal{I} = (t_1^{d_1}, \ldots, t_n^{d_n})$ *such that* $\mathrm{inv}(\mathcal{J}) := (d_1, \ldots, d_n)$ *is maximal possible with respect to the lexicographic order.*

(ii) Consider the weighted blow up $Y' = \mathrm{Bl}_{\mathcal{I}}(Y)$ *and the transform* $\mathcal{J}' = (\mathcal{J}\mathcal{O}_{Y'})(\mathcal{I}\mathcal{O}_{Y'})^{-1}$*. Then* $\mathrm{inv}(\mathcal{J}') < \mathrm{inv}(\mathcal{J})$*.*

3.4. **Justification.** The proof of our main theorem is based on the maximal contact theorem and perhaps can be viewed as its quintessence. In particular, coordinates $(t_1, \ldots, t_n)$ are just iterative maximal contact elements, and the weights are the corresponding (appropriately normalized) orders.

3.5. **Whitney's umbrella revisited.** Returning to the example of Whitney's umbrella given by $\mathcal{J} = (x^2 - zy^2)$, the invariant is $(2, 3, 3)$ and the center is $\mathcal{I} = (x^2, y^3, z^3) = (x^{1/3}, y^{1/2}, z^{1/2})^6$. The weighted blow up along $\mathcal{I}$ indeed decreases the invariant. For example, the z-chart is given by $x' = x/w^3$, $y' = y/w^2$, $z = w^2$, the transform is $(x^2 - zy^2)w^{-6} = x'^2 - y'^2$, and the invariant drops to $(2, 2)$.

3.6. **Destackification.** Similarly to the log principalization of [1, 2], the weighted principalization of [3] principalizes ideals on orbifolds even when it starts with an ideal on a smooth variety. On the level of varieties (or coarse moduli spaces) its output has quotient singularities, but the latter can be easily resolved by combinatorial methods. This can be also obtained by a slightly more general destackification procedure. So, our method produces a classical desingularization as well.

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Tropical geometry and coherent sheaves on the projective plane

Pierrick Bousseau

Tropical geometry is known to be an efficient way to describe holomorphic curves in SYZ Lagrangian torus fibrations. The goal of this talk was to describe an alternative way how tropical geometry can emerge as an answer to algebro-geometric questions.

Following some general conjectural picture due to Kontsevich-Soibelman [KS14] and some previous work of Bridgeland [Bri17] in the context of quivers, we construct a tropical picture, a scattering diagram, in the space of stability conditions on the derived category of coherent sheaves on the complex projective plane

$\mathbb{P}^2$. This scattering diagram provides a new algorithm computing Hodge numbers (for intersection cohomology) of the classical moduli spaces of Gieseker semistable sheaves on $\mathbb{P}^2$.

Moduli spaces of Gieseker semistable sheaves on $\mathbb{P}^2$ form a family of possibly singular projective varieties M_γ, indexed by some $\gamma \in \mathbb{Z}^3$ keeping track of the topological data (Chern classes). Using intersection cohomology, we can define Hodge numbers $Ih^{p,q}(M_\gamma)$, reducing to usual Hodge numbers when M_γ is smooth.

A natural idea to study the Hodge numbers $Ih^{p,q}(M_\gamma)$ is to try to modify the notion of stability, in order to change and possibly simplify the geometry of the moduli spaces of semistable objects. One way to do that is to consider Bridgeland stability conditions on the derived category of coherent sheaves [Bri07]. For surfaces like $\mathbb{P}^2$, there is a standard way to construct stability conditions. The upshot is an upper half-plane U of stability conditions. For every $\sigma \in U$, we have moduli spaces M_γ^σ of σ-semistable objects, which for an appropriate range of σ coincide with the moduli spaces M_γ we wish to study. Along some particular curves in U, called walls, the geometry of the moduli spaces M_γ^σ changes.

In order to get a tropical picture, the main idea is to consider rays L_γ in U defined by the condition that $\sigma \in L_\gamma$ if and only of M_γ^σ is nonempty and the central charge Z_γ^σ is purely imaginary. Furthermore, we do a change of coordinates on U such that the rays L_γ become straight lines: the resulting U becomes the upper-part of a parabola in $\mathbb{R}^2$. To a ray L_γ and a point $\sigma \in M_\gamma$, we attach the numerical data of the Hodge numbers $Ih^{p,q}(M_\gamma)$.

When several rays cross, it means that we are on a wall. An essential point is that knowing the rays with their numerial data on one side of the wall, there is a completely algorithmic way to produce the rays with their numerical data on the other side of the wall, called the Kontsevich-Soibelman wall-crossing formula [KS08]. This formula is expected to be satisfied due to the connection with Donaldson-Thomas theory of the non-compact Calabi-Yau 3-fold $K_{\mathbb{P}^2}$. The proof of this formula in the precise setting we care about is a technical part of the story, and uses previous work of Meinhardt-Reineke [MR17] and Li-Zhao [LZ19b].

If we wish to obtain an algorithm computing all the rays L_γ with their numerical data, it is then enough to find all the rays existing in a particular region of U. We show that it is possible using a quiver description of the derived category of coherent sheaves on $\mathbb{P}^2$.

We can use our algorithm to get concrete results such that a new proof of the fact that $Ih^{p,q}(M_\gamma) = 0$ if $p \neq q$ (a previous proof follows from the work of Manschot-Mozgovoy [MM18]), or a proof that the Euler characteristic of M_γ for γ corresponding to dimension one degree d sheaves of holomorphic Euler characteristic one, is divisible by $3d$ (a result previously conjectured by Choi).

Finally, we remark that the tropical picture we obtain in fact coincides with a previously known tropical picture, but coming from the more traditional perspective of holomorphic curves in SYZ Lagrangian torus fibrations. More precisely, this tropical picture is the Gross-Siebert picture for $\mathbb{P}^2$ relative to a smooth cubic

E [CPS10], and so the scattering diagram is expected to compute relative Gromov-Witten invariants of $(\mathbb{P}^2, E)$. This expectation is proved in some recent work of Gabele.

The fact that the same tropical picture has two very different interpretations, as computing relative Gromov-Witten invariants and as computing numerical invariants of moduli spaces of sheaves, makes possible to use it as a bridge to transfer information from the sheaf side to the Gromov-Witten side and vice-versa. In particular, using knowledge on the sheaf side, it becomes possible to prove a roughly 15 years old conjecture due to Takahashi [Tak01] on the multicovering structure of relative Gromov-Witten invariants of $(\mathbb{P}^2, E)$.

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Obstructions to deforming maps from curves to surfaces

Takeo Nishinou

Nearly 100 years ago, Severi [4, 5] proved that a compact smooth complex curve C on a compact smooth complex surface S is unobstructed in the sense that its first order infinitesimal deformation can be extended to any higher order, provided the curve is *semiregular*. The curve C is called semiregular if the natural restriction map $H^0(S, K_S) \to H^0(C, \iota^* K_S)$ is surjective, here K_S is the canonical sheaf of S.

The notion of semiregularity and the associated result were generalized by Kodaira and Spencer [2] to smooth divisors on higher dimensional complex manifolds. Later, Bloch [1] extended the notion of semiregularity to local complete intersection subvarieties, and related it to the smoothness of the Hilbert scheme at the corresponding point as well as to variation of Hodge structures. In particular, the semiregularity of a local complete intersection subvariety guarantees the vanishing of the obstruction to the deformation.

Although these results are striking, often it is not easy to check whether a given subvariety is semiregular or not, and even if we can check it, there is little control of the deformation beyond its existence. On the other hand, since the obstruction to deformations is in principle determined by the information of a neighborhood of the subvariety, it would not be too optimistic to expect that we can extend the notion of semiregularity from the original cohomological (in other words, global) condition to a more local one.

We show that this is in fact the case for curves on surfaces, and we extend the notion of semiregularity to maps rather than subvarieties, see [3].

Theorem 1. *Let $\varphi\colon C \to X$ be a map from a reduced complete complex curve to a smooth complex surface, which is locally embedding. Assume that φ is semiregular, that is, the natural map $H^0(X, K_X) \to H^0(C, \varphi^* K_X)$ is surjective. Then the map φ is unobstructed in the sense that any first order deformation can be extended to arbitrary higher order.*

Here, a map $\varphi\colon C \to X$ which is locally embedding is called semiregular if the natural map $H^0(S, K_S) \to H^0(C, \varphi^* K_S)$ is a surjection. In this theorem, the surface S need not be compact, reflecting the local nature of its proof.

In practical situations, starting from a few embedded curves on a surface for which the classical semiregularity condition holds, we may construct new curves by putting these together in a simple combinatorial way. Then the constructed curves, seen as the images of suitable maps, often satisfy the (extended) semiregularity again, and we can deform such curves on the surface. Note that although φ is locally embedding, the image $\varphi(C)$ need not be reduced.

The case where Theorem 1 is most effective would be when the target X has the trivial canonical sheaf. In this case, any reduced curve on X is semiregular. Thus, any map φ from a reduced curve which is locally embedding is unobstructed. Based on this observation, we can prove the following.

Corollary 1. *A generic complex polarized K3 surface contains infinitely many g dimensional families of irreducible nodal curves of geometric genus g, for any positive integer g.*

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Chirality of real cubic fourfolds

Sergey Finashin
(joint work with Viatcheslav Kharlamov)

In our previous work [FK1] we have classified real non-singular cubic hypersurfaces in the 5-dimensional projective space up to equivalence that includes both real projective transformations and continuous variations of coefficients preserving the hypersurface non-singular. Here, we perform a finer classification giving a full answer to the chirality problem: which of real non-singular cubic hypersurfaces can not be continuously deformed to their mirror reflection.

Both deformation equivalence relations emerge naturally in the study of real non-singular projective hypersurfaces in the framework of 16th Hilbert's problem. More precisely, the *pure deformation equivalence* assigns hypersurfaces to the same equivalence class if they can be joined by a continuous path (called a *real deformation*) in the space of real non-singular projective hypersurfaces of some fixed degree. Another one is the *coarse deformation equivalence*, in which real deformations are combined with real projective transformations.

If the dimension of the ambient projective space is even, then the group of real projective transformations is connected, and the above equivalence relations coincide. By contrary, if the dimension of the ambient projective space is odd, this group has two connected components, and some of coarse deformation classes may split into two pure deformation classes. The hypersurfaces in such a class are not pure deformation equivalent to their mirror images and are called *chiral*. The hypersurfaces in the other classes are called *achiral*, since each of them is pure deformation equivalent to its mirror image.

The first case where a discrepancy between pure and coarse deformation equivalences shows up is that of real non-singular quartic surfaces in 3-space (achirality of all real non-singular cubic surfaces is due to F. Klein [Kl]). In this case it was studied in [Kh1, Kh2], where it was used to upgrade the coarse deformation classification of real non-singular quartic surfaces obtained by V. Nikulin [N] to a pure deformation classification.

Real non-singular cubic fourfolds is a next by complexity case. Their deformation study was launched in [FK1], where we classified them up to coarse deformation equivalence. Then in [FK2] we began studying of the chirality phenomenon and gave complete answers for cubic fourfolds of maximal, and almost maximal, topological complexity. The approach, which we elaborated and applied in [FK2] relies on the surjectivity of the period map for cubic fourfolds established by R. Laza [La] and E. Looijenga [Lo].

Recall that according to [FK1] there exist precisely 75 coarse deformation classes of real non-singular fourfold cubic hypersurfaces $X \subset P^5$ (throughout the paper X stands both for the variety itself and for its complex point set, while $X_{\mathbb{R}} = X \cap P^5_{\mathbb{R}}$ denotes the real locus). These classes are determined by the isomorphism type of the pairs $(\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X), h \in \mathbb{M}(X))$ where $\mathbb{M}(X) = H^4(X;\mathbb{Z})$ is considered as a lattice, $h \in \mathbb{M}(X)$ is the *polarization class* that is induced from the standard generator of $H^4(P^5;\mathbb{Z})$, and conj^* is induced by complex conjugation $\mathrm{conj} : X \to X$. This result can be simplified further and expressed in terms of a few simple numerical invariants. Namely, it is sufficient to consider the sublattice $\mathbb{M}^0_+(X) \subset \mathbb{M}(X)$, $\mathbb{M}^0_+(X) = \{x \in \mathbb{M}(X) : \mathrm{conj}^* x = x, xh = 0\}$, and to retain only the following three invariants: the rank ρ of $\mathbb{M}^0_+$, the rank d of the 2-primary part $\mathrm{discr}_2 \mathbb{M}^0_+$ of the discriminant $\mathrm{discr}\, \mathbb{M}^0_+$, and the type, even or odd, of the discriminant form on $\mathrm{discr}_2 \mathbb{M}^0_+$.

Thus, to formulate the pure deformation classification of real non-singular cubic fourfolds, it is sufficient to list the triples of invariants (ρ, d, parity) which specify the coarse deformation classes and to indicate which of the coarse classes are chiral, and which ones are achiral.

Theorem 1. *Among the 75 coarse deformation classes precisely 18 are chiral, and, thus, the number of pure deformation classes is 93. The chiral classes have pairs (ρ, d) satisfying $\rho+d \le 12$. The only achiral classes with $\rho+d \le 12$ are three classes with $4 \le \rho = d \le 6$ and one class with $(\rho, d) = (8, 4)$ and even pairity.*

A complete description of the pure deformation classes is presented in Table 1, where the coarse deformation classes are marked by letters c and a: by c, if the class is chiral, and by a, if it is achiral. We use ρ and d as Cartesian coordinates and employ bold letters to indicate even parity, while keeping normal letters for odd. For some pairs (ρ, d) there exist two coarse deformation classes, one with even discriminant form, and another with odd, and in this case, we put the even one in brackets.

In fact, the values of ρ and d determine the topology of the real locus of the cubic fourfold and are determined by it. Namely, for all pairs (ρ, d) except one the real locus of the fourfold is diffeomorphic to $\mathbb{RP}^4 \# a(S^2 \times S^2) \# b(S^1 \times S^3)$, where $a = \frac{1}{2}(\rho - d), b = \frac{1}{2}(22 - \rho - d)$. The exception is $(\rho, d, \mathrm{parity}) = (12, 10, \mathrm{even})$, in which case the real locus is diffeomorphic to $\mathbb{RP}^4 \sqcup S^4$ (see [FK3]). Comparing this with Table 1 we come to the following conclusion.

Corollary 1. *Chirality of a cubic $X \subset P^4$ is determined by the topological type of its real locus $X_{\mathbb{R}}$ unless $X_{\mathbb{R}} = \mathbb{RP}^4 \# 2(S^2 \times S^2) \# 5(S^1 \times S^3)$, or equivalently, $(\rho, d) = (8, 4)$. If $(\rho, d) = (8, 4)$, then X is achiral in the case of even parity, and chiral in the case of odd.* □

TABLE 1. Pure deformation classification via chirality

d \ ρ	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
11										a											
10									a		a(**a**)										
9								a		a		a									
8							a(**a**)		a		a(**a**)		a								
7						a		a		a		a		a							
6					a		a(**a**)		a		a(**a**)		a		a						
5				a		c		a		a		a		a		a					
4			a		c		c(**a**)		a		a(**a**)		a		a(**a**)		a				
3		c		c		c		c		a		a		a		a		a			
2	c		c(**c**)		c		**c**		c		a(**a**)		a		**a**		a		a(**a**)		
1		c		c						c		a						a		a	
0			**c**								**c**								**a**		

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Tropical refined curve counting

LOTHAR GÖTTSCHE

(joint work with Florian Block, Franziska Schroeter, Vivek Shende)

In the talk I reviewed our work on tropical refined curve counting and a number of related developments.

1. REFINED CURVE COUNTING

Let S be a smooth projective surface and L a line bundle on S. The Severi degree $n_{(S,L),\delta}$ is the number of δ-nodal curves in $|L|$ through $\dim |L| - \delta$ points in S, where $|L|$ is the complete linear system of L. In [9] a conjectural generating function for the Severi degrees $n_{(S,L),\delta}$ was given, valid whenever L is sufficiently ample with respect to δ. In [10] refined curve counting invariants $N^{(S,L),\delta}$ were defined in terms of the χ_y-genera of the relative Hilbert schemes of points $\mathcal{C}^{[n]}$ of the universal curve over $|L|$. These are symmetric Laurent polynomials in y, which for L sufficiently ample with respect to δ interpolate between the Severi degrees $n_{(S,L),\delta}$ (at $y = 1$) and the (totally real) Welschinger numbers counting real algebraic curves though configurations or real points (at $y = -1$). In [10] also a conjectural generating function for these refined invariants is given. In [1] these refined curve counting invariants are interpreted as refined K-theoretic Donaldson-Thomas invariants of the total space of the canonical line bundle on S, which is a local Calabi-Yau threefold.

2. Refined tropical curve counting

In tropical geometry curves on toric surfaces S in a linear system $|L|$ can be counted by piecewise linear objects in $\mathbb{R}^2$, the tropical curves Γ (the pair (S, L) is encoded in the directions of the unbounded edges of Γ). These curves Γ are counted with certain vertex multiplicities $\mu(v)$, associated to every vertex v of Γ, the multiplicity of $\mu(\Gamma)$ is the product of $\mu(v)$ over all vertices v of Γ. Counting tropical curves through $\dim |L| - \delta$ general points in $\mathbb{R}^2$ using the Mikhalkin multiplicity $m(v)$ as vertex multiplicities, one obtains the tropical Severi degrees $n^{trop}_{(S,L),\delta}$, which coincide with the Severi degrees $n_{(S,L),\delta}$ for toric surfaces. In the same way using the Welschinger multiplicities as vertex multiplicities, one obtains the totally real tropical Welschinger invariants, which coincide with the totally real Welschinger numbers for suitable point configurations on S.

In [3] a new polynomial vertex multiplicity is introduced, the quantum version $[m(v)]_y = \frac{y^{m(v)/2} - y^{-m(v)/2}}{y^{1/2} - y^{-1/2}}$ of the Mikhalkin multiplicity $m(v)$. We define the refined tropical Severi degres $N^{trop}_{(S,L),\delta}(y)$ as the count of curves with this multiplicity. They interpolate between the tropical Severi degrees (at $y = 1$) and the (totally real) tropical Welschinger invariants (at $y = -1$). Furthermore it is conjectured that if L is sufficiently ample, the refined Severi degrees coincide with the refined curve counting invariants of [10]. There is an approach towards proving this conjecture via non-Archimedian motivic integration [14]. In [12] it is shown that the refined tropical Severi degrees are indeed tropical invariants, i.e. independent of the tropical configuration of points. In [13] it is shown that under special conditions the tropical refined curve count can be interpreted as counting real curves C, counting each C as a monomial $y^{w(C)}$, with the weight $w(C)$ expressed in terms of the signed area of the amoeba of C.

3. Fock space

The refined Severi degrees can for many toric surfaces (given by h-transversal lattice polygons) be computed in terms of the operation of a Heisenberg algebra on a Fock space. This was shown in [4] motivated by the work in [6] for the usual

Severi degrees. The Heisenberg algebra H is generated by operators a_n, b_n for $n \in \mathbb{Z}$ with commutation relations

$$[a_n, a_m] = 0 = [b_n, b_m], \quad [a_n, b_m] = [n]_y \delta_{n,-m}, \quad [n]_y = \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}.$$

The corresponding Fock space $F(H)$ is the space of all polynomials with coefficients in $\mathbb{Q}[y^{\pm 1/2}]$ in the creation operators a_{-i} b_{-j} with $i, j > 0$. The refined Severi degrees are obtained as vacuum expectation values of certain operators in H on $F(H)$. The reason for this result is the following. The vacuum expectation values can be computed in terms of Feynman diagrams, counted with certain multiplicities. On the other hand the refined tropical Severi degrees can be computed in terms of floor diagrams, which encode in a simplified way the combinatorics of the tropical curves. One can show that the Feynman diagrams and the floor diagrams are the same, and both are counted with the same multiplicities.

4. Logarithmic Gromov-Witten invariants with λ-classes

Bousseau [5] relates refined Severi degrees to log-Gromov-Witten invariants with λ-classes of toric surfaces S. The refined Severi degree corresponding to the count of genus g curves is obtained from the generating function of Gromov-Witten invariants with λ-classes of curves of all genera $g' \geq g$ by a change of variables. The expected dimension of the moduli space $M = M_{g',n}(S, L)$ of genus g' maps to S is too large to obtain a finite count of curves by $g' - g$. One integrates against the Chern class $\lambda_{g'-g} = c_{g'-g}(\Omega_{C/M})$ of the relative dualizing sheaf of the universal curve over M to obtain a class of the right dimension.

5. Refined descendent invariants

The refined tropical Severi degrees interpolate between the Severi degrees and the totally real Welschinger invariants counting real curves through configurations of real points. More general Welschinger invariants count real genus 0 curves through configurations of real points and pairs of complex conjugated points. For toric surfaces S these numbers can be computed via tropical geometry. For simplicity I restrict to the case that $S = \mathbb{P}^2$, then $W^0_{d,r,s}$ counts degree d curves through r real points and s pairs of complex conjugated points. In [11] a refinement $N^{0,trop}_{d,r,s}(y)$ of this tropical invariant is given. Roughly speaking the definition is as follows: the same tropical curves are counted as for the usual tropical Severi degrees, however the point conditions are changed: Let P be a configuration of r thin and s fat points in $\mathbb{R}^2$. We say that a tropical curve Γ of degree d passes through P if the thin points of P lie on Γ and the fat points of P are vertices of Γ. These tropical curves are again counted with a vertex multiplicity. A vertex of Γ which is not a fat point of P is counted with the refined multiplicity $[m(v)]_y = \dfrac{y^{m(v)/2} - y^{-m(v)/2}}{y^{1/2} - y^{-1/2}}$, where $m(v)$ is the Mikhalkin multiplicity and a vertex containing a fat point with the new multiplicity $\{m(v)\}_y = \frac{y^{m(v)/2} + y^{-m(v)/2}}{y^{1/2} + y^{-1/2}}$.

This is in general only a rational function in $y^{1/2}$, but it is shown that the multiplicity of any tropical curve is a Laurent polynomial in y. Furthermore one obtains in this way a tropical invariant, which coincides with the Welschinger invariant $W^0_{d,r,s}$ at $y = -1$, and its value at $y = 1$ are the primary descendent Gromov Witten invariants $\int_{M_{0,n}(\mathbb{P}^2,d)} \psi_1^{a_1} ev_1^*(pt) \cdots \psi_n^{a_n} ev_n^*(pt)$ with $a_i \in \{0, 1\}$. Here $M_{0,n}(\mathbb{P}^2, d) = \{f : (C, x_1, \dots, x_n) \to \mathbb{P}^2\}$ is the moduli space of genus 0 stable maps and $\psi_i = c_1(L_i)$, with L_i the line bundle on $M_{0,n}(\mathbb{P}^2, d)$ with fibre T^*_{C,x_i} at $(f, C, x_1, \dots, x_n)$. In [2] higher order refined tropical descendent invariants are defined. These are again polynomial invariants in a variable y, which at $y = 1$ specialize to the higher order descendents $\int_{M_{0,n}(\mathbb{P}^2,d)} \psi_1^{a_1} ev_1^*(pt) \cdots \psi_n^{a_n} ev_n^*(pt)$ with a_i arbitrary.

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