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Differentialgeometrie im Großen

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Abstract. The topics discussed at the meeting reflected current trends in global differential geometry. These topics included complex geometry, Einstein metrics, geometric flows, metric geometry and manifolds satisfying curvature bounds.

Mathematics Subject Classification (2010): 53C.

Introduction by the Organizers

The workshop *Differentialgeometrie im Großen* was held June 30 - July 5, 2019. The participants were specialists in differential geometry and its neighboring fields, covering a broad spectrum of subareas which are in the focus of current developments.

The lectures during the five days of the meeting were roughly organized according to different themes.

The first day of the meeting began with talks on Einstein metrics and metrics with conical singularities. On the second day, the theme of the morning was complex geometry while in the afternoon we saw talks on Higgs bundles and curves on hyperbolic surfaces.

Wednesday morning's talks included topics on geodesic metric spaces with bounded curvature and the actions of isometry groups. In the afternoon we had the traditional hike.

The theme of Thursday was geometric flows, and in particular we heard about some of the latest advances in Ricci flow. We ended the workshop on Friday morning with three talks, whose topics included homogeneous Einstein metrics and metrics of nonnegative sectional curvature.

The meeting gave a good overview of the current developments in differential geometry, and highlighted some of the important developments in the field. The workshop was attended by researchers from around the world, ranging from graduate students to scientific leaders in their areas.

The atmosphere during the meeting was lively and open, and greatly benefited from the ideal environment at Oberwolfach.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Workshop: Differentialgeometrie im Großen

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Abstracts

On groups of isometries preserving multiple horospheres

Grigori Avramidi
(joint work with Tâm Nguyễn-Phan)

Let X be an n-dimensional Hadamard manifold and let Γ be a group acting on X by covering space transformations. Suppose that Γ preserves some horospheres. To keep track of these, denote by $\mathrm{Fix}^0(\Gamma)$ the set of points at infinity whose horospheres are preserved by Γ. This talk explained a relation between the dimension of Γ and $\mathrm{Fix}^0(\Gamma)$ in the case when $\Gamma = \mathbb{Z}^r$ is a free abelian group of rank r. To formulate this relation, we introduced certain simplices in the ideal boundary of X, called Busemann simplices, which are constructed using convex combinations of Busemann functions. Here is the main result discussed in the talk.

Theorem. If $\mathrm{Fix}^0(\mathbb{Z}^r)$ contains a non-degenerate Busemann k-simplex, then

$$\dim X \geq k + 1 + r.$$

The same result holds for general groups Γ if the rank r is replaced by the homological dimension of Γ. (Theorem 8 of [1]).

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Marked length spectrum rigidity for actions on CAT(0) cube complexes

Jonas Beyrer
(joint work with Elia Fioravanti)

Given an action $G \curvearrowright X$ of a finitely generated group G on a metric space (X, d), the *marked length spectrum* $\ell_{G \curvearrowright X}$ or sometimes just ℓ_X is the function from the group to $\mathbb{R}_{\geq 0}$ that assigns each element its translation length, i.e. $\ell_X : G \to \mathbb{R}_{\geq 0}, \ell_X(g) \mapsto \inf_{x \in X} d(x, g \cdot x)$.

If restricting to particular spaces and actions the marked length spectrum is a natural candidate to tell when spaces are equivariantly isometric and when not, which is captured in the following question.

Question. Given two actions $G \curvearrowright X, G \curvearrowright Y$ such that both actions and both spaces belong to a certain class. Are then X and Y equivariantly isometric?

If the answer to this question is yes, we speak of *marked length spectrum rigidity.*

Let us explain two specific cases of this: First, assume that G is torsion free, the action on X, Y is proper and cocompact and X, Y are negatively curved manifolds. In this case the quotients $G\backslash X$ and $G\backslash Y$ are compact negatively curved manifolds. Then the question is actually a notoriously hard conjecture from the 1980s - see [3]. For long time only the case when X, Y are surfaces was known to be true [6]. However, very recently big progress has been made in the general case [5].

Second, assume that $G \curvearrowright T_1, T_2$, where T_i are $\mathbb{R}$-trees (sometimes also called metric trees) and the action is *non-elementary* which means that the induced action on the tree union the space of ends $G \curvearrowright T_i \cup \mathrm{Ends}(T_i)$ has no finite orbits. Then a classical result of Culler and Morgan says that in this case we have marked length spectrum rigidity [4].

For us, the goal is to consider this question for actions on CAT(0) cube complexes. In recent years cube complexes have become somewhat ubiquitous within geometric group theory, as there are many groups acting nicely on them and such an action allows to derive strong algebraic properties of the group. Probably most prominently has been their use for the proofs of the virtual Haken and virtual fibering conjectures for 3-manifolds.

Note that CAT(0) cube complexes have two natural metrics. We will consider the length functions with respect to the ℓ^1 metric (which is not CAT(0)). Then combining the results from [1, 2] we have the following marked length spectrum rigidity result for actions on CAT(0) cube complexes with repsect to the ℓ^1 metric.

Theorem. (B. - Fioravanti) *Let X, Y be irreducible* CAT(0) *cube complexes and $G \curvearrowright X, Y$ such that $\ell_X = \ell_Y$. Then X and Y are equivariantly isomorphic if one of the following holds*

(1) G is hyperbolic, the actions are proper and cocompact and X, Y are essential and hyperplane essential,
(2) the actions are non-elementary and essential, X, Y have no free faces and $\mathrm{Aut}(X), \mathrm{Aut}(Y)$ *contain uniform lattices.*

Irreducible means that the cube complexes do not split as a non-trivial product and *non-elementary* means that the action on the visual compactification $X \cup \partial_\infty X$ has no finite orbits. For *essential, hyperplane-essential* and *no free faces* see [2].

We want to remark that irreducible, essential and hyperplane-essential are necessary conditions for length spectrum rigidity to hold. In particular in point (1) of the theorem we have optimal assumptions on the cube complex; though the assumption on the group and the group action are a bit restrictive. In point (2) however, the assumption on the group and the action are very little (probably optimal), at the cost of stronger assumptions on the cube complex.

An application of the theorem is to generalize the compactification of the Culler-Vogtmann outer space associated to free groups, to the 'untwisted' outer space associated to an irreducible right angled Artin group. For more details see [2].

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Homogeneous Einstein metrics on Euclidean spaces are Einstein solvmanifolds

Christoph Böhm

(joint work with Ramiro Lafuente)

A Riemannian manifold (M^n, g) is called homogeneous, if its isometry group $\mathrm{Isom}(M^n, g)$ acts transitively on M^n, and it is called Einstein, if its Ricci tensor satisfies $\mathrm{Ric}_\mathrm{g} = \lambda \cdot \mathrm{g}$, for some Einstein constant $\lambda \in \mathbb{R}$.

Theorem 1. *Homogeneous Einstein metrics on Euclidean spaces are isometric to Einstein solvmanifolds.*

A simply-connected Riemannian solvmanifold is a simply-connected solvable Lie group endowed with a left-invariant metric. Theorem 1 was known for Ricci-flat homogeneous spaces, for homogeneous $\mathbb{R}$-bundles over irreducible Hermitian symmetric spaces and in dimensions $n \leq 5$ and $n = 7$ [1].

Recall now one of the main results [2] in the field due to Lauret, who proved that Einstein solvmanifolds are standard. The standard condition is an algebro-geometric condition introduced and studied in [3] by Heber. For standard Einstein solvmanifolds of fixed dimension, Heber showed for instance finiteness of the eigenvalue type of the *modified Ricci curvature*, see below, a result intimately related to the finiteness of critical values of a (real) moment map. Even though a classification of standard Einstein solvmanifolds seems difficult, Theorem 1 together with the work of Lauret and Heber would yield a very precise understanding of *all* non-compact homogeneous Einstein manifolds, provided the Alexseevskii conjecture holds true.

Turning to the proof of Theorem 1, we would like to mention that a purely algebraic proof is elusive at the moment. To overcome this, we prove that homogeneous Euclidean Einstein spaces admit cohomogeneity-one actions by non-unimodular subgroups with orbit space $\mathbb{R}$. We then show *periodicity* of the corresponding foliation by orbits, meaning that after passing to a homogeneous quotient the orbit space becomes S^1, and that the integral of the mean curvature of the orbits over the orbit space vanishes, a condition we call *integral minimality.* This then reduces the proof of Theorem 1 essentially to the following

Theorem 2. *Suppose that (M^n, g) admits an effective, cohomogeneity-one action of a Lie group $\bar{G}$ with closed, integrally minimal orbits and $M^n/\bar{G} = S^1$. If in addition (M^n, g) is* orbit-Einstein *with negative Einstein constant, then all orbits are standard homogeneous spaces.*

A cohomogeneity-one manifold (M^n, g) is called *orbit-Einstein* with negative Einstein constant, if for $\lambda < 0$ we have $\mathrm{Ric}_{\mathrm{g}}(\mathrm{X},\mathrm{X}) = \lambda \cdot \mathrm{g}(\mathrm{X},\mathrm{X})$ for all vectors X tangent to orbits. Generalizing [3], we say that a homogeneous space $(\bar{G}/\bar{H}, \bar{g})$ is *standard*, if the Riemannian submersion induced by the free isometric action of the maximal connected normal nilpotent subgroup $\bar{N} \leq \bar{G}$ on $\bar{G}/\bar{H}$ has *integrable* horizontal distribution.

We turn now to the proof of Theorem 2. Since the $\bar{G}$-orbits form a family of equidistant hypersurfaces in M^n, we write $g = dt^2 + \bar{g}_t$, $t \in \mathbb{R}$, for a smooth curve of homogeneous metrics $\bar{g}_t$ on $\bar{M} = \bar{G}/\bar{H}$. By the Gauß and the Riccati equation the orbit-Einstein equation on (M^n, g) with Einstein constant -1 is equivalent to the "Einstein flow" $\bar{L}_t' + (\mathrm{Tr}L_t) \cdot \bar{L}_t = \mathrm{Ric}_{\bar{\mathrm{g}}_t} + \bar{\mathrm{g}}_t$. Here $\bar{L}_t$ denotes the shape operator of an orbit and $\mathrm{Ric}_{\bar{\mathrm{g}}_t}$ the Ricci-endomorphism of $(\bar{M}, \bar{g}_t)$.

We decompose the Ricci tensor $\mathrm{ric}_{\bar{\mathrm{g}}}$ of a homogeneous space $(\bar{M} = \bar{G}/\bar{H}, \bar{g})$ as a sum of two tensors, one tangent to the $\mathrm{Diff}_{\bar{G}}(\bar{M})$-orbit through $\bar{g}$, and another one, the *modified Ricci curvature* $\mathrm{ric}^*_{\bar{\mathrm{g}}}$, orthogonal to it. Here, $\mathrm{Diff}_{\bar{G}}(\bar{M})$ denotes the set of automorphisms of $\bar{G}$ preserving $\bar{H}$. We set $h(\bar{g}) := \frac{1}{2} \cdot (\mathrm{scal}^*_{\bar{\mathrm{g}}} - \mathrm{scal}_{\bar{\mathrm{g}}}) \geq 0$.

Using the orbit-Einstein equation for (M^n, g) and the compactness of the orbit space we establish a maximum principle for the real-valued function $h(t) := h(\bar{g}_t)$, which yields an upper bound for $2h$ given by $\mathrm{tr}\beta^+ = \mathrm{n} - 1 - 1/\|\beta\|^2 \geq 0$. Here β is the stratum label of the homogeneous space $\bar{G}/\bar{H}$, a self-adjoint endomorphism, coming from the Morse-type, Kirwan-Ness stratification of the space of Lie brackets on $T_e\bar{G}$ introduced by Lauret [2]. From this a priori estimate for h we establish the existence of a Lyapunov function for the orbit-Einstein equation. Using that the orbits are integrally minimal it follows that this Lyapunov function is periodic, hence constant. As a consequence, several inequalities become equalities and Theorem 2 follows.

Finally, we indicate how Theorem 2 implies Theorem 1. Using the standard condition for the given homogeneous Einstein space $(\mathbb{R}^n = G/H, g)$ and all the codimension-one orbits, one shows that the simple factors of a Levi factor $L \leq G$ are pairwise orthogonal. Together with the condition $G/H \simeq \mathbb{R}^n$, this implies that the induced metric on L/H is *awesome*, that is, it admits an orthogonal Cartan decomposition. This leads to a contradiction by [4], unless the Levi factor is trivial, in which case G is solvable.

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Magical nilpotents, higher Teichmüller spaces and Higgs bundles

Brian Collier

(joint work with Steve Bradlow, Oscar Garcia-Prada, Peter Gothen, André Oliveira)

Let S be a closed, connected and orientable surface of genus $g \geq 2$ and let G be a real or complex simple, connected Lie group. Associated to this fixed data is a moduli space $\mathcal{X}(S, G)$ called the G character variety of S. This moduli space parameterizes conjugacy classes of reductive representations $\rho : \pi_1 S \to G$ of the fundamental group of S into G. Namely,

$$\mathcal{X}(S, G) = Hom^+(\pi_1 S, G)/G,$$

where $Hom^+(\pi_1 S, G)$ is the set of group homomorphisms which are completely reducible after post composition with the adjoint representation.

When G is *compact or complex*, the set of connected components of $\mathcal{X}(S, G)$ is in bijection with the cohomology group $H^2(S, \pi_1 G)$. This was proven in [19] for $G = SU(n)$, in [20] for all compact groups, and in [18] for complex groups. Since a simple Lie group G is homotopy equivalent to a compact subgroup $K < G$, for G complex, every representation $\rho : \pi_1 S \to G$ can be deformed to a compact representation, i.e., one of the form $\tilde{\rho} : \pi_1 S \to K \to G$. For real Lie groups, the component count of $\mathcal{X}(S, G)$ is more subtle.

One motivation for studying character varieties comes from hyperbolic geometry. Given a representation $\rho : \pi_1 S \to PSL_2\mathbb{R}$ which is injective and has discrete image (discrete-faithful), the quotient $\mathbb{H}^2/\rho(\pi_1 S)$ of the hyperbolic plane $\mathbb{H}^2$ by ρ is homeomorphic to S. Moreover, identifying $PSL_2\mathbb{R}$ with the orientation preserving isometries of $\mathbb{H}^2$, the surface S inherits a hyperbolic metric. In this way, the Teichmüller space $\mathcal{T}(S)$ of isotopy classes of hyperbolic metrics on S is identified with the set of conjugacy classes of discrete-faithful representations $\rho : \pi_1 S \to PSL_2\mathbb{R}$. In fact, under this identification, $\mathcal{T}(S) \subset \mathcal{X}(S, PSL_2\mathbb{R})$ is an open and closed set consisting entirely of discrete-faithful representations. In particular, $\mathcal{T}(S)$ does not contain compact representations.

For higher rank Lie groups, there is a class of representations called *Anosov representations* which generalize many features of discrete-faithful representations into $PSL_2\mathbb{R}$. In particular, Anosov representations define *open* subsets of the character variety consisting entirely of discrete-faithful representations with many nice geometric and dynamical properties, see [17, 11, 15, 9]. A notable difference with $\mathcal{T}(S)$ is that these open sets are not necessarily closed. One way to construct Anosov representations is to deform $\mathcal{T}(S)$ into character varieties of higher rank Lie groups. Such deformations arise from nilpotent elements of Lie algebras.

Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$. An element $e \in \mathfrak{g}$ is nilpotent if $ad_e : \mathfrak{g} \to \mathfrak{g}$ is a nilpotent endomorphism. The Lie group G acts on the set of nilpotents with finitely many orbits. Moreover, there is a unique open and dense orbit; such nilpotent elements are called *principal*. When $G = SL_n\mathbb{C}$, this is a consequence of the Jordan decomposition theorem.

The Jacobson-Morozov theorem defines a bijection between conjugacy classes of nonzero nilpotents and conjugacy classes of $\mathfrak{sl}_2\mathbb{C}$ subalgebras of $\mathfrak{g}$. Namely, a nonzero nilpotent $e \in \mathfrak{g}$ can be completed to a triple $\{f, h, e\}$ satisfying

$$[e, f] = h \ , \qquad [h, e] = 2e \quad \text{and} \quad [h, f] = -2f \ .$$

Thus, associated to e are embeddings

$$\iota_e : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{g} \qquad \text{and} \qquad \iota_e : \mathfrak{sl}_2\mathbb{R} \to \mathfrak{g}^{\mathbb{R}},$$

for some (not necessarily unique) real form $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$. On the Lie group level there are maps $\iota_e : SL_2\mathbb{C} \to G$ and $\iota_e : SL_2\mathbb{R} \to G^{\mathbb{R}}$. Finally, post composing representations in $\mathcal{T}(S)$ with ι_e defines a map

$$\iota_e : \mathcal{T}(S) \to \mathcal{X}(S, G^{\mathbb{R}}),$$

where we lift representations in $\mathcal{T}(S)$ to $SL_2\mathbb{R}$ if $\iota_e(SL_2\mathbb{R}) \cong SL_2\mathbb{R}$.

For all nonzero nilpotents, the representations in $\iota_e(\mathcal{T}(S))$ are Anosov. Thus, by deforming, there is an open neighborhood of $\iota_e(\mathcal{T}(S))$ in $\mathcal{X}(S, G^{\mathbb{R}})$ (and in $\mathcal{X}(S, G)$) of Anosov representations. However, for most nilpotents e, the representations in $\iota_e(\mathcal{T}(S))$ can be deformed in $\mathcal{X}(S, G^{\mathbb{R}})$ to compact representations. In particular, this open set of Anosov representations is not closed. However, this is not always the case. When $e \in \mathfrak{g}$ is a principal nilpotent, the real form $\mathfrak{g}^{\mathbb{R}}$ is the *split real form* [16], for example the split real form of $\mathfrak{sl}_n\mathbb{C}$ is $\mathfrak{sl}_\mathfrak{n}\mathbb{R}$. For a principal nilpotent e and $G^{\mathbb{R}} < G$ the split real form, Hitchin showed that the representations in the connected components $Hit(G^{\mathbb{R}}) \subset \mathcal{X}(S, G^{\mathbb{R}})$ containing $\iota_e(\mathcal{T}(S))$ have many nice properties [14]. In particular, $Hit(G^{\mathbb{R}})$ does *not* contain compact representations.

The representations in $Hit(G^{\mathbb{R}})$ are now called Hitchin representations. To prove his results, Hitchin exploited the the nonabelian Hodge correspondence which defines a homeomorphism between the character variety $\mathcal{X}(S, G^{\mathbb{R}})$ and a moduli space of holomorphic objects on a Riemann surface called *Higgs bundles*. This correspondence is combination of works of Hitchin [13], Simpson [21], Donaldson [8] and Corlette [7]. The moduli space of Higgs bundles has more structure than the character variety, and this structure provides powerful tools to study the topology of the moduli space. Two important tools in Hitchin's work [14] are the Hitchin fibration and the properness of a certain Morse function.

Unfortunately, it is very difficult to translate most geometric properties of representations into a language adapted to Higgs bundles. As a result, the use of Higgs bundles in [14] provided little information about the geometry of Hitchin representations. In fact, Labourie developed the notion of Anosov representations in [17] exactly to understand the geometry of Hitchin representations. In particular, he proved that every Hitchin representation is Anosov. Thus, like $\mathcal{T}(S)$, $Hit(G^{\mathbb{R}}) \subset \mathcal{X}(S, G^{\mathbb{R}})$ is an open and closed set consisting entirely of discrete-faithful representations. As a result, $Hit(G^{\mathbb{R}})$ is now called a higher Teichmüller space. Other known examples of higher Teichmüller spaces arise from so called maximal representations into Hermitian Lie groups of tube type [4].

Recently, Guichard-Labourie-Wienhard [10] have developed a refinement of the Anosov condition which aims to characterize all higher Teichmüller spaces. Roughly, a parabolic subgroup $P_\Theta < G^{\mathbb{R}}$ of a real Lie group $G^{\mathbb{R}}$ has a Θ-positive structure if triples of pairwise disjoint transverse points in $G^{\mathbb{R}}/P_\Theta$ admit a cyclic order. For such pairs $(G^{\mathbb{R}}, P_\Theta)$, one defines a set of Θ-positive Anosov representations. This set is open and conjectured to also be closed (see [12]).

The theorems below prove the analogue of Hitchin's results about $Hit(G^{\mathbb{R}})$ for what should be all higher Teichmüller spaces. They will appear in [3]. The proofs of these results also use Higgs bundles, and the main new idea is the notion of *magical nilpotents*. This is a Lie theoretic mechanism which is adapted to the language of Higgs bundles. Using the properties of magical nilpotents we are able to prove properties about the resulting Higgs bundles, then translate these properties into statements about the character variety. Finally, we establish a bijection between magical nilpotents and Θ-positive structures.

We briefly recall Hitchin's method for finding components of the moduli space $\mathcal{M}(G^{\mathbb{R}})$ of Higgs bundle. There is a function $f : \mathcal{M}(G^{\mathbb{R}}) \to \mathbb{R}^{\geq 0}$ which is proper [13]. Thus, f attains a local minimum on every connected component. Furthermore, the global minima of f have $f = 0$ and, via the nonabelian Hodge correspondence, are in bijection with compact representations. Thus, if there are connected components of the character variety $\mathcal{X}(S, G^{\mathbb{R}})$ with no compact representations, f must have additional local minima which are nonzero.

Magical nilpotents are a special class of nilpotents in a *complex* simple Lie group $\mathfrak{g}$. We will not give the technical definition here, but a magical nilpotent defines a canonical real form $\mathfrak{g}^{\mathbb{R}}$ and various decompositions of the complex Lie algebra. Using this decomposition data, we construct nonzero local minimum of the above function f and build connected components of the moduli space of Higgs bundles.

Theorem 1. *Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$ and $e \in \mathfrak{g}$ be a magical nilpotent with canonical real form $G^{\mathbb{R}} < G$. Let S be a closed orientable surface of genus $g \geq 2$ and $\mathcal{X}(S, G^{\mathbb{R}})$ be the $G^{\mathbb{R}}$ character variety. Then, there exists a nonempty open and closed subset $\mathcal{H}_e(S, G^{\mathbb{R}}) \subset \mathcal{X}(S, G^{\mathbb{R}})$ which contains $\iota_e(\mathcal{T}(S)) \subset \mathcal{H}_e(X, G^{\mathbb{R}})$ and does not contain compact representations. Moreover, the centralizer of a representation $\rho \in \mathcal{H}_e(S, G^{\mathbb{R}})$ is compact thus, there is no proper parabolic subgroup $P < G^{\mathbb{R}}$ such that $\rho : \pi_1 S \to P \to G^{\mathbb{R}}$.*

Magical nilpotents can be equivalently defined for real Lie algebras. The following theorem relates the magical nilpotents and Θ-positivity. Recall that there is a parabolic subgroup P_e associated to a nilpotent e.

Theorem 2. *Let $G^{\mathbb{R}}$ be a real simple Lie group, $e \in \mathfrak{g}^{\mathbb{R}}$ be a nilpotent and $P_e < G^{\mathbb{R}}$ the associated parabolic subgroup. Then e is a magical nilpotent if and only if $(G^{\mathbb{R}}, P_e)$ has a Θ-positive structure. In particular, $G^{\mathbb{R}}$ must be split, hermitian of tube type, locally isomorphic to $SO(p, q)$ with $1 < p \leq q$, or a real form of F_4, E_6, E_7, E_8 with reduced root system F_4. Moreover, the sets $\mathcal{H}_e(S, G^{\mathbb{R}})$ each contain an open set of positive Anosov representations.*

Theorem 1 recovers Hitchin representations for split groups [14], maximal representations for Hermitian Lie groups of tube type (see for example [2, 5]), and the components described in [1] and [6] for $SO(p, q)$.

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Log-concavity of volume

Eleonora Di Nezza
(joint work with Tamás Darvas, Chinh H. Lu.)

1. Introduction

Let (X, ω) be a compact Kähler manifold of complex dimension n and fix θ a smooth closed $(1,1)$-form which represents a big cohomology class. By definition a function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called quasi-psh if it is locally written as $u = \varphi + g$, where g is a smooth function and φ is a psh function. A function u is called θ-psh on X if it is quasi-psh and $\theta + dd^c u \geq 0$ in the sense of currents. We let $\mathrm{PSH}(X, \theta)$ denote the set of all integrable θ-psh functions on X. The bigness of θ then means that there exists $\psi \in \mathrm{PSH}(X, \theta - \varepsilon\omega)$ for some $\varepsilon > 0$ small enough.

Given $u, v \in \mathrm{PSH}(X, \theta)$ we say that u is more singular than v (and write $u \preceq v$) if there exists a constant $C > 0$ such that $u \leq v + C$ on X. We say that u has the same singularities as v if $u \preceq v$ and $v \preceq u$. And in this case we wirte $[u] = [v]$, where $[u]$ denotes the *singularity class of* u.
A function $u \in \mathrm{PSH}(X, \theta)$ has minimal singularities if it is less singular than any other θ-psh functions. One can construct such potentials by taking the envelope:

$$V_\theta(x) := \sup\{u(x) \,:\, u \in \mathrm{PSH}(X, \theta),\ u \leq 0\}.$$

It then follows that the upper smi-continuous regularization V_θ^* is a θ-psh function which is ≤ 0, hence contributes to its definition. Therefore $V_\theta^* = V_\theta$.

Given $\theta_1, ..., \theta_p$ big forms and $u_j \in \mathrm{PSH}(X, \theta_j)$, the non-pluripolar product $(\theta_1 + dd^c u_1) \wedge ... \wedge (\theta_p + dd^c u_p)$ is defined in [BEGZ10], and the resulting (p,p)-current is closed and positive. In particular, when $p = n$, we obtain the mixed Monge-Ampère measure of $u_1, ..., u_n$ (with respect to $\theta_1, ..., \theta_n$). When $u_1 = ... = u_n = u$ and $\theta_1 = ... = \theta_n = \theta$ this process gives the non-pluripolar Monge-Ampère measure of u which will be denoted by θ_u^n. By construction one always has $\int_X \theta_u^n \leq \int_X \theta_{V_\theta}^n$ and the last term is called the volume $\mathrm{Vol}(\theta)$ of θ (when θ is the curvature of a big holomorphic line bundle, this volume is the same as the volume defined by Boucksom [Bo02]).
For details about quasi-psh fucntion (and much more) we refer to [GZ17].

The main result we present is the following which confirms a conjecture by [BEGZ10]:

Theorem 1. *Log-concavity of volume is true. More precisely, given $\theta_1, \cdots, \theta_n$ big forms and $u_1, \cdots, u_n$ quasi-psh functions such that u_j is θ_j-psh $j \geq 1$ we have*

$$\int_X \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n} \geq \left(\int_X (\theta^1_{u_1})^n\right)^{1/n} \cdots \left(\int_X (\theta^n_{u_n})^n\right)^{1/n}$$

where $\theta^j_{u_j} := \theta_j + dd^c u_j$. *In particular the map* $\theta_u \to \log \int_X \theta_u^n$ *is concave.*

As we will show in the following the proof of the above theorem deeply relies on the resolution of Monge-Ampère equations with prescribed singularity:

Theorem 2. *The Monge-Ampère equation in a given singularity class is uniquely solvable.*

2. Preliminaries

Through all the Section(s), $\theta_j, j \in \{1, \ldots, n\}$ are smooth closed and real $(1,1)$-forms on X whose cohomology classes are big.

2.1. **Relative finite energy class.** We start with two results about the mononicity of the volume of (mixed) non-pluripolar Monge-Ampère measures:

Proposition 1. *Let* $u_j, v_j \in \mathrm{PSH}(X, \theta_j)$ *such that* $[u_j] = [v_j]$, $j \in \{1, \ldots, n\}$. *Then*

$$\int_X \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n} = \int_X \theta^1_{v_1} \wedge \ldots \wedge \theta^n_{v_n}.$$

The proof of the above result is basically due to [WN17].

Theorem 3. *Let* $u_j, v_j \in \mathrm{PSH}(X, \theta^j)$ *such that* u_j *is less singular than* v_j *for all* $j \in \{1, \ldots, n\}$. *Then*

$$\int_X \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n} \geq \int_X \theta^1_{v_1} \wedge \ldots \wedge \theta^n_{v_n}.$$

Now, fixing $\phi \in \mathrm{PSH}(X, \theta)$ one can consider only θ-psh functions that are more singular than ϕ. Such potentials form the set $\mathrm{PSH}(X, \theta, \phi)$. Thanks to Theorem 3, the map $[u] \to \int_X \theta_u^n$ is monotone increasing (but not strictly increasing!). It is then natural to consider the set of ϕ-relative *full mass potentials*:

$$\mathcal{E}(X, \theta, \phi) := \left\{ u \in \mathrm{PSH}(X, \theta, \phi) \text{ such that } \int_X \theta_u^n = \int_X \theta_\phi^n \right\}.$$

2.2. **Envelopes.** Naturally, when $v \in \mathrm{PSH}(X, \theta, \phi)$ we only have $\int_X \theta_v^n \leq \int_X \theta_\phi^n$. As pointed out in [DDL2], when studying the potential theory of the above space, the following well known envelope constructions will be of great help:

$$\mathrm{PSH}(X, \theta) \ni \psi \to P_\theta(\psi, \chi),\ P_\theta[\psi](\chi),\ P_\theta[\psi] \in \mathrm{PSH}(X, \theta).$$

These were introduced by Ross and Witt Nyström [RWN14] in their construction of geodesic rays, using slightly different notation. Given $\psi, \chi \in \mathrm{PSH}(X, \theta)$, the starting point is the "rooftop envelope" $P_\theta(\psi, \chi) := (\sup\{v \in \mathrm{PSH}(X, \theta),\ v \leq \min(\psi, \chi)\})^*$. This allows to introduce

$$P_\theta[\psi](\chi) := \Big(\lim_{C \to +\infty} P_\theta(\psi + C, \chi)\Big)^*.$$

It is easy to see that $P_\theta[\psi](\chi)$ only depends on the singularity type of ψ. When $\chi = V_\theta$, we will simply write $P_\theta[\psi] := P_\theta[\psi](V_\theta)$ and refer to this potential as the *envelope of the singularity type* $[\psi]$. Such envelopes, even if they are a priori less singular than the function we start with, preserve the mass:

Proposition 2. *If $u \in \mathrm{PSH}(X,\theta)$ then $\int_X \theta_u^n = \int_X \theta_{P_\theta[u]}^n$. More generally, given $u_j \in \mathrm{PSH}(X,\theta_j)$ we have*

$$\int_X \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n} = \int_X \theta_{P_{\theta 1}[u_1]} \wedge \cdots \theta_{P_{\theta n}[u_n]}.$$

2.3. **Model potentials.** Potentials ϕ that satisfy $\phi = P[\phi]$ are called *model potentials*, and play and important role in finite energy pluripotential theory. The connection with a model type singularity $[u]$ (defined in the introduction) is as follows. In case $\int_X \theta_u^n > 0$, it was proved in [DDL2, Theorem 3.12] that $P_\theta[P_\theta[u]] = P_\theta[u]$. Informally, this means that every model type singularity with non-vanishing mass admits a model potential representative.

3. Monge-Ampère equations in a given singularity class

We focus our attention on the existence and uniqueness of solutions of the Monge-Ampère equation

$$\theta_u^n = \mu, \;\; u \in \mathcal{E}(X,\theta,\phi), \tag{1}$$

where μ is a given non-pluripolar Borel measure on X and ϕ is a θ-psh function on X such that

$$P[\phi] = \phi \;\; \text{and} \;\; \int_X \theta_\phi^n = \mu(X) > 0.$$

The result we were able to prove states as follows:

Theorem 4. *Let μ and ϕ as above. Then there exists a unique solution $u \in \mathcal{E}(X,\theta,\phi)$ of (1). Moreover, in the particular case when $\mu = f\omega^n$ for some $f \in L^p(X,\omega)$, $p > 1$, u additionally satisfies $[u] = [\phi]$.*

4. Proof of the Log-concavity of volume

In this Section, we give the proof of Theorem 1 as a direct consequence of solvability of complex Monge-Ampère equations with prescribed singularity type.

Proof. We can assume that all the masses are non-zero, otherwise the right-hand side of the inequality to be proved is zero.
After rescaling, we can assume that $\int_X \omega^n = \int_X (\theta^j_{u_j})^n = 1$, $j \in \{1,\ldots,n\}$. Set $\phi_j := P_{\theta_j}[u_j]$ and observe that it is a model potential (see Section 2.3).

For each $j = 1,...,n$ Theorem 2 ensures existence of $\varphi_j \in \mathcal{E}(X,\theta_j,\phi_j)$ such that $\left(\theta^j_{\varphi_j}\right)^n = \omega^n$ and $[\varphi_j] = [\phi_j]$. A combination of Propositions 1 and 2 then gives

$$\int_X \theta^1_{\varphi_1} \wedge ... \wedge \theta^n_{\varphi_n} = \int_X \theta^1_{P_{\theta_1}[u_1]} \wedge ... \wedge \theta^n_{P_{\theta_n}[u_n]} = \int_X \theta^1_{u_1} \wedge ... \wedge \theta^n_{u_n}.$$

Finally, an application of [BEGZ10, Proposition 1.11] gives that $\theta^1_{\varphi_1} \wedge \ldots \wedge \theta^n_{\varphi_n} \geq \omega^n$. The conclusion follows after we integrate this last estimate. □

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Mirzakhani's Curve Counting

Viveka Erlandsson
(joint work with Juan Souto)

Let S be a closed, orientable surface of genus $g \geq 2$ and $\mathrm{Map}(S)$ its mapping class group. By a *curve* γ on S we mean a free homotopy class of an immersed, essential, closed curve and by a *multi-curve* a formal finite sum $\sum_{i=1}^{n} a_i\gamma_i$ where $a_i \in \mathbb{Z}_{\geq 0}$ and γ_i a curve. We say two multi-curves are of the same *type* if they lie in the same mapping class group orbit. Putting a hyperbolic structure on S, Mirzakhani [5, 6] studied the growth of the number of curves of a given type of length bounded by L. She showed that this number is asymptotic to a constant times L^{6g-6}. More precisely:

Theorem 1 (Mirzakhani [5, 6]). *Let X be a hyperbolic structure on S and γ_0 a multi-curve on S. Then*

$$\lim_{L\to\infty} \frac{\#\{\gamma \text{ of type } \gamma_0 \,|\, \ell_X(\gamma) \leq L\}}{L^{6g-6}} = \frac{c_{\gamma_0}}{m_g} B(X)$$

where $B(X) = m_{Th}(\{\lambda \in \mathcal{ML}(S) \,|\, \ell_X(\lambda) \leq 1\})$ is the Thurston measure of the set of measured laminations of X-length at most 1, $m_g = \int B(X)$ is the Weil-Petersson integral of $B(X)$ over the moduli space, and c_{γ_0} is a positive rational number depending only on γ_0.

Mirzakhani proved the above theorem first in the case when γ_0 is simple in [5] and later, using different methods, for a general curve in [6]. Here we give a new, unified proof of the theorem in both cases using a very different approach from that of Mirzakhani.

The main tool for the new proof is the space of *geodesic currents* $\mathcal{C}(S)$ introduced by Bonahon [1, 2], consisting of $\pi_1(S)$-invariant Radon measure on the space of geodesics in the universal cover of S and equipped with the weak*-topology. The set of all curves on S can be naturally identified with a subset of $\mathcal{C}(S)$ and, by allowing for positive weights, is in fact dense in this space. Moreover, Bonahon proved that the geometric intersection pairing of curves extends to a continuous, bi-linear, Map(S)-invariant form $\iota : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}_{\geq 0}$. The space of measured laminations $\mathcal{ML}(S)$ can be identified with the subset $\{\lambda \mid \iota(\lambda, \lambda) = 0\} \subset \mathcal{C}(S)$. Finally, given a hyperbolic structure X on S there is a unique current, the Liouville current, associated to X such that $\iota(X, \gamma) = \ell_X(\gamma)$ for every curve γ on S. Hence we can also identify the Teichmüller space of S with a subset of $\mathcal{C}(S)$.

Now, fix a multi-curve γ_0 and for each $L > 0$ consider the following measure on $\mathcal{C}(S)$

$$m^L_{\gamma_0} = \frac{1}{L^{6g-6}} \sum_{\gamma} \delta_{\frac{1}{L}\gamma} \tag{1}$$

where the sum is taken over all γ of type γ_0 and δ_x denotes the Dirac measure centered at x. Let σ be any filling current (one which intersects every other current positively) and consider $b(\sigma) = \{\lambda \in \mathcal{C}(S) \mid \iota(\sigma, \gamma) \leq 1\}$, i.e. the unit ball in $\mathcal{C}(S)$ with respect to "σ-length". Then

$$m^L_{\gamma_0}(b(\sigma)) = \frac{\#\{\gamma \text{ of type } \gamma_0 \mid \iota(\gamma, \sigma) \leq L\}}{L^{6g-6}}.$$

Hence, finding the asymptotic growth of the number of curves of type γ_0 boils down to studying the asymptotics of the family of measures $(m^L_{\gamma_0})_L$. We prove that these measures converge as $L \to \infty$:

Theorem 2. *Let γ_0 be a multi-curve on S. Then*

$$\lim_{L \to \infty} m^L_{\gamma_0} = \frac{c_{\gamma_0}}{\kappa} m_{Th}.$$

Here c_{γ_0} is the same constant as in Theorem 1 and

$$\kappa = \sum_{\alpha \in \mathcal{ML}_{\mathbb{Z}}(S)/\operatorname{Map}(S)} m_{Th}(\{\lambda \mid \iota(\alpha, \lambda) \leq 1\}/\operatorname{Stab}(\alpha))$$

where $\mathcal{ML}_{\mathbb{Z}}(S)$ stands for the set of all simple multicurves.

As a consequence it follows that, for any filling multi-curve σ (in particular for a hyperbolic structure X) we have

$$\lim_{L \to \infty} \frac{\#\{\gamma \text{ of type } \gamma_0 \mid \iota(\sigma, \gamma) \leq L\}}{L^{6g-6}} = \frac{c_{\gamma_0}}{\kappa} B(\sigma) \tag{2}$$

where $B(\sigma) = m_{Th}(\{\lambda \in \mathcal{ML}(S) \mid \iota(\lambda, \sigma) \leq 1\})$. Note that when $\sigma = X$ this gives Theorem 1. In fact, $\iota(\sigma, \cdot)$ in (2) can be replaced with any notion of length on curves which extends to a continuous and homogeneous function on $\mathcal{C}(S)$.

There are two main ingredients in the proof of Theorem 2:

(I) A previous result by the authors showing that the family of measures (1) is precompact and any accumulation point is a positive multiple of the Thurston measure (see [3, Proposition 4.1]).

(II) A new result giving an effective way to estimate the Thurston measure using a finite number of types of simple multi-curves (see [4, Proposition 3.2]).

Combining these results one can compute the multiple of the Thurston measure for each accumulation point. As it turns out, this constant is the same for every accumulation point, and hence there is only one and the measures converge.

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Rigidity of certain Einstein 4-manifolds

Joel Fine

(joint work with Kirill Krasnov, Michael Singer)

My talk describes joint work with Kirill Krasnov (University of Nottingham) and Michael Singer (University College London). We are interested in the Einstein equation for a Riemannian manifold (M, g):

$$\mathrm{Ric}(g) = \Lambda g, \quad \Lambda \in \mathbb{R}$$

When considered modulo diffeomorphisms, this is a non-linear elliptic PDE for g. The linearisation (again modulo diffeomorphisms) is a Laplace-type operator. In particular it has index zero. This means naively that one might expect the moduli space of Einstein metrics on M with Einstein constant Λ to be zero dimensional. Our work gives a curvature condition on an Einstein 4-manifold which ensures that this actually happens.

It is perhaps worth noting that this naive expectation is often false! The moduli space of Einstein metrics on a surface of genus $g > 1$ has dimension $6g - 6$. More generally Kähler–Einstein metrics typically have moduli, corresponding to the complex moduli of the underlying manifold. There is, however, a theorem due to Koiso which guarantees that for negatively curved manifolds of dimension $n \geq 3$ the moduli space is zero dimensional:

Theorem 1 (Koiso's Local Rigidity Theorem (1978))**.** *Let (M^n, g) be a compact orientable Einstein metric with negative curvature (i.e. all sectional curvatures are negative), where $n \geq 3$. Then g is an isolated point in the moduli space of Einstein metrics. In other words, if g_t is a path of Einstein metrics on M with $g_0 = g$ then there exists a path f_t of diffeomorphisms of M with $f_t = \mathrm{Id}$ and $f_t^* g_t = g$.*

Our main result is a 4-dimensional "chiral" generalisation of Koiso's Theorem. To explain it, first recall the additional decomposition of the curvature tensor in 4-dimensions. On an oriented Riemannian 4-manifold (M^4, g) the Hodge star decomposes the 2-forms $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ into ± 1 eigenspaces. This means that the curvature splits as a 2-by-2 block:

$$\mathrm{Rm} = \begin{pmatrix} R_{++} & R_{-+} \\ R_{+-} & R_{--} \end{pmatrix}$$

Here $R_{++} \colon \Lambda^+ \to \Lambda^+$ is self-adjoint and similarly for R_{--} whilst $R_{-+} \colon \Lambda^- \to \Lambda^+$ and $R_{+-} = R^*_{-+}$. One can identify R_{-+} with the trace free Ricci curvature and so for an Einstein metric the curvature is "diagonal": $\mathrm{Rm} = R_{++} \oplus R_{--}$.

An alternative way to think of this decomposition is to consider the Levi-Civita connection on Λ^+. The curvature is a 2-form with values in $\mathfrak{so}(\Lambda^+)$ which is isomorphic to Λ^+ itself (via the cross product). One can check that the curvature of Λ^+ is $R_{++} \oplus R_{--} \in (\Lambda^+ \oplus \Lambda^-) \otimes \Lambda^+$. In particular, g is Einstein precisely when the Levi-Civita connection on Λ^+ is a self-dual instanton.

We can now formulate negative curvature in this notation. Write λ_+ for the maximal eigenvalue of R_{++} and λ_- for the maximal eigenvalue of R_{--}. (These are continuous functions on M.) One can check that asking for an Einstein metric to have negative sectional curvatures is equivalent to $\lambda_+ + \lambda_- < 0$. In particular at least one of R_{++} or R_{--} is negative definite.

Theorem 2 (F.–Krasnov–Singer)**.** *Let (M^4, g) be a compact oriented Einstein 4-manifold and suppose that either R_{++} or R_{--} is negative definite. Then g is an isolated point in the moduli space of Einstein metrics.*

It suffices to prove the result for R_{++} since switching orientation exchanges R_{++} and R_{--}. It is important to note that the inequality $R_{++} < 0$ is very different from negative curvature. For example there are complete Einstein 4-manifolds with $R_{++} < 0$ and $\pi_2 \neq 0$ (D. Calderbank and M. Singer, *Einstein metrics and complex singularities*, Invent. math. 156:2 405–433 (2004)).

The proof of Koiso's Local Rigidity Theorem uses a clever Weitzenböck argument to show that the gauge-fixed linearised Einstein equations are actually a positive operator. The Weitzenböck remainder is positive because the sectional curvatures are all negative. Our proof is very different. It relies on an alternative formulation of Einstein's equations, which works only in 4D, due to Plebanski in 1977. Even if you're not so interested in our Theorem, which is a small improvement on a result which is over 40 years old, then you should still have time for Plebanski's work, which despite being even older than Koiso's Theorem is seemingly little known or exploited in mathematics.

Plebanski gave a variational formulation of Einstein 4-manifolds, as critical points of a functional now called the *Plebanski action*. What's truly ingenious about the action is that it is not at first sight an action on the space of metrics. Under certain conditions, it becomes an action on the space of connections on an auxilliary SO(3)-bundle E which makes it look a little like gauge theory.

One of the consequences of this is that it avoids one of the pitfalls of the Einstein–Hilbert action. The Einstein–Hilbert action has Einstein metrics as critical points but they are saddles at which the Hessian has infinitely many positive and negative eigenvalues. The Plebanski action (when converted to its "pure connection" form) has *elliptic* Hessian (modulo gauge), with only a finite dimensional space of negative eigenvalues. The key to proving our rigidity result is to show that when R_{++} the Hessian is actually positive definite modulo gauge.

The Plebanski action is defined as follows. Let $E \to M$ be an SO(3) bundle over a compact oriented 4-manifold. Let A be an SO(3)-connection in E, Σ a section of $\Lambda^2 \otimes \mathfrak{so}(E)$ and Ψ a section of $S^2_0 E$, the bundle of trace-free symmetric endomorphisms of E. Set

$$S(A, \Sigma, \Psi) = \int \left[\mathrm{Tr}(F_A \wedge \Sigma) - \frac{1}{2} \mathrm{Tr} \left(\left(\Psi + \frac{\Lambda}{3} \right) \Sigma \wedge \Sigma \right) \right]$$

Varying with respect to Ψ, one sees that at a critical point of S, the symmetric matrix of 4-forms $\Sigma \wedge \Sigma$ is a multiple of the identity. This implies that there is a unique metric on M for which Σ self-dual with $\Sigma\colon \mathfrak{so}(3)^* \to \Lambda^+$ an isometry. Varying with respect to Σ one sees that $F_A = (\Psi + \Lambda/3)\Sigma$ and so A is a self-dual instanton. Finally varying with respect to A one sees that $d_A \Sigma = 0$. This means that if we use $\Sigma\colon \mathfrak{so}(E)^* \to \Lambda^+$ to push A forward from a connection on $\mathfrak{so}(E)^*$ to a metric connection on Λ^+ it agrees with the Levi-Civita connection. As we said above, asking that the Levi-Civita connection on Λ^+ be a self-dual instanton is equivalent to the metric being Einstein. Notice that $\Psi + \Lambda/3$ becomes identified with R_{++}. In particular, $\mathrm{Tr}(R_{++}) = \Lambda$ and this means Λ is the Einstein constant.

We now come to an important observation due to Krasnov (K. Krasnov. *Pure Connection Action Principle for General Relativity.* Phys. Rev. Lett. 106:25, p. 251103 (2011)). When $\Psi + \Lambda/3$ is invertible, we can recover Σ from A via $\Sigma = (\Psi + \Lambda/3)^{-1} F_A$. This means that we can eliminate Σ from the story. Similarly, Ψ is determined by the requirement that $\Sigma \wedge \Sigma \sim \mathrm{Id}$. This leaves A as the only variable. This "pure connection" formalism captures all Einstein metrics for which R_{++} is definite.

It is in this setting that the Hessian of $S(A)$ becomes elliptic modulo gauge. Moreover a careful computation combined with a judicious choice of gauge fixing shows that when R_{++} is negative definite the Hessian is strictly positive modulo gauge. From here the local rigidity theorem follows from a slice theorem for the gauge action.

Mass, Kähler Manifolds, and Symplectic Geometry

Claude LeBrun

Let (M, g) be a complete, non-compact, connected Riemannian n-manifold, where $n \geq 3$. We say that (M, g) is *asymptotically Euclidean* (or *AE*) if there is a compact subset $\mathcal{K} \subset M$ such that $M - \mathcal{K}$ consists of finitely many components, each of which is diffeomorphic to the complement of a closed ball $D^n \subset \mathbb{R}^n$ in such a manner that g becomes the standard Euclidean metric plus terms that fall off "sufficiently quickly" at infinity. More generally, (M, g) is said to be *asymptotically locally Euclidean* (or *ALE*) if the complement of some compact set $\mathcal{K}$ consists of finitely many components, each of which is diffeomorphic to $(\mathbb{R}^n - D^n)/\Gamma_j$ for some finite subgroup $\Gamma_j \subset O(n)$, in such a way that the metric g again differs from the background Euclidean metric by terms that fall off "sufficiently quickly" at infinity. The components of $M - \mathcal{K}$ are called the *ends* of M, and, because we have assumed that $n \geq 3$, the Γ_j are just the fundamental groups of the corresponding ends.

However, to turn this rough idea into a genuine mathematical definition, we still need to precisely explain what we mean by saying that g falls off "sufficiently quickly" to the Euclidean metric in the given asymptotic coordinates. Here, the literature offers only tenuous guidance as to how to proceed, because too many authors have tweaked the definition in order to accommodate the use of their specific techniques. However, the weakest standard fall-off hypotheses that suffice to prove compelling results are the ones introduced by Chruściel [3], who just assumed that

(i) the metric g is of class C^2, with scalar curvature s in L^1; and
(ii) in some asymptotic chart at each end of M^n, and for some $\varepsilon > 0$, the components of the metric and their first partial derivatives satisfy

$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon}), \qquad g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}).$$

With these very weak hypotheses, Chruściel proved that the *mass*

$$\mathfrak{m}(M, g) := \lim_{\varrho \to \infty} \frac{\mathbf{\Gamma}(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{S_\varrho/\Gamma_i} [g_{k\ell,k} - g_{kk,\ell}]\, \mathbf{n}^\ell d\mathfrak{a}_E$$

of an ALE manifold (M, g) at a given end is both well-defined and invariant under a large class of changes of asymptotic coordinate system. Here, commas indicate partial derivatives in the given asymptotic coordinates, summation over repeated indices is understood, S_ϱ is the Euclidean coordinate sphere of radius ϱ, Γ_i is the fundamental group of the relevant end, $\mathbf{\Gamma}$ is the Euler Gamma function, $d\mathfrak{a}_E$ is the $(n-1)$-dimensional volume form induced on this sphere by the Euclidean metric, and $\vec{\mathbf{n}}$ is the outward-pointing Euclidean unit normal vector.

The above notion of mass originated in gravitational physics, not in geometry, and its geometrical meaning unfortunately remains rather enigmatic. However, in the special case where (M, g) is a *Kähler* manifold of real dimension $n = 2m$, my earlier work with Hajo Hein [5] deciphered the meaning of the mass $\mathfrak{m}(M, g)$ by showing that, up to a constant depending only on the dimension, it actually equals an explicit topological term plus the integral of the scalar curvature. However, while our proofs only required Chruściel fall-off in complex dimension ≥ 3, to make our proofs work in complex dimension 2 we unfortunately needed to either assume that $\varepsilon > \frac{1}{2}$ in (ii), or else to replace (ii) with a stronger, Bartnik-type [1] fall-off hypothesis like

$$g_{jk} - \delta_{jk} \in C^{2,\alpha}_{-1-\varepsilon}, \quad \varepsilon > 0,$$

in order to obtain our results in the special case of complex dimension $m = 2$. Fortunately, however, I have more recently shown [8] that, in fact,

> *Chruściel fall-off suffices to imply all the main results of* [5]*, even in real dimension four.*

This was accomplished by providing new proofs in real dimension four that, in contrast to our previous approach, primarily depend on results in symplectic geometry. In particular, Chruściel fall-off (i)–(ii) suffices to imply all the following key results from [5], even in complex dimension $m = 2$:

Lemma. *Any ALE Kähler manifold has only* **one end***.*

This seemingly innocuous assertion is a corner-stone of what follows, because it gives the mass of an ALE Kähler manifold an unambiguous meaning — no choice of an end is involved. This paves the way for the key mass formula, in conjunction with another topological fact that we will discuss next.

If M is a smooth manifold, recall that one defines the compactly-supported deRham cohomology $H^k_c(M)$ as the compactly-supported closed k-forms modulo exterior derivatives of compactly-supported $(k-1)$-forms. There is thus a natural map $H^2_c(M) \to H^2(M)$ induced by the inclusion of compactly-supported forms into all differential forms; and for any ALE manifold of real dimension $n \geq 4$, this map can be shown to actually be an *isomorphism.* This entitles us to define

$$\clubsuit : H^2(M) \to H^2_c(M)$$

to be the inverse of this natural isomorphism.

Theorem C. *Let (M, g, J) be an ALE Kähler manifold of complex dimension m, where the metric is merely assumed to have Chruściel fall-off* (i)–(ii) *in some real asymptotic coordinate system. Then the mass of (M, g, J) is given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

where s_g and $d\mu_g$ are respectively the scalar curvature and volume form of g, c_1 is the first Chern class of (M, J), $[\omega]$ is the Kähler class, and $\langle\ ,\ \rangle$ is the duality pairing between $H^2_c(M)$ and $H^{2m-2}(M)$.

While the above mass formula first appears in my paper with Hein [5], the new result is that this assertion holds even if one only assumes Chruściel metric fall-off, no matter what the dimension. Theorems **A** and **B** of [5] are immediate corollaries of Theorem **C** concerning the scalar-flat case; however, the ellipticity of the equation $s = 0$ for a Kähler metric actually implies that an ALE scalar-flat Kähler metric have much faster fall-off than Chruściel, so the methods under discussion do not lead to meaningful improvements of either Theorem **A** or **B**.

On the other hand, these new methods [8] do lead to a genuine improvement of another key result from [5]:

Theorem E (Penrose Inequality for Kähler Manifolds)**.** *Let (M^{2m}, g, J) be an AE Kähler manifold with scalar curvature $s \geq 0$, where g is merely assumed to satisfy the Chruściel fall-off conditions. Then (M, J) carries a numerically canonical divisor $K = \sum \mathbf{n}_j D_j$, where the D_j are compact complex hypersurfaces, the n_j are positive integers, and where $\bigcup_j D_j \neq \varnothing$ unless M is diffeomorphic to $\mathbb{R}^{2m}$. Consequently,*

$$\mathfrak{m}(M, g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum_j \mathbf{n}_j \, Vol(D_j)$$

with equality iff the Kähler manifold (M, g, J) is scalar-flat.

Thus, in the Kähler context, volumes of certain real-codimension-two minimal submanifolds provide a lower bound for the mass, in a manner analogous to the usual Penrose inequality [2, 6, 13], involving areas of minimal real hypersurfaces. Here it must be emphasized that this result only concerns AE manifolds, and not to the more general ALE spaces. This is perhaps best clarified by pointing out a direct consequence of Theorem **E**:

Theorem D (Positive Mass Theorem for Kähler Manifolds)**.** *Any AE Kähler manifold with $s \geq 0$ has $\mathfrak{m}(M, g) \geq 0$, with equality iff (M, g) is Euclidean space.*

Without the Kähler condition, the positive mass theorem for AE Riemannian manifolds is actually well-established in the low-dimensional [14] and spin [9, 17] settings, and a proof in full generality has been announced in a recent preprint [15]. However, despite conjectures [4] to the contrary, the corresponding assertion fails for general ALE manifolds. Indeed, the mass is negative [7] for many ALE Kähler manifolds with $s \equiv 0$, a fact that is now best understood as a systematic consequence [5, Corollary 5.8] of the mass formula of Theorem **C**.

Many of the analytic subtleties encountered in the 4-dimensional case are subtly intertwined with the fact that the complex structure of an ALE Kähler surface need *not* be standard at infinity. However, the symplectic structure at infinity of such a manifold *is* always standard, even when one merely assumes Chruściel metric fall-off. This allows one to construct a compact symplectic 4-manifold $\hat{M}$ from (M, ω) by truncating each end at some large radius, and then gluing in standard plugs. However, this construction introduces an immersed symplectic 2-sphere with positive normal bundle into each capped-off asymptotic region, and so allows one to show that M can only have one end. Indeed, general results of McDuff [11, 12] regarding compact symplectic 4-manifolds show that the existence a symplectic 2-sphere with positive normal bundle implies that $b_+(\hat{M}) = 1$, whereas the existence of a surface of positive self-intersection in each capped-off end forces $b_+(\hat{M})$ to be at least as large as the number of ends of M. Theorem **C** is then deduced by combining this observation with arguments previously developed in [5]. The 4-dimensional case of Theorem **E** is then proved by using ideas of Taubes [10, 16] to first construct the divisor K as a pseudo-holomorphic curve relative to any compatible almost-complex structure on $\hat{M}$, but then applying this more general fact to an almost-complex structure carefully chosen so that the corresponding almost-Kähler metric on the truncated version of the M admits a distance-non-increasing retraction map onto an "interior" region where the almost-complex structure coincides with the original integrable complex structure J. A calibrated geometry argument then shows that the constructed pseudo-holomorphic cycle K must actually be contained within the interior region, and so is actually a union of genuine holomorphic curves in the ALE Kähler manifold (M, g, J).

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Ancient solutions in Lagrangian mean curvature flow

Jason D. Lotay
(joint work with Ben Lambert and Felix Schulze)

Mean curvature flow is the negative gradient flow of the volume functional on submanifolds, and its critical points are minimal submanifolds. An important well-known fact is that if one works in a Kähler–Einstein manifold M, then mean curvature flow preserves the class of Lagrangian submanifolds; i.e. (real) submanifolds L of half the dimension of M on which the Kähler form vanishes.

In the particular case of Calabi–Yau manifolds, which are Ricci-flat Kähler manifolds, (connected) minimal Lagrangians are called special Lagrangian and have the property of being volume-minimizing, since they are characterised as the Lagrangians whose Lagrangian angle is constant. Hence, Lagrangian mean curvature flow in Calabi–Yau manifolds has the attractive feature of only having absolute minima as its critical points. Moreover, if one starts with a zero Maslov Lagrangian (one for which the Lagrangian angle is single-valued), then Lagrangian mean curvature flow is a Hamiltonian isotopy, and so potentially gives a tool for understanding the following challenging problem.

Question. *In a Calabi–Yau manifold, which Hamiltonian isotopy classes of compact zero Maslov Lagrangians admit a special Lagrangian representative?*

A key point when studying any geometric flow is the possibility of singularities. Singularities of (Lagrangian) mean curvature flow are modelled on ancient solutions to the flow in Euclidean space: i.e. solutions to the flow which are defined for all negative times. It is therefore important to understand which ancient solutions can arise in order to understand singularity formation, possible conditions under which certain singularities can be ruled out, and potential surgeries to overcome singularities.

Understanding singularities in Lagrangian mean curvature flow is particularly important since pioneering work of Neves [4] shows that singularities of the flow are, in a sense, unavoidable. Specifically, for any compact Lagrangian L of dimension at least 2 in a Calabi–Yau manifold M, there is another compact Lagrangian L' in M, which is Hamiltonian isotopic and C^0-close to L, such that Lagrangian mean curvature flow starting at L' develops a finite time singularity. However, it is important to notice that the L' constructed is never almost calibrated; that is, the Lagrangian angle θ of L' cannot satisfy $\cos\theta \geq \epsilon$ for some $\epsilon > 0$.

The almost calibrated condition is natural since it implies the zero Maslov condition, it is preserved by Lagrangian mean curvature flow, and one can always assume one is trying to flow to a special Lagrangian with zero Lagrangian angle. Moreover, it arises in the well-known Thomas–Yau Conjecture [6], which asserts that long-time existence and convergence of almost calibrated Lagrangian mean curvature flow is equivalent to a "stability" condition on the initial Lagrangian L, defined in terms of the Lagrangian angle and possible ways of decomposing L as a connect sum of almost calibrated Lagrangians. This stability condition is motivated by Mirror Symmetry for Calabi–Yau manifolds, which should relate special Lagrangians to Hermitian–Yang–Mills connections. In this way, the Thomas–Yau Conjecture can be interpreted as an analogue of the well-known relationship between stable bundles and long-time existence and convergence of Hermitian–Yang–Mills flow.

On the back of these considerations, we were motivated to classify ancient solutions for almost calibrated Lagrangian mean curvature flow in $\mathbb{C}^n$. The most obvious ancient solutions are self-shrinkers (solutions which simply shrink under dilations along the flow), but smooth zero Maslov self-shrinkers which could arise as singularity models do not exist. As a consequence, there are no Type I singularities for zero Maslov Lagrangian mean curvature flow, so one must look at Type II blow-ups at a singularity: these are smooth ancient (in fact, eternal) solutions to Lagrangian mean curvature flow in $\mathbb{C}^n$ which give a finer description of the structure of the singularity.

The Type II blow-up a priori has no particular structure, other than solving the flow, and there are many possible examples, the simplest other than self-shrinkers being special Lagrangians and translators: solutions which are either stationary or just translate under the flow, respectively. However, we showed in [2], following work in [5], that any blow-down of a zero Maslov ancient solution to Lagrangian mean curvature flow is a finite union of special Lagrangian cones: informally, a blow-down of an ancient solution describes the ancient solution's asymptotic behaviour at spatial infinity as time goes to negative infinity. This is analogous to the structure theory of Neves [3], who showed that any tangent flow at a finite-time singularity of zero Maslov Lagrangian mean curvature flow is a finite union of special Lagrangian cones. Of course, one has the following natural, and potentially related, open questions.

Question. *When can one relate tangent flows to blow-downs of Type II blow-ups in zero Maslov Lagrangian mean curvature flow?*

Question. *When are tangent flows or blow-downs of Type II blow-ups in zero Maslov Lagrangian mean curvature flow unique?*

The situation in $\mathbb{C}^2$ is particularly amenable to study since special Lagrangian cones are necessarily planes. Therefore the simplest examples where one can have a non-trivial Type II blow-up are when one of its blow-downs is a pair of special Lagrangian planes, possibly with multiplicity. For our main classification result, we observe (by work in [2]) that any Type II blow-up for zero Maslov Lagrangian mean curvature flow is necessarily exact in $\mathbb{C}^n$: that is, any primitive for the standard Kähler form on $\mathbb{C}^n$ must be exact when restricted to the blow-up. This is useful because it is known [1] that there is a unique (up to scale) exact, embedded, special Lagrangian asymptotic to a pair of transverse special Lagrangian planes with the same Lagrangian angle, known as a Lawlor neck. Another useful fact in $\mathbb{C}^2$ is that one can perform a hyperkähler rotation to identify special Lagrangians with a given Lagrangian angle with complex curves.

In our classification result, we will implicitly assume certain natural, mild properties for ancient solutions which must be satisfied by Type II blow-ups. In all cases, we show that the ancient solutions are in fact stationary.

Theorem. *Let L_t be an exact, almost calibrated, ancient solution to Lagrangian mean curvature flow in $\mathbb{C}^2$ with a blow-down given by a pair of special Lagrangian planes P_+, P_-.*

(a) If $P_+ \cap P_- = \{0\}$ and $P_\pm$ have different Lagrangian angles, then $L_t = P_+ \cup P_-$ for all t.
(b) If $P_+ \cap P_- = \{0\}$ and $P_\pm$ have the same Lagrangian angle, then either $L_t = P_+ \cup P_-$ or L_t is a Lawlor neck for all t.
(c) If $P_+ = P_- = P$, so the blow-down is a plane P with multiplicity two, then, for all t, either $L_t = P$ or L_t is, after hyperkähler rotation, the graph of $z \mapsto cz^2$ for $z \in \mathbb{C}$, for some $c \in \mathbb{C} \setminus \{0\}$.

Parts (a) and (b) extend to $\mathbb{C}^n$ for all $n \geq 2$. However, part (c) is special to $n = 2$. As a follow-up to this theorem, Wood [7] has given explicit examples where an exact, almost calibrated, Lagrangian mean curvature flow in $\mathbb{C}^2$ develops a finite-time singularity at the origin, the tangent flow at 0 is a pair of transverse special Lagrangian planes with the same Lagrangian angle, and the Type II blow-up is a Lawlor neck whose blow-down is the same pair of special Lagrangian planes.

There is clearly a missing case in the theorem, which is when the planes P_+ and P_- intersect in a line. In that case, the planes cannot have the same Lagrangian angle and so one does not expect the ancient solution to necessarily be stationary. We therefore have a final open question.

Question. *Can we classify the exact, almost calibrated, ancient solutions to Lagrangian mean curvature flow in $\mathbb{C}^2$ which have a blow-down given by a pair of planes intersecting in a line?*

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Construction of negatively curved complex submanifolds

Jean-Paul Mohsen

The Donaldson-Auroux asymptotic technics are a theory which has been used in symplectic geometry. They use an analogy between symplectic geometry and complex geometry. Donaldson noticed that they also provide new results in complex projective geometry. In this talk, I give new examples of such complex results.

On moduli spaces of spherical surfaces with conical points

Gabriele Mondello
(joint work with Dmitri Panov)

Metrics of positive curvature (with conical singularities of prescribed angles $2\pi\vartheta_1$, $2\pi\vartheta_2$, ..., $2\pi\vartheta_n$) on a surface S behave quite differently than flat or hyperbolic ones. In general, even if the obvious Gauss-Bonnet constraint

$$\chi(S) + \sum_i (\vartheta_i - 1) > 0$$

is satisfied, existence and uniqueness of spherical (i.e. $K = 1$) metrics with prescribed angles at the conical singularities in a given conformal class is not granted. More precisely, existence is known in the subcritical case (Troyanov 1991) and in many supercritical cases (Bartolucci-De Marchis-Malchiodi 2011). On the other hand, uniqueness is known for angles smaller than 2π (Luo-Tian 1992), which can only occur in genus 0.

In this talk I report about my joint work [1]-[2] with Dmitri Panov (King's College of London) on a number of features of the moduli spaces $\mathcal{MS}ph_{g,n}(\boldsymbol{\vartheta})$ of spherical metrics on compact oriented surfaces of genus g with n conical singularities $x_1, \ldots, x_n$ of prescribed angles $2\pi \cdot (\vartheta_1, \ldots, \vartheta_n)$ and of the map $F_{g,n,\boldsymbol{\vartheta}}$: $\mathcal{MS}ph_{g,n}(\boldsymbol{\vartheta}) \to \mathcal{M}_{g,n}$ that sends a spherical surface with conical points to the underlying Riemann surface with marked points.

We assume $\chi(S) - n < 0$ and the Gauss-Bonnet condition $\chi(S) + \sum_i (\vartheta_i - 1) > 0$ in what follows. Here are some of the results I reported about.

Theorem A. If $g > 0$, then $\mathcal{MS}ph_{g,n}(\boldsymbol{\vartheta}) \neq \emptyset$.

If $g = 0$, then

$$d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) > 1 \implies \mathcal{MS}ph_{0,n}(\boldsymbol{\vartheta}) \neq \emptyset$$
$$d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) < 1 \implies \mathcal{MS}ph_{0,n}(\boldsymbol{\vartheta}) = \emptyset$$

where $\mathbf{1} = (1, 1, \ldots, 1)$, $\mathbb{Z}^n_o = \{\boldsymbol{\eta} \in \mathbb{Z}^n \,|\, \eta_1 + \cdots + \eta_n \text{ odd}\}$ and $d_1(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{i=1}^n |\alpha_i - \beta_i|$.

Theorem B. Fix a positive integer m and take $m_1, m_2, m_3 \geq m$ integers. For $\varepsilon_1, \ldots, \varepsilon_m > 0$ small enough, and for $\boldsymbol{\vartheta} = (m_1 + \frac{1}{2}, m_2 + \frac{1}{2}, m_3 + \frac{1}{2}, \varepsilon_1, \ldots, \varepsilon_m)$, the moduli space $\mathcal{MS}ph_{0,3+m}(\boldsymbol{\vartheta})$ has at least 3^m connected components. The same holds for the image inside $\mathcal{M}_{0,3+m}$ of the forgetful map $F_{0,3+m,\boldsymbol{\vartheta}}$.

Theorem H. Take $\boldsymbol{\vartheta} = \left(\frac{1}{2},\, \frac{1}{2},\, \frac{3}{2},\, \eta\right)$ with $\eta \in \left(0,\, \frac{1}{2}\right)$. Then $\mathcal{MSph}_{0,4}(\boldsymbol{\vartheta}) \cong (0,\pi)_c \times S^1_\phi$ and $F_{0,4,\boldsymbol{\vartheta}}$ is not holomorphic.

Let
$\mathrm{NB}_{\boldsymbol{\vartheta}}(g,n) := \min\left\{ \big| \|\boldsymbol{\vartheta}_I\|_1 - \|\boldsymbol{\vartheta}_{I^c}\|_1 + 2b - \chi(S) + n \,\big|\, I \subseteq \{1,2,\dots,n\} \text{ and } b \in \mathbb{Z}_{\geq 0} \right\}$.

Theorem E. If $\mathrm{NB}_{\boldsymbol{\vartheta}}(g,n) > 0$, then the forgetful map $F_{g,n,\boldsymbol{\vartheta}}$ is proper.

Theorem E was also proven by Bartolucci-Tarantello (2002). The following result is a quantitative version of it.

Theorem C. Assume that $\dot{S} = S \setminus \{x_1,\dots,x_n\}$ is not a 3-punctured sphere and suppose that $\mathrm{NB}_{\boldsymbol{\vartheta}}(g,n) \geq \varepsilon \in \left(0, \frac{1}{2}\right)$. Then

$$\mathrm{Extsys}(\dot{S}) \geq \frac{2\pi\|\boldsymbol{\vartheta}\|_1}{\log(1/\varepsilon)} \quad \Longrightarrow \quad \mathrm{sys}(S,\boldsymbol{x}) \geq \left(\frac{\varepsilon}{4\pi\|\boldsymbol{\vartheta}\|_1}\right)^{-3\chi(S)+2n}$$

where $\mathrm{sys}(S,\boldsymbol{x})$ is $\frac{1}{2}$ times the length of a shortest geodesic arc on S with ends in $\boldsymbol{x} = \{x_1,\dots,x_n\}$ and $\mathrm{Extsys}(\dot{S})$ is the minimum of the extremal lengths of loops on $\dot{S}$ for the underlying conformal structure.

Among the global properties, we discuss non-emptiness and we show that such moduli spaces can have an arbitrarily large number of connected components. Furthermore, we show that no spherical metric in a given conformal class exists if one angle is too small. Such result relies on an explicit systole inequality which relates metric invariants (systole) and conformal invariants (extremal systole) of spherical surfaces, and that can be of independent interest.

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Ricci flow and diffeomorphism groups

Bruce Kleiner

(joint work with Richard Bamler)

The lecture discussed recent joint work with Richard Bamler, in which Ricci flow through singularities was use to prove some conjectures about diffeomorphism groups and moduli spaces of metrics. The main results are:

- (Generalized Smale Conjecture) If g_0 is a metric of constant sectional curvature ± 1 on a compact connected 3-manifold M, then the isometry group $\mathrm{Isom}(M,g_0)$ is a deformation retract of the diffeomorphism group $\mathrm{Diff}(M)$. This includes new proofs of the original Smale Conjecture (when $M = S^3$) and the GSC for hyperbolic manifolds.
- If M is a non-Haken 3-manifold with a *Nil* metric g_M, then the Weak Smale Conjecture holds for M: the identity component $\mathrm{Isom}^0(M,g_M)$ of $\mathrm{Isom}(M,g_0)$ is a deformation retract of the identity component $\mathrm{Diff}^0(M)$.

Together with the GSC, this completes the structure theory of diffeomorphism groups of prime 3-manifolds, after earlier work of Hatcher, Ivanov, Gabai, Hong-Kalliongis-McCullough-Rubinstein, McCullough-Soma.

- If M is a spherical space form, then the space $\mathrm{Met}_{\mathrm{PSC}}(M)$ of metrics of positive scalar curvature on M is contractible. This extends Marques theorem that $\mathrm{Met}_{\mathrm{PSC}}(M)$ is connected.

On the regularity of Ricci flows coming out of metric spaces

Felix Schulze
(joint work with Alix Deruelle, Miles Simon)

In Ricci flow, the long-standing desire to be able to start the flow with an initial manifold without any bounded curvature assumption is starting to be fulfilled, thanks to recent advances by Hochard [3], Simon-Topping [4, 5], and Bamler-Cabezas-Rivas-Wilking [1], amongst others. The work of Simon-Topping has led to the resolution of the Anderson-Colding-Cheeger-Tian conjecture concerning the topological structure of non-collapsed Ricci limit spaces in dimension 3.

Consider a sequence of pointed Riemannian manifolds (M_i^3, g_i, x_i) with Ricci curvature bounded from below by minus one and $\mathrm{Vol}(B_{g_i}(x_i, 1)) \geq v_0 > 0$ such that $(M_i^3, d_{g_i}, x_i) \to (X, d_0, x_0)$ in the Gromov-Hausdorff sense. In the proof of [5], Simon-Topping consider a family of Ricci flows $(M_i^3, (g_i(t))_{0\leq t<T_i}$ starting at $g_i(0) = g_i$ and show that the flows satisfy the following uniform estimates

(1) $T_i \geq T(v_0) > 0\,,$
(2) $|\mathrm{Rm}(g_i(t))| \leq \frac{C(v_0)}{t}$ for $0 < t < T(v_0)$,
(3) $\mathrm{Ric}(g_i(t)) \geq -k(v_0)$ for $0 < t < T(v_0)$,

for some $T(v_0) > 0$ and $k(v_0) \geq 0$. These estimates imply that the manifolds M_i are locally diffeomorphic and that the distances behave well under Ricci flow, i.e. locally

$$e^{t-s} d_s(x, y) \geq d_t(x, y) \geq d_s(x, y) - C(v_0)\sqrt{t-s}$$

for all $x, y \in M_i = M$ and $0 < s < t < T(v_0)$. Simon-Topping use these estimates to show that there exists a Ricci flow $(M, (g(t))_{0<t<T}$, starting form (X, d_0) as $t \searrow 0$ in the sense above, satisfying the estimates (1), (2) and (3). It also follows that X is locally homeomorphic to M. But it is known that the distance d_0 on the limit space (X, d_0) is not necessarily everywhere locally induced by a smooth metric g_0 on M. This suggest the following natural initial regularity question for a Ricci flow $(M, (g(t))_{0<t<T}$ as above:

Question. Assume (X, d_0) is smooth locally around a point $p_0 \in X$, i.e. the distance d_0 is locally around p_0 induced by a smooth metric g_0 on a neighbourhood U_0 of p_0. Does $(U_0, (g(t))$ converge locally smoothly to (U, g_0)?

We show that this is true under the above assumptions, see [2].

Theorem (Deruelle-Schulze-Simon 2019). *Let* $(M, (g(t))_{0<t<T}$ *be a Ricci flow, such that*

$$Ric(g(t)) \geq -1 \qquad and \qquad |Rm(g(t))| \leq \frac{c}{t}$$

for all $t \in (0,1)$*, and let*

$$(B_{d_0}(p_0, R), d_0) = \lim_{t \searrow 0}(B_{g(t)}(p_0, R), d_t)\,.$$

Assume $(B_{d_0}(p_0, R), d_0)$ *is smooth (in the above sense) around* p_0*. Then there exists a neighbourhood* U_0 *of* p_0 *such that* $g_t \to g_0$ *as* $t \searrow 0$*, where* g_0 *is a smooth Riemannian metric on* U_0 *which locally induces* d_0 *around* p_0*.*

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Constant mean curvature hypersurfaces in Minkowski space

Peter Smillie

(joint work with Francesco Bonsante, Andrea Seppi)

The first part of this talk is on a general existence and uniqueness result for entire spacelike hypersurfaces with constant mean curvature (CMC) in $n+1$ dimensional Minkowski space $\mathbb{R}^{n,1}$. In the 1980s, Treibergs and Choi-Treibergs proposed to study these surfaces in terms of their blowdowns, and proved existence results for a large class of blowdown data [Tre82, CT90]. Other authors [Li95, GJS06] studying constant negative Gaussian curvature hypersurfaces in Minkowski space proved analogous existence results, but replaced the notion of blowdown with something we call the null support function. Briefly, an entire spacelike surface in $\mathbb{R}^{n,1}$ is the graph of an entire 1-Lipschitz function on $\mathbb{R}^n$, and the null support function is the restriction of the Legendre transform of this function to the unit sphere. It is a lower semicontinuous function on the sphere, valued in $\mathbb{R} \cup \{\infty\}$, which describes the collection of null (i.e. coisotropic) affine planes in $\mathbb{R}^{n+1}$ that are asymptotic to the surfaces.

Our first result is a complete classification of entire CMC spacelike hypersurfaces in terms of their null support function:

Theorem 1. *The null support function of an entire CMC spacelike hypersurface is finite at at least two points. Conversely, every lower semicontinuous function on the $n-1$ sphere valued in $\mathbb{R} \cup \{\infty\}$ that is finite on at least two points is the null support function of a unique entire spacelike hypersurface in $\mathbb{R}^{n,1}$ with mean curvature 1.*

An exmple of Bonsante and Fillastre [BF17] shows that we can't hope for such a clean result for the classification of constant Gaussian curvature hypersurfaces in dimension at least 3+1. This remains an interesting question! However, there is a lot more you can say about this problem in $\mathbb{R}^{2,1}$, and this is the focus of the second part of the talk.

In 2+1 dimensions, the classification of constant Gaussian curvature surfaces is equally clean:

Theorem 2 ([BSS19])**.** *The null support function of an entire spacelike hypersurface with constant negative Gaussian curvature in $\mathbb{R}^{2,1}$ is finite at at least three points. Conversely, every lower semicontinuous function on the circle valued in $\mathbb{R} \cup \{\infty\}$ that is finite on at least three points is the null support function of a unique entire spacelike surface in $\mathbb{R}^{2,1}$ with Gaussian curvature equal to -1.*

If we normalize the curvature to be equal to -1, then such surfaces are locally isometric to the hyperbolic plane. Since they are graphs over $\mathbb{R}^2$, they are also simply connected. However, even though they are properly embedded, they need not be complete (in particular, they need not be globally isometric to the hyperbolic plane). Indeed, there are many functions on the circle for which the corresponding surface is incomplete. For instance, we have

Theorem 3. *Suppose $s : S^1 \to \mathbb{R} \cup \{\infty\}$ is the null support function of a constant Gaussian curvature surface Σ. If there exists a direction $\theta \in S^1$ such that*

$$\liminf_{\theta' \to \theta} s(\theta') > s(\theta)$$

then Σ is incomplete.

Due to the presence of examples like this, we are not yet able to describe all isometric embeddings of the hyperbolic plane into $\mathbb{R}^{2,1}$. However, we have the following result in the other direction:

Theorem 4. *Suppose $s : S^1 \to \mathbb{R} \cup \{\infty\}$ is the null support function of a constant Gaussian curvature surface Σ. If for all $\theta \in S^1$, there exists a sequence $\theta_i \to \theta$ and a large enough constant M such that for all i,*

$$s(\theta_i) - s(\theta) < M|\theta_i - \theta|$$

then Σ is complete.

For instance, this theorem applies to a null support function which is finite on a Cantor set, and Lipschitz on that set. At the moment we know of no complete surface that does not satisfy this condition. Finally, I'll describe a nice consequence of the proof of this last theorem:

Theorem 5. *Every entire spacelike surface with constant Gaussian curvature in* $\mathbb{R}^{2,1}$ *is intrinsically isometric to the interior of a simply connected surface with geodesic boundary.*

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On geometric properties of log Kähler-Einstein metrics.

Cristiano Spotti

(joint work with Martin de Borbon, Patricio Gallardo, Jesus Martinez Garcia)

We are interested in studying geometric properties of log Kähler-Einstein (KE) metrics with conical singularities on *pairs* $(X, D = \sum(1-\beta_i)D_i)$ given by a smooth complex manifold X and a mildly singular (klt) divisor D. Roughly speaking, such metrics are smooth KE metrics on $X \setminus D$, which are modeled on cones with angles $2\pi\beta_i$ in the normal direction to *smooth points* of D. What should happen near $Sing(D)$ is more subtle (see later). Such metrics provide a way to interpolate between different complete geometries, and they are deeply related to moduli spaces of algebraic varieties.

By pluripotential theory, it is known that *weak* KE metrics exist on any such pairs provided that the natural topological constrain on the log first Chern class are satisfied and K-stability holds in the log Fano case (even if this case has not been proved in full generality yet.) However, these results do not give information on the geometric behavior near the divisor D, and only for D smooth or normal crossing the asymptotic of the metric to the models is known thanks to work of Jeffres-Mazzeo-Rubinstein and Guenancia-Paun.

In [1], together with M. de Borbon, we showed:

Theorem 1. [1]. *Weak Ricci flat metrics on log Calabi-Yau surfaces have conical behavior near smooth points of D and they are polynomially asymptotic to polyhedral Kähler cones near $Sing(D)$, provided the singularities of the pair are stable.*

For example, D can have a singularities modeled on the cusp $x^2 = y^3$ with cone angle in the stable range $\beta \in (\frac{1}{6}, \frac{5}{6})$. For bigger values of β we expect jumping of tangent cone, as the theory developed in the non-conical setting by Donaldson-Sun, and algebraically by Chi Li and others, suggests. These asymptotic results suggest an optimal Miyaoka-Yau inequality in this setting. The proof of the theorem is based on the Yau's original continuity path (in certain weighted spaces) after having constructed a good background metric with control on the bisectional curvature.

In [2] we have constructed, in particular, analogous of Kronheimer's ALE spaces with singularities along the exceptional sets.

Theorem 2. [2]. *Let $\pi : X \to \mathbb{C}^2/\Gamma$ be the minimal resolution of any isolated two dimensional quotient singularity. Then in every Kähler class there is a unique ALE Ricci flat Kähler metric with cone singularities (with cone angles fixed by the resolution) along the normal crossing exceptional set.*

Finally in [3], together with P. Gallardo and J. Martinez Garcia, we have investigated the first examples of (compact) moduli spaces of log KE pairs in the Fano situation. These results generalize previous ones in the absolute case by Odaka-Spotti-Sun, and their proofs are based on a continuity method strategy combined with the control of the normalized volume of singularities.

Theorem 3. [3]. *For $\beta > \frac{\sqrt{3}}{2}$ the Gromov-Hausdorff compactification $\overline{M}_\beta$ of the moduli of log KE pairs $(X, (1-\beta)H)$ given by a cubic surface X and a smooth hyperplane section H is identified with a natural explicit GIT quotient.*

Similarly, for $n = 2, 3$ and $\beta > \beta_0$ explicit, the existence of log KE metric on singular pairs $(\mathbb{P}^n, (1-\beta)H_d)$, with H_d degree $d > n+1$ hypersurface is equivalent to GIT stability for the standard action of $SL(n+1)$ on $Sym^d(\mathbb{C}^{n+1})$.

These last results rise the following question:

Question 1. *Is it true that, for any K-polystable (KE Fano) variety X and for β big enough, log K-polystability (existence of log KE metrics) for $(X, (1-\beta)D)$ with D corresponding to a plurianticanonical section is equivalent to the GIT stability for the natural action of the reductive automorphism group $Aut(X)$ on the space of sections $H^0(K_X^{-l})$?*

For small β it is clear that the above is not true, as the case of quartic curves in the projective plane with cone $\beta = \frac{1}{2}$ shows.

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Conformal deformations of metric spaces

Stephan Stadler
(joint work with Alexander Lythchak)

Conformal changes are a basic tool in Riemannian geometry which allow to deform a given metric in a controlled way. Geometric quantities of the resulting metric are in principle computable in terms of a single function, the conformal factor. The definition makes sense in a metric setting. We have two results in this direction.

(1) In spaces of curvature at most Λ (in the sense of Alexandrov), small open balls carry complete metrics of curvature at most $-\Lambda$.
(2) If a metric space has dimension at most two and Ricci curvature bounded below (in a syntactic sense), then it is an Alexandrov space.

This joint work with Alexander Lytchak.

Calabi-Yau metrics on $\mathbf{C}^n$

Gábor Székelyhidi

Suppose that $u : \mathbf{R}^n \to \mathbf{R}$ is convex, and satisfies $\det(u_{jk}) = 1$, where u_{jk} are the components of the Hessian of u. The well known Jörgens-Calabi-Pogorelov theorem then implies that the Hessian u_{jk} is constant. A basic question going back to Calabi is to study the complex analog of this problem, namely to classify plurisubharmonic functions $\phi : \mathbf{C}^n \to \mathbf{R}$ which satisfy the equation

$$\det(\phi_{j\bar{k}}) = 1. \tag{1}$$

Here $\phi_{j\bar{k}} = \partial_{z_j}\bar{\partial}_{z_k}\phi$ is the complex Hessian of ϕ. It is not difficult to find solutions of this equation for which $\phi_{j\bar{k}}$ is not constant, and it is more natural to ask for a geometric classification of the induced Kähler metrics with Kähler form $\omega = \sqrt{-1}\partial\bar{\partial}\phi$. The equation (1) then says that the volume form of ω equals the Euclidean volume form. Since such an ω is a Calabi-Yau metric, i.e. a Ricci flat Kähler metric, one could also more generally ask to classify Calabi-Yau metrics on $\mathbf{C}^n$. Some prior results are as follows.

- The Taub-NUT metric on $\mathbf{C}^2$ is non-flat, and has the same volume form as the Euclidean metric (see LeBrun [6]).
- If ω on $\mathbf{C}^2$ is a complete Calabi-Yau metric with maximal volume growth, i.e. $\mathrm{vol}\, B_\omega(0,r) > cr^4$ for all r and a fixed $c > 0$, then ω is flat (see Tian [10]).
- If $n > 2$, then $\mathbf{C}^n$ admits complete Calabi-Yau metrics with maximal volume growth that are not flat (see Li [7], Conlon-Rochon [3] and [8]). These metrics have the same volume form as the Euclidean metric.

A specific example of the third type is a Calabi-Yau metric ω_0 on $\mathbf{C}^n$ with tangent cone $\mathbf{C} \times A_1$ at infinity. Here

$$A_1 = \{x_1 + \ldots + x_n = 0\} \subset \mathbf{C}^{n-1}$$

is the $(n-1)$-dimensional A_1-singularity, equipped with the Stenzel cone metric $\sqrt{-1}\partial\bar{\partial}|x|^{2\frac{n-2}{n-1}}$. Recall that the tangent cone at infinity is obtained as a pointed Gromov-Hausdorff limit of the sequence $(\mathbf{C}^n, k^{-1}\omega_0, 0)$ as $k \to \infty$.

The main result of [9] is the following.

Theorem. If ω is a Calabi-Yau metric on $\mathbf{C}^n$ with tangent cone $\mathbf{C}\times A_1$ at infinity, then $\omega = aF^*\omega_0$, where $a > 0$ and $F : \mathbf{C}^n \to \mathbf{C}^n$ is a biholomorphism.

Let us remark that Conlon-Hein [2] classified all Calabi-Yau manifolds with certain tangent cones at infinity, such as A_1. In addition there are many related classification results of geometric objects with prescribed asymptotics, for example minimal hypersurfaces. Compared to these, the main new difficulty in our result is that the tangent cone $\mathbf{C} \times A_1$ has singular rays, and near these rays we have limited control of the convergence of the metric ω to the tangent cone.

We now describe the main ingredients in the proof. The first goal is to show that for large $R > 0$ there are biholomorphisms $F_R : \mathbf{C}^n \to \mathbf{C}^n$, numbers $a_R > 0$, and $u_R : B_{\omega_0}(0, R) \to \mathbf{R}$ such that

- $a_R F_R^*\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}u_R$,
- $\sup_{B_{\omega_0}(0,R)} |u_R| \leq \epsilon_R R^2$, with $\epsilon_R \to 0$ as $R \to \infty$,
- the u_R satisfy the Monge-Ampère equation

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}u_R)^n = \omega_0^n. \tag{2}$$

The construction of such biholomorphism relies on Donaldson-Sun's theory [4] of polynomial growth holomorphic functions on Calabi-Yau manifolds with maximal volume growth. In our setting we can use their results to show that for large R, $\mathbf{C}^n$ has an embedding $\mathbf{C}^n \to \mathbf{C}^{n+1}$ as the hypersurface $az + x_1^2 + \ldots + x_n^2 = 0$ for some $a > 0$, such that in addition on $B_\omega(0, R)$ we have a Kähler potential ϕ satisfying $\omega = \sqrt{-1}\partial\bar{\partial}\phi$ and $\phi \sim |z|^2 + |x|^{2\frac{n-2}{n-1}}$. Applying the same results also to ω_0 we can find the required biholomorphisms.

Given this result, we need to study solutions of the Monge-Ampère equation (2). Scaling down by a factor of R, this amounts to studying solutions u of the Monge-Ampère equation on a unit ball that is close in the Gromov-Hausdorff sense to the unit ball in the tangent cone $\mathbf{C} \times A_1$, such that $|u|_{L^\infty}$ is very small. The main result is the following.

Proposition. There exist $\epsilon, \lambda > 0$ with the following property. Suppose that $\eta = c\omega_0$ for $c < \epsilon$, and $u : B_\eta(0, 1) \to \mathbf{R}$ satisfies

- $(\eta + \sqrt{-1}\partial\bar{\partial}u)^n = \eta^n$,
- $\sup_{B_\eta(0,1)} |u| < \epsilon$.

Then there exists $\beta > 0$, an automorphism $g \in \mathrm{Aut}(\mathbf{C}^n)$ such that $g(0) = g$, and $u' : B_\eta(0, 1) \to \mathbf{R}$ such that

- $\beta g^*(\eta + \sqrt{-1}\partial\bar{\partial}u) = \eta + \sqrt{-1}\partial\bar{\partial}u'$,
- $\sup_{B_\eta(0,\lambda)} |u'| \leq \lambda^2 \sup_{B_\eta(0,1)} |u|$,
- $(\eta + \sqrt{-1}\partial\bar{\partial}u')^n = \eta^n$.

The proof of this result relies on a dichotomy:

(i) Suppose that u concentrates near the singular set $\mathbf{C} \times \{0\}$ in the tangent cone, in the L^∞ sense. In this case, we can use the maximum principle with suitable sub- and supersolutions of the Monge-Ampère equation to show that

$$\sup_{B_\eta(0,1/2)} |u| \ll \sup_{B_\eta(0,1)} |u|.$$

A key point here is that while the geometry of η degenerates as $\epsilon \to 0$, we can still find good Kähler potentials for η, which can be used to build sub- and supersolutions.

(ii) If u does not concentrate near the singular set, then we can show that u is close, in an L^2-sense, to a harmonic function on the unit ball of the tangent cone $\mathbf{C} \times A_1$. Results of Conlon-Hein [1] and Hein-Sun [5] show that any harmonic function on $\mathbf{C} \times A_1$ of degree at most 2 is either pluriharmonic, or corresponds to an automorphism of $\mathbf{C} \times A_1$. Using this we can replace u with an "equivalent" potential u' which has faster than quadratic growth, and this leads to the result. Note that not all automorphisms of the tangent cone give rise to automorphisms of the hypersurface $az + x_1^2 + \ldots + x_n^2 = 0$. This is the source of the additional scaling factor β above.

The proof of the Theorem is obtained by letting $R \to \infty$, and iterating the decay property given by the Proposition. Note that the method of proof can likely be used more generally to classify $\partial\bar{\partial}$-exact Calabi-Yau manifolds with prescribed tangent cones.

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Asymptotically Locally Euclidean Kähler Manifolds

Ioana Suvaina
(joint work with Hans-Joachim Hein, Rares Rasdeaconu)

An asymptotically locally Euclidean (ALE) Kähler manifolds (M, J, g) is a complete Kähler manifold with ends modeled by quotients of the Euclidean spaces. In particular, each connected component of the complement of a compact set can be identified via a diffeomorphism, f, with $(\mathbb{C}^n \setminus B_R(0), g_{Euc})/\Gamma$, with Γ a finite subgroup of $U(n)$ acting freely. For each multi-index $\mathcal{I}$ of order $|\mathcal{I}|$

$$\partial^{\mathcal{I}}\left(f_*(g) - g_{Euc}\right) = O(R^{-\tau-|\mathcal{I}|}),$$

as $R \to \infty$, where g_{Euc} denotes the Euclidean metric, and $\tau > 0$ is a real number called the order of g. In general, the diffeomorphism f does not have to be compatible with complex structure, but it will have a similar asymptotic decay to the standard complex structure as the metric g. Depending on the situation that one wants to study, there are different orders of decay that one can consider [5, 6, 4].

Simply connected ALE Ricci-flat Kähler surfaces (hyperkähler 4-manifolds) were constructed by Eguchi-Hanson, Gibbons-Hawking, Hitchin [7], and Kronheimer [10], and were classified by Kronheimer [11]. The non-simply connected surfaces, which are free quotients of some of Hitchin's examples [7], were classified by Şuvaina [17] and Wright [19]. We have the following unified classification theorem, which rephrases Kronheimer's results and draws an analogy with a special class of quotient singularities:

Theorem 1 (Kronheimer, 1989; Suvaina, 2011)**.** *Let (M, J, g, ω_g) be a smooth ALE Ricci-flat Kähler surface, asymptotic to $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $U(2)$ acting freely on $\mathbb{C}^2 \setminus \{0\}$. Then the complex manifold (M, J) can be obtained as the minimal resolution of a fiber of a one-parameter $\mathbb{Q}$-Gorenstein deformation of the quotient singularity $\mathbb{C}^2/\Gamma$. Given the Kähler class $\Omega = [\omega_g] \in H^2(M, \mathbb{R})$ then g is the unique ALE Ricci-flat Kähler metric in this class.*

Conversely, any complex surface (M, J) obtained by the above construction admits a unique ALE Ricci-flat Kähler metric in any Kähler class Ω.

In particular, the manifold M is either a $A_*, D_*, E_{6,7,8}$-type surface or a quotient of an A_{dn}-surface by a finite cyclic group $\mathbb{Z}_n$, with asymptotics either $\mathbb{C}^2/\Gamma$ with $\Gamma \subset SU(2)$ or $\mathbb{C}^2/\frac{1}{dn^2}(1, dnm-1), (n, m) = 1$, respectively.

In higher dimensions, ALE Ricci-flat Kähler metrics were constructed by Calabi [1], Joyce [9] and Tian-Yau [18].

There are currently many known non-Ricci-flat ALE Kähler manifolds which admit a preferred metric, of zero scalar curvature. In complex dimension two, such metrics were constructed by LeBrun [12, 13], Joyce [8], Calderbank-Singer [2], Lock-Viaclovsky [15] and Han-Viaclovsky [3, 4]. Non-Ricci-flat ALE scalar flat Kähler metrics in higher dimensions were found by Simanca [16].

While the geometry of a Ricci-flat Kähler manifold is quite restrictive, the canonical line bundle or one of its powers has to be trivial, a generic ALE Kähler manifold is believed to admit scalar flat Kähler metrics. The first step towards the classification of ALE (scalar flat) Kähler manifolds is to understand the underlaying complex structure. A common feature of all the known examples is that the underlying complex structure is a resolution of a deformation of a quotient singularity. In joint work with Hein and Rasdeaconu [6] we are able to prove this:

Theorem 2. *Any ALE Kähler manifold asymptotic to* $\mathbb{C}^n/\Gamma$ *is isomorphic to a resolution of a deformation of the isolated quotient singularity* $(\mathbb{C}^n/\Gamma, 0)$.

The proof of Theorem 2 is based on the construction of a suitable analytic divisorial compactification of ALE Kähler manifolds due to Hein and LeBrun [5] and Li [14]. The divisor at infinity is isomorphic to $\mathbb{CP}^{n-1}/\Gamma$ and the ring of sections of its normal bundle is isomorphic to the ring of invariants of the isolated quotient singularity $\mathbb{C}^n/\Gamma$. This suffices to show that the starting ALE Kähler manifold is in fact a resolution of an affine algebraic variety. To identify this affine algebraic variety we appeal to the "sweeping out the cone with hyperplane sections" technique of Pinkham.

The deformation theory of isolated quotient singularities in complex dimension two is well-understood and it allows us to prove the following result:

Theorem 3. *For every finite subgroup* $\Gamma \subset U(2)$ *containing no reflections, there exist only finitely many diffeomorphism types underlying minimal ALE Kähler surfaces which are asymptotic to* $\mathbb{C}^2/\Gamma$.

In dimension at least three, by Schlessinger's theorem, isolated quotient singularities are rigid under deformation. Hence, Theorem 2 immediately implies:

Corollary 1. *If* $n \geq 3$, *every ALE Kähler manifold asymptotic to* $\mathbb{C}^n/\Gamma$ *is biholomorphic to a resolution of the isolated singularity* $\mathbb{C}^n/\Gamma$.

This recovers and refines the rigidity results of Hein-LeBrun in [5, Section 2] via a different method. In contrast to dimension two, in higher dimensions it is a difficult task to impose a minimality condition on a resolution. For this reason we do not attempt here to explore a generalization of Theorem 3 in higher dimensions.

The next step towards a complete classification is to prove existence of ALE scalar flat Kähler metrics, with significant work done recently by Han-Viaclovsky [4]. The most challenging part of such a program would be to prove uniqueness of the scalar flat Kähler metric in a given Kähler class. This is a completely open problem.

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Ricci flow from open manifolds with lower curvature bounds

Peter M. Topping

(joint work with Miles Simon, Andrew McLeod)

In this talk we took a look at research from the past few years, and ongoing research, into the problem of starting the Ricci flow with initial data that is something more general than a closed Riemannian manifold (Hamilton [Ham82]) and more general than an open manifold that has been constrained to be regular at infinity by having bounded curvature (Shi [Shi89]).

The first case we are interested in is starting the flow with an open (i.e. complete, noncompact) Riemannian manifold with an appropriate notion of positive curvature, but no overall curvature bound, and no constraint on the collapsing behaviour of the manifold at infinity. In the situations in which we are interested, it is highly unlikely that one can start the flow in general if we weaken the positive curvature hypothesis to almost-positive curvature.

The second case we are interested in is starting the flow with a space that is rougher than a Riemannian manifold. It should be a metric space with some sort of extra geometric structure, and we consider Ricci limit spaces in 3D, and variants of these in higher dimensions. One has to pose carefully what we mean by a Ricci flow having such initial data because one is not given an underlying manifold on which to work, and the Ricci flow should create this.

It turns out that these two topics are highly interlinked, since in both cases we need similar results about existence of Ricci flows starting with *smooth* initial data, that satisfy optimal estimates.

There is then the question of which type of positive curvature to consider, and which sort of limit spaces to take. In the talk we made the case for the curvature condition arising from the so-called PIC1 condition introduced in 1988 by Micallef and Moore [MM88]. PIC1 coincides with positive Ricci curvature in 3D, and implies positive Ricci curvature in higher dimensions, but is much more general than, for example, positive complex sectional curvature. This condition interacts extremely nicely with Ricci flow.

An interesting open question, that has been previously considered in 3D is the following, where the noncompact case is the issue:

Conjecture 1. *If (M, g_0) is a smooth, complete, n-dimensional Riemannian manifold, $n \geq 3$, satisfying the PIC1 condition, then there exists a smooth PIC1 Ricci flow $g(t)$ on M for $t \in [0, T)$, some $T > 0$, such that $g(0) = g_0$.*

Although this conjecture does not allow the positive curvature to be weakened in general, in the talk we considered what happens if we weaken to *curvature bounded below* but look for a different type of solution, namely so-called *Pyramid Ricci flows* [MT18], which are inspired by Hochard's notion of partial Ricci flows [Hoc16]. They are Ricci flows defined on a subset of space-time that includes the initial time slice.

Combining work and ideas from Hochard [Hoc16], M.Simon and the author [ST17], McLeod and the author [MT18], Hochard's thesis [Hoc19], the author [T10], Y.Lai [Lai18] and Bamler–Cabezas-Rivas–Wilking [BCRW19], in [MT19], with McLeod, we proved:

Theorem 1 (***Global pyramid Ricci flows***)**.** *Let $\alpha_0, v_0 > 0$, $n \in \mathbb{N}$ with $n \geq 3$. Suppose that (M, g_0) is an n-dimensional complete Riemannian manifold with $\mathrm{K}_{\mathrm{IC}_1}[g_0] \geq -\alpha_0$ throughout, and $\mathrm{Vol}\mathbb{B}_{g_0}(x_0, 1) \geq v_0$ for some $x_0 \in M$. Then there exist increasing sequences $C_j \geq 1$ and $\alpha_j > 0$ and a decreasing sequence $T_j > 0$, all defined for $j \in \mathbb{N}$, and depending only on n, α_0 and v_0, for which the following is true.*

There exists a smooth Ricci flow $g(t)$, defined on a subset of spacetime that contains, for each $j \in \mathbb{N}$, the cylinder $B_{g_0}(x_0, j) \times [0, T_j]$, satisfying that $g(0) = g_0$ throughout M, and further that, again for each $j \in \mathbb{N}$,

$$(1) \qquad \begin{cases} \mathrm{K}_{\mathrm{IC}_1}[g(t)] \geq -\alpha_j & B_{g_0}(x_0, j) \times [0, T_j] \\ |\mathrm{Rm}|_{g(t)} \leq \dfrac{C_j}{t} & B_{g_0}(x_0, j) \times (0, T_j]\,. \end{cases}$$

Here, $\mathrm{K}_{\mathrm{IC}_1} \geq -\alpha_0$ refers to a lower curvature bound, where $\mathrm{K}_{\mathrm{IC}_1} > 0$ would correspond to PIC1. See [MT19] for further details.

The theorem can equally well be stated with initial data that is an IC_1-limit space, i.e. a pointed Gromov-Hausdorff limit of a sequence of manifolds as in the theorem, all using the same constants n, α_0 and v_0. It turns out that an application is that an IC_1-limit space is homeomorphic to a smooth manifold via a homeomorphism that is locally bi-Hölder. A slightly weaker version of this in 3D was proved by M. Simon and the author [ST17] in order to prove the Anderson-Cheeger-Colding-Tian conjecture in 3D.

A key ingredient in the proof is the so-called *pyramid extension lemma*, and we gave an exposition of this, as proved in [MT19], extending ideas from [MT18].

One possible application of Conjecture 1 in the future could be the following:

Conjecture 2. *An open n-dimensional PIC1 Riemannian manifold ($n \geq 3$) is diffeomorphic to $\mathbb{R}^n$.*

The 3D case of this conjecture is a result of Schoen-Yau [SY82] proved using minimal surface techniques. Current technology can only address restricted cases; in particular He and Lee [HL18] handle the case of maximal volume growth.

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Constant curvature conical metrics

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(joint work with Rafe Mazzeo, Bin Xu)

We are interested in the following "singular uniformization" problem: given a compact Riemann surface M, a collection of distinct points $\mathfrak{p} = \{p_1, \ldots, p_k\} \subset M$ and a collection of positive real numbers $\beta_1, \ldots, \beta_k$, is it possible to find a metric g on M with constant curvature and with conic singularities with prescribed cone angles $2\pi\beta_j$ at the points p_j? If there is a solution, the sign of its curvature is the same as that of the conic Euler characteristic

$$\chi(M, \vec{\beta}) = \chi(M) + \sum_{j=1}^{k} (\beta_j - 1). \tag{1}$$

This problem has a long history bringing together several different areas of mathematics. When $\chi(M, \vec{\beta}) \leq 0$, existence and uniqueness of solutions for any $\vec{\beta} \in \mathbb{R}_+^k$ is easy to prove using barrier arguments [8]. In the spherical case, for all cone angles lying in $(0, 2\pi)$, Troyanov [11] discovered an auxiliary set of linear inequalities on the β_j which are necessary and sufficient for existence; later, Luo and Tian [5] proved uniqueness of the solution in this angle regime. The spherical case when some of the cone angles are bigger than 2π has proved more challenging, and a lot of new phenomena emerge: uniqueness fails [1, 3, 4] and existence is not guaranteed [9, 2]; smooth deformation is not always possible [13] and the moduli space is expected to have singular strata [10].

In [6], we constructed the extended configuration spaces $\mathcal{E}_K$, as well as the associated extended configuration families $\mathcal{C}_K$. Each $\mathcal{E}_K$ is a manifold with corners which is a compactification of the open set in M^K consisting of all *distinct* ordered K-tuples $\{p_1, \ldots, p_K\}$. $\mathcal{C}_K$ is a universal curve over this configuration space in the sense that it too is a manifold with corners equipped with a singular fibration over $\mathcal{E}_K$. Each singular fiber in $\mathcal{C}_K$ is given by a tower of hemispheres attached to the surface M. We obtained a new regularity result on the degenerating family of metrics with merging cone points.

Theorem 1 ([6]). *When $\chi(M, \vec{\beta}) \leq 0$, or $\chi(M, \vec{\beta}) \leq 0$ with all cone angles less than 2π, hence solutions exist for all choices of data sets, then the solution families are polyhomogeneous, i.e., maximally smooth, as a family of fiber metrics on $\mathcal{C}_k$.*

We then apply this machinery to study the existence and deformation theory for spherical conic metrics with some or all of the cone angles greater than 2π.

Deformations are obstructed precisely when the number 2 lies in the spectrum of the Friedrichs extension of the Laplacian. There are many spherical cone metrics for which 2 does lie in the spectrum [12].

Our main result is that, even if 2 does lie in the spectrum, there is an unobstructed deformation space if we allow for more drastic deformations which permit the individual points p_j to 'splinter' into a collection of conic points with smaller cone angles. More precisely, we have the following trichotomy theorem:

Theorem 2 ([7]). *Let (M, g_0) be a spherical conic metric. Denote*

$$K = \sum_{j=1}^{k} \max\{[\beta_j], 1\}.$$

Let ℓ be the multiplicity of the eigenspace of Δ_{g_0} with eigenvalue 2. There are three cases:

(1) (Local freeness) If $2 \notin \operatorname{spec}(\Delta_{g_0})$, then g_0 has a smooth neighborhood parametrized by conformal structure, cone positions and angles.
(2) (Partial rigidity) If $1 \leq \ell < 2K$, then for any nearby admissible angles and conformal structures, there exists a $2K - \ell$ dimensional p-submanifold X that parametrizes nearby cone metrics.
(3) (Complete rigidity) If $\ell = 2K$, then there is no nearby spherical cone metric obtained by moving or splitting the conic points of g_0.

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Torus action over rationally elliptic manifolds of positive curvature

Burkhard Wilking

We study positively curved manifolds which admit an isometric effective action by a torus $\mathbb{T}^d$ of dimension $d = 5, 6, 7, 8$. We show that if the underlying manifold is closed even dimensional and rational elliptic, then its rational homotopy type corresponds to a rational homotopy type of connected rank one symmetric space.

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