Mathematisches Forschungsinstitut Oberwolfach

Report No. 37/2019

DOI: 10.4171/OWR/2019/37

C*-Algebras

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11 August – 17 August 2019

Abstract. The subject of Operator Algebras is a flourishing broad area of mathematics which has strong ties to many other areas in mathematics including Functional/Harmonic Analysis, Topology, (non-commutative) Geometry, Geometric Group Theory, Dynamical Systems, Descriptive Set Theory, Model Theory, Random Matrices and many more. The goal of the Oberwolfach meeting is to give its participants a global view of the subject to maintain and strengthen contacts between researchers from these different directions, making it possible for the most important developments and techniques to be disseminated.

Mathematics Subject Classification (2010): 46LXX.

Introduction by the Organizers

The 2019 C*-algebras workshop at Oberwolfach showcased some of the most important breakthroughs in Operator Algebras made in the past years and set the agenda for numerous potential future developments. A key element in most of the lectures and discussions was the constant interaction between the two major research directions within Operator Algebras, namely C*-algebras and von Neumann algebras, and with other fields of mathematics like quantum information theory, ergodic theory, geometric group theory, etc.

Since the early days of Operator Algebras in the work of Murray and von Neumann in the 1940s, the classification of all "sufficiently small" C*-algebras and von Neumann algebras has been a constant major research direction. On the von Neumann algebra side, this culminated in the complete classification of all amenable von Neumann algebras in the period 1975-1985 by Connes and Haagerup. For C*-algebras, this led to Elliott's classification program aiming to classify simple nuclear (i.e. amenable) C*-algebras by K-theoretic invariants.

During the past decades, an enormous progress has been made in this classification program. A necessary condition for classifiability emerged from counterexamples of Rørdam-Villadsen and through the Toms-Winter conjecture: one has to restrict to C*-algebras with finite nuclear dimension, which is a non-commutative analogue of having finite covering dimension. We now know that simple nuclear separable unital C*-algebras with finite nuclear dimension and satisfying the Universal Coefficient Theorem (UCT) are completely classified by K-theory and traces. This relies on the combination of a huge amount of work approaching the problem from two sides: the "abstract" side proving that all such nuclear C*-algebras can be approximated by simpler building blocks and the "concrete" side classifying these limits of building blocks by K-theory and traces. The first two lectures of the workshop by Tikuisis and White highlighted the ongoing work to directly prove the classification theorem in an "abstract" manner, directly deducing isomorphism of C*-algebras from isomorphism of the Elliott invariant.

The only hypothesis in the classification theorem that could still hold automatically is the UCT. Lin's lecture showed how extension theory of C*-algebras might lead to a proof that every nuclear C*-algebra satisfies the UCT. Also the natural question which "easy constructions" exhaust all classifiable C*-algebras was addressed during the workshop. Strung explained the range of the Elliott invariant for C*-algebras given by minimal dynamical systems and Li showed that all classifiable C*-algebras may be obtained from twisted étale groupoids. The natural next step in our understanding of nuclear C*-algebras A will be the classification of actions of amenable groups on A. At this moment, there are not even conjectural statements of how such a classification may look like, but the first steps of such a program were presented in Szabó's lecture.

There is no "disintegration" theory that will allow one to extend the classification of simple nuclear C*-algebras to the non-simple case. Nonetheless there are plenty, although still sporadic, classification results for non-simple nuclear C*-algebras. Elliott's classification of AF-algebras from the 1970s is one such example. James Gabe gave a lecture on his new and simplified proof of Kirchberg's classification (from the 1990s) of the non-simple strongly purely infinite nuclear C*-algebras in terms of an ideal related KK-theory, which, as a corollary, yields a complete classification of the $\mathcal{O}_2$-stable nuclear C*-algebras in terms of their primitive ideal spaces.

A central point in the recent progress on the classification program is the usage of von Neumann algebraic (i.e. measurable) methods in the study of C*-algebras, which are inherently topological objects. Also the converse is of increasing importance. Topological boundary actions of discrete groups are being used to prove structural properties of von Neumann algebras and their dynamics. A highlight in this direction was presented in the lectures by Boutonnet and Houdayer, who obtained a non-commutative version of the Nevo-Zimmer factor theorem, providing a striking dichotomy for actions of higher rank lattices as $\mathrm{SL}(n, \mathbb{Z})$ on arbitrary von Neumann algebras equipped with a stationary state.

Driven by the long standing Connes Embedding Problem, asking whether any finite number of elements in a II_1 factor can be approximated in noncommutative moments by matrices, there is an increasingly intense interaction between Operator Algebras and Quantum Information Theory. The lectures of Musat and Paulsen demonstrated how operator algebraic insights lead to new results in quantum information theory and how, conversely, progress in quantum information theory might one day lead to solving the Connes embedding problem.

Another major driving force for von Neumann algebra research is the Free Group Factor Problem: do the group von Neumann algebras of the free groups $\mathbb{F}_n$ depend on n? An important role in unraveling the structure of such non-amenable II_1 factors M is played by the different embeddings of the hyperfinite II_1 factor R into M. Popa's lecture provided an overview of the usage of these embeddings and of new results on the existence of coarse embeddings $R \hookrightarrow M$. Conjecturally, all maximal amenable subfactors $R \subset L(\mathbb{F}_n)$ are coarse and possibly even freely complemented, meaning that $L(\mathbb{F}_n)$ can be written as the free product of R and a complement P. In his lecture, Jekel presented new non-commutative transport of measure techniques, generalizing work of Guionnet-Shlyakhtenko, that lead in particular to such free complementation results and other isomorphism results between free group factors and factors generated by specific families of non-commutative random variables.

One of the main technical tools to understand the structure of a given operator algebra are central sequences, i.e. bounded sequences of elements that are asymptotically central. The concept has been introduced by Murray and von Neumann and they used it to prove the non-isomorphism of three II_1 factors: the hyperfinite II_1 factor R, the free group factor $L(\mathbb{F}_2)$ and their tensor product $R \overline{\otimes} L(\mathbb{F}_2)$. Central sequences were used by McDuff in her characterization of II_1 factors M that tensorially absorb the hyperfinite II_1 factor, which is a key ingredient for Connes' classification of amenable II_1 factors, as well as for the current classification results for nuclear C*-algebras. McDuff observed that all known examples of central sequences in a II_1 factor M arise from a so-called residual decreasing sequence of subalgebras $A_n \subset M$. Ioana provided in his lecture the first examples of II_1 factors without such a residual sequence of subalgebras.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Elizabeth Gillaspy, Matthew Kennedy and Jesse D. Peterson in the "Simons Visiting Professors" program at the MFO.

Workshop: C*-Algebras

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Abstracts

Classifying *-homomorphisms

Aaron Tikuisis and Stuart White
(joint work with Jorge Castillejos, Samuel Evington, and Wilhelm Winter, and with José Carrión, James Gabe, and Christopher Schafhauser)

For close to 30 years, the Elliott programme to identify those separable nuclear C*-algebras which can be reasonably classified by K-theoretic data has been a major focus of C*-algebras research. This major project can be viewed as the C*-analogue of the celebrated work of Connes on the structure of injective II_1 factors leading to the complete classification of separably acting amenable von Neumann algebras.[1] Combining the work of large numbers of researchers over decades, we now have a definitive theorem in the simple case.

Theorem (The classification theorem)**.** *Regular, simple, separable, and nuclear* C*-*algebras in the UCT class are classified by K-theory and traces.*

Moreover, the range of the invariant is understood. That is all possible K-groups (including the position of the unit in the unital case), traces and the pairings between them[2] are known, and there is a construction of a classifiable[3] C*-algebra for each. Thus we have a book of classifiable C*-algebras which we can read to uncover structure, even in cases where computation of the invariant might be out of reach. For but one spectacular example, see Xin Li's work on Cartan masas in classifiable C*-algebras described elsewhere in these proceedings.

Before proceeding further, let us discuss the adjectives in the classification theorem, and put them into context by considering dynamical examples. Given an action $\alpha : G \curvearrowright X$ of a countable discrete group on a compact Hausdorff space, when is the crossed product C*-algebra $C(X) \rtimes_\alpha G$ classifiable? For classification to be generally applicable, we must be able to tackle questions of this nature for various natural constructions, giving nice sufficient conditions in terms of the underlying (in this case dynamical) data.

- **Separability**. This is completely necessary for classification, which relies on a back-and-forth argument — the Elliott intertwining argument — and is entirely analogous to the separable predual assumption required in von Neumann classification results.[4] Our assumptions on G and X ensure $C(X) \rtimes_\alpha G$ is always separable.

[1] Using Murray and von Neumann's uniqueness of the hyperfinite II_1 factor, and finished by Haagerup's celebrated uniqueness of the hyperfinite III_1 factor.

[2] We do not explicitly need the order structure on K_0 in the invariant, only the pairing with traces. Taking this revisionist view on the Elliott invariant, it follows that A and $A \otimes \mathcal{Z}$ always have the same Elliott invariant for all simple separable nuclear A.

[3] Henceforth we refer to a C*-algebra covered by the classification theorem as 'classifiable'

[4] Indeed, even Murray and von Neumann's uniqueness of the hyperfinite II_1 factor uses an intertwining type argument.

- **Simplicity**. This corresponds to studying von Neumann algebra factors. However, every (separably acting) von Neumann algebra is a direct integral of factors, so from the factor case one obtains complete results for all separably acting injective von Neumann algebras. This is not the case in the C*-setting, so much more work will be required to obtain classification and structural results without simplicity. In the setting of actions, when α is free and minimal (i.e. all orbits are dense) the reduced crossed product will always be simple.[5]
- **Nuclearity**. This is the appropriate notion of amenability for C*-algebras, corresponding to injectivity of von Neumann algebras and readily testable in examples. $C(X) \rtimes_\alpha G$ is nuclear when the action is amenable (as happens when G is amenable).
- **The UCT class**. This is the class of C*-algebras KK-equivalent to an abelian C*-algebra. It is a fundamental challenge — arguably the fundamental challenge — in the theory of nuclear C*-algebras to determine whether all separable nuclear C*-algebras are in the UCT class. But every example that has been concretely written down is. In particular, Tu's work shows that $C(X) \rtimes_\alpha G$ is in the UCT class when the action is amenable ([14]).
- **Regularity**. Examples of Villadsen, Rørdam and Toms in the early 2000s show that one cannot expect to classify all separable simple nuclear C*-algebras by reasonably computable invariants of a K-theoretic nature.[6] The obstruction is perforation (in the Cuntz semigroup), and is shown to occur in certain inductive limits of algebras of the form $M_{n_i}(C(X_i))$, where the dimension of the spaces X_i grow much faster than n_i. To exclude this phenomena it became apparent that *regularity conditions* must be included in the classification theorem.

  The study of these regularity conditions has been driven over the last decade by the Toms–Winter conjecture, which predicts that three very different looking conditions — finite nuclear dimension (a non-commutative generalisation of finite covering dimension), $\mathcal{Z}$-stability (tensorial absorption of the Jiang–Su algebra $\mathcal{Z}$), and strict comparison (unperforation in the Cuntz semigroup) — should all be equivalent for simple, separable, non-elementary, nuclear C*-algebras. By now, most of this conjecture is a theorem (see [18] for an overview of the state of the art). In particular, finite nuclear dimension and $\mathcal{Z}$-stability are equivalent for simple, separable, non-elementary, nuclear C*-algebras (the last steps being taken in [2, 1]); these are the algebras we call regular.

[5] In fact, one only needs the action to be topologically free, i.e., the action is free on a dense set of points.

[6] As George Elliott points out, it is not inconceivable that separable simple nuclear C*-algebras can be classified by invariants derived from or in the spirit of Cuntz semigroup; though outside the regular setting, such invariants will be very difficult to compute precisely in examples.

While the regularity hypotheses of finite nuclear dimension and $\mathcal{Z}$-stability are equivalent, establishing them tends not to be equivalently easy. In relatively straightforward situations it is often easier to prove finite nuclear dimension than $\mathcal{Z}$-stability. Indeed, for $\mathcal{Z}$ itself, establishing that $\dim_{\mathrm{nuc}}(\mathcal{Z}) = 1$ is fairly straightforward from the construction, but proving that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$ is not.[7] For our crossed product examples, direct computation of the nuclear dimension is carried out through either the Rohklin dimension (a coloured extension of the Rohklin theorem of Ornstein and Weiss), or dynamical asymptotic dimension, and has been a major area of research [7, 12, 6]. A perhaps-near-optimal result for a direct computation is that the crossed product $C(X) \rtimes_\alpha G$ will have finite nuclear dimension when X has finite covering dimension and G is finitely generated and nilpotent [19]. The estimate on the nuclear dimension will depend on both the covering dimension of X and on coarse type properties of G; in particular for $G = \mathbb{Z}^d$, the estimates go to infinity with d.

For more complex examples, new techniques make $\mathcal{Z}$-stability obtainable when direct nuclear dimension calculations seem out of reach. Indeed, Kerr's breakthrough work on almost finiteness [8] (building on Matui's earlier concept for groupoids with zero-dimensional unit space) now provides a general tool for obtaining $\mathcal{Z}$-stability for crossed products, without having to pass through finite nuclear dimension. Combining this with various developments in tiling theory of amenable groups one has $\mathcal{Z}$-stability for crossed products $C(X) \rtimes G$ where X has finite covering dimension, and all finitely generated subgroups of G have sub-exponential growth [9]; for example $\mathbb{Z}^\infty = \lim_{d\to\infty} \mathbb{Z}^d$ is covered by this method (as are finitely generated groups of intermediate growth), but not by Rohklin dimension approaches. Moreover, work of Elliott and Niu [4] (and very recent further work of Niu) shows that even when the space X is not finite dimensional, one can still hope to obtain $\mathcal{Z}$-stability for the crossed product when the action has mean dimension zero.

Using works of Kirchberg, Rørdam, Winter, and Zacharias, regularity gives rise to dichotomy: a simple C*-algebras which is either of finite nuclear dimension, or $\mathcal{Z}$-stable, is necessarily either stably finite, or purely infinite. The classification theorem respects this dichotomy, in that the purely infinite and stably finite cases are currently handled separately, with the former established by Kirchberg and by Phillips (using Kirchberg's absorption theorems) in the '90s. We have nothing new to say on that subject, so for the remainder of the report, we focus on the stably finite case.

[7] The meta-reason is that $\mathcal{Z}$ is naturally one-dimensional, whereas the construction of $\mathcal{Z} \otimes \mathcal{Z}$ is naturally two-dimensional. Accordingly Jiang and Su's proof that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$ is one of the major highlights of the large body of work classifying inductive limits in the '90s; an alternative approach to this is currently being developed by Schemaitat.

For stably finite C*-algebras, the unital version of the classification theorem was obtained in 2015, with the last ingredients being [5, 3, 13]. The key ingredient is Lin's concept of internal approximations by nice building blocks up to a small error in trace (inspired by Popa's proof of injectivity implies hyperfiniteness). Very roughly speaking, [5] classifies unital, simple separable nuclear C*-algebras whose tensor products with UHF-algebras have tracial approximations of a particular form. Maps between UHF-stable algebras of this class are classified up to a strong form of *asymptotic* unitary equivalence, which allows access to Winter's localisation technique [16], and gives rise to classification up to $\mathcal{Z}$-stability. The required tracial approximations are obtained in [3] using Winter's classification-by-embeddings [17] in the presence of enough quasidiagonality: quasidiagonality of all traces. This last condition was shown to be automatic in the presence of the UCT in [13]. In all but one of these steps, when regularity is used it is through $\mathcal{Z}$-stability; the exception is the classification-by-embeddings theorem which uses finite nuclear dimension, and so Winter's long and difficult proof [15] that finite nuclear dimension implies $\mathcal{Z}$-stability also forms a component of the classification theorem as proven in 2015. As it turns out, the methods used to prove $\mathcal{Z}$-stability imply finite nuclear dimension in [2] also enable one to access classification-by-embeddings directly from $\mathcal{Z}$-stability, so $\mathcal{Z}$-stability should now be viewed as the main regularity hypothesis for classification.

In the remainder of the report, we discuss a new abstract approach to the classification theorem growing out of Schafhauser's breakthrough work [10, 11], aiming to compare and contrast it with tracial approximation methods. This arose through trying to better understand the quasidiagonality ingredient in classification. Indeed, a certain 'stable uniqueness' theorem of Dadarlat–Eilers plays a major role in the quasidiagonality theorem of [13]; this was inspired by the use of stable uniqueness theorems in the work of Eilliott, Gong, Lin and Niu. Subsequently Schafhauser developed an abstract conceptual approach to the quasidiagonality theorem, by rephrasing it as a lifting problem and using KK-theory much more directly (through a Weyl–von Neumann theorem of Elliott and Kuracovsky). These new techniques are central in the abstract approach to classification.

Thus we've collectively come full circle: starting with a classification argument, extracting a key ingredient from it to obtain quasidiagonality, simplifying and abstracting the argument there, and then extending the ideas back to give a new approach classification. Quasidiagonality has twice proven to be exactly the right level of difficulty to enable the transfer of ideas between researchers to take place.

The route to classifying separable C*-algebras is through the Elliott intertwining argument, built from both existence and uniqueness theorems for *-homomorphisms between them (classification of morphisms). The point is that building a *-isomorphism between C*-algebras A and B directly is nigh on impossible, so one constructs these by better and better approximations to *-isomorphisms. Of course, directly constructing a *-homomorphism $A \to B$ isn't much easier, so one allows for approximate *-homomorphisms,[8] which are easier to obtain. But by enlarging the class of morphisms to approximate *-homomorphisms, it becomes harder to establish the uniqueness component of classification — for this we must enlarge the invariant. Fortunately for us, a detailed analysis of the invariant needed to classify *-homomorphisms was undertaken by Gong, Lin and Niu for maps between simple nuclear, UHF-stable C*-algebras with good tracial approximations in [5]. We use exactly their invariant, but work in a very general setting. Indeed, in our approach to the classification of maps $A \to B$, we divide the hypotheses into three groups:

- **Domain side**. Hypotheses on A,
- **Morphism**. Hypotheses on the allowed maps, and
- **Co-domain side**. Hypotheses on B.

Classification of algebras will then be obtained by symmetrising assumptions, i.e. using the class of algebras satisfying both the domain and co-domain side hypotheses, and for which the identity map satisfies the morphism hypotheses.

Approximately multiplicative maps can be notationally cumbersome, and are best handled with ultrapowers or sequence algebras. Let ω be either an ultrafilter or the co-finite filter on $\mathbb{N}$. Given a C*-algebra B, set

$$B_\omega := \ell^\infty(B)/\{(x_n)_{n=1}^\infty \in \ell^\infty(B) : \lim_{n\to\omega} \|x_n\| = 0\}. \tag{1}$$

In this way a sequence $(\phi_n)_{n=1}^\infty$ of approximate *-homomorphisms from A to B corresponds to an exact *-homomorphism $A \to B_\omega$. Moreover, when A is separable, *-homomorphisms $A \to B_\omega$ will be approximately unitarily equivalent precisely when they are unitarily equivalent. So we aim to classify, in as much generality as possible, *-homomorphisms $A \to B_\omega$ up to unitary equivalence. Whenever this is done,[9] reindexing arguments give classification results for *-homomorphisms $A \to B$.

The first step in the process uses von Neumann algebraic methods. We now restrict to co-domains B such that the collection $T(B)$ of tracial states is non-empty and compact,[10] for example unital stably finite exact C*-algebras. Each trace $\tau \in T(B)$ induces a 2-norm $\|x\|_{2,\tau} := \tau(x^*x)^{1/2}$ on B, and we obtain the uniform trace norm as $\|x\|_{2,u} := \sup_{\tau\in T(B)} \|x\|_{2,\tau}$. Define the uniform trace sequence algebra or ultrapower by

$$B^\omega := \ell^\infty(B)/\{(x_n)_{n=1}^\infty \in \ell^\infty(B) : \lim_{n\to\omega} \|x_n\|_{2,u} = 0\}. \tag{2}$$

Notice that B_ω naturally quotients onto B^ω (as $\|\cdot\|_{2,u} \leq \|\cdot\|$). The extensive use of uniform tracial ultrapowers and their interplay with the norm ultrapower was initiated in the ground breaking work of Matui and Sato on the Toms–Winter

[8] *-linear maps which are approximately multiplicative on specified finite sets.

[9] By an invariant behaving suitably.

[10] This will be a standing hypothesis in our classification of *-homomorphisms. However, when we reach the classification of algebras, we can weaken this assumption to stable finiteness by working with suitable hereditary subalgebras.

conjecture, and has been crucial to the use of von Neumann technique in C*-algebras ever since. A key example arises when B has a unique trace τ; in this case (and when ω is an ultrafilter) B^ω is naturally identified with the ultrapower of the II_1 factor $\pi_\tau(B)''$. In general, B^ω is not a von Neumann algebra, however the key new ingredient in passing from $\mathcal{Z}$-stability to finite nuclear dimension in [2] also gives enough structure for B^ω in order to be able to use von Neumann like methods.

Theorem 1. *For B nuclear and $\mathcal{Z}$-stable, B^ω 'behaves like' an ultrapower of finite von Neumann algebras.*

There is a precise meaning of 'behaves like', namely that B has complemented partitions of unity as set out in [2].

A consequence of Connes' theorem is that if A is a separable nuclear C*-algebra, and $\mathcal{M}$ a finite von Neumann algebra, then *-homomorphisms $A \to \mathcal{M}^\omega$ are classified by traces. This follows as such morphisms will factor through the finite part of the bidual of A, which is hyperfinite by Connes, and so can be approximated by finite dimensional C*-algebras; then, the classification of maps from finite dimensional C*-algebras into finite von Neumann algebras amounts simply to Murray and von Neumann's classification of projections by traces. Our first use of Theorem 1 is that when B^ω 'behaves like' a finite von Neumann algebra we will have a similar classification of maps by traces. We now want to work towards a classification of maps into B_ω, by 'lifting' the classification of maps back to B^ω. The rest of this report outlines our ongoing joint work with Carrión, Gabe and Schafhauser which achieves this.

Given separable, unital, nuclear C*-algebras A and B such that B is $\mathcal{Z}$-stable, and maps $\phi, \psi : A \to B_\omega$ which agree on traces, what extra information is needed to deduce that ϕ and ψ are unitarily equivalent? Writing $q : B_\omega \to B^\omega$ for the canonical surjection, which has kernel J_B (the *trace-kernel ideal*) it follows that $q\phi$ and $q\psi$ agree on traces, so are unitarily equivalent. But, as B^ω behaves like a von Neumann algebra, the unitary witnessing this equivalence can be written as an exponential, and hence lifted to B_ω. After conjugating one of the maps by this unitary, we may assume that $q\phi = q\psi$. In this way (ϕ, ψ) defines a Cuntz-pair and hence an element of $KK(A, J_B)$.[11] Combining work of Dadarlat–Eilers with the Elliott–Kucerovsky theorem and using $\mathcal{Z}$-stability crucially to destabilise, we obtain the following uniqueness result.

Theorem 2. *Use notation as above, and assume in addition that ϕ, ψ are full (i.e., that $\phi(a)$ and $\psi(a)$ each generate B_ω as an ideal, for any nonzero $a \in A$). Then ϕ and ψ are unitarily equivalent if and only if the class $[\phi, \psi]$ in $KK(A, J_B)$ vanishes.*

[11] There is a large detail here — J_B is not stable, nor σ-unital, but nevertheless it is 'stable enough' for KK-computations to be performed. The correct notion, identified by Schafhauser in his pioneering work on these methods [10, 11] is that of separable stability, and we can get this from enough comparison on B.

The next step is to find a way of computing the KK-class in Theorem 2. It is notable that we do not need to assume that A is in the UCT class in this theorem; however, the UCT assumption becomes necessary in order to compute this KK-class in terms of the following functorial, K-theoretic invariant (also used by Gong–Lin–Niu in their classification of maps between rationally TAF C*-algebras.)

Definition 1. Let A be a unital C*-algebra. Then

$$\mathrm{inv}(A) := (\underline{K}(A), T(A), \hat{K}_1^{alg}(A), [1_A]_0, \rho_A, \mathrm{Th}_A, \mathfrak{a}_A)$$

where:

- $\underline{K}(A) = \bigoplus_{i=0,1} \bigoplus_{n=0}^{\infty} K_i(A; \mathbb{Z}/n)$ is total K-theory, as a graded module over the ring of Bockstein operators, and $[1_A]_0 \in K_0(A; \mathbb{Z})$.
- $T(A)$ is the set of tracial states on A,
- $\hat{K}_1^{alg}(A) = \bigcup_{n=1}^{\infty} U(M_n(A))/\overline{CU(M_n(A))}$, a type of Hausdorffized algebraic K_1, (where, for a unital C*-algebra B, $CU(B) = \langle uvu^*v^* : u, v \in U(B) \rangle$ is the commutator subgroup of the unitary group $U(B)$ of B).
- The remaining parts are maps as follows:

$$K_0(A) \xrightarrow{\rho_A} \mathrm{Aff}(T(A)) \xrightarrow{\mathrm{Th}_A} K_1^{alg}(A) \xrightarrow{\mathfrak{a}_A} K_1(A);$$

  ρ_A is the pairing map, Th_A is the Thomsen map, and $\mathfrak{a}_A$ is the natural quotient map.

If A, E are unital C*-algebras, a *morphism* of invariants $\Phi : \mathrm{inv}(A) \to \mathrm{inv}(E)$ consists of maps

$$\begin{aligned} \Phi_{\underline{K}} = (\Phi_{i,n})_{i=0,1; n\in\mathbb{N}} &: \underline{K}(A) \to \underline{K}(E) && \text{a graded Bockstein morphism,} \\ \Phi_T &: T(E) \to T(A) && \text{a continuous affine function,} \\ \Phi_{alg} &: \hat{K}_1^{alg}(A) \to \hat{K}_1^{alg}(E) && \text{a group morphism} \end{aligned}$$

such that $\Phi_{0,0}([1_A]_0) = [1_E]_0$ and

$$\begin{array}{ccccccc} K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}\, T(A) & \xrightarrow{\mathrm{Th}_A} & \hat{K}_1^{alg}(A) & \xrightarrow{\mathfrak{a}_A} & K_1(A) \\ \downarrow{\scriptstyle \Phi_{0,0}} & & \downarrow{\scriptstyle (\Phi_T)^*} & & \downarrow{\scriptstyle \Phi_{alg}} & & \downarrow{\scriptstyle \Phi_{1,0}} \\ K_0(E) & \xrightarrow{\rho_E} & \mathrm{Aff}\, T(E) & \xrightarrow{\mathrm{Th}_E} & \hat{K}_1^{alg}(E) & \xrightarrow{\mathfrak{a}_E} & K_1(E) \end{array}$$

commutes.

For ϕ, ψ as in Theorem 2, and when A is in the UCT class, the class of $[\phi, \psi]$ in $KK(A, B_\infty)$ amounts to the difference between $\underline{K}(\phi)$ and $\underline{K}(\psi)$. The information contained in the class in $KK(A, J_B)$ (and not seen in terms of $\underline{K}$) is captured by the difference between $\hat{K}_1^{alg}(\phi)$ and $\hat{K}_1^{alg}(\psi)$, essentially using Lin's rotation map.

In this fashion, Theorem 2 can be turned into a uniqueness theorem using inv as the invariant. Combining it with the corresponding existence theorem which we do not discuss here, gives rise to our main classification of *-homomorphisms result, which we state in fuller generality than we did for Theorem 2. Indeed we state the result more generally than we defined the invariant above!

Theorem 3. *Let A be a separable, exact* C^**-algebra in the UCT class. Let B be a nuclear, $\mathcal{Z}$-stable* C^**-algebra, such that $T(B)$ is compact and nonempty, and every densely defined trace on B is bounded. Then for any faithful amenable*[12] *morphism $\Phi : \mathrm{inv}(A) \to \mathrm{inv}(B_\omega)$, there exists a full *-homomorphism $\phi : A \to B_\omega$ such that $\mathrm{inv}(\phi) = \Phi$. The map ϕ is unique up to approximate unitary equivalence.*

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Infinite dimensional aspects of the analysis of quantum information theory

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(joint work with Mikael Rørdam)

Factorizable quantum channels, introduced by C. Anantharaman-Delaroche in [1] within the framework of operator algebras, have recently found important applications in the analysis of quantum information theory, revealing new infinite dimensional phenomena, and leading also to reformulations of the Connes Embedding Problem.

A factorizable channel on $M_n(\mathbb{C})$ is a completely positive trace-preserving map (UCPT, for short) into $M_n(\mathbb{C})$ that *factors* through a finite tracial von Neumann algebra (M, τ_M) via unital *-homomorphisms $\alpha, \beta : M_n(\mathbb{C}) \to M$, so that $T = \beta^* \circ \alpha$, where β^* is the adjoint of β. In previous work with U. Haagerup, [6], these channels were equivalently characterized as arising from an ancillary tracial von Neumann algebra (N, τ_N) and a unitary u in $M_n(\mathbb{C}) \otimes N$ such that $T(x) = (\mathrm{id}_n \otimes \tau_N)(u(x \otimes 1_N)u^*)$, for all $x \in M_n(\mathbb{C})$. It was shown in [6] that there are non-factorizable channels in all dimensions $n \geq 3$, and that each is a counterexample to a conjectured restoration in the asymptotic limit of the classical Birkhoff theorem, the so-called *asymptotic quantum Birkhoff conjecture*, due to J. Smolin, F. Verstraete and A. Winter [14]. In the current preprint [8], joint additionally with M. B. Ruskai, we provide a recipe for constructing large classes of non-factorizable channels in all dimensions $n \geq 3$, and we further analyze the convex structure of unital quantum channels. In dimension $n = 3$, we exhibit examples of (factorizable) extreme points in the UCPT class, which are not extreme in either the UCP or the CPT class. The study of extreme points of the convex set of factorizable channels is an interesting problem, currently being investigated.

Given a factorizable channel, the ancilla and its *size* are not uniquely determined! For example, if S_n is the completely depolarizing channel in dimension $n \geq 2$, $S_n(x) = \mathrm{tr}_n(x)1_n$, $x \in M_n(\mathbb{C})$, where tr_n is the normalized trace on $M_n(\mathbb{C})$, and 1_n is the identity $n \times n$ matrix, then $\mathbb{C}^{n^2}$, $M_n(\mathbb{C})$, but also (a corner of) the von Neumann algebra free product $(M_n(\mathbb{C}), \mathrm{tr}_n) * (M_n(\mathbb{C}), \mathrm{tr}_n)$ are all possible ancillas. While it was shown in [11] that in dimension $n \geq 3$, there are channels not admitting a full matrix algebra as an ancilla, it was an open question whether every factorizable channel does admit a finite dimensional ancilla. In the recent work [12], we show that von Neumann algebras of type II_1 are, indeed, needed to describe such channels (at least) in dimensions $n \geq 11$, thus witnessing infinite dimensional phenomena in the analysis of quantum information theory.

The proof uses analysis of correlation matrices arising from projections, respectively, unitaries, in tracial von Neumann algebras. Namely, we consider the set $\mathcal{D}(n)$ of $n \times n$ matrices arising from second-order moments of n-tuples of projections in finite von Neumann algebras with a (normal, faithful) tracial state, and its subset $\mathcal{D}_{\mathrm{fin}}(n)$ consisting of those matrices that arise similarly from n-tuples of projections in finite-dimensional von Neumann algebras. Using a theorem of S.

A. Kruglyak, V. I. Rabanovich and Y. S. Samoilenko, [10], describing which scalar multiples of the identity operator on a (finite dimensional) Hilbert space arise as the sum of n projections, we prove that $\mathcal{D}_{\text{fin}}(n)$ is not closed, whenever $n \geq 5$. As an application, we give a more direct proof of the main result from [4] that the set of synchronous quantum correlation matrices $C_q^s(n,2)$ is non-closed when $n \geq 5$.

In his seminal paper [9], E. Kirchberg reformulated the Connes Embedding Problem (CEP) in terms of the set $\mathcal{G}(n)$ of $n \times n$ matrices of correlations arising from unitaries in finite von Neumann algebras with a (normal, faithful) tracial state. In the further refinement by K. Dykema and K. Juschenko, [3], a positive answer to CEP is shown to be equivalent to the statement that $\mathcal{F}(n) = \mathcal{G}(n)$, for all $n \geq 3$, where $\mathcal{F}(n)$ is the closure of the set of $n \times n$ matrices of correlations arising from unitaries in full matrix algebras. A trick originating in ideas of O. Regev, W. Slofstra and T. Vidick, which we carry out in the setting of finite von Neumann algebras, allows us to conclude that the set $\mathcal{F}_{\text{fin}}(2n+1)$ of matrices of correlations arising from unitaries in finite dimensional von Neumann algebras is non-closed, whenever $\mathcal{D}_{\text{fin}}(n)$ is non-closed, i.e., for all $n \geq 5$.

Finally, the non-closure of the sets $\mathcal{F}_{\text{fin}}(2n+1)$, for $n \geq 5$, together with the connection between $\mathcal{G}(n)$ and the set of factorizable Schur multipliers on $M_n(\mathbb{C})$, established in [7], yield the existence of factorizable Schur multipliers with no finite dimensional (or, even stronger, no type I) ancilla, in each dimension ≥ 11. We provide concrete such examples. A result of N. Ozawa (cf. the Appendix in [12]) shows that the construction in [10] of an n-tuple of projections with sum equal to a multiple α of the identity can be realized in the hyperfinite II_1 factor $\mathcal{R}$, for all admissible values of α, except, possibly, for two extremal ones. This implies that the factorizable Schur multipliers with no finite dimensional ancilla found above admit the hyperfinite II_1 factor $\mathcal{R}$ as an ancilla (except, possibly, for the cases corresponding to the above mentioned extremal values of α).

In the very recent paper [13], we recast the description of factorizable maps on $M_n(\mathbb{C})$ in terms of traces on the unital universal free product $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$. This new viewpoint leads to central questions in C*-algebra theory.

More precisely, we use the Choi matrix to relate a factorizable quantum channel in dimension $n \geq 2$ to a certain matrix of correlations, further shown to be parameterized by a trace on the free unital product $M := M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$. We show that factorizable channels admitting finite dimensional ancilla are parameterized by finite dimensional traces on M, and factorizable channels that can be approximated by ones possessing a finite dimensional ancilla are parameterized by traces in the closure of the finite dimensional ones.

The C*-algebra M is known to be residually finite dimensional (RFD), [5], and semiprojective, [2]. It is not generally the case that the set of finite dimensional traces on an RFD C*-algebra A necessarily is weak*-dense in the whole trace simplex of A. For the special case of the C*-algebra M, we prove that the closure of the finite dimensional traces is equal to the set of hyperlinear traces. This shows that CEP has an affirmative answer if and only if all traces on M are hyperlinear, for all $n \geq 3$. We finally show that each metrizable Choquet simplex is a face of the simplex of tracial states on $M_n(\mathbb{C}) *_{\mathbb{C}} M_n(\mathbb{C})$. We leave open whether this is the Poulsen simplex.

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II_1 factors with exotic central sequence algebras

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(joint work with Pieter Spaas)

A uniformly bounded sequence (x_k) in a II_1 factor M is called *central* if it satisfies $\lim_k \|x_k y - y x_k\|_2 = 0$, for every $y \in M$. Central sequences have played a fundamental role in the study of II_1 factors since the beginning of the subject with Murray and von Neumann's property Gamma [13]. A separable II_1 factor M has *property Gamma* if it admits a central sequence (x_k) which is not trivial, in the sense that $\inf_k \|x_k - \tau(x_k)1\|_2 > 0$. Murray and von Neumann proved that the hyperfinite II_1 factor has property Gamma, while the free group factor $L(\mathbb{F}_2)$ does not, and thus gave the first example of two non-isomorphic separable II_1 factors [13]. In the late 1960s, binary properties of central sequences (e.g., whether any two central sequences commute) were used to provide additional examples of non-isomorphic separable II_1 factors in [3, 5, 17, 19]. Shortly after, McDuff used a refined analysis of central sequences to construct a continuum of such factors [9, 10]. A key property of these factors is the existence of a residual sequence, in the following sense introduced in [12, Definition 2]:

Definition 1. A von Neumann subalgebra A of a separable II_1 factor M is called *residual* if $\lim_k \|x_k - E_A(x_k)\|_2 = 0$, for any central sequence (x_k) in M. A sequence $(A_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of M is called a *residual sequence* if

(1) $A_{n+1} \subset A_n$, for any n,
(2) A_n is residual in M, for any n, and
(3) if $x_k \in A_k$ and $\|x_k\| \leq 1$, for any k, then $(x_k) \subset M$ is a central sequence.

Remark. Let ω be a free ultrafilter on $\mathbb{N}$. A decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of M is residual if and only if the central sequence algebra, $M' \cap M^\omega$ [11], is equal to the "tail algebra" $\cap_{n\in\mathbb{N}} A_n^\omega$.

In [12], McDuff noted that it was unknown whether every II_1 factor admits a residual sequence. The main goal of this work is to provide the first examples of II_1 factors with no residual sequences. Before stating our results, we note that several large, well-studied classes of II_1 factors admit a residual sequence:

- Any II_1 factor without property Gamma.
- The hyperfinite II_1 factor R.
- Any II_1 factor M which is strongly McDuff, i.e., of the form $M = N\bar{\otimes}R$, where N is a II_1 factor without property Gamma. If $(R_n)_{n\in\mathbb{N}}$ is any residual sequence of R, then Connes' characterization of property Gamma [4, Theorem 2.1] implies that $(1 \otimes R_n)_{n\in\mathbb{N}}$ is a residual sequence of M.
- Any (finite or infinite) tensor product $M = \bar{\otimes}_{k=1}^K M_k$, where $K \in \mathbb{N}\cup\{\infty\}$, and for every k, M_k is a II_1 factor admitting a residual sequence.

We are now ready to state our main result which gives examples of II_1 factors with no residual sequences, and thereby settles McDuff's question [12].

Theorem 1. *Let Γ be a countable non-amenable group. For every $k \in \mathbb{N}$, let $\pi_k : \Gamma \to \mathcal{O}(\mathcal{H}_k)$ be an orthogonal representation such that $\pi_k^{\otimes l}$ is weakly contained in the left regular representation of Γ, for some $l = l(k) \in \mathbb{N}$, and there exist unit vectors $\xi_k \in \mathcal{H}_k$ such that $\|\pi_k(g)(\xi_k) - \xi_k\| \to 0$, as $k \to \infty$, for every $g \in \Gamma$.*

Let $\Gamma \curvearrowright (B_k, \tau_k)$ be the Gaussian action associated to $\pi_k^{\oplus\infty}$, $(B, \tau) := \bar{\otimes}_{k\in\mathbb{N}}(B_k, \tau_k)$ and $\Gamma \curvearrowright (B, \tau)$ be the diagonal product action. Define $M = B \rtimes \Gamma$.

Then the II_1 factor M does not admit a residual sequence of von Neumann subalgebras.

The proof of Theorem 1 relies on Popa's deformation/rigidity theory [16]. A key ingredient in the proof is Boutonnet's work [1, 2], which generalizes some of Popa's work in the context of Bernoulli actions [14, 15], and provides strong structural information about subalgebras Q of II_1 factors M associated to Gaussian actions that have a "large" relative commutant, $Q' \cap M$.

To provide examples to which Theorem 1 applies, let $\Gamma = \mathbb{F}_2$ be the free group on two generators. Denote by $|g|$ the word length of an element $g \in \Gamma$ with respect to a free set of generators. By [6], the function $\varphi_k : \Gamma \to \mathbb{R}$ given by $\varphi_k(g) = e^{-|g|/k}$

is positive definite. Then the GNS orthogonal representations π_k associated to φ_k satisfy the hypothesis of Theorem 1.

In [7, Theorems E and F], the authors settled in the negative a problem of Jones and Schmidt [8, Problem 4.3] by providing examples of countable ergodic p.m.p. equivalence relations $\mathcal{R}$ on a probability space (X,μ) such that the inclusion $(A\subset M) := (L^\infty(X)\subset L(\mathcal{R}))$ satisfies the following: $M'\cap A^\omega$ is not equal to $\cap_n B_n^\omega$, for any decreasing sequence of von Neumann subalgebras $(B_n)_{n\in\mathbb{N}}$ of A with $B_{n+1}\subset B_n$ of *finite index* for every $n\in\mathbb{N}$. Theorem 1 allows us to strengthen the negative solution to [8, Problem 4.3] given in [7] by providing examples of equivalence relations $\mathcal{R}$ for which $M'\cap A^\omega$ cannot be written as $\cap_n B_n^\omega$, for *any* decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of A.

Next, let us consider the following "lifting problem": if $P,Q\subset M^\omega$ are separable commuting von Neumann subalgebras, do there exist commuting von Neumann subalgebras $P_n,Q_n\subset M$, for every $n\in\mathbb{N}$, such that $P\subset\prod_\omega P_n$ and $Q\subset\prod_\omega Q_n$? This problem has a positive answer if P or Q is amenable. The answer is also positive if $P=M$ and M has a residual sequence, $(A_n)_{n\in\mathbb{N}}$. Indeed, in this case, for any separable subalgebra $Q\subset M'\cap M^\omega$, we can find a sequence of integers $n_k\to\infty$ such that $Q\subset\prod_\omega A_{n_k}$ and $M\subset\prod_\omega(A'_{n_k}\cap M)$.

On the other hand, the above lifting problem has a negative answer in general. Thus, we prove that if Γ is not inner amenable (e.g., $\Gamma=\mathbb{F}_2$), then the conclusion of Theorem 1 holds when we replace Gaussian actions by free Bogoljubov actions [18] in its hypothesis. Moreover, there is a separable von Neumann subalgebra Q of $M'\cap M^\omega$ such that there is no sequence $(A_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of M satisfying $Q\subset\prod_\omega A_n\subset M'\cap M^\omega$. Consequently, the lifting problem has a negative answer for the commuting subalgebras $P=M$ and Q of M^ω.

Acknowledgment. The author was supported in part by NSF Career Grant DMS #1253402 and NSF FRG Grant #1854074.

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Quantum Markov semigroups on q-Gaussian algebras

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(joint work with Martijn Caspers and Yusuke Isono)

A quantum Markov semigroup on a finite von Neumann algebra (M, τ) is a one-parameter semigroup $(T_t)_{t\geqslant 0}$ of normal, ucp (unital, completely positive), trace-preserving maps $T_t \colon M \to M$, which is continuous in the ultrastrong topology. We moreover assume that the GNS implementations $T_t \colon L^2(M) \to L^2(M)$, mapping $x\Omega$ to $T_t(x)\Omega$, are self-adjoint. In this case we can write $T_t = e^{-t\Delta}$ for a certain positive operator $A \colon L^2(M) \to L^2(M)$, usually unbounded, which we call the *generator* of the semigroup; most of the information about the semigroup will be contained in the spectrum of Δ.

Existence of nice quantum Markov semigroups can provide information about approximation properties of von Neumann algebras. In fact, amenability (see [CS17]) and Haagerup property (see [JM04]) can always be witnessed by a quantum Markov semigroup, whose generator has discrete spectrum going to infinity (sufficiently fast in the case of amenability). An appropriate semigroup of Fourier multipliers was also critical in Haagerup's proof of the metric approximation property of the reduced C^*-algebra of the free group (see [Haa79]).

Strong solidity, introduced by Ozawa and Popa in [OP10], is an important structural property of nonamenable von Neumann algebras – it implies primeness (no nontrivial factorisation as a tensor product) and lack of Cartan subalgebras. It turns out that in a number of cases strong solidity can be approached via quantum Markov semigroups.

The key to obtaining results in this direction is a construction by Cipriani and Sauvageot (see [CS03]) of a derivation ∂ into a bimodule (which we call the *gradient bimodule*) H over M such that $\Delta = \partial^*\partial$; it should be viewed as an analogue of the formula $\triangle = \operatorname{div} \circ \operatorname{grad}$ for the Laplacian.

Of particular importance is whether this bimodule is weakly contained in the so-called coarse correspondence $L^2(M) \otimes L^2(M)$; it means that the homomorphism from $M \otimes M^{\mathrm{op}}$ to $B(H)$, describing the left and right actions, extends to the minimal tensor product. The application to strong solidity can made via the property AO^+ – there exists a locally reflexive, ultraweakly dense C*-subalgebra $A \subset M$ and a ucp map $\theta\colon A \otimes_{\min} A^{\mathrm{op}} \to B(L^2(M))$ such that $\theta(x \otimes y^{\mathrm{op}}) - xy^{\mathrm{op}}$ is compact for all $x \in A$ and y^{op} in A^{op}. Isono proved in [Iso15] that property AO^+, when combined with the completely bounded approximation property, implies strong solidity. Weak containment provides a $*$-homomorphism from $A \otimes_{\min} A^{\mathrm{op}}$ to $B(H)$ and we need a way to transfer it to $B(L^2(M))$.

This can be done using the derivation ∂, assuming that the generator Δ satisfies certain conditions. In this case we can construct an isometry $S\colon L^2(M) \to H$, given by $e_i \mapsto \frac{\partial e_i}{\|\partial e_i\|}$ for a particular choice of an orthonormal basis. Conjugation by S yields a map from $B(H)$ to $B(L^2(M))$ and additional constraints allow us to conclude that property AO^+ is satisfied – this is the main result of [CIW19].

Theorem 1. *Let (M, τ) be a finite von Neumann algebra and let $(T_t)_{t \geqslant 0}$ be a quantum Markov semigroup with generator Δ. If Δ is filtered and has subexponential growth, and the gradient bimodule H is weakly contained in the coarse bimodule, then M satisfies the property* AO^+ *(assuming that we can ensure the local reflexivity condition).*

We need to explain the conditions appearing in the statement of the theorem. We say that Δ has *subexponential growth* if its spectrum is discrete, tends to infinity and $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$, where $\{\lambda_0 < \lambda_1 < \dots\}$ is the complete set of eigenvalues of Δ, without multiplicities. It is *filtered* if

- the resolvent of Δ is compact;
- for each eigenvalue λ there is a subspace $\mathcal{A}(\lambda) \subset M$ such that $\mathcal{A}(\lambda)\Omega \subset L^2(M)$ is the eigenspace corresponding to λ;
- $\mathcal{A} := \bigoplus_{k=0}^{\infty} \mathcal{A}(\lambda_n)$ is ultraweakly dense in M and we have $\mathcal{A}(\lambda_n)\mathcal{A}(\lambda_m) \subset \bigoplus_{k=|n-m|}^{n+m} \mathcal{A}(\lambda_k)$.

In order to apply this result, we still need a criterion for weak containment of the gradient bimodule in the coarse bimodule. It turns out that looking at properties of the map

$$L^2(M) \ni x\Omega \mapsto T_t(\Delta(axb) - \Delta(ax)b - a\Delta(xb) + a\Delta(x)b)\Omega \in L^2(M),$$

where $a, b \in \mathcal{A}$, is key. Namely, if this map is Hilbert-Schmidt for all $t > 0$ (we say then that the semigroup is *immediately gradient Hilbert-Schmidt*), then the gradient bimodule is weakly contained in the coarse bimodule.

The canonical example to which this result applies is the free group factor $L(\mathbb{F}_n)$ with the semigroup given by $\lambda_g \mapsto e^{-t|g|}\lambda_g$, where $|g|$ denotes the word length of g; this reproves one of the main results of [OP10].

In the case of q-Gaussian algebras $\Gamma_q(H_{\mathbb{R}})$ (where $q \in (-1, 1)$), new results are obtained. The semigroup in this case is constructed using the so-called second quantisation procedure and is analogous to the semigroup considered in the case of free group factors; it is called the Ornstein-Uhlenbeck semigroup. Checking that the generator is filtered and has subexponential growth is simple, only the Hilbert-Schmidt estimates are challenging. The main result is the following.

Theorem 2. *The Ornstein-Uhlenbeck semigroup on the q-Gaussian algebra* $\Gamma_q(H_{\mathbb{R}})$ *is immediately gradient Hilbert-Schmidt for* $|q| \leqslant (\dim H_{\mathbb{R}})^{-\frac{1}{2}}$. *Hence* $\Gamma_q(H_{\mathbb{R}})$ *satisfies property* AO^+ *in this case.*

For small values of $\dim H_{\mathbb{R}}$ it is an improvement of a result by Shlyakhtenko from [Shl04].

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Full factors and co-amenable inclusions

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(joint work with Jon Bannon and Amine Marrakchi)

Property Γ was introduced by Murray and von Neumann for a type II_1 factor to distinguish the hyperfinite factor and the free group factor. Since then, it turns out that property Γ is intimately connected to various important aspects of the structure of II_1 factors. Recently property Γ has also been introduced for tracial C*-algebras and played a significant role in the classification theory of simple amenable C*-algebras. By Connes's characterization [4] (and Marrakchi [5] in the type III setting), a factor does not have property Γ if and only if it is *full* in the sense that $M' \cap M^\omega = \mathbb{C}1$ for any non-principal ultrafilter ω.

In this talk, I give an outline of our solution to Popa's conjecture from 1986 [8]: Let $N \subset M$ be a co-amenable subfactor of type II_1. If M is full, so is N. This conjecture has been solved in the cases where $N \subset M$ has finite index [6] and where $M = N \rtimes \Gamma$ with Γ amenable and $\Gamma \curvearrowright N$ an outer (cocycle) action [7, 2, 3].

In fact, Pimsner and Popa [6] have obtained a stronger result that $N' \cap M^\omega = \mathbb{C}1$ for any finite-index irreducible subfactor $N \subset M$ with M full. This is what we will generalize.

Theorem 1 ([1]). *Let M be a full factor of any type and $N \subset M$ be a co-amenable von Neumann subalgebra with a faithful normal conditional expectation. Then, there is a non-zero projection $p \in N' \cap M$ such that $p(N' \cap M^\omega)p = \mathbb{C}p$. In particular, N is full if it is a factor.*

This has a cute corollary which strengthens a recent result of Tomatsu [9].

Corollary 2. *Let $G \curvearrowright M$ be an outer action of a compact group G on a full factor M. Then, $M^G \subset M$ is a co-amenable inclusion with $(M^G)' \cap M^\omega = \mathbb{C}1$. In particular, $G \curvearrowright M$ is minimal and M^G and $M \rtimes G$ are full.*

We outline here the proof of Theorem 1 for the case M is a factor of type II_1. The first step is the simple observation that if M is full, then every M-M bimodule H that is weakly equivalent to $L^2(M)$ must contain $L^2(M)$ (the converse is also true). Since $N \subset M$ is co-amenable, one has

$$(*) \qquad {}_M L^2(M)_M \preceq {}_M L^2(M) \underset{N}{\otimes} L^2(M)_M$$

by definition. Suppose that $\succeq$ also holds in $(*)$. Then $\subseteq$ holds by the above observation. Thanks to Popa's intertwining bimodule method, $\subseteq$ is equivalent to existence of a nonzero $p \in N' \cap M$ such that $pN \subset pMp$ is an irreducible subfactor of finite index. Now the above-mentioned Pimsner–Popa theorem yields that $p(N' \cap M^\omega)p = \mathbb{C}p$. However, it is not clear when $\succeq$ holds in $(*)$. We will see it holds if we replace N with $P := (N' \cap M^\omega)' \cap M$. This is enough for the proof of Theorem 1. Indeed, since $N \subset P \subset M$, the inclusion $P \subset M$ is again co-amenable. Also, $N' \cap M^\omega = P' \cap M^\omega$ and $P = (P' \cap M^\omega)' \cap M$.

Lemma 1. *If $P = (P' \cap M^\omega)' \cap M$, then ${}_M L^2(M) \underset{P}{\otimes} L^2(M)_M \preceq {}_M L^2(M)_M$.*

Sketch of Proof. Let E_P denote the trace preserving conditional expectation of M onto P. It suffices to show: For every finite subset $F \subset M$ and $\epsilon > 0$, there are unitary elements $u_1, \ldots, u_k \in \mathcal{U}(M)$ such that $E_P(x) \approx_\epsilon \frac{1}{k} \sum u_i x u_i^*$ for all $x \in F$. (This amounts to $\langle x(1 \otimes_P 1)y, 1 \otimes_P 1 \rangle = \tau(E_P(x)y) \approx \frac{1}{k} \sum_i \langle x \hat{u}_i y, \hat{u}_i \rangle$.) In fact, we prove a stronger assertion: For every $x \in M$ the unique element

$$z \in \Omega := \overline{\mathrm{conv}}^{\|\cdot\|_2} \{uxu^* : u \in \mathcal{U}(P' \cap M^\omega)\} \subset M^\omega$$

that attains the minimal 2-norm in Ω belongs to M. This will imply that $z \in (P' \cap M^\omega)' \cap M = P$ and $z = E_P(x)$. Let's write $z = (z(n))_n \in M^\omega$. It is left to show $z(n)$ is ω-convergent w.r.t. the 2-norm. Suppose for a contradiction that $z(n)$ is non-convergent. Then by a re-indexing trick, one can find $(z(n_l))_l$ and $(z(n_l'))_l$ in Ω such that $\|z(n_l) - z(n_l')\|_2 \geq \epsilon_0 > 0$. It follows that $\frac{1}{2}(z(n_l) + z(n_l'))$ in Ω has smaller 2-norm than z; A contradiction. □

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Commutants mod normed ideals

Dan-Virgil Voiculescu

To Alain Connes' noncommutative geometry, the normed ideals of compact operators are purveyors of infinitesimals. A numerical invariant associated to an n-tuple of hermitian operators and a normed ideal, the modulus of quasicentral approximation, plays a key role in the study of perturbations from these ideals.

Recently, the commutant mod the normed ideal of the n-tuple of hermitian operators has brought new structure to this area. These are Banach algebras of operators, with respect to a natural norm, and there is a closed ideal of compact operators and a "Calkin algebra" of the commutant (the quotient by the ideal). The connection with the modulus of quasicentral approximation appears in that: the compact ideal has a contractive approximate unit iff the modulus is zero and the compact ideal has a bounded approximate unit iff the modulus is finite.

For the Calkin algebra of the commutant we get that vanishing of the modulus implies the Calkin algebra is a C^*-algebra (isometrically) , while if the modulus is only bounded, then the Calkin algebra is isomorphic to a C^*-algebra (but the isomorphism may change the norm). In case the normed ideal is the ideal of compact operators and the C^*-algebra of the n-tuple of operators does not contain non-zero compact operators, the Calkin algebra of the commutant mod the normed ideal is the Paschke dual of the C^*-algebra of the n-tuple of operators.

However, the Calkin algebras of commutants are quite far from being smooth subalgebras of Paschke duals, as has been shown by results on the K-theory of commutants mod normed ideals. The K-theory results have been obtained using perturbation theory facts, which have connections with the modulus of quasicentral approximation, in particular from invariance of absolutely continuous spectra, like classical consequences of the Kato-Rosenblum theorem and older results of myself generalizing the Kato-Rosenblum theorem to n-tuples of commuting operators.

On the other hand, under suitable assumptions, there are many similarities of the compact ideal, commutant, and Calkin algebra of the commutant with the usual compact operators, bounded operators, and usual Calkin algebra (the case of the "normed ideal zero"). These similarities include Banach space duality properties, coronas, and more.

Recently, I have found that the perturbation machinery based on the modulus of quasicentral approximation and commutants mod normed ideals can be generalized to deal with hybrid normed ideal perturbations, that is, instead of one normed ideal we have an n-tuple of normed ideals and the perturbation of each of the components of the n-tuple of operators is from the corresponding ideal in the n-tuple of ideals. Surprisingly, this continues to produce sharp results for perturbations of n-tuples of commuting hermitian operators, the main part of the proof being about singular integrals of mixed homogeneity.

References to the original papers can be found in my survey paper: "Commutants mod normed ideals" arXiv:1810.12497.

Strong Property (T) for $\tilde{A}_2$-buildings

Mikael de la Salle
(joint work with Jean Lécureux, Stefan Witzel)

Let $\Gamma = \langle S \rangle$ be a finitely generated group, and $\ell\colon \Gamma \to \mathbb{N}$ the corresponding length function.

We start by recalling the following characterization of Kazhdan's property (T): Γ has property (T) if $C^*(\Gamma)$, the full C*-algebra of Γ, carries a *Kazhdan projection*, that is an idempotent P which is invariant ($\gamma P = P$ for every $\gamma \in \Gamma$) and belongs to the closure of $\{f \in \mathbb{C}\Gamma \mid \sum_\gamma f(\gamma) = 1\}$. Such a P is unique, and corresponds to the orthogonal projection on the space of invariant vectors for the universal represention of Γ, or alternatively the spectral projection relative to $\{0\}$ of the Laplacian $\Delta = \sum_{s \in S \cup S^{-1}} (s-1)^*(s-1)$.

Vincent Lafforgue was led to introduce a strengthening, allowing to handle not only unitary representations, but also representations with small exponential growth of the norms.

Definition 1. Γ has strong property (T) if there exists $\varepsilon > 0$ such that, for every $C > 0$, the Banach algebra $C_{s\ell+C}(G)$ carries a Kazhdan projection, where $C_{s\ell+C}(G)$ is the completion of $\mathbb{C}\Gamma$ for the norm

$$\sup\{\|\pi(f)\| \mid (\pi, \mathcal{H}) \text{ Hilbert space representation}, \|\pi(\gamma)\| \leq e^{s\ell(\gamma)+C}\ \forall \gamma \in \Gamma\}.$$

Lafforgue's definition was motivated by his work on the Baum-Connes conjecture, as strong property (T) is a natural obstruction to apply the approach he developped for the Baum-Connes conjecture (as in [12]), see [11]. But in the same way as Kazhdan's property (T) has turned to be central in many different areas of mathematics because of the ubiquity of unitary representations, there are many natural contexts where non unitary representations of group occur. And Lafforgue's strong property (or his variants) have also several consequences, among other to coarse geometry (obstruction to embedding discrete graphs in Banach spaces [9, 10, 8]), to operator algebras [13, 4, 6, 5, 8] and very recently to dynamics [1, 2].

Until recently, the only groups known to have strong property (T) where higher rank semisimple algebraic groups or their close relatives (eg lattices) [9, 14, 7, 18].

The purpose of the talk was to provide examples not coming from algebraic groups, namely groups acting geometrically on locally finite $\tilde{A}_2$-buildings. A possible definition of a locally finite $\tilde{A}_2$-building is a simply connected 2-dimensional simplicial complex where the link of every vertex is the incidence structure of a finite projective plane. It is more enlightening to think of it as a collection of triangle tesselations of the plane glued together in a tree-like way. The prototypical examples are the Bruhat-Tits buildings of $\mathrm{PGL}_3(\mathbf{F})$ for a non-archimedean locally compact field $\mathbf{F}$, but there are many other examples [17, 15].

Theorem 1. *Let X be a locally finite $\tilde{A}_2$-building. Every group admitting an action by isometries on X with finitely many orbits of vertices and finite stabilizers has strong property (T).*

By the essence of strong property (T), the proof has at the same time to involve local analysis on the group (as the respresentation is a priori unbounded on the group), but at the same time at larger and larger scales (as locally, a representation with small exponential growth rate s but large constant C cannot be distinguished from a representation with large exponential growth rate). So we try to adapt the method of [9] to our setting where we do not have an ambient locally compact group (in the typical case, the automorphism group is discrete [16]). Indeed, the idea in [9] was to derive strong property (T) from a study of the local behaviour of K-finite matrix coefficients of representations of G, where $K = \mathrm{PGL}_3(\mathcal{O})$ is the maximal compact subgroup of $G = \mathrm{PGL}_3(\mathbf{F})$, for $\mathcal{O}$ the ring of integers of $\mathbf{F}$. For K-invariant matrix coefficients, there is a rather direct translation in our setting : we are trying to understand *locally but at infinitely many different scales* the harmonic analysis of the Hecke algebra introduced in [3], which plays the rôle of the algebra of K-biinvariant functions of G. The tools of [3] are of no use in the setting of non-unitary representations, so we have to developp a more local study based on the geometry of some structure that we call biaffine Hjelmslev planes, which encode the geometry of large balls in $\tilde{A}_2$-buildings. Non K-invariant matrix coefficients of G do not make sense in our general setting, so we replace its study by an analysis of a phenomenon *Hecke-harmonic implies constant* for representations of Γ.

We also generalize the results of [13] and prove that groups as in Theorem 1 do not have the approximation property of Haagerup and Kraus, and more generally that the non-commutative L_p space of the von Neumann algebra of Γ does not have the operator space approximation property for $p \notin [\frac{4}{3}, 4]$. What happens for

$p \in [\frac{4}{3}, 4]$ remains an intriguing open question, in particular in the Bruhat-Tits case.

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Noncommutative Choquet theory

Matthew Kennedy

(joint work with Kenneth R. Davidson)

We introduce a new and extensive theory of noncommutative convexity along with a corresponding theory of noncommutative functions. We establish noncommutative analogues of the fundamental results from classical convexity theory, and apply these ideas to develop a noncommutative Choquet theory that generalizes much of classical Choquet theory.

The central objects of interest in noncommutative convexity are noncommutative convex sets. The category of compact noncommutative sets is dual to the category of operator systems, and there is a robust notion of extreme point for a noncommutative convex set that is dual to Arveson's notion of boundary representation for an operator system.

We identify the C*-algebra of continuous noncommutative functions on a compact noncommutative convex set as the maximal C*-algebra of the operator system of continuous noncommutative affine functions on the set. In the noncommutative setting, unital completely positive maps on this C*-algebra play the role of representing measures in the classical setting.

The continuous convex noncommutative functions determine an order on the set of unital completely positive maps that is analogous to the classical Choquet order on probability measures. We characterize this order in terms of the extensions and dilations of the maps, providing a new perspective on the structure of completely positive maps on operator systems.

We also establish a noncommutative generalization of the Choquet-Bishop-de Leeuw theorem asserting that every point in a compact noncommutative convex set has a representing map that is supported on the extreme boundary. In the separable case, we obtain a corresponding integral representation theorem.

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A non-nuclear C*-algebra with the Weak Expectation Property and the Local Lifting Property

Gilles Pisier

In [2] Kirchberg gave the first example of a non-nuclear C*-algebra A such that

$$A \otimes_{\min} A^{\rm op} = A \otimes_{\max} A^{\rm op}. \tag{1}$$

In other words, there is a unique C*-norm on the algebraic tensor product $A \otimes A^{\rm op}$, (but there is some C*-algebra B for which the latter uniqueness does not hold for $A \otimes B$. In the first lines of that paper [2], he observed that this could be viewed as the analogue for C*-algebras of the author's construction in [5] of an infinite-dimensional Banach space X such that the completions of $X \otimes X$ for the projective and injective tensor norms coincide. It was thus tempting for the author to try to adapt the Banach space approach in [5] to the C*-algebra setting to produce new examples satisfying (1). In some sense the present paper is the result of this quest but it started to be more than wishful thinking only a few years ago.

Kirchberg ([3] see also [6]) proved that if A has the Weak Expectation Property (WEP) and B the Local Lifting Property (LLP), then

$$A \otimes_{\min} B = A \otimes_{\max} B. \tag{2}$$

Thus if a C*-algebra A has both WEP and LLP, then (2) holds with $A = B$, and in fact since both LLP and WEP remain valid for A^{op} we have (1).

Kirchberg also proved [3] that $\mathrm{C}^*(\mathbb{F}_\infty)$ has the LLP. This is in some sense the prototypical example of LLP, just like $B(H)$ is for the WEP.

The WEP, originally introduced by Lance [4], has drawn more attention recently because of Kirchberg's work [2] and in particular his proof that the Connes embedding problem is equivalent to the assertion that $A = \mathrm{C}^*(\mathbb{F}_\infty)$ (or $A = \mathrm{C}^*(\mathbb{F}_2)$) satisfies (1) or equivalently that it has the WEP.

The main result of this paper is the construction of a non-nuclear (and even non exact) separable C*-algebra A with both WEP and LLP. This answers a question that, although it seems to have remained implicit in Kirchberg's work, was clearly in the back of his mind when he produced the A satisfying (1). But since at the time he conjectured the equivalence of WEP and LLP, the question did not seem so natural until the latter equivalence was disproved in [1].

While we cannot prove (1) for $A = \mathrm{C}^*(\mathbb{F}_\infty)$, our algebra A has the same collection of finite-dimensional operator subspaces as $\mathrm{C}^*(\mathbb{F}_\infty)$. Thus our construction might shed some light, one way or the other, on the Connes (-Kirchberg) embedding problem, which is equivalent to the question whether $A = \mathrm{C}^*(\mathbb{F}_2)$ (or $A = \mathrm{C}^*(\mathbb{F}_\infty)$), which is known to have the LLP, also has the WEP.

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Isoperimetry, Littlewood functions, and unitarisability of groups

Maria Gerasimova
(joint work with Dominik Gruber, Nicolas Monod, Andreas Thom)

Definition 1. Let us assume that Γ is a discrete group.

- A representation $\pi : \Gamma \to B(H)$, where H is a Hilbert space, is called *uniformly bounded* if $\sup_{g\in\Gamma} \|\pi(g)\| < \infty$.
- A representation $\pi : \Gamma \to B(H)$ is called *unitarisable* if there exists an operator $S : H \to H$ such that $S^{-1}\pi(g)S$ is a unitary operator for all $g \in \Gamma$.
- A group Γ is called *unitarisable* if every uniformly bounded representation is unitarisable.

The first question which arises in this context if all groups are unitarisable. The answer to this is negative and the main non-example is a non-abelian free group $\mathbb{F}_n$ with n generators for $n \geq 2$ [4]. The next classical result says that amenable groups are unitarisable. It has been open ever since whether this is a characterization of unitarisability (this question is called Dixmier's problem [2]). The question remains open only for non-amenable groups without free subgroups.

One of the approaches to study unitarisability and amenability is related to the space of the Littlewood functions $T_1(\Gamma)$. The latter is the space of all functions $f\colon \Gamma \to \mathbb{C}$ admitting a decomposition

$$f(x^{-1}y) = f_1(x, y) + f_2(x, y) \qquad \forall x, y \in \Gamma$$

with $f_i\colon \Gamma \times \Gamma \to \mathbb{C}$ such that

$$\sup_x \sum_y |f_1(x, y)| < \infty \quad \text{and} \quad \sup_y \sum_x |f_2(x, y)| < \infty.$$

The connection is as follows:

(1) Γ is amenable if and only if $T_1(\Gamma) \subseteq \ell^1(\Gamma)$ as shown in [5].
(2) If Γ is unitarisable, then $T_1(\Gamma) \subseteq \ell^2(\Gamma)$ as shown in [1].
(3) If Γ contains a non-abelian free subgroup, then $T_1(\Gamma) \nsubseteq \ell^p(\Gamma)$ for all $p < \infty$.

It turns out that we can say something more about non-amenable groups.

Theorem 1 (GGMT, [3]). *For every non-amenable group Γ there exists $p > 1$ such that*

$$T_1(\Gamma) \nsubseteq \ell^p(\Gamma).$$

This result inspired us to define the Littlewood exponent $\mathrm{Lit}(\Gamma) \in [0, \infty]$ of a group Γ as follows:

$$\mathrm{Lit}(\Gamma) := \inf \{p : T_1(\Gamma) \subseteq \ell^p(\Gamma)\}.$$

The main results about the Littlewood exponent are listed in the theorem below.

Theorem 2 (GGMT, [3]).

(1) $\mathrm{Lit}(\Gamma) = 0$ *if and only if* Γ *is finite.*
(2) $\mathrm{Lit}(\Gamma) = 1$ *if and only if* Γ *is infinite amenable.*
(3) $\mathrm{Lit}(\Gamma) \leq 2$ *if* Γ *is unitarisable.*
(4) $\mathrm{Lit}(\Gamma)$ *is outside the interval* $(1, 2)$ *if* Γ *has the rapid decay property.*
(5) $\mathrm{Lit}(\Gamma) = \infty$ *if* Γ *contains a non-abelian free subgroup.*

Unfortunately, the last statement is not a characterisation of the existence of a free non-abelian subgroup.

Theorem 3 (GGMT, [3]). *There exists a torsion group* Λ *with* $\mathrm{Lit}(\Lambda) = \infty$.

There is also a connection between $\mathrm{Lit}(\Gamma)$ and the geometry of a group Γ, more precisely, between $\mathrm{Lit}(\Gamma)$ and the asymptotics of isopermetric quantities attached to Γ as follows. Given a finite symmetric subset $S \subset \Gamma$, consider the (possibly disconnected) Cayley graph $\mathrm{Cay}(\Gamma, S)$. Recall that the *Cheeger constant* $h(\Gamma, S)$ is defined by

$$h(\Gamma, S) = \inf_F \frac{|\partial_S(F)|}{|F|},$$

where the infimum runs over all non-empty finite subsets $F \subset \Gamma$. Define the relative maximal average degree $e(\Gamma, S)$ by

$$e(\Gamma, S) = 1 - \frac{h(\Gamma, S)}{|S|}.$$

Finally, our asymptotic invariant is

$$\eta(\Gamma) = -\liminf_S \frac{\ln e(\Gamma, S)}{\ln |S|},$$

where the *limes inferior* is taken over all symmetric finite subsets S of Γ.

Then we can prove the following result.

Theorem 4 (GGMT, [3]). *For any group* Γ, *we have* $\eta(\Gamma) = 1 - \dfrac{1}{\mathrm{Lit}(\Gamma)}$.

This result allows us to construct a group Γ with a nontrivial Littlewood exponent $1 < \mathrm{Lit}(\Gamma) < \infty$. It also allows us to estimate this invariant for some complicated groups (e.g. for Burnside groups of large exponent) and find some geometric applications.

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Abelian sub-C*-algebras and dynamic dimension

Wilhelm Winter
(joint work with Kang Li, Hung-Chang Liao)

For a given nuclear C*-algebra all relevant information is encoded in its systems of completely positive approximations. However, extracting such information in an efficient manner is often not easy.

For C*-algebras associated to topological dynamical systems, it is in general not possible to recover the underlying dynamics without giving additional data, e.g. specifying a suitable abelian subalgebra (often a Cartan MASA, or even a diagonal). In this talk I described ways of keeping track of such subalgebras in systems of cp approximations in order to extract dynamic dimension type information.

In particular, the diagonal dimension, defined in joint work with Kang Li and Hung-Chang Liao, is the C*-algebraic analogue of Kerr's fine tower dimension for free actions of amenable groups. For uniform Roe C*-algebras it recovers the asymptotic dimension of the underlying coarse metric space, and it is closely related to the dynamic asymptotic dimension of a principal étale groupoid.

For the canonical diagonals of irrational rotation algebras, the value of the diagonal dimension is at least two, as opposed to one for the nuclear dimension.

When there is isotropy the situation becomes more subtle, but there is evidence that the basic idea behind diagonal dimension can be expanded to also cover not necessarily free actions (or not necessarily principal groupoids). The resulting notion is related to the colourable amenability condition of Bartels–Lück–Reich. For the standard MASA in the Toeplitz algebra, this dimension takes the "right" value, namely 2, whereas the nuclear dimension is 1, and the diagonal dimension is infinity. There is a number of further examples illustrating the difference of all these notions, and the need to come up with them in the first place.

C*-algebras and Games

Vern Paulsen

We are interested in a certain type of memoryless, finite input-output games, where two cooperating non-communicating players try to maximize their probability of giving winning responses by using quantum resources. For a certain family of these games, we prove that for each game there is an affiliated C*-algebra whose properties tell us about the existence of certain perfect strategies. This talk is based on work with several different sets of coauthors [2, 3, 1]

We shall refer to the two players as Alice and Bob. By a *two person finite input-output game* we mean a tuple, $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$, where I_A, I_B, O_A, O_B are finite sets and $\lambda : I_A \times I_B \times O_A \times O_B \to \{0,1\}$ is a function. The sets I_A and I_B are finite input sets for Alice and Bob, respectively. Often it is convenient to think of these as sets of questions. The sets O_A and O_B represent the sets of possible replies, or answers that Alice and Bob can return. If they are given the pair of questions (x,y), respectively, and they return the pair of answers (a,b), then $\lambda(x,y,a,b) = 0$ means that the pair of answers (a,b) is an *incorrect*, or *losing* reply to the pair of questions (x,y) while $\lambda(x,y,a,b) = 1$ means that the pair of answers (a,b) is a *correct* or *winning* reply to the pair of questions (x,y).

For each *round* of the game, a third party, often called the Referee, selects an input pair (x,y), gives x to Alice and y to Bob and they reply with, respective answers, (a,b). Their common goal is to try to give a reply such that $\lambda(x,y,a,b) = 1$. This is what is meant by saying that they are *cooperating players*. At the start of the game, Alice and Bob both know the sets I_A, I_B, O_A, O_B and the function λ, but during the round, Alice and Bob must both give their replies without knowing what question the other player was asked or what reply the other player gave. This is what is meant by *non-communicating*. Often the Referee selects the pair (x,y) at random according to some density on $I_A \times I_B$ and in this case the players also know the distribution on questions. These games are *memoryless* in the sense that there is no requirement that if they receive the same input pair (x,y) at two different rounds, that they need to reply with the same output pair.

A *random strategy* for such a game produces a conditional probability density $p(a,b|x,y)$ which represents the probability that they produce output (a,b) when given input (x,y). A random strategy is called *perfect* if the probability that it produces a losing output pair is 0, i.e.,

$$\lambda(x,y,a,b) = 0 \implies p(a,b|x,y) = 0.$$

There are many ways to produce such probability densities. One can use classical random variables, and the set of densities that one can obtain this way are called the *local* densities. Alternatively, Alice(and similarly Bob) could each have a quantum measurement system, one for each input, perform the measurement and report the outcome for her reply. If the quantum states that Alice and Bob are measuring are not entangled, then the densities that they will obtain are the same as one can obtain with classical random variables. But when the quantum states are entangled then they can obtain a larger set of conditional densities than in the classical setting.

Consequently games exist that have no perfect classical strategies, but have perfect quantum strategies. However, there are at least three, possibly different, mathematical models describing these densities. It is known that two of these models are the same if and only if *Connes' embedding problem* has a positive answer. Thus, it is possible that by finding a game that has a perfect strategy in one model but not the other, one could disprove this conjecture.

In this talk we introduce *synchronous games*. For each synchronous game there is an affiliated *-algebra[2, 3], whose representation theory tells us if the game has a prefect strategy of each of the four types.

For each pair of graphs, there is a particular synchronous game, known as the *graph isomorphism game* and in this case the affiliated *-algebra is a representation of the *quantum permutation group.* Using the theory of quantum groups, we are able to show that any time this *-algebra is non-zero, then the corresponding game isomorphism game has a perfect strategy in the largest set of "quantum probability densities".

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Decompositions and K-theory

Rufus Willett

Our broad goal is to get information about K-theory using 'local' information about a C*-algebra. We were particularly inspired by work of Oyono-Oyono and Yu as for example in [2] and [3]. The main tools in their context are filtered C*-algebras and controlled K-theory. Here, however, we aim to do without filtrations, and work only with the usual K-theory groups.

Let A be a C*-algebra, and let C and D be two C*-subalgebras. Define group homomorphisms

$$\iota : K_*(C \cap D) \to K_*(C) \oplus K_*(D), \quad \alpha \mapsto (\alpha, -\alpha)$$

and

$$\sigma : K_*(C) \oplus K_*(D) \to K_*(A), \quad (\alpha, \beta) \mapsto \alpha + \beta.$$

These maps sometimes fit into a six-term exact sequence, part of which looks like

$$\longrightarrow K_1(C) \oplus K_1(D) \xrightarrow{\sigma} K_1(A) \overset{\partial}{\dashrightarrow} K_0(C \cap D) \xrightarrow{\iota} K_0(C) \oplus K_0(D) \longrightarrow$$

Here the dashed arrow labeled ∂ may or may not exist.

For example, if C and D are ideals in A such that $A = C + D$, then the dashed arrow can indeed be filled in, in such a way that the sequence above is always exact. In order to explain our local methods, let us first recall one way to construct ∂ under these strong assumptions. Assume for simplicity that A is unital, and let $h \in C$ be a positive contraction such that $1 - h$ is in D. Let u be an invertible element in $M_n(A)$ for some n, and define

$$a := h + (1 - h)u, \quad b := h + (1 - h)u^{-1}$$

and (using a variant of the 'Whitehead trick')

$$v := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, with $\widetilde{B}$ denoting the unitisation of B, v has the following properties:

(1) v is in $M_{2n}(\widetilde{D})$;

(2) $v\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ is in $M_{2n}(\widetilde{C})$;

(3) $v\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is in $M_{2n}(C \cap D)$.

We then define

$$\partial[u] := \left[v\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}v^{-1}\right] - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in K_0(C \cap D);$$

this defines a map $\partial : K_1(A) \to K_0(C \cap D)$ that makes the sequence above exact.

Going back to general C*-subalgebras C and D, given $u \in M_n(A)$ as above, we say an invertible $v \in M_{2n}(A)$ is a *lift* for v if it satisfies (1), (2), and (3) above; in this case, we can again define a class (but not a homomorphism!) by

$$\partial_v(u) := \left[v\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}v^{-1}\right] - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in K_0(C \cap D).$$

Lifts can be used to give exactness type results. For example, one always has the following fact with no further assumptions.

Lemma 1. *Say $\alpha \in K_0(C \cap D)$ is such that $\iota(\alpha) = 0$. Then there is an invertible $u \in M_n(A)$ for some n and a lift v for u such that $\partial_v(u) = \alpha$.*

To get exactness at other positions, we need to be able to construct lifts. We use a form of 'local decomposability' for a C*-algebra as defined below.

Definition 1. Let A be a C*-algebra, and let $\mathcal{C}$ be a class of pairs of C*-subalgebras of A. We say A *decomposes over* $\mathcal{C}$ if for all finite $\mathcal{F} \subseteq A$ and $\epsilon > 0$ there is a positive contraction $h \in M(A)$ and a pair $(C, D) \in \mathcal{C}$ such that:

(i) $\|[h, a]\| < \epsilon$ for all $a \in \mathcal{F}$;

(ii) $h\mathcal{F}$ (respectively, $(1-h)\mathcal{F}$) is contained in C (respectively, D) up to ϵ error.

The C*-algebra A *excisively decomposes over* $\mathcal{C}$ if it satsifies the property above, and so that in addition we may guarantee that h and (C, D) satisfy

(iii) $h(1-h)\mathcal{F}$ and $h^2(1-h)\mathcal{F}$ are contained in $C \cap D$ up to ϵ-error.

Finally, A *strongly excisively decomposes over* $\mathcal{C}$ if it decomposes over $\mathcal{C}$, and if in addition for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $(C, D) \in \mathcal{C}$ and all C*-algebras B, if $a \in A \otimes B$ is within δ of both $C \otimes B$ and $D \otimes B$, then a is within ϵ of $(C \cap D) \otimes B$.

This allows one to prove exactness properties at other positions in our '$\sigma - \partial - \iota$' sequence. In particular, one can use this to prove the following two results; see [4] for details.

Theorem 1. *Say A excisively decomposes over a class $\mathcal{C}$ such that for all $(C, D) \in \mathcal{C}$, $C \cap D$, C, and D all have trivial K-theory. Then A has trivial K-theory.*

This is useful for applications to the Baum-Connes conjecture, in the sprirt of [1] (but giving more general results, and with simpler proofs).

Theorem 2. *Say A strongly excisively decomposes over a class $\mathcal{C}$ such that for all $(C, D) \in \mathcal{C}$, $C \cap D$, C, and D all satisfy the Künneth formula. Then A satisifies the Künneth formula*

This can be used to give new examples of C*-algebras satisfying the Künneth formula (a weak form of the UCT).

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(Non)-uniqueness of C*-norms on group rings of amenable groups

Vadim Alekseev
(joint work with David Kyed)

The interplay between group theory and operator algebras dates back to the seminal papers by Murray and von Neumann. By choosing different completions of a discrete countable group Γ one obtains interesting analytic objects; for instance the Banach algebra $\ell^1(\Gamma)$, the full and reduced C*-algebras $C^*(\Gamma)$ and $C^*_r(\Gamma)$, and the group von Neumann algebra $L\Gamma$. In general there are many norms on, say, $\ell^1(\Gamma)$ such that the completion with respect to this norm gives a C*-algebra, and the question of when the C*-completion is unique (in which case Γ is said to be *C*-unique*) has been studied by various authors [5, 3, 2]. A C*-unique discrete group is evidently amenable and it is, to the best of the authors' knowledge, an open question whether the converse is true, although it is known to be false in the more general context of locally compact groups [5].

More recently, Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam [4] put emphasis on the question of when the complex group algebra $\mathbb{C}\Gamma$ has a unique C*-completion. Of course, this question is interesting only for amenable groups, as non-amenable groups trivially have different C*-norms on their group rings. As is easily seen [4, Proposition 6.7], if Γ is locally finite (i.e. if every finitely generated subgroup is finite) then $\mathbb{C}\Gamma$ has a unique C*-completion, and [4, Question 6.8] asked if the converse is true.

In our work, we were able to prove that the following classes of non-locally finite groups have several C*-completions.

Theorem 1. *The class of countable groups* Γ *for which* $\mathbb{C}\Gamma$ *does not have a unique C*-norm includes the following:*

(i) *Infinite groups of polynomial growth.*
(ii) *Torsion free, elementary amenable groups with a non-trivial, finite conjugacy class.*
(iii) *Groups with a central element of infinite order.*

The key to the proof of this result is the tension between the so-called strong Atiyah conjecture which predicts a concrete restriction on the von Neumann dimension of kernels of elements in the complex group algebra under the left regular representation and the existence of "small" projections in the centre of $L\Gamma$.

More precisely, if one introduces the *torsion denominator* of Γ as

$$\theta(\Gamma) = \frac{1}{\mathrm{lcm}\{|H| \mid H \leqslant \Gamma \text{ finite}\}} \in [0,1],$$

then the strong Atiyah conjecture predicts that the von Neumann dimension of kernels of elements in $\mathbb{C}\Gamma$ are multiples of $\theta(\Gamma)$.

On the other hand, one can consider the *central mesh* of $L\Gamma$,

$$\sigma(\Gamma) = \inf\{\tau(p) \mid p \in \mathrm{Proj}(Z(L\Gamma)),\, p \neq 0\} \in [0,1],$$

where τ denotes the canonical trace on $L\Gamma$. The key observation is that the inequality $\sigma(\Gamma) < \theta(\Gamma)$ implies the existence of a non-trivial projection $p \in Z(L\Gamma)$ with the property that the map $a \mapsto a(1-p)$ is injective on $\mathbb{C}\Gamma$. This gives a proper quotient of $C^*_r(\Gamma)$ to which $\mathbb{C}\Gamma$ injects, thus constructing a non-trivial norm on $\mathbb{C}\Gamma$.

During the meeting in Oberwolfach it was pointed out by Narutaka Ozawa that the original question of Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam [4, Question 6.8] actually has negative answer: the group ring of the lamplighter group $\Gamma = \mathbb{Z}/2 \wr \mathbb{Z}$ has unique C*-norm. We provide his argument here for the sake of completeness: setting $H = \bigoplus_{\mathbb{Z}} \mathbb{Z}/2$, we see that $C^*(\Gamma) \cong C^*(H) \rtimes \mathbb{Z}$ is a crossed product of the Bernoulli action which is topologically free, so by the Archbold–Spielberg theorem [1, Theorem 1] every ideal I in $C^*(\Gamma)$ intersects $C^*(H)$ nontrivially. But the group H is locally finite, so by C*-uniqueness I intersects its group ring nontrivially.

In view of this result, it would be of interest to exactly characterise amenable groups Γ which have a unique C*-norm on $\mathbb{C}\Gamma$. In particular, it would be interesting to know whether there is a torsion-free amenable group with a unique C*-norm on its complex group ring.

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Amalgamated products of RFD C*-algebras

Kristin Courtney
(joint work with Tatiana Shulman)

A C*-algebra is called residually finite dimensional or RFD when it has a separating family of finite dimensional representations. Taking the direct sum of these representations yields an embedding of the algebra into a direct sum of matrix algebras, and hence RFD C*-algebras can be thought of as block-diagonalizable C*-algebras with finite dimensional blocks. This useful finite dimensional approximation property finds itself at the heart of several major open problems in operator algebras. In particular, Kirchberg has famously proved ([8]) that Connes' Embedding Problem is equivalent to asking whether the full group C*-algebra $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ is RFD. Residual finite dimensionality is a C*-algebraic analogue to maximal almost periodicity for locally compact Hausdorff groups or even residual finiteness for discrete groups, and these are more than mere analogies. It is not hard to see that if the full group C*-algebra of a group is RFD then the group is maximally almost periodic, and Malcev's theorem extends this fact to finitely generated residually finite groups. By work of Bekka and Bekka-Louvet, we know that the converse holds exactly when the group is amenable.

In the interest of finding more examples (and non-examples) of RFD C*-algebras, mathematicians have explored when residual finite dimensionality is preserved under usual C*-algebraic constructions, such as full amalgamated products. In full generality, this question is quite difficult; indeed, $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ can be written as an amalgamated product of RFD C*-algebras. Nonetheless, a complete answer is known when the amalgam is finite dimensional: Exel and Loring showed in [6] that the (unital) amalgamated product of any RFD C*-algebras are RFD. When the C*-algebras are finite dimensional, Armstrong, Dykema, Exel, and Li showed in [1] that the only obstruction is whether or not the two algebras have faithful traces that agree when restricted to the amalgam. Later Li and Shen extended these ideas in [10] to characterize when the amalgamated product of any pair of RFD C*-algebras over a finite dimensional amalgam is again RFD.

However, surprisingly little is known when the amalgam is infinite dimensional. For some direction, we turn to the well-traversed analogue of our question in group theory. In [2], Baumslag proved that any two non-abelian finitely generated nilpotent groups have a non-residually finite amalgamated product over some common subgroup. Consequently, there are quite tame RFD C*-algebras whose amalgamated product is no longer RFD. However, Baumslag also proved in [2] that the amalgamated product of any pair of polycyclic groups over a common central amalgam is residually finite. In the locally compact setting, Khan and Morris have shown in [7] that the amalgamated product of any pair of maximally almost periodic groups over a common open compact central subgroup is again maximally almost periodic. These results together point to central amalgams as the next frontier in the C*-setting, and indeed Korchargin has showed in [9] that any amalgamated product of commutative C*-algebras is RFD. Until now, this was the only result in C*-algebras for infinite dimensional amalgams.

In [5], we prove that the amalgamated product of two separable RFD C*-algebras over a common central subalgebra is again RFD, provided that the two algebras are strongly RFD (also known as completely RFD), which means that all of their quotients are RFD. The class of strongly RFD C*-algebras includes commutative C*-algebras as well as just-infinite RFD C*-algebras and C*-algebras whose irreducible representations are all finite dimensional. Using results of Moore ([11]), this latter class allows us to give new examples of maximally almost periodic groups with maximally almost periodic amalgamated products.

However, it is cumbersome to define a property of a group solely in terms of its group C*-algebra. This leads to the following question: can we characterize which (discrete) groups will have strongly RFD group C*-algebras?

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Constructions in minimal amenable dynamics and applications to classification of C^*-algebras

Karen R. Strung
(joint work with Robin J. Deeley and Ian F. Putnam)

The interactions between dynamics and operator algebras go all the way back to the group measure space construction in the original papers of Murray and von Neumann. The subsequent classification of von Neumann algebra factors in the 70's and 80's was intimately linked to ergodic theory: the von Neumann algebra associated to *any* probability measure preserving ergodic transformation is the hyperfinite II_1 factor $\mathcal{R}$. In fact, one can construct $\mathcal{R}$ from any countable amenable Borel equivalence relation and the celebrated Connes–Feldman–Weiss theorem says that any such equivalence relation is orbit equivalent to one given by a single transformation [4]. In the study of C^*-algebras the analogous interplay—between minimal dynamical systems, minimal amenable equivalence relations, and Elliott's classification program—is less well understood.

A *minimal dynamical system* (X, φ) consists of a compact metric space X and a homeomorphism $\varphi : X \to X$ such that, for every nonempty closed subset $Y \subset X$, if $\varphi(Y) \subset Y$, then $Y = X$. When (X, φ) has *mean dimension zero* in the sense of Gromov and Lindenstrauss, the associated crossed product $C(X) \rtimes_\varphi \mathbb{Z}$ is a simple, separable, unital C*-algebra with finite nuclear dimension and stable rank one. In this case, $C(X) \rtimes_\varphi \mathbb{Z}$ is in the class of C*-algebras classified by Elliott invariants, that is, K-theory, traces, and a pairing between them [7]. In particular, (X, φ) has mean dimension zero whenever X is finite dimensional.

The principal tool for determining the K-theory of $C(X) \rtimes_\varphi \mathbb{Z}$ is the Pimsner–Voiculescu exact sequence, and from this one can see that the K_1-class of the canonical unitary $u \in C(X) \rtimes_\varphi \mathbb{Z}$ which implements the action (in the sense that $ufu^* = f \circ \varphi^{-1}$ for every $f \in C(X)$), will always be nonzero. This means, for example, that a crossed product by a minimal homeomorphism will never be isomorphic to an AF algebra, and can never be isomorphic to the Jiang–Su algebra, $\mathcal{Z}$—a simple, separable, unital, infinite-dimensional C*-algebra with $K_*(\mathcal{Z}) \cong K_*(\mathbb{C})$. In particular, the class of single transformations will not exhaust the class of C*-algebras analogous to $\mathcal{R}$, that is, those which are simple, separable, unital, infinite dimensional with finite nuclear dimension and stable rank one. Thus the question arises: what is the range of the invariant for these C*-algebras?

Given a nonempty closed subset $Y \subset X$ meeting every φ-orbit at most once, we can define a so-called *orbit-breaking* equivalence relation, which is a minimal amenable equivalence relation $\mathcal{E}_Y \subset X \times X$ obtained by splitting any orbit through Y into two equivalence classes given by the forward and backward orbit from Y. Note that in the measure-theoretical setting, breaking an orbit at such a set Y would have no effect, as it will necessarily be of measure zero on all invariant Borel probability measures. Unlike in the ergodic setting, minimal amenable equivalence relations turn out to be much more general than the orbit equivalence relation. For example, by choosing a suitable Cantor minimal system and breaking the orbit at a single point, one can construct the groupoid C*-algebra to obtain any simple unital AF algebra. It is not known whether the C*-algebras of such equivalence relations can exhaust the Elliott invariant, and answering this question is closely related to understanding the range of crossed products by minimal homeomorphisms: their tracial state spaces will be affinely homeomorphic to the simplex of φ-invariant measures, and their K-theory can be calculated from the K-theory of the containing crossed product and the space Y at which the orbit is broken using a six-term exact sequence of Putnam [11].

In both cases, determining the range of the invariant seems to lead us to the highly nontrivial question of whether or not a given infinite compact metric space X admits a minimal homeomorphism. A characterization of such spaces remains a difficult open problem in topological dynamics. In general, it is easier to produce "no-go" results for certain classes of well-behaved spaces. For example, it is well known that any homeomorphism of the 2-sphere will either fix or swap the poles, and hence will have either fixed or periodic points. More generally, any compact manifold with nonzero Euler characteristic cannot admit a minimal homeomorphism [8].

From the point of view of C*-algebras, however, it only matters what the dynamical system looks like in K-theory. Thus the question becomes whether or not, for a given infinite compact metric space X, one can find a second space Y admitting a minimal homeomorphism and which has the same K-theory as X. In this case, we can go quite far:

Theorem 1 ([6] Theorem 2.3)**.** *Let W be a finite connected CW-complex. Then there exists a infinite compact metric space X that admits a minimal homeomorphism and there are isomorphisms*

$$H^*(X) \cong H^*(W), \quad K^*(X) \cong K^*(W)$$

of Čech cohomology and K-theory.

The proof relies on the existence of homeomorphisms on point-like spaces constructed by the authors in previous work [5], together with existence results for skew product systems due to Glasner and Weiss [9]. The space X turns out to be a product of the original CW-complex W, the Hilbert cube Q, and a point-like space Z. We also show that generically, such systems have mean dimension zero. The K-theory of the associated crossed product $A := C(Z \times W \times Q) \rtimes \mathbb{Z}$ is then given by

$$K_*(A) \cong K^0(W) \oplus K^1(W),$$

and by breaking the orbit at a set of the form $Y := \{z\} \times W \times Q$, the associated orbit-breaking algebra A_Y has K-theory given by

$$K_0(A_Y) \cong K^0(W), \qquad K_1(A_Y) \cong K^1(W).$$

For the orbit-breaking constructions, as can already be seen above, we have more control over what K-theory can be realised. In fact, we can do even better and also avoid any worries about mean dimension by restricting to finite-dimensional spaces. Again, the key ingredient is the "point-like" spaces originally constructed by the authors in [5]. Since the space Z can be constructed to have arbitrary large but finite dimension, for any finite-dimensional metric space Y we embed Y into large enough such Z in such a way that each orbit of the corresponding minimal homeomorphism $\zeta : Z \to Z$ meets Y at most once. Since any orbit-breaking subalgebra has tracial state space affinely homeomorphic to the containing crossed product, we can also use the results of [5] to arrange for the tracial state space to be any finite Choquet simplex. Using this we show the following:

Theorem 2 ([6] Theorem 6.3)**.** *Let G_0 and G_1 be arbitrary countable abelian groups. and let Δ be any finite Choquet simplex. There exists an orbit-breaking subalgebra* C*-*algebra* A_Y *satisfying*

$$K_0(A_Y) \cong K^0(Y) \cong \mathbb{Z} \oplus G_0, \qquad K_1(A_Y) \cong K^1(Y) \cong G_1, \qquad T(A_Y) \cong \Delta.$$

Similar work in this direction is done [10] (also see [1, 2, 3]), where the main result is that every stably finite classifiable C*-algebra can be realized as the C*-algebra of a *twisted* principal étale groupoid. Rather than focusing on constructions coming from dynamics, Li mimics known inductive limit constructions at the level of the groupoids. The twist on the groupoid is only nontrivial when the K_0-group of the corresponding C*-algebra is torsion free. Here, although our constructions do not realize the Elliott invariant of every classifiable C*-algebra, we are able to have torsion in K-theory without requiring any twists.

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Ergodic embeddings of R into factors

Sorin Popa

It has been shown in [P81] that the hyperfine II_1 factor R can be embedded ergodically into any separable II_1 factor. More generally, if $M \subset \mathcal{M}$ is an irreducible inclusion of separable II_1 factors, then M contains an "R-direction" that's ergodic in $\mathcal{M}$. In other words, here exists $R \hookrightarrow M$ such that $R' \cap \mathcal{M} = \mathbb{C}$. We will call R-*ergodicity* this strengthened form of ergodicity for an inclusion of factors.

Producing "large" ergodic copies of R inside arbitrary factors, and more generally inside irreducible inclusions of factors $M \subset \mathcal{M}$, i.e., establishing R-ergodicity from mere ergodicity, turns out to be of crucial mportance for a multitude of problems, notably in proving vanishing cohomology results. This is because once having $R \subset M$ that's ergodic in some appropriate "augmentation" $\mathcal{M}$ of M, the amenability of R can be used to "push" any $x \in \mathcal{M}$ into $R' \cap \mathcal{M} = \mathbb{C}1$, by averaging over unitaries in R, via the Ad-action. When applied to suitable x, this amounts to "untwisting" a cocycle.

Along these lines, we present in this talk a result showing that any separable II_1 factor M contains a *coarse* hyperfinite II_1 subfactor, i.e., a subfactor $R \subset M$ such that the Hilbert R-bimodule L^2M decomposes as the direct sum of a copy of the trivial R-bimodule, L^2R, and a multiple of the coarse R-bimodule, $L^2R\overline{\otimes}L^2R^{\mathrm{op}}$. Moreover, R can be taken so that to satisfy several other "constraints", such as being contained in an irreducible subfactor $P \subset M$ and being almost orthogonal and coarse with respect to a given subalgebra $Q \subset M$ satisfying $P \not\prec_M Q$ (the pair R, Q is coarse if ${}_RL^2M_Q$ is a multiple of $L^2R\overline{\otimes}L^2Q^{op}$).

The coarse subfactor $R \subset M$ is constructed as an inductive limit of dyadic matrix algebras, through an iterative technique that we have much used in the past. But while in all previous work the resulting bimodule structure ${}_RL^2M_R$ remained "blind", the big novelty of this proof is that we are able to construct embeddings $R \hookrightarrow M$ with complete control of the bimodule decomposition at the end of the iterative process.

Coarseness of a subalgebra is in some sense the "most random" position it may have in the ambient II_1 factor. It automatically entails mixingness, which in turn implies strong malnormality, a property that's in dichotomy with the weak quasi regularity of the subalgebra. Altogether, our main result shows the following [P18]:

Theorem 1. *Any separable* II_1 *factor M contains a hyperfinite factor $R \subset M$ that's coarse in M (and thus also mixing and strongly malnormal in M). Moreover, given any irreducible subfactor $P \subset M$, any von Neumann subalgebra $Q \subset M$ satisfying $P \not\prec_M Q$ and any $\varepsilon > 0$, the coarse subfactor $R \subset M$ can be constructed so that as to be contained in P, be coarse with respect to Q, and satisfy $R \perp_\varepsilon Q$.*

The condition $P \not\prec_M Q$ for two subalgebras of the II_1 factor M is in the sense of our "intertwining by bimodules" formalism and means that there exists no nonzero *intertwiner* from P to Q (i.e., $x \in M$ with $\dim(L^2(PxQ)_Q) < \infty$). This automatically implies that Q has uniform infinite index in M, i.e., given any nonzero projection $p \in Q' \cap M$, the index of the inclusion $Qp \subset pMp$ is infinite. When

Q is an irreducible subfactor of M, it amounts to $Q \subset M$ having infinite Jones index.

A subalgebra $B \subset M$ is *mixing* if the action $\mathrm{Ad} : \mathcal{U}(B) \curvearrowright M$ is mixing relative to B, i.e., $\lim_u \|E_B(xuy)\|_2 = 0$, for all $x, y \in M \ominus B$, where the limit is over $u \in \mathcal{U}(B)$ tending weakly to 0. The subalgebra $B \subset M$ is *strongly malnormal* if any $x \in M$ that's a *weak intertwiner* for B, i.e., any x that satisfies $\dim(L^2(A_0 x B)_B) < \infty$, for some diffuse $A_0 \subset B$, lies in B.

One can show that if $R \subset M$ is coarse then $R \subset M$ is mixing. In turn, the mixing property implies very strong absorption properties for $R \subset M$, meaning that R is strongly malnormal in the above sense. In particular, any maximal abelian *-subalgebra (abbreviated hereafter as *MASA*) A of R is a MASA in M, with all its weak intertwiners contained in R. So the above theorem implies:

Corollary 2. *Any separable* II_1 *factor* M *has a coarse MASA* $A \subset M$*, which in addition is strongly malnormal and mixing, with infinite multiplicity. Moreover, given any irreducible subfactor* $P \subset M$*, any von Neumann subalgebra* $Q \subset M$ *such that* $P \not\prec_M Q$ *and any* $\varepsilon > 0$*, the coarse MASA* $A \subset M$ *can be constructed inside* P*, coarse to* Q*, and satisfying* $A \perp_\varepsilon Q$*.*

The problems of whether any separable II_1 factor contains malnormal MASAs and MASAs with infinite multiplicity, both of which are strengthening of singularity, have been open for some time.

The same type of iterative technique used to prove Theorem 1 is used to prove the following [P19]:

Theorem 3. *Any separable continuous factor* $\mathcal{M}$ *contains an ergodic copy of the hyperfinite* II_1 *factor,* $R \hookrightarrow \mathcal{M}$*, and can be embedded ergodically into the unique AFD* II_∞ *factor,* $\mathcal{M} \hookrightarrow R^\infty = R \overline{\otimes} \mathcal{B}(\ell^2\mathbb{N})$*.*

This result complements the results along these lines in [P81], which covered the case $\mathcal{M}$ is II_1 or III_λ, $0 < \lambda < 1$, of the above corollary, as well as results we have obtained in 1984 showing that if $\mathcal{M}$ is III_0 or III_1, then it contains an irreducible AFD type III factor ([P84]; see also [L84]).

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Ultrapowers vs. asymptotic sequence algebras

Ilijas Farah

Suppose that B is a C*-algebra and $\mathcal{F}$ is a filter on $\mathbb{N}$. Then

$$c_{\mathcal{F}}(B) = \{(a_n) \in \ell_\infty(B) : \limsup_{n\to\mathcal{F}} \|a_n\| = 0\}$$

is a (norm-closed, two-sided, and therefore self-adjoint) ideal of $\ell_\infty(B)$. The quotient algebra $\ell_\infty(B)/c_{\mathcal{F}}(B)$ is the *reduced power*. If $\mathcal{F}$ contains the *Fréchet filter* of all cofinite subsets of $\mathbb{N}$ and B is nontrivial, then $B_{\mathcal{F}}$ is nonseparable.

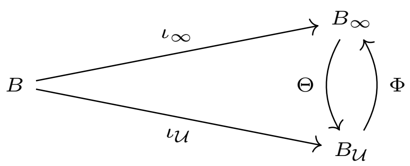
The two most important instances of such quotients are the *asymptotic sequence algebra* B_∞, corresponding to the case when $\mathcal{F}$ is the Fréchet filter and *ultrapowers*, corresponding to the case when $\mathcal{F} = \mathcal{U}$ is a *nonprincipal ultrafilter*, i.e., a filter which includes the Fréchet filter and is maximal under the inclusion. Although B_∞ and $B_{\mathcal{U}}$ are typically quite different—e.g., the center of B_∞ includes ℓ_∞/c_0 while $B_{\mathcal{U}}$ is primitive whenever B is primitive—the algebras B_∞ and $B_{\mathcal{U}}$ often behave similarly.

The algebra B is routinely identified with its image in $B_{\mathcal{F}}$ under the diagonal embedding $\iota_{\mathcal{F}}$ that sends a to the constant representing sequence $(a, a, \dots)$.

Functorial classification results for C*-algebras have two components, existence and uniqueness. Given a functor $F\colon \text{C*-alg} \to \mathbb{K}$, the *existence* asserts that for separable C*-algebras A and B every morphism $\alpha\colon F(A) \to F(B)$, is realized by a *-homomorphism $\Phi\colon A \to B$ such that $F(\Phi) = \alpha$. An intermediate step in proving the existence is often to realize α by a *-homomorphism from A into $B_{\mathcal{F}}$ for some $\mathcal{F}$ (recall that B is identified with its diagonal image inside $B_{\mathcal{F}}$). This is typically much easier in the case of ultrapowers (essentially, thanks to Łoś's Theorem, [7, Theorem 2.3.1]), both in the classification of Kirchberg algebras ([14]) and in the stably finite case, when one takes direct advantage of the *tracial ultrapower* of A whose fibres are ultrapowers of II_1 factors (see e.g., [13], [12], [3], [15], and White's and Tikuisis's contributions to the present proceedings).

On the other hand, B_∞ is better suited to the next stage of a typical existence proof. This is because of the "reindexing technique" ([14, Proposition 1.37], [9, Theorem 4.3]). Every permutation f of $\mathbb{N}$ induces an automorphism Φ_f of B_∞ whose restriction to B is equal to the identity. Every *-homomorphism Ψ from a separable C*-algebra A into B_∞ such that $\Phi_f \circ \Psi$ is unitarily equivalent to Φ is unitarily equivalent to a *-homomorphism of A into B. One does not expect a reasonable analog of reindexing to work for ultrapowers because if $\mathcal{U}$ is an ultrafilter and f is a permutation of $\mathbb{N}$, then f is either equal to the identity on a set in $\mathcal{U}$ or it sends a set in $\mathcal{U}$ into its complement. The following enables easy transfer between ultrapowers and asymptotic sequence algebras.

Theorem 1. *If the Continuum Hypothesis holds, $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$, and B is a separable and unital C*-algebra, then there are unital *-homomorphisms Φ and Θ such that the following diagram commutes and $\Phi \circ \Theta = \mathrm{id}_{B_{\mathcal{U}}}$.*



Theorem 1 cannot be proved in ZFC (see the upcoming joint paper with S. Shelah). It is a consequence of a more elaborate 'ZFC-variant' stated in terms of a *σ-complete back-and-forth system* of separable approximations to the above diagram (see [5, Theorem 4.2]). It allows for a transfinite intertwining argument, strong enough to imply the following.

Theorem 2. *Suppose $F\colon \mathrm{C}^*\text{-}alg \to \mathbb{K}$ is a functor. For unital and separable C^*-algebras A and B and a morphism $\alpha\colon F(A) \to F(B)$ the following are equivalent.*

(1) The morphism α is realized by a *-homomorphism $\Phi\colon A \to B_\mathcal{U}$ for some (any) nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.
(2) The morphism α is realized by a *-homomorphism $\Phi\colon A \to B_\infty$.

Since $B_\mathcal{U}$ is a quotient of B_∞, (2) trivially implies (1). The conclusion of Theorem 2 when F is any of the standard K-theoretic functors follows from Theorem 1 by a standard absoluteness argument (see e.g., [1, Appendix 2]). Theorem 2 answers a question of Chris Schafhauser and Aaron Tikuisus. Its instance when F consists of total K-theory, algebraic K_1, and the tracial simplex was proved by Schafhauser by a delicate argument.

The following is included for its entertainment value.

Theorem 3. *Assume the Continuum Hypothesis, let B be a separable and unital C^*-algebra, and let $\mathcal{U}$ and $\mathcal{V}$ be nonprincipal ultrafilters on $\mathbb{N}$.*

(1) $B_\mathcal{U} \cong B_\mathcal{V} \cong (B_\mathcal{U})_\mathcal{V} \cong ((B_\mathcal{V})_\mathcal{U})_\mathcal{V} \cong \dots$
(2) $B_\infty \cong (B_\infty)_\infty \cong (B_\infty)_\infty)_\infty \cong (B_\infty)_\mathcal{U} \cong (B_\mathcal{U})_\infty \cong \dots$
(3) If the filter generated by $\mathcal{F}$ and some $X \subseteq \mathbb{N}$ is an ultrafilter, then $B_\mathcal{F}$ has $B_\mathcal{U}$ as a direct summand.
(4) If the assumption of (3) *fails then $(B_\mathcal{F})_\mathcal{U} \cong B_\infty$ or even $B_\mathcal{F} \cong B_\infty$.*

All of these isomorphisms are equal to the identity on the diagonal copy of B.

More precisely, (1) and (2) state that any finite iteration of the operations of taking asymptotic sequence algebra and taking an ultrapower associated to a nonprincipal ultrafilter on $\mathbb{N}$ results in an algebra isomorphic to B_∞ if the asymptotic sequence algebra was taken at least once and $B_\mathcal{U}$ otherwise. Item (3) is easy, and (1) was first proved in [10]. The ZFC variant of the following, stated in terms of σ-complete back-and-forth systems, is our main result.

Theorem 4. *Assume that the Continuum Hypothesis holds and B is a unital and separable C^*-algebra. Then (K denotes the Cantor space) $(B \otimes C(K))_\mathcal{U} \cong B_\infty$ via an isomorphism that sends $b \otimes 1_{C(K)}$ to b for all $b \in B$.*

The case when $B \cong C(K)$ follows from [4, Theorem 5.14 and p. 2659]. The proofs use logic of metric structures ([2], [7]). The analog of Łoś's Theorem fails for $A_{\mathcal{F}}$ in general, and it has to be replaced by Ghasemi's metric Feferman–Vaught theorem ([11, Theorem 3.3]). The other component in the proofs is the countable saturation of asymptotic sequence algebras ([8, Theorem 1.1]) and ultraproducts (a classical result of Keisler). Simplified proofs of Ghasemi's theorem and countable saturation of B_∞ can be found in [6, Chapter 16]. Proofs of Theorems 1–4 appear in [5] and the alternative proofs sketched in my talk will appear elsewhere.

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Matrix product operator algebras, anyons, and subfactors

Yasuyuki Kawahigashi

In a recent work of Bultinck-Mariëna-Williamson-Şahinoğlu-Haegemana-Verstraete [1] in condensed matther physics, they use certain tensor networks and these tools look similar to combinatorial ones used in subfactor theory. We give precise mathematical relations between their machinery and Ocneanu's flat connections as in [3].

Let $N \subset M$ a subfactor of type II_1 finite Jones index and finite depth. Let

$$\cdots \subset M_{-2} \subset M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \cdots$$

be the Jones tower/tunnel where $M_{-1} = N$ amd $M_0 = M$. Choose a minimal projection p in $M'_{-1} \cap M_{2j+1}$ and consider the sequence of commuting squares

$$\begin{array}{ccccccc}
(M'_{-1} \cap M_{-1})p & \subset & (M'_{-3} \cap M_{-1})p & \subset & (M'_{-5} \cap M_{-1})p & \subset \cdots \\
\cap & & \cap & & \cap & \\
p(M'_{-1} \cap M_{2j+1})p & \subset & p(M'_{-3} \cap M_{2j+1})p & \subset & p(M'_{-5} \cap M_{2j+1})p & \subset \cdots.
\end{array}$$

This series of commuting squares is described with Ocneanu's flat connection [2]. We have a finite family of such flat connections since this flat connection depends only on the simple summand of $M'_{-1} \cap M_{2j+1}$ to which p belongs and we have identification of such summands for different j. In this way, we have a family of the flat symmetric bi-unitary connections from such a subfactor. We show that they satisfy all the requirements in Bultinck-Mariëna-Williamson-Şahinoğlu-Haegemana-Verstraete [1] up to the normalization constants and the resulting anyon algebra in [1] and Ocneanu's tube algebra for the subfactor in [2] are isomorphic. In particular, the tensor categories arising from these two algebras are the equivalent, hence both are modular tensor categories and the Verlinde formula holds for the former.

The matrix product operator algebras arising from tensors corresponding to possibly non-flat symmetric bi-unitary connections also fall in ths framework. The tensor category for such tensors is described with the "flat part" of the non-flat symmetric bi-unitary connection in subfactor theory.

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Dynamical alternating groups, property Gamma, and inner amenability

David Kerr

(joint work with Robin Tucker-Drob)

The topological full group of a continuous action $\Gamma \curvearrowright X$ of a countably infinite group on the Cantor set naturally contains many embedded copies of the ordinary alternating group A_3, and together these copies of A_3 generate what is defined by Nekrashevych in [4] to be the *alternating group* of the action, denoted by $\mathsf{A}(\Gamma, X)$. As shown in [4], the group $\mathsf{A}(\Gamma, X)$ is simple when the action is minimal, and if Γ is finitely generated and the action $\Gamma \curvearrowright X$ is expansive and has no orbits of cardinality less than 5 then $\mathsf{A}(\Gamma, X)$ is finitely generated. By a theorem of

Juschenko and Monod, the alternating group is amenable in the case that $\Gamma = \mathbb{Z}$ and the action is free and minimal [3]. On the other hand, for $\Gamma = \mathbb{Z}^2$ Elek and Monod gave an example of an expansive free minimal action on the Cantor set for which the alternating group is nonamenable [2], while Szőke showed that such expansive free minimal examples always exist whenever Γ is amenable and not virtually cyclic [5].

We prove that when Γ is amenable and the action is topologically free, then the alternating group has property Gamma, and in particular is inner amenable (i.e., there exists an atomless finitely additive probability measure on $G \setminus \{1_G\}$ which is invariant under the action of G by conjugation). In conjunction with the work of Elek–Monod and Szőke, this gives a positive answer to the question of whether there exist simple finitely generated groups which are inner amenable but not amenable.

We also show that if Γ is either (i) torsion-free, ICC, and residually finite or (ii) of the form $\Gamma_0 \times \mathbb{Z}$ where Γ_0 is nontrivial and torsion-free then there is an uncountable family of topologically free expansive minimal actions $\Gamma \curvearrowright X$ on the Cantor set such that $\mathsf{A}(\Gamma, X)$ is C*-simple (and in particular nonamenable) and such that the groups $\mathsf{A}(\Gamma, X)$ for different α are pairwise nonisomorphic. The proof uses the topological version of Austin's result on the invariance of entropy under bounded orbit equivalence [1].

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Classification of $\mathcal{O}_\infty$-stable C*-algebras

James Gabe

The Kirchberg–Phillips theorem, proved independently by Kirchberg [Kir95] and Phillips [Phi00] in the mid 90's, is one of the early major successes in the Elliott classification programme giving a complete K-theoretic classification of all separable, nuclear, $\mathcal{O}_\infty$-stable, simple C*-algebras satisfying the universal coefficient theorem (UCT). Here $\mathcal{O}_\infty$-stability means that the C*-algebra A is isomorphism to the tensor product $A \otimes \mathcal{O}_\infty$ where $\mathcal{O}_\infty$ denotes the Cuntz algebra generated by infinitely many isometries with mutually orthogonal range projections.

In celebrated work of Kirchberg, this classification theorem was generalised to not necessarily simple C*-algebras by classifying with the more complicated invariant known as ideal–related KK-theory. The proof of this result of Kirchberg has never been published, but an outline of the proof strategy was published [Kir00]. Kirchberg is writing a book for which this is the main result.

To describe the theorem, let $\mathcal{I}(A)$ denote the lattice of two-sided, closed ideals in the C*-algebra A. Given separable C*-algebras A and B, and an order preserving map $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ one constructs the groups $KK(\Phi; A, B)$ exactly as classical KK-theory but by only considering Kasparov modules intertwining Φ in a suitable sense. Kirchberg's classification theorem is as follows.

Theorem 1 (Kirchberg). *Let A and B be separable, nuclear, unital, $\mathcal{O}_\infty$-stable* C*-*algebras. Then $A \cong B$ if and only if there is an order isomorphism $\Phi\colon \mathcal{I}(A) \xrightarrow{\cong} \mathcal{I}(B)$ and an invertible element $\alpha \in KK(\Phi; A, B)$ with $\alpha_0([1_A]_0) = [1_B]_0 \in K_0(B)$.*

The classification above is strong in the sense that there is an isomorphism $\phi\colon A \xrightarrow{\cong} B$ such that $\phi(I) = \Phi(I)$ for all $I \in \mathcal{I}(A)$, and $KK(\Phi; \phi) = \alpha$.

The theorem is usually stated slightly differently (using actions by topological spaces instead of the order isomorphism Φ) but every competent mathematician should be able to turn Kirchberg's original statement into the above theorem.

Recently I have found a new and very different proof of this important theorem of Kirchberg. The main intermediate theorem (in a special case) is the following.

Theorem 2 ([Gab19]). *Let A be a separable, nuclear, unital* C*-*algebra, let B be a unital, $\mathcal{O}_\infty$-stable* C*-*algebra, and let $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ be map preserving suprema, compact containment and satisfying $\Phi(A) = B$. Suppose that $\alpha \in KK(\Phi; A, B)$ is an element satisfying $\alpha_0([1_A]_0) = [1_B]_0$ in $K_0(B)$. Then there exists a unital $*$-homomorphism $\phi\colon A \to B$, unique up to asymptotic unitary equivalence, such that $\overline{\operatorname{span}}\, B\phi(I)B = \Phi(I)$ for all $I \in \mathcal{I}(A)$, and $KK(\Phi; \phi) = \alpha$.*

A new proof of a special case of Kirchberg's classification was already previously obtained in [Gab18a].

Theorem 3 ([Gab18a, Theorem B]). *Let A and B be separable, nuclear, unital* C*-*algebras. Then $A \otimes \mathcal{O}_2 \cong B \otimes \mathcal{O}_2$ if and only if $\mathcal{I}(A) \cong \mathcal{I}(B)$.*

The methods developed in [Gab18a] have already had far reaching application such as the following which is joint work with Bosa, Sims, and White.

Theorem 4 ([BGSW19, Theorem A]). *Separable, nuclear, $\mathcal{O}_\infty$-stable* C*-*algebras have nuclear dimension one.*

Another consequence of [Gab18a], which uses deep results from lattice theory, is the following projectionless version of Kirchberg's $\mathcal{O}_2$-embedding theorem.

Theorem 5 ([Gab18b, Theorem A]). *A separable, exact* C*-*algebra A embeds into a zero-homotopic* C*-*algebra (which can be taken to be $C_0((0,1], \mathcal{O}_2)$ or Rørdam's ASH algebra $\mathcal{A}_{[0,1]}$) if and only if the primitive ideal space of A has no (non-empty) compact, open subsets.*

Consequently one obtains applications to AF embeddability and to Connes and Higson's picture of KK-theory through asymptotic $*$-homomorphisms. In the following, a C*-algebra is traceless if it has no non-trivial lower semicontinuous, extended quasitraces. Examples of such are all $\mathcal{O}_\infty$-stable C*-algebras.

Theorem 6 ([Gab18b, Corollary C]). *Separable, exact, traceless* C*-*algebras are AF embeddable if and only if they are quasidiagonal. This is characterised by their primitive ideal space having no compact, open subsets.*

Finally there is an application to KK-theory. Let $[[A, B]]$ denote the homotopy classes of asymptotic $*$-homomorphisms in the sense of Connes and Higson. Using recent results of Dadarlat and Pennig [DP17] one obtains the following.

Theorem 7 ([Gab18b, Corollary E]). *Let A be a separable, nuclear* C*-*algebra. Then $KK(A, B) \cong [[A, B \otimes \mathcal{K}]]$ for all separable* C*-*algebras B if and only if the primitive ideal space of A has no compact, open subsets.*

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The stable uniqueness theorem for equivariant Kasparov theory

Gábor Szabó

(joint work with James Gabe)

The overarching goal of the research underpinning this talk is to employ ideas from the Elliott program in order to classify natural classes of C*-dynamical systems up to cocycle conjugacy. A clear starting point for the ordinary classification program is given by the Elliott intertwining argument, even though it is usually the last ingredient in the proof of virtually every abstract classification theorem. As was explained in other talks at the beginning of this workshop, this motivates an in-depth study of $*$-homomorphisms between C*-algebras, in particular with regard to *uniqueness* and *existence* theorems relative to a predetermined functorial invariant. In the *usual* way of viewing C*-dynamical systems over a locally compact group G as a category, however, it is unclear what the correct analog of this phenomenon should be. In the first part of my talk, I presented the basic framework of the so-called cocycle category [5]:

Definition 1. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on C*-algebras. A (non-degenerate) *cocycle representation* is a pair $(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$, where $\varphi : A \to \mathcal{M}(B)$ is a non-degenerate $*$-homomorphism and $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(B))$ is a strictly continuous β-cocycle satisfying $\mathrm{Ad}(\mathbb{u}_g) \circ \beta_g \circ \varphi = \varphi \circ \alpha_g$ for all $g \in G$. If we furthermore assume $\varphi(A) \subseteq B$, then we call $(\varphi, \mathbb{u})$ a cocycle morphism.

After coming up with the composition formula given by

$$(\psi, \mathbb{v}) \circ (\varphi, \mathbb{u}) := (\psi \circ \varphi, \psi(\mathbb{u}_\bullet)\mathbb{v}_\bullet),$$

the above definition introduces the arrows in a suitable category of C*-dynamical systems over G. From the definition it is clear that a cocycle conjugacy in the known sense translates to isomorphism in this category. Indeed, every unitary $v \in \mathcal{U}(\mathcal{M}(B))$ gives rise to a special kind of *inner* cocycle morphism given by $\mathrm{Ad}(v) := (\mathrm{Ad}(v), v\beta_\bullet(v)^*)$, which gives rise to a canonical notion of (approximate/asymptotic) unitary equivalence. The obvious desirable Elliott intertwining theorem then holds within this category as follows.

Theorem 1. *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable* C*-*algebras. Suppose that*

$$(\phi, \mathbb{u}) : (A, \alpha) \to (B, \beta) \quad \textit{and} \quad (\psi, \mathbb{v}) : (B, \beta) \to (A, \alpha)$$

are two cocycle morphisms such that their mutual compositions

$$(\psi, \mathbb{v}) \circ (\phi, \mathbb{u}) \quad \textit{and} \quad (\phi, \mathbb{u}) \circ (\psi, \mathbb{v})$$

are approximately inner. Then $(\phi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are approximately unitarily equivalent to mutually inverse cocycle conjugacies between (A, α) and (B, β).

This simple fact indeed yields the appropriate framework for considering existence and uniqueness theorems in the aforementioned sense. In the main part of the talk, I gave a reminder regarding the *Cuntz–Thomsen* picture of equivariant KK-theory [3, 6], with the motivation to extract useful information in order to attack the uniqueness problem. In that picture, an element in the group $KK^G(\alpha, \beta)$ is represented by an equivariant Cuntz pair, which is a pair of cocycle representations

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$$

with the additional requirement that the pointwise differences $\psi - \varphi$ and $\mathbb{v} - \mathbb{u}$ take value in B. As the main result in the talk, I presented an equivariant generalization of the stable uniqueness theorem due to Lin and Dadarlat–Eilers [1, 2], which goes as follows. Both A and B are assumed to be separable below.

Theorem 2. *Assume that*

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$$

forms an equivariant Cuntz pair. Then its associated element in $KK^G(\alpha,\beta)$ vanishes if and only if there exists a cocycle representation $(\theta,\mathbb{x}):(A,\alpha)\to(\mathcal{M}(B),\beta)$ and a norm-continuous map $v:[0,\infty)\to\mathcal{U}(1+B)$ such that

$$v_t(\varphi\oplus\theta)(a)v_t^* \overset{t\to\infty}{\longrightarrow} (\psi\oplus\theta)(a)$$

for all $a\in A$, and

$$\max_{g\in K}\|(\mathbb{v}_g\oplus\mathbb{x}_g)-v_t(\mathbb{u}_g\oplus\mathbb{x}_g)\beta_g(v_t)^*\| \overset{t\to\infty}{\longrightarrow} 0$$

for all compact sets $K\subseteq G$.

I proceeded to give a very rough outline of some of the non-trivial ingredients entering in the proof of this result. Finally, I formulated a conjecture in the cocycle category framework that ought to be the correct statement of an equivariant Kirchberg–Phillips theorem [4] concerning outer actions of amenable groups on Kirchberg algebras and their classification via equivariant KK-theory. I speculated that the main result of this talk may be exploited to obtain a solution to said conjecture.

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Paschke Duality and C*-algebra Extensions

Huaxin Lin

This informal talk is to discuss the Paschke duality, it relation with C*-algebra extension theory and its relation with the Universal Coefficient Theorem for amenable C*-algebras.

Proposition 1. *Let A be a separable amenable* C*-*algebra, let $C\subset A$ be an amenable* C*-*subalgebra and let B be a σ-unital stable* C*-*algebra.*

(1) Suppose that $\tau:A\to M(B)/B$ is an essential trivial absorbing extension. Then $\tau|_C$ is an essential trivial absorbing extension.

If A is unital, τ is a unital trivial absorbing extension, and $1_C=1_A$, then $\tau|_C$ is unital trivial absorbing extension; if $\tau|_C$ is not unital, then $\tau|_C$ is absorbing.

(2) *Suppose that $\tau_C : C \to M(B)/B$ is an essential trivial absorbing extension. Then there is a trivial absorbing extension $\tau : A \to M(B)/B$ such that $\tau|_C = \tau_C$.*

If C is unital, $1_C = 1_A$, and τ_C is a unital essential trivial absorbing extension, then there is a unital trivial absorbing extension $\tau : A \to M(B)/B$ such that $\tau|_C = \tau_C$.

Definition 1. Let A be a separable C*-algebra, B be a non-unital but σ-unital C*-algebra, and let $\phi : A \to M(B)/B$ be an essential extension. Define

$$\phi(A)' := \{x \in M(B)/B : x\phi(a) = \phi(a)x \text{ for all } a \in A\},$$

and define

$$\begin{aligned} I_\phi := \phi(A)^\perp &= \{x \in \phi(A)' : x\phi(a) = 0 \text{ for all } a \in A\} \\ &= \{x \in M(B)/B : x\phi(a) = 0 \text{ for all } a \in A\}. \end{aligned}$$

Define $\phi(A)^{I_\phi} = \{x \in M(B)/B : yx, xy \in I_\phi \text{ for all } y \in I_\phi\}$. Note that $\phi(A)^{I_\phi}$ is a C*-subalgebra of $M(B)/B$ which contains I_ϕ as a (closed two-sided) ideal. Moreover, both $\phi(A)$ and $\phi(A)'$ are in $\phi(A)^{I_\phi}$. Denote by $\pi_{I_\phi} : \phi(A)^{I_\phi} \to \phi(A)^{I_\phi}/I_\phi$ the quotient map. Then $\pi_{I_\phi} \circ \phi : A \to \phi(A)^{I_\phi}/I_\phi$ is injective. Define $\phi(A)^c = \phi(A)'/I_\phi$.

The Paschke duality (Proposition 3 of [25], see also Remark 2.8 of [21]) states that there is an isomorphism from $K_i(\phi(A)^c) \cong KK^{i+1}(A, B)$, $i = 0, 1$. The case that A is unital was proved by Valette [25] and generalized by Skandalis in [21]. In fact, Valette also discussed the non-unital cases (see Corollary 4 of [25]).

Fix a trivial essential absorbing extension $\phi : A \to M(B)/B$. Define $D = \overline{\phi(A)\phi(A)^{I_\phi}\phi(A)}$. Note that D is σ-unital (as A is separable) and $D \perp I_\phi$. There is an injective homomorphism $j_D : \pi_{I_\phi}(D) \to D$ such that $\pi_{I_\phi} \circ j_D = \mathrm{id}_{\pi_{I_\phi}(D)}$ and $j_D \circ (\pi_{I_\phi})|_D = \mathrm{id}_D$.

First assume that A is not unital. Let $s \in j_o(M_2(O_2))$, where $j_o : O_2 \to \phi(1_A)\phi(A)'\phi(1_A)$ is such that $s^* \operatorname{diag}(1,1)s = \operatorname{diag}(1,0)$ and $ss^* = \operatorname{diag}(1,1)$. Let $e = s^* \operatorname{diag}(0,1)s \in j_o(O_2)$.

Let $p' \in \phi(A)^c$ be a non-zero projection and let $x' \in \phi(A)'_+$ be such $\pi_{I_\phi}(x') = p'$. Define $p_0 = \pi_{I_\phi}(s^* \operatorname{diag}(x', 0)s)$. Since $s \in j_o(O_2) \subset \phi(A)'$, $p_0 \in \phi(A)^c$ and $[p_0] = [p']$ in $K_0(\phi(A)^c)$. Let $p = p_0 + \pi_{I_\phi}(e) = \pi_{I_\phi}(s^* \operatorname{diag}(x', 1)s)$. It is a projection. Since $[\operatorname{diag}(0,1)] = 0$, $[p_0] = [p]$. Choose $x := s^* \operatorname{diag}(x', 1)s \in \phi(A)'_+$. Then $\pi_{I_\phi}(x) = p$.

Define $\bar{\phi}^p : A \to \pi_{I_\phi}(D)$ by $\bar{\phi}^p(a) = (\pi_{I_\phi} \circ \phi(a))p$ for all $a \in A$. Then define $\phi^p = j_D \circ \bar{\phi}_p$. Note that

$$\phi(a)s^* \operatorname{diag}(0,1)s = \phi(a)e \text{ for all } a \in A$$

which gives an injective homomorphism. Thus,

$$\phi^p(a) = \phi(a)x = \phi(a)(s^* \operatorname{diag}(x', 0)s) \oplus \phi(a)s^* \operatorname{diag}(0,1)s \text{ for all } a \in A,$$

which is an absorbing essential extension.

For any $a, b \in A$, we have $\phi(ab)x^2 = \phi(a)x\phi(b)x = \phi^p(ab) = \phi(ab)x$. It follows that

$$\phi(a)x = \phi(a)x^2 \text{ for all } a \in A.$$

Suppose that $q \in \phi(A)^c$ is another projection such that there is a $w \in \phi(A)^c$ with $w^*w = \mathrm{diag}(p, 1)$ and $ww^* = \mathrm{diag}(q, 1)$. Note that $\phi^{1_{\phi(A)^c}} = \phi$. Since ϕ is absorbing, to simplify notation, replacing p by $p \oplus 1$ and q by $q \oplus 1$, we may assume without loss of generality that $w^*w = p$ and $ww^* = q$. It follows that there is a continuous path $\{p_t : t \in [0, 1]\}$ of projections in $M_2(\phi(A)^c)$ such that $p_0 = \mathrm{diag}(p, 0)$ and $p_1 = \mathrm{diag}(q, 0)$. Define $\phi_t : A \to M_2(M(B)/B)$ by

$$\phi_t(a) = (j_D \otimes \mathrm{id}_{M_2})(\mathrm{diag}(\pi_{I_\phi}(\phi(a)), \pi_{I_\phi}(\phi(a)))p_t) \text{ for all } a \in A.$$

Note that $\phi_0 = \phi^p$ and $\phi_1 = \phi^q$. It follows that $[\phi_0] = [\phi_1]$ in $KK^1(A, B)$.

If A is unital, then $\phi(A)^c = \phi(1_A)\phi(A)'\phi(1_A)$. Let us choose $x' = p'$. Replacing 1 by $\phi(1_A)$ above, a simpler argument leads to the same conclusion above.

Define $\Lambda_0 : K_0(\phi(A)^c) \to KK^1(A, B)$ by $\Lambda_0([p]) = [\phi^p]$. This defines a homomorphism from $K_0(\phi(A)^c)$ to $KK^1(A, B)$. *We remind the reader that ϕ^p is assumed to be absorbing.*

If C is unital, $1_C = 1_A$, and τ_C is a unital essential trivial absorbing extension, then there is a unital trivial absorbing extension $\tau : A \to M(B)/B$ such that $\tau|_C = \tau_C$

Theorem 2 (Proposition 3 of [25] and see also Remark 2.8 of [21])**.** *Let A be a separable amenable* C**-algebra and let B be a σ-unital stable* C**-algebra. Fix a trivial essential absorbing extension $\phi : A \to M(B)/B$. Then the map $\Lambda_0 : K_0(\phi(A)^c) \to KK^1(A, B)$ is a group isomorphism.*

Proof. Similar to before, fix an essential trivial absorbing extension $\phi : A \to M(B)/B$. If A is unital, then $\phi(1_A) \neq 1_{M(B)/B}$ and $[\phi(1_A)] = 0$.

Without loss of generality, by 1, we may assume that there exists a trivial absorbing extension $\widetilde{\phi_O} : \tilde{A} \otimes O_2 \to M(B)/B$ (where $\tilde{A}$ is the unitization of A, or $\tilde{A} = A$, if A is unital) such that $\widetilde{\phi_O}|_{A \otimes 1} = \phi$. We may assume that $j_o = \widetilde{\phi_O}|_{1_{\tilde{A}} \otimes O_2}$. Write $O_2 = O_2 \otimes O_2$. Choose $h_0 \in j_o(O_2 \otimes 1_{O_2})_+$ with ${}^{(}h_0) = [0, 1]$. Denote by $\phi^{\sim} : A \otimes C(\mathbb{T}) \to M(B)/B$ such that

$$\phi^{\sim}(a \otimes f) = \widetilde{\phi_O}(a \otimes f(e^{i2\pi h_0})) \tag{1}$$

for all $a \in A$ and $f \in C(\mathbb{T})$. Note that $\phi^{\sim}$ is a trivial absorbing extension, and if A is unital, $\phi^{\sim}$ is a unital trivial absorbing extension. Note also that $h_0 \in \phi(A)'$.

Let $u' \in \phi(A)^c$ be a unitary and let $y' \in \phi(A)'$ such that $\pi_{I_\phi}(y') = u'$ and let $\imath_{u'} : C(\mathbb{T}) \to \phi(A)^c$ be the injective homomorphism defined by $\imath_{u'}(f) = f(u')$ for all $f \in C(\mathbb{T})$. Note that, in the case that A is unital, $I_\phi = (1 - \phi(1_A))(M(B)/B)(1 - \phi(1_A))$. In this case, we choose $y' = u' \in \phi(1_A)\phi(A)'\phi(1_A)$.

Let $s \in M_2(j_o(1 \otimes O_2))$ be as in the preceding discussion and let $e = s^* \mathrm{diag}(0, 1)s$ (if A is unital, let $e = s^* \mathrm{diag}(0, \phi(1_A))s$). Choose a unitary $z_0 \in ej_0(O_2)e$ with ${}^{(}z_0) = \mathbb{T}$. Let $y_0 = s^* \mathrm{diag}(y', 0)s$ and $y = y_0 + z_0$. Then $u := \pi_{I_\phi}(y_0 + z_0) = \pi_{I_\phi}(s^*y's + z_0) \in \phi(A)^c$ is a unitary such that $[u] = [u']$ as $z_0 \in j_o(O_2)$.

Define $\phi'_u : A \otimes C(\mathbb{T}) \to D/I_\phi$ by $\phi'_u(a \otimes f) = \pi_{I_\phi}(\phi(a))f(u)$ for all $a \in A$ and $f \in C(\mathbb{T})$ (recall $D = \overline{\phi(A)\phi(A)^{I_\phi}\phi(A)}$, see the discussion preceding the theorem). Define $\phi_u := j_D \circ \phi'_u : A \otimes C(\mathbb{T}) \to D \subset M(B)/B$. Recall $S = \{f \in C(\mathbb{T}) : f(1) = 0\} \cong C_0((0,1))$. Define $\phi_{u,s} := \phi_u|_{SA} : SA \to D \subset M(B)/B$. Note that

$$\phi_u(a \otimes f)e = e\phi_u(a \otimes f) = \phi^\sim(a \otimes f)e \text{ for all } a \in A \text{ and } f \in C(\mathbb{T})$$
$$\phi_{u,s}(a \otimes f)e = e\phi_{u,s}(a \otimes f) = \phi^\sim(a \otimes f)e \text{ for all } a \in A \text{ and } f \in S.$$

It follows from 1 that $\phi^\sim(a \otimes f)e$ defines a trivial absorbing extension. Therefore, ϕ_u and $\phi_{u,s}$ are essential absorbing extension of $A \otimes C(\mathbb{T})$ and of SA by B, respectively.

Note that $yd, dy \in D$ for all $d \in D$. Therefore, there exists $y_1 \in M(D)$ such that $y_1 d = yd$ and $dy_1 = dy$ for all $d \in D$. Note that, since $y \in \phi(A)'$,

$$y_1\phi(a) = y\phi(a) = \phi(a)y = \phi(a)y_1 \text{ for all } a \in A. \tag{2}$$

There exist $c_1, c_2 \in I_\phi$ such that $y^*y = 1 + c_1$ and $yy^* = 1 + c_2$. For any $d \in D$,

$$\begin{aligned} y_1^*y_1d &= y^*yd = (1 + c_1)d = d, \\ d(y_1^*y_1) &= d(y^*y) = d(1 + c_1) = d, \\ y_1y_1^*d &= yy^*d = (1 + c_2)d = d, \\ dy_1y_1^* &= dyy^* = d(1 + c_2) = d. \end{aligned}$$

It follows that y_1 is a unitary in $M(D)$. By (2), for non-commutative polynomial P of y and y^*,

$$P(y_1, y_1^*)\phi(a) = P(y, y^*)\phi(a) = \phi(a)P(y, y^*) = \phi(a)P(y_1, y_1^*). \tag{3}$$

This implies that, for any $f \in S$,

$$f(y_1)\phi(a) = \phi(a)f(y_1). \tag{4}$$

We have

$$\phi_{u,s}(a \otimes f) = \phi(a)f(y_1) \text{ for all } a \in A \text{ and } f \in S. \tag{5}$$

Now suppose that $v \in \phi(A)^c$ is another unitary such that $[u] = [v]$ in $K_1(\phi(A)^c)$ and $\phi_{v,s}$ is an essential absorbing extension. Then there exists a continuous path $\{u(t) : t \in [0,1]\} \subset M_2(\phi(A)^c)$ such that $u(0) = \mathrm{diag}(u, 1)$ and $u(1) = \mathrm{diag}(v, 1)$. Define $\overline{\Psi}_t : SA \to \pi_{I_\phi}(D)$ by

$$\overline{\Psi}_t(a \otimes f) = \mathrm{diag}(\pi_{I_\phi}(\phi(a)), \pi_{I_\phi}(\phi(a))) \cdot f(u(t)\,\mathrm{diag}(1, e^{2\pi i(\pi_{I_\phi}(h_0))}))$$

for all $a \in A$, $f \in S$, and $t \in [0,1]$. Define $\Psi_t : SA \to D \subset M(B)/B$ by

$$\Psi_t = (j_D \otimes \mathrm{id})_{M_2} \circ \bar{\Psi}_t.$$

Then for all $a \in A$ and $f \in S$,

$$\begin{aligned} \Psi_0(a \otimes f) &= \mathrm{diag}(\phi(a), \phi(a)) \cdot f(u(0)\,\mathrm{diag}(1, e^{2\pi i h_0})) \\ &= \mathrm{diag}(\phi(a), \phi(a)) \cdot f(\mathrm{diag}(u, e^{2\pi i h_0})) \\ &= \mathrm{diag}(\phi_{u,s}(a \otimes f), \phi^\sim(a \otimes f)). \end{aligned}$$

Similarly,

$$\Psi_1 = \psi_{v,s} \oplus \phi^{\sim}|_{SA}.$$

Since $\phi^{\sim}$ is trivial and $\phi_{u,s}$ and $\phi_{v,s}$ are absorbing, in $KK^1(SA, B)$,

$$[\phi_{u,s}] = [\Psi_0] = [\Psi_1] = [\phi_{v,s}].$$

Thus, one obtains a group homomorphism $\Lambda_1 : K_1(\phi(A)^c)$ to $KK^1(SA, B)$. $\square$

Define $\overline{\psi}_{u'} : A \otimes C(\mathbb{T}) \to D/I_\phi$ by $\psi'_{u'}(a \otimes f) = \pi_{I_\phi}(\phi(a))f(u')$ for all $a \in A$ and $f \in C(\mathbb{T})$, and define $\psi_{u'} = j_D \circ \bar{\psi}_{u'}$. Set $\psi_{u',s} = \psi_{u'}|_{SA}$. Consider the absorbing extension $\lambda_{u'} = \psi_{u'} \oplus \phi^{\sim}$ and $\lambda_{u',s} = \lambda_{u'}|_{SA}$. The argument above also shows that $[\phi_{u,s}] = [\lambda_{u',s}]$.

Theorem 3 (Proposition 3 of [25], Remark 2.8 of [21], and Theorem 3.2 of [22])**.** *Let A be a separable amenable* C*-*algebra and let B be a σ-unital stable* C*-*algebra. Then $\Lambda_1 : K_1(\phi(A)^c) \to KK^1(SA, B)$ is a group isomorphism.*

We use the Paschke duality above to study C*-algebra extension theory. This informal talk also outlines a strategy to study the UCT.

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Stationary characters on lattices in semi-simple groups

Rémi Boutonnet
(joint work with Cyril Houdayer)

This talk aims to study unitary representations of lattices in higher rank semi-simple Lie groups and the structure of the C*-algebras generated by such representations. Our main results can be seen as generalizations of Margulis normal subgroup theorem, (see [8, Theorem IV.4.10]), in the framework of unitary representations.

Typically, we will study an irreducible lattice Γ in a connected semi-simple Lie group G of rank at least 2, having no-compact simple factor. For simplicity we assume that G has trivial center, and our main results will need to require that in fact every simple factor of G has rank at least 2. In this context, the Margulis normal subgroup theorem states that any non-trivial normal subgroup of Γ is of finite index. Later, Stuck and Zimmer generalized this theorem to a measurable setting, proving that any ergodic probability measure-preserving action of such a lattice Γ is either essentially free, or factors through an action of a finite quotient of Γ, see [11].

More recently, an operator algebraic generalization was proved by Peterson [10], answering a question of Connes. Inspired by Margulis superrigidity theorem, and the measure theoretic arguments used in its proof, Connes suggested (see [5]) that the embedding of the lattice Γ in its von Neumann algebra $L\Gamma$ should be rigid, in the sense that any group homomorphism from Γ into a tracial von Neumann algebra M should either range into a finite dimensional subalgebra of M, or should extend to a von Neumann algebraic morphism $L\Gamma \to M$. Here one needs to make an ergodicity assumption on the morphism: its range should generate a factor inside M. Thanks to the GNS construction, such a statement is naturally phrased in terms of *characters* on the group, see Theorem 1 below. Recall that a character on Γ is a positive definite function $\varphi : \Gamma \to \mathbb{C}$ which is conjugacy invariant and such that $\varphi(1) = 1$. Aternatively, a character can be seen as a tracial state on the universal C*-algebra $C^*(\Gamma)$ of Γ. The set of all characters is then a compact convex subset of the dual of $C^*(\Gamma)$. We say that a character is *almost periodic* if the corresponding GNS representation is finite dimensional.

Theorem 1 (Peterson, [10])**.** *Any extremal character of* Γ *as above is either equal to the Dirac function* δ_e *at the rival element* e *or it is almost periodic.*

Given a unitary representation π of Γ, we will denote by $C^*_\pi(\Gamma)$ the C*-algebra generated by $\pi(\Gamma)$. If τ is a trace on $C^*_\pi(\Gamma)$, then $\tau \circ \pi$ is a character on Γ. Thus the above result can be seen as a classification of all tracial states on such algebras $C^*_\pi(\Gamma)$. But in general, there is a priori no reason for such C*-algebras to admit a trace at all. So in order to study general representations of Γ, we will consider a stationary version of traces. Given a probability measure μ_0 on Γ, a μ_0-*stationary state* on $C^*_\pi(\Gamma)$ is a state φ such that

$$\sum_{g\in\Gamma} \mu_0(g)\varphi(\pi(g)^{-1}x\pi(g)) = \varphi(x), \text{ for all } x \in C^*_\pi(\Gamma).$$

The function $\varphi \circ \pi$ is then called a μ_0-*character*. This is inspired by the work of Hartman and Kalantar [6], where some connexions between this notion and C*-simplicity are explored.

The strength of stationary states is that they always exist, as can be seen from an averaging argument. The following theorem then implies that traces on $C^*_\pi(\Gamma)$ always exist, for arbitrary representations π.

Theorem 2. *Assume that every simple factor of* G *has rank at least* 2*. There exists a probability measure* μ_0 *on* Γ *such that every* μ_0*-character on* Γ *is conjugation invariant (so it is actually a character).*

Combining this fact with Peterson's result on characters, we derive the following structural result. This is a far reaching generalization of C*-simplicity and the unique trace property for such groups (see [7, 2]).

Corollary 3. *Under the assumption of Theorem 2, any weakly mixing representation* π *of* Γ *weakly contains the left regular representation* λ*. Moreover, if we denote by* $\Theta_{\pi,\lambda} : C^*_\pi(\Gamma) \to C^*_\lambda(\Gamma) : \pi(\gamma) \mapsto \lambda_\Gamma(\gamma)$ *the corresponding surjective unital* $*$*-homomorphism, then*

(1) $\tau_\Gamma \circ \Theta_{\pi,\lambda}$ *is the unique tracial state on* $C^*_\pi(\Gamma)$*.*
(2) $\ker(\Theta_{\pi,\lambda})$ *is the unique proper maximal ideal of* $C^*_\pi(\Gamma)$*.*

We point out that similar results for some specific quasi-regular representations have been obtained recently in [1].

Going back to our main result above, Theorem 2, let us point out that we choose the measure μ_0 as constructed by Furstenberg, [4]: we require that the Poisson boundary of (Γ, μ_0) is equal to the Poisson boundary of (G, μ) for some adapted probability measure μ on G. We will call such a measure μ_0 a *special measure*.

Our proof of Theorem 2 is based on a combination of C*-algebraic techniques imported from the recent approach to C*-simplicity, and the following result, which is a *non-commutative factor theorem* in the spirit of a theorem of Nevo and Zimmer [9].

Theorem 4. *Assume that every simple factor of G has rank at least 2. Let μ_0 be a special measure on Γ. Consider an action of Γ on a von Neumann algebra M, whose fixed point algebra is trivial: $M^\Gamma = \mathbb{C}$. Assume that ϕ is a normal state on M, which is μ_0-stationary. Then the following dichotomy holds:*

- *either ϕ is Γ-invariant,*
- *or there exists a proper parabolic subgroup $Q \subsetneq G$ and a Γ-equivariant normal unital $*$-embedding $\theta : L^\infty(G/Q) \to M$.*

There are several major differences with the initial result of Nevo and Zimmer [9], both in the statement and in the techniques of proof, which greatly improve the range of applications. This will be explained in a separate talk by Cyril Houdayer.

To conclude, let us mention that Theorem 4 also implies Peterson's result on character rigidity. At first glance, this may seem surprising since we already know that a character is conjugacy invariant, so there is no point in applying directly Theorem 4 to the GNS von Neumann algebra generated by a character. Instead, we apply it to the so-called *non-commutative Poisson boundary* considered by Peterson (see also [3]). The conclusion then follows from C*-algebraic techniques in a similar fashion to the proof of Theorem 2.

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A noncommutative Nevo–Zimmer theorem

Cyril Houdayer
(joint work with Rémi Boutonnet)

In this talk, based on our joint work with Rémi Boutonnet (see [1]), I will explain the main technical novelty that provides a structure theorem for stationary actions of lattices on von Neumann algebras. Before doing so, let me first introduce some notation and terminology.

- Let G be any connected semisimple Lie group with finite center and no nontrivial compact factor, all of whose simple factors have real rank at least two. Choose a maximal compact subgroup $K < G$ and a minimal parabolic subgroup $P < G$, so that $G = KP$. For instance, for every $n \geq 3$, let $G = \mathrm{SL}_n(\mathbb{R})$ and choose $K = \mathrm{SO}_n(\mathbb{R})$ and $P < G$ the subgroup of upper triangular matrices.
- We denote by $\nu_P \in \mathrm{Prob}(G/P)$ the unique K-invariant Borel probability measure on the homogeneous space G/P. More generally, if $P \subset Q \subset G$ is a parabolic subgroup, we denote by $\nu_Q \in \mathrm{Prob}(G/Q)$ the unique K-invariant Borel probability measure on the homogeneous space G/Q. Observe that for every parabolic subgroup $P \subset Q \subset G$, the probability measure $\nu_Q \in \mathrm{Prob}(G/Q)$ is G-quasi-invariant.

The following concept is central in our talk.

Definition 1. Let G be as above. Let $\Gamma < G$ be any lattice. We say that a probability measure $\mu_0 \in \mathrm{Prob}(\Gamma)$ is *special* if the following three conditions are satisfied:

(i) The support of μ_0 is equal to Γ;
(ii) $\mu_0 * \nu_P = \nu_P$, that is, ν_P is μ_0-stationary;
(iii) The space $(G/P, \nu_P)$ is the Poisson–Furstenberg boundary associated with the simple random walk on Γ with law μ_0 (see [3, 2]).

By a result of Furstenberg [4, Theorem 3] (see also [2, Theorem 2.21]), there always exists a special probability measure $\mu_0 \in \mathrm{Prob}(\Gamma)$. Moreover, for every parabolic subgroup $P \subset Q \subset G$, ν_Q is the unique μ_0-stationary measure on the homogeneous space G/Q (see [5, Corollary VI.3.9]).

Let M be any von Neumann algebra, $\phi \in M_*$ any normal state and $\sigma : \Gamma \curvearrowright M$ any action. We simply write $\gamma\phi = \phi \circ \sigma_\gamma^{-1} \in M_*$ for every $\gamma \in \Gamma$. We say that the action $\sigma : \Gamma \curvearrowright M$ is *ergodic* if the fixed-point subalgebra $M^\Gamma = \{x \in M \mid \sigma_\gamma(x) = x, \forall \gamma \in \Gamma\}$ satisfies $M^\Gamma = \mathbb{C}1$. We say that the state $\phi \in M_*$ is μ_0-*stationary* if $\sum_{\gamma\in\Gamma} \mu_0(\gamma)\, \gamma\phi = \phi$. If the action $\sigma : \Gamma \curvearrowright M$ is ergodic and the state $\phi \in M_*$ is

μ_0-stationary, we say that (M, ϕ) is an ergodic (Γ, μ_0)-von Neumann algebra. Our main result is the following structure theorem for stationary actions of lattices on von Neumann algebras.

Theorem 1. *Let G be as above. Let $\Gamma < G$ be any lattice and $\mu_0 \in \mathrm{Prob}(\Gamma)$ any special probability measure. Let (M, ϕ) by any ergodic (Γ, μ_0)-von Neumann algebra. Then the following dichotomy holds.*

- *Either ϕ is Γ-invariant.*
- *Or there exists a proper parabolic subgroup $P \subset Q \subsetneq G$ and a Γ-equivariant normal unital $*$-embedding $\theta : L^\infty(G/Q, \nu_Q) \to M$ such that $\phi \circ \theta = \nu_Q$.*

It is worth pointing out that Theorem 1 is new even in the case of stationary actions of lattices on *abelian* von Neumann algebras. The proof of Theorem 1 relies on two novel aspects.

Firstly, for any ergodic (Γ, μ_0)-von Neumann algebra (M, ϕ), we construct a μ-stationary normal state φ on the induced von Neumann algebra $\mathrm{Ind}_\Gamma^G(M) = L^\infty(G/\Gamma)\overline{\otimes} M$, where $\mu \in \mathrm{Prob}(G)$ is a K-invariant admissible Borel probability measure. This is where we use that the probability measure $\mu_0 \in \mathrm{Prob}(\Gamma)$ is *special* and we exploit the fact that the Γ-space $(G/P, \nu_P)$ is also a G-space. This construction is new even in the case of stationary actions of lattices on measure spaces.

Secondly, we prove the following noncommutative analogue of Nevo–Zimmer structure theorem for arbitrary ergodic (G, μ)-von Neumann algebras (see [6, Theorem 1] for the case of ergodic (G, μ)-spaces).

Theorem 2. *Let G be as above. Let $\mu \in \mathrm{Prob}(G)$ be any K-invariant admissible Borel probability measure. Let $(\mathcal{M}, \varphi)$ by any ergodic (G, μ)-von Neumann algebra. Then the following dichotomy holds.*

- *Either φ is G-invariant.*
- *Or there exists a proper parabolic subgroup $P \subset Q \subsetneq G$ and a G-equivariant normal $*$-embedding $\Theta : L^\infty(G/Q, \nu_Q) \to \mathcal{M}$ such that $\varphi \circ \Theta = \nu_Q$.*

The proof of Theorem 2 constitutes the most technical part of our work. The generalization of Nevo–Zimmer theorem to the noncommutative setting presents both technical and conceptual difficulties and is not a mere adaptation of their original proof. To prove Theorem 1, we then apply our noncommutative Nevo–Zimmer theorem to the induced (G, μ)-von Neumann algebra $(\mathrm{Ind}_\Gamma^G(M), \varphi)$. To descend back to (M, ϕ), we use a disintegration argument for representations of C*-algebras.

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Non-commutative transport of measure and free complementation of some subalgebras of $L(\mathbb{F}_d)$

David Jekel

Introduction. W*-algebras are often viewed as non-commutative measure spaces. Tracial W*-algebras can be considered as non-commutative probability spaces, and their self-adjoint elements can be viewed as random variables. A tuple of such random variables $(X_1, \dots, X_d)$ has a *law* (or *distribution*), which is defined to be the map $\mu : \mathbb{C}\langle x_1, \dots, x_d \rangle \to \mathbb{C}$ given by $p(x) \mapsto \tau(p(X))$. If $\|X_j\| \leq R$. Moreover, every such law can be realized by random variables in a tracial W*-algebra through the GNS construction.

In the commutative setting, any two standard Borel probability spaces are isomorphic as measure spaces, so for any such probability measure μ and ν, there is some "transport map" f such that $f_*\mu = \nu$. Classical transport theory studies the "most efficient" choice of f, regularity properties of f, and so forth. But in the the non-commutative setting, transport might not exist to begin with. Indeed, McDuff showed that there are uncountably many non-isomorphic II_1 factors [8]. We don't know whether $L(\mathbb{F}_n)$ and $L(\mathbb{F}_m)$ are isomorphic for distinct $n, m > 1$.

However, for certain non-commutative laws that arise naturally from random matrix theory, we can show that the associated non-commutative random variables $X_1, \dots, X_d$ generate a W*-algebra that is isomorphic to the free group factor $L(\mathbb{F}_d)$. The free group factor is a natural model for non-commuting random variables since it is the W*-algebra generated by freely independent semicircular variables $S_1, \dots, S_d$, which are the analogue of independent Gaussians in free probability.

Setup and Examples. Consider a d-tuple of $N \times N$ self-adjoint random matrices $(X_1^{(N)}, \dots, X_d^{(N)})$ given by the probability density

$$d\mu^{(N)}(x) = (1/Z^{(N)}) e^{-N^2 V^{(N)}(x)}\, dx,$$

where $V^{(N)} : M_N(\mathbb{C})_{sa}^d \to \mathbb{R}$ is a "potential" function such that $V^{(N)}(x) - (c/2)\|x\|_2^2$ is convex and $V^{(N)}(x) - (C/2)\|x\|_2^2$ is concave. We also assume that the gradient $DV^{(N)}$ is "asymptotically approximable by trace polynomials," which means roughly speaking that $DV^{(N)}$ has a well-defined limit as $N \to \infty$ in a certain space of "functions in non-commutative real variables." In this case, there exist non-commutative random variables $X_1, \ldots, X_d$, such that for every non-commutative polynomial p,

$$\tau_N(p(X^{(N)})) \to \tau(p(X)) \text{ in probability,}$$

where τ_N denotes the normalized trace on $M_N(\mathbb{C})$. See [6, Theorem 4.1] as well as the similar earlier results [2], [3]

The canonical example of such a sequence of potentials is $V^{(N)}(x) = (1/2)\|x\|_2^2$, in which the the large N limit is precisely a family of free semicircular variables. The convex setting also includes the generators of the q-Gaussian algebra when q is small, as shown by [1] [4]. It also includes the case when $X = S + \epsilon p(S)$, where $p = (p_1, \ldots, p_d)$ is a tuple of self-adjoint non-commutative polynomials and ϵ is sufficiently small (depending on p).

Transport to a Free Semicircular Family. In the setting of convex potentials described above, we have $\mathrm{W}^*(X_1, \ldots, X_d) \cong \mathrm{W}^*(S_1, \ldots, S_d)$. One method to prove this is to study classical transport theory in the large N limit. It is well-known that solving a certain PDE will produce a function $f^{(N)}$ that pushes forward $\mu^{(N)}$ to the Gaussian measure (see e.g. [9, §4]). We can show (and this is the technical crux) that $f^{(N)}$ has a well-defined large N limit f in an appropriate space of functions, such that $f(X)$ is well-defined in $\mathrm{W}^*(X_1, \ldots, X_d)$. Since $f^{(N)}(X^{(N)})$ is a Gaussian tuple of matrices, then $f(X)$ is a semicircular family and hence $\mathrm{W}^*(X)$ contains $\mathrm{W}^*(S) \cong L(\mathbb{F}_d)$. Since the same analysis holds for the inverse function for $f^{(N)}$, we get $\mathrm{W}^*(X) \cong \mathrm{W}^*(S)$.

The existence of transport for convex potentials under broadly similar hypotheses was known before [4] [5]. But the author's techniques also yielded the more general result that the transport could be constructed in a "lower triangular" manner [7, Theorem 8.11].

Theorem 1. *Let $(X_1, \ldots, X_d)$ be non-commutative random variables given by uniformly convex, semi-concave sequence of potentials as above. Then there exists an isomorphism $\phi : \mathrm{W}^*(X_1, \ldots, X_d) \to \mathrm{W}^*(S_1, \ldots, S_d)$ such that*

$$\phi(\mathrm{W}^*(X_1, \ldots, X_k)) = \mathrm{W}^*(S_1, \ldots, S_k) \text{ for } k = 1, \ldots, d.$$

Sketch of proof. We iteratively transport each generator *conditioned* on the previous ones. The first step is to construct f such that

$$(X_1, \ldots, X_{d-1}, f(X)) \sim (X_1, \ldots, X_{d-1}, S_d).$$

In the $N \times N$ matrix approximation, $X_d^{(N)}$ has a nice conditional probability density given $(X_1^{(N)}, \ldots, X_{d-1}^{(N)})$. For every, $(x_1, \ldots, x_{d-1})$, we use classical methods to construct $f^{(N)}(x_1, \ldots, x_{d-1}, -)$ that pushes forward the conditional density of $X_d^{(N)}$ to the Gaussian. This "fiberwise" transport results in the global behavior that

$$\left(X_1^{(N)}, \ldots, X_{d-1}^{(N)}, f^{(N)}(X^{(N)})\right) \sim \left(X_1^{(N)}, \ldots, X_{d-1}^{(N)}, S_d^{(N)}\right) \text{ in law,}$$

where $S_d^{(N)}$ is an independent Gaussian. By Voiculescu's asymptotic freeness theorem [12, Theorem 2.4], $S_d^{(N)}$ becomes a freely independent semicircular in the large N limit. By showing that $f^{(N)}$ has a reasonable large N limit, we obtain

$$\mathrm{W}^*(X_1, \dots, X_d) \cong \mathrm{W}^*(X_1, \dots, X_{d-1}, S_d) = \mathrm{W}^*(X_1, \dots, X_{d-1}) * \mathrm{W}^*(S_d).$$

Then we apply ths same argument to $(X_1, \dots, X_{d-1})$ and so forth. □

Applications and Questions. The theorem implies that $\mathrm{W}^*(X_1)$ is a freely complemented MASA in $\mathrm{W}^*(X_1, \dots, X_d)$, and it is even maximal amenable, because we know that this is true for $\mathrm{W}^*(S_1)$ thanks to Popa [11]. Thus, our result re-proves maximal amenability for the generator MASA in the q-Gaussian case for a certain range of q depending on d (however, this was shown for a d-independent range of q values in [10]).

Our result also holds for small polynomial perturbations of S. This leads us to ask, under what conditions on $p = (p_1, \dots, p_d)$, is it true that $p_1(S)$ generates a freely complemented MASA? Is this true "generically"? Are "most" MASA's in $L(\mathbb{F}_d)$ maximal amenable? And if a MASA in $L(\mathbb{F}_d)$ is maximal amenable, then is it freely complemented? (Coarseness of such maximable amenable MASA's was conjectured by Ben Hayes, and as mentioned by Sorin Popa, this relates to the Peterson-Thom conjecture.)

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C*-algebras of stable rank one and their Cuntz semigroups

Hannes Thiel
(joint work with Ramon Antoine, Francesc Perera, Leonel Robert)

Stable rank one. Recall that a unital C*-algebra is said to have *stable rank one* if its invertible elements are norm-dense. This strong finiteness condition was introduced by Rieffel, [Rie83], to study nonstable K-theory. First examples of C*-algebras of stable rank one include II_1-factors and commutative C*-algebras $C(X)$ with X of covering dimension at most one.

Stable rank one has very nice permanence properties: It passes to ideals, quotients, hereditary sub-C*-algebras, directs sums, inductive limits, matrix amplifications and stabilizations. Using this, it follows for instance that the class of stable rank one C*-algebras includes all AF- and A$\mathbb{T}$-algebras.

In [Put90], Putnam proved that all irrational rotation algebras have stable rank one. Later, is was shown by Elliott-Evans, [EE93], that these algebras are even A$\mathbb{T}$-algebras. These results were generalized in two ways: First, it was shown that large classes of simple C*-algebras are ASH-algebras; see for example [LP10]. Second, it was studied when simple ASH-algebras have stable rank one; see for example [DNNP92], [EHT09].

The picture was clarified with the discovery of the Jiang-Su algebra $\mathcal{Z}$, [JS99], which is a unital, simple, nonelementary C*-algebra that is KK-equivalent to $\mathbb{C}$. One says that a C*-algebra A is *$\mathcal{Z}$-stable* if $A \cong A \otimes \mathcal{Z}$. This important regularity property is the C*-algebraic analog of the McDuff-property for von Neumann algebras. Rørdam, [Rør04], showed that $\mathcal{Z}$-stable, simple, stably finite, C*-algebras have stable rank one. In [Tom11], Toms proved that simple ASH-algebras with slow dimension growth are $\mathcal{Z}$-stable, and thus have stable rank one.

There are also many simple, non-$\mathcal{Z}$-stable C*-algebras of stable rank one. Amenable examples were constructed by Villadsen, [Vil98], and Toms, [Tom08]. Nonamenable examples are given by Dykema-Haagerup-Rørdam, [DHR97]: the reduced group C*-algebra $C^*_\lambda(G_1 * G_2)$ is simple and has stable rank one whenever G_1 and G_2 are nontrivial groups that are not both isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Question 1. *Let G be a discrete group such that its reduced group C*-algebra $C^*_\lambda(G)$ is simple. Does $C^*_\lambda(G)$ have stable rank one?*

The Cuntz Semigroup. Let A be a C*-algebra, and let $A \otimes \mathbb{K}$ denote its stabilization. Recall that the *Murray-von Neumann semigroup* $V(A)$ is defined through equivalence classes of projections in $A \otimes \mathbb{K}$. Orthogonal addition turns $V(A)$ into a commutative monoid, which is naturally isomorphic to the set of isomorphism classes of finitely generated, projective A-modules. If A is unital, then its K_0-group is isomorphic to the Grothendieck completion of $V(A)$.

Given $a, b \in A_+$, we write $a \precsim_{\mathrm{Cu}} b$ if there is a sequence $(c_n)_n$ in A with $\lim_n \|a - c_n b c_n^*\| = 0$. Further, a and b are *Cuntz equivalent*, denoted $a \sim_{\mathrm{Cu}} b$, if $a \precsim_{\mathrm{Cu}} b$ and $b \precsim_{\mathrm{Cu}} a$. These relations were introduced by Cuntz, [Cun78], in his study of dimension functions on C*-algebras. The *Cuntz semigroup* of A is

$$\mathrm{Cu}(A) := (A \otimes \mathbb{K})_{+/\sim_{\mathrm{Cu}}}.$$

Again, orthogonal addition turns $\mathrm{Cu}(A)$ into a commutative monoid, and the relation $\precsim_{\mathrm{Cu}}$ induces an additive order on $\mathrm{Cu}(A)$. There is a picture of $\mathrm{Cu}(A)$ using countably generated, Hilbert A-modules, [CEI08]. If A has stable rank one, then $\mathrm{Cu}(A)$ is naturally isomorphic to the isomorphism classes of such modules.

The following table contains some examples of the considered invariants. It becomes apparent that $\mathrm{Cu}(A)$ contains more information than $V(A)$:

A	$V(A)$	$K_0(A)$	$\mathrm{Cu}(A)$
$\mathbb{C}$ or $M_n(\mathbb{C})$	$\mathbb{N}$	$\mathbb{Z}$	$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$
II_1-factor	$[0,\infty)$	$\mathbb{R}$	$[0,\infty) \sqcup (0,\infty]$
$C([0,1])$	$\mathbb{N}$	$\mathbb{Z}$	$\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$
$\mathcal{Z}$ or $C^*_\lambda(\mathbb{F}_\infty)$	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{N} \sqcup (0,\infty]$

In [CEI08], Coward-Elliott-Ivanescu introduced the category **Cu** of abstract Cuntz semigroups, and they showed that the Cuntz semigroup defines a functor from C*-algebras to **Cu**. A systematic study of **Cu** with applications to the structure theory of C*-algebras was conducted in [APT18] and [APT17]. In particular, it was shown that **Cu** admits a natural tensor product construction that gives it the structure of a closed, symmetric, monoidal category.

Theorem 1 ([APT19]). *The category* **Cu** *is complete and cocomplete, and the Cuntz semigroup functor preserves inductive limits and direct sums. Further, a scaled version of the Cuntz semigroup preserves products and ultraproducts.*

The Cuntz semigroup is an invariant that contains a lot of information about the C*-algebra, including its lattice of ideals and its simplex of (quasi)traces. Thus, the results in [APT19] allow us to access the ideal lattice and the quasitraces of products and ultraproducts of C*-algebras.

The rank problem. Let A be a simple, stably finite, exact C*-algebra. Then $\mathrm{Cu}(A)$ decomposes as $\mathrm{Cu}(A) = V(A)^\times \sqcup \mathrm{Cu}(A)_{\mathrm{soft}}$, where $V(A)^\times$ contains the classes of nonzero projections, and where $\mathrm{Cu}(A)_{\mathrm{soft}}$ consists of classes of elements with connected spectrum. Every trace $\tau\colon A \to \mathbb{C}$ induces a dimension function:

$$d_\tau\colon \mathrm{Cu}(A) \to [0,\infty], \quad d_\tau([a]) = \lim_{n\to\infty} \tau(a^{1/n}).$$

One calls $d_\tau([a])$ the 'rank' of a with respect to τ. The rank map is

$$\alpha\colon \mathrm{Cu}(A)_{\mathrm{soft}} \to \mathrm{LAff}(T(A))_{++}, \quad \alpha([a])(\tau) = d_\tau(a).$$

One says that A has *strict comparison* if α is an order-embedding, and there are examples when this fails. The *rank problem* for A is to determine the range of α. There are no examples known when α is not surjective.

Question 2. *Given a separable, unital, simple, exact* C*-*algebra, and given* $f \in \mathrm{LAff}(T(A))_{++}$, *is there* $a \in (A \otimes \mathbb{K})_+$ *with* $d_\tau([a]) = f(\tau)$ *for all* $\tau \in T(A)$?

If A is $\mathcal{Z}$-stable, then α is an isomorphism. If A is amenable, then the Toms-Winter conjecture predicts that A is $\mathcal{Z}$-stable if and only if A has strict comparison. Thus, α is predicted to be an isomorphism whenever it is an order-embedding. This establishes a connection between the Toms-Winter conjecture and the rank problem.

Theorem 2 ([Thi17]). *If A has stable rank one, then α is surjective.*

Corollary 3. *If A has stable rank one and strict comparison, then*

$$\mathrm{Cu}(A) \cong V(A) \sqcup \mathrm{LAff}(T(A))_{++} \cong \mathrm{Cu}(A \otimes \mathcal{Z}).$$

If, additionally, A has locally finite nuclear dimension (for example A is an ASH-algebra), then it follows that A is $\mathcal{Z}$-stable.

Riesz interpolation. In [APRT18] we unveil new structure in the Cuntz semigroup of C*-algebras of stable rank one. The main result is:

Theorem 4 ([APRT18]). *If A has stable rank one, then* $\mathrm{Cu}(A)$ *has Riesz interpolation: If $x_j \leq z_k$ for $j,k = 1,2$, then there is y with $x_1, x_2 \leq y \leq z_1, z_2$.*

Corollary 5. *If A is separable and of stable rank one, then* $\mathrm{Cu}(A)$ *is semilattice.*

These results allow us to apply semilattice theory to study C*-algebras of stable rank one. Using this method, we confirm a conjecture of Blackadar-Handelman and we solve the Global Glimm Halving Problem in this context.

Given a unital C*-algebra, we use $\mathrm{DF}(A)$ to denote the compact convex set of (not necessarily continuous) dimension functions on A. For a compact, Hausdorff space X, the set $\mathrm{DF}(C(X))$ can be identified with the finitely additive probability measures on X, which is a Choquet simplex. In [BH82], Blackadar-Handelman conjectured that $\mathrm{DF}(A)$ is always a Choquet simplex. We prove:

Theorem 6. *If A has stable rank one, then* $\mathrm{DF}(A)$ *is a Choquet simplex.*

The Global Glimm Halving Problem, [KR02], seeks to characterize when a C*-algebra A has no finite-dimensional irreducible representations. A sufficient condition is that A admits a $*$-homomorphism $M_k(C_0((0,1])) \to A$ with full image for each $k \in \mathbb{N}$, and the problem is to show that this criterion is also sufficient. Under the assumption of stable rank one, we show an even stronger result:

Theorem 7. *Let A have stable rank one, and let $k \in \mathbb{N}$. Then A has no irreducible representation of dimension $< k$ if and only if there exists a $*$-homomorphism $M_k(C_0((0,1])) \to A$ with full image.*

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Constructing Cartan subalgebras in classifiable C*-algebras

Xin Li

In my talk, I explained how to construct Cartan subalgebras in all classifiable C*-algebras. Let us now present the precise statements.

First, let us deal with the unital case, where the statement says:

Theorem 1. *Given*

- *a weakly unperforated, simple scaled ordered countable abelian group* (G_0, G_0^+, u),
- *a non-empty metrizable Choquet simplex* T,
- *a surjective continuous affine map* $r: T \to S(G_0)$,
- *a countable abelian group* G_1,

there exists a twisted groupoid (G, Σ) *such that*

- G *is a principal étale second countable locally compact Hausdorff groupoid,*
- $C^*_r(G, \Sigma)$ *is a simple unital* C^**-algebra which can be described as the inductive limit of subhomogeneous* C^**-algebras whose spectra have dimension at most* 3*,*
- *the Elliott invariant of* $C^*_r(G, \Sigma)$ *is given by*

$$\Big(K_0(C^*_r(G,\Sigma)), K_0(C^*_r(G,\Sigma))^+, [1_{C^*_r(G,\Sigma)}], T(C^*_r(G,\Sigma)), r_{C^*_r(G,\Sigma)}, K_1(C^*_r(G,\Sigma))\Big)$$
$$\cong \quad (G_0, G_0^+, u, T, r, G_1).$$

In the stably projectionless case, the statement reads as follows:

Theorem 2. *Given*

- *countable abelian groups* G_0 *and* G_1*,*
- *a non-empty metrizable Choquet simplex* T*,*
- *a homomorphism* $\rho : G_0 \to \mathrm{Aff}(T)$ *which is weakly unperforated in the sense that for all* $g \in G_0$*, there is* $\tau \in T$ *with* $\rho(g)(\tau) = 0$*,*

there exists a twisted groupoid (G, Σ) *such that*

- G *is a principal étale second countable locally compact Hausdorff groupoid,*
- $C^*_r(G, \Sigma)$ *is a simple stably projectionless* C^**-algebra with continuous scale which can be described as the inductive limit of subhomogeneous* C^**-algebras whose spectra have dimension at most* 3*,*
- *the Elliott invariant of* $C^*_r(G, \Sigma)$ *is given by*

$$\Big(K_0\big(C^*_r(G,\Sigma)\big), K_0\big(C^*_r(G,\Sigma)\big)^+, T\big(C^*_r(G,\Sigma)\big), \rho_{C^*_r(G,\Sigma)}, K_1\big(C^*_r(G,\Sigma)\big)\Big)$$
$$\cong \quad (G_0, \{0\}, T, \rho, G_1).$$

For more details, as well as more general results (covering all possible Elliott invariants for classifiable C^*-algebras), the reader may consult [3]. The connection between Cartan subalgebras and groupoids is built by work of Kumjian [1] and Renault [4].

I gave an overview of the idea of the proof of these statements. A key ingredient is a construction of Cartan subalgebras in inductive limit C^*-algebras. This can be found in [3] and builds on [2].

Finally, in the last part of the talk I discussed the particular example of the Jiang-Su algebra in detail, including a description of the spectra of (some of) the Cartan subalgebras we construct, as well as a result saying that our construction gives rise to continuum many pairwise non-conjugate Cartan subalgebras (in other words, the underlying groupoids are pairwise non-isomorphic).

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