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Geometric, Algebraic, and Topological Combinatorics

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Abstract. The 2019 Oberwolfach meeting "Geometric, Algebraic and Topological Combinatorics" was organized by Gil Kalai (Jerusalem), Isabella Novik (Seattle), Francisco Santos (Santander), and Volkmar Welker (Marburg). It covered a wide variety of aspects of Discrete Geometry, Algebraic Combinatorics with geometric flavor, and Topological Combinatorics. Some of the highlights of the conference included (1) Karim Adiprasito presented his very recent proof of the g-conjecture for spheres (as a talk and as a "Q&A" evening session) (2) Federico Ardila gave an overview on "The geometry of matroids", including his recent extension with Denham and Huh of previous work of Adiprasito, Huh and Katz.

Mathematics Subject Classification (2010): 05Exx, 52Bxx, 13Dxx, 13Fxx, 14Txx, 52Cxx, 55xxx, 57Qxx, 90Cxx.

Introduction by the Organizers

The 2019 Oberwolfach meeting "Geometric and Algebraic Combinatorics" was organized by Gil Kalai (Hebrew University, Jerusalem), Isabella Novik (University of Washington, Seattle), Francisco Santos (University of Cantabria, Santander), and Volkmar Welker (Philipps-Universität Marburg, Marburg).

The conference consisted of one 1-hour lecture by Adiprasito on the g-theorem plus thirty-three shorter talks, ranging from 30 to 45 minutes. On Thursday evening there was a problem session and on Monday, Tuesday and Wednesday evenings there were several informal sessions: Adiprasito offered two "Question and Answer" sessions on the technical ingredients of his recently announced proof of the g-theorem for spheres, and a group of participants had a discussion on

Helly-type results in topology and combinatorics. There were many other small group discussions, some of which initiated new collaborations. All together it was a very productive and enjoyable week.

The conference treated a broad spectrum of topics from Topological Combinatorics (face vectors of spheres and manifolds, configuration spaces, embeddability, Helly-type and Borsuk-Ulam theorems) Geometric Combinatorics (tropical geometry, polytope theory, matroid polytopes, positroids) and Algebraic Combinatorics (Lefschetz theorems, web algebras, amplituhedra, semistable reduction).

What follows tries to summarize the richness and depth of the work and the presentations, concentrating on some of the highlights.

The first Monday lecture, by Federico Ardila, served two purposes. On one hand, it provided a nice introduction to matroid theory for non-experts, focusing on geometric models for matroids such as the matroid polytopes. On the other hand, it introduced the recent formalism of *conormal fans* by Ardila, Denham and Huh, with which they have been able to extend previous results of Adiprasito, Huh and Katz related to the long-standing Heron-Rota-Welsh conjecture from the sixties.

Another long-standing conjecture was the topic of the first afternoon talk, by Adiprasito. The g-theorem, proved by Stanley, Billera and Lee in 1980, gives a complete characterization of the possible f-vectors of simplicial polytopes in arbitrary dimension. It has been conjectured since then that the same characterization extends to triangulated spheres; that is, that every triangulated sphere has the same f-vector as some simplicial polytope (even though the class of the boundary complexes of simplicial polytopes forms a very tiny part of the class of triangulated spheres). The proof of this conjecture was announced by Adiprasito in December 2018 and is one of the most outstanding results in topological and geometric combinatorics of the past 20 years. Adiprasito gave a 1-hour overview of the main ingredients of his (highly technical) proof, and offered two more informal evening sessions where participants asked specific questions.

The rest of Monday was devoted to other results on topics close to these two talks, namely, matroids and f-vectors, respectively. For example, Hailun Zheng described the construction of centrally symmetric neighborly spheres, settling another problem that stood open for a few decades.

On Tuesday morning we had several talks related to *positivity* in Grassmann and flag varieties. Thomas Lam's opening talk described the amplituhedron as an extension of cyclic polytopes, continued with the even more general Grassman polytopes, and concluded with a concrete question about ordinary polytopes that is motivated by a physical scattering amplitude computation. The session continued with very interesting combinatorics coming form plabic graphs and subdivisions of hypersimplices, with motivations coming from the positive Grassmannian. Tuesday afternoon was devoted to topological combinatorics, in particular to hypertrees and configuration spaces.

Wednesday morning started with two lectures with connections to deep algebraic objects: Web algebras and Kazhdan-Lusztig-Stanley polynomials of matroids. They were followed by talks with a touch from the theory of computing – one on semidefinite programming and the other on the complexity of f-vector problems.

On Thursday the main topic was the interactions between algebraic geometry and combinatorics, including several talks on tropical geometry (one of the most successful and active of these interactions) and a proof, based on triangulations of polytopes, of the important "semistable reduction conjecture" (Liu's talk).

On Friday morning we came back to topology, the highlight perhaps being Wagner's proof that embeddability of d-dimensional simplicial complexes in $\mathbb{R}^n$ is undecidable for certain pairs (d, n), and a close inspection of the complexity of the question for (almost) every pair. On Friday afternoon we had two open-ended talks by two of the pioneers in our field: one on continuous matroids (Anders Björner, joint work with L. Lovázs) and another on the hyperplane arrangement of all 0-1 linear equalities in real space (Lou Billera, joint with Florian Frick).

It bears repeating that quite a few breakthrough results were announced and presented during the conference. These include the g-theorem, the semistable reduction theorem, and the extension by Ardila et al. of the Adiprasito-Huh-Katz theorem. The timeliness and novelty of the talks presented is witnessed by the following arXiv preprints, all at most three months old at the time of the workshop (and two uploaded after the workshop) that were presented in it:

arXiv:1905.02287 (Tran's talk)
arXiv:1906.05859 (follow-up to Adiprasito's preprint from Dec. 18)
arXiv:1907.06115 (Zheng's talk)
arXiv:1906.03501 (Lam's talk)
arXiv:1909.05435 (Postnikov's talk)
arXiv:1906.05764 (Olarte's talk)
arXiv:1908.04241 (Kahle's talk)
arXiv:1909.08937 (Averkov's talk)
arXiv:1908.09628 (Nevo's talk)
arXiv:1906.1095 (Joswig's talk)
arXiv:1907.06276 (Frick's talk)
arXiv:1907.05055 (Tancer's talk)

There was also a number of talks devoted to fundamental concepts of theories that are important but perhaps not that familiar to most of the audience. This includes the talks by Rincón, Lam, Mikhalkin, Elias, Proudfoot, Payne, and Blagojević. Last but not least, there was a lively problem session; at the end of this report is the list of problems posed.

We are extremely grateful to the Oberwolfach institute, its directorate and to all of its staff for providing a perfect setting for an inspiring, intensive week of "Geometric, Algebraic, and Topological Combinatorics".

Gil Kalai, Isabella Novik, Francisco Santos, Volkmar Welker
Jerusalem/Seattle/Santander/Marburg, September 2019

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Workshop: Geometric, Algebraic, and Topological Combinatorics

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Abstracts

The geometry of matroids

Federico Ardila

1. Introduction

Matroid theory is a combinatorial theory of independence which has its origins in linear algebra and graph theory, and turns out to have deep connections with many other fields. In particular, the geometric roots of the field have since grown much deeper, bearing many new fruits. We discuss three geometric models of a matroid: the basis polytope, the Bergman fan, and the conormal fan. They elucidate the geometric nature of these objects, and lead to the solution of long-standing questions at the intersection of combinatorics, algebra, and geometry.

A *matroid* $M = (E, \mathcal{B})$ on a finite set E is a collection $\mathcal{B} \neq \emptyset$ of subsets of E, called the *bases*, such that:

$$\text{If } A, B \in \mathcal{B} \text{ and } a \in A - B, \text{ then } (A - a) \cup b \in \mathcal{B} \text{ for some } b \in B - A.$$

The prototypical example is that of a *linear matroid*: if E is a set of vectors spanning a vector space V, the subsets of E which are bases of V form a matroid.

2. The matroid polytope

Let M be a matroid on the ground set E. Its *matroid polytope* is

$$P_M = \operatorname{conv}\{e_B : B \text{ is a basis of } M\},$$

where $\{e_i : i \in E\}$ is the standard basis of $\mathbb{R}^E$ and we write $e_B = e_{b_1} + \cdots + e_{b_r}$ for $B = \{b_1, \ldots, b_r\}$.

This construction arises naturally in combinatorial optimization [4] and algebraic geometry [6]. It leads to the following beautiful combinatorial characterization of matroids.

Theorem 1. *(Edmonds, 1970,* [4]*, Gelfand-Goresky-MacPherson-Serganova, 1987,* [6]*) A collection $\mathcal{B}$ of subsets of $[n]$ is the set of bases of a matroid if and only if every edge of the polytope $P_{\mathcal{B}} := \operatorname{conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$ is a translate of $e_i - e_j$ for some i, j.*

One could *define* a matroid to be a subpolytope of the cube $[0, 1]^n$ that only uses these vectors as edges. Notice that from this polytopal point of view, even if one only cares about linear matroids, all matroids are equally natural. Matroid theory provides the correct level of generality.

3. The Bergman fan

Our second model comes from tropical geometry. This is a powerful technique that turns an algebraic variety V into a simpler, piecewise linear space $\operatorname{Trop} V$ that still contains geometric information about V. Many difficult problems in algebraic geometry can be solved by turning them into combinatorial problems in tropical geometry. Tropical varieties are simpler than algebraic varieties, but they are still very intricate. An important example to understand is the following: What is the tropicalization of a linear subspace V of $\mathbb{C}^n$?

The *flats* of M are the subsets $F \subseteq E$ such that $r(F \cup e) > r(F)$ for all $e \notin F$. We say F is *proper* if it does not have rank 0 or r. The flats of a linear matroid M are the (subsets of E contained in the) subspaces spanned by E.

Definition/Theorem 2. *(Ardila-Klivans, 2006,* [3]*)*
1. The Bergman fan Σ_M *of a matroid M on E is the polyhedral complex in $\mathbb{R}^E / \langle e_E \rangle$ consisting of the cones*

$$\sigma_{\mathcal{F}} = \operatorname{cone}\{e_F \,:\, F \in \mathcal{F}\}$$

for each flag $\mathcal{F} = \{F_1 \subsetneq \cdots \subsetneq F_l\}$ of proper flats of M. Here $e_F := e_{f_1} + \cdots + e_{f_k}$ for $F = \{f_1, \ldots, f_k\}$.
2. The tropicalization of a linear subspace V of $\mathbb{C}^n$ is the Bergman fan of its matroid:

$$\operatorname{Trop} V = \Sigma_{M(V)}.$$

3. The Bergman fan Σ_M is a cone over a wedge of $|\mu(M)|$ $(r-2)$-spheres, where $\mu(M)$ is the Möbius number.

Tropical varieties have a natural notion of *degree*, which is analogous to and compatible with the notion of the degree of an algebraic variety. We have the following remarkable characterization.

Theorem 3. *(Fink, 2013,* [5]*) A tropical variety has degree* 1 *if and only if it is the Bergman fan of a matroid.*

One could *define* a matroid to be a tropical variety of degree 1; this is the tropical analog of a linear space. Notice that, although Σ_M only arises via tropicalization when M is a linear matroid, one should really consider the Bergman fans of all matroids. Again, matroid theory really provides the correct level of generality.

The theorems above explain the importance of matroids in tropical geometry. On the one hand, they provide a useful testing ground for general results. On the other hand, they are fundamental building blocks; a *tropical manifold* is a tropical variety that locally looks like a (Bergman fan of a) matroid.

4. The conormal fan

We now introduce another polyhedral model of M that leads to stronger inequalities for matroid invariants. Recall that the *dual matroid* $M^\perp$ is the matroid on E with bases $\mathcal{B}^\perp = \{E - B \,:\, B \in \mathcal{B}\}$. A *biflag* consists of a flag $\mathcal{F} = \{F_1 \subseteq \cdots \subseteq F_l\}$

of nonempty flats of M and a flag $\mathcal{G} = \{G_1 \supseteq \cdots \supseteq G_l\}$ of nonempty flats of $M^\perp$ such that $F_i \cup G_i = E$ for all i and $F_i \cup G_{i+1} \neq E$ for some i.

Definition 4. *(Ardila-Denham-Huh, 2017,* [2]*) The* conormal fan $\Sigma_{M,M^\perp}$ *of a matroid M is the polyhedral complex in $\mathbb{R}^E / \langle e_E \rangle \times \mathbb{R}^E / \langle f_E \rangle$ consisting of the cones*

$$\sigma_{\mathcal{F},\mathcal{G}} = \text{cone}\{e_{F_i} + f_{G_i} \,:\, 1 \leq i \leq l\}$$

for each biflag $(\mathcal{F}, \mathcal{G})$. Here $\{e_i : i \in E\}$ and $\{f_i : i \in E\}$ are bases for two copies of $\mathbb{R}^E$.

It would be interesting to find an intrinsic characterization of conormal fans, in analogy with Theorems 1 and 3.

5. Chow rings, Hodge theory, unimodality and log-concavity.

The *Chow ring* of the Bergman fan Σ_M is defined to be

$$A^*(\Sigma_M) := \mathbb{R}[x_F \,:\, F \text{ proper flat of } M]/(I_M + J_M),$$

where $I_M = \langle x_{F_1} x_{F_2} \,:\, F_1 \subsetneq F_2 \text{ and } F_1 \supsetneq F_2 \rangle$ and

$$J_M = \left\langle \sum_{F \ni i} x_F - \sum_{F \ni j} x_F \,:\, i, j \in E \right\rangle.$$

The Chow ring of the conormal fan $\Sigma_{M,M^\perp}$ has a similar presentation. When M is linear over $\mathbb{C}$ they have cohomological interpretations; but surprisingly, these Chow rings *always* behave like the cohomology ring of a smooth projective variety:

Theorem 5. *(Adiprasito-Huh-Katz, 2018; Ardila-Denham-Huh, 2019) The Chow rings of the Bergman fan* [1] *and conormal fan* [2] *of a matroid M satisfy Poincaré duality, the hard Lefschetz theorem, and the Hodge-Riemann relations.*

Theorem 5 is the main geometric tool in the proofs of the following series of long-standing conjectures by Read (1968), Rota-Heron-Welsh (1970), Mason (1972), Hoggar (1974), Brylawski (1982), and Dawson (1983):

Theorem 6. *(Adiprasito-Huh-Katz, 2018; Ardila-Denham-Huh, 2019) The following sequences associated to a matroid M are unimodal and log-concave:*
- *the f-vector $f(M) = (f_0, \ldots, f_r)$, where f_d is the number of independent sets of size d* [1]*,*
- *the Whitney numbers of the first kind $w(M)$* [1]*,*
- *the h-vector $h(M)$,* [2] *and.*
- *the h-vector of the broken circuit complex* [2]*.*

This research program was initiated by Huh [7, 8] for realizable matroids. Adiprasito, Huh, and Katz subsequently extended it to all matroids by introducing this *Hodge theory of matroids*. Ardila, Denham, and Huh further strengthened this machinery by developing a *Lagrangian theory of matroids*. For further details and precise statements of these results, see [1, 2].

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The matroid based valuation conjecture

Ngoc Mai Tran

This talk is based on a paper of the same name [Tra19]. It concerns the matroid-based valuation conjecture of Ostrovsky and Paes Leme [OPL15].

Let n be a natural number and $(\mathbb{R}, \oplus, \odot)$ be the max-plus tropical algebra, with $a \oplus b := \max(a, b)$, $a \odot b := a + b$. Fix a lattice polytope $P \subset \mathbb{Z}^n$. Unless otherwise stated, all lattice functions $u : P \cap \mathbb{Z}^n \to \mathbb{R}$ considered in this proposal will be concave, that is, it is equal to the restriction to lattice points of a concave function $\bar{u} : P \to \mathbb{R}$. Such a function gives rise to the tropical Laurent polynomial $f_u : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_u(p) = \bigoplus_{a \in P \cap \mathbb{Z}^n} u(a) \odot p^{\odot a} = \max_{a \in P \cap \mathbb{Z}^n} (u(a) + \langle p, a \rangle). \tag{1}$$

For each $p \in \mathbb{R}^n$, the cell $\sigma_u(p) \subset P \cap \mathbb{Z}^n$ is the set of $a \in P \cap \mathbb{Z}^n$ that achieves the maximum in (1). Convex hulls of the cells are called faces. The collection of faces $\{\mathrm{conv}(\sigma_p) : p \in \mathbb{R}^n\}$ fit together to form a polyhedral complex called the regular subdivision of P induced by u, denoted Δ_u. Say that a lattice polytope $P \subset \mathbb{Z}^n$ is $M^\natural$*-convex* if its set of edges are parallel to $\{e_i - e_j, e_i : i = 1, \ldots, n\}$.

Definition 1 (Gross substitutes)**.** *A concave function $u : P \cap \mathbb{Z}^n \to \mathbb{R}$ has the gross substitutes (GS) property if every face of Δ_u is $M^\natural$-convex, or equivalently, that the edges of Δ_u are in $\{e_i - e_j, e_i : i, j \in [n], i \neq j\}$.*

Gross substitute valuations arise in economics as follows. Roughly speaking, an auction is a mechanism to sell goods. In a product-mix or multi-unit combinatorial auction, multiple agents can make simultaneous bids on multiple subsets of goods at once. Auctions where agents bid with GS functions have many desirable economics and computational properties [BM97, GS99, KJC82, BK16, TY19].

However, specifying an arbitrary GS bid function defined on n different good types requires at least $2^n/\operatorname{poly}(n)$ values [Haj08, Lem17]. Thus a very important question in economics is:

Problem. Find a generative description of gross substitute functions.

A solution to this problem would enable new ways to generate GS bids, lifting a fundamental obstacle in implementing GS auctions in practice.

This problem generated quite some interest in economics [HM05, Haj08, KTY14, OPL15, Lem17, Mil17, BPL18]. The general idea is to start with a class of known gross substitutes valuations, and close it up under operations that preserve gross substitutability. Two natural operations with simple economics interpretations are merging and endowments. These operations generate matroid union and contraction, respectively. With these operations, Hatfield and Milgrom [HM05] conjectured that unit demand valuations would generate all GS functions (the EAV conjecture). Ostrovsky and Paes Leme disproved this, and conjectured that all weighted ranks of all matroids on a finite set $[m]$ would be a generating set. The resulting class, matroid-based valuations (MBVs), is conjectured to be equal to the set of gross substitutes. By definition, one needs $m \geq n$. The larger $m(n)$ is relative to n, the more complex the starting class of weighted ranks one must start with, and thus the less attractive it is to represent gross substitutes valuations as matroid based valuations. The case $m = n$ is the most interesting. The author [Tra19] disproved this conjecture when $m = n$ for $n \geq 4$ (and showed that it holds for $m = n$ for $n \leq 3$. It is open whether for each n, ther exists some finite m such that the MBVs on $[m]$ would generate all the GS on $[n]$.

The paper [Tra19] contains more results on connections between this conjecture and Welsh's problem of characterizing irreducible matroids.

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The Hopf monoid of ordered matroids and shuffles

Jose A. Samper

(joint work with Federico Castillo and Jeremy Martin)

The seminal work of Aguiar and Ardila [1] on Hopf monoids and generalized permutahedra carries within a lesson for matroid theory: the algebraic structure of a matroid is better understood through the lens of matroid polytopes and their geometry. Indeed, the Hopf monoid of matroids and its corresponding Hopf algebras have invariants that admit very clean geometric descriptions and quite messy combinatorial ones.

One common technique that is widely used to understand certain matroid invariants is to order the underlying set of the matroid, and then exploit the order structure to associate (order-dependent) objects whose invariants depend only on the matroids. Examples of this include, for example, broken circuit complexes and activity theories, which are order-dependent objects that aid our understanding of matroid invariants such as the characteristic and Tutte polynomials.

We propose an algebraic model that aims to capture the combinatorial subtleties of oredered matroids in the realm of Hopf monoids. The approach is to combine the Hopf monoid of generalized permutahedra with the Hopf monoid $\mathbf{L}^*$ of linear orders, with product given by shuffles. In this case, the algebraic structure of $\mathbf{L}^*$ yields a coproduct structure that is compatible with ordered deletions and contractions. On the geometric side of the story it replaces global geometry of the matroid polytope with local geometry of the normal cone of a vertex of the matroid polytope and its relationship to the braid fan (the normal fan of the permutahedron).

Some of the main results of the paper include:

(1) A cancellation-free formula for the antipode of the Hadamard product $\mathbf{OGP} = \mathbf{GP} \times \mathbf{L}^*$ of generalized permutahedra with linear orders, that elucidates how geometry and total orders are combined in matroid theory. The formula extends to the larger Hopf monoid $\mathbf{OGP}_+$ of possibly-unbounded generalized permutahedra equipped with a compatible linear order.

(2) A characterization of *order-decomposable simplicial complexes*, the largest class of pure ordered complexes that admits a Hopf monoid structure compatible with shuffles and restrictions. Well-known subclasses of order-decomposable complexes include ordered matroids, shifted complexes, and broken circuit complexes.

(3) A description of a class of ordered simplicial complexes coming from unbounded generalized permutahedra whose vertices are 0,1-vectors. This class strictly contains ordered matroids. These complexes behave similarly to matroids, and form a Hopf submonoid of $\mathbf{OGP}_+$.

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Tropical ideals

Felipe Rincón

(joint work with Jan Draisma and Diane Maclagan)

An ideal in a polynomial ring over a field with the trivial valuation gives rise to a polyhedral fan called its tropical variety, by taking all weight vectors whose initial ideals do not contain a monomial. In the middle of this construction sits a tropical ideal, obtained by recording the supports of all polynomials in the ideal. This tropical ideal is a purely combinatorial object, and it contains more information than the tropical variety itself. For these reasons, tropical ideals, axiomatised in [2], were proposed as the correct combinatorial/algebraic structures on which to build a theory of tropical schemes. Concretely, if K is an infinite field, a classical ideal $J \subset K[x_1, \ldots, x_n]$ gives rise to the tropical ideal

$$I = \operatorname{trop}(J) := \{\operatorname{supp}(F) : F \in J\} \subseteq 2^{N^n},$$

P where the support $\operatorname{supp}(F)$ of a polynomial $F = \sum_{\mathbf{u} \in \mathbb{N}^n} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in K[x_1, \ldots, x_n]$ is given by $\operatorname{supp}(F) := \{\mathbf{u} \in \mathbb{N}^n : c_{\mathbf{u}} \notin 0\}$. A tropical ideal arising in this way is called realisable (over the field K). In general, a (possibly non-realisable) *tropical ideal* in the variables $x_1, \ldots, x_n$ is a non-empty collection I of finite subsets of $\mathbb{N}^n$ satisfying the following conditions:

- $\emptyset \in I$
- If $S, T \in I$ then $S \cup T \in I$.

- If $S \in I$ then $S + e_i := \{v + e_i : v \in S\} \in I$ for any $1 \leq i \leq n$, where $\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{Z}^n$.
- Monomial elimination axiom: If $S, T \in I$ and $\mathbf{u} \in S \cap T$ then there is $U \in I$ such that $S \Delta T \subset U \subset (S \cup T) \setminus \{u\}$, where Δ denotes symmetric difference.

Tropical ideals are in this way combinatorial abstractions of the possible collections of subsets of $\mathbb{N}^n$ that arise as the supports of all polynomials in a fixed ideal over a field. The monomial elimination axiom is basically an instance of the cycle elimination axiom for matroids. While 'most' tropical ideals are expected to be non-realisable, there are essentially very few examples known so far of non-realisable tropical ideals – one of them, for instance, can be found in [2, Example 2.8]. Generalising the notion of tropical variety for a classical ideal, the *variety* of an arbitrary tropical ideal I is defined to be

$$V(I) := \{\mathbf{x} \in \mathbb{R}^n : \forall S \in I, \min_{\mathbf{u} \in S}(\mathbf{x} \cdot \mathbf{u}) \text{is attained by at least two terms}\}.$$

Tropical ideals turn out to have very nice algebraic and geometric properties. It was proved in [2] that tropical ideals, while not finitely generated as ideals – nor in any sense that we know of! – have a rational Hilbert series, satisfy the ascending chain condition and the weak Nullstellensatz, and have varieties that are finite weighted polyhedral fans. This leads to the following realisability question.

Question 1. *Which weighted polyhedral fans are the variety of some tropical ideal?*

When the tropical ideal records the supports of the polynomials in a classical prime ideal J, then the tropical variety is a pure-dimensional and balanced polyhedral fan [3, Theorem 3.3.5]. Conversely, the question of which balanced polyhedral complexes are realised by classical ideals has received much attention, especially in the case of curves. But for general tropical ideals, very little is known about 1: no natural algebraic criterion that ensures that the variety is pure-dimensional is known, nor has their top-dimensional part been proved to be balanced. In fact, until recently we had no intuition as to whether tropical ideals are flexible enough that they can realise basically any balanced polyhedral fan, or rather more rigid, like algebraic varieties. In view of the following theorem, we now lean towards the latter intuition.

Theorem 2 ([1])**.** *There exists no tropical ideal whose tropical variety is the Bergman fan of the direct sum of the Vámos matroid V_8 and the uniform matroid $U_{2,3}$ of rank two on three elements, with all maximal cones having weight 1.*

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Combinatorial Lefschetz Theorems beyond positivity

Karim Adiprasito

The hard Lefschetz theorem, in almost all cases that we know, is connected to rigid algebro-geometric properties. Most often, it comes with a notion of an ample class, which not only induces the Lefschetz theorem but the induced bilinear form satisfies the Hodge-Riemann relations as well, which give us finer information about its signature (see for instance Voisin, CUP 2002).

Even in the few cases that we have the hard Lefschetz without the Hodge-Riemann relations, they are often at least conjecturally present in some form, as for instance in the case of Grothendieck's standard conjectures and Deligne's proof of the hard Lefschetz standard conjecture. This connection is deep and while we understand Lefschetz theorems even for singular varieties, to this day, we have no way to understand the Lefschetz theorem without such a rigid atmosphere for it to live in.

Our goal and result in arxiv:1812.10454 is to provide a different criterion for varieties to satisfy the hard Lefschetz theorem that goes beyond positivity, and abandons the Hodge-Riemann relations entirely (but not the associated bilinear form); instead of finding Lefschetz elements in the ample cone of a variety, we give general position criteria for an element in the first cohomology group to be Lefschetz. The price I pay for this achievement is that the variety itself has to be sufficiently "generic".

For the current results I therefore turn to toric varieties, which allow for a sensible notion of genericity without sacrificing all properties of the variety, most importantly, without changing its Betti vector. Specifically, I consider varieties with a fixed equivariant cohomology ring, and allow variation over the Artinian reduction, i.e., the variation over the torus action. The main result can be summarized as follows:

Theorem [A, arxiv:1812.10454] Consider a PL $(d-1)$-sphere Σ, or more generally a PL rational homology sphere of that dimension, and the associated graded commutative face ring $\mathbb{R}[\Sigma]$ (see Stanley, Birkhäuser Prog. in Math. 1996). Then there exists an open dense subset of the Artinian reductions $\mathcal{R}$ of $\mathbb{R}[\Sigma]$ and an open dense subset $\mathcal{L} \subset A^1(\Sigma)$, where $A(\Sigma) \in \mathcal{R}$, such that for every $k \leq \frac{d}{2}$, we have:

(1) *Generic Lefschetz theorem:* For every $A(\Sigma) \in \mathcal{R}$ and every $\ell \in \mathcal{L}$, we have an isomorphism

$$A^k(\Sigma) \xrightarrow{\cdot \ell^{d-2k}} A^{d-k}(\Sigma).$$

(2) *Hall-Laman relations:* The *Hodge-Riemann bilinear form*

$$\begin{array}{rccccc} \mathrm{Q}_{\ell,k}: & A^k(\Sigma) & \times & A^k(\Sigma) & \longrightarrow & A^d(\Sigma) \cong \mathbb{R} \\ & a & & b & \longmapsto & \deg(ab\ell^{d-2k}) \end{array}$$

is nondegenerate when restricted to any squarefree monomial ideal in $A(\Sigma)$, as well as the annihilator of any squarefree monomial ideal.

The Lefschetz theorem is therefore as announced valid for generic Artinian reductions. A slightly weaker form applies to general, non PL triangulations. In particular, the more algebrao-geometric reader may consult the following Corollary for easier visualization.

Corollary Consider $\mathfrak{F}$ a complete simplicial fan in $\mathbb{R}^d$. Then, after perturbing the rays of $\mathfrak{F}$ to a suitable rational fan $\mathfrak{F}'$, the Chow ring of the toric variety $X_{\mathfrak{F}'}$ satisfies the hard Lefschetz theorem with respect to a generic degree one element, while the equivariant Chow ring remains unchanged from $X_{\mathfrak{F}}$ to $X_{\mathfrak{F}'}$.

These results have a myriad of consequences, among them:

(1) *g-conjecture, McMullen Isr. J. Math. 1971:* It proves that the f-vector, i.e. the number of vertices, edges, two-dimensional faces etc. of a simplicial sphere is also the f-vector of some simplicial polytope.
(2) *Grünbaum conjecture, J. Comb. Theor. 1970:* It generalizes a result of Déscartes: If Δ is a simplicial complex of dimension d that allows a PL embedding into $\mathbb{R}^{2d}$ then

$$f_d(\Delta) \;\leq\; (d+2)f_{d-1}(\Delta)$$

Balanced non-partitionable and non-shellable complexes

Lorenzo Venturello
(joint work with Martina Juhnke-Kubitzke)

There is a number of different notions of how to decompose a simplicial complex. Various classes of decomposable complexes usually enjoy desirable properties, from a combinatorial and topological point of view, but it is often unclear how they relate to other families. In the first part of my talk I will focus on the relation between partitionability and Cohen-Macaulyness. Duval, Goeckner, Klivans and Martin [2] recently disproved a conjecture of Stanley stating that every Cohen-Macaulay simplicial complex is partitionable. However, several interesting subcases of this conjecture remained open. I will discuss a joint work with Martina Juhnke-Kubitzke [3] where we extend the original counterexample to Cohen-Macaulay complexes which are balanced (i.e., their skeleton can be minimally colored). These class contains (strictly) the order complexes of Cohen-Macaulay posets, but our counterexample does not fall into this subclass. In the second part I will discuss the problem of obtaining balanced 3-spheres which are not shellable on few vertices. In [4] we design a computer program based on the balanced analog of bistellar flips, called cross-flips. Applying these local moves to barycentric subdivisions of 3-spheres containing knots which are sufficiently complicated (see [1]) we obtain a balanced 3-sphere which is not shellable (with f-vector (1,28,204,352,176)) and one which is shellable but not vertex-decomposable (with f-vector 1,22,136,228,114).

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The upper bound theorem for centrally symmetric simplicial spheres

Hailun Zheng

(joint work with Isabella Novik)

A simplicial complex is centrally symmetric (cs, for short) if it possesses a free simplicial involution. A polytope P is cs if $P = -P$. This talk reviewed the results concerning the upper bound problems of cs polytopes and cs simplicial spheres.

In a complete analogy with the notion of neighborliness, we define cs-neighborliness as follows: a cs simplicial complex Δ is cs-ℓ-neighborly if every set of ℓ of its vertices, no two of which are antipodes, is a face of Δ. Adin [1] and Stanley (unpublished) proved that among all cs simplicial spheres of dimension $d-1$ and with $2n$ vertices, a cs-$\lfloor d/2 \rfloor$-neighborly sphere simultaneously maximizes all the face numbers, *assuming such a sphere exists.*

Unfortunately, for fixed $d \geq 4$ and $n \geq d$, the cs $2n$-vertex simplicial d-polytope that are cs-$\lfloor d/2 \rfloor$-neighborly do not always exist. McMullen and Shephard [4] proved that while there do exist cs d-polytopes with $2(d+1)$ vertices that are cs-$\lfloor d/2 \rfloor$-neighborly, a cs d-polytope with $2(d+2)$ vertices cannot be more than cs-$\lfloor (d+1)/3 \rfloor$-neighborly. Moreover, Linial and Novik [3] showed that a cs d-dimensional polytope with more than 2^d vertices cannot be even cs-2-neighborly. So far, there is no plausible upper bound conjecture for cs polytopes.

The existence of cs-2-neighborly simplicial 3-spheres with $2n$ vertices for any $n \geq 4$ was confirmed by Jockusch [2] in 1995. In a joint work with Novik [5] we presented a generalization of Jockusch's construction.

Theorem 1. *For any $d \geq 4$ and $n \geq d$, there exists a cs simplicial $(d-1)$-sphere with $2n$ vertices that is cs-$\lfloor \frac{d}{2} \rfloor$-neighborly.*

This result combined with work of Adin and Stanley completely resolves the upper bound problem for cs simplicial spheres. The construction is by induction on both d and n. In particular, the key idea in the inductive step is to find a certain pair of antipodal $(d-1)$-balls in the cs simplicial $(d-1)$-sphere that is both $\lfloor \frac{d}{2} \rfloor$-stacked and cs-$\lfloor \frac{d}{2} \rfloor$-neighborly and then replace these balls with the cones over their boundary complexes.

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Grassmann polytopes and amplituhedra

Thomas Lam

(joint work with Nima Arkani-Hamed, Yuntao Bai, Pavel Galashin, Steven Karp)

I reported on joint works with Arkani-Hamed and Bai [ABL], and with Galashin and Karp [GKL]. Let $1 \leq k \leq n$ be positive integers. The *totally nonnegative Grassmannian* $\mathrm{Gr(k,n)}_{\geq 0}$ is the subspace of the real Grassmannian representable by $k \times n$ matrices all of whose $k \times k$ minors are nonnegative. This remarkable topological space was first defined by Lusztig (in a different way), and later by Postnikov. Its face structure was studied in detail by Postnikov and by Rietsch. We showed recently [GKL] with Galashin and Karp that $\mathrm{Gr(k,n)}_{\geq 0}$ is a regular CW complex homeomorphic to a closed ball.

Now let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ be a linear map of full rank, where $k+m \leq n$. It induces a rational map $Z : \mathrm{Gr(k,n)} \to \mathrm{Gr(k,k+m)}$ sending a subspace $V \subset \mathbb{R}^n$ to the subspace $Z(V) \subset \mathbb{R}^{k+m}$ (if $Z(V)$ has dimension k, otherwise Z is not defined at V). When Z is defined on the whole of $\mathrm{Gr(k,n)}_{\geq 0}$, we call the image $Z(\mathrm{Gr(k,n)}_{\geq 0}) \subset \mathrm{Gr(k,k+m)}$ a *Grassmann polytope* [Lam]. If in addition the image of Z in $\mathrm{Gr(k+m,n)}$ lies in $\mathrm{Gr(k+m,n)}_{\geq 0}$, we call $Z(\mathrm{Gr(k,n)}_{\geq 0})$ the *amplituhedron*, defined by physicists Arkani-Hamed and Trnka [AT]. It is an important open problem to study the topology and combinatorics of Grassmann polytopes in analogy with usual convex polytopes.

Another fundamental problem is to show that Grassmann polytopes are *positive geometries*, a notion introduced in joint work with Arkani-Hamed and Bai [ABL]. Roughly speaking, a positive geometry $X_{\geq 0}$ is a compact semialgebraic space equipped with a meromorphic top form $\Omega(X_{\geq 0})$ with the property that the polar structure of $\Omega(X_{\geq 0})$ reflects the combinatorics of the boundary structure of $X_{\geq 0}$. We conjecture that Grassmann polytopes are positive geometries. In the case of the amplituhedron, the corresponding canonical form Ω called the "amplituhedron form" should essentially be the $N = 4$ super Yang-Mills tree amplitude.

We show that usual convex polytopes are always positive geometries. The canonical form $\Omega(P)$ of a convex polytope P is already quite interesting. In particular, a non-trivial result is that $\Omega(P)$ takes constant sign in the interior of P.

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Higher Secondary polytopes and regular plabic graphs

Alexander Postnikov

(joint work with Pavel Galashin and Lauren Williams)

For a configuration A of n points in R^d, we introduce the higher secondary polytopes $\Sigma_{A,1}, \dots, \Sigma_{A,n-d}$, such that $\Sigma_{A,1}$ is the secondary polytope of Gelfand-Kapranov-Zelevinsky, while the Minkowski sum of these polytopes is Billera-Sturmfels' fiber zonotope associated with A.

In a special case when $d = 3$, we refer to our polytopes as higher associahedra. They turn out to be related to the theory of total positivity, specifically, to certain combinatorial objects called plabic graphs that appear in the study of the totally positive Grassmannian. These graphs also appear as on-shell diagrams in the study of scattering amplitudes in N=4 SYM theory, and as contour plots of soliton solutions to the KP equation.

We define a subclass of regular plabic graphs and show that they correspond to the vertices of the higher associahedron $\Sigma_{A,k}$, while square moves connecting them correspond to the edges of $\Sigma_{A,k}$.

Hypersimplicial Subdivisions

Jorge Alberto Olarte

(joint work with Francisco Santos)

The protagonists of this talk are hypersimplicial subdivisions, defined as follows. Let A be a set of n points affinely spanning $\mathbb{R}^d$. Let Δ_n be the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$. Consider the linear projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ sending the vertices of Δ^n to the points in A. (We implicitly consider the points in A labelled by $[n]$, so that π sends e_i to the point labelled by i). Let $\Delta_n^{(k)} := k\Delta_n \cap [0,1]^n$ be the standard hypersimplex and $A^{(k)}$ the image of the vertices of $\Delta_n^{(k)}$ under π (so that points in $A^{(k)}$ are labelled by k-subsets of $[n]$). A *hypersimplicial subdivision* of $A^{(k)}$ is a polyhedral subdivision of $\operatorname{conv}(A^{(k)})$ such that every face of the subdivision is the image of a face of $\Delta_n^{(k)}$ under π. Put differently, we call hypersimplicial subdivisions the π-induced subdivisions of the projection $\pi : \Delta_n^{(k)} \to \operatorname{conv}(A^{(k)})$, as introduced in [BS92, BKS94] (see also [Rei99, DLRS10]).

One reason to study such subdivisions comes from the case where $A \subset \mathbb{R}^2$ are the vertices of a convex polygon. Galashin [Gal18] shows that in this case fine hypersimplicial subdivisions, which we call *hypertriangulations*, are in bijection with maximal collections of chord-separated k-sets. These, in turn, correspond to reduced plabic graphs, [OPS15] which are a fundamental tool in the study of the positive Grassmannian [Pos06, Pos19]. Moreover, the poset of hypersimplicial subdivisions is isomorphic to the poset of complete reduced Grassmannian graphs introduced in [Pos19].

It is of interest the more general case where A are the vertices of a cyclic polytope $\mathcal{C}_{n,d} \subset \mathbb{R}^d$. (The n-gon is the case $d = 2$). In [Pos19, Problem 10.3] Postnikov asks the *generalized Baues problem* for this scenario; that is, he asks whether the poset of hypersimplicial subdivisions of $\mathcal{C}_{n,d}^{(k)}$ has the homotopy type of a $(n-d-2)$-sphere. For $k = 1$ this was shown to have a positive answer by Rambau and Santos [RS00]. For $d = 2$, Balitskiy and Wellman show the poset to be simply connected and again ask the Baues question for it ([BW19, Theorem 6.4 and Question 6.1]). We here give the answer to this:

Theorem 1. *Let P_n be the vertices of any convex n-gon. The poset of hypersimplicial subdivisions $\mathcal{B}(\Delta_n^{(k)} \to P_n^{(k)})$ retracts onto the poset of* coherent *hypersimplicial subdivisions. In particular, it has the homotopy type of an $(n-4)$-sphere.*

[Pos19, Problem 10.3] also asks for which values of the parameters can all hypersimplicial subdivisions of $\mathcal{C}_{n,d}^{(k)}$ be lifted to zonotopal tilings of the cyclic zonotope. This was already known to be false for $d = 1$ [Pos19, Example 10.4] and we generalize the counterexamples to every odd dimension:

Theorem 2. *Consider the cyclic polytope $\mathcal{C}_{n,d} \subset \mathbb{R}^d$ for odd d and $n \geq d+3$. Then, for every $k \in [2, n-2]$ there exist hypersimplicial subdivisions of $\mathcal{C}_{n,d}^{(k)}$ that do not extend to zonotopal tilings of the cyclic zonotope $Z(\mathcal{C}_{n,d})$.*

In contrast, Galashin [Gal18] showed that the answer to Postnikov's question is positive in dimension two for *hypertriangulations*, a result that was generalized to all hypersimplicial *subdivisions* by Balitskiy and Wellman [BW19, Lemma 6.3].

The poset of coherent hypersimplicial subdivisions of any A is isomorphic to the face poset of a polytope, a particular case of a fiber polytope. When $k = 1$ this is just the secondary polytope of A, so for $k > 1$ we call it the *k-th hypersecondary polytope of A*. We study hypersecondary polytopes for any $A \subset \mathbb{R}^d$ and $k \leq d+1$. Specifically, we show that these polytopes are normally equivalent to the Minkowski sum of certain faces of the secondary polytope of A. By symmetry, an analogue statement holds for $n - d - 1 \leq k < n$.

Theorem 3. *Let $A \subseteq \mathbb{R}^d$ be a configuration of size n and $k \in [d+1]$. Let $s = \max(n-k+1, d+2)$. The hypersecondary polytope $\mathcal{F}^{(k)}(A)$ is normally equivalent to the Minkowski sum of the secondary polytopes of all subsets of A of size s.*

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Fibers of maps to totally nonnegative spaces

Patricia Hersh

Sergey Fomin and Michael Shapiro proved that the totally nonnegative real part of the unipotent radical of a Borel in a semisimple, simply connected algebraic group has a cell decomposition with Bruhat order as its poset of closure relations, They conjectured that (after deconing) this was a regular CW complex homeomorphic to a closed ball.

Much of the interest in these spaces comes from their interpretation as the images of maps $f_{(i_1,\dots,i_d)}$ for $(i_1,\dots,i_d)$ a reduced word. Specifically, these maps are related to Lusztig's theory of canonical bases as well as connections to mutation in cluster algebras. In type A, they are defined as follows. Let $x_i(t) = I_n + E_{i,i+1}$ for I_n the $n \times n$ identity matrix and $E_{i,i+1}$ the $n \times n$ matrix with entires of 0 everywhere except for a 1 in row i and column $i+1$. The map $f_{(i_1,\dots,i_d)}$ sends any $(t_1,\dots,t_d) \in \mathbb{R}^d_{\geq}$ to $x_{i_1}(t_1)\cdots x_{i_d}(t_d)$ where $\mathbb{R}_{\geq}$ denotes the nonnegative real numbers. The matrix $x_i(t)$ may be interpreted as an exponentiated Chevalley generator; this leads to the nonnegative real relations amongst exponentiated Chevalley generators precisely corresponding to pairs of points in $\mathbb{R}^d_{\geq 0}$ being in the same fiber of $f_{(i_1,\dots,i_d)}$.

The Fomin-Shapiro conjecture was indeed proven about 5 years ago in [3]. In new joint work with Jim Davis and Ezra Miller in [2], we now study the fibers of these maps. The project from [2] under discussion here may be seen as the follow up to [3], turning attention now from the images of these maps now to the fibers.

We show that the poset of closure relations for each fiber is that of a known regular CW complex, namely the interior dual block complex of a subword complex This builds upon an earlier connection to subword complexes in [1]. The subword complexes were introduced by Knutson and Miller in conjunction with their work on matrix Schubert varieties in [4] and proven in [5] to be shellable topological manifolds with boundary that are all balls or spheres. We prove for subword complexes that are balls (rather than spheres) that their interior dual block complexes are contractible. We also show that all of the subword complexes whose interior dual block complexes describe fibers are ones that are balls rather than spheres. Thus, we obtain a good topological understanding of the combinatorial model for the fibers, suggestive of the structure of the fibers themselves.

As our main topological result, we give a cell decomposition of each fiber $f^{-1}_{(i_1,\ldots,i_d)}(M)$. We do this in a way that parametrizes the points in any given open cell in this cell decomposition of the fiber. More specifically, for each strata σ we give a homeomorphism between $[0,1)^{\dim \sigma}$ and the union of open cells that are all in the closure of σ and all have a fixed vertex in $\overline{\sigma}$ in their closure. We conjecture the much stronger statement that these cell decompositions are regular CW complexes homeomorphic to closed balls.

In the course of proving our results, we also develop many new properties of the Demazure product, or equivalently of the product in the 0-Hecke algebra of a finite Coxeter group. Just as one has reduced and nonreduced words in a Coxeter group, these same notions exist when using the Demazure product on simple reflections. Under the Demazure product, multiplying w on the right by s_i which would increase length yields ws_i whereas multiplying w on the right by s_i that decreases length yields w; put another way, simple reflection which would decrease length under the usual Coxeter-theoretic product instead do nothing under the Demazure product. In a (reduced or nonreduced) word, one may speak of the last letter i_d in a word $(i_1,\ldots,i_d)$ as being redundant if the Demazure product of $s_{i_1}\cdots s_{i_{d-1}}$ with s_{i_d} equals $s_{i_1}\cdots s_{i_{d-1}}$; we call the last letter nonredundant otherwise.

Our topological cell decomposition proofs are by induction. We prove that multiplying on the right by a nonredundant letter i_{d+1} yields a fiber $f^{-1}_{(i_1,\ldots,i_d,i_{d+1})}(M)$ homeomorphic to a fiber $f^{-1}_{(i_1,\ldots,i_d)}(M')$ of strictly shorter length. On the other hand, we show for i_{d+1} that the last parameter can take a half-open interval $[0,k_{d+1})$ of values. All these results require considerable understanding of the Demazure product. As the base case of our induction, for a reduced word $(i_1,\ldots,i_d)$ each nonempty fiber consists of a single point. This in fact leads to a very natural connection to subword complexes, since choosing a subword of $(i_1,\ldots,i_d)$ amounts to choosing which of the parameters in $(t_1,\ldots,t_d)$ should be positive (as opposed to 0). It turns out that the subwords yielding points in a fiber $f^{-1}_{(i_1,\ldots,i_d)}(M)$ for M in the open cell indexed by $u\in W$ will be exactly those subwords with the same Demazure product u that indexes the open cell of M.

Finally, we suggest the question of whether analogous results hold for other maps of interest including Postnikov's measurement map to the totally nonnegative real part of the Grassmannian.

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Hypertrees

Nathan Linial
(joint work with Amir Dahari, Roy Meshulam, Ilan Newman, Yuval Peled, Yuri Rabinovich and Mishael Rosenthal)

It is an elementary theorem that the following three conditions are equivalent for an n-vertex graph G with $n-1$ edges: (i) G is connected, (ii) It is acyclic. (iii) It is collapsible (see below).

In search of a d-dimensional counterpart of this, we consider a n-vertex d-dimensional simplicial complex X with a full $(d-1)$-dimensional skeleton. The analog of connectivity, resp. acyclicity is the vanishing of X's $(d-1)$-st resp. d-th homology. The largest number of d-faces in a $\mathbb{Q}$-acyclic n-vertex d-dimensional complex with a full $(d-1)$-skeleton is $\binom{n-1}{d}$. A *hypertree*, as defined in 1983 by Kalai [3] is such a complex with $\binom{n-1}{d}$ d-faces.

An *elementary collapse* in a graph is a step in which we remove a vertex of degree 1 and the unique edge that contains it. A graph is *collapsible* if it is possible to eliminate all its edges in a series of elementary collapses. If some $(d-1)$-dimensional face τ in X is contained in a unique d-face σ, then in the corresponding *elementary collapse* we remove both τ and σ from X. We say that X is *d-collapsible* if it is possible to eliminate all its d-faces in a series of elementary collapses. It is easy to see that for all $d \geq 1$, a d-dimensional d-collapsible complex is acyclic, but for $d \geq 2$ the reverse implication does not hold. This suggests the following:

Task 1. *Quantify the difference between d-collapsibility and acyclicity?*

The following appealing problem from Kalai's original paper is still open. This conjecture is supported by ample numerical evidence.

Conjecture 2. *For $d \geq 2$ only a vanishingly small (in n) fraction of the n-vertex d-hypertrees are collapsible.*

The Linial-Meshulam $X_d(n,p)$ model [4], generates n-vertex d-dimensional simplicial complexes with a full $(d-1)$-skeleton by picking every d-face independently with probability p (For $d = 1$ this coincides with Erdős Rényi's random $G(n,p)$ graphs). We have dedicated several papers to the study of these random complexes (see [7] for a survey). Among our main discoveries, we showed that the threshold for d-collapsibility is $p = (1 + o_d(1))\frac{\log d}{n}$, much below $(1 + o_d(1))\frac{d}{n}$, the threshold for d-acyclicity [8]. This expresses the difference between d-collapsibility and acyclicity. We ask:

Problem 3. *Find explicitly defined families of non-collapsible hypertrees.*

The smallest, and probably the best known such example is the 6-point triangulation of the projective plane. In [5] we introduced a new infinite family of hypertrees called *sum complexes*. It contains infinitely many collapsible members as well as infinitely many non-collapsible ones, and both sub-families are well-characterized, and collapsibility in this model is rare.

The *d-star* on vertex set $[n]$ is arguably the simplest hypertree. Its d-faces are all sets of size $d + 1$ that contain the vertex n. A d-star is clearly collapsible. It is an old truism in combinatorics that contrary to popular belief, it is not always easy *to find hay in a haystack*. Indeed, while randomly generated hypertrees do not tend to be collapsible, we still do not have a satisfactory supply of non-collapsible hypertrees.

The n-vertex trees are the bases of the graphic matroid of the complete graph K_n. Likewise, the d-dimensional hypertrees are also the bases of a matroid. In contrast, we know that this is *not* the case for collapsible trees. We wonder whether collapsible trees exhibit some good matroid-like behavior. Also, in light of the recent advances in random generation of matroid bases [1, 2], it is interesting to seek efficient ways to sample uniformly collapsible hypertrees.

Even very basic things regarding hypertrees are still unknown. For example - how many they are. We certainly do not expect an exact answer like Cayley's formula for the number of trees, but there are many concrete questions that suggest themselves here, see [9].

The analogy with 1-dimensional trees is very suggestive. E.g., is there a good notion of a d-hyperpath? With my MSc student A. Dahari, we study the following n-vertex d-complexes with a full $(d-1)$-skeleton. Fix some $c \neq 0$ and declare the $(d+1)$-set $\{x_1, \ldots, x_d, y\}$ a d-face iff $\sum x_i + cy \equiv 0 \pmod{n}$. It is easy to see that no $(d-1)$-face is covered more than $(d+1)$ times, and for many choices of n and c this yields a hypertree. The full picture still eludes us.

The *shadow* of a d-complex X is the set of all those d-faces, not in X whose addition to X creates new d-cycles. This notion plays a crucial role in [8]. Surprisingly, as discovered in [6], if n is prime and 2 is a primitive element mod n, then there exists a 2-dimensional *almost tree* F, i.e., an acyclic 2-complex whose number of 2-faces is $\binom{n-1}{2} - 1$, that *has no shadow.* Namely, by adding to F any

2-face not in it, a 2-hypertree is attained. Much remains unknown, in particular the situation in dimensions $d \geq 3$ is completely obscure at present.

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Configuration spaces of disks in an infinite strip

Matthew Kahle

(joint work with Hannah Alpert and Robert MacPherson)

We study the topology of configuration spaces $\mathcal{C}(n, w)$ of n non-overlapping disks of unit diameter in an infinite strip of width w. In other words, for integers $n, w \geq 0$ we define

$$\begin{aligned}\mathcal{C}(n, w) = \{&(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n} : \\ &(x_i - x_j)^2 + (y_i - y_j)^2 \geq 1 \text{ for every } i \neq j, \text{ and} \\ &1/2 \leq y_i \leq w - 1/2 \text{ for every } i.\}\end{aligned}$$

Our main result gives fairly sharp estimates for the Betti numbers as the number of disks tends to infinity. We use the notation $f = \Theta(g)$ to indicate that there exist positive constants c_1, c_2 such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all sufficiently large n.

Theorem 1 (Asymptotic rate of growth of the Betti numbers as $n \to \infty$)**.**

(1) If $w \geq 2$ and $0 \leq j \leq w - 2$ then the inclusion map $i : \mathcal{C}(n, w) \to \mathcal{C}(n, \mathbb{R}^2)$ induces an isomorphism on homology

$$i_* : \; H_j[\mathcal{C}(n, w)] \to H_j[\mathcal{C}(n, \mathbb{R}^2)].$$

So if $n \to \infty$ then the asymptotic rate of growth is given by

$$\beta_j \left[\mathcal{C}(n, w)\right] = \Theta\left(n^{2j}\right).$$

(2) *If $w \geq 2$ and $j \geq w-1$ then write $j = q(w-1)+r$ with $q \geq 1$ and $0 \leq r < w-1$. Then we have that*

$$\beta_j\left[\mathcal{C}(n,w)\right] = \Theta\left((q+1)^n n^{qw+2r}\right).$$

If $w = 1$ and $j = 0$, then $\beta_0 = n!$.

(3) *If either $w = 0$, or $w = 1$ and $j \geq 1$, then $\beta_j = 0$.*

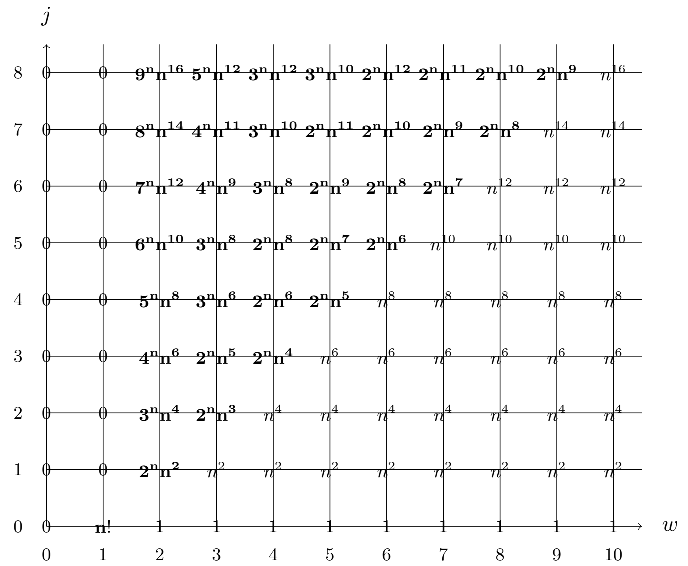

FIGURE 1. Theorem 1 describes the rate of growth of $\beta_j[\mathcal{C}(n,w)]$, for fixed j and w, as $n \to \infty$. The results are up to a constant factor; e.g. $\beta_8[\mathcal{C}(n,3)] = \Theta\left(5^n n^{12}\right)$.

Configuration spaces of disks arise naturally as the phase space of a 2-dimensional "hard-spheres" system, so are of interest in physics as well. See, for example, the discussion of hard disks in a box by Diaconis in [6], and the review of the physics literature by Carlsson et al. in [5].

Inspired by the statement of Theorem 1, we introduce the notion of "homological solid, liquid, and gas" regimes in the (w, j) plane.

- We define the "homological solid" phase to be wherever homology is trivial. The motivation for this definition is that one expects in a crystal phase, things are fairly rigid and that the configuration space is simple.
- We define the "homological gas" phase to be where homology agrees with the configuration space of points in the plane. In other words, through the lens of this homology group, the particles are indistinguishable from points, corresponding to the assumption of atoms acting as point particles in an ideal gas.
- Finally, we define the "homological liquid" phase to be everything else. This is the most interesting regime topologically, and we were somewhat surprised to find that there is a lot of homology. Another physical metaphor for the homological liquid regime, suggested to us by Jeremy Mason, is a turbulent fluid.

Most of the work is in estimating the Betti numbers in the homological liquid regime. For lower bounds, we use the duality between the homology of $\mathcal{C}(n, w)$ and its homology with closed support. For upper bounds, we first prove that $\mathcal{C}(n, w)$ is homotopy equivalent to a cell complex $\text{cell}(n, w)$, and then apply discrete Morse theory. In the end, the lower and upper bounds match up to a constant factor.

The following theorem describes the shapes of the regimes for every n. We note that the boundary between solid and liquid regimes is more interesting for finite n than it appears to be in Theorem 1.

Theorem 2 (The phase portrait for every n)**.**

(1) If $w \geq 2$ and $0 \leq j \leq w-2$, then the inclusion map $i : \mathcal{C}(n, w) \to \mathcal{C}(n, \mathbb{R}^2)$ induces an isomorphism on homology

$$i_* : \ H_j[\mathcal{C}(n, w)] \to H_j[\mathcal{C}(n, \mathbb{R}^2)].$$

If $w \geq n$, then $\mathcal{C}(n, w)$ is homotopy equivalent to $\mathcal{C}(n, \mathbb{R}^2)$.

(2) If $1 \leq w \leq n-1$ and $w-1 \leq j \leq n - \lceil n/w \rceil$ then $H_j(\mathcal{C}(n, w)) \neq 0$, but the inclusion map $i : \mathcal{C}(n, w) \to \mathcal{C}(n, \mathbb{R}^2)$ does not induce an isomorphism on homology

$$i_* : \ H_j[\mathcal{C}(n, w)] \to H_j[\mathcal{C}(n, \mathbb{R}^2)].$$

(3) If either $w = 0$, or $w \geq 1$ and $j \geq n - \lceil n/w \rceil + 1$, then

$$H_j[\mathcal{C}(n, w)] = 0.$$

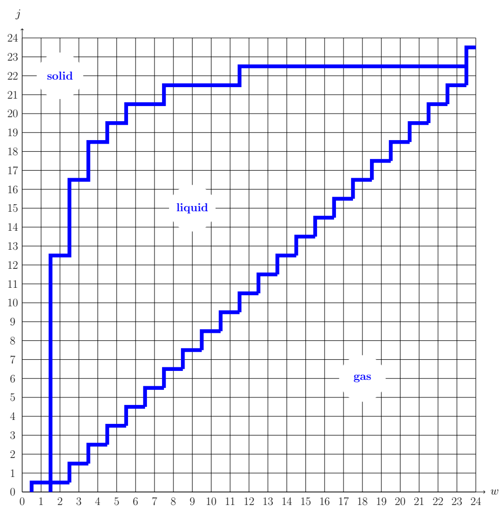

FIGURE 2. Theorem 2 describes the shapes of the homological solid, liquid, and gas regimes for every n. We illustrate the case $n = 24$.

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Diameter bounds on cocircuit graphs of oriented matroids

Steven Klee

(joint work with Ilan Adler, Jesús A. De Loera, and Zhenyang Zhang)

Let $\mathcal{M}$ be an oriented matroid of rank r on a ground set with n elements. The canonical example of such a structure comes from a set $E = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of nonzero vectors that span $\mathbb{R}^r$. If $\mathbf{c} \in \mathbb{R}^r$ is an arbitrary cost vector, we can record the signs of the inner products of $\mathbf{c}$ with the vectors $\mathbf{v}_i$ to obtain a *signed covector* $X = (\text{sign}(\mathbf{c}^T\mathbf{v}_i))_{i=1}^n$. The signed covectors of minimal nonempty support are called *signed cocircuits*, which we denote as $\mathcal{C}^*$.

The combinatorial structure of the pair $\mathcal{M} = (\mathcal{C}^*, E)$ is formalized in the definition of an oriented matroid. We refer to the books of Björner et al. [1] and Ziegler [5] for precise definitions.

Geometrically, let H_i be the hyperplane whose normal vector is $\mathbf{v}_i$ and orient H_i so that its positive side contains $\mathbf{v}_i$. Intersecting the hyperplanes $\{H_i\}_{i=1}^n$ with the unit sphere $\mathbb{S}^{r-1}$ gives a regular cell decomposition of the sphere whose faces are in bijective correspondence with the signed covectors of $\mathcal{M}$ and whose vertices are in bijective correspondence with the signed cocircuits of $\mathcal{M}$.

Folkman and Lawrence [3] showed that for any arbitrary oriented matroid of rank r on n elements, there exists an arrangement of *pseudospheres* $s_1, \ldots, s_n$ on $\mathbb{S}^{r-1}$ in which the cells of the induced decomposition correspond to signed cocircuits and the vertices correspond to signed covectors. The *cocircuit graph* of an oriented matroid $\mathcal{M}$, which we denote as $G^*(\mathcal{M})$, is the graph of the 1-skeleton of this pseudosphere arrangement.

In the remainder of this abstract, we will use the following notation. If X is a signed cocircuit in a uniform oriented matroid $\mathcal{M}$, then $X^+ = \{i \ : \ X_i = +\}$. We define X^- and X^0 similarly. If X and Y are signed cocircuits, the *separating set* $S(X,Y) = \{i \ : \ \{X_i, Y_i\} = \{+,-\}\}$. In other words, $S(X,Y)$ is the set of pseudospheres (or hyperplanes) that separate X from Y.

1. The problem

We are motivated by the following problem, introduced by Fukuda and Terlaky.

Conjecture 1. *Let $\mathcal{M}$ be an oriented matroid of rank r on n elements. Then*

$$\operatorname{diam}(G^*(\mathcal{M})) \leq n - r + 2. \tag{1}$$

This conjecture bears a striking resemblance to the famous Hirsch conjecture (which was disproved by Santos [4]), and with good reason. If $P \subseteq \mathbb{R}^d$ is a polytope with n facets, we can lift P into the $x_{d+1} = 1$ coordinate hyperplane of $\mathbb{R}^{d+1}$ so that its supporting (affine) hyperplanes become central hyperplanes. Intersecting this hyperplane arrangement with $\mathbb{S}^d$ gives an oriented matroid (or pseudosphere arrangement) in which one of the cells is isomorphic P itself.

However, substituting $r = d+1$ into Eq. (1) gives $n - r + 2 = n - d + 1$, which differs from the conjectured Hirsh bound by 1. The reason for this is that each signed cocircuit X has an antipodal cocircuit $-X$, which cannot be reached in fewer than $n - r + 2$ steps, as we will see below.

2. Results

One of the first reductions made in studying the Hirsch conjecture was a reduction to simple polytopes. These are d-polytopes in which each vertex is supported by exactly d facets. We make a similar reduction to *uniform* oriented matroids in this context. More precisely, $\mathcal{M}$ is uniform if $|X^0| = r - 1$ for each signed cocircuit $X \in \mathcal{C}^*$.

Lemma 2. *Let $\mathcal{M}$ be an oriented matroid of rank r on n elements. Then there exists a uniform oriented matroid $\mathcal{M}'$ of rank r on n elements such that*

$$\operatorname{diam}(G^*(\mathcal{M})) \leq \operatorname{diam}(G^*(\mathcal{M}')).$$

We begin with a few incremental results.

Lemma 3. *Let $\mathcal{M}$ be a uniform oriented matroid of rank r on n elements, and let $X, Y \in \mathcal{C}^*(\mathcal{M})$. Then*

$$\operatorname{dist}(X,Y) \geq \begin{cases} |S(X,Y)| + |X^0 \setminus Y^0| & \text{if } X \neq -Y \\ n - r + 2 & \text{if } X = -Y. \end{cases} \tag{2}$$

Moreover, the inequality in Eq. (2) is an equality if $|X^0 \setminus Y^0| \leq 1$ (and in particular if $X = -Y$).

This lemma shows that the conjectured upper bound in Eq. (1) cannot be improved. Optimistically, one could hope that perhaps $\mathrm{dist}(X, Y) \leq n - r + 1$ if X and Y are not antipodal cocircuits, but that is quickly shown to be false by considering Santos's counterexample to the Hirsch conjecture.

Now we move on to our main results. We begin with oriented matroids in which n and r are small.

Theorem 4. *Let $\mathcal{M}$ be a uniform oriented matroid of rank $r \leq 3$ on n elements. Then* $\mathrm{diam}(G^*(\mathcal{M})) \leq n - r + 2$.

We have verified the main conjecture for small oriented matroids using the database of Finschi and Fukuda [2]

Theorem 5. *Let $\mathcal{M}$ be a uniform oriented matroid of rank r on n elements. If $n \leq 8$ or $n = 9$ and $r \neq 5$, then*

$$\mathrm{diam}(G^*(\mathcal{M})) \leq n - r + 2.$$

We conclude with our main result, which gives a quadratic bound on the diameter of the cocircuit graph of an oriented matroid. This bears a stark contrast to the best-known upper bounds on polytope diameters, which are linear in fixed dimension but exponential in the dimension.

Theorem 6. *Let $\mathcal{M}$ be a uniform oriented matroid of rank r on n elements. Let $X, Y \in \mathcal{C}^*(\mathcal{M})$ such that $X \neq -Y$. Then*

$$\mathrm{dist}(X, Y) \leq \left\lceil \frac{|X^0 \setminus Y^0|}{2} \right\rceil (n - r + 1) \leq \left\lceil \frac{r-1}{2} \right\rceil (n - r + 1).$$

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Introduction to web algebras and local intersection forms

Ben Elias

The goal of this talk is to explain and motivate my recent Clasp Conjecture. This conjecture is situated in the realm of combinatorics: certain numbers, which can be computed by manipulating labeled graphs, are conjectured to be equal to products of ratios of quantum numbers, where the product is indexed by various positive roots in a root system. However, each of these numbers happens to be the entry is a 1×1 matrix describing a local intersection form of interest in representation theory.

Let us give the simplest example, which relates to the representation theory of $\mathfrak{sl}_2$. Given $m, n \in \mathbb{Z}_{\geq 0}$, an (n, m) *crossingless matching* is an unoriented 1-manifold with boundary embedded in the planar strip $\mathbb{R} \times [0, 1]$, with no closed components (i.e. circles), and whose boundary consists of n points on the bottom $\mathbb{R} \times \{0\}$ and m points on the top $\mathbb{R} \times \{1\}$. Each component of the 1-manifold has exactly two points on its boundary, which are considered to be matched to each other. When two points on the top boundary are matched it creates a *cup*, while two points on bottom produce a *cap*; when a point on bottom matches a point on top it creates a *thru-strand*. A crossingless matching is a *cap diagram* if it has no cups: each of the m points on top is matched to some point on the bottom.

One can place a form on the $\mathbb{Z}$-span of the (n, m) cap diagrams by defining $\langle a, b \rangle$ as follows: flip b upside-down to obtain a cap diagram, then stack a on top of b to obtain a 1-manifold with both m top and bottom boundary points. If this manifold does not have m thru-strands, set $\langle a, b \rangle = 0$. If it has m thru-strands, then set $\langle a, b \rangle = (-2)^{\text{number of circles}}$. We encourage the reader to compute the form on the space of $(5, 3)$ cap diagrams, obtaining minus the Cartan matrix of A_4; this particular form is negative definite! Meanwhile, the form on the space of $(5, 1)$ cap diagrams is positive definite.

Let us motivate this mysterious-looking form. If L is a simple object in a $\mathbb{C}$-linear abelian category, and X is some other object, then one can consider the *local intersection pairing* of L inside X. It is the pairing between vector spaces $\mathrm{Hom}(L, X)$ and $\mathrm{Hom}(X, L)$ given by composition:

$$\mathrm{Hom}(X, L) \times \mathrm{Hom}(L, X) \to \mathrm{End}(L) = \mathbb{C}.$$

Dual bases for this pairing give the inclusion and projection maps of orthogonal copies of L living as direct summands inside X. Thus the multiplicity of L as a summand of X is the rank of the local intersection pairing. In the presence of a duality functor which gives an isomorphism $\mathrm{Hom}(L, X) \cong \mathrm{Hom}(X, L)$, this pairing can be transferred to a form on $\mathrm{Hom}(X, L)$, the *local intersection form*.

In the talk we explain why the form on $(5, 3)$ cap diagrams is actually an example of a local intersection form. One can situate this form within the Temperley-Lieb category, a monoidal category which describes all morphisms between tensor products of the standard two-dimensional representation V of $\mathfrak{sl}_2$. The $(5, 3)$ cap diagrams correspond to a basis of $\mathrm{Hom}(V^{\otimes 5}, S^3 V)$, and the rank of this form is the multiplicity of the simple $S^3 V$ inside the tensor product $V^{\otimes 5}$. In a semisimple category the local intersection form must always be non-degenerate (because the dimension of $\mathrm{Hom}(X, L)$ is equal to the number of copies of L inside X), as we saw in this example. The definiteness properties we also saw come from the geometric Satake equivalence and the Hodge-Riemann bilinear relations, and I do not know of any combinatorial proofs.

However, the form on cap diagrams was a $\mathbb{Z}$-bilinear form, and one can ask about its rank in various positive characteristics. This actually describes an analogous multiplicity in modular representation theory, where the answers (beyond $\mathfrak{sl}_2$) are largely unknown and are a major focus of modern research in the field. Finding a combinatorial description of the invariant factors of various local intersection forms is a huge open problem.

Let us return to the $\mathbb{C}$-linear setting. Let $L_k := S^k V$ denote the simple representation of $\mathfrak{sl}_2$ with highest weight k. The for all $k \geq 1$ we have

$$L_k \otimes V \cong L_{k+1} \oplus L_{k-1}.$$

Thus there is a one-dimensional Hom space $\mathrm{Hom}(L_k \otimes V, L_{k-1})$ (which happens to have a preferred basis) whose local intersection form is a 1×1 matrix, i.e. a number. We demonstrate how to compute this number inductively: it is $\frac{-(k+1)}{k}$. Knowing this particular local intersection form gives enough data to inductively define the *Jones-Wenzl projectors* or *clasps*, which are the idempotents inside $V^{\otimes k}$ which project to the top summand L_k. It is still not entirely obvious what the combinatorics are which lead to the answer $\frac{-(k+1)}{k}$.

There is a general philosophy: if you have an algebraic category of interest (example: complex representations of $\mathfrak{sl}_2$) then you should choose (wisely) a combinatorial subcategory (example: tensor products of V). Now there is hope of finding a presentation for this combinatorial subcategory by generators and relations (example: the Temperley-Lieb category), and finding an explicit basis (example: crossingless matchings). There is a systematic way of doing this, related to the combinatorics of decomposing tensor products. Once you have generators and relations, you have an integral form of the category, which can be specialized to finite characteristic where the situation is much harder and more interesting. This philosophy has worked in numerous settings: semisimple lie algebras of rank two due to pioneering work of Kuperberg, representations of $\mathfrak{gl}_n$ due to Cautis-Kamnitzer-Morrison with the basis due to myself, and other categories involved in the categorification of quantum groups and Hecke algebras, due to many authors. For references and more historical discussion see [1]. The limits of this technique have not yet been reached. Many of the same questions we asked above about crossingless matchings have precise analogs in other contexts.

Now we pose our clasp conjecture, which relates to the representation theory of $\mathfrak{gl}_n$. Let $V = \mathbb{C}^n$ be the standard representation. Then $\Lambda^k V$ is a fundamental representation for any $0 < k < n$. Let μ be a weight appearing in the weight decomposition of $\Lambda^k V$, and let λ be an arbitrary dominant weight. If $\lambda + \mu$ is also dominant then $L_{\lambda+\mu}$ is a direct summand of $L_\lambda \otimes \Lambda^k V$. Thus there is a one-dimensional local intersection form of $L_{\lambda+\mu}$ inside $L_\lambda \otimes \Lambda^k V$, which (after a preferred choice of basis) becomes a number that we denote $\kappa_{\lambda,\mu}$. We conjecture that

$$\kappa_{\lambda,\mu} = \Pi \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \lambda + \rho + \mu, \alpha \rangle}.$$

Here, ρ is the half sum of the positive roots, and the product is a product over certain positive roots α depending on μ: they are precisely those roots for which the denominator is exactly one less than the numerator. For example, when $n = 2$ we have $\kappa_{k,-1} = \frac{k+1}{k}$, which matches the local intersection form computed above (the sign is accounted for by a difference between $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$). We would love to see a combinatorial explanation for this conjecture.

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The algebraic geometry of Kazhdan-Lusztig-Stanley polynomials

Nicholas Proudfoot

Let P be a finite poset. Let

$$I(P) := \prod_{x \leq y} \mathbb{Z}[t].$$

For any $f \in I(P)$ and $x \leq y \in P$, let $f_{xy}(t) \in \mathbb{Z}[t]$ denote the corresponding component of f. The group $I(P)$ admits a ring structure with product given by convolution:

$$(fg)_{xz}(t) := \sum_{x \leq y \leq z} f_{xy}(t) g_{yz}(t).$$

Let $r : P \to \mathbb{Z}$ be a function with the property that, if $x < y$, then $r_{xy} := r(y) - r(x) > 0$. Let $\mathscr{I}(P) \subset I(P)$ denote the subring of functions f with the property that the degree of $f_{xy}(t)$ is less than or equal to r_{xy} for all $x \leq y$. The ring $\mathscr{I}(P)$ admits an involution $f \mapsto \bar{f}$ defined by the formula

$$\bar{f}_{xy}(t) := t^{r_{xy}} f_{xy}(t^{-1}).$$

An element $\kappa \in \mathscr{I}(P)$ is called a ***P*-kernel** if $\kappa_{xx}(t) = 1$ for all $x \in P$ and $\kappa^{-1} = \bar{\kappa}$. Let

$$\mathscr{I}_{\frac{1}{2}}(P) := \left\{ f \in \mathscr{I}(P) \;\middle|\; f_{xx}(t) = 1 \text{ for all } x \in P \text{ and } \deg f_{xy}(t) < r_{xy}/2 \text{ for all } x < y \in P \right\}.$$

Various versions of the following theorem appear in [Sta92, Corollary 6.7], [Dye93, Proposition 1.2], and [Bre99, Theorem 6.2]; see [Pro18, Theorem 2.2] for this precise statement.

Theorem 1. *If $\kappa \in \mathscr{I}(P)$ is a P-kernel, there exists a unique pair of functions $f, g \in \mathscr{I}_{\frac{1}{2}}(P)$ such that $\bar{f} = \kappa f$ and $\bar{g} = g\kappa$.*

The polynomials $f_{xy}(t)$ and $g_{xy}(t)$ are called right and left Kazhdan-Lusztig-Stanley polynomials, or **KLS-polynomials** for short. There are a number of special cases in which these polynomials have been studied.

- Let W be a Coxeter group, equipped with the Bruhat order and the rank function given by the length of an element of W. The classical R-polynomials $\{R_{vw}(t) \mid v \leq w \in W\}$ form a W-kernel, and the classical Kazhdan-Lusztig polynomials $\{f_{xy}(t) \mid v \leq w \in W\}$ are the associated right KLS-polynomials. If W is finite, then there is a maximal element $w_0 \in W$, and $g_{vw}(t) = f_{(w_0 w)(w_0 v)}(t)$.

- Let P be the poset of faces of a polytope Δ, with weak rank function given by relative dimension (where $\dim \emptyset = -1$). Then the function $\kappa_{xy}(t) = (t-1)^{r_{xy}}$ is a P-kernel, and $g_{\emptyset\Delta}(t)$ is called the g-polynomial of Δ [Sta92, Example 7.2]. The dual polytope Δ^* has the property that its face poset is opposite to P, and this implies that $f_{\emptyset\Delta}(t)$ is equal to the g-polynomial of Δ^*.
- For any P, define $\zeta \in \mathscr{I}(P)$ by the formula $\zeta_{xy}(t) = 1$ for all $x \leq y \in P$. Then the characteristic polynomial $\chi := \zeta^{-1}\bar{\zeta}$ is a P-kernel. The associated left KLS-polynomials are identically 1, but the right KLS-polynomials can be very interesting! In particular, each coefficient of $f_{xy}(t)$ can be expressed as alternating sums of multi-indexed Whitney numbers for the interval $[x, y] \subset P$ [PXY18, Theorem 3.3]. If P is the lattice of flats of a matroid M with the usual rank function, with minimum element 0 and maximum element 1, then $f_{01}(t)$ is called the Kazhdan-Lusztig polynomial of M [EPW16].

Each of these families of examples has a subfamily in which the KLS-polynomials have a cohomological interpretation.

- Let G be a split reductive algebraic group. Let $B, B^* \subset G$ be Borel subgroups with the property that $T := B \cap B^*$ is a maximal torus. Let $W := N(T)/T$ be the Weyl group. For all $w \in W$, let
$$V_w := \{gB \mid g \in BwB\}$$
be the corresponding Schubert cell in the flag variety G/B. For any $v \leq w$, the Kazhdan-Lusztig polynomial $f_{v,w}(t)$ is equal to the Poincaré polynomial for the cohomology of the stalk of the intersection cohomology sheaf IC_{V_w} at a point of V_v [KL80, Corollary 4.8].
- Let Δ be a rational polytope with associated projective toric variety $X(\Delta)$, and let $Y(\Delta)$ denote the affine cone over $X(\Delta)$. Then the g-polynomial $g_{\emptyset\Delta^*}(t) = f_{\emptyset\Delta}(t)$ is equal to the Poincaré polynomial for the intersection cohomology of $Y(\Delta)$ [DL91, Theorem 6.2], [Fie91, Theorem 1.2], or equivalently the Poincaré polynomial for the stalk of $\mathrm{IC}_{Y(\Delta)}$ at the cone point.
- Let $\mathcal{A}$ be a collection of nonzero linear forms on a vector space V, and let M be the associated matroid. Let $R_{\mathcal{A}}$ be the Orlik-Terao algebra, which is the subalgebra of rational functions on V generated by the reciprocals of the linear forms. Then the Kazhdan-Lusztig polynomial of M is equal to the Poincaré polynomial for the intersection cohomology of $\operatorname{Spec} R_{\mathcal{A}}$ [EPW16, Theorem 3.10], or equivalently the Poincaré polynomial for the stalk of $\mathrm{IC}_{R_{\mathcal{A}}}$ at the cone point.

Each of these statements was proved independently, but it is in fact possible to prove all three in a uniform way. Suppose that we have a variety Y over $\mathbb{F}_q$ and a stratification
$$Y = \bigsqcup_{x \in P} V_x.$$

We define a partial order on P by putting $x \leq y \iff V_x \subset \bar{V}_y$ and a rank function $r(x) = \dim V_x$. Suppose that, for each $x \in P$, we have a **conical slice** $C_x \subset Y$ to the stratum V_x (see [Pro18, Section 3.1] for a precise definition of a conical slice). Finally, suppose that there exists an element $\kappa \in \mathscr{I}(P)$ such that $|C_x(\mathbb{F}_{q^s}) \cap V_y(\mathbb{F}_{q^s})| = \kappa(q^s)$ for all $s > 0$.

Theorem 2. [Pro18, Theorem 3.6] *The element $\kappa \in \mathscr{I}(P)$ is a P-kernel, and for any $x \leq y$, the associated right KLS-polynomial $f_{xy}(t)$ is equal to the Poincaré polynomial for the ℓ-adic étale cohomology of the stalk of $\mathrm{IC}_{\bar{V}_y}$ at a point of V_x.*

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Semidefinite approaches to polynomial optimization: Power and limitations of the SOS cones

Gennadiy Averkov

We give an overview on how semidefinite programming is used in polynomial optimization and also discuss the limitations of the current approach based on the SOS cones. Let's start by introducing conic and semidefinite programming. General conic programming with respect to a closed convex cone K is the problem

$$\inf \left\{ c^\top x \,:\, Ax = b,\ x \in K \right\}$$

of optimization of a linear objective function subject to a system $Ax = b$ of linear and inequalities and the condition that the vector x of the optimization variables is in the cone K. The case $\mathbb{R}^n_+$ gives linear programming, while the case

$$K = \mathcal{S}^k_+ := \{k \times k \text{ symmetric psd matrices over } \mathbb{R}\}$$

gives *semidefinite programming (SDP)*. One can express SDP in terms of the so-called linear matrix inequalities (LMIs). Consider a $k \times k$ symmetric matrix

$$A(x) := \Big(a_{ij}(x)\Big)_{i,j=1,\dots,k}$$

with entries $a_{ij}(x)$ being affine functions in $x \in \mathbb{R}^n$. The condition

$$A(x) \in \mathcal{S}_+^k$$

is called a *linear matrix inequality (LMI)* of size k on n real-valued variables $x \in \mathbb{R}^n$, while the respective set

$$\left\{x \in \mathbb{R}^n \,:\, A(x) \in \mathcal{S}_+^k\right\}$$

is called a *spectrahedron*. *Semidefinite programming* is optimization of a linear function subject to finitely many LMIs [WSV00, AL12]. SDP is efficiently solvable using interior-point methods under mild assumptions. *But* if you can avoid LMIs of large size, you should really do that because of the running-time and numerical-stability issues.

A nice thing about SDP is that some very basic classes of algorithmic and optimization problems can naturally be phrased as a special case of SDP. Linear programming is a subset of SDP, since linear constraints are LMIs of size 1. Determination of the maximum eigenvalue of a symmetric matrix is a semidefinite problem with an LMI on one variable:

$$\min\left\{\lambda \,:\, \lambda I - A \in \mathcal{S}_+^n\right\}.$$

LMIs frequently allow to convexify non-convex problems of algebraic nature so that afterwards SDP can be used to solve underlying optimization problems. Numerous application areas of SDP include problems in probability and statistics, coding theory, systems and control theory and combinatorial optimization [WSV00].

We say that a set C has an *extended formulation* with m LMIs of size k if C is a linear image of a spectrahedron described by m LMIs of size k. A standard way to reduce optimization of a linear function over a given semi-algebraic convex set C is by providing a semidefinite extended formulation of C and lifting the underlying optimization problem over C to an optimization problem over the respective spectrahedron.

In what follows, we deal with polynomial with real coefficients. Polynomial optimization is optimization of a polynomial objective function subject to finitely many polynomial inequalities. An approach of Lasserre to solving polynomial optimization problems is based on Positivstellensätze (which describe positivity of polynomials in terms of sum-of-squares certificates) and semidefinite formulations of the so-called sum-of-squares cones [Mar08, Lau09, Las15]. A polynomial is called a *sum of squares* if it can be represented as a sum of squares of finitely many polynomials. For given positive integers n and d, we introduce the *sum-of-squares cone* $\Sigma_{n,2d}$ to be the cone of n-variate sum-of-squares polynomials of degree at most $2d$. The cone $\Sigma_{n,2d}$ is known to have a semidefinite extended formulation with one LMI of size $\binom{n+d}{n}$ [Las15, §2.1]. To approximate the optimal value of a polynomial optimization problem following Lasserre's approach, one establishes a hierarchy of SDPs which based on the mentioned semidefinite extended formulations of $\Sigma_{n,2d}$ with growing values of d. The approach allows to obtain strong approximations of polynomial optimization problems at a very high computational cost due to the LMIs of a very large size that are used in the hierarchy of the SDPs.

So far, it has not been clear if the known semidefinite formulation of $\Sigma_{n,2d}$ is optimal in terms of the size of the LMIs. We present a theorem that allows to confirm that the known semidefinite extended formulation of $\Sigma_{n,2d}$ is best possible.

We consider the *semidefinite extension complexity* of a set C (denoted as $\operatorname{sxc}(C)$), which is the smallest k such that C has a semidefinite extended formulation with one LMI of size k, and introduce the *semidefinite extension degree* of C (which we denote as $\operatorname{sxdeg}(C)$) to be the smallest k such that C has a semidefinite extended formulation with finitely many LMIs of size k.

Theorem 1 (Main theorem)**.** *Let $X \subseteq \mathbb{R}^n$ be a set with non-empty interior. Let C be a closed convex cone in the space of n-variate polynomials of degree at most $2d$ such that every polynomial in C is non-negative on X and there exist finite subsets S of X of arbitrarily large cardinality with the following property:*

(∗) *For every k-element subset T of S, some polynomial f in the cone C is equal to zero on T and is strictly positive on $S \setminus T$.*

Then $\operatorname{sxdeg}(C) > k$.

Using Theorem 1, we obtain

Corollary 2. $\operatorname{sxdeg}(\Sigma_{n,2d}) = \operatorname{sxc}(\Sigma_{n,2d}) = \binom{n+d}{n}$.

Corollary 2 shows that the known semidefinite formulation of $\Sigma_{n,2d}$ is best possible in terms of both the size and the number of the LMIs.

The case $d = 1$ of Corollary 2 yields the semidefinite extension degree of $\mathcal{S}_+^k$:

Corollary 3. $\operatorname{sxdeg}(\mathcal{S}_+^k) = k$.

Corollary 3 implies that the expressive power of the semidefinite optimization grows strictly with the growth of the size k of the underlying LMIs. In other words, the family of all convex semialgebraic sets that have a semidefinite extended formulation (we call such sets *semidefinitely representable*) can be decomposed into the hierarchy of the families

$$\operatorname{SDR}(k) := \{S \subseteq \mathbb{R}^n \,:\, n \in \mathbb{N},\ \operatorname{sxdeg}(S) \le k\}$$

with each level of the hierarchy being strictly larger than the previous one. The lowest level SDR(1) of the hierarchy is just the family of all polyhedra. The family SDR(1) corresponds to linear optimization. The next level SDR(2) corresponds to the second-order cone programming.

Corollary 3 covers the result $\operatorname{sxdeg}(\mathcal{S}_+^3) = 3$ of Fawzi [Faw19] as a special case.

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Complexity yardsticks for f-vectors of polytopes and spheres

Eran Nevo

We consider geometric and computational measures of complexity for sets of integer vectors, asking for a qualitative difference between f-vectors of simplicial and general d-polytopes, as well as flag f-vectors of d-polytopes and regular CW $(d-1)$-spheres, for $d \geq 4$.

The face numbers of simplicial d-polytopes are characterized by the celebrated g-theorem, conjectured by McMullen [6] and proved by Stanley [12] and Billera-Lee [2]. In contrast, the f-vector, and the finer flag f-vector, of general d-polytopes of dimension $d \geq 4$ are not well understood, despite considerable effort, see e.g. Grünbaum's book [5, Ch.10]; likewise for regular and strongly regular CW spheres. Are there "qualitative" differences between these sets of vectors? We suggest geometric measures to make this question precise. The computational complexity aspect is also considered. For other measures of complexity in dimension 4, like *fatness*, see e.g. Ziegler's ICM paper [14], and e.g. [16, 11] for general d.

1.1. **Geometric complexity.** Let $\mathcal{F}$ be a family of graded posets of rank $d+1$ with a minimum and a maximum. For instance denote by $\mathcal{F} = \mathcal{P}^d$ (resp. $\mathcal{P}^d_s$) the face lattices of all (resp. simplicial) d-polytopes. Let $f(\mathcal{F})$ be the set of f-vectors of elements in $\mathcal{F}$, counting the number of elements in each rank i, denoted f_i, for $1 \leq i \leq d$. (Note the shift of index by 1 with respect to the dimension convention.)

For a subset T of $\mathbb{R}^d$ and $t \in T$ let $\overline{\mathrm{Conv}(T)}$ (resp. $\overline{\mathrm{Cone}_t(T)}$) be the minimal closed convex set (resp. cone with apex t) containing T. Let σ^d denote the d-simplex.

The following (1–3) are geometric consequences of the g-theorem, and (4) a consequence of the Generalized Lower Bound Theorem [7, 8].

Theorem 1. *(1)* Convex hull*:* $C_d := \overline{\mathrm{Conv}(f(\mathcal{P}^d_s))} = \overline{\mathrm{Cone}_{f(\sigma^d)}(f(\mathcal{P}^d_s))}$ *is a simplicial cone of dimension* $\lfloor d/2 \rfloor$*.*

(2) Density of rays*: for any $\epsilon > 0$ and any $x \in C_d$ there exists a simplicial polytope $P \in \mathcal{P}_s^d$ such that the angle between $x - f(\sigma^d)$ and $f(P) - f(\sigma^d)$ is less than ϵ.*

(3) Density of points*: for any $x \in C_d$ there exists a simplicial polytope $P \in \mathcal{P}_s^d$ such that in the l_1-norm $||x - f(P)||_1 = O(||x||_1^{1-\frac{1}{\lfloor d/2 \rfloor}}) = o(||x||_1)$. (The $O(\cdot)$ estimate is tight; $d \geq 2$.)*

(4) Boundary polytopes*: the only polytopes $P \in \mathcal{P}_s^d$ with $f(P)$ on the boundary of C_d are the k-stacked polytopes for some $k \leq \frac{d}{2} - 1$; only the 1-stacked polytopes have $f(P)$ on an extremal ray, all are on the same ray.*

When $d \geq 4$, all analogous statements for $\mathcal{P}^d$ seem open.

Problem 2. *(1)* Convex hull. *Is* $\overline{\mathrm{Conv}(f(\mathcal{P}^d))} = \overline{\mathrm{Cone}_{f(\sigma^d)}(f(\mathcal{P}^d))}$*?*

(1') Finite generation. *Is* $\overline{\mathrm{Cone}_{f(\sigma^d)}(f(\mathcal{P}^d))}$ *finitely generated?*

(2) Ray density. *Are the rays from $f(\sigma^d)$ through $f(P)$ for all $P \in \mathcal{P}^d$ dense in* $\overline{\mathrm{Cone}_{f(\sigma^d)}(f(\mathcal{P}^d))}$*?*

(3) Point density. *Is it true that for any $x \in \overline{\mathrm{Conv}(f(\mathcal{P}^d))}$ there exists $P \in \mathcal{P}^d$ such that $||x - f(P)||_1 = o(||x||_1)$?*

(4) Boundary. *For which polytopes $P \in \mathcal{P}^d$ does $f(P)$ lie on the boundary of* $\overline{\mathrm{Conv}(f(\mathcal{P}^d))}$*? Of* $\overline{\mathrm{Cone}_{f(\sigma^d)}(f(\mathcal{P}^d))}$*?*

For $d = 4$ Ziegler [16] showed that the limits of the rays spanned by $f(\mathcal{P}^d)$ in $\overline{\mathrm{Cone}_{f(\sigma^d)}(f(\mathcal{P}^d))}$ form a convex set; this is open for $d > 4$. Possibly all rays in $\overline{\mathrm{Cone}_{f(\sigma^d)}(f(\mathcal{P}^d))}$ are limit rays, which is equivalent to a YES answer to (1,3); and just to (1) if restricting to the extremal rays.

As for (1') for $d = 4$, it is not known if the *fatness* parameter $\frac{f_1+f_2}{f_0+f_3}$ is bounded above by some constant C. If not, then $\overline{\mathrm{Cone}_{f(\sigma^4)}(f(\mathcal{P}^4))}$ would be determined, with exactly 5 facets [1, 4]. Ziegler [15] showed that if C exists then $C \geq 9$.

Similar questions to those in Problem 2 can be asked about the set of *flag f-*vectors of d-polytopes and again are open for $d \geq 4$ (and known for $d \leq 3$ by Steinitz [13]; there the flag f-vector is determined by the f-vector).

1.2. **Computational complexity.** Computational complexity gains importance in Enumerative Combinatorics in recent years, see Pak's ICM paper [9] for a recent survey. Yet, this perspective is still largely missing in f-vector theory.

Fix d and consider the following decision problems: given a vector $v \in \mathbb{Z}_{\geq 0}^d$ (resp. $v \in \mathbb{Z}_{\geq 0}^{2^d}$), does $v = f(P)$ (resp. $v = \mathrm{flag(P)}$) for some $P \in \mathcal{F}$?

For $\mathcal{F} = \mathcal{P}^d$ this is decidable, by finding all combinatorial types of d-polytopes with n vertices – see Grünbaum's book [5, Sec.5.5] for a proof using Tarski's elimination of quantifiers theorem. Using the existential theory of the reals, e.g. [3, 10], gives an algorithm that runs in time double exponential in size of the encoding of v (in binary, on a deterministic Turing machine).

For $\mathcal{F} = \mathcal{P}_s^d$ this is *effectively* decidable, namely: For a vector $v = (v_1, \ldots, v_d) \in \mathbb{Z}_{\geq 0}^d$ denote $N(v) := \sum_{i=1}^d \lceil \lg_2(v_i) \rceil$, the number of bits in its encoding in binary. Then,

Theorem 3. *Deciding if $v \in f(\mathcal{P}_s^d)$ can be done in polynomial time in $N(v)$.*

Problem 4. *Can deciding it $v \in f(\mathcal{P}^d)$ be done in polynomial time in $N(v)$?*

Recognizing the cone $\overline{\mathrm{Cone}_{f(\sigma^d)}(f(\mathcal{P}^d))}$ may turn out undecidable:

Problem 5. *Fix $d \geq 4$. Is the following problem decidable?: given a hyperplane H through $f(\sigma^d)$, does it support the cone $\overline{\mathrm{Cone}_{f(\sigma^d)}(f(\mathcal{P}^d))}$, or contain an interior ray of it?*

As mentioned, for $d = 4$, if fatness of 4-polytopes is unbounded then the decision problem is easy. The analogs of Problems 4 and 5 for flag-f vectors of d-polytopes are open; likewise for Problem 4 for regular CW $(d-1)$-spheres.

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Graph complexes

Sam Payne

Start with a rational vector space K generated by pairs (G, ω), where G is a finite graph in which all vertices have valence at least 3, and ω is a total ordering of the edge set $E(G)$. Impose the relation that $(G, \omega) = \operatorname{sgn}(\sigma)(G', \omega')$ for each isomorphism $G \cong G'$ inducing a permutation σ on the edge orderings. This means, in particular, that $(G, \omega) = 0$ in K if G has an automorphism that acts by an odd permutation on $E(G)$.

The differential is defined by

$$d(G, \omega) = \sum_i (-1)^i (G/e_i, \omega_{\hat{i}}).$$

Here, G/e_i is the graph obtained by contracting the ith edge in the ordering ω, and $\omega_{\hat{i}}$ is the induced ordering on the remaining edges. Note that $d^2 = 0$. The pair (K, d) is Kontsevich's *graph complex*.

Kontsevich initially defined six different graph complexes, corresponding two three different operads, each with two different systems of coefficients [3]. This (K, d) is the commutative graph complex with even coefficients. Many other variants are possible, and a number of have been studied in the literature.

The genus of the graph $g(G) = \#E(G) - \#V(G) + 1$ is preserved by d. Then K splits as a direct sum of finite dimensional subcomplexes $K = \bigoplus_g K^{(g)}$.

Let $K_i^{(g)} \subset K^{(g)}$ be the subspace generated by graphs of genus g with i edges. Then d restricts to

$$d_i : K_i^{(g)} \to K_{i-1}^{(g)},$$

and we write

$$H_i(K^{(g)}) = \frac{\ker(d_i)}{\operatorname{im}(d_{i+1})}$$

for the corresponding *graph homology* groups.

In this talk, I presented the definition of the graph complex, illustrated with examples, and stated a few basic theorems highlighting its essential properties, relations to other branches of mathematics, and open problems that may be suitable for direct combinatorial study.

I highlighted, in particular, the existence of a natural isomorphism from $H_i(K^{(g)}$ to the top graded piece of the weight filtration on the cohomology of the moduli space of curves M_g [2]. From this, plus known results on the cohomology of M_g, one deduces that $H_i(K^{(g)})$ vanishes for $i < 2g$. Another natural isomorphism, due to Willwacher [4], identifies $\prod_g (H_{2g}(K^{(g)})^\vee$ with the Grothendieck-Teichmüller Lie algebra. Applying results of Brown from Grothendieck-Teichmüller theory [1], one deduces that $\dim H_{2g}(K^{(g)})$ grows exponentially with g.

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Iterated discriminants

Sandra di Rocco

(joint work with A. Dickenstein and R. Morrison)

Let K be an algebraically closed field and $A \subset \mathbb{Z}^n$ be a finite lattice subset. A polynomial with prescribed monomials in A, $p(x) = \sum_{m_i \in A} c_{m_i} x^{m_i} \in K[x_1, \ldots, x_n]$, is called an A-polynomial. The theory of discriminants of an A-polynomial, called an A-discriminant, was introduced in [GKZ94] and has been extensively studied both from a geometric point of view, in connection with toric projective duality, and from a computational viewpoint.

The A-discriminant is a polynomial $D_A \in K[c_{m_i}]_{m_i \in A}$ whose roots correspond to A-polynomials having some multiple solution $x \in (K^*)^n$, or equivalently to singular hyperplane sections of the monomial embedding defined by A :

$$\phi_A : (K^*)^n \to \mathbb{P}^{|A|-1}, \phi(x) = (\ldots, x^{m_i}, \ldots)_{m_i \in A}.$$

If the A-polynomial is multi-homogeneous, meaning that $\mathrm{Conv}(A) = \Delta_{n_1} \times \ldots \times \Delta_{n_l}$ where Δ_s denotes the unimodular simplex of dimension s, then the A-discriminant corresponds to the hyperdeterminant of size $(n_1+1) \times \ldots \times (n_l+1)$ [GKZ94, Chapter 14], which we denote by $H_{(n_1+1)\times \ldots \times (n_l+1)}$.

Consider now a system of $(r+1)$ polynomial equations of type $A_0, \ldots, A_r \subset \mathbb{Z}^n$:

$$p_{A_0}(x) = p_{A_1}(x) = \ldots = p_{A_r}(x) = 0. \tag{1}$$

$$P_{A_j} = \sum_{a \in A_j} c_{j,a} x^a$$

In the case where $r+1 = n$, Bernstein's theorem says that for a generic choice of coefficients of the polynomials p_{A_i} the system has a finite number of solutions in $(K^*)^n$ equal to the mixed volume $MV(P_0, \ldots, P_{n-1})$, where $P_i = \mathrm{Conv}(A_i)$ are the Newton polytopes of the polynomials p_{A_i}. For special choices of coefficients, two or more of these solutions may come together and create a point of higher multiplicity. For $r+1 < n$, points of higher multiplicity correspond to singularities of the variety cut out by the hypersurfaces $p_{A_i} = 0$. Work by Salmon [Sal58], Bromwich [Bro71] and more recently Farouki et al. [FNO89] classifies singular intersections of two quadric surfaces. A generalization of their result to higher dimensional quadric hypersurfaces is given in [Ott13].

The basic idea of this line of results was already pursued by Cayley in connection with tangent intersections of conics in $\mathbb{C}^2$. Consider two conics, given in matrix form by

$$\begin{bmatrix} x & y & 1 \end{bmatrix} M_i \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0, \quad i = 1, 2.$$

A tangential intersection (that is, a multiple root of the system) is a point sharing the same normal line to the two curves. One sees geometrically that systems intersecting tangentially are the roots of the hyperdeterminant of type $2\times3\times3$, i.e. solutions to $H_{2\times3\times3}(M_1, M_2) = 0$. The hyperdeterminant is in fact a polynomial in the entries of the matrices M_1 and M_2. Using the Schäfli decomposition method [Sch53, GKZ94], one obtains

$$H_{2\times3\times3}(M_1, M_2) = D_3(\det(M_1 + tM_2)), \tag{2}$$

where D_3 is the univariate discriminant of the degree 3 polynomial giving by $\det(M_1 + tM_2)$ considered as a polynomial in t. For generic matrices this is a polynomial of degree 4.

We consider Equation 2 to be an *iterated process*, as we are computing the discriminant of a discriminant.

Definition 1. *We call an isolated solution $u \in (K^*)^n$ a* non-degenerate multiple root *for the system 1 if the gradient vectors $\nabla_x f_i(u), i = 0, \ldots, r$ are linearly dependent.*

We now introduce a generalisation of discriminant from one polynomial to a polynomial system, called the *mixed discriminant*, we refer to [CCD$^+$13] for more details. We define the *mixed discriminantal variety* to be the closure of the locus of coefficients for which the system has a non-degenerate multiple root. If this variety is a hypersurface, it is defined by a single irreducible polynomial which we call the *mixed discriminant*, denoted by $MD_{A_0,\ldots A_r}$. If the mixed discriminant is not a hypersurface, we call the system *defective* and set $MD_{A_0,\ldots A_r} = 1$. More concretely:

Definition 2. *The mixed discriminant associated to a polynomial system of type $A_0, \ldots, A_r$ is the irreducible polynomial $MD_{A_0,\ldots A_r} \in K[\{c_{j,a}\}]_{j=0,\ldots r; a\in A_j}$ in the coefficients of the $r + 1$ polynomials, vanishing if the corresponding system has tangential solutions. If such a polynomial does not exist we set $MD_{A_0,\ldots A_r} = 1$.*

Notice that if such a polynomial exists then it is irreducible and unique up to scalar as it corresponds to the defining polynomial of a hypersurface.

Note moreover that $M_{\Delta_{n_0},\ldots,\Delta_{n_r}} = H_{(n_0+1)\times\ldots\times(n_l+1)}$ and thus $M_{\Delta_{n_0},\ldots,\Delta_{n_r}} = D_{\Delta_{n_0}\times\cdots\times\Delta_{n_r}}$.

Our first result is a generalization of the above, namely the characterization of the mixed discriminant in terms of the discriminant of a single polynomial. There is a close relationship between the mixed discriminant and the polytope in $\mathbb{R}^{n+r}$ generated by the A_i's, called the *Cayley polytope*.

Definition 3. *Let the matrix C have as columns the lifted configurations $e_i \times A_i \in \mathbb{Z}^{n+r}$ for $i = 0, \ldots, r$, where $e_0 = 0$ and e_i is the standard i^{th} basis vector for $\mathbb{Z}^r$ for $i \geq 1$. The matrix C is called the* Cayley matrix *of the system $A_0, \ldots, A_r$, and its convex hull is the* Cayley polytope *associated to the polytopes $Conv(A_0), \ldots, Conv(A_r)$.*

Notice that when $A_0 = A_1 = \ldots = A_r = A$ then $Conv(C) = \Delta_r \times Conv(A)$.

Proposition 4. [DDRM+19] *Assume $D_C \neq 1$. Then*

$$MD(A_1, \ldots, A_r) = D_C$$

This characterization leads to the following definition of *multivariate iterated discriminant* of order r. We introduce $(r+1)$ new variables $\lambda_0, \ldots, \lambda_r$ and encode the initial system by one auxiliary polynomial with support in C:

$$\phi(x, y) = \lambda_0 p_0(x) + \ldots + \lambda_r p_r(x) = \sum_0^r \lambda_j \sum_{a \in A_j} c_{j,a} x^a.$$

We will often use the abuse of notation of denoting this polynomial and its tuple of coefficients by F_λ where $\lambda = (\lambda_o, \ldots, \lambda_r)$. Let D_C be the C-discriminant, which is the unique (up to sign) irreducible polynomial with integer coefficients in the unknowns $c_{j,a}$ which vanishes whenever the hypersurface defined by ϕ in $(K^*)^{n+r+1}$ is not smooth.

For simplicity of exposition we will consider the case when $A_0 = \ldots = A_r = A$ and use the notation $MD_{r,A} := MD_{A,\ldots,A}$. Notice that $D_A(F_\lambda)$ is a homogeneous polynomial of degree $\delta_A = \deg(D_A)$ in $\lambda_0, \ldots, \lambda_r$.

Definition 5. *The* m*ultivariate iterated discriminant of order* r *is the polynomial*

$$ID_{r,A}(f_0, \ldots, f_r) := D_{\delta_A \Delta_r}(D_A(F_\lambda)).$$

It is worth noting that in the classical case of $r = 0$, all these polynomials coincide: we have $MD_{0,A} = ID_{0,A} = D_A$ by definition, and $D_{\delta\Delta_0}(D_A(\lambda f_A)) = D_A$ because the discriminant (in the variable λ) of the monomial $D_A \lambda^\delta$ is the coefficient D_A, as observed in [Jou91]. Moreover, when $A = \Delta_i$ we have that $ID_{r,A}$ coincides with the hyperdeterminant Shäfli decomposition. Our second result is a precise relation between $MD_{r,A}$ and $ID_{r,A}$. The advantage of relating $MD_{r,A}$ with $ID_{r,A}$ is that the iterated discriminant is computationally more accessible.

Theorem 6. [DDRM+19] *Assuming $D_A \neq 1$, $ID_{r,A}$ is a polynomial which in the cases when the codimension of the singular locus of the dual variety X_A^* is higher than $r+1$ is irreducible and coincides with $MD_{r,A}$. If the codimension is exactly $r+1$ then it is not necessarily irreducible and its factors are $MD_{r,A}$ and powers of the Chow form of the irreducible components of $sing(X_A^*)$.*

A corollary of the above results is that an iterated method for a characterization of singular complete intersections is unfortunately only possible for two hypersurfaces in degree 2, as in [Ott13]:

Proposition 7. [DDRM+19] *Let $1 < d$ and $1 \leq r \leq n$. Then* $\deg(MD_{r,d\Delta_n}) = \deg(ID_{r,d\Delta_n})$ *if and only if $r = 1$ and $d = 2$.*

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Semistable reduction in characteristic zero

Gaku Liu

(joint work with Karim Adiprasito, Michael Temkin)

In 2000 Abramovich and Karu [1] proved that any dominant morphism $f : X \to B$ of varieties of characteristic zero can be made weakly semistable by replacing B by a smooth alteration B' and replacing the proper transform of X by a modification X'. In the language of log geometry this means that $f' : X' \to B'$ is log smooth and saturated for appropriate log structures. Moreover, Abramovich and Karu formulated a stronger conjecture that $f' : X' \to B'$ can be even made semistable, which amounts to making X' smooth as well, and explained why this is the best resolution of f one might hope for. In this talk, we outline a solution to the semistable reduction conjecture.

To do this, we prove a generalization of the following combinatorial theorem by Knudesn, Mumford, and Waterman [3]: For every lattice polytope P, there is a dilation of P which admits a unimodular triangulation. Here, a unimodular triangulation is a triangulation in which every simplex is a lattice polytope of volume $1/(\dim P)!$. This theorem was used by Kempf, Knudsen, Mumford, and Saint-Donat [3] to prove semistable reduction over bases of dimension one. Our generalization considers maps of polytopes $P \to Q$ and shows that the map can be "dilated" so that both polytopes admit unimodular triangulations which are consistent with the map.

Finally, we use our construction to prove the following strengthening of Knudesn-Mumford-Waterman: For every lattice polytope P, there exists an integer $c(P)$

such that for all integers $c \geq c(P)$, the polytope cP admits a unimodular triangulation.

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Cone valuations, Gram's relation, and flag-angles

Raman Sanyal

(joint work with Spencer Backman, Sebastian Manecke)

The *Euler–Poincaré relation* is a cornerstone of geometric combinatorics. It states that the face numbers $f_i(P)$, $i = 0, \ldots, d-1$ of a convex d-dimensional polytope $P \subset \mathbb{R}^d$ are not independent of each other but satisfy the linear relation

$$f_0(P) - f_1(P) + \cdots + (-1)^{d-1} f_{d-1}(P) \ = \ 1 - (-1)^d \, .$$

This simple linear relation is the starting point for Eulerian posets, Euler characteristics, and more. Höhn [6] showed in his PhD thesis that the Euler–Poincaré relation is the only linear relation on face numbers, up to scaling.

A natural semi-discrete invariant that associates a weight to every face is given by the interior angles. For a face $F \subseteq P$ and an arbitrary point q in the relative interior of F one defines the *tangent cone* or the *cone of feasible directions* of P at F as

$$T_F P \ := \ \{ u : q + \epsilon u \in P \text{ for all } \epsilon > 0 \text{ sufficiently small} \} \, .$$

This is a polyhedral cone that is independent of the choice of q and that captures the local structure of P around F. For a convex cone $C \subseteq \mathbb{R}^d$ with apex at the origin, the *standard cone angle* is

$$\alpha(C) \ := \ \frac{vol(C \cap B_d)}{vol(B_d)} \, , \tag{1}$$

where B_d is the unit ball. The *interior angle* of P at F is defined as $\widehat{\alpha}(F, P) := \alpha(T_F P)$ and the k-th *interior angle* of P is $\widehat{\alpha}_k(P) := \sum_F \widehat{\alpha}(F, P)$, where the sum is over all faces of P of dimension k. Finally, the *interior angle vector* is $\widehat{\alpha}(P) = (\widehat{\alpha}_0(P), \widehat{\alpha}_0(P), \ldots, \widehat{\alpha}_{d-1}(P))$. Gram [3] studied linear relations on interior angles of 3-dimensional polytopes and Sommerville and Höhn generalized his findings to all dimensions d. The following linear relation is known as *Gram's relation* (see [5, Chapter 14.4] for more on the history)

$$\widehat{\alpha}_0(P) - \widehat{\alpha}_1(P) + \cdots + (-1)^{d-1} \widehat{\alpha}_{d-1}(P) \ = \ (-1)^{d+1} \, . \tag{2}$$

A simple geometric proof was found by Perles and Shephard [11] and Höhn [6] showed that Gram's relation is the unique-up-to-scaling linear relation on interior angles of d-dimensional polytopes.

In (1) the definition of α, the unit ball can be replaced by any other convex body, which gives rise to anisotropic notions of cone angles; see, for example, [4, 8, 9]. More generally, we define a *cone angle* as a simple and normalized valuation on polyhedral cones in $\mathbb{R}^d$. That is, α satisfies

$$\alpha(C \cup C') \;=\; \alpha(C) + \alpha(C') - \alpha(C \cap C'),$$

whenever $C, C', C \cup C'$ are polyhedral cones and, additionally, $\alpha(C) = 0$ for $\dim C < d$ and $\alpha(\mathbb{R}^d) = 1$.

It is clear that interior angles of polytopes can be defined with respect to any cone angle and we prove the following strengthenings of Höhn's results.

Theorem 1 ([1]). *Let α be a cone angle. Then the associated interior angles of any d-dimensional polytope $P \subset \mathbb{R}^d$ satisfy Gram's relation* (2) *and it is the unique linear relation, up to scaling.*

In order to prove that result, we consider the class of *zonotopes* or, more generally, the class of *belt polytopes*. Zonotopes are Minkowski sums of finitely many segments. The latter, introduced by Baladze [2], are polytopes whose normal fans correspond to linear hyperplane arrangements. For a k-dimensional face $F \subseteq P$, denote by $L(F)$ the k-dimensional linear subspace parallel to F. The collection $\mathcal{L}(P) = \{L(F) : F \subseteq P \text{ face}\}$ of linear subspaces is partially ordered by inclusion with top element $\widehat{1} = \mathbb{R}^d$. If P is a belt polytope with corresponding hyperplane arrangement $\mathcal{H}$, then $\mathcal{L}(P)$ is exactly the lattice of flats of $\mathcal{H}$. The *cocharacteristic polynomial* of $\mathcal{L}(P)$

$$\psi_{\mathcal{L}(P)}(t) \;:=\; \sum_{L \in \mathcal{L}(P)} |\mu_{\mathcal{L}(P)}(L, \widehat{1})| t^{d - \dim L}\,,$$

where $\mu_{\mathcal{L}(P)}$ is the Möbius function of $\mathcal{L}(P)$, was introduced by Novik, Postnikov, and Sturmfels [10]. We prove the following result.

Theorem 2. *Let α be any cone angle and P a d-dimensional belt polytope. Then*

$$\widehat{\alpha}_0(P) + \widehat{\alpha}_1(P)t + \cdots + \widehat{\alpha}_d(P)t^d \;=\; \psi_{\mathcal{L}(P)}(t)\,.$$

For the standard cone angle, this was first shown by Klivans and Swartz [7]. Thus, the interior angle vector of a belt polytope P does not depend on the chosen cone angle and is determined by the combinatorics of P. So, in order to prove Theorem 1, we construct d-dimensional belt polytopes $P_1, \ldots, P_d \subset \mathbb{R}^d$ for every $d \geq 1$ such that the corresponding cocharacteristic polynomials are affinely independent.

In the talk we also explain connections between *exterior angles* of belt polytopes and *rank vectors* of lattice of flats. We closed with a brief description of *flag-angle vectors*, a suitable analog of *flag-f-vectors*.

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Tropical Voronoi diagrams

Michael Joswig

(joint work with Francisco Criado, Francisco Santos)

The *tropical distance* between two points $a, b \in \mathbb{R}^{d+1}$ is

$$\operatorname{dist}(a,b) \ := \ \max_{i\in[d+1]}(a_i - b_i) - \min_{j\in[d+1]}(a_j - b_j) \ = \ \max_{i,j\in[d+1]}(a_i - b_i - a_j + b_j) \ .$$

It does not depend on choosing min or max as the tropical addition. The map dist : $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}$ induces a metric on the *tropical projective d-torus* $\mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$, where $\mathbf{1} = (\mathbf{1}, \ldots, \mathbf{1})$. Let S be a finite subset of $\mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$, whose elements we call *sites*. The *tropical Voronoi region* of a site $a \in S$ is the set

$$V(a) \ = \ \left\{ x \in \mathbb{R}^{d+1}/\mathbb{R}\mathbf{1} \ \middle| \ \operatorname{dist}(x,a) \leq \operatorname{dist}(x,b) \text{ for all } b \in S \setminus \{a\} \right\} \ .$$

It turns out that the Voronoi regions are homeomorphic to full-dimensional balls, and the resulting cell decomposition is the *tropical Voronoi diagram* $\operatorname{Vor}(S)$; see Figure 1 for a planar example.

Theorem. If S is in general position then each tropical Voronoi region of S is the star convex union of finitely many (possibly unbounded) ordinary polyhedra.

We discuss several procedures for computing tropical Voronoi diagrams. Our best method for general but fixed d is a randomized incremental algorithm with expected running time $O(n^d \log n)$, where $n = \#S$.

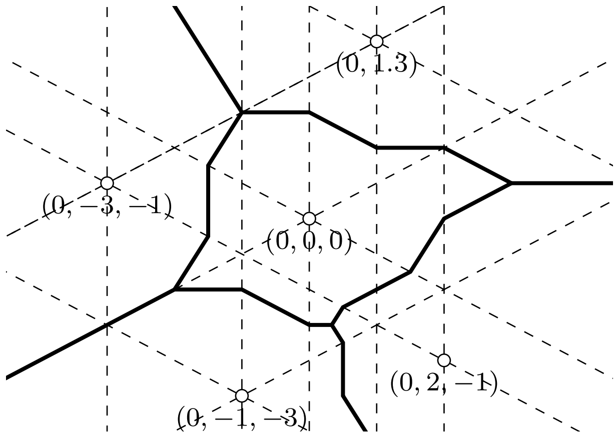

FIGURE 1. Tropical Voronoi diagram of five points in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$

The centrally symmetric polytope

$$\mathbb{B}^d \;:=\; \left\{ x \in \mathbb{R}^{d+1}/\mathbb{R}\mathbf{1} \;\middle|\; \operatorname{dist}(x,0)=1 \right\} \;=\; \frac{1}{2}\operatorname{conv}\left(\{\pm 1\}^{d+1} \setminus \{\pm\mathbf{1}\}\right) + \mathbb{R}\mathbf{1}$$

is the *tropical unit ball.* It follows that dist yields a polyhedral norm [2, Sect. 7.2]. The set bisector$(A) := \bigcap_{a\in A} V(a)$ is called the *bisector* of $A \subset S$. The bisector of k points in general position is either empty or pure of dimension $d+1-k$ [3, Cor. 3.6]. In particular, for more than $d+1$ points it is empty. Moreover, we have a formula for the Betti numbers of bisectors of three points [3, Thm. 3.23]. The results on bisectors hold even for arbitrary polyhedral norms, not only in the tropical setting.

This study is motivated by a rising interest in metric properties in (real) tropical geometry. We close this note with a list of open problems, whose solutions could be a first few steps toward a version of *tropical differential geometry.*

(1) What is the topological structure of the tropical bisectors of k points for all k and d? For $k=2$ this is [3, Thm. 4.6], and for $k=3$ we have some data from [3, Thm. 3.23]. For $k=4$ and $d=3$ we have various examples, which we could not derive a pattern from. Note that the tropical bisector of two points in general position lies in a tropical hypersurface od degree $d+1$; cf. [3, Prop. 4.1].
(2) Which combinatorial properties do *semi-polytropes* have? These are the ordinary polyhedra which naturally subdivide the Voronoi regions. Can the expected running time of $O(n^d \log n)$ for our randomized incremental algorithm be improved?
(3) What is a good notion of a "tropical curvature" for arbitrary polygonal space curves? The ad-hoc definition in [1] was useful for obtaining new results about the complexity of the interior point method of linear programming. Does this fit, at least in the plane, with a "tropical medial axis" construction and tropical unit balls "osculating" at points on closed

polygonal curves? If so, it should be possible to study this via sequences of tropical Voronoi diagrams, where the number of sites tends to infinity.

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Higher connectivity of tropicalizations

Diane Maclagan

(joint work with Josephine Yu)

Balinski's theorem [Bal61] states that the vertex-edge graph of a d-dimensional polytope P is d-connected. This can be generalized to consider the graph whose vertices are the k-dimensional faces of P, and whose edges connect k-dimensional faces that live in a common $(k+1)$-dimensional face. The connectivity of this graph was studied by Sallee [Sal67] and Athanasiadis [Ath09]. In [Ath09, Theorem 1.1], Athanasiadis shows that this graph of d-connected if $k = d-2$, and $(k+1)(d-k)$ connected otherwise.

An alternative approach is to consider the *hypergraph* whose vertices are the k-dimensional faces of P, and whose hyperedges consist of all k-dimensional faces in a given $(k+1)$-dimensional face. We then have the following result.

Corollary 1 (M-Yu)**.** *Let P be a rational polytope. Then the k-face hypergraph is $(d-k)$-connected.*

Here we regard a hypergraph as connected if there is a path between any two vertices where each step connects vertices that live in a common hyperedge, and a hypergraph is s-connected if removing any $s-1$ vertices and their adjacent hyperedges leaves the hypergraph connected. The hypothesis that the polytope P is rational is a consequence of the proof techniques, and probably not necessary.

This a corollary of a theorem about higher connectivity of tropicalizations. The tropicalization of a subvariety X of the algebraic torus $(K^*)^n$ defined over a valued field is

$$\mathrm{trop}(X) = \mathrm{cl}(\mathrm{val}(X(L))) \subseteq \mathbb{R}^n,$$

where the closure is in the usual Euclidean topology, and L/K is a nontrivially valued algebraically closed field extension. By the Structure Theorem for Tropical Geometry (see, for example, [MS15, Theorem 3.3.5]), this is the support of polyhedral complex. When X is irreducible, this complex is pure of dimension $\dim(X)$, and *connected through codimension one.* This latter condition, which was originally observed in [BJSST07], and given in refined and corrected form in [CP12], means that the facet-ridge hypergraph, with a vertex for every maximal-dimensional polyhedron in the complex, and a hyperedge for every ridge, is connected.

A polyhedral complex is $\mathbb{R}$-rational if the facet normals of every polyhedron in the complex are rational. Our main result, which is Theorem 1 of [MY19], is

Theorem 2. *Let K be a field of characteristic 0 which is either algebraically closed, complete, or real closed with convex valuation ring. Let X be a d-dimensional irreducible subvariety of $(K^*)^n$. Fix a a pure d-dimensional $\mathbb{R}$-rational polyhedral complex Σ with an ℓ-dimensional lineality space such that $\operatorname{trop}(X)$ is the support $|\Sigma|$ of Σ. The facet-ridge hypergraph of Σ is $(d-\ell)$-connected.*

Remark 1.

(1) Balinski's theorem is a special case of this theorem, as $\mathbb{R}^n = \operatorname{trop}((K^*)^n)$, and the vertex-edge graph of a polytope is the facet-ridge hypergraph of the normal fan. The requirement in the theorem that the polyhedral complex be $\mathbb{R}$-rational can be relaxed in this case.

(2) The k-face hypergraph of a d-dimensional polytope P is the facet-ridge hypergraph of the $(d-k)$-skeleton of the normal fan of P. When P is rational this equals $\operatorname{trop}(X)$ for X a complete intersection of k polynomials with Newton polytope P.

(3) This gives a necessary condition for a polyhedral complex to be the tropicalization of an irreducible variety. There are examples of pure balanced polyhedral complexes that do not satisfy this extra connectedness requirement.

(4) The tropicalization a linear space is the Bergman fan of the corresponding (realizable) matroid. The fine fan structure on the Bergman fan of an arbitrary matroid of rank r is an r-dimensional fan with a one-dimensional lineality space. This is $r-1$-connected. It would be interesting to extend this to other fan structures on the Bergman fan, and to more general tropical linear spaces.

The method of proof involves induction on dimension, slicing with a general classical hyperplane. This uses a new Tropical Bertini theorem, which is based on a recent toric Bertini theorem of Fuchs, Mantova, and Zannier [FMZ18], and extensions of Amoroso and Sombra [AS17]. An earlier version of this work was also presented at MFO Workshop 1918 on Tropical Geometry: new directions [IMMS19].

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Rational homology spheres and tropical curves

Grigory Mikhalkin

Spatial tropical curves are graphs embedded to $\mathbb{R}^3$ in a certain piecewise-linear way. In particular, we require that all edges are straight and parallel to integer vectors as well as a balancing condition at every point, see e.g. [1]. A tropical curve $\Gamma \subset \mathbb{R}^3$ with κ ends has *toric degree* which is the collection $Z = \{\zeta_j\}_{j=1}^{\kappa}$ of integer vectors $\zeta_j \in \mathbb{Z}^3$ parallel to the corresponding leaves of Γ. The balancing condition implies that $\sum\limits_{j=1}^{\kappa} \zeta_j = 0$. The tropical curve is rational if the graph Γ is a tree.

A tropical enumerative problem is defined by a collection $\Lambda = \{l_j\}_{j=1}^{\kappa}$ of affine lines $l_j \subset \mathbb{R}^3$ parallel to integer vectors in $\mathbb{R}^3$. If a toric degree Z is fixed, and the lines in Λ are chosen generically (with respect to translations in $\mathbb{R}^3$) then there are finitely many tropical curves $\Gamma \subset \mathbb{R}^3$ passing through each line l_j. Furthermore each l_j must pass through its own leaf of Γ. The curve Γ acquires the combinatorial multiplicity $\mu_\Lambda(\Gamma)$, as well as the vertex multiplicity $\mu_{\text{vert}}(\Gamma)$ independent of Λ so that $\mu_\Lambda(\Gamma)$ is always an integer multiple of $\mu_{\text{vert}}(\Gamma)$, see [2].

The sum $\sum\limits_{C} \mu_\Lambda(\Gamma)$ computes the answer to a complex enumerative problem in $(\mathbb{C}^\times)^3$ corresponding to the toric degree Z and the configuration Λ. In particular, it is independent of moving the lines l_j by translations. According to the seminal Stominger-Yau-Zaslow conjecture [3] this correspondence should have a mirror counterpart.

Suppose that $\Delta \subset \mathbb{R}^3$ is a complex polyhedron while X_Δ is the toric 3-fold corresponding to Δ. Then the continuations of the edges of Δ are lines $l_j \subset \mathbb{R}^3$ with integer slopes. If we define Λ as a suitable subcollection of these edge continuations then Λ can be made generic by a small deformation of Δ.

It turns out that a reasonable mirror correspondence is provided by lifting the intersection $\Gamma \cap \Delta$ to a smooth Lagrangian variety L_Γ as in [2]. It is possible if Γ intersect $\partial\Delta$ nicely, i.e. only along the edges and in an integer *bissectrice* way. Then $L_\Gamma \subset X_\Delta$ is a smooth 3-dimensional graph-manifold (in the sense of Waldhausen). If Γ is rational then L_Γ is a rational homology sphere whose first homology group $H_1(L_\Gamma)$ is the torsion group of order $\mu_\Lambda(\Gamma)/\mu_{\text{vert}}(\Gamma)$. Furthermore, Γ admits $\mu_{\text{vert}}(\Gamma)$ many of Hamiltonianly non-equivalent lifts. Thus the total contribution of the Lagrangians from the symplectic side is again $\sum_C \mu_\Lambda(\Gamma)$, i.e. it coincides with the contribution from the complex enumerative side.

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A combinatorial version of the fractional Helly theorem

Andreas Holmsen

Variations on Helly's theorem. Among the fundamental theorems of combinatorial convexity is Helly's theorem [7]. It states that *if the members of a finite family of convex sets in $\mathbb{R}^d$ have empty intersection, then there are some $d+1$ or fewer members of the family whose intersection is empty.* Variations of Helly's theorem are intensively studied and have applications far beyond their field of origin. An important example is the *fractional Helly theorem* due to Katchalski and Liu [13].

Theorem 1 (Fractional Helly). *For every $d \geq 1$ and $c_1 \in (0,1]$ there exists a $c_2 \in (0,1]$ with the following property. Let F be a family of $n \geq d+1$ convex sets in $\mathbb{R}^d$ where at least $c_1\binom{n}{d+1}$ of the $(d+1)$-tuples of F have non-empty intersection. Then there are at least $c_2 n$ members of F whose intersection is non-empty.*

The optimal value for c_2 (in terms of c_1 and d) was later found by Kalai [10], and independently by Eckhoff [4]. In particular, they showed that $c_2 \to 1$ as $c_1 \to 1$, which gives us Helly's theorem.

These optimal values are consequences of more general results concerning bounds on the f-vectors of certain classes of simplicial complexes. More recently, the work by Alon et al. [1] shows that the fractional Helly theorem is closely related to the construction of *weak ε-nets* and to the notion of *χ^*-boundedness* in graph and hypergraph theory. Another noteworthy example is the *colorful Helly theorem*, discovered by Lovász, and independently (in a dual form) by Bárány in [3].

Theorem 2 (Colorful Helly). *Let $F_1, \dots, F_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$ where $\bigcap_{i=1}^{d+1} C_i \neq \emptyset$ for any choice $C_1 \in F_1, \dots, C_{d+1} \in F_{d+1}$. Then for some $i \in [d+1]$ we have $\bigcap_{C \in F_i} C \neq \emptyset$.*

Note that we get Helly's theorem by setting $F_1 = \cdots = F_{d+1}$. In its dual form, the colorful Helly theorem has several applications in discrete geometry (see [14, Chapters 8 and 9]).

An axiomatic approach. Let $H = (V, E)$ be a k-uniform hypergraph on a finite vertex set V and edge set $E \subset \binom{V}{k}$. The *non-edges* of H are the elements of $\binom{V}{k} \setminus E$. We say that H satisfies *non-edge transversal property* if the following holds:

For any non-edges $\tau_1, \ldots, \tau_k$ that are pairwise disjoint, there exists a non-edge τ such that $|\tau \cap \tau_i| = 1$ for every $i \in [k]$.

Let $\mathcal{H}_k$ denote the class of all k-uniform hypergraphs that satisfy the non-edge transversal property. Our goal is to investigate this class in terms of various combinatorial parameters.

Clique complexes. Given a k-uniform hypergraph $H = (V, E)$ we can form a simplicial complex on the vertex set V whose faces consists of those subsets of vertices that are contained in a clique (complete subhypergraph) in H. This is the *clique complex* of H. By the clique complexes construction, we can relate the class $\mathcal{H}_k$ to several commonly studied classes of simplicial complexes. We denote by

$\mathcal{R}(d)$ – the class of d-representable complexes,
$\mathcal{C}(d)$ – the class of d-collapsible complexes,
$\mathcal{L}(d)$ – the class of d-Leray complexes,
$\mathcal{H}(d)$ – the class of clique complexes of hypergraphs in $\mathcal{H}_{d+1}$.

For details on the first three classes (including precise definitions) we recommend the survey by Tancer [16]. We have the following inclusions.

$$\mathcal{R}(d) \subset \mathcal{C}(d) \subset \mathcal{L}(d) \subset \mathcal{H}(d).$$

Let us point out that all these inclusions are strict. The first two inclusions are due to Wegner [17], and the last inclusion is a consequence of a result of Kalai and Meshulam [12].

Linear size cliques. The *clique number* of a k-uniform hypergraph H, denoted by $\omega(H)$, is the maximum number of vertices in a clique in H. A class of k-uniform hypergraphs Γ satisfies the *fractional Helly property* if there exists a function $f : (0, 1] \to (0, 1]$ such that

$$|E| \geq c\binom{|V|}{k} \implies \omega(H) \geq f(c)n$$

for every $H = (V, E) \in \Gamma$. If $\lim_{x \to 1} f(x) = 1$ we say Γ satisfies the *strong* fractional Helly property. The following was shown in [8].

Theorem 3. *For every $k \geq 2$, the class $\mathcal{H}_k$ satisfies the strong fractional Helly property.*

This generalizes a result of Gyárfás, Hubenko and Solymosi [6] who established the case $k = 2$. Note that the fractional Helly theorem is a consequence of Theorem 1.

χ^*-boundedness. The *clique cover number* of a k-uniform hypergraph H, denoted by $\chi^*(H)$, is the smallest number of parts needed to partition H into cliques. (This is the same as the *chromatic number* of the complement of H.) The *independence number* of H, denoted by $\alpha(H)$, is the maximum cardinality of a subset of vertices that does not contain any edge. A class of k-uniform hypergraphs Γ is called χ^*-*bounded* if there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that

$$\chi^*(H) \leq g(\alpha(H))$$

for every $H \in \Gamma$. By combining results from [1] and [9] we can show the following.

Theorem 4. *For every $k \geq 2$, the class $\mathcal{H}_k$ is χ^*-bounded.*

The case $k = 2$ follows from a theorem of Gyárfás [5]. We note that the celebrated (p, q) theorem due to Alon and Kleitman [2] is a consequence of Theorem 2.

Future directions. Our results indicate that many of the geometric Helly-type theorems are manifestations of purely combinatorial results concerning basic hypergraph parameters. Recently, Patáková [15] has found other far-reaching geometric applications. We believe that further study of the class $\mathcal{H}_k$ will be useful to attack the Kalai–Meshulam conjecture on the *homological VC dimension* and other problems on χ-boundedness in graph theory.

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Data-Classification Complexes and Tverberg-type theorems

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This is a report of research done in joint work with Deborah Oliveros (UNAM), Tommy Hogan, Dominic Yang (UC Davis). This work was supported by NSF. Full paper with proofs is available in the Arxiv.

The classical Tverberg's theorem says that a set with sufficiently many points in R^d can always be partitioned into m parts so that the (m - 1)-simplex is the (nerve) intersection pattern of the convex hulls of the parts. Motivated by questions from the performance of data classification algorithms, such as multi-class logistic regression method, we investigated other versions of Tverberg's theorem where the nerve complex describing the intersections of classes is not a simplex.

To state our results precisely we begin with some terminology and notation typical of geometric topological combinatorics (Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of convex sets in $\mathbb{R}^d$. The *nerve* $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ is the simplicial complex with vertex set $[m] := \{1, 2 \dots, m\}$ whose faces are $I \subset [m]$ such that $\cap_{i \in I} F_i \neq \emptyset$.

Given a collection of points $S \subset \mathbb{R}^d$ and an n-partition into n color classes $\mathcal{P} = S_1, \dots, S_n$ of S, we define *the nerve of the partition*, $\mathcal{N}(\mathcal{P})$ to be the nerve complex $\mathcal{N}(\{\mathrm{conv}(S_1), \dots, \mathrm{conv}(S_n)\})$, where $\mathrm{conv}(S_i)$ is the convex hull of the elements in the color class i. Note that this construction is also considerSimilarly, given a partition $\mathcal{P}$, we define the *intersection graph of the partition*, denoted $\mathcal{N}^1(\mathcal{P})$, as the 1-skeleton of the nerve of $\mathcal{P}$.

Given a simplicial complex K, and a finite set of points S in $\mathbb{R}^d$, we say that K is *partition induced on* S if there exists a partition $\mathcal{P}$ of S such that the nerve of the partition is isomorphic to K. We say that K is *d-partition induced* if there exists at least one set of points $S \subset \mathbb{R}^d$ such that K is partition induced on S. It was shown by G. Y. Perelman [7] that every d-dimensional simplicial complex is $(2d+1)$-partition induced on some point set. This result is in fact optimal, because the barycentric subdivision of the d-skeleton of a $(2d+2)$-dimensional simplex is not $2d$-partition induced, see [9] and [8] for details.

The key contribution of our paper is to generalize the classical Tverberg's theorem by showing that similar theorems exist where other simplicial complexes –not just simplices– appear as the nerve of the partition. We introduce a terminology for these special complexes:

Definition 1. *A simplicial complex K is d-*Tverberg *if there exists a constant* $\mathrm{Tv}(K, d)$ *such that K is partition induced on all point sets $S \subset \mathbb{R}^d$ in general position with $|S| > \mathrm{Tv}(K, d)$. The minimal such constant* $\mathrm{Tv}(K, d)$ *is called the* Tverberg number for K in dimension d.

First of all, note one can re-state the classical Tverberg's theorem as follows:

Theorem 2 (Tverberg's theorem rephrased)**.** *The* $(m-1)$*-simplex is a* d*-Tverberg complex for all* $d \geq 1$*, with Tverberg number* $(d+1)(m-1)+1$*.*

Finally, before stating our first result, recall that the *k-hypergraph Ramsey number* $R_k(m)$ is the least integer N such that every red-blue 2-coloring of all k-subsets of an N-element set contains either a red set of size m or a blue set of size m, where a set is called red (blue) if all k-subsets from this set are red (or respectively blue).

Theorem 3. *All trees and cycles are* d*-Tverberg complexes for all* $d \geq 2$*.*

- *(A) Every tree* T_n *on* n *nodes, is a* d*-Tverberg complex for* $d \geq 2$*. The Tverberg number* $\mathrm{Tv}(T_n, d)$ *exists and it is at most* $R_{d+1}((d+1)(n-1)+1)$*. More strongly,* $\mathrm{Tv}(T_n, 2)$ *is at most* $\binom{4n-4}{2n-2}+1$*.*
- *(B) Every* n*-cycle* C_n *with* $n \geq 4$ *is a* d*-Tverberg complex for* $d \geq 2$*. The Tverberg number exists and* $\mathrm{Tv}(C_n, d)$ *is at most* $nd+n+4d$*.*

The proof of Theorem 3 relies on several powerful non-constructive tools such as the Ham-Sandwich theorem, a characterization of oriented matroids of cyclic polytopes [3], and the multi-dimensional version of Erdös-Szekeres theorem (this is due to Grünbaum [5] and Cordovil and Duchet [3], see also Chapter 9 of [2], and the survey [6]). These tools are enough to show the existence of a Tverberg number $\mathrm{Tv}(T_n, d)$, but the bounds are far from tight.

We can prove the following general lower bound for the Tverberg numbers:

Lemma 4. *For any connected simplicial complex* K *with* $n \geq 2$ *vertices, if it exists, then* $\mathrm{Tv}(K, d) \geq 2n$*.*

In addition to this general lower bound, we show that the upper bounds of Theorem 3 can indeed be improved by giving better bounds on the Tverberg numbers of *caterpillar trees.* Caterpillar trees are those in which all the vertices are within distance one of a central path; these include paths and stars.

Theorem 5. *If a tree* T_n *is a caterpillar tree with* n *nodes, then* T_n *is* d*-Tverberg complex for all* d*, and its* d*-Tverberg number* $\mathrm{Tv}(T_n, d)$ *is no more than* $(d+1)(n-1)+1$*.*

In terms of intersection properties caterpillar graphs have been shown to be precisely the trees that are also interval graphs by Eckhoff [4]. In other words, the previous theorem implies that a tree T_n is also 1-Tverberg if and only if T_n is a caterpillar tree.

Furthermore, in dimension two we can give some exact Tverberg numbers for trees:

Theorem 6.

- *(A) The* 2*-Tverberg numbers* $\mathrm{Tv}(S_n, 2)$ *for a star tree with* n *nodes equals* $2n$*.*
- *(B) The* 2*-Tverberg numbers of the path and cycle with four nodes are* $\mathrm{Tv}(P_4, 2) = 9$ *and* $11 \leq \mathrm{Tv}(C_4, 2) \leq 13$*.*

The proof of Theorem 6 (B) requires exhaustive computer enumeration of all possible partitions, over all possible order types of point sets with fewer than ten points. Luckily, these order types were classified in [1].

The proofs require an investigation of oriented matroid types. We also present results on the distribution of simplicial complexes arising from the classification of data.

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What is . . . a *p*-adic Gromov–Borsuk–Ulam theorem?

Pavle Blagojević

(joint work with Nevena Palić, Roman Karasev)

Michail Gromov, in his 2003 landmark paper *Isoperimetry of waists and concentration of maps* [2], proved the following celebrated result.

Waist of the sphere theorem. *Let $n \geq k \geq 1$ be integers, and let $f\colon S^n \longrightarrow \mathbb{R}^k$ be a continuous map. Then there exists a point $z \in \mathbb{R}^k$ such that for every real number $\varepsilon > 0$ holds that*

$$\operatorname{vol}(f^{-1}(z) + \varepsilon) \geq \operatorname{vol}(S^{n-k} + \varepsilon),$$

where S^{n-k} denotes the equatorial $(n-k)$-dimensional sphere in the sphere S^n.

Here $S^{n-k} + \varepsilon$ stands for the ε-neighborhood of S^{n-k} in S^n with respect to the Euclidean distance of $\mathbb{R}^{n+1}$, and vol is the spherical n-volume.

In particular, for the projection map $p\colon S^n \longrightarrow \mathbb{R}^k$ onto the first k coordinates $(x_1,\ldots,x_{n+1}) \longmapsto (x_1,\ldots,x_k)$ we have that the fiber $p^{-1}(0)$ is exactly the equatorial sphere S^{n-k}. Consequently, every continuous map $S^n \longrightarrow \mathbb{R}^k$ has a fiber that is "at least as big as the largest fiber of the projection p".

In the case when $k = n$, according to the Borsuk–Ulam theorem, every continuous map $f\colon S^n \longrightarrow \mathbb{R}^n$ has a fiber containing the equatorial S^0. Thus, the Borsuk–Ulam implies the waist of sphere theorem in this situation. A natural question one can ask at this point is: Does the waist of sphere theorem for $k = n$ implies the Borsuk–Ulam theorem? The answer in general is no.

The waist of the sphere theorem, being a weakening of a Borsuk–Ulam theorem itself, emerged from a new dyadic result of the Borsuk–Ulam type, the following so-called Gromov–Borsuk–Ulam theorem.

Gromov–Borsuk–Ulam theorem. *Let $n \geq k \geq 1$ be integers, and let $f\colon S^n \longrightarrow \mathbb{R}^k$ be a continuous map. For every natural number i there exists a partition of the sphere S^n into 2^i open convex sets of equal volumes with all the center points of the elements of the partition having the same image in $\mathbb{R}^k$.*

Gromov's waist of sphere theorem, and in particular its proof, motivated a lot of new extended, modified and even discretized results, for an overview consult [3].

In this talk we aske the following question: *What would be a p-adic Gromov–Borsuk–Ulam theorem?* More precisely, what obstacle did we have to overcome in order to prove the following p-adic version of the waist of sphere theorem for a class of $\mathbb{Z}/p$-equivariant maps. Will a restriction on the class of maps imply some extra information about which fiber is the "biggest"?

A p-adic waist of sphere theorem. *Let $n > k \geq 1$ be integers and let p be an odd prime. Let the cyclic group $\mathbb{Z}/p$ be a subgroup of $\mathrm{SO}(n+1)$ acting orientation preserving and freely on the sphere S^n. Furthermore, let V be an arbitrary real $\mathbb{Z}/p$-representation of dimension k such that its Euler class is not contained in the Fadell–Husseini index of $\mathrm{SO}(n+1)$, that is $\mathfrak{e}(V) \notin \mathrm{Index}_{\mathbb{Z}/p}(\mathrm{SO}(n+1);\mathbb{F}_p)$. If $f\colon S^n \longrightarrow V$ is a $\mathbb{Z}/p$-equivariant map, then for every real number $\varepsilon > 0$*

$$\mathrm{vol}(f^{-1}(0)+\varepsilon) \geq \mathrm{vol}(S^{n-k}+\varepsilon).$$

The theorem we are looking for, a p-adic Gromov–Borsuk–Ulam theorem that implies the stated p-adic waist of sphere theorem, claimes the following.

A p-adic Gromov–Borsuk–Ulam theorem. *Let $n > k \geq 1$ be integers and let p be an odd prime. Let the cyclic group $\mathbb{Z}/p$ be a subgroup of $\mathrm{SO}(n+1)$ acting orientation preserving and freely on the sphere S^n, and let $c\colon \mathcal{CO}(S^n) \longrightarrow S^n$ be a center map. Furthermore, let V be an arbitrary real $\mathbb{Z}/p$-representation of dimension k such that its Euler class is not contained in the Fadell–Husseini index of $\mathrm{SO}(n+1)$, that is $\mathfrak{e}(V) \notin \mathrm{Index}_{\mathbb{Z}/p}(\mathrm{SO}(n+1);\mathbb{F}_p)$. If $f\colon S^n \longrightarrow V$ is a $\mathbb{Z}/p$-equivariant map, then for every real number $\varepsilon > 0$ there exists an integer i_ε with the property that for every $i \geq i_\varepsilon$ there exists a $(\mathbb{Z}/p \times \mathbb{Z}/p^{li})$-invariant convex partition Π of the sphere S^n into p^{i+1}-pieces such that:*

(1) every (convex) element S of the partition Π is a (k,ε)-pancake,
(2) $f(c(S)) = 0 \in V$ for every element S of the partition Π, and
(3) $\mathrm{vol}(S_1) = \mathrm{vol}(S_2)$ for all elements S_1, S_2 of the partition Π.

Here a subset S of the sphere S^n is assumed to be convex if S is not contained in any closed hemisphere of S^n and in addition the cone spanned by S with apex at the origin is a convex subset of $\mathbb{R}^{n+1}$. With $\mathcal{CO}(S^n)$ we denoted the metric space of all open convex subsets of S^n where the metric is assumed to be the Hausdorff metric. Then a center map is just a continuous map $\mathcal{CO}(S^n) \longrightarrow S^n$. For a definition of a (k,ε)-pancake see [4, Def. 3.1]. A (convex) partition of the sphere S^n into r pieces is an ordered collection of r its open (convex) sets $(S_1, \dots, S_r)$ such that

— $S_1 \cup \cdots \cup S_r = S^n$,
— $S_i \neq \emptyset$ for all $1 \leq i \leq r$, and
— $S_i \cap S_j = \emptyset$ for all $1 \leq i < j \leq r$.

The proof we sketch uses methods of Fadell–Husseini ideal index theory and equivariant obstruction theory in combination with the cellular model of Blagojević and Ziegler for the classical configuration space [1]. For more details of the proof consult the PhD thesis on Nevena Palić [5].

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Generalizations of the Borsuk–Ulam theorem

Florian Frick
(joint work with Henry Adams and Johnathan Bush)

The Borsuk–Ulam theorem states that any odd map $f\colon S^n \to \mathbb{R}^n$, that is, a map satisfying $f(-x) = -f(x)$ for all $x \in S^n$, has a zero. This result has found numerous applications across mathematics—many of them in combinatorics and discrete geometry, such as Lovász' proof of Kneser's conjecture or the Ham–Sandwich theorem. In these applications one parametrizes (a subspace of) possible solutions by the n-sphere S^n in a symmetric way and then constructs the map f as measuring n equations that commute with the antipodal symmetry. Invoking the Borsuk–Ulam theorem then finishes the proofs.

If fewer equations need to be balanced, and thus f is a map to $\mathbb{R}^k$ with $k < n$, we expect more solutions. Several suitable such strengthenings of the Borsuk–Ulam theorem exist. For example, it is a consequence of a result of Yang [5] that any odd map $f\colon S^n \to \mathbb{R}^k$ has an $(n-k)$-dimensional subspace of zeros.

Here we address the question, what can be shown if the system of equations is overdetermined, for an odd map $f\colon S^n \to \mathbb{R}^k$ with $k > n$? Certainly a generic map f will not have any zeros. In joint work with Henry Adams and Johnathan Bush [2], we prove:

Theorem 1. *(a) If $f\colon S^{2n-1} \to \mathbb{R}^{2kn+2n-1}$ is odd and continuous, then there is a subset $X \subset S^{2n-1}$ of diameter at most $\frac{2\pi k}{2k+1}$ such that* $\operatorname{conv}(f(X))$ *contains the origin.*

(b) If $f\colon S^n \to \mathbb{R}^{n+2}$ is odd and continuous, then there is a subset $X \subset S^n$ of diameter at most the diameter of the regular $(n+1)$-simplex inscribed in S^n such that $\operatorname{conv}(f(X))$ *contains the origin.*

Here S^n carries the instrinsic metric, where each closed geodesic has length 2π. Part (a) for $k = 0$ is the usual Borsuk–Ulam theorem for maps from odd-dimensional spheres. It should be emphasized that the proof of Theorem 1 uses the Borsuk-Ulam theorem. The new ingredient of [1, 2] is a lower bound for the topology of a certain configuration space of nearby points: Given a metric space X and scale parameter $\varepsilon > 0$, the metric thickening X_ε is the space of all probability measures in X with finite support of diameter less than ε. We equip this space with the 1-Wasserstein metric, or metric of optimal transport. Lower bounding the homotopical connectivity of the metric thickening of spheres at varying scale parameters ε, together with the classical Borsuk–Ulam theorem, yields Theorem 1.

Here we outline one application of this result. A trigonometric polynomial is an expression of the form $p(t) = c + \sum_{k=1}^n a_k \cos(kt) + b_k \sin(kt)$, inducing a map $S^1 \to \mathbb{R}$. In the case that $c = 0$, we call p a *homogeneous* trigonometric polynomial. The set $S \subset \{1, \dots, n\}$ of integers k with $a_k \neq 0$ or $b_k \neq 0$ is called the *spectrum* of p, and the largest integer in S is the *degree* of p. The spectrum of p constrains the set of roots of p; for example, if p is homogeneous of degree n then it has a root on any closed circular arc of length $\frac{2\pi n}{n+1}$; see [3, 4].

If the spectrum of p consists only of odd integers, then p is called a *raked* trigonometric polynomial. We show the following structural result about the roots of raked trigonometric polynomials:

Theorem 2. *Let $X \subset S^1$ be such that* $\operatorname{diam}(X) < \frac{2\pi k}{2k+1}$. *Then there is a raked homogeneous trigonometric polynomial of degree $2k-1$ that is positive on X. Moreover, there is a set $X \subset S^1$ of diameter $\frac{2\pi k}{2k+1}$ such that no raked homogeneous trigonometric polynomial of degree $2k-1$ is positive on X.*

Apply part (a) of Theorem 1 for $n = 1$ to the symmetric trigonometric moment curve

$$\gamma\colon S^1 \to \mathbb{R}^{2k}, \quad t \mapsto (\sin(t), \cos(t), \sin(3t), \cos(3t), \dots, \cos((2k-1)t).$$

There is a set $X \subset S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that the convex hull of $\gamma(X)$ captures the origin. In particular, no hyperplane can separate $\gamma(X)$ from the origin and thus the inner product $\langle z, \gamma(X)\rangle$ has to change sign for every $z \in \mathbb{R}^{2k} \setminus \{0\}$. The inner products of a non-zero vector z with γ range over all non-zero raked homogeneous trigonometric polynomials of degree at most $2k-1$, which proves the second part of Theorem 2.

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Embeddability of Simplicial Complexes is Undecidable

Uli Wagner

(joint work with Marek Filakovský, Stephan Zhechev)

We consider the following decision problem $\text{EMBED}_{k\to d}$ in computational topology (where $k \le d$ are fixed positive integers): Given a finite simplicial complex K of dimension k, does there exist a (piecewise-linear) embedding of K into $\mathbb{R}^d$?

The special case $\text{EMBED}_{1\to 2}$ is *graph planarity*, which is decidable in linear time, by the well-known algorithm of Hopcroft and Tarjan [8]; the same is true for $\text{EMBED}_{2\to 2}$ [6].

For $d = 3$, the problems $\text{EMBED}_{2\to 3}$ and $\text{EMBED}_{3\to 3}$ are known to be algorithmically decidable [13] as well as NP-hard as well [5]; the precise computational complexity of these problems remains unresolved.

In higher dimensions, the problem $\text{EMBED}_{k\to d}$ is polynomially-time solvable for any fixed pair (k, d) in the so-called *metastable range* $d \ge \frac{3(k+1)}{2}$. This follows from the celebrated *Haefliger–Weber theorem* [7, 14] (which characterizes embeddability in the metastable range in terms of a suitable *deleted product obstruction*) together with a series of results by Čadek et al. [1, 3, 9, 4] in computational homotopy theory (which show that the deleted product obstruction can be efficiently computed).

By contrast, we prove that the embeddability problem is undecidable for almost all pairs of dimensions outside the metastable range:

Theorem 1. $\text{EMBED}_{k\to d}$ *is algorithmically undecidable for* $8 \le d < \lfloor \frac{3(k+1)}{2} \rfloor$.

This almost completely resolves the decidability vs. undecidability of the embeddability problem in higher dimensions and establishes a sharp dichotomy between polynomial-time solvability and undecidability.

Theorem 1 strengthens, in a wide range of dimensions, earlier results of Matoušek et al. [12], who showed that $\mathrm{EMBED}_{k\to d}$ is undecidable for $4 \le k \in \{d-1, d\}$, and NP-hard for all remaining pairs (k, d) outside the metastable range and satisfying $d \ge 4$. Moreover, our result complements recent work of Manin and Weinberger [10], who showed that the question of whether a k-dimensional manifold with boundary admits a *smooth* embedding into $\mathbb{R}^d$ is algorithmically undecidable if $d-k$ is *even* and $11k \ge 10d+1$ (the manifolds they construct are always piecewise-linearly embeddable).

Our proof builds on work by Čadek et al. [2], who showed how to encode an arbitrary system of Diophantine equations into a homotopy-theoretic *extension problem* (whence the latter is undecidable by Matiyasevich's [11] negative solution to Hilbert's Tenth Problem); we turn their construction into an embeddability problem, using techniques from piecewise-linear topology due to Penrose, Whitehead, Zeeman, Irwin, Lickorish, and others.

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Continuous matroids

Anders Björner
(joint work with László Lovász)

This talk reviews work done in the mid-1980s in collaboration with László Lovász. Our main concern at that time was to provide conditions that make it possible to pass to the limit of a class of finite matroids. See [2] for a recent update.

The characteristic property of a continuous matroid is the existence of a rank function taking as values the full real unit interval. Known examples of such rank functions include Lebesgue measure on the unit interval and the dimension function of certain von Neumann algebras. In both these cases the lattice property of modularity plays a crucial role. A more general concept, *pseudomodularity*, makes possible the construction of e.g. continuous field extensions (algebraic matroids) and continuous partition lattices (graphic matroids).

In the papers [1, 3] we presented conditions guaranteeing, for certain classes $\mathcal{C}$ of finite matroid lattices, the existence of embedding schemes that allow passing to the limit $\mathcal{C}_\infty$. A basic and beautiful example of the kind we have in mind is when $\mathcal{C}$ is the class of finite Boolean lattices. Then $\mathcal{C}_\infty$ is the sigma algebra of Lebesgue measurable subsets of the unit interval, and the rank function is Lebesgue measure. Another example is the class $\mathcal{PG}$ of finite projective geometries over a finite field, for which $\mathcal{PG}_\infty$ is the corresponding hyperfinite von Neumann geometry. These measure-theoretic and dimension-theoretic interpretations suggest probabilistic and geometric connections.

From a combinatorial point of view, these examples, both due to von Neumann, can be thought of as continuous analogues of free matroids and linear matroids. We show that also partition lattices and field extensions have such continuous analogs, corresponding to the classes of graphic and algebraic matroids, respectively.

The main problem one faces when trying to construct embedding schemes and limits for matroids is the absence of a key technical property: modularity. We introduce a somewhat weaker notion, called *pseudomodularity*, that does the job for us in several cases where modularity is missing.

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Even maps, the Colin de Verdiére number, and representations of graphs

Martin Tancer
(joint work with Vojtěch Kaluža)

In 1990 Colin de Verdière [CdV90] introduced a graph parameter $\mu(G)$. It arises from the study of the multiplicity of the second smallest eigenvalue of certain matrices associated to a graph G (discrete Schrödinger operators); however, it turns out that this parameter is closely related to geometric and topological properties of G. In particular, this parameter is minor monotone, and moreover, it satisfies:

(i) $\mu(G) = 0$ if and only if G embeds in $\mathbb{R}^0$;
(ii) $\mu(G) \leq 1$ if and only if G embeds in $\mathbb{R}^1$;
(iii) $\mu(G) \leq 2$ if and only if G is outer planar;
(iv) $\mu(G) \leq 3$ if and only if G is planar; and
(v) $\mu(G) \leq 4$ if and only if G admits a linkless embedding into $\mathbb{R}^3$.

The characterization up to the value 3 as well as the minor monotonicity of μ was shown by Colin de Verdière [CdV90, CdV91]. The characterization of graphs with $\mu(G) \leq 4$ was established by Lovász and Schrjver [LS98]. Beyond this, any description is known only for the classes of graphs with $\mu(G) \geq |V(G)| - k$ for $k = 1, 2, 3$ and partial results are known also for $k = 4, 5$; see [KLV97]. Due to the aforementioned properties, the study of μ gained a lot of popularity.

Later, in 2009, van der Holst and Pendavingh [vdHP09] introduced another minor monotone parameter $\sigma(G)$, whose definition is much closer to the topological properties of G. Roughly speaking, $\sigma(G)$ is defined as a minimal integer k such that every CW-complex $\mathcal{C}$ whose 1-skeleton is G admits a so-called even mapping into $\mathbb{R}^k$. This is a mapping f such that whenever ϑ and τ are disjoint cells of $\mathcal{C}$, then $f(\vartheta) \cap f(\tau) = \emptyset$ if $\dim \vartheta + \dim \tau < k$, and $f(\vartheta)$ and $f(\tau)$ cross in an even number of points if $\dim \vartheta + \dim \tau = k$. For a precise definition, we refer to [vdHP09].

It turns out that $\sigma(G) \leq k$ if and only if $\mu(G) \leq k$ for $k \in \{0, 1, 2, 3, 4\}$. In addition, van der Holst and Pendavingh [vdHP09, Conj. 43] conjectured that this is true also for $k = 5$. However, in general, σ and μ differ. They provide an example of a graph with $\mu(G) \leq 18$, but $\sigma(G) \geq 20$ based on a previous work of Pendavingh [Pen98]. On the other hand, van der Holst and Pendavingh [vdHP09, Cor. 41] proved that $\mu(G) \leq \sigma(G) + 2$, while they conjectured that $\mu(G) \leq \sigma(G)$. We confirm this conjecture.

Theorem 1. *For any graph G, $\mu(G) \leq \sigma(G)$.*

Our tools that we use for the proof of Theorem 1 also allow us to show that the gap between μ and σ appears at much smaller values.

Theorem 2. *There is a graph G such that $\mu(G) \leq 7$ and $\sigma(G) \geq 8$.*

We remark here that adding a new vertex to a graph G and connecting it to all vertices of G increases both $\mu(G)$ and $\sigma(G)$ by exactly one (unless G is the complement of K_2); see [vdHLS99, Thm. 2.7] and [vdHP09, Thm. 28]. Consequently, Theorem 2 immediately implies that for every $k \in \mathbb{N}, k \geq 7$ there is a graph G_k with $\mu(G_k) \leq k$ and $\sigma(G_k) \geq k+1$.

The key step in the proof of Theorem 2 is to provide a lower bound on σ; otherwise we follow [Pen98]. We remark that the example of G with $\mu(G) \leq 18$ but $\sigma(G) \geq 20$ coming from [vdHP09, Pen98] is highly regular Tutte's 12-cage. The important property is that the second largest eigenvalue of the adjacency matrix of Tutte's 12-cage has very high multiplicity. We use instead the incidence graphs of finite projective planes, which enjoy the same property. Namely, if H_q is the incidence graph of a finite projective plane of order q, we will show that $\mu(H_3) \leq 9$, whereas $\sigma(H_3) \geq 11$. Then, by further modification of this graph, we obtain the graph from Theorem 2.

As a complementary result, based on properties of finite projective planes, we also show that the gap between μ and σ is asymptotically large.

Theorem 3. *Let $q \in \mathbb{N}$ be such that a finite projective plane of order q exists. Then $\mu(H_q) \in O(q^{3/2})$, while $\sigma(H_q) \geq \lambda(H_q) \geq q^2$, where λ is the graph parameter from* [vdHLS95].

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On the real matroid of all 0-1 vectors

Louis Billera

(joint work with Florian Frick)

A collection $\mathcal{C} \subset 2^{[n]}$ of subsets of the set $[n] = \{1, 2, \ldots, n\}$ is said to be balanced if, considered as vertices of the n-cube $[0,1]^n$, their convex hull meets the diagonal $[(0,\ldots,0),(1,\ldots,1)]$ in the cube. Otherwise $\mathcal{C}$ is unbalanced. Thus $\{\{1,2\},\{1,3\},\{2,3\}\}$ is balanced, while $\{\{1\},\{2\},\{1,2\}\}$ is unbalanced. We seek to enumerate maximal unbalanced collections. These each contain $2^{n-1} - 1$ sets. The study of balanced collections arose in economics over 50 years ago, while the question of enumerating unbalanced collections arose more recently in quantum physics.

The maximal unbalanced collections in $[n+1]$ are in bijection with the regions cut out by the hyperplane arrangement $\mathcal{A}_n$ consisting of all hyperplanes in $\mathbb{R}^n$ having 0-1 vectors as normals. Thus we are led to study the real matroid M_n on the set of all 0-1 n-vectors. We show the coefficient of t^{n-2} in the characteristic polynomial of M_n is $(4^n - 3^n - 2^n + 1)/2$.

Additionally, we show, jointly with F. Frick, that the family of all unbalanced collections in $[n]$ is a simplicial complex of dimension $2^{n-1}-2$ having the homotopy type of the $(n-2)$-sphere.

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Open Problems

collected by Joseph Doolittle

1. Positivity of Matroids

(Jesús De Loera)

Let $P(M)$ be the matroid polytope of M.

Conjecture 1.
- *The coefficients of the Ehrhart polynomial of $P(M)$ are positive.*
- *The h^*-vector of $P(M)$ is unimodal.*

The conjecture is true for all uniform matroids of rank $leq2$, all uniform matroids with $|E| \leq 75$, and all matroids with $|E| \leq 9$.

2. Cubical polytopes and almost interesecting triangles

(Eran Nevo)

The following two questions are related.

Let $C(d, n)$ be the maximum number of facets of a cubical d-polytope with n vertices.

Question 2. *How large is $C(d, n)$?*

An observation of Kalai is that $C(d, n) = O(n^2)$. Is $C(d, n) = o(n^2)$ for all d?

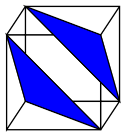
Let $t(n)$ be the maximum number of triangles in $\mathbb{R}^3$ with vertices from a point set of size n such that the intersection of any two triangles is either a vertex of both triangles or empty.

Question 3 (Kalai's almost disjoint triangles)**.** *How large is $t(n)$?*

Previous observations are that $t(n) = \Omega(n^{\frac{3}{2}})$ and $t(n) = O(n^2)$.

A new observation is that $t(n) \geq \frac{2}{7}C(4, n)$.

This can be seen by taking any cubical 4-polytope, and subdividing $\mathbb{R}^3$ by the Schlegel diagram of the polytope. Then in one of the cubes, the following triangles are created.



All 6 cubes adjacent to this one and this cube itself are removed from consideration. This process is iterated, and upon completion, $\frac{1}{7}$ of the cubes will have had two triangles drawn in them. No pair of triangles created this way can share anything but a common vertex. This gives the $\frac{2}{7}$ coefficient in the observation above.

3. The complex of totally mixed faces

(Sam Payne)

Let P_1 and P_2 be polytopes in $\mathbb{R}^n$. Say a face F of the Minkowski sum $P_1 + P_2$ is *totally mixed* if $F = F_1 + F_2$ with $F_i \in P_i$ such that:

- $dim(F_i) > 0$
- $dim(F) = dim(F_1) + dim(F_2)$

The totally mixed faces of $P_1 + P_2$ forms a poset. Is this poset shellable? Does its geometric realization have the rational homologz of a wedge of $n - 2$-spheres?

The motivation for this question comes from the tropicialization of complete intersections.

4. Spanning trees of bipartite graphs

(Steve Klee)

Let G be a bipartite graph with vertices $V(G) = X \sqcup Y$.

Ehrenborg made the followin conjecture about the number of spanning trees of G.

Conjecture 4.

$$\tau(G) \leq \frac{\prod_{v \in V(G)} \deg(v)}{|X| \cdot |Y|}$$

Ehrenborg and van Willigenburg showed the bound is tight for Ferrers graphs.

The presenter now devlops an equivalent conjecture.

Assume $X = [n]$. For $S \subset [n]$ nonempty, let $X_S = \#\{y \in Y : N(y) = S\}$. Construct the n by n matrix M with entries

$$M_{ii} = \sum_{S : i \in S} |S|$$

$$M_{ij} = \sum_{S : i \in S, j \notin S} \frac{X_S}{|S|}.$$

Conjecture 5. $\det(M) \leq \det(\operatorname{diag}(M))$.

There are results like this in linear algebra, but they generally need M symmetric and PSD. M is not symmetric.

$\det(M)$ and $\det(\operatorname{diag}(M))$ are invariant under permuting the labels in X. So there are polynomials in $\mathbb{Q}[X_S : S \subset [n] nonempty]$ fixed by $\mathcal{S}_n$. Is there a nice basis in symmetric function theorey to represent these polynomials?

Experiments for small n seem to indicate that

$$\det(\operatorname{diag}(M)) - \det(M) = \sum c_\mu x^\mu + \sum c_{\mu\nu}(x^\mu - x^\nu)^2$$

with the c_μ and $c_{\mu\nu}$ non-negative. Is this more precise statement true?

5. A curious hyperplane arrangement

(Nati Linial)

Let H be the hyperplane arrangement in $\mathbb{R}^{\binom{n}{2}}$ whose hyperplanes are indexed by unordered triples of integers in $[n]$. The hyperplane $H_{i,j,k}$ is given by the equation $x_{i,j} + x_{j,k} = x_{i,k}$. Is anything known about this arrangement?

6. Lattice Polytope generators

(Benjamin Nill)

Given $A \subset \mathbb{Z}^d$ finite, define the affine lattice

$$\operatorname{aff}(A) := \left\{ \sum_{a \in A} K_a \cdot a : K_a \in \mathbb{Z}, \sum_{a \in A} K_a = 1 \right\}$$

$$SR(A) := \min\{|B| : B \subset A, \operatorname{aff}(B) = \operatorname{aff}(A)\}$$

Clearly, $SR(A)$ is unbounded. Now we consider convexity.

Let $P := \operatorname{conv}(A)$ a d-dimensional lattice polytope with $\operatorname{aff}(P \cap \mathbb{Z}^d) = \mathbb{Z}^d$.

$$SR(P) := \min\{|B| : B \subset P \cap \mathbb{Z}^d, \operatorname{aff}(B) = \mathbb{Z}^d\}$$

If P contains affine lattice basis, then $SR(P) = d + 1$.

Theorem 6 (Averkov, Hofscheier, Nill)**.** *There exists* $f : SR(P) \leq f(d)$.

$$SR(d) := \max\{SR(P) : P \text{ a } d\text{-dimensional lattice polytope}\}$$

$SR(1) = 2$, $SR(2) = 3$

Theorem 7 (Blanco, Santos, 2017)**.** $SR(3) = 5$

Question 8. *What is* $SR(4)$*? What is* $O(SR(d))$*?*

7. Equivariant Log Concavity

(Nicholas Proudfoot)

Let M be a matroid and W be a finite group of symmetries of M. Let $OS^*(M)$ be the Orlik-Soloman algebra, which is a graded ring with an action of W.

Conjecture 9. *For all i, $OS^i(M)^{\otimes 2}$ contains $OS^{i-1}(M) \otimes OS^{i+1}(M)$ as a subrepresentation.*

Example 10. *If W is the trivial group, this was proved by Adiprasito-Huh-Katz.*

Example 11. *Let M be the braid matroid and $W = \mathcal{S}_n$. Then $OS^*(M) \cong H^*(\mathrm{Conf}(n, \mathbb{R}^2))$. This was verified by computer up to $n = 10$.*

8. Equivariant Gal phenomenon

(Michelle Wachs)

The question as to whether there is an equivariant version of Gal's conjecture was raised in a paper with John Shareshian (arXiv:1702.0666). Let P be a flag simplicial d-dimensional polytope on which a finite group G acts simplicially. The action of G induces a graded representation of G on cohomology of the associated toric variety $X(P)$. We say that (P, G) exhibits the *equivariant Gal phenomenon* if there exist G-modules $\Gamma_{P,k}$ such that

$$\sum_{j=0}^{d} H^{2j}(X(P))t^j = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \Gamma_{P,k}\ t^k(1+t)^{d-2k}.$$

This reduces to γ-positivity of the h-polynomial of P when G is the trivial group. Gal's conjecture says that the h-polynomial of every flag simplicial polytope is γ-positive.

The dual permutohedron P_n^* and the dual stellohedron St_n^* are examples of flag simplicial polytopes. There is a natural simplicial action of the symmetric group $\mathfrak{S}_n$ on P_n^* and also one on St_n^*. Shareshian and I showed that $(P_n^*, \mathfrak{S}_n)$ and $(St_n^*, \mathfrak{S}_n)$ both exhibit the equivariant Gal phenomenon. We also showed that not all pairs (P, G) where P is a flag simplicial polytope and G is a group acting simplicially on P, exhibit the equivariant Gal phenominon. We posed the following problem: For nontrivial G, find classes of (P, G) beyond $(P_n^*, \mathfrak{S}_n)$ and $(St_n^*, \mathfrak{S}_n)$ that exhibit the equivariant Gal phenomenon.

9. Zero point suspensions

(Francisco Santos)

For any simplicial complex S one can define the *suspension (or bipyramid) of S* as the join of S with two points v_1 and v_2. One can also reduce by one the number of vertices in the result as follows: consider any vertex v of the original S. Since the edges $\{v, v_1\}$ and $\{v, v_2\}$ in the suspension of S have the same link, one can make a $(2,1)$-*bistellar flip* that merges these two edges (and every pair of faces $\sigma \cup \{v, v_1\}$ and $\sigma \cup \{v, v_2\}$ in their stars) into a single one. This operation is called *one-point* *suspension of* S because it is homeomorphic to the suspension, it contains S as a subcomplex, and it increases the vertex set by one instead of two.

Question 12. *Given a simplicial complex* S *that is a combinatorial* d*-sphere with* $n \geq d+3$ *vertices, does there always exist a combinatorial* $(d+1)$*-sphere* S' *with the same vertex set and with* $S \subset S'$*?*

It would follow from the follwing false statement: There is a vertex v in S such that the ball $S \backslash \star(v)$ can be completed to a d-sphere without additional vertices. A counterexample to this statement appears in Altshuler's "A peculiar triangulation of the 3-sphere". This example has 10 vertices and has vertex-transitive symmetry. It is also neighborly. However, this 3-sphere is contained in a 4-sphere with 10 vertices, so is not a counterexample to the question.

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