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Calculus of Variations (hybrid meeting)

Organized by
Alessio Figalli, Zürich
Robert V. Kohn, New York
Tatiana Toro, Seattle
Neshan Wickramasekera, Cambridge UK

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Abstract. Calculus of Variations touches several interrelated areas. In this workshop we covered several topics, such as minimal submanifolds, mean curvature and related flows, free boundary problems, variational models of interacting dislocations, defects in physical systems, phase transitions, etc.

Mathematics Subject Classification (2010): 49-xx, 35Jxx, 53Cxx, 58-xx.

Introduction by the Organizers

The workshop *Calculus of Variation* took place in a very challenging moment, as it was one of the first workshops after the lockdown of the institute due to Covid-19. In particular, this has been a hybrid workshop, with approximately 20 local participants and other 25 connected online. Despite the difficulties posed by such a new format, the workshop has been a success, at least for what concerns the talks (of course, the interaction has been much more difficult for the participants online). Indeed, we managed to have a full schedule, with 4 to 5 talks per day, covering essentially all the topics discussed in the proposal.

We can group the talks according to their subject area, and in chronological order in each area, as follows:

- *Minimal surfaces and curvature flows*: Gerhard Huisken presented joint work with Markus Wolff on anisotropic inverse mean curvature flow and its applications to general relativity; Zihui Zhao presented her joint work with Camillo De Lellis on the boundary regularity of multivalued minimizers of the Dirichlet energy for analytic boundary data; on a topic with close parallels with the theory of minimal hypersurfaces, Juncheng Wei presented a series of results, with Yong Liu and Kelei Wang, concerning the structure of finite Morse index solution of the Allen-Cahn equation, proving stability of saddle solutions in dimensions 8, 10, and 12 and thereby settling a conjecture of Cabré; Giada Franz discussed joint work with Alessandro Carlotto and Mario Schultz constructing free boundary minimal surfaces of arbitrary genus in the unit ball in $\mathbb{R}^3$; Costante Bellettini presented his work establishing that in any compact manifold of positive Ricci curvature, minimal hypersurfaces arising from minmax solutions to the Allen–Cahn equation have multiplicity 1; Simone Steinbrüchel discussed boundary regularity of area minimizing hypersurfaces, showing how to adapt the classical work of Hardt–Simon to the setting of Riemannian ambient spaces; Yoshihiro Tonegawa described a result, with Lami Kim and Salvatore Stuvard, on the existence of a weak-mean curvature flow (i.e. Brakke flow) starting from closed rectifiable sets.

- *Vectorial variational problems*: Angkana Rüland discussed a series of results, joint with Pierluigi Cesana, Francesco Della Porta, Jamie Taylor, Christian Zillinger and Barbara Zwicknagl, on the symmetric and the dynamics of shape-memory alloys; Radu Ignat showed a recent result, with Antonin Monteil, on the symmetry of transition layers in some variational systems; Yves van Gennip described a series of works with Yoshikazu Giga, Jun Okamoto, Blaine Keetch and Jeremy Budd et al. on the study of Ginzburg-Landau type energy on graphs, and their use to define gradient flows for applications in data analysis and image processing; Cyrill Muratov presented work with A. Bernand-Mantel and T. M. Simon on the minimization of micromagnetic energies, giving a quantitative description of skyrmion solutions in the regime when the exchange is the dominant term in the energy; Christof Melcher discussed joint work with Zisis N. Sakellaris, where they investigated how classical spin-orbit coupling emerges from reduced rotational symmetry in a purely geometric framework, showing an existence result, at suitable energy levels, of smooth solutions for an anisotropic versions of the harmonic map problem.
- *Other*: Maria Colombo presented a series of results with Antonio De Rosa, Andrea Marchese, Paul Pegon and Antoine Prouff on branched optimal transport, showing the stability of optimizers with respect to variations of the sources and target measures; Xavier Fernarndez-Real described two results with Y. Jhaveri and X. Ros-Oton on the study of the free boundary regularity in the fractional obstacle problem, showing that generically free boundaries are much nicer than one would expect from explicit examples; Antonin Chambolle described the variational approach to fracture, corresponding to a quasi-static evolution process introduced to generalize the classical theory of Griffith for predicting crack growth in brittle material,

and described a series of new results on the existence of minimizers; Selim Esedoglu presented work with Alexander Zaitzeff and Krishna Garikipati investigating ways of discretizing gradient flows, showing methods to improve the convergence of the discrete solution to the continuous one, and how to apply it the mean curvature flow.

Workshop (hybrid meeting): Calculus of Variations

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Abstracts

Anisotropic inverse mean curvature flow and applications in General Relativity

Gerhard Huisken
(joint work with Markus Wolff)

We consider closed embedded hypersurfaces $F : \Sigma^2 \to (N^3, g, K)$ bounding a region in an asymptotically flat Riemannian manifold (N^3, g) that carries an additional symmetric bilinear form K. Such manifolds arise as initial data sets for the Einstein equations in Lorentzian manifolds (L^4, h) modelling isolated gravitating systems such as stars, black holes or galaxies. The deformation of such hypersurfaces by inverse mean curvature flow

$$\frac{d}{dt}F = \frac{1}{H}\,\nu, \quad \text{(Here H is the mean curvature,} \, \nu \, \text{is the external unit normal)}$$

with respect to g was used in [1] to prove the Riemannian version of the Penrose inequality, further regularity properties were proven in [2].

In the lecture a new flow is explained where inverse mean curvature flow is replaced by the "flow along inverse space-time mean curvature ". The space-time inverse mean curvature is given by $\sqrt{H^2 - P_{\Sigma^2}}$, where P_Σ is the two-trace of K on Σ, motivated by the fact that this yields the Lorentz-norm of the mean curvature vector if the 2-surface $\Sigma^2 \subset (N^3, g, K) \subset (L^4, h)$ arises as a co-dimension 2 surface in a Lorentzian 4-manifold. In [4] Cederbaum and Sakovich showed that the Lorentz invariance of this quantity allows for the construction of foliations by surfaces of constant space-time mean curvature under weaker asymptotic conditions than necessary for constant mean curvature surfaces.

Solutions of the smooth flow equation

$$\frac{d}{dt}F = \frac{1}{\sqrt{H^2 - P_{\Sigma^2}}}\,\nu$$

develop singularities in general, so one needs to look for weak solutions constructed as level-sets of an 'arrival time' function $u : N^3 \to R$ satisfying a zero-Dirichlet condition on the initial surface and a quasi-linear degenerate elliptic PDE in the exterior of the initial surface:

$$D_i\Big(\frac{D_i u}{|Du|}\Big) = \sqrt{|Du|^2 + ((g^{ij} - \frac{\nabla^i u \nabla^j u}{|\nabla u|^2})K_{ij})^2}$$

which is equivalent to the original equation if the solution is smooth with non-vanishing gradient. A concept of weak solutions needs to allow "jumps" of the surface corresponding to regions of zero gradient in the solution u - and there the anisotropy of the righthand side creates major difficulties since in these jump-regions a "unit normal" is not well defined.

To overcome this difficulty we add an extra dimension and look for a function U together with a vectorfield ν on $N^3 \times R$ (which equals the unit normal to the level-sets in smooth regions) such that the sublevel-sets $E_t := \{U < t\}$ minimize the functional

$$\mathcal{J}^{\widetilde{K}}_{U,\nu}(F) := |\partial^* F \cap \widetilde{K}| - \int_{F\cap\widetilde{K}} \sqrt{|\nabla U|^2 + P_\nu^2}, \tag{1}$$

on compact sets $\widetilde{K} \subset N^3 \times R$, where $P_\nu := \left(g^{ij} - \nu^i\nu^j\right) K_{ij}$ is the 2-trace of K orthogonal to the vectorfield ν.

It is shown how to construct solutions in this weak sense for triples (N^3, g, K) satisfying $\mathrm{tr}_{N^3} K \equiv 0$ under very general asymptotic conditions on the initial data and the ambient manifold using elliptic regularisation, compare [5]. It is shown that in these solutions the level sets are of class $C^{1,\alpha}$ and the cylinders above jump regions are foliated by $C^{2,\alpha}$- hypersurfaces satisfying the equation $H = |P_\nu|$. Details will appear elsewhere.

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Symmetry, (Ir-)Regularity and Avalanching Dynamics in Shape-Memory Alloys

Angkana Rüland

(joint work with Pierluigi Cesana, Francesco Della Porta, Jamie M Taylor, Christian Zillinger, Barbara Zwicknagl)

This talk is based on [DPR20, CDPRZZ20, DPRTZ20, RTZ19].

Shape-memory alloys undergo a solid-solid diffusionless phase transformation in which symmetry is reduced in the passage from the high temperature phase (austenite) to the low temperature phase (martensite). They are often very successfully modelled variationally by the phenomenological theory of martensite [BJ89, Bal04, Bha03]. Due to the highly non-(quasi-)convex and non-linear structure of the energies, a striking dichotomy between rigidity and flexibility arises in their mathematical analysis: At (relatively) high regularity the solutions are often rather rigid, obeying certain characteristic equations, while at low regularity a plethora of wild, highly fractal solutions exist [DM95, MŠ99, DM12]. Already the differential inclusions modelling exactly stress-free structures display this behaviour. In this talk, we hence consider the differential inclusion

$$\nabla u \in K \text{ in } \Omega \subset \mathbb{R}^n,$$

with $u : \Omega \to \mathbb{R}^n$ denoting the deformation of the material with respect to the austenite reference configuration. The set $K = K(\theta) \subset \mathbb{R}^{n\times n}$ is typically of the form

$$K(\theta) = \begin{cases} \alpha(\theta)SO(n)Id & \text{for } \theta > \theta_c, \\ \bigcup\limits_{m=1}^{N} SO(n)U_j(\theta_c) \cup \alpha(\theta_c)SO(n)Id & \text{for } \theta = \theta_c, \\ \alpha(\theta) \bigcup\limits_{m=1}^{N} SO(n)U_j(\theta) & \text{for } \theta \leq \theta_c. \end{cases}$$

Here $\theta : \Omega \to \mathbb{R}_+$ denotes (normalised) temperature with θ_c being the critical temperature, the matrices $U_j(\theta) \in \mathbb{R}^{n\times n}_{sym,+}$ model the variants of martensite and $\alpha(\theta)$ denotes the thermal expansion of the material. In this talk, we discuss several aspects of this dichotomy. We seek to study the following questions:

- Which conditions on the set K give rise to solutions with high symmetry and low (elastic and surface) energy? Physically, these structures could be regarded as candidates for nucleation mechanisms. Mathematically, they are also of interest as building block structures for convex integration schemes at (relatively) high regularity (similar to the origami type structures from [DMP08]).
- Given a set K, can we characterize all possible solutions to the associated differential inclusion? In particular, are there wild solutions which also enjoy higher Sobolev regularity to geometrically nonlinear models such as the geometrically nonlinear two-well problem? An important question – which however is still open – is the analysis of the threshold behaviour between rigidity and flexibility for these settings. Is there a critical threshold separating the rigid and the flexible regime? Can upper bounds on the maximal regularity of irregular solutions be found?
- Experimentally, it is found (both by acoustic and optic measurements) that nucleation processes in shape-memory alloys are highly irregular and display strongly intermittent behaviour for various important quantities. Thus, recently, simplified, geometrically constrained models have been introduced by [CH18, BCH15, TIVP17] seeking to capture this behaviour. These are, however, purely scalar. Is it possible to extend these to vectorial problems incorporating compatibility conditions? Can convex integration – which has a natural time-dependent interpretation – be related to this? Can the associated random processes be analysed?

In this talk, reporting on [CDPRZZ20], we discuss these aspects by first returning to the structures from [CKZ17] (see also [Pom10]) and extending them to a general number of wells. This allows to connect the setting of shape-memory alloys to that of nematic elastomers and in the continuum limit recovers structures which had been proposed in [ADMD15] in the setting of nematic elastomers.

Also, a careful analysis of layer solutions consisting of outer nucleation rings provide an interpretation of the experimentally observed disclination in tripole star structures in certain shape-memory alloys which undergo a hexagonal-to-rhombic transformation [KK91]. Important open questions involve the possibility of producing similar structures in three dimensions and the scaling of the structures with prescribed disclination angle.

In the second part we present the results from [DPR20] which provide the first example of flexible solutions to a geometrically nonlinear phase transformation in which $K^{lc} \neq \mathrm{conv}(K)$. This thus requires a careful analysis of the geometric and physical nonlinearities in the system. We illustrate how this is complemented by the scaling result of [RTZ19].

Finally, presenting the ideas from [DPRTZ20], we propose a connection between the simplified, geometrically constrained models on nucleation from [CH18, BCH15, TIVP17] and convex integration processes. We analyse the solutions' regularity and the associated random processes. Compatibility naturally enters in this model providing a full vectorial version of the models from [CH18, BCH15, TIVP17]. The building block structures used rely on the "Conti-constructions" from the second part of the talk. The derivation of stochastic PDEs describing the resulting microstructure would be an interesting open problem.

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Boundary regularity of area-minimizing currents: a linear model with analytic interface

Zihui Zhao

(joint work with Camillo De Lellis)

The Plateau's problem has a rich history in the theory of calculus of variations. It asks the following question: Given a boundary curve Γ, what is the surface T that minimizes area among all surfaces spanning Γ? The first major breakthrough was Douglas's (as well as Rado's) study of minimizers among surfaces which are images of disks or images of surfaces with fixed genera in the 1930s. In [13] Federer and Fleming introduced the notion of *integral current*, which models a countable union of orientable submanifolds, and they proved minimizers exist among the class of integral currents with prescribed boundary. Compared to Douglas's parametric approach, the class of integral current does not restrict to a fixed topological type (in particular it allows for minimizers with infinite topology), and it allows us to study not just surfaces but also submanifolds of arbitrary dimension. We are interested in studying the regularity of area-minimizing integral currents. More precisely:

Regularity Problem. *Suppose $\Gamma \subset \mathbb{R}^{m+n}$ is a closed integral current of dimension $(m-1)$ (with suitable a priori regularity), and an integral current T minimizes the area among all m-dimensional integral currents satisfying $\partial T = \Gamma$. Then is the support of the current T regular? (Regularity means* $\operatorname{spt}(T)$ *is locally contained in a smooth embedded submanifold.) And if not, how big is the singular set?*

The question above is well-understood when the current T in question has codimension one in the ambient Riemannian manifold or Euclidean space (that is, $n = 1$): in the interior a minimizer has a singular set of Hausdorff dimension at most $(m - 7)$ (see the survey [4, Chapters 3 and 4] for references therein); and every boundary point is regular (see [15]). The analogous question in higher codimensions (that is, $n \geq 2$) is more difficulty due to the possibility of branch singularity. For example, the holomorphic curve $C = \{(z, w) \in \mathbb{C}^2 : z^2 = w^3\}$ is a 2-dimensional area-minimizing current in $\mathbb{R}^4$. The origin is a branched singular point and the tangent of C at the origin is a flat plane with multiplicity 2. In Almgren's seminal work (see [2] and also [6, 7, 8]) he proved in the case of codimensions $n \geq 2$, the singular set of a minimizer has Hausdorff dimension at most $(m - 2)$ in the interior. In particular, he approximated area-minimizing current locally by multi-valued functions (the number of sheets given by the pointwise density of the current) which minimize the Dirichlet energy. This way, quantifying the size of the singular set for Dirichlet energy-minimizers can be thought of as a linear model to quantify the singular set for area-minimizing currents. (Modeled on the definition of regular points for currents, we say a point in the domain of a multi-valued function f is regular, if locally f is merely a multiple copy of a single-valued function.) Moreover, in the special case of two-dimensional currents, Chang (see [3, 9, 10, 11]) proved that in the interior singular points for a minimizer are isolated, and the minimizer is locally a classical minimal surface (à la Douglas-Rado) with possible branchings.

Much less is known about the regularity of minimizers at the boundary (see [1]). In 2018, De Lellis et al. made a breakthrough and proved that the regular set is relatively open and dense at the boundary, see [5]. In an attempt to further quantify the size of the boundary singular set, they also study the case when the prescribed boundary Γ is a smooth curve and construct the following example:

Example. *There are a C^∞ simple closed curve $\Gamma \subset \mathbb{R}^4$ and a (unique) two-dimensional area-minimizing current T with $\partial T = \Gamma$, such that the boundary singular set of T has an accumulation point.*

That is to say, Chang's result does not hold at the boundary, even for smooth simple closed curve. This is due to the failure of unique continuation for analytic functions at the boundary. On the other hand, when the boundary curve is *analytic*, classical minimal surfaces have no branched singular points at the boundary, see [14, 16]. Therefore we conjecture that when the prescribed boundary curve $\Gamma \subset \mathbb{R}^{2+n}$ is analytic, the branched singular points for a minimizer T are isolated and locally T has finite topology at the boundary. This work is ongoing, and in this workshop I present our first result [12] towards that goal, which is the study of the linear model to area-minimizing current with prescribed boundary.

To state the result more precisely, we first write the analytic curve Γ as (γ, φ), where γ is the projection of Γ onto the two-dimensional flat tangent plane π (identified with $\mathbb{R}^2$) of T, and the graph of the function $\varphi : \mathbb{R}^2 \to \mathbb{R}^n$ is exactly Γ. Then we approximate the current T using a pair of multi-valued functions $f = (f^+, f^-)$, where

- $f^+ \in W^{1,2}(\pi^+, \mathcal{A}_Q)$ and $f^- \in W^{1,2}(\pi^-, \mathcal{A}_{Q-1})$, with π^+, π^- being two connected components of π separated by γ and $Q \in \mathbb{N}$ denoting the density of T on one side of the boundary;
- and they satisfy the boundary condition $f^+|_\gamma = f^-|_\gamma + \varphi$.

This way, the study of branched singular points at a boundary of the minimizing current is reduced to the study of Dirichlet energy-minimizers satisfying corresponding boundary conditions (c.f. [2]). We prove the following theorem:

Theorem. *Given an analytic boundary (γ, φ), suppose $f = (f^+, f^-)$ minimizes the Dirichlet energy among all competitors with the prescribed boundary condition. Then the singular set of f is discrete.*

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Stability of Saddle Solutions to Allen-Cahn Equation in $\mathbb{R}^{8,10,12}$

Juncheng Wei
(joint work with Yong Liu, Kelei Wang)

We consider rigidity results for bistable Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^N. \tag{1}$$

The well-known De Giorgi's Conjecture states that at least when $N \leq 8$ the only **monotone** solutions are one-dimensional, i.e. $u = \tanh(\frac{\mathbf{a}\cdot\mathbf{x}+b}{\sqrt{2}})$ for some unit vector $\mathbf{a}$. This conjecture has been almost completely solved by Ghoussoub-Gui 1998 ($N = 2$), Ambrosio-Cabré [1] 2000 ($N = 3$), Savin 2003 ($N \leq 8$; under an extra condition), and del Pino-Kowalczyk-Wei 2011 ($N \geq 9$).

On the other hand a complete classification of **global minimizers**, i.e. solutions satisfying the property that

$$J(u) \leq J(u+\phi), \forall \phi \in C_0^1(\mathbb{R}^N), \text{ where } J(u) = \int (\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2)$$

has been achieved recently: Savin 2003 ($N \leq 7$); Liu-Wang-Wei 2017 ($N \geq 8$).

For stable solutions to Allen-Cahn equation, the classification is far from complete: for $N = 2$ it follows from Berestycki-Caffarelli-Nirenberg [2]; while for $N \geq 8$ there are counterexamples (Pacard-Wei 2013). The cases for $3 \leq N \leq 7$ are left open. We mention a recent striking result of Figalli-Serra [5] who proved that stable solutions to half-laplacian Allen-Cahn $\sqrt{-\Delta} u = -u + u^3$ in $\mathbb{R}^3$ are one-dimensional.

Next we consider classification of "not-too-unstable" solutions, i.e. finite Morse index solutions. When $N = 2$, the following result gives partial answer.

Theorem ([11]). *Finte Morse index implies finite-ended.*

Here a solution to Allen-Cahn equation in $\mathbb{R}^2$ is called **finite-ended** if there exists a compact set $K \subset \mathbb{R}^2$ and k-oriented half lines $\{\mathbf{a}_j \cdot \mathbf{x} + b_j = 0\}$, $j = 1, \ldots, k$ (for some choice of $\mathbf{a}_j \in \mathbb{R}^2$, $|\mathbf{a}_j| = 1$ and $b_j \in \mathbb{R}$) such that

$$|u(\mathbf{x}) - \sum_{j=1}^{k} (-1)^j \tanh(\frac{\mathbf{a}_j \cdot \mathbf{x} + b_j}{\sqrt{2}})| \leq Ce^{-c|\mathbf{x}|} \text{ in } \mathbb{R}^2 \setminus K.$$

The set of $2m-$ended solutions is denoted by $\mathcal{M}_{2m}$. (By definition the number of ends must be even.) There are still many unanswered questions related to classification of $\mathcal{M}_{2m}$: is it connected? what is its dimension? what is the Morse index? All these questions are largely open to Allen-Cahn equation. However using the fact that finite Morse index implies finite ends, and also integrable system theory, Liu and Wei [9] can explicitly write down all the solutions of multiple-ended solutions to another double-well potential (elliptic sine-Gordon)

$$\Delta u + \sin(\pi u) = 0, \ |u| < 1 \text{ in } \mathbb{R}^2.$$

Theorem ([9]). *(1) $\mathcal{M}_{2m}$ is a $2m-$dimensional smooth connected manifold; (2) Each solution in $\mathcal{M}_{2m}$ is nondegenerate, i.e. the bounded kernel is $2m-$dimensional; (3) Each solution in $\mathcal{M}_{2m}$ has exactly Morse index $\frac{m(m-1)}{2}$.*

We present two new rigidity results. The first one is the half space theorem. It is well-known that the study of Allen-Cahn has parallels in minimal surfaces (Kohn-Sternberg [7], Tonegawa-Wickramasekera [10]). For minimal surfaces, Hoffman and Meeks (1990) proved that a connected, proper, nonplanar minimal surface in $\mathbb{R}^3$ is not contained in a half space. They also proved Strong Half space theorem: Two proper, connected minimal surfaces in $\mathbb{R}^3$ must intersect, unless they are parallel planes. In the following we prove a parallel half space theorem.

Theorem ([6]). *In $\mathbb{R}^2$ and $\mathbb{R}^3$, suppose u is an entire solution of the Allen-Cahn equation whose zero level set is contained in a half space. Then u is one-dimensional.*

Remark: The result is not true if zero level set is replaced by other level set. For instance, consider the radial solutions.

A corollary of the above half space theorem is

Corollary ([6]). *Suppose u_1, u_2 are two solutions of the Allen-Cahn equation in $\mathbb{R}^2$, with $u_1 < u_2$. Then they are both one-dimensional.*

The main tool to prove the Half Space Theorem is the following Pohozaev type identity: Let u be a solution of the Allen-Cahn equation, X be a constant vector field in $\mathbb{R}^n$, and ν be the outward normal vector of $\partial\Omega$. Then

$$\int_{\partial\Omega}\left[\left(\frac{1}{2}\left|\nabla u\right|^2+\frac{1}{4}(1-u^2)^2\right)X\cdot\nu-\left(\nabla u\cdot X\right)\left(\nabla u\cdot\nu\right)\right]d\sigma=0.$$

Finally we discuss the stability of saddle solutions. Saddle solutions to Allen-Cahn are constructed on the so-called Simons' cone in $\mathbb{R}^{2m}$:

$$S_m := \{(x_1,\cdots,x_m)\in\mathbb{R}^{2m} : \sqrt{x_1^2+...+x_m^2}=\sqrt{x_{m+1}^2+...+x_{2m}^2}\}.$$

Introducing the variables (s,t) : $s=\sqrt{x_1^2+...+x_m^2}, t=\sqrt{x_{m+1}^2+...+x_{2m}^2}$, Allen-Cahn equation becomes

$$u_{ss}+\frac{m-1}{s}u_s+u_{tt}+\frac{m-1}{t}u_t+u-u^3=0.$$

In [3], Cabré and Terra constructed the so-called saddle solutions to Allen-Cahn equation in $\mathbb{R}^{2m}$ satisfiing

$$u\left(s,t\right)=-u\left(t,s\right),$$
$$u\left(s,t\right)>0,\ s>t>0.$$

Moreover Cabré [4] showed that this saddle solution is *unique.* (When $m=1$, this saddle solution was first constructed by Dang, Fife and Peletier in 1991. Schatzman proved that Morse index equals 1 when $m=1$.))

The stability of Cabré-Terra saddle solution was classified by Cabré in the following theorem

Theorem ([4]). *The saddle solution is stable for $m \geq 7$ and unstable for $m \leq 3$. Furthermore for $m = 2, 3$ the Morse index equals ∞.*

Recall that in minimal surface theory it is known that for $m \geq 4$ Simons' Cone is strict area-miniming. Cabré [4] conjectured that the saddle solutions are also stable if $m = 4, 5, 6$ and furthermore are global minimizers.

Recently we solved the stability part of Cabré's Conjecture:

Theorem ([8]). *The Cabré-Terra Saddle solutions are stable in $\mathbb{R}^8, \mathbb{R}^{10}$ and $\mathbb{R}^{12}$.*

A key ingredient of the proof is the construction of suitable super-solutions for the linearized operator around the saddle solution u (in $\mathbb{R}^8$):

$$L\phi := \phi_{ss} + \phi_{tt} + \frac{3}{s}\phi_s + \frac{3}{t}\phi_t + \left(1 - 3u^2\right)\phi,$$

in the set $\Omega := \{s > t > 0\}$. In [4] Cabré's used test function $\frac{u_s}{t^\alpha} - \frac{u_t}{s^\alpha}$ in $\mathbb{R}^{2m}, m \geq 7$. Instead our test function in $\mathbb{R}^{2m}, m \geq 4$ is

$$\left[\tanh\left(\frac{s}{t}\right)\frac{\sqrt{2}s}{\sqrt{s^2+t^2}} + \frac{1}{4.2}\left(1 - e^{-\frac{s}{2t}}\right)\right](s+t)^{-2.5}\,u_s$$
$$-\left[\tanh\left(\frac{t}{s}\right)\frac{\sqrt{2}t}{\sqrt{s^2+t^2}} + \frac{1}{4.2}\left(1 - e^{-\frac{t}{2s}}\right)\right](s+t)^{-2.5}\,u_t.$$

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A De Giorgi type conjecture for elliptic systems under the divergence constraint

Radu Ignat
(joint work with Antonin Monteil)

We analyse the symmetry of transition layers in some variational models arising in physics where the order parameter is a vector field of vanishing divergence. We develop a theory of calibrations in order to prove that one-dimensional transition layers are the unique global minimisers in these models.

This question resembles to the famous De Giorgi conjecture for minimal surfaces: if $u : \mathbb{R}^N \to \mathbb{R}$ is a C^2 solution to $\Delta u = \frac{dW}{du}(u)$ in $\mathbb{R}^N$ with $W(u) = \frac{1}{4}(1-u^2)^2$ such that $-1 < u < 1$ and $\partial_1 u > 0$ in $\mathbb{R}^N$, then u is one-dimensional ($1D$) provided that $N \leq 8$. After important contributions of Ghoussoub-Gui, Ambrosio-Cabré etc., Savin proved the conjecture under the additional boundary condition

$$\lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1 \quad \text{for every } x' \in \mathbb{R}^{N-1}. \tag{1}$$

Finally, Del Pino-Kowalczyk-Wei gave a counter-example satisfying (1) for $N \geq 9$. Lately, an intensive research was developed for vector-valued solutions $u : \mathbb{R}^N \to \mathbb{R}^d$ to the elliptic system $\Delta u = \nabla W(u)$ in $\mathbb{R}^N$ for potentials $W : \mathbb{R}^d \to \mathbb{R}_+$. The typical potential arising in phase separation models (such as Bose-Einstein condensates with two components, i.e., $d = 2$) is $W(u_1, u_2) = \frac{1}{2}u_1^2 u_2^2 + \Lambda(1 - |u|^2)^2$ for $\Lambda \geq 0$. Under certain boundary conditions, one-dimensional symmetry of solutions has been shown provided monotonicity / growth / stability conditions on solutions in certain dimensions N.

Periodic strip. In the following, we focus on the infinite strip domain $\Omega = \mathbb{R} \times \mathbb{T}^{N-1}$ in x_1 direction where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the flat torus. For $d \geq 1$ and nonnegative continuous potentials $W : \mathbb{R}^d \to \mathbb{R}_+$, we set the energy functional

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u)\, dx, \quad u : \Omega \to \mathbb{R}^d.$$

In this context, the boundary condition (1) becomes a necessary condition for finite energy configurations:

Lemma 1 ([2]). *Assume that W has a finite number of zeros and $\liminf_{|u|\to\infty} W(u) > 0$. If $E(u) < \infty$, then there exist two zeros $u^\pm \in \mathbb{R}^d$ of W such that*

$$\lim_{x_1 \to \pm\infty} u(x_1, \cdot) = u^\pm \quad \textit{in } L^2 \textit{ and a.e. in } \mathbb{T}^{N-1}. \tag{2}$$

Thus, for a.e. $x' \in \mathbb{T}^{N-1}$, $x_1 \in \mathbb{R} \mapsto u(x_1, x')$ represents a curve connecting $u^\pm$ in $\mathbb{R}^d$ endowed with the singular metric $g_W = 2W g_0$ where g_0 is the Euclidean metric. Let $\mathrm{geod}_W(u^-, u^+) = \inf\{\int_{-1}^1 \sqrt{2W(\gamma(t))}\, |\dot\gamma|\, dt \,:\, \gamma \in \mathrm{Lip}(-1,1), \gamma(\pm 1) = u^\pm\}$.

Corollary 2. *If a minimal geodesic connecting two zeros $u^\pm$ of W in $(\mathbb{R}^d, g_W)$ exists, then any global minimiser of E connecting $u^\pm$ is $1D$, i.e., $u = u(x_1)$.*

Proof. If $u : \Omega \to \mathbb{R}^d$ satisfies (2), we have:

$$E(u) = \int_\Omega \frac{1}{2}|\nabla' u|^2 + \frac{1}{2}|\partial_1 u|^2 + W(u)\, dx \geq \int_\Omega \frac{1}{2}|\nabla' u|^2\, dx + \mathrm{geod}_W(u^-, u^+). \tag{3}$$

It proves that optimal $1D$ transition layers connecting $u^\pm$ are global minimisers; moreover, if u is a global minimiser of E, then $\nabla' u = 0$ a.e. in Ω, so $u = u(x_1)$. □

Our model. From now on, we assume that $N = d$ and $u : \Omega \to \mathbb{R}^N$ is divergence-free. This constraint is natural in certain asymptotic regimes in liquid crystals, elasticity, ferromagnetism etc. In particular, if $\bar{u}(x_1) = \int_{\mathbb{T}^{N-1}} u(x_1, x')\, dx'$ is the x'-average of u and $E(u) < \infty$, then $\bar{u}$ is continuous with constant first component, i.e., $\bar{u}_1 = a$ in $\mathbb{R}$. Moreover, as in Lemma 1, if the set $\{W(a,\cdot) = 0\}$ is finite and $\liminf_{u_1 \to a,\, |u'| \to \infty} W(u_1, u') > 0$, then there are two zeros $u^\pm$ of $W(a,\cdot)$ such that (2) holds (see [2, Theorem 1.3]). We want to determine $W : \mathbb{R}^N \to \mathbb{R}_+$ such that

$$\inf\{E(u)\ :\ u : \Omega \to \mathbb{R}^N, \nabla \cdot u = 0 \text{ with } (2)\} \tag{4}$$

has only one-dimensional global minimisers.[1] They satisfy the nonlinear Stokes system $-\Delta u + \nabla W(u) = \nabla p$ for some pressure p (due to the constraint $\nabla \cdot u = 0$).

Calibrations. Inspired by the beautiful paper of Jin-Kohn [3], our strategy is to construct calibrations $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\int_\Omega \nabla \cdot [\Phi(u)]\, dx \leq E(u) \quad \text{for every } u : \Omega \to \mathbb{R}^N \text{ with } \nabla \cdot u = 0 \text{ and (2)}, \tag{5}$$

$$\text{there exists } u^* \text{ satisfying} \int_\Omega \nabla \cdot [\Phi(u^*)]\, dx = E(u^*),\ \nabla \cdot u^* = 0 \text{ and (2)}. \tag{6}$$

Note that (5) & (6) yield u^* is a global minimiser in (4). The aim is to prove that u^* is $1D$ and that the equality in (5) is achieved only for $1D$ vector fields u. Roughly speaking, this is related with the equipartition of the energy density for any minimiser u in (4), i.e., $\frac{1}{2}|\nabla u|^2 = W(u)$ in Ω.

Results for $N = 2$. The typical example is given by the Aviles-Giga model where $W(u) = \frac{1}{4}(1 - |u|^2)^2$; note that $\{W(a,\cdot) = 0\}$ is finite[2] for every $a \in \mathbb{R}$. Jin-Kohn proved that optimal $1D$ transition layers are global minimizers in (4) (for the reverse implication, see Theorem 3 below). Moreover, they proved for the potential $W_\delta(u) = \frac{1}{4}(1 - \delta u_2^2 - u_1^2)^2$ with $\delta > 0$ small enough that $1D$ transition layers are no longer global minimizers in (4). Our main result is the following:

Theorem 3 ([1]). *Assume that $W = \frac{1}{2}w^2$ where $w \in C^2(\mathbb{R}^2)$ solves the Tricomi equation $\partial_{11} w - f(u_1)\partial_{22} w = 0$ for every $u \in \mathbb{R}^2$ with $f \in C^1(\mathbb{R})$ satisfying $|f| \leq 1$. If $u^\pm = (a, u_2^\pm)$ are two zeros of W such that $W(a,\cdot) > 0$ in the interval (u_2^-, u_2^+), then any minimizer $u \in L^\infty$ in (4) is $1D$.*

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[1] Note that for divergence-free maps u, (3) holds if additionally $u_1 = a$ in Ω.

[2] Without the constraint $\nabla \cdot u = 0$, no global minimisers of E exist when $W(u) = \frac{1}{4}(1 - |u|^2)^2$ since two zeros $u^\pm$ can be connected within the curve $\{W = 0\}$, so the infimum of E vanishes.

The assumption $u \in L^\infty$ can be dropped out provided that $|\nabla w(u)| \leq Ce^{\beta|u|^2}$ for every $u \in \mathbb{R}^2$ for some $C, \beta > 0$. If $f = 1$, then w solves the wave equation, e.g., W is the Aviles-Giga potential. If $f = -1$, then w is an harmonic function, e.g., $w(u) = u_2^2 - u_1^2$. If $f = \frac{1}{\delta}$ with $\delta \geq 1$, we recover the potential W_δ (in a different regime of δ than in [3]). The proof of Theorem 3 is based on constructing a calibration Φ such that $\nabla\Phi(u) = \begin{pmatrix} \alpha(u) & w(u) \\ f(u_1)w(u) & \alpha(u) \end{pmatrix}$ for some function α.

Results for general $N \geq 2$. We present two strategies to construct calibrations Φ. We denote by Π_0 (resp. Π^+) the projection on traceless $N \times N$ matrices (resp. the projection on symmetric matrices). For divergence-free $u : \Omega \to \mathbb{R}^N$, we have:

$$\nabla \cdot [\Phi(u)] = \nabla\Phi(u) : \Pi_0 \nabla u^T = \Pi_0 \nabla\Phi(u) : \nabla u^T \leq \frac{1}{2}(|\nabla u|^2 + |\Pi_0 \nabla\Phi(u)|^2).$$

Strategy 1. We impose that $|\Pi_0 \nabla\Phi(u)|^2 \leq 2W(u)$ for every $u \in \mathbb{R}^N$. In particular, (5) holds. This strategy yields the following result (see [1, Theorem 2.11]): Assume that $X = \{x_0, \ldots, x_N\}$ is an affine basis in $\mathbb{R}^N$ and ρ is a (prescribed) metric on X. Then there exists a potential W such that $X = \{W = 0\}$, $\rho = \mathrm{geod}_W$ on $X \times X$ and any minimiser of (4) connecting $u^\pm \in X$ in a periodic strip in direction $\nu \perp (u^+ - u^-)$ is $1D$. The proof is based on the calibration $\Phi = \varphi\nu$ where $\varphi \in \mathrm{Lip}(\mathbb{R}^N)$ satisfies $|\varphi(u^+) - \varphi(u^-)| = \rho(u^+, u^-)$ for every $u^\pm \in X$, yielding the potential $W = \frac{1}{2}|\nabla\varphi|^2$.

Strategy 2. We impose that $\nabla\Phi(u)$ is symmetric and $|\Pi_0 \nabla\Phi(u)|^2 \leq 4W(u)$ for every $u \in \mathbb{R}^N$ (the constant 4 is crucial here). Then for divergence-free u,

$$\nabla \cdot [\Phi(u)] = \Pi_0 \nabla\Phi(u) : \nabla u^T = \Pi_0 \nabla\Phi(u) : \Pi^+ \nabla u^T \leq |\Pi^+ \nabla u|^2 + \frac{1}{4}|\Pi_0 \nabla\Phi(u)|^2.$$

It yields (5) since $\|\nabla u\|^2_{L^2(\Omega)} = 2\|\Pi^+ \nabla u\|^2_{L^2(\Omega)}$ (see [1, Proposition 4.12]). This strategy yields a class of potentials W for which (4) has only $1D$ global minimisers (see [1, Theorem 2.10]). The proof is based on calibrations such that $\nabla\Phi = \nabla^2\Psi$ with Ψ solving the wave equation in any two directions x_i and x_j, i.e., $\partial_{ii}\Psi = \partial_{jj}\Psi$ in $\mathbb{R}^N$; the potential is then given by $W = \frac{1}{2}\sum_{i<j} |\partial_j \Phi_i|^2$. In particular, we recover the following extension of the Aviles-Giga potential in dimension $N \geq 3$: $W(u) = \frac{1}{4}(1 - |u|^2)^2 + |u''|^2(u_1^2 + u_2^2)$ corresponding to $\Psi(u) = -\frac{u_1 u_2}{\sqrt{2}}(\frac{u_1^2 + u_2^2}{3} + |u''|^2 - 1)$ for every $u = (u_1, u_2, u'') \in \mathbb{R}^N$ with $u'' = (u_3, \ldots, u_N)$.

Perspective. Motivated by micromagnetics, a future problem is to extend this study for divergence-free vector fields u satisfying the nonconvex constraint $|u| = 1$.

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Free boundary minimal surfaces with connected boundary in the unit ball

Giada Franz
(joint work with Alessandro Carlotto, Mario Schulz)

Let B^3 be the three-dimensional unit ball in the Euclidean space $\mathbb{R}^3$ and let $\Sigma^2 \subset B^3$ be a compact surface that is properly embedded, i.e., it holds $\partial\Sigma = \Sigma \cap \partial B^3$. We say that Σ is a *free boundary minimal surface* in B^3 if it is a critical point of the area functional with respect to variations that constrain its boundary to the boundary ∂B^3 of the unit ball.

If we compute the first derivative of the area along a variation $\{\Phi_t(\Sigma)\}_{t\in(-\varepsilon,\varepsilon)}$, where $\Phi_t \colon \overline{B^3} \to \overline{B^3}$ is a continuous family of diffeomorphisms with $\Phi_0 = \mathrm{id}$, we obtain that

$$\frac{d}{dt}\Big|_{t=0} \mathrm{area}(\Phi_t(\Sigma)) = -\int_\Sigma \langle H, X\rangle + \int_{\partial\Sigma} \langle \nu, X\rangle,$$

where $X = \partial\Phi_t/\partial t$ is the variation vector field, H is the mean curvature of Σ and ν is the outward unit co-normal to $\partial\Sigma$. Hence we have that Σ is a free boundary minimal surface if and only if it has mean curvature identically zero and it intersects the boundary of the unit ball orthogonally.

The study of free boundary minimal surfaces goes back at least to Courant (cf. [2]), but for many years the equatorial disc and the critical catenoid (see Figures 1 and 2, from [14]) were the only known examples in B^3.

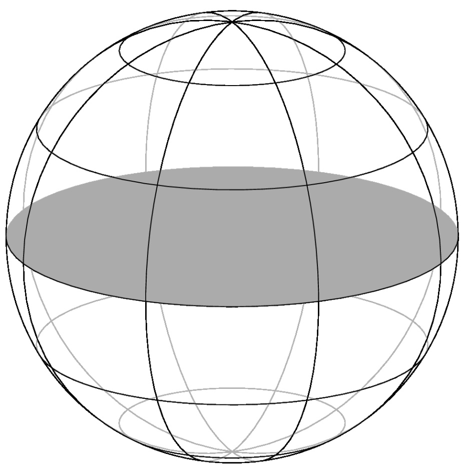

Figure 1. Equatorial disc

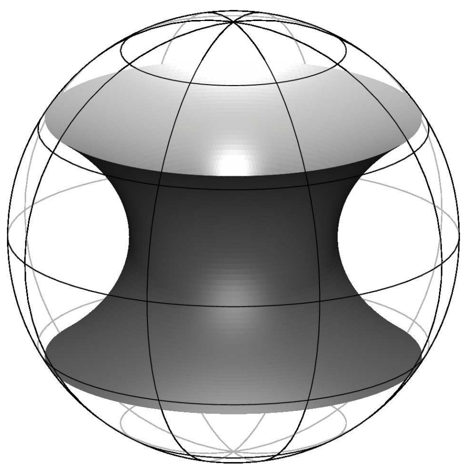

Figure 2. Critical catenoid

The work of Fraser–Schoen [5, 6, 7] on extremals for Steklov eigenvalues breathed new life into the study of these objects. Now the pictures of examples in front of us is much richer: we have a family of examples with genus zero and arbitrary number of boundary components ([7], see also [4, 11]), with genus one and sufficiently large number of boundary components ([4]), with one boundary component and sufficiently large genus ([9]), and with three boundary components and sufficiently large genus ([11, 8]).

However, we still do not have a complete answer to the existence question for any given topological type (see also Open Question 1 in [13]). Indeed, before our work it was not even known an example of a free boundary minimal surface of genus one and connected boundary. In our paper we address the problem for this low topological type and, more in general, for connected boundary and we are able to prove the following result.

Theorem 1 (Theorem 1.1 in [1]). *For each $g \geq 1$ there exists an embedded free boundary minimal surface M_g in B^3 with connected boundary and genus g.*

The idea of the proof is to use an equivariant version of Almgren–Pitts min-max theory, developed by Ketover in [10] and [11]. More precisely, the surface M_g is obtained by a one-parameter min-max procedure imposing $\mathbb{D}_{g+1}$ dihedral symmetry, where $\mathbb{D}_{g+1}$ is the group (of order $2(g+1)$) of isometries of the unit ball B^3 generated by rotations of angle π around the $(g+1)$ horizontal axes $\xi_k := \{(r\cos(k\pi/(g+1)), r\sin(k\pi/(g+1)), 0) \mid r \in [-1,1]\}$ for $k = 1, \ldots, g+1$.

Imposing this dihedral symmetry enables us to control the topology of the resulting surface, which is typically a delicate point when performing any min-max procedure. What we obtain is indeed a free boundary minimal surface M_g (see Figures 3 and 4 for $g = 1$ and $g = 2$, from [14]) with:

- dihedral symmetry $\mathbb{D}_{g+1}$;
- connected boundary, which follows from a nontrivial application of Simon's Lifting Lemma (cf. Proposition 2.1 in [3]);
- genus g, as a consequence of the lower semicontinuity of the genus (cf. Theorem 9.1 in [12]) and the dihedral symmetry of the first point.

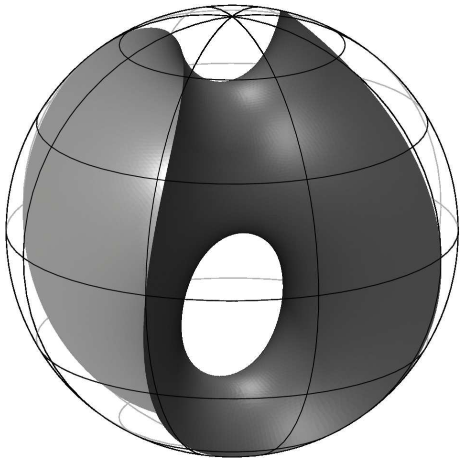

FIGURE 3. Surface M_1

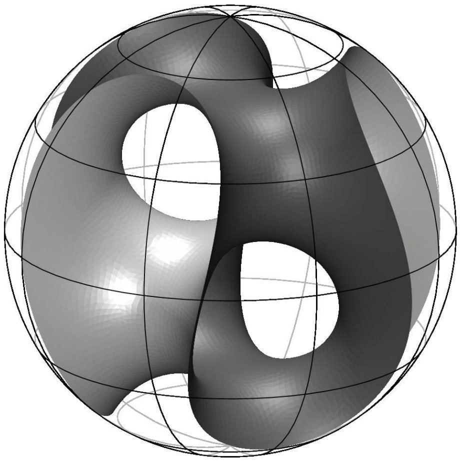

FIGURE 4. Surface M_2

We conclude by remarking that, besides the existence question (which is still unknown in many cases), it would also be very interesting to understand for which topological types there is a unique (up to isometries) free boundary minimal surface with that given topology. So far, it is only known that the equatorial disc is unique in its topological class and that in general uniqueness fails (see the examples of Kapouleas-Wygul with connected boundary and sufficiently large genus in [9]).

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Allen-Cahn minmax and multiplicity-1 minimal hypersurfaces in positive Ricci

Costante Bellettini

A fundamental geometric question is whether there exists a closed minimal hypersurface in any given compact Riemannian manifold N, whose dimension we denote by $n+1$ with $n \in \mathbb{N} \setminus \{0\}$. The original (affirmative) answer was obtained in the late 1970s and early 80s, through the combined efforts of Almgren, Pitts, Schoen, Simon, Yau ([1], [8], [10], [9]), after that Birkhoff had addressed, in pioneering work, the case $n = 1$. These works provided the fields of differential geometry and analysis of PDEs with a large number of highly interesting related questions, and with techniques that have first become important tools in geometric analysis, and then have been further refined in the past decade to produce dramatic progress in the field (starting with the resolution of the long-standing Willmore conjecture by Marques–Neves [6]).

In very recent years, a new approach to the same existence question has been successfully implemented. This relies on an approximation of the area functional by means of the so-called Allen–Cahn energy $\mathcal{E}_\epsilon(u) = \frac{1}{2\sigma}\int_N \epsilon\frac{|\nabla u|^2}{2} + \frac{W(u)}{\epsilon}$, where N is the ambient manifold, $\epsilon \in (0,1)$, $u \in W^{1,2}(N)$ and $W : \mathbb{R} \to \mathbb{R}$ is a "double well" potential, i.e. a C^2 non-negative function with two non-degenerate minima at ± 1, and $\sigma > 0$ is a normalizing constant depending only on W. The idea that the area functional could be approximated by means of the Allen–Cahn energy was formalized in the case of minimizers by Modica–Mortola ([7]), following an idea of De Giorgi, in the framework of Γ-convergence.

For the existence question that we are discussing, minimization does not provide an effective framework in general. For example, on the standard sphere S^{n+1} direct minimization cannot produce a minimal hypersurface: all critical points of the area functional, for example an equatorial S^n, can be continuously deformed to points. In order to detect such critical points, rather than appealing to minimization, one needs a suitable minmax construction, that allows the detection of "saddle-type" critical points. The approach developed in the late 70s is usually referred to as minmax à la Almgren–Pitts and involves a careful and lengthy construction in a space of integral varifolds. The Allen–Cahn approach developed in recent years involves (i) a minmax for $\mathcal{E}_\epsilon$ at fixed ϵ, to produce a solution u_ϵ with Morse index ≤ 1; (ii) a suitable strategy to take the $\epsilon \to 0^+$ limit of the u_ϵ and thereby produce the desired minimal hypersurface. Step (i) is completed in the work by Guaraco [4], while step (ii) is carried out in [4] thanks to the works of Hutchinson–Tonegawa, Tonegawa, Tonegawa–Wickramasekera, Wickramasekera ([5], [11], [12], [14]). The set up in (i) only requires the use very classical PDE minmax tools, such as the verification of the Palais–Smale condition. For (ii), roughly speaking, for each u_ϵ one considers an associated varifold V^ϵ, that is a Radon measure on the Grassmannian bundle $G_n(N)$. The varifold V^ϵ is build by keeping track of the Allen–Cahn energy density of u_ϵ and of the direction of ∇u_ϵ (wherever $\nabla u_\epsilon \neq 0$). This operation amounts, geometrically, to endowing the collection of level sets of u_ϵ with a suitable weight, so to form a "diffused interface": as $\epsilon \to 0^+$ this interface collapses onto a sharp one, supported on an n-dimensional integral varifold V that is minimal in the generalized sense of geometric measure theory. The Morse index information allows to reduce the study of V locally to that of stable minimal integral varifolds, for which optimal smoothness results ([12], [14]) imply that V is, except for a small singular set of dimension at most $n-7$, given by a sum of smooth minimal hypersurfaces, each endowed with a constant integer multiplicity.

The possible appearance, in the Allen–Cahn minmax leading to V, of multiplicity on V is the question addressed in [2], on which we focus here. The possibility that multiplicity higher than 1 may appear on V causes, first of all, challenging difficulties in completing the proof described above, specifically in obtaining the smoothness result for V (as in [14]); this is due to the fact that a priori any point with multiplicity higher than one may be singular. Once smoothness of spt$V$ is known (up to a possible singular set of dimension $\leq n-7$), it easily follows that multiplicity is a constant integer on each smooth connected component of V. The knowledge that the multiplicity is in fact 1 on smooth portions of V would lead to further geometric consequences that are otherwise prevented.

The multiplicity issue is ubiquitous in geometric analysis, and its resolution, in several contexts, always leads to extra geometric and analytic information. It is not true in full generality that minmax constructions, including multi-parameter ones, lead to a varifold with unit multiplicity on its smooth portions. In the context of codimension-1 minmax constructions, natural classes of Riemannian manifolds in which it is plausible to expect multiplicity-1 on smooth portions of the resulting varifolds are given by manifolds of dimension $n+1 \geq 3$, whose metrics have either positive Ricci curvature or are generic in a suitable sense (for example, bumpy metrics introduced by White [13] are generic for $n \leq 6$). Indeed, for multi-parameter minmax constructions within the Almgren–Pitts framework, the multiplicity-1 conclusion is obtained by Zhou ([15]), for said types of Riemannian manifolds, with $n \leq 6$; this confirms a conjecture by Marques–Neves. A similar conjecture for minmax constructions carried out by means of an Allen–Cahn approximations is natural.

We prove:

Theorem 1. *Let N^{n+1}, $n \geq 2$, is a compact Riemannian manifold endowed with a metric that has positive Ricci curvature. Then the Allen–Cahn (one-parameter) minmax construction developed in* [4] *produces a multiplicity-*1 *integral varifold V, that is smoothly embedded away from a set Σ of dimension $\leq n-7$.*

Moreover, the (smooth) minimal hypersurface $M = spt(V) \setminus \Sigma$ is connected, has Morse index equal to 1 *and is two-sided.*

As a concrete example, the minmax procedure on $\mathbb{RP}^3$ will not yield an equatorial $\mathbb{RP}^2$ with multiplicity 2. In fact, it is known that it yields a Clifford torus with multiplicity 1.

The substantial content of Theorem 1 is the multiplicity information. The regularity of V was already known (as described earlier). The geometric information of two-sidedness follows immediately once the multiplicity-1 conclusion is achieved. Connectedness and Morse index information are also easily seen to hold because of the positiveness of the Ricci curvature of N and of the two-sidedness of M.

The case $n = 2$ of Theorem 1 is a special case of the result by Chodosh–Mantoulidis ([3]). In this work the authors show that if $n = 2$ and N has a bumpy metric or a metric with positive Ricci curvature, then sequences of Allen-Cahn solutions u_{ϵ_i} for $\epsilon_i \to 0^+$ such that $\mathcal{E}_{\epsilon_i}(u_{\epsilon_i})$ and the Morse index of u_{ϵ_i} are both bounded above independently of ϵ_i, have the following property: the associated varifolds converge (subsequentially) to a multiplicity-1 integral varifold, supported on a smooth minimal surface. This confirms the Allen–Cahn counterpart of Marques–Neves conjecture (alluded to above) for $n = 2$. (It also leads to establishing, in this dimension, a generic version of Yau's conjecture on the existence of infinitely many minimal surfaces; the same remark applies to [15].)

The proof of Theorem 1 is carried out in [2]. It relies on the direct exploitation of the minmax characterisation of the critical points u_ϵ, rather than on the (weaker) information that their Morse index is at most 1. The class of continuous paths in $W^{1,2}(N)$ with endpoints at the constant -1 and at the constant $+1$ is the admissible class of paths used for the minmax construction in [4]. Given any minimal hypersurface M in N such that $\dim_{\mathcal{H}}\left(\overline{M} \setminus M\right)$, the proof in [2] produces, for each sufficiently small ϵ, a continuous path γ in $W^{1,2}(N)$ with endpoints at the constant -1 and at the constant $+1$ and such that the maximum of $\frac{1}{2\sigma}\mathcal{E}_\epsilon$ on γ is at most $2\mathcal{H}^n(M) - \beta$, where $\beta > 0$ is a fixed geometric quantity, depending only on $M \subset N$. The fact that $\frac{1}{2\sigma}\mathcal{E}_\epsilon(u_\epsilon)$ converges to the mass of V as $\epsilon \to 0^+$ gives a contradiction to the fact that M may appear in $\text{spt}(V)$ with multiplicity ≥ 2. The arbitaryness of M concludes. The construction of the path γ exploits the major flexibility afforded by the Allen–Cahn approach; for example, an operation such as cutting off a hole in a hypersurface has a continuous analogue in the Allen–Cahn framework.

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Rigidity estimates for incompatible fields and an application to plasticity

Adriana Garroni
(joint work with Sergio Conti)

Rigidity à la Friesecke James and Müller [5, 6] and their classical linear counter part, i.e., the Korn inequality, are fundamental tools for the analysis of variational models in elasticity, allowing for well posedness and compactness in asymptotic limits (as linearization, dimension reduction, etc.).

The rigidity estimates are the quantitative version of very classical statements where differential constraints for a function u translate in strong pointwise conditions. The case we have in mind is Liouville's theorem which states that if a C^1 function $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\nabla u \in \mathrm{SO}(n)$ in Ω, i.e., its gradient is a rotation at each point of Ω, then it should be constant. In other words u must be a rigid transformation. The linear version of this statement is that if a function u satisfies $\nabla u(x) + \nabla u^T(x) = 0$ for all $x \in \Omega$, then there exists a constant skew symmetric matrix A (a so-called infinitesimal rotation) such that $\nabla u = A$.

The corresponding quantitative estimates are then, for all $u \in H^1(\Omega, \mathbb{R}^n)$,

- the classical Korn inequality: there exists a skew symmetric matrix A such that
$$\int_\Omega |\nabla u - A|^2 dx \le C \int_\Omega |\nabla u + \nabla u^T|^2 dx$$
- the rigidity estimate of Friesecke James and Müller: there exists a constant rotation $R \in \mathrm{SO}(n)$ such that
$$\int_\Omega |\nabla u - R|^2 dx \le C \int_\Omega \mathrm{dist}(\nabla u, \mathrm{SO}(n))^2 dx$$

where the constants C depend only on Ω.

When we interpret the function u as a deformation of an elastic body, the corresponding elastic energy degenerates in $\mathrm{SO}(n)$ due to frame indifference assumption, which in a linear theory gives rise to a quadratic energy that degenerates on skew symmetric matrices. Therefore the above estimates turn out to be crucial. In order to deal with variational problems with growth other than 2, there are several generalisations of these two estimates in the literature, for instance assuming that u has gradient in L^p, $p > 1$, or in the weak L^p spaces (see e.g. [4]).

In the presence of plastic deformations, and then beyond the elastic regime mentioned above, the variational models should account for plastic slips that occur at microscopic scale and for the presence of topological defects which induce elastic distortion. Therefore, the relevant variable is not the gradient of a deformation but a more general matrix-valued field, the elastic strain, which may not be a gradient and may have a non trivial curl, possibly concentrated on low-dimensional sets. In particular in dimension 3 the curl can be concentrated along a line dislocation, or, at larger scales on a surface, for instance in the case of dislocation walls and grain boundaries.

It is then clear that the associated variational theories need the corresponding estimates for fields that may be incompatible, in the sense that are not gradients, but for which one knows that the curl is a bounded measure.

Given a matrix-valued function $\beta \in L^1(\Omega;\mathbb{R}^{n\times n})$, $\operatorname{curl}\beta$ denotes the tensor-valued distribution whose rows are the curl, in the sense of distributions, of the rows of β, and $\mathcal{M}(\Omega;\mathbb{R}^k)$ denotes the space of Radon measures in Ω with values in $\mathbb{R}^k$. In dimension 3, $\operatorname{curl}\beta$ can as usual be identified with a matrix-valued distribution $\alpha_{il} := \sum_{j,k} \epsilon_{ljk} \operatorname{curl}\beta_{ijk}$. The natural space is then the one obtained by the Sobolev-conjugate exponent $1^* := n/(n-1)$.

The general statements for the rigidity estimate and for Korn's inequality with incompatible fields, respectively, are then the following:

1. Given an open, bounded, connected, and Lipschitz set Ω in $\mathbb{R}^n$, with $n \geq 2$, there exists a constant $C = C(n,\Omega) > 0$ such that for every $\beta \in L^1(\Omega;\mathbb{R}^{n\times n})$ with $\operatorname{curl}\beta \in \mathcal{M}(\Omega;\mathbb{R}^{n\times n\times n})$, there exists a rotation $R \in \mathrm{SO}(n)$ such that
$$\|\beta - R\|_{L^{1^*}(\Omega)} \leq C \left(\|\operatorname{dist}(\beta,\mathrm{SO}(n))\|_{L^{1^*}(\Omega)} + |\operatorname{curl}\beta|(\Omega) \right).$$
2. Given an open, bounded, connected, and Lipschitz set Ω in $\mathbb{R}^n$, with $n \geq 2$, there exists a constant $C = C(n,\Omega) > 0$ such that for every $\beta \in L^1(\Omega;\mathbb{R}^{n\times n})$ with $\operatorname{curl}\beta \in \mathcal{M}(\Omega;\mathbb{R}^{n\times n\times n})$ there exists an antisymmetric matrix A such that
$$\|\beta - A\|_{L^{1^*}(\Omega)} \leq C \left(\|\beta + \beta^T\|_{L^{1^*}(\Omega)} + |\operatorname{curl}\beta|(\Omega) \right).$$

Here $|\operatorname{curl}\beta|(\Omega)$ denotes the total variation of the measure $\operatorname{curl}\beta$. In the case when $\operatorname{curl}\beta$ is absolutely continuous with respect to the Lebesgue measure it is simply given by the L^1 norm of $\operatorname{curl}\beta$. We remark that the main point in this estimate is that it provides an estimate with the total variation of the measure $\operatorname{curl}\beta$ and therefore allows for concentrated incompatibilities, which correspond to concentrated defects as for example dislocations.

In 2 dimensions we have $1^* = 2$ and Korn and rigidity estimates for incompatible field (1. and 2. with $n = 2$) have been obtained in [9] and [8], respectively, while in higher dimension a recent result by Lauteri and Luckhaus, [7], shows that the rigidity estimate can be obtained in the Lorentz space $L^{1^*,\infty}$, but their results do not cover the case $p = 1^*$ in the results above.

In [10], we show that the result in dimension 3 and above can be obtained via Hodge decomposition using a sharp regularity result due to Bourgain and Brezis for the div-curl system [2, 3] (see also [11]).

The key step in the proof is then the use of the Hodge decomposition, with sharp estimates in L^{1^*}, which permits to apply the estimates in L^p with $p = 1^*$ to the gradient part, when the domain is a cube. In order to recover the case of a general Lipschitz domain we use a Whitney type covering argument combined with a weighted Poincaré inequality.

These estimates have been recently used by Garroni and Marziani and by Conti, Garroni, and Ortiz in upscaling of three-dimensional models for dislocations in the geometrically non-linear regime and in the linear discrete setting, respectively.

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On the well posedness of branched transport

Maria Colombo

(joint work with Antonio De Rosa, Andrea Marchese, Paul Pegon, Antoine Prouff)

Branched structures are employed to describe several supply-demand systems such as the structure of the nerves of a leaf, the cardiovascular systems, as well as in other phenomena, such as superconductivity models [5]. They also appear in the most diverse problems, such as the weak approximability of Sobolev maps between manifolds by smooth maps [4].

Given a flow that transports a given source onto a given target measure along a 1-dimensional network, the transportation cost per unit length is supposed in our model to be proportional to a concave power $\alpha \in (0,1)$ of the intensity of the flow. This was first formulated for discrete source and target measures in terms of weighted oriented graphs; later two different formulations of the problem have been given. More precisely, let μ^- and μ^+ be probability measures on $\mathbb{R}^d$, representing the source and the target measure. A transport plan $P \in \mathcal{P}(\mathrm{Lip}_1)$ is a probability on the space of 1-Lipschitz curves and satisfies

$$(e_0)_\# P = \mu^-, \qquad (e_\infty)_\# P = \mu^+$$

(here e_0 and e_∞ are the evaluations of a Lipschitz curve at its initial and final point). At each point $x \in \mathbb{R}^d$, the transport density is described by

$$\theta_P(x) = P(\gamma : x \in \gamma([0,\infty))$$

and the energy of the plan is

$$\mathcal{E}^\alpha(P) = \int_{\mathrm{Lip}_1} \int_0^\infty \theta_P(\gamma(t))^{\alpha-1} |\dot\gamma(t)| \, dt \, dP(\gamma). \tag{1}$$

We are interested in the minimization problem

$$\mathcal{E}^\alpha(\mu^-,\mu^+) = \min\{\mathcal{E}^\alpha(P) : (e_0)_\# P = \mu^-, \ \ (e_\infty)_\# P = \mu^+\}.$$

Xia [6] introduced instead an Eulerian description of the problem by means of transport paths

$$T = [E,\tau,\theta] = \theta\tau\mathcal{H}^1|_E$$

where E is (a subset of) a countable family of C^1 curves, τ is a unitary vector specifying the orientation of the curves, and $\theta : E \to \mathbb{R}^+$ is the multiplicity. The initial and final measures are specified through $\mathrm{div}(T) = \mu^+ - \mu^-$.

The natural generalization of the energy is

$$\mathbb{E}^\alpha(T) = \int_E \theta^\alpha \, d\mathcal{H}^1 \qquad \text{if } T = [E,\tau,\theta],$$

which is also the lower semicontinuous envelope of the same energy restricted to finite graphs. This implies the existence of minimizers of

$$\mathbb{E}^\alpha(\mu^-,\mu^+) = \min\Big\{\mathbb{E}^\alpha(T) : T \text{ transport path with } \mathrm{div}T = \mu^+ - \mu^-\}$$

Through the lagrangian description, we can formulate different branched transport problems, such as the mailing problem. It has the same energy (1) but a different boundary condition, that allows to specify which part of μ^- goes to a given part of μ^+. More precisely, the coupling between the initial particles and their target is described through a prescribed coupling $\pi \in \mathcal{P}(B_1 \times B_1)$ and admissible transport plans are required to satisfy

$$(e_0, e_1)_\# P = \pi.$$

The talk focuses on the stability for optimal transport plans, with respect to variations of the source and target measures, which was previously known only when α is bigger than a critical threshold.

Theorem 1 ([2, 3]). *Let $\alpha \in (0,1)$, let μ^+ and μ^- mutually singular and let $\mu_n^\pm \rightharpoonup \mu^\pm \in \mathcal{P}(B_1)$. Let P_n be an optimal plan connecting μ_n^- to μ_n^+ satisfying*

$$\sup_{n\in\mathbb{N}} \mathcal{E}^\alpha(P_n) < \infty, \qquad P_n \rightharpoonup P.$$

Then P is optimum between μ^- and μ^+.

The two formulations differ on several aspects, and taking advantage of both point of views turns out to be important to develop a proof of the stability result:

- On one side, the description by means of transport plans is more accurate: it is indeed easy to associate to each transport plan a canonical transport path, whereas associating a transport plan to a transport path is not canonical and can be done under certain assumptions via Smirnov's decomposition theorem.
- On the other side, localizing transport paths to a certain open domain in $\mathbb{R}^d$, slicing them, as well as joining transport paths with a common marginal is easy, while the corresponding definitions for transport plans are less immediate. For this reason, the proof of the theorem above goes through the analogous result for transport paths.

Further steps in this theory may include the study of optimizers of the mailing problem: due to the loss of tree structure (acyclic), little is known about minimizers. For instance [1, Problem 15.5 and 15.6], given an optimal transport plan for the mailing problem, does it have a finite graph structure in any ball outside of the support of the initial and target measures? Is the number of branches at a branching point uniformly bounded?

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Emergent spin-orbit coupling in a spherical magnet

Christof Melcher
(joint work with Zisis N. Sakellaris)

A key feature of spin-orbit coupled condensed matter systems such as chiral magnets and multicomponent Bose-Einstein condensates is a loss of independent rotational symmetry in the domain and the target space. The corresponding energy terms typically have the form of helicity, thus transform as a differential one-form and are therefore sensitive to rotations and reflections in a sign-reversing manner. This form of chiral symmetry breaking induces modulation and is responsible for the occurrence of topological patterns and solitons, called chiral skyrmions [1]. In this talk we discussed how conversely classical spin-orbit coupling emerges from reduced rotational symmetry in a purely geometric framework [2]. To this end we examined variational principles for fields $\boldsymbol{m} : \mathbb{S}^2 \to \mathbb{S}^2$ based on an anisotropic versions of the harmonic map problem

$$E(\boldsymbol{m}) = \frac{1}{2} \int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} \boldsymbol{m}|^2 + \kappa \left(1 - (\boldsymbol{m} \cdot \boldsymbol{\nu})^2\right) \, \mathrm{d}\sigma$$

where $\boldsymbol{\nu}$ is the outer unit normal and κ is a positive constant that is supposed to be large - interpreted as a model for a spherical magnet. The energy is invariant with respect to joint rotations, i.e. $E(\boldsymbol{m}_R) = E(\boldsymbol{m})$ for $\boldsymbol{m}_R(x) = R\boldsymbol{m}(R^\dagger x)$ with $R \in SO(3)$ arbitrary, which may be seen as a principle of frame-indifference. The minimal energy configuration for large enough κ is the hedgehog $\boldsymbol{m} = \pm\boldsymbol{\nu}$, see [3], with $E(\pm\boldsymbol{\nu}) = 4\pi$ and topoglocal degree $Q(\pm\boldsymbol{\nu}) = \pm 1$ given by

$$Q(\boldsymbol{m}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \boldsymbol{m}^* \omega_{\mathbb{S}^2}$$

where $\omega_{\mathbb{S}^2}$ is the standard volume form on $\mathbb{S}^2$. On the other hand, localized skyrmionic structures interpolate between the hedgehog map $\boldsymbol{m} = \boldsymbol{\nu}$ near south pole and its strongly localized inversion near the north pole. Local minimizers $\boldsymbol{m}_0$ of this form satisfying

$$E(\boldsymbol{m}_0) < 8\pi \quad \text{and} \quad Q(\boldsymbol{m}_0) = 0 \tag{1}$$

occur in the same regime of large anisotropy. This is in sharp contrast to the flat counterpart given by

$$F(\boldsymbol{u}) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \boldsymbol{u}|^2 + \kappa \left(1 - (\boldsymbol{u} \cdot \hat{\boldsymbol{e}}_3)^2\right) dx \quad \text{for} \quad \boldsymbol{u} : \mathbb{R}^2 \to \mathbb{S}^2 \tag{2}$$

featuring independent $O(2)$ symmetry. A simple scaling argument rules out the existence of finite energy equilibria other than the constant states $\boldsymbol{u} = \pm\hat{\boldsymbol{e}}_3$. However, non-trivial solitonic structure in all non-trivial homotopy classes can be stabilized dynamically via spin rotation in the framework of the corresponding Landau-Lifshitz equation $\partial_t \boldsymbol{u} + \boldsymbol{u} \times (\Delta \boldsymbol{u} + \kappa u_3 \hat{\boldsymbol{e}}_3) = 0$. The Hamiltonian structures enables a variational approach to periodic solutions of the form $\boldsymbol{u}(x,t) = R(\omega t)\boldsymbol{u}_0(x)$ based on minimization under an additional constraint on the total spin $S(\boldsymbol{u}) = \int_{\mathbb{R}^2} (1 - u_3) \, dx$ playing the role of angular momentum. Rotational invariance suggests a co-rotational ansatz $\boldsymbol{u}_0(re^{i\varphi}) = e^{ik\varphi} h(r)$ of arbitrary degree $k \in \mathbb{Z} \setminus \{0\}$ with a radial profile h. Existence and orbital stability have been proven in [4].

Due to the lack of independent rotational symmetry and the principle of energy conservation, such precessional solutions are not possible in the spherical case, where one rather expects rotating solutions of the form

$$\boldsymbol{m}(x,t) = R(\omega t)\boldsymbol{m}_0(R(-\omega t)x). \tag{3}$$

Aiming for a variational approach and to elucidate the emergence of a spin-orbit phenomenon, we exploit the Hamiltonian formulation of Landau-Lifshitz dynamics. We equip the configuration space $\mathcal{M} = C^\infty(\mathbb{S}^2;\mathbb{S}^2)$ with a Poisson bracket

$$\{F,G\} \quad = \quad \Big\langle \boldsymbol{\nabla} F(\boldsymbol{m}), \boldsymbol{m} \times \boldsymbol{\nabla} G(\boldsymbol{m}) \Big\rangle$$

where $\boldsymbol{\nabla} = \boldsymbol{\nabla}_{L^2_\sigma}$ denotes the functional gradient with respect to the L^2_σ-metric. In this framework the Landau-Lifshitz equation reads

$$\partial_t \boldsymbol{m} + \{\boldsymbol{m}, E\} = 0. \tag{4}$$

It is clear that an appropriate form of spin angular momentum S giving rise to spin precession in the flat case cannot be conserved independently. Moreover, the embedding of the domain $\mathbb{S}^2$ into $\mathbb{R}^3$ and rotational invariance calls for vectorial momenta. Therefore we were led to introduce the total angular momentum vector $\boldsymbol{J} = \boldsymbol{S} + \boldsymbol{L} \in \mathbb{R}^3$ as the sum of spin and orbital angular momentum given by the following functionals

$$\boldsymbol{S}(\boldsymbol{m}) = \int_{\mathbb{S}^2} \boldsymbol{m}\, d\sigma \quad \text{and} \quad \boldsymbol{L}(\boldsymbol{m}) = \int_{\mathbb{S}^2} \boldsymbol{\nu}\, (\boldsymbol{m}^* \omega_{\mathbb{S}^2}).$$

Among the key properties underpinning the physical notion of angular momenta are frame indifference $\boldsymbol{S}(\boldsymbol{m}_R) = R\boldsymbol{S}(\boldsymbol{m})$ and $\boldsymbol{L}(\boldsymbol{m}_R) = R\boldsymbol{L}(\boldsymbol{m})$, and the canonical commutation relations

$$\{S_i, S_j\} = \epsilon_{ijk} S_k \quad \text{and} \quad \{L_i, L_j\} = \epsilon_{ijk} L_k.$$

The total angular momentum is an integral of motion $\{\boldsymbol{J}, E\} = 0$ i.e. conserved for smooth solutions to the Landau-Lifshitz equation, while spin and orbital angular momenta are not conserved independently. Geometrically, $\boldsymbol{S}$ and $\boldsymbol{L}$ are generators of rotations in spin and coordinate space, respectively, i.e.

$$\{\boldsymbol{m}, S_3\} = \hat{\boldsymbol{e}}_3 \times \boldsymbol{m} \quad \text{and} \quad \{\boldsymbol{m}, L_3\} = -\partial_\chi \boldsymbol{m}.$$

Note that by virtue of frame indifference we can always fix a rotation axis along $\hat{\boldsymbol{e}}_3$. We then say a field $\boldsymbol{m} : \mathbb{S}^2 \to \mathbb{S}^2$ is equivariant iff $\boldsymbol{m}_R = \boldsymbol{m}$ for all $R \in \mathrm{SO}(3)_{\hat{\boldsymbol{e}}_3}$ the isotropy group of $\hat{\boldsymbol{e}}_3$. Equivariant fields $\boldsymbol{m}$ are precisely the critical points of J_3 and satisfy the following quantization property

$$|\boldsymbol{J}(\boldsymbol{m})| = \begin{cases} 4\pi & \text{for } Q = 0 \\ 0 & \text{for } Q = \pm 1. \end{cases}$$

Note that other topological degrees are not possible under the assumption of equivariance. We are particularly interested in the skyrmionic case $Q(\boldsymbol{m}_0) = 0$ such that $|\boldsymbol{J}(\boldsymbol{m}_0)| \neq 4\pi$.

Theorem 1. *For every $\kappa > 0$, there exists $\varepsilon > 0$ such that if $\boldsymbol{J}_0 \in \mathbb{R}^3$ satisfies $4\pi < |\boldsymbol{J}_0| < 4\pi + \varepsilon$ then*

$$\inf\{E(\boldsymbol{m}) : Q(\boldsymbol{m}) = 0 \text{ and } \boldsymbol{J}(\boldsymbol{m}) = \boldsymbol{J}_0\}$$

is attained by a smooth field $\boldsymbol{m}_0 : \mathbb{S}^2 \to \mathbb{S}^2$ with $E(\boldsymbol{m}_0) < 8\pi$ which is not equivariant. Moreover, $\boldsymbol{m}_0$ is the profile of a jointly rotating (3) *solution of the Landau-Lifshitz equation* (4).

The result is based on a concentration compactness argument in the spirit of [1]. The key is an energy estimate as in (1) that rules out the collapse of topological charge. For the construction it is useful to perform a change of coordinates in the target representing $\boldsymbol{m} = R\boldsymbol{u}$ where R is a $SO(3)$ valued field associated to an orthonormal moving frame $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\nu}\}$. The result is a gauged version of (2) where ∇ is replaced by the covariant gradient $\nabla + A$ with $A = R^\dagger \nabla R$. Cartan's calculus reveals a geometric realization of spin-orbit terms of the form

$$[\boldsymbol{u}(\nabla \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}]_3$$

coming from the second fundamental form. The energy bound can then be deduced from a construction as in [1]. An important open problem is to understand how minimal energies depend on the size of the angular momentum in order to estimate the rotation frequency ω, and to ascertain the obtained solution to the Landau-Lifshitz equation is non-static. This is of course closely related to the notoriously difficult problem of proving symmetry of minimizing skyrmions.

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Boundary Regularity for minimal Surfaces

Simone Steinbrüchel

Minimal surfaces have been studied in the last two centuries by various mathematicians. In the 1930's, T. Radó [16] and J. Douglas [12] proved the existence of 2-dimensional minimal surfaces in $\mathbb{R}^3$ and for this work, Douglas was awarded the Fields medal. Since then a lot of progress has been made and moreover, a new language was invented in order to understand the higher dimensional case. The language that we use in this work is the one of Geometric Measure Theory, where we see surfaces as currents supported in a Riemannian manifold and area minimizing currents are those having least mass among all currents sharing the same boundary. The existence of such minimizers has been proven by H. Federer and W. Fleming [14] in the 1960's. However, such a minimizing (integral) current is supported on a rectifiable set and thus a priori can have many singularities.

A posteriori singularities are rare. In his Ph.D. thesis [1], W. Allard proved that, in case the boundary is contained in the boundary of a uniformly convex set and the ambient manifold is the euclidean space, then all boundary points are regular. This means, in a neighborhood of a boundary point, the support of the current is a regular manifold with (possibly empty) boundary. Later, R. Hardt and L. Simon came to the same conclusion in [15] when having replaced the assumption of the uniform convexity by the fact that the current is of codimension 1. However, as the result of Hardt and Simon is stated and proved only in the Euclidean space, in our work, we provide an adaptation of the arguments to the case of general Riemannian manifolds. We show the following:

Let $U \subset \mathbb{R}^{n+k}$ be open and T an n-dimensional locally rectifiable current in U that is area minimizing in some smooth $(n+1)$-manifold M and such that ∂T is an oriented C^2 submanifold of U. Then for any point $a \in \operatorname{spt}(\partial T)$, there is a neighborhood V of a in U satisfying that $V \cap \operatorname{spt}(T)$ is an embedded $C^{1,\frac{1}{4}}$ submanifold with boundary.

The theorem of Hardt and Simon is then a case of the one stated above, however we follow their strategy of proof with a few modifications in order to deal with additional error terms coming from the ambient manifold in the main estimates.

Notice that the complete absence of singular points only happens at the boundary and only in codimension 1. Indeed, in 2018, C. De Lellis, G. De Philippis, J. Hirsch and A. Massaccesi showed in [7] that in the case of higher codimension and on a general Riemannian manifold, there can be singular boundary points but anyway, the set of regular ones is dense. Moreover, in the interior of an area minimizing current, we know thanks to the works of E. Bombieri, E. De Giorgi, E. Giusti [5], W. Allard [2, 3] and J. Simons [17], that the set of singularities of an n-dimensional current in an $(n+1)$-dimensional manifold is of dimension at most $n-7$. In the case of higher codimension, the sharp dimension bound is $n-2$ which was first proven in Almgren's Big regularity paper [4] and then revisited and shortened by De Lellis and Spadaro in [6, 8, 9, 10, 11].

Overview of the proof. We would like to measure how flat a current T is. Therefore we introduce its excess in a cylinder of radius r and denote it by $E_C(T, r)$. It is the scaled version of the difference between the mass of the current in a cylinder and the mass of its projection. The main ingredient to deduce the boundary regularity is the fact that this excess scales (up to a small rotation) like r assuming that the curvature of both the boundary of the current κ_T and the ambient manifold $\mathbf{A}$ are small. More precisely, we show:

Let M be a smooth manifold and let T be area minimizing in M such that $\max\{E_C(T,1), \mathbf{A}, \kappa_T\} \leq \dfrac{1}{C}$. Then there is an angle η such that for all $0 < r < R$ the following holds

$$E_C(\eta_\# T, r) \leq Cr.$$

In order to prove it, we first analyze the current away from the boundary. There we can use results from the interior regularity theory and find that the current is supported on a union of graphs of functions fulfilling the minimal surface equation. When zooming in (up to rescaling), the boundary (and the ambient manifold) become more flat and therefore, we can find the interior graphs closer to the boundary. The point is then to study what happens in the limit when the graphs on both sides of the boundary grow together. This limiting rescaled functions we call the harmonic blow-ups.

After proving the uniform convergence of the harmonic blow-ups also at boundary points, we show in a first step that in case the harmonic blow-ups are linear, they coincide on both sides of the boundary. In order to drop the assumption of linearity, we blow up the harmonic blow-ups a second time. Then, knowing that the harmonic blow-ups coincide and in fact merge together in a smooth way, we prove the excess decay via a compactness argument: if the excess decay did not hold, there would be a sequence of currents whose blow-ups cannot coincide. Then this decay leads to a $C^{1,\frac{1}{4}}$-continuation up to the boundary of the functions whose graphs describe the current assuming that the excess and the curvatures are sufficiently small. We then collect everything together and deduce that either the current lies only on one side of the boundary or both sides merge together smoothly. In case of a one-sided boundary, Allard's boundary regularity theory [3] covers the result.

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Regularity of the two-phase free boundaries

Bozhidar Velichkov
(joint work with Guido De Philippis and Luca Spolaor)

1. One-phase free boundaries

Let $D \subset \mathbb{R}^d$ be a fixed domain (for simplicity we suppose that D is the unit ball B_1), $\varphi : \partial D \to \mathbb{R}$ be a nonnegative function and $\Lambda > 0$ be a given constant. We consider the one-phase Bernoulli problem

$$\text{(1)} \qquad \textit{Minimize} \quad \int_D |\nabla u|^2\,dx + \Lambda|\{u>0\}\cap D| \quad \textit{among}$$
$$\textit{all functions} \quad u \in H^1(D) \quad \textit{such that} \quad u = \varphi \quad \textit{on} \quad \partial D.$$

1.1. **Example.** In dimension one, if D is the interval $[0,1]$, and if, for instance,

$$\phi(0) = a > 0 \qquad \text{and} \qquad \phi(1) = 0,$$

then it is easy to check that the minimizer should have the form

$$u(t) = \frac{a}{\ell}(\ell - t) \quad \text{for} \quad 0 \le t \le \ell\,, \qquad u(t) = 0 \quad \text{when} \quad t \ge \ell.$$

A simple computation gives that u is optimal, when $\ell = 1$ or $|u'(\ell)| = \dfrac{a}{\ell} = \sqrt{\Lambda}$.

1.2. **Lipschitz continuity.** The optimal regularity for u (in any dimension) was obtained by Alt and Caffarelli. In [1] they showed that if $u : D \to \mathbb{R}$ is a solution of the one-phase problem (1), then it is (locally) Lipschitz continuous, which by the $1D$ example is optimal.

1.3. **Blow-up limits.** If $0 = x_0 \in \partial\{u > 0\} \cap D$, then the family of functions $u_r(x) = \frac{1}{r}u(rx)$ is (locally) uniformly Lipschitz. In particular, every sequence $r_n \to 0$ has a subsequence (still denoted by r_n) such that u_{r_n} converges locally uniformly to some $u_0 : \mathbb{R}^d \to \mathbb{R}$ (which depends on the sequence r_n) and is a non-zero 1-homogeneous local minimizer of the one-phase functional (see [6]).

1.4. **Viscosity solutions.** If u is a solution to (1), then $|\nabla u| = \sqrt{\Lambda}$ on $\partial\{u > 0\}$ in viscosity sense, that is, for every $x_0 \in \partial\{u > 0\} \cap D$ and $\varphi \in C^\infty(D)$,

- if $u(x_0) = \varphi(x_0)$ and $u \le \varphi_+$ in D, then $|\nabla\varphi(x_0)| \ge \sqrt{\Lambda}$;
- if $u(x_0) = \varphi(x_0)$ and $u \ge \varphi_+$ in D, then $|\nabla\varphi(x_0)| \le \sqrt{\Lambda}$.

In [4] De Silva proved the following theorem for viscosity solutions

Theorem (De Silva [4]). *There are $\varepsilon_0 > 0$ and $0 < \rho < 1$ such that if u is Lipschitz, harmonic in $\{u > 0\}$, $|\nabla u| = \sqrt{\Lambda}$ on $\partial\{u > 0\}$ in viscosity sense, $u(0) = 0$ and*

$$\sqrt{\Lambda}\,(x_d - \varepsilon)_+ \le u(x) \le \sqrt{\Lambda}\,(x_d + \varepsilon)_+ \quad \textit{for every} \quad x \in B_1$$

for some $\varepsilon < \varepsilon_0$, then there is a unit vector ν, ε-close to e_d and such that

$$\sqrt{\Lambda}\,(x\cdot\nu - \varepsilon/2)_+ \le \frac{1}{\rho}u(\rho x) \le \sqrt{\Lambda}\,(x\cdot\nu + \varepsilon/2)_+ \quad \textit{for every} \quad x \in B_1.$$

As a consequence, a classical argument gives:
Corollary. *Let u be a solution of* (1). *If at some point $x_0 \in \partial\{u>0\}\cap D$ the function u has a blow-up limit of the form $u_0(x)=\sqrt{\Lambda}\,x_d^+$, then $\partial\{u>0\}$ is $C^{1,\alpha}$ manifold in a neighborhood of x_0.*

The proof of the De Silva's theorem can be divided into two main steps.
Step 1. The first step is to prove the non-rescaled version of the theorem:
Lemma (De Silva [4])**.** *In the hypotheses of the above theorem, if*

$$\sqrt{\Lambda}\,(x_d+A)_+ \le u(x) \le \sqrt{\Lambda}\,(x_d+B)_+ \quad \textit{for every} \quad x\in B_1$$

where $0<B-A<\varepsilon_0$, then there are a,b such that $0<b-a<\frac{1}{2}(B-A)$ and

$$\sqrt{\Lambda}\,(x_d+a)_+ \le u(x) \le \sqrt{\Lambda}\,(x_d+b)_+ \quad \textit{for every} \quad x\in B_\rho.$$

Notice that this statement can be summarized in the following claim:

(2) *If u is ε-close to a solution of the form $\sqrt{\Lambda}\big((x-x_0)\cdot\nu\big)_+$ in B_1, then in B_ρ, u is $\varepsilon/2$-close to $\sqrt{\Lambda}\big((x-y_0)\cdot\nu\big)_+$, where $\nu\in\partial B_1$ is the same as above.*

Proof. If u, calculated in some fixed point (say $\bar{x}=1/5e_d$), is bigger that $x_d^+\sqrt{\Lambda}$ calculated in the same point, then the lower bound on u can be improved in B_ρ. In fact, in [4] it was constructed an increasing family of functions w_t such that

$$\Delta w_t>0,\quad |\nabla w_t|>\sqrt{\Lambda},\quad w_t=u \text{ on } \partial B_1,\quad \text{and}\quad w_t\le u \text{ at } \bar{x},$$

for every $0<t<1$. Since none of w_t can touch u from below, we get that $w_1\le u$. This provides the improvement in B_ρ. □

Step 2. Let ρ be fixed. Suppose by contradiction that such an ε_0 does not exist. Then, there is a sequence u_n of ε_n-flat solutions which are not flatter in B_1. But then, Step 1 gives that the sequence

$$v_n(x)=\frac{u_n(x)-\sqrt{\Lambda}x_d^+}{\varepsilon_n}$$

converges in some suitable sense to a function v. One can prove (see [4]) that the limit v is a solution to some *limit problem.* In the one-phase case v is harmonic in $\{x_d>0\}$ with Neumann boundary condition on $\{x_d=0\}$. Thus, the classical regularity of the harmonic functions gives that, for a dimensional constant $C_d>0$,

$$x\cdot\nabla v(0)-C_d\rho^2 \le v(x)\le x\cdot\nabla v(0)+C_d\rho^2 \quad \text{in} \quad B_\rho.$$

Using the convergence of v_n to v, we get that u_n is flatter in the direction of the vector $e_d+\varepsilon_n\nabla v(0)$. When $\rho>0$ is small enough, this is a contradiction.

2. Two-phase free boundaries - the main result

Given a domain D in $\mathbb{R}^d$, a (sign-changing) function $\varphi:\partial D\to\mathbb{R}$ and constants $\Lambda_+>0$ and $\Lambda_->0$, we consider the following two-phase Bernoulli problem:

(3) *Minimize* $$\int_D |\nabla u|^2\,dx+\Lambda_+|\{u>0\}\cap D|+\Lambda_-|\{u<0\}\cap D|$$

among all functions $u\in H^1(D)$ *such that* $u=\varphi$ *on* ∂D.

This problem was introduced by Alt, Caffarelli and Friedman in [2]. The Lipschitz continuity of the solutions was proved in [2]. Moreover, if x_0 is a two-phase point

$$x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\} \cap D,$$

then every blow-up limit of u at x_0 is of the form

$$u_0(x) = \alpha(x \cdot \nu)_+ - \beta(x \cdot \nu)_-, \tag{4}$$

where ν is a unit vector and α, β are positive constants such that

$$\alpha^2 \geq \Lambda_+, \quad \beta^2 \geq \Lambda_-, \quad \alpha^2 - \beta^2 = \Lambda_+ - \Lambda_-.$$

This leads to a two-phase optimality condition in viscosity sense.

In [3], with De Philippis and Spolaor, we proved the following theorem.

Theorem (De Philippis-Spolaor-V. [3])**.** *Let u be a solution of* (3)*. Then, in a neighborhood of every two-phase point $x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\} \cap D$, both free boundaries $\partial\{u > 0\}$ and $\partial\{u < 0\}$ are $C^{1,\alpha}$ manifolds (in any dimension $d \geq 2$).*

Remark 1. The same regularity result holds for viscosity solutions, which are ε-close to a solution of the form (4), just as in the case of De Silva's Theorem.

Remark 2. In dimension two, this theorem was proved in the earlier paper [5] via an epiperimetric inequality.

The proof of the above theorem follows the main steps from the proof of the De Silva's Theorem, but there are two main differences. The first one is technical and comes from the fact that the limit problem from Step 2 is a thin two-membrane problem, but this does not require a change in the general approach to the problem. The second difference is hidden in Step 1. In fact the statement (2) turns out to be false for the two-phase problem. Even in dimension one. For instance, for every $\varepsilon > 0$, the function

$$u_\varepsilon(t) = \sqrt{\Lambda}_+(t + \varepsilon)_+ - \sqrt{\Lambda}_-(t - \varepsilon)_-$$

is a solution to the two-phase problem and is ε-close in the interval $(-1, 1)$ to the 1-homogeneous global solution

$$\sqrt{\Lambda}_+ t_+ - \sqrt{\Lambda}_- t_-.$$

but this closeness cannot be improved in the smaller interval $(-\rho, \rho)$.

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Brakke flows starting from closed rectifiable sets

Yoshihiro Tonegawa
(joint work with Lami Kim, Salvatore Stuvard)

A time-parametrized family of surfaces is called the mean curvature flow if the velocity of the surface is equal to the mean curvature at each point and time. One can define a generalized notion of mean curvature flow in the framework of varifold called the Brakke flow. The aim of the talk is to present a general existence theorem of Brakke flow starting from closed rectifiable set with fixed boundary condition, and as the application, to give an existence of Brakke flow starting from stationary varifold with expanding hole. The general existence theorem is the following.

Theorem ([3]) *Assume the following* (1)-(4):

(1) *A bounded open set* $\Omega_0 \subset \mathbb{R}^{n+1}$ *is strictly convex with* C^2 *boundary* $\partial\Omega_0$.
(2) *A closed countably* n*-rectifiable set* $\Gamma_0 \subset \Omega_0$ *has finite* n*-dimensional Hausdorff measure, that is,* $\mathcal{H}^n(\Gamma_0) < \infty$.
(3) *There exists a set of mutually disjoint non-empty open sets* $E_{1,0}, \dots, E_{N,0}$ *with* $N \geq 2$ *such that* $\cup_{i=1}^N E_{i,0} = \Omega_0 \setminus \Gamma_0$ *and* $\mathcal{H}^n(\Gamma_0 \setminus \cup_{i=1}^N \partial^* E_{i,0}) = 0$. *Here* $\partial^* E$ *is the reduced boundary of* E.
(4) *There exists a set* $Z \subset \partial\Gamma_0 := \overline{\Gamma_0} \setminus \Omega_0$ *with* $\overline{Z} = \partial\Gamma_0$ *and the following property. Setting* $F_i := \overline{E_{i,0}} \setminus (\Omega_0 \cup \partial\Gamma_0)$ *for each* $i = 1, \dots, N$, *for each* $x \in Z$, *there exists* $i, j \in \{1, \dots, N\}$, $i \neq j$, *such that* $x \in \partial F_i \cap \partial F_j$ *(the closure* $\overline{A}$ *is taken with respect to* $\mathbb{R}^{n+1}$ *topology and* ∂F_i *is as a set within* $\partial\Omega_0$*).*

Then there exists a Brakke flow $\{\mu_t\}_{t\geq 0}$ *such that* $\mathcal{H}^n\lfloor_{\Gamma_0} = \lim_{t\to 0+} \mu_t$ *and* $\overline{\operatorname{spt}\mu_t} \setminus \Omega_0 = \partial\Gamma_0$ *for all* $t \geq 0$. *Moreover, there exists a sequence* $\{t_j\}_{j\in\mathbb{N}}$ *with* $\lim_{j\to\infty} t_j = \infty$ *such that* μ_{t_j} *converges to a stationary integral varifold* μ_∞ *in* Ω_0 *such that* $\overline{\operatorname{spt}\mu_\infty} \setminus \Omega_0 = \partial\Gamma_0$.

This is a "Brakke flow with fixed boundary condition", and the similar result without boundary condition has been previously established in [1]. The last statement shows that we may obtain a solution of the Plateau problem via this existence theorem for given $\partial\Gamma_0$. As a further inquiry, we looked at a case that Γ_0 is stationary, that is, the generalized mean curvature is zero. In this case, time-independent solution $\mu_t = \mathcal{H}^n\lfloor_{\Gamma_0}$ is a Brakke flow, but we find that there exists a genuinely time-dependent Brakke flow under some assumption on the singularity of Γ_0.

Theorem ([4]) *In addition to the assumptions* (1)-(4) *above, assume that there exists a point* $x_0 \in \Gamma_0$ *(and we may assume* $x_0 = 0$ *after a change of variable) with the following.*

(1) $\lim_{r\to 0+} \mathcal{H}^n\lfloor_{(\Gamma_0/r)} = q\,\mathcal{H}^n\lfloor_{\{x_{n+1}=0\}}$ *for some* $q \in \mathbb{N}$ *with* $q \geq 2$.
(2) *There exist* $\alpha > 1/2$, $r_0 > 0$, $c_1 > 0$ *such that, writing* $C(r) := \{(x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : |x'| < r, |x_{n+1}| < r_0\}$,

$$\Gamma_0 \cap C(r) \subset \{(x', x_{n+1}) : |x_{n+1}| \leq c_1 r/(\log(1/r))^\alpha$$

for all $0 < r < r_0$.

Then there exists a Brakke flow $\{\mu_t\}_{t\geq 0}$ *such that* $\mathcal{H}^n\lfloor_{\Gamma_0} = \lim_{t\to 0+}\mu_t$ *with fixed boundary as in the previous theorem. We have* $\mu_t(\Omega_0) < \mu_0(\Omega_0)$ *and* μ_t *is time-dependent.*

In fact, we can show that this μ_t has an expanding hole starting at the origin. Examples satisfying the assumptions are minimal immersion with a branch point and self-intersections. This type of "flat singularity with a high multiplicity" is little understood so far. The proof uses Brakke's expanding hole lemma iteratively. The existence result gives rise to a notion of dynamical stability of singular minimal surfaces and it is interesting to know the regularity property of dynamically stable minimal surfaces.

A regularity property for $n = 1$ Brakke flow constructed in [1, 3] is also described ([2]). For almost all time, $\operatorname{spt}\mu_t$ consists of a finite number of $W^{2,2}$ curves which meet at junctions where curves have specific angles of either 0, 60 or 120 degrees. So far, we have a partial regularity theorem for unit-density Brakke flow, but with this added structure theorem of junctions, almost all space-time partial regularity seems achievable at least for $n = 1$.

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Minimization of elasticity energies with cracks

Antonin Chambolle

The "variational approach to fracture" [10, 1] is a quasi-static evolution process introduced to generalize the classical theory of Griffith [12] for predicting crack growth in brittle material. It relies (to simplify) on the following minimization problem:

$$\min_{\substack{(u,K)\\ u=u^0 \text{ on } \Gamma^D}} \int_{\Omega\setminus K} |e(u)|^2 dx + \mathcal{H}^{n-1}(K) \tag{1}$$

where here, the bounded open domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$ in general) represents a "reference configuration", $u : \Omega \to \mathbb{R}^d$ an infinitesimal (vectorial) displacement which is assumed in the class $H^1(\Omega \setminus K)$ or a local variant, u^0 a (vectorial) boundary load exerted on the Dirichlet boundary $\Gamma^D \subset \partial\Omega$ and K the crack set, that is, a closed, $(n-1)$-dimensional set across which u is allowed to jump and which therefore represents a fracture. In the energy, $e(u) = (\nabla u + \nabla u^T)/2$ is the symmetrized (weak) gradient of u and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure.

The talk was adressing the efforts to show existence of minimizers for problem (1), recently completed under the assumptions that u^0 is Lipschitz and Γ^D is a regular part of the boundary. Some progress had been made earlier, in dimension 2 mostly, in particular in [11] who could readily show the existence of quasistatic evolutions in the sense of [10].

In higher dimension, though, the main breakthrough was a “Korn-Poincaré” inequality, proved in [2], showing the rigidity of displacements u with little jump K, up to a small set whose size is controlled by $\mathcal{H}^{n-1}(K)$. In practice, it is shown that such displacements are close to an “infinitesimal rigid motion”, which is an affine function $a(x) = Ax + b$, with skew-symmetric gradient ($A + A^T = 0$, that is, $e(a) = 0$), outside of the exceptional set.

A first outcome of this result was a compactness result, shown in [4], which establishes the existence of minimizers for a weak variant of (1), where K is replaced with an intrinsic jump set J_u which is merely a countably $(n-1)$-rectifiable set (possibly dense, possibly not closed, etc). Such a jump set J_u is well defined provided $u \in GSBD(\Omega)$, the space of “generalized special functions with bounded deformation”, a set introduced by G. Dal Maso in [8] as the natural energy space for (1).

The proof of the compactness result relies on local projections onto infinitesimal rigid motions (which lie in a finite dimensional set, and hence whose compactness properties are elementary), allowing to show that u with bounded energy is arbitrarily close, in some particular sense, to finite-dimensional manifolds (with a distance controlled by the energy).

This provides existence of weak solutions, but, as in the now standard theory of “free discontinuity problems” and the Mumford-Shah functional [9], one has to prove then that the jump set J_u of a global minimizer is essentially closed, so that letting $K = \overline{J_u} \cap (\Omega \cup \Gamma^D)$, one does not increase the $\mathcal{H}^{n-1}$-measure and (u, K) is a minimizer for (1). This is done, as in [9], by establishing lower density bounds for the jump set J_u.

The first result in this direction was proved, in dimension 2, by S. Conti, M. Focardi and F. Iurlano in [7]. Then, in [3], thanks to the rigidity result of [2], we could extend the work [7] to arbitrary dimension. Eventually in [5] we proved a “Dirichlet” variant of [3], where the lower density for J_u is established also in the possible boundary part of the jump ($J_u \cap \Gamma^D$). This achieved the proof of existence of minimizers for (1).

In the last part of the talk, we observed that the result of [4] is not complete enough to show existence of minimizers for a variant of (1) as simple as

$$\min_{u=u^0 \text{ on } \Gamma^D} \int_\Omega f(x, e(u))dx + \mathcal{H}^{n-1}(J_u)$$

(written as a weak formulation, with $u \in GSBD(\Omega)$ and J_u its jump set). Such energy could arise for instance in a problem where the reference configuration is pre-constrained. We started recently to address such a variant, sticking for now to the weak formulation, and could show a compactness and lower-semicontinuity result yielding a proof of existence of minimizers (with natural assumptions on f) in a recent work with V. Crismale [6].

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Variational models on graphs and related flows for data analysis and image processing

Yves van Gennip

(joint work with Yoshikazu Giga, Jun Okamoto, Blaine Keetch, Jeremy Budd et al.)

Discretisations of variational models on finite graphs have been succesfully used for diverse applications such as data clustering, data classification, community detection, image segmentation, and maximum cut approximations. We focus on the graph Ginzburg–Landau functional, which was introduced in [1], and a variant, the signless graph Ginzburg–Landau functional, introduced in [2]. The former is given by

$$G_\varepsilon(u) := \sum_{i,j\in V} \omega_{ij}|u_i - u_j|^2 + \frac{1}{\varepsilon}\sum_{i\in V} W(u_i),$$

where $u : V \to \mathbb{R}$ is a real-valued function defined on the node set V of a finite, undirected graph with weighted edges. The value of u at node i is denoted by u_i. We assume that the edge weight ω_{ij} is strictly positive when node i and j are connected by an edge and zero otherwise, although some research also allows for negative edges [3]. A typical choice for the potential function W, is the double-well function $W(x) := x^2(x-1)^2$, which is a quartic potential with minima of equal depth located at $x = 0$ and $x = 1$. As we will see, other choices are preferred in different contexts. The parameter $\varepsilon > 0$ is usually small. Consequently, if u_i minimizes G_ε, then u_i is forced to be close to 0 or 1. Moreover, the terms $\omega_{ij}|u_i - u_j|^2$ encourage u to take similar values on strongly connected nodes, i.e. on nodes that are connected by highly weighted edges. This mimics the phase separation behaviour of the continuum Ginzburg–Landau functional [4] on which this graph functional is based.

If we interpret u_i as a label assigned to node i, minimizers of G_ε will group the nodes into two clusters, corresponding to $\{u_i \approx 0\}$ and $\{u_i \approx 1\}$ in such a way that strongly connected nodes are placed in the same cluster. The functional G_ε as given, has the constant functions $u = 0$ and $u = 1$ as global minimizers. In practice, therefore, additional constraints are imposed on the minimization of G_ε. Common ones are a hard mass constraint, $\sum_{i\in V} u_i = M \in \mathbb{R}$, if the required size of the clusters is known a priori, or a soft fidelity constraint in the form of an additional term $\sum_{i\in V} \lambda_i(u_i - f_i)^2$ in the functional. Here f_i encodes a priori known information on a (subset of) the nodes and $\lambda_i = \lambda > 0$ on these nodes and zero otherwise. For example, in data classification or image segmentation, f_i can encode a priori knowledge of the class membership of certain nodes or certain pixels.

If $S \subset V$, the graph cut between S and its complement S^c is given by $\sum_{\substack{i\in S\\ j\in S^c}} \omega_{ij}$. This provides a notion of perimeter on a graph and indeed, analogously to the continuum Ginzburg–Landau functional approximating perimeter for small ε [5], the graph Ginzburg–Landau functional approximates the graph cut when ε is small. These notions are made rigorous using the concept of Γ-convergence [6].

The signless Ginzburg–Landau functional, G_ε^+, is a variant introduced in [2], in which the terms $\omega_{ij}|u_i - u_j|^2$ are replaced by $\omega_{ij}|u_i + u_j|^2$ and the potential W is assumed to have symmetrically placed minima at $x = \pm 1$, e.g. $W(x) = (x^2-1)^2$. A similar Γ-convergence argument as the one referred to above, can be used to show that $-G_\varepsilon^+$ also approximates the graph cut (up to constants) for small ε and hence minimizing G_ε^+ gives approximate solutions to the max-cut problem [2].

The formulations for G_ε and G_ε^+ given above lead to formation of two clusters, but extensions to more than two clusters are possible [7, 8].

In what follows, we will restrict our focus to G_ε. There are two common ways to attempt minimization of G_ε, both inspired by the study of partial differential equations. The first one is to compute the solution to the gradient flow of G_ε, i.e.

$\frac{du}{dt} = -\nabla_{\mathcal{V}} G_\varepsilon(u)$. This gives the graph Allen–Cahn equation:

$$\frac{du_i}{dt} = -\sum_{j\in V} \omega_{ij}(u_i - u_j) - \frac{1}{\varepsilon} W'(u_i).$$

We can recognize the graph Laplacian [9], $(\Delta u)_i := \sum_{j\in V} \omega_{ij}(u_i - u_j)$, an object studied intensively in spectral graph theory [10]. Even though the Allen–Cahn equation is not guaranteed to lead to a global minimizer and can 'get stuck' in a local minimum of G_ε, in practice it works well [1, 11]. Mass or fidelity constraints are easily incorporated into the equation, through the addition of the term $\sum_{j\in V} W'(u_j)/|V|$ or $-2\lambda_i(u_i - f_i)$, respectively.

A second way to attempt minimization of G_ε is via the Merriman–Bence–Osher (a.k.a. threshold dynamics) scheme. Originally introduced in the continuum setting [12] as a way to approximate flow by mean curvature, transscribed to a graph this iterative scheme alternates short time diffusion

$$\frac{du}{dt} = -\Delta u, \quad \text{for } t \in (0,\tau], \qquad u(0) = u^k,$$

with thresholding

$$u^{k+1} = \begin{cases} 1, \text{if } u(\tau) \geq \frac{1}{2}, \\ 0, \text{if } u(\tau) < \frac{1}{2}. \end{cases}$$

Starting from a binary function $u^0 : V \to \{0,1\}$, threshold dynamics generates a sequence of binary functions u^k. It can be shown this process terminates in a finite number of steps after which $u^k = u^{k+1}$. The intuition behind the use of this scheme for minimization of G_ε is that the nonlinear term $-\frac{1}{\varepsilon}W'(u_i)$ in the Allen–Cahn equation has approximately the same effect as the thresholding step in the Merriman–Bence–Osher scheme, that is, forcing u to take values at or near 0 or 1.

The connection between the Allen–Cahn gradient flow and the Merriman–Bence–Osher scheme on graphs has been made rigorous in [15]. A priori two major differences between these dynamics can be observed: Allen–Cahn is continuous in time, while Merriman–Bence–Osher is discrete in time; and solutions to Allen–Cahn are real-valued, while solutions to Merriman–Bence–Osher are binary. The second difference is overcome by replacing the smooth double-well potential $W(x) = x^2(x-1)^2$, by the double-obstacle potential:

$$W(x) := \begin{cases} \frac{1}{2}x(1-x), & \text{if } x \in [0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

Because this W is no longer differentiable, the Allen–Cahn equation has to be changed to a differential inclusion. The first difference is overcome by introducing a semi-discrete implicit Euler discretisation scheme for Allen–Cahn. This leads to the following time-discrete iterative scheme for graph Allen–Cahn with the double obstacle potential and time-step τ:

$$\left(1 - \frac{\tau}{\varepsilon}\right) u_{n+1} - e^{\tau\Delta u_n} + \frac{\tau}{2\varepsilon} = \frac{\tau}{\varepsilon}\beta_{n+1}.$$

By comparing the variational form of this scheme with the variational form of threshold dynamics [13, 14], it is shown that for $\tau = \varepsilon$ this scheme is equivalent to the Merriman–Bence–Osher scheme [15]. By choosing $0 < \tau/\varepsilon < 1$ a relaxed threshold dynamics scheme is obtained. Moreover, by applying the semi-discrete implicit Euler discretisation to Allen–Cahn with a mass or fidelity constraint, threshold dynamics schemes which incorporate those constraints are constructed [16, 17].

The question of consistency of the graph-based methods is addressed via discrete-to-continuum limits. At a variational level, Γ-convergence arguments show that minimizers of G_ε converge to minimizers of a continuum Ginzburg–Landau functional in the limit $|V| \to \infty$ [6]. Using a variational inequality formulation of gradient flows [18] initial results on the discrete-to-continuum convergence of discrete gradient flows derived from G_ε have also been obtained [19]. The research in this area is ongoing.

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Variational extrapolation: Boosting the accuracy of minimizing movements and other schemes for gradient flows

Selim Esedoglu
(joint work with Alexander Zaitzeff, Krishna Garikipati)

For a sufficiently smooth cost function E defined on a Hilbert space H, a typical approach to generating a time discrete approximation to the gradient flow

$$u' = -\nabla_H E(u) \text{ with } u(0) = \phi \tag{1}$$

is the backwards Euler, or minimizing movements, scheme:

$$\begin{aligned} u^{n+1} &= \arg\min_u \left\{ E(u) + \frac{1}{2k}\|u - u^n\|_H^2 \right\} \\ u^0 &= \phi \end{aligned} \tag{2}$$

where $k > 0$ is the time step size, and u^n is the approximation at time $t = nk$. An attractive feature of this approximation is *unconditional energy stability*

$$E(u^{n+1}) \leq E(u^n) \tag{3}$$

for every n, regardless of k. However, even in the simplest, ODE setting of $H = \mathbb{R}^d$, scheme (2) is only first order accurate:

$$\|u^1 - u(k)\| = O(k^2) \text{ as } k \to 0. \tag{4}$$

We investigate whether the variational formulation (2) has a simple analogue that preserves stability condition (3) but improves upon its accuracy (4), with an eye towards applications to dynamics such as motion by mean curvature, which doesn't quite fit the classical formulation of gradient flow, but can at least formally be recognized as such.

To that end, we look for *multi-stage* versions of (2). The M-stage version has the form

$$\begin{aligned} U^0 &= u^n \\ U^m &= \arg\min_u \left\{ E(u) + \frac{1}{2k}\sum_{i=0}^{m-1} \gamma_{m,i}\|u - U^i\|_H^2 \right\} \text{ for } m = 1, 2, \ldots, M \\ u^{n+1} &= U^M. \end{aligned} \tag{5}$$

The sum of quadratic terms in (5) can be combined into one, ensuring that, in practice, this M-stage analogue of (2) entails nothing other than the same type of variational problem as in (2), solved M times per time step.

The question is whether the parameters $\gamma_{m,i}$ in (5) can be chosen so that stability condition (3) can be guaranteed while achieving high order consistency:

$$\|u^1 - u(k)\| = O(k^{p+1}) \text{ as } k \to 0 \text{ with } p \geq 2 \tag{6}$$

as long as the exact solution exists and is sufficiently smooth. In the finite dimensional setting, scheme (5) can be recognized as a Runge-Kutta scheme; as such, conditions on $\gamma_{m,i}$ ensuring consistency (6) to the desired order p are given by standard order conditions for Runge-Kutta schemes. But can (3) be achieved simultaneously? It turns out that, at least for $p = 2$ and $p = 3$, the answer is yes.

In particular, we exhibit an $M = 3$ stage scheme of the form (5) that is consistent to second order ($p = 2$ in (6)), and unconditionally gradient stable (3). One choice for the requisite parameters (which are not unique) that guarantee both conditions is

$$\gamma = \begin{pmatrix} \gamma_{1,0} & 0 & 0 \\ \gamma_{2,0} & \gamma_{2,1} & 0 \\ \gamma_{3,0} & \gamma_{3,1} & \gamma_{3,2} \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ -2 & 6 & 0 \\ -2 & \frac{3}{14} & \frac{44}{7} \end{pmatrix}.$$

We also exhibit an $M = 5$ stage scheme that is consistent to third order ($p = 3$ in (6)) and still unconditionally gradient stable (3), by again providing specific values for the parameters $\gamma_{m,i}$.

To apply the multi-stage variational, discrete time approximation strategy (5) in the context of motion by mean curvature, which is formally gradient flow for the cost function

$$E(\Sigma) = \mathrm{Per}(\Sigma) \text{ for } \Sigma \subset \mathbb{R}^d \tag{7}$$

we turn to an algorithm known as threshold dynamics that was introduced by Merriman, Bence, and Osher [2]. This algorithm generates a discrete time approximation by the two-step procedure

(1) Convolution:

$$u = G_k * \mathbf{1}_{\Sigma^n} \text{ where } G_k(x) = \frac{1}{(4\pi k)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4k}\right) \tag{8}$$

(2) Thresholding:

$$\Sigma^{n+1} = \left\{ x \,:\, u(x) \geq \frac{1}{2} \right\}. \tag{9}$$

It was realized in [1] that Algorithm (8) & (9) can be given the variational formulation:

$$\Sigma^{n+1} = \arg\min_{\Sigma} \left\{ \int_{\Sigma^c} G_k * \mathbf{1}_\Sigma \, dx + \int_{\mathbb{R}^d} (\mathbf{1}_\Sigma - \mathbf{1}_{\Sigma^n}) G_k * (\mathbf{1}_\Sigma - \mathbf{1}_{\Sigma^n}) \, dx \right\} \tag{10}$$

where the first term

$$E_k(\Sigma) = \int_{\Sigma^c} G_k * \mathbf{1}_\Sigma \, dx \tag{11}$$

in energy (10) once normalized can be shown to converge, in the sense of Γ-convergence, to (7), while the second acts to limit movement in analogy with $\|u - u^n\|_H^2$ in (2). This minimizing movements interpretation of threshold dynamics allows extending multi-stage approach (5) to the context of motion by mean curvature. Indeed, we show that the following $M = 5$ stage version of threshold dynamics dissipates the nonlocal energy (11) (and is hence unconditionally stable) and achieves second order consistency ($p = 2$) for $d = 2$:

$$\begin{aligned} \Omega^0 &= \Sigma^n, \\ \Omega^m &= \left\{ x \,:\, G_{\tau k} * \sum_{i=0}^{m-1} \gamma_{m,i} \mathbf{1}_{\Omega^i} \geq \frac{1}{2} \right\} \text{ for } m = 1, 2, \ldots, M, \\ \Sigma^{n+1} &= \Omega^M \end{aligned} \tag{12}$$

for an appropriate choice of the constant τ and the parameters $\gamma_{m,i}$.

In summary, Algorithm (5) boosts the order of accuracy of (2) while maintaining its desirable stability property (3), and appears to be applicable beyond the classical setting of gradient flow on a Hilbert space. It is also phrased entirely in terms of multiple steps of (2). Is there a similar strategy to boost the order of accuracy of other common schemes for gradient flows, such as semi-implicit schemes that choose a decomposition for the energy $E = E_1 + E_2$ and then solve

$$u^{n+1} = \arg\min_u \left\{ E_1(u) + L(u^n, u) + \frac{1}{2k} \|u - u^n\|_H^2 \right\} \tag{13}$$

at every time step, where $L(u^n, \cdot)$ is the linearization of E_2 at u^n? Again, desirable stability properties of (13) should be essentially preserved. We exhibit such strategies.

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The non-regular part of the free boundary for the fractional obstacle problem

Xavier Fernández-Real

(joint work with Y. Jhaveri, X. Ros-Oton)

In this talk we summarize the main regularity results for the thin (or fractional) obstacle problem, with particular emphasis in the currently known results regarding the structure of the free boundary.

Let $\varphi \in C^\infty(B_1')$ and consider an even solution to the thin (or fractional) obstacle problem, with obstacle φ,

$$\text{(1)} \qquad \begin{cases} u \geq \varphi & \text{on } B_1 \cap \{x_{n+1} = 0\} \\ \Delta u = 0 & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{u = \varphi\}) \\ \Delta u \leq 0 & \text{in } B_1. \end{cases}$$

Then, the solution u is $C^{1,1/2}$ on either side of the obstacle. That is, there exists a constant C depending only on n such that

$$\|u\|_{C^{1,1/2}(B_{1/2}^+)} + \|u\|_{C^{1,1/2}(B_{1/2}^-)} \leq C\left(\|\varphi\|_{C^{1,1}(B_1')} + \|u\|_{L^\infty(B_1)}\right).$$

Moreover, if we denote $\Lambda(u) := \{u = \varphi\}$ the contact set, the boundary of $\Lambda(u)$ in the relative topology of $\mathbb{R}^n$, $\partial_{\mathbb{R}^n}\Lambda(u)$, is the free boundary, and can be divided into two sets

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Deg}(u),$$

the set of *regular points*,

$$\mathrm{Reg}(u) := \left\{ x = (x', 0) \in \Gamma(u) : 0 < cr^{3/2} \leq \sup_{B_r'(x')} (u - \varphi) \leq Cr^{3/2}, \quad \forall r \in (0, r_\circ) \right\},$$

and the set of non-regular points or *degenerate points*,

$$\mathrm{Deg}(u) := \left\{ x = (x', 0) \in \Gamma(u) : 0 \leq \sup_{B_r'(x')} (u - \varphi) \leq Cr^2, \quad \forall r \in (0, r_\circ) \right\}.$$

Alternatively, each of the subsets can be defined according to the order of the blow-up (the frequency) at that point. Namely, the set of regular points are those whose blow-up is of order $\frac{3}{2}$, and the set of degenerate points are those whose blow-up is of order κ for some $\kappa \in [2, \infty]$. In a recent work with X. Ros-Oton, [4], we show that while in general, degenerate and regular points could be of the same dimension, generically this is not true.

More precisely, the free boundary can be stratified as

$$\text{(2)} \qquad \Gamma(u) = \Gamma_{3/2} \cup \Gamma_{\text{even}} \cup \Gamma_{\text{odd}} \cup \Gamma_{\text{half}} \cup \Gamma_* \cup \Gamma_\infty,$$

where:

- $\Gamma_{3/2} = \mathrm{Reg}(u)$ is the set of regular points. They are an open $(n-1)$-dimensional subset of $\Gamma(u)$, and it is C^∞ (see [1, 9, 2]).
- $\Gamma_{\text{even}} = \bigcup_{m \geq 1} \Gamma_{2m}(u)$ denotes the set of points whose blow-ups have even homogeneity. Equivalently, they can also be characterised as those points of the free boundary where the contact set has zero density, and they are often called singular points. They are contained in the countable union of C^1 $(n-1)$-dimensional manifolds; see [8]; and up to a lower-dimensional set, the regularity of the manifolds can be upgraded to C^2 as we show in a recent work with Y. Jhaveri, [3]. Generically, however, points in $\Gamma_2(u)$ have dimension at most $n-3$, and points in $\Gamma_{2m}(u)$ have dimension at most $n-2m$ for $m \geq 2$; see [4].

- $\Gamma_{\mathrm{odd}} = \bigcup_{m\geq 1} \Gamma_{2m+1}(u)$ is, a priori, at most $(n-1)$-dimensional and it is $(n-1)$-rectifiable (see [10, 6, 5]), although it is not known whether it exists. Generically, $\Gamma_{2m+1}(u)$ has dimension at most $n-2m$; see [4].
- $\Gamma_{\mathrm{half}} = \bigcup_{m\geq 1} \Gamma_{2m+3/2}(u)$ corresponds to those points with blow-ups of order $\frac{7}{2}$, $\frac{11}{2}$, etc. They are much less understood than regular points, although in some situations they have a similar behaviour. The set Γ_{half} is an $(n-1)$-dimensional subset of the free boundary and it is a $(n-1)$-rectifiable set (see [6, 10, 7]). Generically, the set $\Gamma_{2m+3/2}(u)$ has dimension at most $n-2m-1/2$.
- Γ_* is the set of all points with homogeneities $\kappa \in (2, \infty)$, with $\kappa \notin \mathbb{N}$ and $\kappa \notin 2\mathbb{N} - \frac{1}{2}$. This set has Hausdorff dimension at most $n-2$, so it is always *small*.
- Γ_∞ is the set of points with infinite order (namely, those points at which $u-\varphi$ vanishes at infinite order). For general C^∞ obstacles it could be a huge set, even a fractal set of infinite perimeter with dimension exceeding $n-1$. When φ is analytic, instead, Γ_∞ is empty. Generically, this set is empty; see [4].

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Magnetic skyrmions in the conformal limit

Cyrill B. Muratov
(joint work with A. Bernand-Mantel and T. M. Simon)

Advances in nanotechnology have enabled an unprecedented degree of control and precision in manufacturing of a variety of ferromagnetic materials for applications in spintronics – an emergent field of electronics utilizing both the charge and the spin of an electron [1]. For example, a typical magnetic tunnel junction device would feature a number of adjacent ultrathin film layers of magnetic and non-magnetic materials whose thicknesses may go down to sub-nanometer scale (a few atomic monolayers) and whose lateral extent could reach down to several tens of nanometers [2]. At these scales, the properties of such quasi two-dimensional magnetic nanomaterials are dominated by the interfaces between the adjacent layers, giving rise to new physics that modifies the material's magnetic properties [3]. Specifically, sub-nanometer thick metallic ferromagnetic films exhibit spin reorientation transition favoring out-of-plane magnetization due to the effect of interfacial perpendicular magnetic anisotropy (PMA) [4]. Similarly, an antisymmetric exchange, also referred to as the Dzyaloshinskii-Moriya interaction (DMI), emerges due to spin-orbit coupling and the absence of reflection symmetry in ferromagnet/heavy metal bilayers and multilayers [5]. These additional physical effects, in turn, promote the appearance of novel magnetization configurations, or spin textures, that exhibit winding and chirality, including chiral domain walls, spin spirals and magnetic skyrmions [6].

A suitable modeling framework for studying the emergence of spatial patterns of magnetization in ferromagnetic materials is that of micromagnetics, whereby the magnetization patterns are viewed as local minimizers of a suitable energy functional. In the context of extended ultra-thin film ferromagnets in the presence of PMA and DMI, the appropriate reduced micromagnetic energy functional is given by (after a suitable non-dimensionalization and on a suitable function class):

$$E(m) := \int_{\mathbb{R}^2} \left\{ |\nabla m|^2 + (Q-1)|m'|^2 - 2\kappa m' \cdot \nabla m_3 \right\} dx$$
$$+ \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot m'(x) \nabla \cdot m'(x')}{|x-x'|} dx\, dx' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_3(x) - m_3(x'))^2}{|x-x'|^3} dx\, dx',$$

where $m : \mathbb{R}^2 \to \mathbb{S}^2$ is the magnetization that approaches $-e_3$ sufficiently fast at infinity, $m' = (m_1, m_2)$ and m_3 are the in-plane and out-of-plane components of $m = m(x_1, x_2)$, respectively, $Q > 1$ is the material's quality factor, $\kappa \in \mathbb{R}$ is the dimensionless DMI strength, and $\delta > 0$ is the effective film thickness measured in the units of the exchange length [7, 8]. As we argue in [9], the appropriate admissible class for the minimization of the energy E that gives rise to compact magnetic skyrmion solutions is

$$\mathcal{A} := \left\{ m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2) : \int_{\mathbb{R}^2} |\nabla m|^2 dx < 16\pi,\ m + e_3 \in L^2(\mathbb{R}^2),\ \mathcal{N}(m) = 1 \right\},$$

where

$$\mathcal{N}(m) := \frac{1}{4\pi}\int_{\mathbb{R}^2} m\cdot\big(\partial_1 m\times\partial_2 m\big)\,dx$$

is the topological degree of m. Adapting the arguments of [10, 11], existence of a minimizer can then be shown for $(2|\kappa|+\delta)^2 < 2(Q-1)$, which is not expected to be sharp.

To gain further understanding into the structure of the obtained solutions, we carried out an asymptotic analysis in the regime $\sigma \ll 1$ and $\lambda\in[0,1]$, where

$$\sigma := \frac{\kappa+\delta}{\sqrt{Q-1}} \quad\text{and}\quad \lambda := \frac{\kappa}{\kappa+\delta},$$

of the minimizers of the energy $E_{\sigma,\lambda}$ that agrees with E after a suitable rescaling of space:

$$\begin{aligned} E_{\sigma,\lambda}(m) := \int_{\mathbb{R}^2}|\nabla m|^2\,dx + \sigma^2\int_{\mathbb{R}^2}|m'|^2\,dx - 2\sigma^2\lambda\int_{\mathbb{R}^2} m'\cdot\nabla m_3\,dx \\ + \frac{1}{2}\sigma^2(1-\lambda)\left(\|\nabla\cdot m'\|^2_{\mathring{H}^{-1/2}(\mathbb{R}^2)} - \|m_3\|^2_{\mathring{H}^{1/2}(\mathbb{R}^2)}\right). \end{aligned}$$

In the *conformal limit* of $\sigma\to 0$, one formally obtains the Dirichlet energy, whose minimizers were identified by Belavin and Polyakov [12]. In particular, for $\mathcal{N}=1$ these minimizers are given explicitly by what we call the Belavin-Polyakov profiles, which consist of all translations, rigid rotations and dilations of the stereographic projection with respect to the south pole:

$$\Phi(x) := \left(-\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right) \qquad x\in\mathbb{R}^2.$$

Through a careful analysis of the minimizers of $E_{\sigma,\lambda}$ with degree 1 for $\sigma\ll 1$ and λ fixed, we have established their precise asymptotic behavior. With the notations that S_θ is the rotation by the angle $\theta\in[-\pi,\pi)$ around the x_3-axis,

$$\theta_0 := \begin{cases} 0 & \text{if } \lambda\geq\lambda_c, \\ \arccos\left(\frac{32\lambda}{3\pi^2(1-\lambda)}\right) & \text{else,}\end{cases} \qquad \lambda_c := \frac{3\pi^2}{32+3\pi^2},$$

and

$$\bar{g}(\lambda) := \begin{cases} (8+\frac{\pi^2}{4})\pi\,\lambda - \frac{\pi^3}{4} & \text{if } \lambda\geq\lambda_c, \\ \frac{128\lambda^2}{3\pi(1-\lambda)} + \frac{\pi^3}{8}(1-\lambda) & \text{else.}\end{cases}$$

we have the following main result [13].

Theorem 1. *Let $\lambda\in[0,1]$. Let m_σ be a minimizer of $E_{\sigma,\lambda}$ over $\mathcal{A}$. Then there exist $x_\sigma\in\mathbb{R}^2$, $\rho_\sigma>0$ and $\theta_\sigma\in[-\pi,\pi)$ such that $m_\sigma - S_{\theta_\sigma}\Phi(\rho_\sigma^{-1}(\cdot - x_\sigma))\to 0$ in $\mathring{H}^1(\mathbb{R}^2;\mathbb{R}^3)$ as $\sigma\to 0$, and*

$$\lim_{\sigma\to 0}|\log\sigma|\rho_\sigma = \frac{\bar{g}(\lambda)}{16\pi}, \qquad \lim_{\sigma\to 0}|\theta_\sigma| = \theta_0,$$

as well as

$$\lim_{\sigma\to 0}\left\{\frac{|\log\sigma|^2}{\sigma^2\log|\log\sigma|}\left|E_{\sigma,\lambda}(m_\sigma)-8\pi+\frac{\sigma^2}{|\log\sigma|}\left(\frac{\bar g^2(\lambda)}{32\pi}-\frac{\bar g^2(\lambda)}{32\pi}\frac{\log|\log\sigma|}{|\log\sigma|}\right)\right|\right\}=0.$$

The theorem above provides a quantitative description of skyrmion solutions in the regime when the exchange is the dominant term in the energy, which is helpful in identifying the appropriate material parameters that give rise to physically realizable skyrmion solutions [9].

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Participants

Dr. Costante Bellettini
Department of Mathematics
University College London
25 Gordon Street
London WC1H 0AY
UNITED KINGDOM

Prof. Dr. Lia Bronsard
Dept. of Mathematics
Mc Master University
1280 Main st West
Hamilton ON L8S 4K1
CANADA

Prof. Dr. Jeff Calder
School of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis MN 55455
UNITED STATES

Prof. Dr. Antonin Chambolle
Centre de Mathématiques Appliquées
École Polytechnique
CNRS
91128 Palaiseau Cedex
FRANCE

Dr. Otis Chodosh
Department of Mathematics
Stanford University
Stanford CA 94305
UNITED STATES

Dr. Maria Colombo
Mathematics institute
EPFL Lausanne
Station 8
Lausanne 1015
SWITZERLAND

Prof. Dr. Selim Esedoglu
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Xavier Fernández-Real Girona
Department of Mathematics
ETH Zürich, HG G 28
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Alessio Figalli
Department Mathematik
ETH Zürich
ETH-Zentrum, HG G 45.2
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Irene Fonseca
Department of Mathematical Sciences
Carnegie Mellon University
5000 Forbes Avenue
Pittsburgh, PA 15213-3890
UNITED STATES

Giada Franz
Department of Mathematics
ETH Zürich, HG G 28
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Adriana Garroni
Dipartimento di Matematica
"Guido Castelnuovo"
Università di Roma "La Sapienza"
Piazzale Aldo Moro, 2
00185 Roma
ITALY

Dr. Or Hershkovits
Department of Mathematics
Stanford University
Stanford, CA 94305
UNITED STATES

Prof. Dr. Gerhard Huisken
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Radu Ignat
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Robert V. Kohn
Courant Institute of Mathematical Sciences
New York University
251 Mercer Street
New York NY 10012-1110
UNITED STATES

Prof. Dr. Denis Kriventsov
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Prof. Dr. Nam Le
Department of Mathematics
Indiana University, Bloomington
831 East 3rd Street
Bloomington IN 47405
UNITED STATES

Prof. Dr. Giovanni Leoni
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213-3890
UNITED STATES

Prof. Dr. Fang-Hua Lin
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
UNITED STATES

Stephen Lynch
Fachbereich Mathematik
Universität Tübingen
Rm. C6 P13
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Francesco Maggi
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
UNITED STATES

Prof. Dr. Christof Melcher
Lehrstuhl für Angewandte Analysis
RWTH Aachen
Gebäude 1950, Rm. 216
Pontdriesch 14-16
52062 Aachen
GERMANY

Prof. Dr. Ulrich Menne
Department of Mathematics
National Taiwan Normal University
No.88, Sec.4, Tingzhou Rd. Wenshan Dist.
Taipei 11677
TAIWAN

Prof. Dr. Stefan Müller
Hausdorff Center for Mathematics
Institute for Applied Mathematics
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Cyrill Muratov
Departement of Mathematical Sciences
New Jersey Institute of Technology
606 Cullimore Hall
323 Martin Luther King Jr. Blvd.
Newark, NJ 07102-1982
UNITED STATES

Prof. Dr. Matteo Novaga
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo 5
56127 Pisa
ITALY

Dr. Alessandro Pigati
Departement Mathematik
ETH-Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Prof. Dr. Thomas Pock
Institute of Computer Graphics and Vision
Graz University of Technology
Rm. IE 02022
Inffeldgasse 16/II
8010 Graz
AUSTRIA

Prof. Dr. Tristan Rivière
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Angkana Rüland
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Dr. Joaquim Serra
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Leon M. Simon
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Prof. Dr. Charles K. Smart
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Simone Steinbrüchel
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Dr. Daniel L. Stern
Department of Mathematics
University of Toronto
40 St George Street
Toronto ON M5S 2E4
CANADA

Prof. Dr. László Székelyhidi Jr.
Mathematisches Institut
Universität Leipzig
Postfach 10 09 20
04009 Leipzig
GERMANY

Prof. Dr. Susanna Terracini
Dipartimento di Matematica
Università degli Studi di Torino
Via Carlo Alberto, 10
10123 Torino
ITALY

Prof. Dr. Yoshihiro Tonegawa
Department of Mathematics
Tokyo Institute of Technology
2-12-1 Ookayama, Meguro-ku
Tokyo 152-8551
JAPAN

Prof. Dr. Peter M. Topping
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Tatiana Toro
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Dr. Yves van Gennip
Delft Institute of Applied Mathematics
Delft University of Technology
Building 28
2628 XE Delft
NETHERLANDS

Prof. Dr. Bozhidar Velichkov
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Dr. Kelei Wang
School of Mathematics and Statistics
Wuhan University
Hubei
Wuhan 430072
CHINA

Prof. Dr. Juncheng Wei
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Prof. Dr. Brian White
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Prof. Dr. Neshan Wickramasekera
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Markus Wolff
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Yang Xiang
Department of Mathematics
Hong Kong University of Science &
Techn.
Clear Water Bay
Kowloon
Hong Kong
CHINA

Dr. Zihui Zhao
Department of Mathematics
University of Chicago
5734 S. University Avenue
Chicago, IL 60637
UNITED STATES