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Komplexe Analysis - Algebraicity and Transcendence (hybrid meeting)

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Abstract. This is the report of the Oberwolfach workshop Komplexe Analysis 2020. It was mainly devoted to the transcendental methods of complex algebraic geometry and featured eighteen talks about recent important developments in Hodge theory, moduli spaces, hyperbolicity, Fano varieties, algebraic foliations, algebraicity theorems for subvarieties and their applications to transcendence proofs for numbers. Two talks were more algebraic in nature and devoted to non-commutative deformations and syzygies of secant varieties.

Mathematics Subject Classification (2020): 14C30, 14D23, 14J45, 32G15, 32G20, 32H25, 32J25, 11J99.

Introduction by the Organizers

The workshop *Komplexe Analysis, Algebraicity and Transcendence*, organized by Philippe Eyssidieux (Grenoble), Jun-Muk Hwang (Seoul), Stefan Kebekus (Freiburg) and Mihai Păun (Bayreuth), took place the week starting from the 16^{th} of August 2020.

Initially (i.e. before the end of 2019) we were excited to see a large number of mathematicians having accepted our invitation, as a result of the vitality and the continuous renewal of the core research themes explored in our workshop. Because of the events we are all aware, it had to be converted into a hybrid meeting, attended by over 40 participants from around the world, ranging from young post-doctoral researcher to senior leaders of the field. The participants from Asia and the USA and a sizable proportion of the participants from Europe followed the talks through an efficient videoconference system.

The program featured eighteen lectures, and allowed ample time for discussion and interaction for the participants present in Oberwolfach.

The organizers aimed for a balanced meeting, reflecting the main current developments on the interactions between global aspects of Complex Analysis and Complex Algebraic Geometry.

The following list of talks and subjects is not exhaustive, but illustrates the diversity and the importance of the recent contributions to the field that were the subjects of the talks. The first topic below (concerning the lectures in Hodge theory) was particularly appreciated by the participants -this was largely due to the wonderful conference by C. Schnell-, and we are expecting spectacular outcomes from this side.

Hodge theory. This classical subject has been recently revivified by novel methods and ideas and this trends were apparent in the talks on this subject. Christian Schnell's talk revisited one of its main foundations, Schmid's SL_2-orbit theorem, giving a new and transparent proof of the norm estimates for the Hodge metric on a $\mathbb{Z}$-Variation of Hodge Structures on a punctured disk, as well as a new approach to its several variables generalisation due to Cattani-Kaplan-Schmid. Bruno Klingler's talk gave an approach to the definability over the field of algebraic numbers $\overline{\mathbb{Q}}$ of the Hodge loci of a $\mathbb{Z}$-VHS coming from geometry on a quasiprojective variety defined over $\overline{\mathbb{Q}}$ reducing it to the zero-dimensional case. Benjamin Bakker's talk surveyed a work in collaboration with Tsimermann and Brunebarbe establishing an extension to admissible Variations of Mixed Hodge structures of the quasiprojectivity of the image of the period mapping, the pure case being the recent solution of a long standing conjecture of Griffiths proved by the same authors and Klingler. The last lecture of our conference was given by Colleen Robles, who presented her joint work with Phillip Griffiths and Mark Green. Their results concern the fundamental question of understanding the behavior of period mappings *at infinity*, the guiding model being the classical Satake-Baily-Borel compactification.

Algebraic Foliations. Jorge Pereira's talk surveyed a recent work in collaboration with Lo Bianco, Rousseau and Touzet classifying birational symetries of transversly projective algebraic codimension one foliations. The fact that algebraic rank one foliations can also be used to prove transcendence result for numbers was highlighted in Jean-Benoît Bost's talk which gave a proof using the theory of theta invariants for infinite dimensional lattices of a geometric version of the classical theorem of Schneider-Lang claiming that if an orbit of a rank one foliation defined over some number field K has enough K-rational points then this orbit is closed.

K-stability. Chenyang Xu's talk surveyed the state of the art concerning the construction of projective moduli spaces for K-stable Fano varieties. Thibaut Delcroix's talk was devoted to the combinatorial description of K-stabilty for polarized spherical varieties.

Hyperbolicity. The work presented by Deng Ya proved the Picard hyperbolicity of the quasiprojective base of a $\mathbb{C}$-VHS with unipotent monodromy at infinity if the period mapping has discrete fibers. A very interesting class of varieties on which the Green-Griffiths-Lang conjecture could be tested, symmetric products of specific algebraic varieties, was presented in Frédéric Campana's talk which featured an hyperbolicity statement for low degree symmetric products of a generic projective surface.

Fano manifolds and klt singularities. Daniel Greb's talk featured a characterization of quotients of projective spaces by finite groups using a Chern number inequality reminiscent of the Miyaoka-Yau inequality. Braun Lukas' talk was devoted to his recent results on the finiteness of the regular local fundamental group of a klt singularity or of the regular fundamental group of a singular Fano varieties and on the finite generation of the Cox rings attached to them. Andreas Höring's talk was devoted to the bigness of the tangent bundle of manifolds and concluded with examples of Fano manifolds with non psef tangent bundles.

Other topics. The enumerative geometry of the moduli space of abelian differentials and its natural compactification was the topic of Martin Möller's talk. Christian Lehn's talk outlined a reduction of the singular Kähler case of Beauville's splitting theorem for compact Kähler manifolds with $c_1 = 0$ to a Bogomolov-Tian-Todorov theorem for the locally trivial deformations of these varieties, which should be the outcome of his current work in progress with Bakker and Guenancia.

Workshop (hybrid meeting): Komplexe Analysis - Algebraicity and Transcendence

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Abstracts

Transcendental numbers for complex analytic geometers

Jean-Benoît Bost

This talk presented a new approach to Diophantine geometry and transcendence proofs based on the recent development of some "infinite dimensional geometry of numbers," where one studies infinite dimensional avatars of Euclidean lattices and their invariants, notably those defined in terms of the associated θ-series.

This formalism is systematically developed in the monograph [2], where some applications to Diophantine geometry are given. In this talk, the focus was placed on transcendence proofs, and we sketched a proof of the theorem of Schneider-Lang, formally similar to the proof of classical algebraization results in complex analytic and formal geometry.

1. Consider for instance the theorem of Chow. It asserts that, if $i : X \hookrightarrow \mathbb{P}^N(\mathbb{C})$ is a $\mathbb{C}$-analytic immersion of a compact $\mathbb{C}$-analytic manifold X, then its image $i(X)$ is an algebraic subset of $\mathbb{P}^N(\mathbb{C})$. Using basic results from complex algebraic geometry, it easily follows from the following analytic statement (which, in a primitive form, already appears in the work of Poincaré on Abelian varieties):

Let X be a compact $\mathbb{C}$-analytic manifold of complex dimension n. For every complex analytic line bundle L over X, the space $\Gamma(X, L)$ of its $\mathbb{C}$-analytic sections is finite dimensional. Moreover, when $D \in \mathbb{N}$ goes to infinity, we have:

$$\dim_{\mathbb{C}} \Gamma(X, L^{\otimes D}) = O(D^n). \tag{1}$$

It turns out that classical results of transcendental number theory may be rephrased as "arithmetic algebraization theorems" asserting the algebraicity of geometric objects defined in terms of complex analytic geometry and of formal geometry over a number field. For instance the theorem of Schneider-Lang, in a generalized form, admits the following formulation.

Let us consider K a number field, embedded in $\mathbb{C}$, X a smooth quasi-projective variety over K, and $L \hookrightarrow T_{X/K}$ a sub-vector bundle of rank 1 of its tangent bundle.

By base field extension from K to $\mathbb{C}$ and analytification, we obtain a complex analytic manifold $X^{\mathrm{an}}_{\mathbb{C}}$ and an analytic sub-vector bundle $L^{\mathrm{an}}_{\mathbb{C}} \hookrightarrow T_{X^{\mathrm{an}}_{\mathbb{C}}}$. Since $L^{\mathrm{an}}_{\mathbb{C}}$ has rank 1, it is integrable and defines a $\mathbb{C}$-analytic foliation of $X^{\mathrm{an}}_{\mathbb{C}}$. Consider an analytic leaf $\mathcal{F}$ of this foliation and assume that, for some closed discrete subset Δ of $\mathbb{C}$, we are given an étale analytic map:

$$f : \mathbb{C} \setminus \Delta \longrightarrow \mathcal{F}.$$

The map f defines an analytic map from $\mathbb{C} \setminus \Delta$ into the quasi-projective complex variety $X^{\mathrm{an}}_{\mathbb{C}} \hookrightarrow \mathbb{P}^N(\mathbb{C})$. The map f is meromorphic on $\mathbb{C}$ when it extends to an analytic map $\tilde{f}$ from $\mathbb{C}$ to $\mathbb{P}^N(\mathbb{C})$. When this holds, the meromorphic map f is of finite order at infinity if there exists $\rho \in \mathbb{R}^*_+$ such that:

$$\int_{\mathbb{C}} \log^+ \frac{r}{|z|} f^*\omega = O(r^\rho) \quad \text{when } r \to +\infty, \tag{2}$$

where ω denotes some continuous positive $(1,1)$-form over $\mathbb{P}^N(\mathbb{C})$.

Using this notation, we may formulate the theorem of Schneider-Lang as the following algebraization result:

Theorem SL. *Let $K, X, \mathcal{F}, \Delta$ and f be as above. If the following conditions are satisfied:*

- (i) *f is meromorphic and satisfies the finite order condition* (2)*, and*
- (ii) *there exists a subset A of $\mathbb{C} \setminus \Delta$ such that $f(A) \subset X(K)$, whose cardinality $|A|$ satisfies $|A| > 2\rho[K : \mathbb{Q}]$,*

then $\mathcal{F}$ is algebraic.

Here the algebraicity of $\mathcal{F}$ precisely means that the Riemann surface $\mathcal{F}$, injectively immersed in $X^{\mathrm{an}}_{\mathbb{C}}$, is actually a (necessarily closed and smooth) complex algebraic curve in $X_{\mathbb{C}}$. It is equivalent to the algebraicity of the formal germ $\widehat{\mathcal{F}}_{f(z)}$ of $\mathcal{F}$ through $f(z)$, for any $z \in A$. These formal germ $\widehat{\mathcal{F}}_{f(z)} \hookrightarrow \widehat{X}_{\mathbb{C}, f(z)}$ are actually defined over the number field K, and when conditions (i) and (ii) hold, $\mathcal{F}$ is the set of complex points of some smooth closed K-curve in X.

2. Once they are expressed as arithmetic algebraization theorems like Theorem SL above, transcendence theorems turn out to follow from finiteness results, analogous to the estimates (1), where now the complex vector space of analytic sections $\Gamma(X, L)$ is replaced by some *infinite dimensional Euclidean lattice* $\Gamma(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ of sections of some metrized line bundle $\widetilde{\mathcal{L}}$ on some suitable "formal-analytic variety" $\widetilde{\mathcal{X}}$, and its complex dimension by its *theta-invariant* $h^0_\theta(\Gamma(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}))$.

Let us briefly explain the meaning of these notions, in a simplified framework.

Recall that a *Euclidean lattice* $\overline{E}$ is a pair $(E, \|.\|)$ consisting in a free $\mathbb{Z}$-module E of finite rank and a Euclidean norm $\|.\|$ on the $\mathbb{R}$-vector space $E_{\mathbb{R}} := E \otimes_{\mathbb{Z}} \mathbb{R}$. In Arakelov geometry, Euclidean lattices occur as the so-called Hermitian vector bundles over the scheme $\operatorname{Spec}\mathbb{Z}$ and, as such, appear as the analogues of vector bundles on a smooth projective curve C over some base field k. In this analogy, the role of the dimension

$$h^0(C, E) := \dim_k \Gamma(C, E) \tag{3}$$

of the k-vector space of global sections of E is played by the $\mathbb{R}_+$-valued invariant of Euclidean lattices:

$$h^0_\theta(\overline{E}) := \log \sum_{v \in E} e^{-\pi \|v\|^2}. \tag{4}$$

For every $\delta \in \mathbb{R}$, we may introduce the Euclidean lattice

$$\overline{E} \otimes \overline{\mathcal{O}}(\delta) := (E, e^{-\delta}\|.\|).$$

Then the theta-function $\theta_{\overline{E}}$ of $\overline{E}$, defined by

$$\theta_{\overline{E}}(t) := \sum_{v \in E} e^{-\pi t \|v\|^2} \quad \text{for every } t \in \mathbb{R}_+^*, \tag{5}$$

satisfies:

$$h^0_\theta(\overline{E} \otimes \overline{\mathcal{O}}(\delta)) = \log \theta_{\overline{E}}(e^{-2\delta}),$$

and the invariants $h^0_\theta(\overline{E})$ of Euclidean lattices are essentially special values of their theta-functions.

The analogy between the invariants (3) and (4) attached respectively to vector bundles on curves and to Euclidean lattices goes back to the German school of number theory (Hecke, F.K. Schmidt, Artin). More recently, it appears in Quillen's diary [4], and, in explicit relation with Arakelov geometry, in the work of van der Geer – Schoof [5] and Groenewegen [3]. The key role of special values of the theta series (5), and also of the measure

$$\gamma_{\overline{E}} := \sum_{v \in E} e^{-\pi \|v\|^2} \delta_v$$

over $E_{\mathbb{R}}$ and of its Fourier transform, when investigating the properties of an arbitrary Euclidean lattice $\overline{E}$ has been rediscovered, in a different context, by Banaszczyk [1], whose approach has been extremely influential in the developments of lattice based cryptography during the last two decades.

Let us emphasize that the invariant $h^0_\theta(\overline{E})$ satisfies formal properties remarkably similar to the ones of $h^0(C, E)$, although they are derived by very different methods, the investigation of h^0_θ often involving non-trivial arguments from harmonic analysis. For instance, as first observed by Quillen and Groenewegen, h^0_θ satisfies the following subadditivity property: *for every admissible short exact sequence of Euclidean lattices*[1]

$$0 \longrightarrow \overline{E} \longrightarrow \overline{F} \longrightarrow \overline{G} \to 0,$$

we have:

$$h^0_\theta(\overline{F}) \leq h^0_\theta(\overline{E}) + h^0_\theta(\overline{G}). \tag{6}$$

An *infinite dimensional Euclidean lattice* is defined as a pair $\overline{\widehat{E}} := (\widehat{E}, \|.\|)$ where $\widehat{E}$ is a topological $\mathbb{Z}$-module isomorphic to $\mathbb{Z}^{\mathbb{N}}$ (equipped with the topology product of the discrete topology on each factor $\mathbb{Z}$), and where $\|.\|$ is a lower continuous Euclidean quasi-norm on the Fréchet space

$$\widehat{E}_{\mathbb{R}} := \widehat{E} \hat{\otimes}_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{\mathbb{N}},$$

namely, a lower continuous map

$$\|.\| : \widehat{E}_{\mathbb{R}} \longrightarrow [0, +\infty]$$

[1] If $\overline{E}$, $\overline{E}$, and $\overline{E}$ are three Euclidean lattices, an admissible short exact sequence $0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0$ is a short exact sequence $0 \to E \to F \to G \to 0$ involving the underlying $\mathbb{Z}$-modules such that the Euclidean norms on $E_{\mathbb{R}}$ and $G_{\mathbb{R}}$ defining $\overline{E}$ and $\overline{G}$ are induced, by restriction and quotient, by the Euclidean norm $F_{\mathbb{R}}$ defining $\overline{F}$.

such that

$$E_{\mathbb{R}}^{\mathrm{Hilb}} := \{v \in E_{\mathbb{R}} \mid \|v\| < +\infty\}$$

is a subvector space of $E_{\mathbb{R}}$ and $\|.\|_{|E_{\mathbb{R}}^{\mathrm{Hilb}}}$ is a Euclidean norm on it. Then $(E_{\mathbb{R}}^{\mathrm{Hilb}}, |.\|)$ is a Hilbert space, continuously embedded in $\widehat{E}_{\mathbb{R}}$.

EXAMPLES: (a) To any $R \in \mathbb{R}_+^*$, we may attach the arithmetic Hardy space defined by $\widehat{E} := \mathbb{Z}[[X]]$ and the quasi-norm $\|.\|_{\mathbb{R}}$ on $\widehat{E}_{\mathbb{R}} \simeq \mathbb{R}[[X]]$ such that:

$$\|\sum_{n\in\mathbb{N}} a_n X^n\|_R^2 := \sum_{n\in\mathbb{N}} a_n^2 R^{2n}.$$

(b) Consider an admissible projective system of Euclidean lattices:

$$\overline{E}_\bullet : \overline{E}_0 \xleftarrow{q_0} \overline{E}_1 \xleftarrow{q_1} \cdots \xleftarrow{q_{i-1}} \overline{E}_i \xleftarrow{q_i} \overline{E}_{i+1} \xleftarrow{q_{i+1}} \cdots .$$

To $\overline{E}_\bullet$ is attached its projective limit $\varprojlim \overline{E}_\bullet$, defined as the infinite-dimensional Euclidean lattice $\widehat{\overline{E}} = (\widehat{E}, \|.\|)$ where $\widehat{E} := \varprojlim_i E_i$ and $\|(x_i)_{i\in\mathbb{N}}\| := \lim_{i\to+\infty} \|x_i\|_{\overline{E}_i}$ for every $(x_i)_{i\in\mathbb{N}} \in E_{\mathbb{R}} \simeq \varprojlim_i E_{i,\mathbb{R}} \left(\hookrightarrow \prod_{i\in\mathbb{N}} E_{i,\mathbb{R}}\right)$.

This construction of projective limits actually produces all infinite dimensional Euclidean lattices $\widehat{\overline{E}}$ such that $E_{\mathbb{R}}^{\mathrm{Hilb}}$ is dense in $\widehat{E}_{\mathbb{R}}$, and from now on we consider only those.

It turns out that there is a non-trivial formalism of theta-invariants attached to such objects. This theory is not formal: with the notation of Example (b), the existence and the value of the limit $\lim_{i\to+\infty} h_\theta^0(\overline{E}_i)$ indeed depends on the projective system used to realize $\widehat{\overline{E}}$ as a projective limit $\operatorname{proj}\lim \overline{E}_\bullet$.

It is actually possible to introduce a nice class of infinite dimensional Euclidean lattices — the θ-finite ones — that is suitable for the applications to transcendence theory, and for which these difficulties can be remedied.

With the notation of Example (b), we may introduce the summability condition:

$$\mathbf{Sum}(\overline{E}_\bullet) : \quad \sum_{i\in\mathbb{N}} h_\theta^0(\overline{\ker q_i}) < +\infty.$$

When it holds, the subbaditivity (6) shows that the limit $\lim_{i\to+\infty} h_\theta^0(\overline{E}_i)$ exists in $\mathbb{R}_+$. An infinite dimensional Euclidean lattice $\widehat{\overline{E}}$ is said to be *θ-finite* when, for every $\delta \in \mathbb{R}$, there exists a projective system $\overline{E}_{\delta\bullet}$ as in Example (b) such that $\widehat{\overline{E}} \simeq \varprojlim \overline{E}_{\delta\bullet}$ and $\mathbf{Sum}(\overline{E}_{\delta\bullet} \otimes \overline{\mathcal{O}}(\delta))$ holds. Then, if $\delta' < \delta$, the limit

$$h_\theta^0(\widehat{\overline{E}} \otimes \overline{\mathcal{O}}(\delta')) := \lim_{i\to+\infty} h_\theta^0(\overline{E}_{\delta,i} \otimes \overline{\mathcal{O}}(\delta'))$$

is a well-defined invariant of $\widehat{\overline{E}}$ and satisfies good formal properties (e.g. the subadditivity (6)). This is established by measure theoretic arguments on the Polish space $\widehat{E}$, which involve the family of measures $\gamma_{\overline{E}_i}$ on its successive quotients E_i.

References

[1] W. Banaszczyk. *New bounds in some transference theorems in the geometry of numbers.* Math. Ann., 296(4):625–635, 1993.

[2] J.-B. Bost. *Theta invariants of euclidean lattices and infinite-dimensional hermitian vector over arithmetic curves.* Progress in Mathematics, Vol. 334, Birkhäuser, 2020

[3] R. P. Groenewegen. *An arithmetic analogue of Clifford's theorem.* J. Théor. Nombres Bordeaux, 13(1):143–156, 2001. 21st Journées Arithmétiques (Rome, 2001).

[4] D. Quillen. *Notebooks 1968–2003*, edited by G. Luke and G. Segal. Published online by the Clay Mathematics Institute. `http://www.claymath.org/publications/quillen-notebooks`.

[5] G. van der Geer and R. Schoof. *Effectivity of Arakelov divisors and the theta divisor of a number field.* Selecta Math. (N.S.), 6(4):377–398, 2000.

Projective flatness over klt spaces and characterisation of finite quotients of projective spaces

Daniel Greb

(joint work with Stefan Kebekus, Thomas Peternell)

Let X be a $\mathbb{Q}$-*Fano* n-*fold*; that is, let X be a normal, projective, n-dimensional variety with at worst klt singularities such that $-K_X$ is $\mathbb{Q}$-ample. If the tangent sheaf $\mathscr{T}_X$ is stable with respect to the anticanonical polarisation $-K_X$, then by [3] its first two $\mathbb{Q}$-Chern classes, which are well-defined for all spaces with klt singularities, satisfy the $\mathbb{Q}$-*Bogomolov-Gieseker Inequality*,

$$\frac{n-1}{2n} \cdot \widehat{c}_1\left(\Omega_X^{[1]}\right)^2 \cdot [-K_X]^{n-2} \leq \widehat{c}_2\left(\Omega_X^{[1]}\right) \cdot [-K_X]^{n-2}.$$

In analogy to the case of manifolds with ample canonical bundle and as a generalisation of the Chen-Oguie inequality for Kähler-Einstein Fano manifolds, one expects more, namely a $\mathbb{Q}$-*Miyaoka-Yau Inequality* of the form

$$\frac{n}{2(n+1)} \cdot \widehat{c}_1\left(\Omega_X^{[1]}\right)^2 \cdot [-K_X]^{n-2} \leq \widehat{c}_2\left(\Omega_X^{[1]}\right) \cdot [-K_X]^{n-2}. \tag{7}$$

As classical examples show, even in the smooth case (7) does not hold without imposing an additional (semistability) condition. Moreover, one observes that (7) looks like the Bogomolov-Gieseker inequality for a $-K_X$-slope-semistable sheaf of rank $n+1$.

In my talk I explained that the natural sheaf to consider in this problem is constructed as follows: The first Chern class of the $\mathbb{Q}$-Cartier divisor $-K_X$ defines a non-trivial, locally split extension of the form

$$0 \to \mathscr{O}_X \to \mathscr{E}_X \to \mathscr{T}_X \to 0,$$

which / whose middle term we call the *canonical extension.* For smooth Fano manifolds, this is just the Atiyah extension induced by the anticanonical line bundle. It has been studied in various problems of Kähler geometry, for example by Semmes, Tian, Donaldson and Greb–Wong.

It now follows as in the proof of the $\mathbb{Q}$-Bogomolov-Gieseker inequality above that if the canonical extension $\mathscr{E}_X$ of a $\mathbb{Q}$-Fano n-fold is slope-semistable with respect to the anticanonical polarisation $-K_X$, then (7) holds. The semistability required for this result can be shown in various classes of examples: It holds if X admits a Kähler-Einstein metric ([4], [1]), and more generally if X is smooth and K-semistable (Chi Li), but also for every smooth Fano threefold of Picard number one and every smooth Fano n-fold of Picard number one and index $n-1$ (the last two statements are proven in [2]).

Once an equality of the form (7) has been established, it is interesting to consider those varieties realising equality. A characterisation of these is given by the following result contained in [2], which applies not only to Fano varieties but more generally to those having nef anticanonical divisor.

Theorem 1. *Let X be a projective variety. Assume that X has at most Kawamata log terminal singularities and that its anti-canonical class $-K_X$ is nef. Then, the following statements are equivalent.*

(1) *There exists an ample Cartier divisor H on X such that the canonical extension $\mathscr{E}_X$ is semistable with respect to H and such that*

$$\frac{n}{2(n+1)} \cdot \widehat{c}_1\left(\Omega_X^{[1]}\right)^2 \cdot [H]^{n-2} = \widehat{c}_2\left(\Omega_X^{[1]}\right) \cdot [H]^{n-2}.$$

(2) *The variety X is a quotient of the projective space or of an Abelian variety by the action of a finite group of automorphisms that acts without fixed points in codimension one.*

Coming back to the Fano case, the previous theorem characterises quotients of projective spaces by finite group actions that are free in codimension one. These varieties are simply-connected, and the regular part has fundamental group isomorphic to the given group. While every finite group admits a finite-dimensional complex representation V such that the induced action on $\mathbb{P}(V)$ is free in codimension one, it is also interesting to notice that *any* non-trivial representation of a non-abelian finite simple group automatically has this property, see again [2].

References

[1] S. Druel, H. Guenancia, M. Păun, *A decomposition theorem for $\mathbb{Q}$-Fano Kähler-Einstein varieties*, Preprint `https://arxiv.org/abs/2008.05352`, 2020.

[2] D. Greb, S. Kebekus, T. Peternell, *Projective flatness over klt spaces and uniformisation of varieties with nef anti-canonical divisor*, Preprint `https://arxiv.org/abs/2006.08769`, 2020.

[3] S. Lu, B. Taji, *A characterization of finite quotients of abelian varieties*, Int. Math. Res. Not. IMRN **2018**, no. 1, 292–319.

[4] G. Tian, *On stability of the tangent bundles of Fano varieties*, Internat. J. Math. **3** (1992), 401–413.

Chern classes of the moduli spaces of Abelian differentials

Martin Möller
(joint work with M. Costantini, J. Zachhuber)

Only few aspects of the topology of the moduli spaces of holomorphic or meromorphic Abelian differentials $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ with singularities of type $\mu = (m_1, \ldots, m_n)$ are currently known, such as the connected components [5], and partial information about (quotients of) the fundamental group. In this talk we explained a route taken in [2] to compute the Euler characteristic of these moduli spaces of Abelian differentials.

The moduli spaces of Abelian differentials can be thought of as relatives of the moduli space of curves $\mathcal{M}_{g,n}$, for which the Euler characteristic was computed in [4] using a cellular decomposition (given by the arc complex) and counting of cells. Our strategy here is quite different. While the Euler characteristic is an intrinsic quantity associated to $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$, our strategy heavily uses the compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ constructed in [1] and all its properties that make it quite similar to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$.

Theorem 1 ([1]). *There is a proper smooth Deligne-Mumford stack $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ that contains the projectivized stratum $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ as open dense substack with the following properties.*

(i) *The boundary $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \setminus \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is a normal crossing divisor.*
(ii) *The boundary strata are labeled by enhanced level graphs, dual graphs of stable curves with an additional level structure.*
(iii) *The codimension of a boundary stratum D_Γ in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is equal to the number of horizontal edges plus the number of levels below zero.*

It is well-known that if the k-dimensional orbifold B admits a compactification $\overline{B}$ with normal crossing boundary divisor D, then the Euler characteristic of B can be computed as integral

$$\chi(B) = (-1)^k \int_{\overline{B}} c_k(\Omega^1_B(\log D)) \tag{8}$$

Our strategy is based on the fact that the compactified spaces $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, much like projective space, admit an Euler sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow (\overline{\mathcal{H}}^1_{\mathrm{rel}})^\vee \otimes \mathcal{O}_{\overline{B}}(-1) \longrightarrow \mathcal{O}_{\overline{B}} \longrightarrow 0 \tag{9}$$

thanks to the flat structure given by period coordinates. Here $\overline{\mathcal{H}}^1_{\mathrm{rel}}$ is the Deligne extension of the local system of relative cohomology and $\mathcal{K}$ is a vector bundle that restricts to the cotangent bundle over $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ and whose difference to the cotangent bundle at the boundary can be determined.

To state the result, let K_Γ be the product of the enhancements associated with the edges of a level graph $\Gamma \in \mathrm{LG}_L$, where L denotes the number of levels below zero, and let $N_\Gamma^\top$ be the dimension of the moduli space of Abelian differentials (possibly on disconnected curves) at the top level of such a level graph.

Theorem 2. *The orbifold Euler characteristic of the moduli space $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is the dimension-weighted sum over all level graphs $\Gamma \in \mathrm{LG}_L(\overline{B})$ without horizontal nodes*

$$\chi(\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)) = (-1)^d \sum_{L=0}^{d} \sum_{\Gamma\in \mathrm{LG}_L(\overline{B})} \frac{K_\Gamma \cdot N_\Gamma^\top}{|\mathrm{Aut}(\Gamma)|} \cdot \prod_{i=0}^{-L} \int_{B_\Gamma^{[i]}} \xi_{B_\Gamma^{[i]}}^{d_\Gamma^{[i]}} \tag{10}$$

of the product of the top power of the first Chern class $\xi_{B_\Gamma^{[i]}}$ of the tautological bundle at each level, where $d_\Gamma^{[i]} = \dim(B_\Gamma^{[i]})$ and $d = \dim(\overline{B}) = N - 1$.

This theorem, as well as all the computations in the tautological ring of the moduli spaces $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ necessary to evaluate the expressions, have been implemented in a sage package `diffstrata`, with the algorithms documented in [3].

References

[1] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, M. Möller, *The moduli space of multi-scale differentials*, preprint, arXiv:1910.13492 (2019).

[2] M. Costantini, M. Möller, J. Zachhuber *The Chern classes and the Euler characteristic of the moduli spaces of Abelian differentials*, preprint, arXiv:2006.12803 (2020).

[3] M. Costantini, M. Möller, J. Zachhuber *diffstrata – a sage package for calculations in the tautological ring of the moduli space of Abelian differentials*, preprint, arXiv:2006.12815 (2020).

[4] J. Harer, D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math **85** (1986), 457–485.

[5] M. Kontsevich, A. Zorich, *Comnnected components of the moduli space of Abelian differentials with prescribed singularities*, Invent. Math **153** (2003), 631–678.

Quasiprojectivity of images of mixed period maps

Benjamin Bakker

(joint work with Yohan Brunebarbe, Jacob Tsimerman)

Let X be an algebraic space, $\mathcal{M}$ a moduli space of polarized integral pure Hodge structures, and $\phi : X \to \mathcal{M}$ the period map corresponding to a polarized integral variation of pure Hodge structures. It was conjectured by Griffiths [4] and proven by the authors [2] that the closure of the image of ϕ is a quasiprojective algebraic variety. The aim of this talk is to extend this result to the mixed setting. The proof makes heavy use of o-minimality, particularly the o-minimal GAGA theorem of [2] and the more recent work with B. Klingler [1] associating a $\mathbb{R}_{\mathrm{an,exp}}$-definable structure to mixed period domains and admissible mixed period maps.

More precisely, let $\mathcal{M}$ now be a moduli space of graded-polarized integral mixed Hodge structures. There is a period space $\mathcal{D}$ parametrizing the associated graded objects of the points in $\mathcal{M}$ with a map $\mathcal{M} \to \mathcal{D}$, and to any period map $\phi : X \to \mathcal{M}$ corresponding to a variation of graded-polarized integral mixed Hodge structures there is a period map $\mathrm{gr}\,\phi : X \to \mathcal{D}$ for the associated graded variation. By the above results we have a factorization

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{gr}\,\phi} & \mathcal{D} \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle \epsilon} & \\ & Z & \end{array}$$

where g is algebraic and dominant (meaning $\mathcal{O}_Z \to g_*\mathcal{O}_X$ is injective), ϵ is a closed immersion, and Z is quasiprojective (polarized by the Griffiths bundle).

In this talk I will discuss the following result:

Theorem 1. *Let X be a separated algebraic space of finite type over $\mathbb{C}$ and $\phi : X \to \mathcal{M}$ the period map associated to an admissible variation of (graded-polarized integral) mixed Hodge structures. Then there is a factorization*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \mathcal{M}. \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle \iota} & \\ & Y & \end{array}$$

where f is dominant algebraic and ι is a closed immersion. Moreover, the natural theta bundle on Y is algebraic and relatively ample over the image Z of the period map of the associated graded. In particular, Y is quasiprojective.

The period space $\mathcal{M}$ is naturally a quotient $\Gamma\backslash M$ of a graded-polarized integral mixed period domain M by an arithmetic group Γ, but the same result for the quotient $\Gamma_{\mathrm{mon}}\backslash M$ by the image of the monodromy representation easily follows from the theorem.

As a sample application we obtain as a corollary the following:

Corollary 2. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{C}$ admitting a quasi-finite admissible mixed period map. Then the coarse moduli space of $\mathcal{X}$ is quasi-projective.*

The factorization statement in the main theorem follows easily from the results of [1] and [2], and the main content is the relative ampleness of the theta bundle. This is especially interesting compared to the corresponding result in the pure case as the positivity does not stem from the negative curvature of $\mathcal{M}$; indeed, the fibers of $\mathcal{M} \to \mathcal{D}$ are flat.

Loosely speaking, the theta bundle Θ arises from the fact that the extension data of adjacent-weight graded pieces of an integral mixed Hodge structure are parametrized by an intermediate Jacobian on which a graded polarization naturally induces a line bundle. There are two main difficulties in establishing the relative ampleness of Θ. First, we must show Θ is algebraic. This follows for X smooth by work of Brosnan–Pearlstein [3] and in general by definable GAGA [2]. In fact, we also give a simplified proof of the result of Brosnan–Pearlstein.

Second, the theta bundle only accounts for the compact parts of the extension data, and the rest of the argument is devoted to showing that the remaining extension data is affine. More precisely, there are period maps for which $Y \to Z$ has positive-dimensional fibers but for which all of the "one-step" extensions are locally constant on the fibers, and in this case the theorem requires $\mathcal{O}_Y$ to be relatively ample—i.e., that Y is quasiaffine over Z. This ultimately relies on the geometry of mixed period spaces parametrizing "two-step" extensions and our argument critically uses the formalism of Hodge modules due to Saito.

References

[1] B. Bakker, B. Brunebarbe, B. Klingler, J.Tsimerman, *Definability of mixed period maps*, arXiv:2006.12403.

[2] B. Bakker, B. Brunebarbe, J.Tsimerman, *o-minimal GAGA and a conjecture of Griffiths*, arXiv:1811.12230.

[3] P. Brosnan, G. .Pearlstein, *Jumps in Archimedean height*, Duke Math. J. **168** (2019), 1737–1842.

[4] P. Griffiths, *Periods of Integrals on Algebraic Manifolds: Summary of main results and discussion of open problems*, Bull. Amer. Math. Soc. **75** (1970), 228–296.

A singular BTT theorem for K-trivial varieties and applications

Christian Lehn

(joint work with Benjamin Bakker, Henri Guenancia)

Deformations of compact Kähler manifolds with trivial canonical bundle are unobstructed by the famous theorem of Bogomolov–Tian–Todorov [5, 16, 17], a result which has greatly improved our understanding of these varieties. This in particular applies to Calabi-Yau manifolds and irreducible holomorphic symplectic manifolds. For singular varieties, an example of Gross shows [8] that deformations of Calabi-Yau varieties with canonical singularities can be obstructed. For singular symplectic varieties on the other hand, Namikawa has proven unobstructedness with varying hypotheses (see [12] and references therein) so that for a long time it looked like there was a dichotomy between symplectic and Calabi-Yau varieties. This is a preliminary report about our research which suggests that this dichotomy disappears once we restrict to locally trivial deformations.

The goal of our research project is to prove an unobstructedness result for numerically K-trivial varieties of Fujiki class $\mathscr{C}$. Let us recall that a variety is called numerically K-trivial if the canonical sheaf is ($\mathbb{Q}$-Cartier and) numerically trivial. So far, we can show

Theorem 1. *Let X be a compact variety of Fujiki class $\mathscr{C}$ with log terminal singularities. Suppose that one of the following holds:*

(1) *There is a holomorphic symplectic form $\sigma \in \Gamma(X^{reg}, \Omega^2_X)$.*
(2) *X has quotient singularities and numerically trivial canonical sheaf.*

Then locally trivial deformations of X are unobstructed.

The proof relies on the results of Kebekus–Schnell [11] and on Saito's theory of Hodge modules [14, 15]. Previously, we obtained unobstructedness for certain symplectic varieties in [2, 3].

As an application of unobstructedness of locally trivial deformations, we can prove a common generalization of the Beauville–Bogomolov decomposition theorem [1] and its singular variant by Druel–Greb–Guenancia–Höring–Kebekus–Peternell ([10] and references therein). We also refer to Campana's article recent article [4].

Theorem 2. *Let X be a compact Kähler variety with log terminal singularities and numerically trivial canonical sheaf. Suppose that locally trivial deformations of X are unobstructed. Then there is a finite quasi-étale cover $X' \to X$ such that X' splits as a product*

$$X' \cong T \times \prod_i C_i \times \prod_j S_j$$

where T is a complex torus, the C_i are Calabi–Yau varieties, and the S_j are irreducible symplectic varieties.

Apart from unobstructedness, the proof uses three main ingredients:

- A result about algebraic approximation by Graf–Schwald [9].
- A relative version of the decomposition of the tangent sheaf as in [7] for locally trivial families.
- Singular Kähler–Einstein metrics in the sense of Eyssidieux–Guedj–Zeriahi [6] and regularity results by Păun [13] in the Kähler case.

Given a locally trivial deformation $\mathcal{X} \to S$, we show that a finite base change $\mathcal{X}' = \mathcal{X} \times_S S'$ has a generalized Beauville–Bogomolov decomposition relative over a Zariski open set $U \subset S'$. This open set U is nonempty by algebraic approximation, and we use cycle spaces to obtain integrability of the foliation on X. The limit cycles are controlled using positivity arguments and singular Kähler–Einstein metrics.

References

[1] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983.

[2] Benjamin Bakker and Christian Lehn. A global Torelli theorem for singular symplectic varieties. *Preprint, arXiv:1612.07894*, 2016.

[3] Benjamin Bakker and Christian Lehn. The global moduli theory of symplectic varieties. *Preprint, arXiv:1812.09748*, 2018.

[4] Frederic Campana. The Bogomolov-Beauville-Yau Decomposition for Klt Projective Varieties with Trivial First Chern Class -Without Tears-, *Preprint, arXiv:2004.08261*, 2020.

[5] F. A. Bogomolov. Hamiltonian Kählerian manifolds. *Dokl. Akad. Nauk SSSR*, 243(5):1101–1104, 1978.

[6] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi. Singular Kähler-Einstein metrics. *J. Amer. Math. Soc.*, 22:607–639, 2009.

[7] Daniel Greb, Henri Guenancia, and Stefan Kebekus. Klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups. *Geom. Topol.*, 23:2051–2124, 2019.

[8] Mark Gross. The deformation space of Calabi-Yau n-folds with canonical singularities can be obstructed. In *Mirror symmetry, II*, volume 1 of *AMS/IP Stud. Adv. Math.*, pages 401–411. Amer. Math. Soc., Providence, RI, 1997.

[9] Patrick Graf and Martin Schwald. The kodaira problem for kähler spaces with vanishing first chern class, 2020.
[10] Andreas Höring and Thomas Peternell. Algebraic integrability of foliations with numerically trivial canonical bundle. *Invent. Math.*, 216(2):395–419, 2019.
[11] Stefan Kebekus and Christian Schnell. Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities. *ArXiv e-prints*, 2018.
[12] Yoshinori Namikawa. On deformations of $\mathbb{Q}$-factorial symplectic varieties. *J. Reine Angew. Math.*, 599:97–110, 2006.
[13] Mihai Păun. Regularity properties of the degenerate Monge-Ampère equations on compact Kähler manifolds. *Chin. Ann. Math., Ser. B*, 29(6):623–630, 2008.
[14] Morihiko Saito. Modules de Hodge polarisables. *Publ. Res. Inst. Math. Sci.*, 24(6):849–995 (1989), 1988.
[15] Morihiko Saito. Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.*, 26(2):221–333, 1990.
[16] Gang Tian. Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 629–646. World Sci. Publishing, Singapore, 1987.
[17] Andrey N. Todorov. The Weil-Petersson geometry of the moduli space of $\mathrm{SU}(n \geq 3)$ (Calabi-Yau) manifolds. I. *Comm. Math. Phys.*, 126(2):325–346, 1989.

Finiteness and klt singularities

Lukas Braun

There is a long-lasting interest in Kawamata log terminal (or *klt*) singularities and their global counterparts, weakly Fano pairs, since they behave very well with respect to many aspects of algebraic and complex geometry. Not only is the canonical ring of a projective klt variety known to be finitely generated and thus a vast part of the goals of the minimal model program are achieved [4], in the weakly Fano case much more is known: the Cox ring is finitely generated and thus for *any* divisor on such a variety a minimal model program can be carried out.

The triviality of the fundamental group of a Fano manifold may have been a first hint for a similar well-behaviour of these spaces also from the topological viewpoint. There are several proofs, relying e.g. on Atiyah's L^2-index theorem or rational connectedness. Triviality of the fundamental group of a weakly Fano variety was then proven by Takayama, notably with different methods, using the Γ-reduction or Shafarevich map and Nadel's vanishing theorem [21]. The *finiteness* of the fundamental group of the *smooth locus* emerged as kind of a folklore conjecture out of these works and was confirmed in dimension two and several special cases [25, 16, 17]. In the local setting, Kollár conjectured the finiteness of the so-called *local fundamental group* $\pi_1^{\mathrm{loc}}(X, x)$ - i.e. the fundamental group of a small punctured neighbourhood - for klt singularities [18, 19]. Progress in this direction was made by Xu, who proved both the local and the global conjecture for *étale* fundamental groups [23]. These results were applied and generalized to several settings not only over the field of complex numbers but also to characteristic p [14, 10, 2, 3].

Finally in [22], it was understood that (as in the étale case) one approach would be to prove both conjectures simultaneously by a local-to-global induction. One of the two necessary steps, the proof of finiteness of the local fundamental group of an $(n+1)$-dimensional klt singularity under the assumption that the smooth locus of n-dimensional weakly Fano pairs has a finite fundamental group, was obtained in [22].

The second step was proven by the author in [6]. One of the main insights is that the concept of the local fundamental group has to be replaced by the *regional fundamental group* $\pi_1^{\mathrm{reg}}(X,x)$, the fundamental group of the smooth locus of X intersected with a small euclidean ball around x. Then the second step means deducing finiteness of the fundamental group of the smooth locus of a weakly Fano pair of dimension n from finiteness of π_1^{reg} for klt singularities of dimension n. In fact, it is not clear if this statement could be obtained from finiteness of the *local* fundamental groups as well. Thus in the second part of [6], it was necessary to modify the proof of the first step from [22] appropriately in order to obtain the statement for the regional instead of the local fundamental group. In summary, we have the following theorem on fundamental groups [6].

Theorem 1. *Let X be an algebraic variety defined over the field of complex numbers $\mathbb{C}$.*

(1) *Let $x \in X$ be a klt singularity. Then the regional fundamental group $\pi_1^{\mathrm{reg}}(X,x)$ is finite.*
(2) *Let (X,Δ) be a weakly Fano pair. Then the fundamental group of the smooth locus $\pi_1(X_{\mathrm{reg}})$ is finite.*

Several consequences of these statements have been obtained so far. Combined with the uniqueness of the minimizer of the normalized volume function of a klt singularity obtained in [24], an effective bound on the order of the regional fundamental group in terms of the dimension and the normalized volume is found in [24, Cor. 1.4]. In the same direction, together with Filipazzi, Moraga, and Svaldi, the author proved the so-called *Jordan-property* for regional fundamental groups of klt singularities in [7], meaning that for any dimension n, there is a constant $c(n)$ depending only on n, such that the regional fundamental group of any klt singularity of dimension n has a normal abelian subgroup of rank at most n and order at most $c(n)$. This statement is deduced from the similar global one for the fundamental groups of the smooth loci of weakly Fano pairs, which in turn relies on the Jordan property of automorphism groups of weakly Fano pairs obtained in [20].

Other applications concern more general spaces with klt singularities, e.g. $\mathbb{Q}$-Fano Kähler-Einstein varieties [12], finite quotients of projective space [15], and compact Kähler spaces with vanishing first Chern class and algebraic singularities [11]. In upcoming work, we aim to explore and extend these applications on more general klt spaces in greater detail.

Here, we will concentrate on the interplay with the already mentioned property of having a finitely generated Cox ring - the *Mori Dream Space* property - and the divisor class group. In prior work [5], the author used the finiteness of étale fundamental groups together with the Mori Dream Space property of klt *quasicone* singularities to prove finiteness of the iteration of Cox rings for weakly Fano pairs and klt quasicones. It was shown in [9, 13], that the Cox ring of a weakly Fano pair is not only finitely generated, but in addition its spectrum has klt singularities. Since it is also a quasicone and Gorenstein (when considering the whole divisor class group - possibly including torsion - in the Cox construction), it even has canonical singularites [5]. The only thing that may fail when the divisor class group has torsion is factoriality of the Cox ring. So if this happens, it is natural to consider the Cox ring of $\widehat{X} := \mathrm{SpecCox}(X)$, the spectrum of the Cox ring of X. The point here is that the statement from [4] concerning finiteness of the Cox ring can be directly transferred to the klt quasicone case, i.e. the Cox ring of $\widehat{X}$ will be finitely generated as well, and, moreover, its spectrum is a Gorenstein canonical quasicone [5]. Iterating this construction leads to the so-called *iteration of Cox rings*, introduced in [1] for log-terminal singularities with a torus-action of complexity one. By explicitly computing the occurring Cox rings, it was shown in [1] that in the above special case, the iteration of Cox rings is finite and ends with a factorial *master Cox ring*.

Then in [5], finiteness of the iteration of Cox rings for weakly Fano pairs and klt quasicones was proven by constructing a sequence of *finite* abelian covers corresponding to the torsion in the respective class groups and thus reducing the problem to finiteness of the étale fundamental groups [23, 14]. Lifting the abelian group actions gives a quotient by a reductive solvable group $\overline{X} \to X$, where $\overline{X}$ is the spectrum of the master Cox ring.

It is thus natural to ask if this can be generalized to arbitrary klt singularities. In upcoming work with Joaquín Moraga [8], we aim to complete the picture by first reasonably defining Cox rings, suitable finiteness and other notions for Cox rings of local rings of klt singularities, leading to the concept of *grocal Cox rings*. These graded rings, finitely generated over the degree zero part, which may be either a local ring or the ground field, comprise the Cox rings of all objects we encountered so far. Building on these definitions, we show finiteness of the iteration of Cox rings for klt singularities, with a factorial canonical master Cox ring as in the quasicone and weakly Fano case.

The existence of these two kinds of *covers* - the *universal quasi-étale cover* corresponding to the fundamental group of the smooth locus; and the spectrum of the master Cox ring corresponding to the iterated divisor class groups - the first one simple from the viewpoint of the fundamental group, the other one simple from the viewpoint of the divisor class group - brings us to the question if both can be combined. This is the second task [8] is concerned with. We construct a *unified cover* - essentially the spectrum of the Cox ring of the universal quasi-étale cover - that has both a simply connected regular locus *and* trivial class group. Moreover, any combination of finite quasi-étale covers and Cox constructions will finally end with this unified cover. The simplicity of this covering space will hopefully enable us to prove statements about X by computations on the cover, which will be the object of future investigations. As a first example class, in [8] we aim to determine explicitly the unified covers of all klt singularities with a torus action of complexity one.

References

[1] Ivan Arzhantsev, Lukas Braun, Jürgen Hausen, and Milena Wrobel, *Log terminal singularities, platonic tuples and iteration of Cox rings*, European Journal of Mathematics (1) **4** (2018), 242–312.

[2] Bhargav Bhatt, Javier Carvajal-Rojas, Patrick Graf, Karl Schwede, and Kevin Tucker, *Étale fundamental groups of strongly F-regular schemes*, Int. Math. Res. Not. IMRN **14** (2019), 4325–4339.

[3] Bhargav Bhatt, Ofer Gabber, and Martin Olsson, *Finiteness of étale fundamental groups by reduction modulo p*, preprint (2017), arXiv:1705.07303.

[4] Caucher Birkar, Paolo Cascini, Christopher Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. (2) **23** (2010), 405–468.

[5] Lukas Braun, *Gorensteinness and iteration of Cox rings for Fano type varieties*, preprint (2019), arXiv:1903.07996.

[6] Lukas Braun, *The local fundamental group of a Kawamata log terminal singularity is finite*, preprint (2020), arXiv:2004.00522.

[7] Lukas Braun, Stefano Filipazzi, Joaquín Moraga, and Roberto Svaldi, *The Jordan property for local fundamental groups*, preprint (2020), arXiv:2006.01253.

[8] Lukas Braun and Joaquín Moraga, *Grocal Cox Rings*, in preparation.

[9] Morgan Brown, *Singularities of Cox rings of Fano varieties*, J. Math. Pures Appl. (6) **99** (2013), 655–667.

[10] Javier Carvajal-Rojas, Karl Schwede, and Kevin Tucker, *Fundamental groups of F-regular singularities via F-signature*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), 993–1016.

[11] Benoît Claudon, Patrick Graf, Henri Guenancia, and Philipp Naumann, *Kähler spaces with zero first Chern class: Bochner principle, fundamental groups, and the Kodaira problem*, preprint (2020), arXiv:2008.13008.

[12] Stéphane Druel, Henri Guenancia, and Mihai Păun, *A decomposition theorem for $\mathbb{Q}$-Fano Kähler-Einstein varieties*, preprint (2020), arXiv:2008.05352.

[13] Yoshinori Gongyo, Shinnosuke Okawa, Akiyoshi Sannai, and Shunsuke Takagi, *Characterization of varieties of Fano type via singularities of Cox rings*, J. Algebraic Geom. (1) **24** (2015), 159–182.

[14] Daniel Greb, Stefan Kebekus, and Thomas Peternell, *Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties*, Duke Math. J. (10) **165** (2016), 1965–2004.

[15] Daniel Greb, Stefan Kebekus, and Thomas Peternell, *Projective flatness over klt spaces and uniformisation of varieties with nef anti-canonical divisor*, preprint (2020), arXiv:2006.08769.

[16] Rajendra V. Gurjar and De-Qi Zhang, *π_1 of smooth points of a log del Pezzo surface is finite. I*, J. Math. Sci. Univ. Tokyo **1** (1994), 137–180.

[17] Rajendra V. Gurjar and De-Qi Zhang, *π_1 of smooth points of a log del Pezzo surface is finite. II*, J. Math. Sci. Univ. Tokyo **2** (1995), 165–196.

[18] János Kollár, *New examples of terminal and log canonical singularities*, preprint (2011), arXiv:1107.2864.

[19] János Kollár, *Singularities of the minimal model program*, With a collaboration of Sándor Kovács, Cambridge Tracts in Mathematics **200**, Cambridge University Press, Cambridge, 2013.

[20] Yuri Prokhorov and Constantin Shramov, *Jordan property for groups of birational selfmaps*, Compos. Math. (12) **150** (2014), 2054–2072.

[21] Shigeharu Takayama, *Simple connectedness of weak Fano varieties*, J. Algebraic Geom. (2) **9** (2000), 403–407.

[22] Zhiyu Tian and Chenyang Xu, *Finiteness of fundamental groups*, Compos. Math. (2) **153** (2017), 257–273.

[23] Chenyang Xu, *Finiteness of algebraic fundamental groups*, Compos. Math. (3) **150** (2014), 409–414.
[24] Chenyang Xu and Ziquan Zhuang, *Uniqueness of the minimizer of the normalized volume function*, preprint (2020), arXiv:2005.08303.
[25] De-Qi Zhang, *The fundamental group of the smooth part of a log Fano variety*, Osaka J. Math. **32** (1995), 637–644.

Big Picard theorem for varieties admitting variation of Hodge structures

Ya Deng

In 1972, Armand Borel [3] and Kobayashi-Ochiai [8] proved generalized big Picard theorem for any hermitian locally symmetric variety X: any holomorphic map from the punctured disk to X extends to a holomorphic map of the disk into any projective compactification of X. In particular, they proved that any analytic map from a quasi-projective variety to X is algebraic. Their theorem motivated Ariyan Javenpeykar to propose the notion of "Borel hyperbolicity" for quasi-projective varieties in [7].

Definition 1. *A complex quasi-projective variety Y is* Borel hyperbolic *if any holomorphic map from another quasi-projective variety X to Y is an algebraic morphism.*

Inspired by the above-mentioned generalized big Picard theorem of Borel and Kobayashi-Ochiai, we proposed the following notion of hyperbolicity in [5].

Definition 2. *A complex quasi-projective variety Y is* Picard hyperbolic *if any holomorphic map from the punctured unit disk to Y extends to a holomorphic map from the unit disk to $\overline{Y}$, where $\overline{Y}$ is some (thus any) projective compactification of Y.*

It has been proved in [7] that a complex quasi-projective variety Y is Borel hyperbolic if it is Picard hyperbolic.

Period domains, introduced by Griffiths in 1969, are classifying spaces for Hodge structures. They are transcendental generalizations of hermitian locally symmetric varieties. In the workshop "Oberwolfach Komplex analysis 2017", Ariyan Javenpeykar proposed the following conjecture.

Conjecture 3. *Let Y be a quasi-projective manifold, which admits a polarized variation of Hodge structures, whose period map is quasi-finite. Then Y is Borel hyperbolic.*

He further conjectured that the moduli spaces of polarized manifolds with semi-ample canonical sheaf is Borel hyperbolic.

Conjecture 3 was first investigated by Bakker, Brunebarbe and Tsimerman in their work [1] on the Griffiths conjecture. They proved Conjecture 3 when the monodromy group of the variation of Hodge structures is arithmetic. Their work is based on delicate results in o-minimality. In particular, they have to use the very recent deep theorem by Bakker, Klingler and Tsimerman [2] on the definability of period maps.

In [5], we proved the following generalized big Picard theorem for varieties admitting complex variation of Hodge structures. In particular, we proved Conjecture 3 completely.

Theorem 4. *Let Y be a quasi-projective manifold, which admits a polarized variation of Hodge structures, whose period map is quasi-finite. Then Y is Picard hyperbolic.*

We use purely complex analytic methods to prove Theorem 4. Based on Nevanlinna theory, we first establish some criterion for Picard hyperbolic in [6].

Lemma 5. *Let X be a quasi-projective manifold. Assume that $\gamma : \Delta^* \to X$. If there is a Finsler metric h for $T_{\overline{X}}(-\log D)$ so that $dd^c \log |\gamma'(t)|_h^2 \geq \gamma^* \omega$ where ω is a Kähler form on $\overline{X}$. Then γ extends to $\Delta \to \overline{X}$.*

Such criterion was first applied in [6] to prove the Picard hyperbolicity of moduli spaces of polarized manifolds with semi-ample canonical sheaf. Let us mention that, prior to that, in [4] we proved the Brody hyperbolicity of these moduli spaces, which is indeed a conjecture by Viehweg and Zuo [9].

In [5], we construct some special Higgs bundles over the variety Y in Theorem 4. We then apply these Higgs bundles to construct the desired negatively curved Finsler metric as Lemma 5 so that we can apply the above criterion to prove the Picard hyperbolicity of Y.

Let us mention that our techniques unifies Picard hyperbolicity of hermitian locally symmetric varieties X by Borel and Kobayashi-Ochiai.

References

[1] B. Bakker, Y. Brunebarbe, and J. Tsimerman. *o-minimal GAGA and a conjecture of Griffiths*, *arXiv e-prints* (2018) arXiv:1811.12230.

[2] B. Bakker, B. Klingler, and J. Tsimerman. *Tame topology of arithmetic quotients and algebraicity of Hodge loci*, *arXiv e-prints* (2018) arXiv:1810.04801.

[3] A. Borel. *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, *J. Differential Geometry* **6** (1972), 543–560.

[4] Y. Deng. *On the hyperbolicity of base spaces for maximal variational families of smooth projective varieties.* To appear in J. Eur. Math. Soc. (JEMS).

[5] Y. Deng. *Big Picard theorem and algebraic hyperbolicity for varieties admitting a variation of Hodge structures*, *arXiv e-prints* (2020) arXiv:2001.04426.

[6] Y. Deng, S. Lu, R. Sun, and K. Zuo. *Picard theorems for moduli spaces of polarized varieties*, *arXiv e-prints* (2019) arXiv:1911.02973.

[7] A. Javanpeykar and R. A. Kucharczyk. *Algebraicity of analytic maps to a hyperbolic variety*, *arXiv e-prints* (2018) arXiv:1806.09338.

[8] S. Kobayashi and T. Ochiai. *Satake compactification and the great Picard theorem*, J. Math. Soc. Japan **23** (1971), 340–350.

[9] E. Viehweg and K. Zuo. *On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds*, Duke Math. J. **118 (1)** (2003), 103–150.

Degenerating variations of Hodge structure, revisited

Christian Schnell

My talk was about Schmid's foundational paper [1]. The local results in Schmid's paper concern polarized variations of Hodge structure (VHS) defined on a punctured disk. Given such a VHS, let V denote the complex vector space of multivalued flat sections of the underlying flat vector bundle, and $T \in \mathrm{GL}(V)$ the monodromy operator; by the monodromy theorem, the eigenvalues of T have absolute value 1. Schmid showed the existence of a "limiting mixed Hodge structure" on V, whose weight filtration $W_\bullet$ is uniquely determined by N, the logarithm of the unipotent part of T. Schmid also proved that the weight filtration accurately reflects the asymptotic behavior of the Hodge metric: if $v \in V$ is a multivalued flat section, then $v \in W_\ell \setminus W_{\ell-1}$ if and only if the pointwise Hodge norm of v grows like $L(t)^{\ell/2}$, where $L(t) = -\log|t|^2$. This gives a very nice conceptual explanation for the appearance of the weight filtration and of the limiting mixed Hodge structure; the only problem is that the Hodge norm estimates come at the end of a long sequence of other results, so this explanation is somewhat "after the fact". (To be exact, Schmid only proves these results when the VHS has an underlying real structure, and when all eigenvalues of T are roots of unity.)

In my talk, I presented a simple direct proof for the Hodge norm estimates. It is inspired by Simpson's and Mochizuki's work on harmonic bundles, but ends up being much easier. An intermediate result, interesting in its own right, is the following comparison theorem: Let E_1 and E_2 be polarized variations of Hodge structure on the punctured disk; if the two underlying flat vector bundles are isomorphic, then the Hodge metrics h_1 and h_2 are mutually bounded, up to a constant, as $t \to 0$.

The main reason for revisiting Schmid's work is a joint project with Claude Sabbah – called the "MHM Project" – in which we are trying to generalize Morihiko Saito's theory of Hodge modules to the case of complex Hodge modules (without rational structure). For that, we need all the foundational results in the theory of degenerating variations of Hodge structure without assuming the existence of a rational structure.

References

[1] W. Schmid, *Variation of Hodge Structure: The Singularities of the Period Mapping*, Inventiones Math. **22** (1973), 211–319.

Symmetric powers of complex projective manifolds: specialness and hyperbolicity issues

Frédéric Campana
(joint work with B. Cadorel, E. Rousseau)

Let X be a connected compact complex manifold of dimension n. Let $m > 0$ be an integer. The symmetric group S_m on m letters operates by permutation of the factors of X^m, the product of m copies of X. Let $q_{X,m} = q : X^m \to \mathrm{Sym}^m(X) = X_m := X^m/S_m$ be the quotient map, so that X_m is a normal connected compact complex space. This construction is functorial, any holomorphic map $f : X \to Y$ inducing a holomorphic $f_m : X_m \to Y_m$. The fibre of f_m over $q_{Y,m}(y_1, \dots, y_m)$ is isomorphic to the product of the fibres of f over the $y_i's$ when these are all distinct. The ramification in X^m of $q : X^m \to X_m$ consists of the $(x_1, \dots, x_m)$ such that $x_i = x_j$ for some $i \neq j$, and so has codimension n.

Aim: Compare the birational and hyperbolicity properties of X and X'_m, for any X and m. These are interesting test cases for various conjectures related to those of S. Lang.

When $n = 1$, $X_m, \forall m > 0$ is smooth. But this is no longer true when $n \geq 2$. For example: $(\mathbb{C}^2)_2 \cong \mathbb{C}^2 \times A_1$ by diagonalising the action of S_2. More generally (using the Reid-Tai criterion):

Theorem 1. *([Arapura-Archava 1998]) The singularities of X_m are canonical. Thus $\kappa(X'_m) = m \cdot \kappa(X)$ if $n > 1, \forall m$. Here X'_m is any smooth model of X_m, and so $J_m : X'_m \to (J_X)_m$ is the Iitaka fibration of X'_m if $J : X \to J_X$ is the Iitaka fibration of X with $\kappa(X) \geq 0$.*

This is false when $m = 1$: if X is a curve of genus g, then X'_m is of general type if $m < g$, is birational to $\mathrm{Jac}(X)$ if $m = g$, and to a $\mathbb{P}^{m-g}$-fibration if $m > g$. So $\kappa(X_m) = 1, 0, -\infty$ respectively in these 3 cases.

We next obtain similar statements for the other two fundamental fibrations of birational geometry: the MRC fibration $r : X \to R_X$, and the 'core map' $c : X \to C_X$, which may be seen as $(J \circ r)^n$ (in the orbifold category).

Recall that the MRC is the unique fibration on X with rationally connected (ie: RC) fibres and non uniruled base R_X.

Theorem 2. [1] *The MRC $r'_m : X'_m \to R_{X'_m}$ of X'_m is given by $r_m : X'_m \to (R_X)_m$, except when some finite étale cover of X fibres over a curve of genus $g \geq 1$ and $m \geq g$. In particular: X is RC if and only if some (or equivalently each) X_m is RC.*

The proof is an easy application of the Miyaoka-Mori uniruledness criterion, and of the canonicity of the singularities of X_m.

Remarks.

1. If X is rational, so is X'_m for each $m > 0$ (Mattuck 1969). This raises several questions about converse statements, such as:

2. If X_m is unirational (resp. rational, stably rational) for some (or any large) $m > 1$, is the same true for X? (The converse is obvious.)

3. *Test cases:* smooth cubics in $\mathbb{P}^4$, Artin-Mumford conic bundles, double covers of $\mathbb{P}^3$ ramified over a smooth sextic, conic bundles over $\mathbb{P}^2$ with discriminants smooth of large degrees. In the first two cases (resp. the last two cases), does X_m become rational (resp. unirational) for some (hence infinitely many) m's? Can the Brauer group of X'_m be estimated from the one of X?

Recall that the 'core map' $c_X : X \to C_X$ is the unique fibration on X with fibres 'special' and (orbifold) base of general type, and that X is 'special' if, for any $L \subset \Omega^p_X$ coherent of rank 1, we have: $\kappa(X, L) < p$. If X is RC, or if $\kappa(X) = 0$, X is special.

Theorem 3. [1] *The core map $c_{X'_m} : X'_m \to C_{X'_m}$ is $c_m : X'_m \to (C_X)_m$, except when X fibres over a hyperbolic orbifold curve, with special fibres, and m is (explicitely) sufficiently large. In particular, X is special if and only if X'_m is special for some $m > 0$ ('if' fails exactly in the previous exceptional case).*

The conjectural relationships between birational geometry and hyperbolicity are the following:

Conjectures:

1. (S. Lang) $\kappa(X) = n$ if and only if there exists $W \subsetneq X$, Zariski closed, such that any non-constant holomorphic map $h : \mathbb{C} \to X$ has image contained in W.

2. (F. Campana) X is special if and only if there exists and 'entire curve' $h : \mathbb{P}C \to X$ with dense image.

On the second conjectural statement, we have (by [1]) the following

Examples.

1. Let $X := F \times B$ be the product of two curves, with $g(F) \leq 1, g(B) \geq 2$. Then $\mathrm{Sym}^m(X)$ contains an entire cuve with dense image if and only if $m \geq g(B)$, i.e: if and only if $\mathrm{Sym}^m(X)$ is special. The main reason is that if $p : X \to B$ is the projection, $p_m : X_m \to B_m$ has generic fibres isomorphic to F^m.

2. If X is a $K3$ (hence special) surface with $\mathrm{Pic}(X) \cong \mathbb{Z}$ generated by an ample line bundle L of degree $(2g-2), g > 1$, then $\mathrm{Sym}^m(X)$ is $\mathbb{P}C^{2g}$-dominable (i.e: there exists a meromorphic $H : \mathbb{C}^{2g} \to \mathrm{Sym}^m(X)$ whose image contains a nonempty Zariski open subset). The proof rests on several important results and the relative Jacobian fibration $\mathrm{Sym}^m(X) \to \mathbb{P}^g$ associated to the curves in the linear system $|L|$ on X. This supports Conjecture 2 above in this case.

Concerning the first statement in the above conjecture, we have:

Theorem 4. [1] *Let $X \subset \mathbb{P}^{n+1}$ be a generic hypersurface of degree $d \geq (2n-1)^5 \cdot (2m^2+10n-1)$. Then X_m is hyperbolic.*

Similar statements hold for generic complete intersections of large and explicit multidegrees in $\mathbb{P}^{n+n'}, n' \geq n$, and for quotients of bounded symmetric domains. The proof of the preceding theorem rests on the consideration of the spaces of jet differentials on suitable orbifold structures on resolutions of X_m, and restricting their base loci, combined with previous work on jet differentials on generic hypersurfaces of projective spaces by Demailly, Brotbek, Darondeau, Ya Deng, Berczi-Kirwan. The technique, due to Riedl-Yang, of decreasing the dimension of these by base loci by taking hyperplane sections also enters crucially the proof in order to get the hyperbolicity statement.

Remark that although optimal, in certain sense, this statement does not imply the 'generic hyperbolicity' for all X_m claimed by Lang's conjecture for any hyperbolic X, such as the ones considered in the theorem. The hyperbolicity of X does not imply that of X_m for $m \geq g$. Indeed: if g is the genus of the normalisation of any irreducible curve $C \subset X$, since $C_m \subset X_m$ then contains dense entire curves, X_m is not hyperbolic. As m grows, more and more projective C_m appear on which the Kobayashi pseudodistance vanishes.

A direct consequence of Theorem 0.1 is that if for m generic points of X an irreducible curve $C \subset X$ exists containing them, then $g(C) > m$ if X is of general type. But:

Theorem 5. [1] *Assume Ω^1_X is ample, $m > 1$ given. There exists then a countable family of subvarieties $V_k \subsetneq X_m$ such that any non-hyperbolic curve $C \subset X_m$ is contained in some V_k. Moreover, $\mathrm{codim}_X(V_k) \geq (n-1), \forall k$.*

References

[1] B. Cadorel, F. Campana, E. Rousseau: *Hyperbolicity and Specialness of Symmetric Powers.* https://arxiv.org/abs/2007.07572, preprint (2020).

Fields of definitions of Hodge loci

Bruno Klingler
(joint work with Anna Otwinowska, David Urbanik)

The purpose of this talk was to explain the results on fields of definition of Hodge loci obtained in [8].

Let $(\mathbb{V}, \mathcal{V}, F^\bullet, \nabla)$ be a variation of $\mathbb{Z}$-Hodge structure ($\mathbb{Z}$VHS) on a smooth complex irreducible algebraic variety S. Recall this means the following: $\mathbb{V}$ is a finite rank $\mathbb{Z}_{S^{\mathrm{an}}}$-local system on the complex manifold S^{an} associated to S; and the holomorphic module with integrable connection $(\mathcal{V}^{\mathrm{an}} := \mathbb{V} \otimes_{\mathbb{Z}_{S^{\mathrm{an}}}} \mathcal{O}_{S^{\mathrm{an}}}, \nabla^{\mathrm{an}})$ on S^{an} associated to $\mathbb{V}$ by the Riemann-Hilbert correspondence is endowed with a decreasing filtration $(F^\bullet)^{\mathrm{an}}$ of $D_{S^{\mathrm{an}}}$-modules such that for each $s \in S^{\mathrm{an}}$ the filtration

$F_s^\bullet$ on $\mathbb{V}_s$ is the Hodge filtration of a pure Hodge structure. Following Deligne [5, Theor.5.9] there exists a unique *algebraic* module with regular integrable connection $(\mathcal{V}, \nabla)$ whose analytification is $(\mathcal{V}^{\mathrm{an}}, \nabla^{\mathrm{an}})$. We moreover assume $\mathbb{V}$, and all $\mathbb{Z}$VHSs in this paper, to be polarizable. In this case there exists a unique filtration $F^\bullet$ on the D_S-module $(\mathcal{V}, \nabla)$ whose analytification provides $(F^\bullet)^{\mathrm{an}}$, see [10, (4.13)].

A typical example of such a $\mathbb{Z}$VHS, referred to as "the geometric case", is $(\mathbb{V} := R^{2k} f_*^{\mathrm{an}} \mathbb{Z}(k), \mathcal{V} := R^{2k} f_* \Omega^\bullet_{X/S}, F^\bullet, \nabla)$ associated to a smooth projective morphism of smooth irreducible complex quasi-projective varieties $f : X \to S$. In this case the Hodge filtration $F^\bullet$ is induced by the stupid filtration on the algebraic De Rham complex $\Omega^\bullet_{X/S}$ and ∇ is the Gauß-Manin connection.

From now on we abbreviate the $\mathbb{Z}$VHS $(\mathbb{V}, \mathcal{V}, F^\bullet, \nabla)$ simply by $\mathbb{V}$. Let $\mathbb{V}^\otimes$ be the infinite direct sum of $\mathbb{Z}$VHS $\bigoplus_{a,b \in \mathbb{N}} \mathbb{V}^{\otimes a} \otimes (\mathbb{V}^\vee)^{\otimes b}$, where $\mathbb{V}^\vee$ denotes the $\mathbb{Z}$VHS dual to $\mathbb{V}$. The (tensorial) Hodge locus $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is the subset of points $s \in S^{\mathrm{an}}$ for which the Hodge structure $\mathbb{V}_s$ admits more Hodge tensors than the very general fiber $\mathbb{V}_{s'}$. Following Deligne [6, 7.5] this is a meager subset of S^{an}. While a priori $\mathrm{HL}(S, \mathbb{V}^\otimes)$ has no nice geometric feature, in the geometric case the Hodge conjecture easily implies that $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is a countable union of closed irreducible algebraic subvarieties of S, see [15]. Remarkably, Cattani, Deligne and Kaplan [3] (see also [2] for a simplified proof) proved unconditionnally that for any $\mathbb{Z}$VHS $\mathbb{V}$ on S the Hodge locus $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is a countable union of irreducible algebraic subvarieties of S, called the strict special subvarieties of S for $\mathbb{V}$ (or sometimes "the irreducible components of the Hodge locus $\mathrm{HL}(S, \mathbb{V}^\otimes)$ "). A special subvariety of dimension zero is called a special point. We refer to [14] for a survey on Hodge loci.

Let us now turn to fields of definitions of special subvarieties. In the geometric case, let us suppose that the morphism $f : X \to S$ is defined over a number field K. In that case the filtered algebraic D_S-module $(\mathcal{V}, F^\bullet, \nabla)$ is also defined over K. Again, the Hodge conjecture is easily seen to imply that each special subvariety Y of S for $\mathbb{V}^\otimes$ is defined over a finite extension of K and that each of the finitely many $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-conjugates of Y is a special subvariety of S for $\mathbb{V}$. In fact this follows from the weaker conjecture that Hodge classes are absolute Hodge, see [4, 3.5].

Let us say that a general $\mathbb{Z}$VHS $\mathbb{V}$ is *defined over a number field* $K \subset \mathbb{C}$ if S, $\mathcal{V}$, $F^\bullet$ and ∇ are defined over K: $S = S_K \otimes_K \mathbb{C}$, $\mathcal{V} = \mathcal{V}_K \otimes_K \mathbb{C}$, $F^\bullet \mathcal{V} = (F^\bullet_K \mathcal{V}_K) \otimes_K \mathbb{C}$ and $\nabla = \nabla_K \otimes_K \mathbb{C}$ with the obvious compatibilities. Following Simpson [11, "Standard conjecture" p.372], such a $\mathbb{Z}$VHS defined over a number field ought to be motivic: there should exist a $\overline{\mathbb{Q}}$-Zariski-open subset $U \subset S$ such that the restriction of $\mathbb{V}$ to U is a direct factor of a geometric $\mathbb{Z}$VHS on U. In particular Simpson's "standard conjecture" and the remark above concerning the geometric case implies:

Conjecture 1. *Any special subvariety associated to a $\mathbb{Z}$VHS defined over K is defined over a finite extension L of K and its finitely many* $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$*-conjugates are still special subvarieties of S for $\mathbb{V}$.*

For simplicity of notations, we will refer to this situation by saying that $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$, that special subvarieties are defined over $\overline{\mathbb{Q}}$ and that their Galois conjugates are special subvarieties.

Let us mention the few results in the direction of Conjecture 1 we are aware of:

(1) In [13, Theor. 0.6 (ii)] (see also [14, Theor. 7.8]) Voisin proves that if $f : X \to S$ is defined over $\mathcal{Q}$ and if the special subvariety $Y \subset S$ defined by a Hodge class $\alpha \in H^{2k}(X_0, \mathbb{Z}(k))$ satisfies that any locally constant Hodge substructure $L \subset H^{2k}(X_y, \mathbb{Z}(k))$, $y \in Y^{\mathrm{an}}$, is purely of type $(0,0)$ then Y is defined over $\overline{\mathbb{Q}}$ and its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{Q})$-translates are still special subvarieties of S for $\mathbb{V}^{\otimes}$.

(2) In [9] Saito and Schnell prove that for any $\mathbb{Z}$VHS defined over $\overline{\mathbb{Q}}$ a special subvariety is defined over $\overline{\mathbb{Q}}$ if (and only if) it contains a single $\overline{\mathbb{Q}}$-point of $S_{\overline{\mathbb{Q}}}$; this generalizes the well-known fact that the special subvarieties of Shimura varieties are defined over $\overline{\mathbb{Q}}$ (as any special subvariety of a Shimura variety contains a CM-point, and CM-points are defined over $\overline{\mathbb{Q}}$).

Notice that both Voisin's and Saito-Schnell's criteria are difficult to check in practice as one usually knows very little about the geometry of a special variety Y: in Voisin's case one would need to control the Hodge structure on the cohomology of a smooth compactification of $X_{|Y}$; in Saito-Schnell's case, there is no natural candidate for points over $\overline{\mathbb{Q}}$.

In [8] we provide a geometric criterion for a special subvariety of a $\mathbb{Z}$VHS $\mathbb{V}$ defined over $\overline{\mathbb{Q}}$ to satisfy Conjecture 1. Let us first recall the notion of algebraic monodromy group. Let S be a smooth irreducible complex algebraic variety and $\mathbb{V}$ a local system on S^{an}. Given an irreducible closed subvariety $Y \subset S$, a natural invariant attached to Y and $\mathbb{V}$ is the algebraic monodromy group $\mathbf{H}_Y$ of Y for $\mathbb{V}$: the connected component of the Tannaka group of the category $\langle \mathbb{V}_{|Y^{\mathrm{nor}}} \rangle^{\otimes}_{\mathbb{Q}\mathrm{Loc}}$ of local systems on (the normalisation of) Y tensorially generated by the restriction of $\mathbb{V}$ and its dual; equivalently the connected component of the Zariski-closure of the monodromy of the local system $\mathbb{V}_{|Y^{\mathrm{nor}}}$.

Definition 2. *Let S be a smooth irreducible complex algebraic variety and $\mathbb{V}$ a local system on S^{an}. Let $Y \subset S$ be an irreducible closed subvariety. We say that Y is* weakly non-factor *for $\mathbb{V}$ if it is not contained in a closed irreducible $Z \subset S$ such that $\mathbf{H}_Y$ is a strict normal subgroup of $\mathbf{H}_Z$.*

Our main result in this paper is the following:

Theorem 3. *Let $\mathbb{V}$ be a polarized variation of $\mathbb{Z}$-Hodge structure on a smooth quasi-projective variety S. Suppose that $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$. Then:*

(1) *any special subvariety of S for $\mathbb{V}$ which is weakly non-factor is defined over $\overline{\mathbb{Q}}$;*
(2) *its Galois-translates are special subvarieties of S for $\mathbb{V}$.*

As an explicit corollary we obtain:

Corollary 4. *Let $\mathbb{V}$ be a polarized variation of $\mathbb{Z}$-Hodge structure on a smooth quasi-projective variety S. Suppose that $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$ and that its adjoint generic Mumford-Tate group $\mathbf{G}_S^{\mathrm{ad}}$ is simple. Then any strict special subvariety $Y \subset S$ for $\mathbb{V}$, which is positive dimensional for $\mathbb{V}$ and maximal for these properties, is defined over $\overline{\mathbb{Q}}$, and its Galois-translates are special subvarieties of S for $\mathbb{V}$.*

Theorem 3 also enables to reduce the full Conjecture 1 to the case of points:

Corollary 5. *Special subvarieties for $\mathbb{Z}$VHSs defined over $\overline{\mathbb{Q}}$ are defined over $\overline{\mathbb{Q}}$ if and only if it holds true for special points. Similarly their Galois-translates are special subvarieties if and only if it holds true for special points.*

References

[1] Y. André, *Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part*, Compositio Math. **82** (1992) 1-24
[2] B. Bakker, B. Klingler, J. Tsimerman, *Tame topology of arithmetic quotients and algebraicity of Hodge loci*, `http://arxiv.org/abs/1810.04801`, to appear in J. Amer. Math. Soc.
[3] E. Cattani, P. Deligne, A. Kaplan, *On the locus of Hodge classes.* J. Amer. Math. Soc., **8** (1995), 483-506
[4] F. Charles, C. Schnell, *Notes on absolute Hodge classes* , in Hodge Theory, Math. Notes **49** (2014), 469-530
[5] P. Deligne, *Equations différentielles à points singuliers réguliers*, LNM **163** (1970)
[6] P. Deligne, *La conjecture de Weil pour les surfaces $K3$*, Invent. Math. **15** (1972), 206-226
[7] B. Klingler, Hodge theory and atypical intersections: conjectures, `http://arxiv.org/abs/1711.09387`, to appear in the book *Motives and complex multiplication*, Birkhaüser
[8] B. Klingler, A. Otwinowska, D. Urbanik, *On the fields of definition of Hodge loci*, in preparation
[9] M. Saito, C. Schnell, *Fields of definition of Hodge loci*, in Recent advances in Hodge theory, LMS **427** (2016) 275-291
[10] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211-319
[11] C. Simpson, *Transcendental aspects of the Riemann-Hilbert correspondence*, Illinois J. Math. **34**, no. 2, 368-391 (1990)
[12] E. Ullmo, *Equidistribution de sous-variétés spéciales. II*, J. Reine Angew. Math. **606** 193-216 (2007)
[13] C. Voisin, *Hodge loci and absolute Hodge classes*, Compositio Math. **143**, 945-958, (2007)
[14] C. Voisin, *Hodge loci*, in Handbook of moduli, Vol. III, 507-546, Adv. Lect. Math. 26, Int. Press, Somerville, MA, 2013
[15] A. Weil, *Abelian varieties and the Hodge ring*, Collected papers III, Springer Verlag, 421-429 (1979)

Rational endomorphisms of codimension one foliations

Jorge Vitório Pereira
(joint work with Federico Lo Bianco, Erwan Rousseau, Frédéric Touzet)

Singular holomorphic foliations (from now on, simply foliations) on projective surfaces invariant by birational maps were classified by Cantat and Favre in [2]. They explored the classification of birational transformations of surfaces to give a very precise description of the foliated surfaces with an infinite group of birational transformations. Foliations on projective surfaces invariant by non-invertible rational maps were classified in [3] by exploring the birational classification of foliations on projective surfaces according to their Kodaira dimension, by Brunella, McQuillan, and Mendes.

The analogue problem for foliations on higher dimensional manifolds is much less studied. Some activity on the subject was spurred by a question posed by Guedj in [4, page 103]: if h is a bimeromorphic map of a compact Kähler manifold which is not cohomogically hyperbolic, is it true that h preserves a foliation? A negative answer was provided by Bedford, Cantat, and Kim in [1], by exhibiting families of pseudo-automorphisms of rational 3-folds which are not cohomologically hyperbolic and do not preserve any foliation. Their strategy explores the explicit form of the pseudo-automorphisms they construct, in particular the existence of certain invariant surfaces, and uses the available knowledge about foliations on surfaces invariant by rational maps.

Transversely projective foliations. Together with Lo Bianco, Rousseau, and Touzet we started in [5] a study of codimension one foliations invariant by rational maps. Our focus is on the following question: given a projective manifold X and a dominant rational map $f : X \dashrightarrow X$ preserving a codimension one foliation $\mathcal{F}$, under which conditions some iterate of f preserves each leaf of $\mathcal{F}$? When this happens, we say that the transverse action of f is finite.

Our approach to this question was inspired by a conjecture of Cerveau and Lins Neto, which predicts that codimension one foliations on projective manifolds are either rational pull-backs of foliations on surfaces or admit a (singular) transversely projective structure. A confirmation of this conjecture would reduce the study of rational maps with infinite transverse action on codimension one foliations to the class of transversely projective foliations.

Theorem 1. *Let X be a projective manifold and let $\mathcal{F}$ be a transversely projective foliation of codimension one on X. If $f : X \dashrightarrow X$ is a rational endomorphism of $\mathcal{F}$ with infinite transverse action then $\mathcal{F}$ is virtually transversely additive.*

Roughly speaking, a transversely additive foliation is a foliation which is defined by a closed 1-form with coefficients in a finite algebraic extension of $\mathbb{C}(X)$.

Zariski dense dynamics. The statement of Theorem 1 gives little information about the nature of the rational endomorphism $f : X \dashrightarrow X$. In order to have a precise description of f, we first restrict to the class of purely transcendental foliations (i.e. through a general point of a general leaf there is no positive dimensional algebraic subvariety tangent to it) and to rational endomorphisms with Zariski dense orbits.

Theorem 2. *Let X be a projective manifold and let $\mathcal{F}$ be a transversely projective and purely transcendental foliation of codimension one on X. If $f \in \mathrm{End}(X, \mathcal{F})$ has Zariski dense orbits then f is conjugated to a Lattes-like map.*

A rational endomorphism $f : X \dashrightarrow X$ is a Lattes-like map if there exists an abelian algebraic group A, a cyclic finite group Γ acting on A, and a map $\varphi : A \to A$ which is the composition of a group endomorphisms with a translation such that f is birationally conjugated to the map induced by φ on the quotient A/Γ.

The proof of Theorem 2 relies on reduction of singularities for foliations defined by closed rational 1-forms and on the following result.

Theorem 3. *Let $f : X \dashrightarrow X$ be a rational map on a projective manifold X with a Zariski dense orbit and let D be a simple normal crossing divisor on X. If f^*D has support contained in the support of D then the quasi-albanese morphism $\mathrm{alb}_{(X,D)} : X - D \to \mathrm{Alb}(X, D)$ is a dominant rational map with irreducible general fiber.*

A conjecture. We do believe that the hypothesis on the transverse structure of $\mathcal{F}$ is not necessary, as predicted by Cerveau-Lins Neto conjecture.

Conjecture 4. *Let X be a projective manifold and let $\mathcal{F}$ be a codimension one foliation on X. If the transverse action of $f \in \mathrm{End}(X, \mathcal{F})$ is infinite then $\mathcal{F}$ is virtually transversely additive.*

We prove that in order to verify Conjecture 4, it suffices to consider foliations invariant by rational endomorphisms with Zariski dense orbits. In particular, Conjecture 4 holds true when the Zariski closure of the general orbit of f has dimension at most two.

We also show that the semi-group of rational endomorphisms preserving a foliation of (adjoint) general type with canonical singularities is finite. Extrapolating the picture drawn by the classification of foliations on surfaces according to their adjoint dimension [6], it seems reasonable to expect that purely transcendental codimension one foliations which are not of adjoint general type are transversely projective. If this expectation is confirmed, then our results would confirm Conjecture 4.

References

[1] Eric Bedford, Serge Cantat, and Kyounghee Kim. Pseudo-automorphisms with no invariant foliation. *J. Mod. Dyn.*, 8(2):221–250, 2014.

[2] Serge Cantat and Charles Favre. Symétries birationnelles des surfaces feuilletées. *J. Reine Angew. Math.*, 561:199–235, 2003.

[3] Charles Favre and Jorge Vitório Pereira. Foliations invariant by rational maps. *Math. Z.*, 268(3-4):753–770, 2011.

[4] Vincent Guedj. Propriétés ergodiques des applications rationnelles. In *Quelques aspects des systèmes dynamiques polynomiaux*, volume 30 of *Panor. Synthèses*, pages 97–202. Soc. Math. France, Paris, 2010.
[5] Federico Lo Bianco, Jorge Vitório Pereira, Erwan Rousseau, and Frédéric Touzet. Rational endomorphisms of codimension one holomorphic foliations *arXiv preprint arXiv:2007.12541*, 2020.
[6] Jorge Vitório Pereira and Roberto Svaldi. Effective algebraic integration in bounded genus. *Algebr. Geom.*, 6(4):454–485, 2019.

Fano manifolds such that the tangent bundle is (not) big

Andreas Höring

Let X be a Fano manifold, i.e. a complex projective manifold such that the anticanonical bundle $-K_X = \det T_X$ is ample. One of the most basic questions one can ask about Fanos is whether the positivity of $\det T_X$ implies any positivity properties of T_X. Obviously, one should not expect too much: by Mori's theorem we know that the projective space is the only Fano manifold such that T_X is ample, and, more generally, one expects that semipositivity of the tangent bundle (e.g. metric conditions, T_X is nef, etc.) should only appear for very special geometries. In this project (which is joint work with Jie Liu and Feng Shao) we consider a much weaker condition:

Definition. *Let X be a projective manifold. We say that the tangent bundle is pseudoeffective (resp. big) if the tautological class $c_1(\mathscr{O}_{\mathbb{P}(T_X)}(1))$ on the projectivised bundle $\mathbb{P}(T_X)$ is pseudoeffective (resp. big).*

Examples of manifolds with pseudoeffective tangent bundle are almost homogeneous spaces (or more generally any manifold admitting a non-zero vector field); by a result Hsiao [6] all toric varieties have big tangent bundle. A priori, it is not clear if the property of having a big tangent bundle is very restrictive. In fact big vector bundles can be very pathological: given an ample line bundle $A \to X$ on any projective manifold, the vector bundle $V = A \oplus A^*$ is big, but its determinant is trivial and the restriction to any curve is not nef. More generally the Chern class inequalities that are so useful when dealing with nef vector bundles will not hold for pseudoeffective vector bundles.

As a first step towards a theory of Fano manifolds with big/pseudoeffective tangent bundles, we characterise this property for some of the standard examples:

Theorem 1. *For $n \geq 2$, let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. Then T_X is pseudoeffective if and only if X is a hyperplane or a quadric.*

For the proof we use the theory of Schur functors to compute explicitly the space of global sections of certain twists of the symmetric powers of the tangent bundle. This involves vanishing theorems of Brückmann and Rackwitz [2], Schneider [12] and Bogomolov-de Oliveira [1]. Since these theorems already fail for complete intersections of higher codimension, settling the question for del Pezzo surfaces of degree 4 and 5 requires a different approach: every Fano manifold carries a family of minimal rational curves, i.e. a family of rational curves $f : \mathbb{P}^1 \to X$ that dominates X and such that for a general point $x \in X$, the curves passing through x form a complete family. It is well-known that a general minimal rational curve is standard, i.e. one has

$$f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^p \oplus \mathcal{O}_{\mathbb{P}^1}^{n-p-1}$$

for some $p \in \{0, 1, \ldots, n-1\}$. By a famous theorem of Cho-Miyaoka-Shepherd Barron [4] and Kebekus [9] one has $p \neq n-1$ unless X is the projective space, so we can consider the curves $\tilde{l} \subset \mathbb{P}(T_X)$ corresponding to the trivial quotients

$$f^*T_X \to \mathcal{O}_{\mathbb{P}^1}.$$

Let $\check{\mathcal{C}} \subset \mathbb{P}(T_X)$ be the closure of the locus covered by these curves, then we call $\check{\mathcal{C}}$ the total dual VMRT of the family of minimal rational curves. This terminology is justified by the fact that for $x \in X$ general the fibre $\check{\mathcal{C}}_x$ is the projective dual of the VMRT $\mathcal{C}_x$ that plays a prominent rôle in the theory of Hwang and Mok [7]. The total dual VMRT is a divisor in $\mathbb{P}(T_X)$ unless the VMRT is dual defective, so we can consider that in "many" cases this construction yields a divisor. It turns out that the class of this divisor can be computed in a number of cases, leading to the following result:

Theorem 2. *Let S be a smooth del Pezzo surface of degree $d := (-K_S)^2$. Then one has*

$$T_S \text{ is big if and only if } d \geq 5.$$

Let X be a 3-dimensional del Pezzo manifold, i.e. a smooth Fano threefold such that $-K_X = 2H$ where H is a Cartier divisor. Set $d := H^3$. Then one has

$$T_X \text{ is big if and only if } d \geq 5.$$

In the surface case, related results were obtained by Paris [11], Hosono-Iwai-Matsumura [5] and Mallory [10].

If X is a del Pezzo threefold of degree d, a general element of the linear system $|H|$ is a del Pezzo surface S of degree d. Thus the two parts of Theorem 2 taken together imply that

$$(*) \qquad T_X \text{ is big if and only if } T_S \text{ is big.}$$

This property is very surprising: the tangent bundle T_S is a *subbundle* of $T_X|_S$, so we can not expect any relation between their positivity properties. For example the del Pezzo threefold of degree five is almost homogeneous, so it has many vector fields. However the del Pezzo surface of degree five has no vector fields, hence the inclusion

$$H^0(S, T_S) \hookrightarrow H^0(S, T_X|_S) \neq 0$$

is the zero map. The proof of our theorem gives a hint why the property $(*)$ holds: if one computes the class of the total dual VMRT of a family of lines on a del Pezzo threefold X, the result depends on the number of (-1)-curves in S ! While the details of the proof are somewhat technical, the general strategy is quite classical: we follow the computation of the invariants of the Fano variety of lines of the cubic threefold that appears in the seminal work of Clemens and Griffiths [3].

References

[1] Fedor Bogomolov and Bruno De Oliveira. Symmetric tensors and geometry of $\mathbb{P}^N$ subvarieties. *Geom. Funct. Anal.*, 18(3):637–656, 2008. doi: 10.1007/s00039-008-0666-7.

[2] Peter Brückmann and Hans-Georg Rackwitz. T-symmetrical tensor forms on complete intersections. *Math. Ann.*, 288(4):627–635, 1990. doi: 10.1007/BF01444555.

[3] C. Herbert Clemens and Phillip A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.

[4] Koji Cho, Yoichi Miyaoka, and Nicholas I. Shepherd-Barron. Characterizations of projective space and applications to complex symplectic manifolds. In *Higher dimensional birational geometry (Kyoto, 1997)*, volume 35 of *Adv. Stud. Pure Math.*, pages 1–88. Math. Soc. Japan, Tokyo, 2002.

[5] Genki Hosono, Masataka Iwai, and Shin-Ichi Matsumura. On projective manifolds with pseudo-effective tangent bundle. *ArXiv preprint*, 1908.06421, 2019.

[6] Jen-Chieh Hsiao. A remark on bigness of the tangent bundle of a smooth projective variety and D-simplicity of its section rings. *J. Algebra Appl.*, 14(7):1550098, 10, 2015. doi: 10.1142/S021949881550098X.

[7] Jun-Muk Hwang. Geometry of minimal rational curves on Fano manifolds. In *School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000)*, volume 6 of *ICTP Lect. Notes*, pages 335–393. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[8] Masataka Iwai. Characterization of pseudo-effective vector bundles by singular hermitian metrics. *ArXiv preprint*, 1804.02146, 2018.

[9] Stefan Kebekus. Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron. In *Complex geometry (Göttingen, 2000)*, pages 147–155. Springer, Berlin, 2002.

[10] Devlin Mallory. Bigness of the tangent bundle of del Pezzo surfaces and D-simplicity. *arXiv preprint*, 2002.11010, 2020.

[11] Matthieu Paris. *Quelques aspects de la positivité du fibré tangent des variétés projectives complexes*. Theses, Université Joseph-Fourier - Grenoble I, December 2010.

[12] Michael Schneider. Symmetric differential forms as embedding obstructions and vanishing theorems. *J. Algebraic Geom.*, 1(2):175–181, 1992.

A condition of existence of cscK metrics on spherical manifolds

Thibaut Delcroix

Consider a smooth compact Kähler manifold X. Determining if there exists a cscK metric or not in a given Kähler class is a very hard geometric analysis problem involving a fourth order highly non-linear PDE. The aim of the Yau-Tian-Donaldson conjecture is to translate this geometric analysis problem into an algebro-geometric problem, involving a condition of K-stability inspired by GIT stability and the Kobayashi-Hitchin correspondence for Hermite-Einstein metrics. It is not apparent in general that the K-stability condition is easier to check than the construction of cscK metrics. The resolution of the Yau-Tian-Donaldson in the special case of the anticanonical class however allowed different authors to solve the question of existence of Kähler-Einstein metrics on large classes of manifolds that seemed out of reach from analytic techniques, and we had the pleasure to present such a result [2] at Oberwolfach in 2017.

In the case of general polarizations, arguably the best result in the direction of the YTD conjecture was obtained by Donaldson in a series of papers starting in 2002 [3] and ending in 2009 [4]. There he showed the YTD conjecture holds for toric surfaces. A toric manifold X is a n-dimensional complex manifold equipped with an effective $(\mathbb{C}^*)^n$ making it almost-homogeneous. It was later realized (see for example [5]) that thanks to big progress in the domain [1], Donaldson's work could actually be used to prove the *uniform* YTD conjecture for toric manifolds of arbitrary dimension. It should be remarked that Donaldson's result for surfaces is more precise and actually allows to determine whether a toric surface admits a cscK metric in simple enough cases, whereas the higher-dimensional uniform YTD result is not a priori as effective.

The work we presented at the workshop was initiated by a remark of Odaka to the effect that a very recent alternative proof of the uniform YTD conjecture for toric manifolds, obtained by Chi Li [6], applies as well to spherical manifolds. Spherical manifolds are a wide generalization of toric manifolds in the following sense. A complex manifold X equipped with an action of a connected reductive group G is spherical if any Borel subgroup of G acts on X with an open orbit. In order to apply this solution of the uniform YTD conjecture to determine explicitly when a spherical manifold admits a cscK metric, we first translate the condition of uniform K-stability of spherical varieties as a convex geometric problem, as was done by Donaldson for toric manifolds. By studying the type of convex geometric problem arising from this condition, we obtain a combinatorial sufficient condition of uniform K-stability which amounts to a finite number of conditions to be checked. This is not the only application we have in mind to the convex-geometric translation of uniform K-stability condition for spherical varieties. Our hope is that it can be used to upgrade the more precise YTD result of Donaldson for toric surfaces to spherical manifolds of dimension three and rank two (the rank of a spherical variety being the maximal dimension of a torus orbit).

References

[1] X. Chen and J. Cheng, *On the constant scalar curvature Kähler metrics, general automorphism group*, arXiv:1801.059070, 2018.
[2] T. Delcroix, *K-stability of Fano spherical varieties*, Ann. Sci. Éc. Norm. Supér. (4), 3(53):615–662, 2020.
[3] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom., 62(2):289–349, 2002.
[4] S. K. Donaldson, *Constant scalar curvature metrics on toric surfaces*, Geom. Funct. Anal., 19(1):83–136, 2009.
[5] T. Hisamoto, *Stability and coercivity for toric polarizations*, arXiv:1610.07998v3, 2020.
[6] C. Li, *Geodesic rays and stability in the cscK problem* arXiv:2001.01366v3, 2020.

Singularities and Syzygies of Secant Varieties of Smooth Projective Curves

Lawrence Ein
(joint work with Wenbo Niu, Jinhyung Park)

In this report, I would describe my joint work with Wenbo Niu and Jihgyung Park [7] on the singularities and syzygies of secant varieties of nonsingular projective curves. We study the secant varieties of a nonsingular projective curve embedded in a projective space by a sufficiently very ample line bundle. Throughout the paper, we work over an algebraically closed field k of characteristic zero. Let

$$C \subseteq \mathbb{P}(H^0(C, L)) = \mathbb{P}^r$$

be a nonsingular projective curve of genus $g \geq 0$ embedded by the complete linear system of a very ample line bundle L on C. For an integer $k \geq 0$, the *k-th secant variety*

$$\Sigma_k = \Sigma_k(C, L) \subseteq \mathbb{P}^r$$

to the curve C is defined to be the Zariski closure of the union of $(k+1)$-secant k-planes to C in $\mathbb{P}^r$. One has the natural inclusions

$$C = \Sigma_0 \subseteq \Sigma_1 \subseteq \cdots \subseteq \Sigma_{k-1} \subseteq \Sigma_k \subset \mathbb{P}^r.$$

If $\deg L \geq 2g + 2k + 1$, then

$$\dim \Sigma_k = 2k + 1 \quad \text{and} \quad \mathrm{Sing}(\Sigma_k) = \Sigma_{k-1}.$$

Note that Σ_{k-1} has co-dimension two in Σ_k. The geometric consequence of the condition $\deg L \geq 2g + 2k + 1$ is that any effective divisor on C of degree $k+1$ spans a k-plane in $\mathbb{P}^r$.

The strategy is the following. We would like to induct on k. One of the difficulties is that Σ_{k-1} is of co-dimension two in Σ_k. This leads us to study the singularities of the pair (Σ_k, Σ_{k-1}). From the classical work, we know Σ_k has a natural smooth model. $\beta_k : B^k(L) \to \Sigma_k$, where $B^k(L)$ is a $\mathbb{P}^k$- bundle over C_{k+1}, the $k+1$ folds symmetric product of C. We prove that $\beta_k^{-1}(\Sigma_{k-1} = Z_{k-1}$ is an irreducible divisor in $B^k(L)$ and the pair $(B^k(L), Z_{k-1})$ is a log-canonical pair. By the work of Bertram [2], the fiber of β_k over a point $p \in \Sigma_m - \Sigma_{m-1}$ is isomorphic to C_{k-m}. Furthermore, one can compute the co-normal sheaf of the fiber. As in [4], one can use the formal functions theorem to show that Σ_k is normal. More recently, Ullery has used a similar argument to show that Σ_1 is normal in [17]. Furthermore,the pair has property similar to a pair with rational singularities. Again using the formal functions theorem, we show that

$$R\beta_k(\mathcal{O}(-Z_{k-1})) = I_{\Sigma_{k-1}/\Sigma_k}.$$

By induction Σ_{k-1} has Du Bois singularities. It follows from a theorem of Kollár and Kovács that the condition implies that Σ_k also has Du Bois singularities.

There has been a great deal of work on the secant varieties in the last three decades. The major part of the research focused on local properties, defining equations, and syzygies. Recently, classical questions on secant varieties find interesting applications to moduli of stable rank 2 bundles on C, algebraic statistics and algebraic complexity theory. However, a lot of problems in this area are still widely open, and not much is known about general pictures. the embedding line bundle is the better geometry the secant variety would have. For the first secant variety of a curve, investigation has been conducted in a series of work by Vermeire [18], [19], [20] and the work with his collaborator Sidman [15], [16]. Among other things, the issue whether secant varieties are normal attracted special attention, as normality is critical in establishing many other important properties. Only for the first secant variety, the normality problem was settled by Ullery [17] fairly recently for a nonsingular projective variety of any dimension under suitable conditions on the embedding line bundle. Soon afterwards Chou and Song [3] further showed that the first secant variety has Du Bois singularities under the setting of Ullery's study.

On the other hand, the classical questions on the projective normality and the defining equations of secant varieties are the initial case of a more general picture involving higher syzygies, under the frame of Green's pioneering work ([9]). Keeping in mind that the curve can be viewed as its zeroth secant variety, the fundamental *Green's* $(2g+1+p)$*–theorem* (see [9]) asserts that if the embedding line bundle L has $\deg L \geq 2g+1+p$, then $C \subseteq \mathbb{P}^r$ is projectively normal and satisfies the property $N_{2,p}$, i.e., the curve is cut out by quadrics and the first p steps of its minimal graded free resolution are linear. This result sheds the lights on understanding the full picture of syzygies of arbitrary order secant varieties.

In this paper, we give a thorough study on singularities and syzygies of the k-th secant variety Σ_k of the curve C for arbitrary integer $k \geq 0$. The general philosophy guiding our research can be summarized as that singularities and syzygies interact each other in the way that the singularities of Σ_k determine its syzygies while the syzygies of Σ_{k-1} determine the singularities of Σ_k, and so on and so forth. It turns out that all the sufficient conditions that guarantee each basic property of secant varieties are satisfied if the embedding line bundle is positive enough beyond an effective bound. each basic properties of secant varieties, and show the interaction between them. It turns out all those conditions are satisfied if the line bundle is positive enough beyond an effective bound.

The first main result of the paper describes that the possible singularities of secant varieties are mild ones naturally appearing in birational geometry.

Theorem 1. ([7]) Let C be a nonsingular projective curve of genus g, and L be a line bundle on C. For an integer $k \geq 0$, suppose that $\deg L \geq 2g+2k+1$. Then $\Sigma_k = \Sigma_k(C,L)$ has normal Du Bois singularities. Furthermore, one has the following:

(1) $g = 0$ if and only if Σ_k is a Fano variety with log terminal singularities.
(2) $g = 1$ if and only if Σ_k is a Calabi–Yau variety with log canonical singularities but not log terminal singularities.

(3) $g \geq 2$ if and only if there is no boundary divisor Γ on Σ_k such that (Σ_k, Γ) is a log canonical pair.

The above theorem therefore completely solves the normality problems mentioned in ([17]), and answers Chou–Song's question ([3]) for curves.

The second main result gives a description of the syzygiies of the secant variety. It reveals one full picture hiding in the Green's $(2g+1+p)$–theorem aforementioned.

Theorem 2. ([7]) Let $C \subseteq \mathbb{P}(H^0(C, L)) = \mathbb{P}^r$ be a nonsingular projective curve of genus g embedded by the complete linear system of a very ample line bundle L on C. For integers $k, p \geq 0$, suppose that

$$\deg L \geq 2g + 2k + 1 + p.$$

Then one has the following:

(1) $\Sigma_k = \Sigma_k(C, L) \subseteq \mathbb{P}^r$ is arithmetically Cohen–Macaulay.
(2) $\Sigma_k \subseteq \mathbb{P}^r$ satisfies the property $N_{k+2,p}$.
(3) $\mathrm{reg}(\mathcal{O}_{\Sigma_k}) = 2k + 2$ unless $g = 0$, in which case $\mathrm{reg}(\mathcal{O}_{\Sigma_k}) = k + 1$.
(4) $h^0(\omega_{\Sigma_k}) = \dim K_{r-2k-1,2k+2}(\Sigma_k, \mathcal{O}_{\Sigma_k}(1)) = \binom{g+k}{k+1}$.

The results in the theorem were conjectured by Sidman–Vermeire ([15], [16]). The conjectures were quite wide open. For $g \leq 1$, the conjectures were settled by Graf von Bothmer–Hulek ([10]) and Fisher ([8]). By work of Vermeire ([18], [19] [20]), Sidman–Vermeire ([15],[16]) and Yang ([22]), the question about $N_{3,p}$ was finally settled for the first secant variety Σ_1.

Theorem 2 gives a complete picture for syzygies of arbitrary order secant varieties of curves. If $\deg L \geq 2g + 2k + 1$, then $\Sigma_k \subseteq \mathbb{P}^r$ is indeed projectively normal. If $\deg L \geq 2g + 2k + 2$, then Σ_k is ideal-theoretically cut out by the hypersurfaces of degree $k + 2$ Furthermore, if $\deg L \geq 2g + 2k + 1 + p$, then the first p steps of the minimal graded free resolution of Σ_k are linear.

References

[1] A. Aropdu and C. Voisin, Green-Lazarsfeld's conjecture for generic curves of large gonality, C. R. Math. Sci. Paris **336** (2003), 335-339.

[2] A. Bertram, Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective space, J. Differential Geom., 35(2) (1992), 429–469.

[3] C. Chou and L. Song, Singularities of secant varieties, Int. Math. Res. Not. IMRN, (9) (2018), 2844–2865.

[4] L. Ein, Normal sheaves of liner systems on curves, Contemporary Math., (1991), 116, 9-18.

[5] L. Ein and R. Lazarsfeld, Asymptotic syzygies of algebraic varieties, Invent. Math.,190, (2012), 603–646.

[6] L. Ein and R. Lazarsfeld, The gonality conjecture on syzygies of algebraic curves of large degree, Publ. Math. Inst. Hautes Études Sci., (2015) 122: 301–313.

[7] Lawrence Ein, Wenbo Niu and Jinhyung Park, Sinbularities and syzygies of secant varieites of nonsingular projective curves, to appear in Invent. Math.

[8] T. Fisher, The higher secant varieties of an elliptic normal curve, preprint, 2006.

[9] M. Green, Koszul cohomology and geometry of projective varieties, J. Diff. Geom. 19, (1984),125-171.

[10] Hans-Christian Graf von Bothmer and Klaus Hulek, Geometric syzygies of elliptic normal curves and their secant varieties, Manuscripta Math., 113 (1), (2004) 35–68.

[11] János Kollár. Singularities of the minimal model program, Cambridge Tracts in Mathematics, (2013).
[12] R. Lazarsfeld, Cohomology on symmetric products, syzygies of canonical curves, and a theorem of Kempf, in Einstein metrics and Yang-Mills connections (Sanda (1990), volume 145 of Lecture Notes in Pure and Appl. Math., (1993), 89–97. Dekker, New York.
[13] J. Rathmann. An effective bound for the gonality conjecture, preprint, arXiv:1604.06072.
[14] C. Soulé, Secant varieties and successive minima, J. Algebraic Geometry, 13: 323–341, 2004.
[15] J. Sidman and P. Vermeire, Syzygies of the secant variety of a curve, Algebra Number Theory, 3(4): 445–465, 2009.
[16] J. Sidman and P. Vermeire, Equations defining secant varieties: geometry and computation, in Combinatorial aspects of commutative algebra and algebraic geometry, volume 6 of Abel Symp., (2011), pages 155–174. Springer, Berlin.
[17] B. Ullery, On the normality of secant varieties, Adv. Math., 288, (2016), 631–642.
[18] P. Vermeire, Some results on secant varieties leading to a geometric flip construction, Compositio Math., 125(3), (2001), 263–282.
[19] P. Vermeire, On the regularity of powers of ideal sheaves, Compositio Math., 131(2): 161–172, 2002.
[20] P. Vermeire, Regularity and normality of the secant variety to a projective curve, J. Algebra, 319(3), (2008), 1264–1270.
[21] P. Vermeire, Equations and syzygies of the first secant variety to a smooth curve Proc. Amer. Math. Soc., 140(8), (2012), 2639–2646.
[22] R. Yang, A letter about syzygies of secant varieties, (2016).

An algebraic construction of K-moduli spaces of Fano varieties

Chenyang Xu

It has been once beyond algebraic geometers' imagination that there could be an intrinsic construction of moduli to parametrize Fano varieties, especially if one wants to find a compactification. One main difficulty is that the separatedness of the functor fails miserably. Nevertheless, in the past a few years, it has become clear that K-stability provides an excellent condition for constructing a well behaved moduli space of Fano varieties, called *the K-moduli of Fano varieties.*

K-stability was first invented by differential geometers to characterize whether a Fano variety admits a Kähler-Einstein metric. It has recently been well received that, by combining with the machinery of higher dimensional geometry, the purely algebraic study of K-stability of Fano varieties became a new field of algebraic geometry, and a number of deep problems has experienced a tremendous progress. One of the major problems is to construct the K-moduli.

While the connection with the existence of Kähler-Einstein metric provides a philosophic justification, technically it seemed bold at the beginning to expect K-stability could be a key ingredient in constructing moduli spaces of Fano varieties, as its definition is somewhat complicated and remote from any known approaches of constructing moduli.

The recently established valuative criterion of K-stability turns out to be quite useful for the construction of K-moduli. We first see how, combined with deep works from the minimal model program, this implies the existence of K-moduli stack, i.e. we know the moduli functor $\mathfrak{X}^{\rm Kss}_{n,V}$ of n-dimensional K-semistable $\mathbb{Q}$-Fano varieties of volume V, which sends $S \in \mathsf{Sch}_k$ to

$$\mathfrak{X}^{\rm Kss}_{n,V}(S) = \left\{ \begin{array}{c} \text{Flat proper morphisms } X \to S \text{, whose fibers are} \\ n\text{-dimensional K-semistable klt Fano varieties with} \\ \text{volume } V\text{, satisfying Kollár's condition} \end{array} \right\}$$

is represented by an Artin stack $\mathfrak{X}^{\rm Kss}_{n,V}$ of finite type.

This statement contains two parts. The boundedness is achieved by showing that the fixed volume and K-semistability assumption imply the local singularities have a more uniform numerical restriction than just being general klt singularities (see [6, 9]), and hence we can invoke the boundedness theorems of Fano varieties in birational geometry. Then we know K-semistable Fano varieties form an open locus in a family of Fano varieties, showed in [3, 7].

Then a more delicate but also more distinguished feature of K-stability is given by the following theorem proved in [4, 1]: The K-moduli stack $\mathfrak{X}^{\rm Kss}_{n,V}(S)$ admits a *separated* good moduli space

$$\phi\colon \mathfrak{X}^{\rm Kss}_{n,V} \to X^{\rm Kps}_{n,V},$$

whose closed points are in bijection with n-dimensional K-polystable $\mathbb{Q}$-Fano varieties of volume V, i.e., $X^{\rm Kps}_{n,V}$ is the desired K-moduli space.

The proof of the above used profound birational geometry construction, as well as a general abstract criterion on when a good moduli space of a finite type Artin stack exists, established in [2].

The major remaining challenge is the following conjecture:

Conjecture: The moduli good space $X^{\rm Kps}_{n,V}$ is *proper*.

Finally, by [5, 8], we also have a projectivity result: any proper subspace of $X^{\rm Kps}_{n,V}$ whose points parametrize reduced uniformly K-stable Fano varieties, is projective.

References

[1] J. Alper, H. Blum, D. Halpern-Leistner, C. Xu, *Reductivity of the automorphism group of K-polystable Fano varieties*, to appear in Invent. Math., arXiv:1906.03122 (2019).

[2] J. Alper, D. Halpern-Leistner, J. Heinloth, *Existence of moduli spaces for algebraic stacks*, arXiv:1812.01128 (2018).

[3] H. Blum, Y. Liu, C. Xu, *Openness of K-semistability for Fano varieties*, arXiv:1907.02408 (2019).

[4] H. Blum, C. Xu, *Uniqueness of K-polystable degenerations of Fano varieties*, Annals of Math. **190** (2019), Issue 2, 609-656.

[5] G. Codogni, Z. Patakfalvi, *Positivity of the CM line bundle for families of K-stable klt Fano varieties*, to appear in Invent. Math., arXiv:1806.07180 (2018).

[6] C. Jia, *Boundedness of $\mathbb{Q}$-Fano varieties with degrees and alpha-invariants bounded from below*, to appear in Ann. Sci. Éc. Norm. Supér., arXiv:1705.02740 (2017)

[7] C. Xu, *A minimizing valuation is quasi-monomial*, Annals of Math. **191** (2020), Issue 3, 1003-1030.

[8] C. Xu, Z. Zhuang, *On positivity of the CM line bundle on K-moduli spaces.* arXiv:1912.12961 (2019).
[9] C. Xu, Z. Zhuang, *Uniqueness of the minimizer of the normalized volume function.* arXiv:2005.08303 (2020).

Non-commutative deformations of perverse coherent sheaves

Yujiro Kawamata

The deformation theory becomes sometimes richer when we allow *associative algebras* instead of commutative algebras to be the parameter algebras of deformation families. There are more NC (not necessarily commutative) deformations than C (commutative) deformations. There are examples where the parameter algebras of C deformations are only finite dimensional while those of NC deformations are infinite dimensional. The most essential invariants are sometimes recovered only from NC deformations. For example, *Donovan-Wemyss conjecture* predicts that some 3-dimensional hypersurface singularities with small resolutions are recovered from the NC deformations of the exceptional curves of the resolutions.

The existence of the *versal NC deformations* $\hat{F}$ is proved in the same way as the versal C deformations, and the abelianization $\hat{R}^{ab}$ of the *NC deformation algebra* $\hat{R}$ (the parameter algebra of $\hat{F}$) is the C deformationn algebra. The main question of this talk is to determine $(\hat{R}, \hat{F})$ in some examples, and we will see that they are related to interesting geometric questions in some cases.

A *simple collection* is a direct sum of objects $F = \bigoplus_{i=i}^r F_i$ in an abelian category such that $End(F) \cong k^r$ for the base field $k = \mathbf{C}$. In this case, the versal NC deformation $\hat{F}$ is obtained by a succession of *universal extensions* (under some finiteness conditions). We set $F^0 = F$ and construct F^m inductively by exact sequences $0 \to Ext^1(F^m, F)^* \otimes_{k^r} F \to F^{m+1} \to F^m \to 0$. Then $\hat{F} = \varprojlim F^m$ is the versal NC deformation over the NC deformation algebra $\hat{R} = \varprojlim R_m$ for $R_m = End(F^m)$.

The versal NC deformation $\hat{F}$ over $\hat{R}$ can be described by using A^∞*-algebra* in the following way. The cohomology group $B = \bigoplus_{i=0}^\infty Ext^i(F, F)$ has an A^∞-algebra structure, and we have $\hat{R} = (\prod_{i=0}^\infty \bigotimes_{k^r}^i B_1^*)/(m^* B_2^*)$, where $m = \sum_{i=2}^\infty m_i : \bigoplus_{i=2}^\infty \bigotimes_{k^r}^i B_1 \to B_2$ is a map constructed from the higher multiplications. We have also a formula for $\hat{F}$. It is natural to ask about the convergence of the formal power series and about the globalization. It is also interesting to consider NC deformations of varieties.

We can define a category of *perverse coherent sheaves* when there is a tilting bundle P for a projective morphism $f : Y \to X$. It is the category of objects in the derived category $D^b(coh(Y))$ which correspond to the modules over the associative algebra $End(P)$ under *Bondal-Rickard equivalence*. It is more algebraic than the geometric category of coherent sheaves.

If $X = Spec(S)$ for a complete local ring S with residue field k, then there are only finitely many *simple objects* $\{s_i\}_{i \in I}$ in the category of perverse coherent sheaves. We can describe the versal NC deformation and the NC deformation algebra for any partial collection $\{s_j\}_{j\in J}$ $(J \subset I)$ of simple objects using the corresponding set of *indecomposable projective objects* $\{P_i\}_{i\in I}$ in the category of perverse coherent sheaves. We have $\hat{F} = cone(Hom(P_{J^c}, P) \otimes_{End(P_{J^c})} P_{J^c} \to P)$ for $P = \bigoplus_{i\in I} P_i$ and $P_{J^c} = \bigoplus_{i\notin J} P_i$, and $\hat{R} = End(P)/I$ for a two-sided ideal $I = (g \mid P \to P_{J^c} \to P)$.

We assume further that $f : Y \to X$ is a projective birational morphism from a smooth 3-fold whose exceptional locus C is an irreducible curve and such that $(K_Y, C) = 0$. Such flopping contractions are induced from *universal flopping contractions* $\tilde{f} : \tilde{Y}_l \to \tilde{X}_l$ of *length* l ($l = 1, 2, 3, 4, 5$ or 6) by base change morphisms $X \to \tilde{X}_l$. We can calculate NC deformation algebras of exceptional fibers of $\tilde{f} : \tilde{Y}_l \to \tilde{X}_l$. We also calculate NC deformation algebras of $f : Y \to X$ in the case of deformations of *Laufer's flops* ($l = 2$). For integers $n \ge 1$ and $0 \le i \le 2n$, X is defined by the following equation in k^4: $x^2 + y^3 + z^2w + yw^{2n+1} = 0$ for $i = 0$ (this is Laufer's flop), and $x^2 + y^3 + y^2(-w)^i + z^2w + yw^{2n+1} - (-w)^{i+2n+1} = 0$ for $1 \le i \le 2n$. Then $\hat{R} = k\langle\langle a, b\rangle\rangle/(ab + ba, a^2 + b^{2n+1})$ for $i = 0$, and $=$ $k\langle\langle a, b\rangle\rangle/(ab+ba, a^2+b^{2n+1}+b^{2i})$ for $1 \le i \le 2n$. We can confirm Donovan-Wemyss conjecture in this case. We note that the abelianizations $\hat{R}^{ab}$ for $n + 1 \le i \le 2n$ are isomorphic, but the non-commutative algebras $\hat{R}$ are not isomorphic.

References

[1] Kawamata, Yujiro. *On multi-pointed non-commutative deformations and Calabi-Yau threefolds.* Compositio Math. 154 (2018), 1815–1842. doi:10.1112/S0010437X18007248.

[2] Kawamata, Yujiro. *Non-commutative deformations of simple objects in a category of perverse coherent sheaves.* to appear in Selecta Math. DOI: 10.1007/s00029-020-00570-w.

[3] Kawamata, Yujiro. *On non-commutative formal deformations of coherent sheaves on an algebraic variety.* arXiv:1908.11483.

[4] Kawamata, Yujiro. *Non-commutative deformations of perverse coherent sheaves and rational curves.* arXiv:2006.09547

Functional Transcendence for Quotients of Bounded Symmetric Domains

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Finite-volume quotients of bounded symmetric domains Ω, which are naturally quasi-projective varieties, are objects of immense interest to Several Complex Variables, Algebraic Geometry, Arithmetic Geometry and Number Theory, and an important topic revolves around functional transcendence in relation to universal covering maps of such varieties (in analogy to the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$). While a lot has already been achieved in the case of Shimura varieties (such as the moduli space $\mathscr{A}_g$ of principally polarized Abelian varieties) by means of methods of Diophantine Geometry, Model Theory, Hodge Theory and Complex Differential Geometry, techniques for the general case of not necessarily arithmetic quotients $\Omega/\Gamma =: X_\Gamma$ have just begun to be developed. For instance, uniformization problems for subvarieties of products of arbitrary compact Riemann surfaces of genus ≥ 2 have hitherto been intractable by existing methods. We will explain a differential-geometric approach leading to characterization results for totally geodesic subvarieties of X_Γ for the universal covering map $\pi : \Omega \to X_\Gamma$. Especially, we will explain how uniformization theorems for bi-algebraic varieties can be proven by transcendental methods involving the Poincaré-Lelong equation (joint work with S.-T. Chan), generalizing earlier results of Ullmo-Yafaev in 2011 in the case of arithmetic quotients. More generally, we will consider the Zariski closures of images of algebraic sets under the universal covering map $\pi : \Omega \to X_\Gamma$. In the arithmetic case, Klingler-Ullmo-Yafaev (2016) has confirmed the hyperbolic Ax-Lindemann Conjecture (which is one of the two major components for the confirmation of the André-Oort Conjecture for Shimura varieties) ascertaining that such Zariski closures are weakly special (equivalently totally geodesic). I will explain how the arithmeticity condition can be dropped in the rank-1 case by a completely different proof using foliation theory, Chow schemes and Kähler geometry.

References

[1] S.-T. Chan; N. Mok: *Asymptotic total geodesy of local holomorphic curves exiting a bounded symmetric domain and applications to a uniformization problem for algebraic subsets, accepted for publication in* J. Differential Geom.

[2] N. Mok: *Zariski closures of images of algebraic subsets under the uniformization map on finite-volume quotients of the complex unit ball*, Compositio Math. **155** (2019), 2129–2149.

Completions of period mappings

Colleen Robles

(joint work with Mark Green, Phillip Griffiths)

Given the data $(\overline{B}, Z; \Phi)$ of a smooth projective variety $\overline{B}$ with reduced normal crossing divisor, and period map $\Phi : B = \overline{B} \backslash B \to \Gamma\backslash D$ with image $\wp = \Phi(B)$, we ask: What are natural completions $\Phi^{\mathrm{c}} : \overline{B} \to \overline{\wp}$ to they have? What properties do they have?

Prior results on completions. In the *classical case* that D is Hermitian and Γ is arithmetic (the situation arising when studying moduli of principally polarized abelian varieties and K3 surfaces), the quotient $\Gamma\backslash D$ admits the projective Satake–Baily–Borel (SBB) compactification $\overline{\Gamma\backslash D}^{\mathsf{S}}$ [2], and its toroidal normalizations $\overline{\Gamma\backslash D}^{\mathsf{T}}$ [1]. For either of these we may take $\overline{\wp}$ to be the closure of $\wp$ in $\overline{\Gamma\backslash D}$. And in the case of SBB we have an extension $\Phi^{\mathsf{S}} : \overline{B} \to \overline{\Gamma\backslash D}^{\mathsf{S}}$ [4]. (The existence of the extension $\Phi^{\mathsf{T}} : \overline{B} \to \overline{\Gamma\backslash D}^{\mathsf{T}}$ is a more subtle business [12, 7].)

We are primarily interested in the non-classical case. In general, $\Gamma\backslash D$ admits no algebraic structure [8, 16]; this means that one does not expect to have a reasonable candidate for $\overline{\Gamma\backslash D}$. Nonetheless, the image $\wp$ is quasi-projective, and the Hodge line bundle $\Lambda = \otimes\det(\mathcal{F}^p) \to B$ descends to an ample line bundle on $\wp$ [3]; so one might still look for a projective compactification $\overline{\wp}$ of the image $\wp$.

Here one expects a *relative* construction, depending on the triple $(\overline{B}, Z; \Phi)$. (One may think of $\overline{B}$ as taking the role played by $\overline{\Gamma\backslash D}$ in the classical case.)

In [14] a compact, Hausdorff *topological* generalization $\Phi^{\mathsf{S}} : \overline{S} \to \overline{\wp}^{\mathsf{S}}$ of SBB is defined. It is conjectured that $\overline{\wp}^{\mathsf{S}}$ admits the structure of an analytic variety. The conjecture is proved for $\dim B = 2$. And it is shown that the the conjecture implies the canonical extension $\Lambda_{\mathrm{e}} = \otimes \det(\mathcal{F}^p_{\mathrm{e}}) \to \overline{B}$ descends to an ample bundle on $\overline{\wp}$.

The perspective of this talk. In the course of pursuing the generalization (above) of SBB, it became clear that a better understanding of the *global* properties of the period map at infinity is needed. (We contrast this with the *local* properties of the period map at infinity, a topic of considerable interest, beginning with the works [19, 10], with significant applications including the Iitaka conjecture [21, 20, 17] and the arithmeticity of Hodge loci [9].) The divisor Z admits a stratification, and the nilpotent orbit theorem yields a variation of (equivalence classes of) limiting mixed Hodge structures (VLMHS) along the strata. Here "global" properties refers the properties of this variation along the compact fibres A of Φ^{S}. The level one extension data of the LMHS defines a map whose restriction to A takes value in an abelian variety that is polarized by theta line bundles. One of the main results of this work relates the pullback of those line bundles to the normal bundles of the irreducible components $Z_i \subset Z$. This relates the geometry *along* A to the geometry *normal* to $A \subset \overline{B}$. It is perhaps the *fundamental geometric property* of the period mapping at infinity.

A toroidal candidate. Let $B \xrightarrow{\hat{\Phi}} \hat{\wp} \to \wp$ be the Stein factorization of Φ. We prove: *The complex analytic variety $\hat{\wp}$ is Zariski open in a compact, complex analytic variety $\hat{\wp}^{\mathsf{T}}$, and the map $\hat{\Phi} : B \to \hat{\wp}$ admits a proper holomorphic completion* $\hat{\Phi}^{\mathsf{T}} : \overline{B} \to \hat{\wp}^{\mathsf{T}}$. We conjecture: *$\hat{\wp}^{\mathsf{T}}$ is a projective completion of $\hat{\wp}$.*

We also give a conjectural definition of $\overline{\wp}^{\mathsf{T}}$ itself. As a set $\overline{\wp}^{\mathsf{T}}$ parameterizes Γ–equivalence classes of limiting mixed Hodge structures, and as such encodes the maximal amount of Hodge theoretic information. (At the other extreme the set $\overline{\wp}^{\mathsf{S}}$ parameterizes Γ–equivalence classes of polarized Hodge structures on the associated graded. The map $\overline{\wp}^{\mathsf{T}} \to \overline{\wp}^{\mathsf{S}}$ quotients-out the extension data of the limiting mixed Hodge structures parameterized by $\overline{\wp}^{\mathsf{T}}$. It is in this sense that $\overline{\wp}^{\mathsf{S}}$ retains the minimal amount of meaningful Hodge theoretic information.)

Properties of line bundles. The canonical extension $\Lambda_{\mathrm{e}} \to \overline{B}$ is nef, [15, Proposition 7.15] (for $\Lambda = \Lambda_{\mathrm{e}}|_B$) and [14, Theorem 1.4.1]. Under a general local Torelli assumption there is an extensive body of literature establishing various other properties such as: the Hodge line bundle $\Lambda \to B$ is big if and only if Φ satisfies generic local Torelli [14]; and the pair $(\overline{B}, Z)$ is of log general type, $K_{\overline{B}} + [Z]$ is big [22]. There are also a number of results on the hyperbolicity of B, including [5, 6, 11].

Here we are predominately interested in establishing conditions under which natural line bundles on $\overline{B}$ are free and ample. We conjecture: *Under suitable local Torelli-type assumptions, there exist integers $0 \le a_i \in \mathbb{Z}$ and m_0 so that*

$m\Lambda_{\rm e} - \sum a_i[Z_i]$ *is ample for* $m \geq m_0$. (This would refine results of [3, 14].) The conjecture is proved in the special case that Z is irreducible. And in dimension two we have the stronger theorem: *Suppose that* $\dim B = 2$ *and assume that the differential of* $\Phi : B \to \Gamma\backslash D$ *is everywhere injective. Then there exists* $a_i \geq 0$ *so that the line bundle* $m\Lambda_{\rm e} - \sum a_i[Z_i]$ *is ample for* $m \gg 0$ (strengthening a result of [14]).

The conjecture above alludes to "suitable local Torelli-type assumptions". This is a condition on the triple $(\overline{B}, Z; \Phi)$. Specifically the Gauss–Manin connection induces a natural map $\Psi : T_{\overline{B}}(-\log Z) \to \mathrm{End}(\mathrm{Gr}^{\bullet}_{\mathcal{F}_{\rm e}})$; we say that the triple $(\overline{B}, Z; \Phi)$ satisfies the *local Torelli property* if the map is injective.

We conjecture: *generic local Torelli for* Φ *implies* $K_{\overline{B}} + [Z]$ *is nef and big, and local Torelli implies* $K_{\overline{B}} + [Z]$ *is free.* Under the stronger hypothesis of local Torelli for $(\overline{B}, Z; \Phi)$ we prove: *The line bundle* $K_{\overline{B}} + [Z]$ *is nef and big.* (The Base Point Free Theorem [18] then implies some multiple $m\,(K_{\overline{B}} + [Z])$ *is free.*) We also identify an additional condition (expressed in terms of the level one extension data map alluded to in our discussion of "perspective") under which $K_{\overline{B}} + [Z]$ *is ample.*

References

[1] A. Ash, D. Mumford, M. Rapoport, and Y. Tai. *Smooth compactification of locally symmetric varieties*. Math. Sci. Press, Brookline, Mass., 1975. Lie Groups: History, Frontiers and Applications, Vol. IV.

[2] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.

[3] Benjamin Bakker, Yohan Brunebarbe, and Jacob Tsimerman. o-minimal GAGA and Hodge Theory. arXiv:1811.12230, 2018.

[4] Armand Borel. Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem. *J. Differential Geometry*, 6:543–560, 1972. Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays.

[5] Damian Brotbek and Yohan Brunebarbe. Arakelov-Nevanlinna inequalities for variations of Hodge structures and applications. arXiv:2007.12957, 2020.

[6] Yohan Brunebarbe. Increasing hyperbolicity of varieties supporting a variation of hodge structures with level structures. arXiv:2007.12965, 2020.

[7] Sebastian Casalaina-Martin, Samuel Grushevsky, Klaus Hulek, and Radu Laza. Extending the Prym map to toroidal compactifications of the moduli space of abelian varieties. *J. Eur. Math. Soc. (JEMS)*, 19(3):659–723, 2017.

[8] James A. Carlson and Domingo Toledo. Compact quotients of non-classical domains are not Kähler. In *Hodge theory, complex geometry, and representation theory*, volume 608 of *Contemp. Math.*, pages 51–57. Amer. Math. Soc., Providence, RI, 2014.

[9] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995.

[10] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. Degeneration of Hodge structures. *Ann. of Math. (2)*, 123(3):457–535, 1986.

[11] Ya Deng, Steven Lu, Ruiran Sun, and Kang Zuo. Picard theorems for moduli spaces of polarized varieties. arXiv:1911.02973, 2019.

[12] Robert Friedman and Roy Smith. Degenerations of Prym varieties and intersections of three quadrics. *Invent. Math.*, 85(3):615–635, 1986.

[13] Wushi Goldring. The Griffiths bundle is generated by groups. *Math. Ann.*, 375(3-4):1283–1305, 2019.

[14] M. Green, P. Griffiths, R. Laza, and C. Robles. Period mappings and properties of the Hodge line bundle. arXiv:1708.09523, 2020.
[15] Phillip A. Griffiths. Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping. *Inst. Hautes Études Sci. Publ. Math.*, pages 125–180, 1970.
[16] Phillip Griffiths, Colleen Robles, and Domingo Toledo. Quotients of non-classical flag domains are not algebraic. *Algebr. Geom.*, 1(1):1–13, 2014.
[17] János Kollár. Subadditivity of the Kodaira dimension: fibers of general type. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 361–398. North-Holland, Amsterdam, 1987.
[18] János Kollár. Effective base point freeness. *Math. Ann.*, 296(4):595–605, 1993.
[19] Wilfried Schmid. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, 22:211–319, 1973.
[20] Eckart Viehweg. Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 329–353. North-Holland, Amsterdam, 1983.
[21] Eckart Viehweg. Weak positivity and the additivity of the Kodaira dimension. II. The local Torelli map. In *Classification of algebraic and analytic manifolds (Katata, 1982)*, volume 39 of *Progr. Math.*, pages 567–589. Birkhäuser Boston, Boston, MA, 1983.
[22] Kang Zuo. On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. *Asian J. Math.*, 4(1):279–301, 2000. Kodaira's issue.

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