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New Developments in Representation Theory of p-adic Groups

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Abstract. The representation theory of p-adic groups has played an important role in the Langlands program. It has seen significant progress in the past two decades, including various instances of the local Langlands correspondences, construction of supercuspidal representations and questions on periods and distinction. This workshop explored new ideas and further developments in this subject.

Mathematics Subject Classification (2010): 22E50.

Introduction by the Organizers

One of the most important objects of study in modern number theory are automorphic representations of a reductive group and their L-functions, as well as their relation to Galois representations, as encapsulated in the Langlands program. Automorphic representations decompose into their local components, which are representations of reductive groups over a local field (which may be $\mathbb{R}$, $\mathbb{C}$ or a nonarchimedean local field). As such, it is important to have a good understanding of the representations of such (p-adic) Lie groups. This subject has been actively studied for more than half a century, beginning with the work of Harish-Chandra, Gelfand, Mautner, Piatetski-Shapiro, Bernstein, Jacquet, Howe and many others. In recent years, there has been substantial progress on several fronts and also some unforeseen interactions with other fields. The aim of this workshop was to take stock of some of these new developments and to explore new ideas and directions for research.

More precisely, the workshop focused on the following 3 main themes:

(1) *Construction of Supercuspidal Representations and Types.*

Supercuspidal representations are the basic building blocks of the representation theory of a p-adic group G: all other irreducible representations can be built from them by the process of parabolic induction. As such, the construction and classification of these building blocks is a central problem in the subject.

Beginning with the work of Howe, there are two constructions of supercuspidal representations by the process of compact induction from open, compact (modulo center) subgroups. One is the construction of Bushnell–Kutzko for GL_n, which has been extended to the setting of classical groups by S. Stevens and to inner forms of GL_n by Sécherre–Stevens. The second is a construction of J.-K. Yu for general G, but only for tame data. The first construction has been shown to be exhaustive for GL_n and its inner forms and for $p \neq 2$ in the case of classical groups. The second construction has been shown to be exhaustive for p sufficiently large. Reconciling the two constructions was one of main themes of this workshop. Further, it was discovered by L. Spice that Yu's proof that his construction yields supercuspidal representations was relying on a misprinted (and therefore false) statement in a reference used. Though an alternative argument that proves that Yu's construction nevertheless yields supercuspidal representations was provided by J. Fintzen in the few months leading up to the workshop. The question of the uniqueness of the inducing data (i.e. unicity of types) in the construction of supercuspidal representations by compact induction is also a basic question.

Another basic question is the harmonic analysis of supercuspidal representations, i.e. the understanding of their Harish-Chandra characters. This typically requires a good understanding of structural aspects of regular semisimple or nilpotent conjugacy classes or conjugacy classes of elliptic tori. For example, a classical result of DeBacker on the domain of validity for the local character expansion of irreducible representations relied on his parametrization of nilpotent conjugacy classes in terms of data obtained from the Bruhat–Tits building.

In another direction, there has been ongoing work to provide geometric construction of supercuspidal representations analogous to the construction of Deligne–Lusztig for finite reductive groups. This strategy has seen substantial impetus and progress in the past two years.

The lectures of J. Fintzen, S. DeBacker, P. Latham, A. Mayeux, J.-J. Ma, C. Chan, and J.-L. Waldspurger discussed the above questions on the construction and understanding of supercuspidal representations.

(2) *Local Langlands Correspondences.*

The local Langlands correspondence (LLC) is a conjectural classification of the set $\mathrm{Irr}(G)$ of equivalence classes of smooth, irreducible representations of the p-adic group G in terms of L-parameters, which are local Galois representations valued in the L-group ${}^LG$ of G. Though unknown in full generality, there are many important classes of groups for which it is now known, such as GL_n and classical groups. The proof of the LLC for these groups involved serious global inputs, such as the trace formula or the cohomology groups of Shimura varieties. As such, the LLC is still considered somewhat mysterious, even if it is known for classical groups.

Therefore one might ask for an explicit local construction of the LLC. For a general group G, one may consider a subclass of representations (such as unipotent representations or regular supercuspidal representations) and ask if one could explicitly attach L-parameters to this subclass of representations. Further, one may ask if certain prominent structures on one side of the LLC can be described directly on the other side. One example is how the notion of the cuspidal support of a representation and the Bernstein decomposition of the category of smooth representations can be manifested on the Galois side. Another example is whether the ramification behaviour of the local Galois representations (i.e. L-parameters) can be read off from the representation theory side. In another direction, one might be interested in a local geometric realization of the LLC that would yield a proof of the LLC beyond classical groups. Finally, there is an ongoing attempt, initiated by Emerton and Helm, to formulate and prove a version of LLC in families, which will include the case of LLC modulo ℓ.

The lectures of C. Bushnell, R. Kurinczuk, J.-F. Dat, M. Oi, N. Imai, J.-L. Waldspurger, A.M. Aubert and M. Solleveld addressed a range of such topics at the frontier of this subject.

(3) *Periods, Distinction and Hecke algebras.*

If $H \subset G$ is a subgroup, $\pi \in \mathrm{Irr}(G)$ is an irreducible representation of G and χ a character of H, then one says that π is (H,χ)-distinguished if $\mathrm{Hom}_H(\pi,\chi) \neq \{0\}$. For interesting classes of subgroups, such as symmetric subgroups or more generally spherical subgroups, one would like to classify and understand those representations which are (H,χ)-distinguished, determine the dimension of the above Hom space (the space of (H,χ)-periods) and understand the relative characters of distinguished representations. Far-reaching conjectures have been formulated by Sakellaridis and Venkatesh in the setting of spherical subgroups. Many branching problems can be formulated as a distinction problem as above, such as the restriction of representations of GL_{n+1} to GL_n (a special case of the Gan–Gross–Prasad conjecture) or the theta correspondence. One may also consider L^2-version of these problems, such as the spectral decomposition of $L^2(H\backslash G)$. An example is the basic affine space $L^2(U\backslash G)$ where U is a maximal unipotent subgroup and another is $L^2(\mathrm{Sp}_{2n}\backslash \mathrm{GL}_{2n})$. There is also a recent conjecture of D. Prasad about Ext-versions of these period problems, where one considers the higher derived functors $\mathrm{Ext}^*(\pi, \chi)$.

A useful technique in understanding representations of p-adic groups is the use of Hecke algebras. Indeed, the Bernstein components of the category of smooth representations of a p-adic group are equivalent to the module categories of appropriate affine Hecke algebras. Because of this, many questions about representation theory can often be reduced to equivalent problems about Hecke algebras. This provides another perspective on the LLC, and similarly one can also treat some distinction problems using Hecke algebra techniques.

The lectures of V. Secherre, M. Suzuki, M. Hanzer, G. Savin, C. Wan, E. Lapid, Y. Sakellaridis, N. Gurevich, D. Ciutobaru and J. Adler concerned various aspects of the subject of periods, distinction and Hecke algebras.

All in all, there were 24 lectures, consisting of 6 30-minute talks and 18 one-hour talks. As is evident from the above description, the representation theory of p-adic groups is a vibrant research area with connections to many different parts of mathematics, such as number theory, arithmetic geometry and harmonic analysis. A wide range of research topics were represented in the talks, showing that the field is expanding its boundary to form productive link-ups with neighbouring subjects. The participants of the workshop include well-known senior mathematicians who are leaders of the fields as well as many young graduate students and postdoctoral researchers. We are amazed and greatly encouraged by the quality and talent of the new generation of researchers entering the field. All participants were happy to have the opportunity to interact in close quarters with people doing cutting edge work in their area. There is no doubt that this subject area will see further striking developments in the coming years and as the organizers, we hope that this workshop will serve as a platform for new collaborations and a launching pad for some of these future breakthroughs.

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Workshop: New Developments in Representation Theory of p-adic Groups

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Abstracts

No worries!

Jessica Fintzen

In 2001, Yu ([14]) proposed a construction of smooth, complex, supercuspidal representations of p-adic groups that since then has been widely used, as is evident by other talks of this workshop. However, Spice noticed recently that Yu's proof relies on a misprinted[1] (and therefore false) statement in one of the references, and it became uncertain whether the representations constructed by Yu are irreducible and supercuspidal. I was asked to report on the current state of affairs.

Let F be a non-archimedean local field of residual characteristic p, and let G be a (connected) reductive group over F. For more than 40 years, mathematicians have tried to construct all (complex, irreducible) supercuspidal representations of $G(F)$. Building on earlier work of Carayol ([4]), Howe ([4]) and Moy ([17]), in the 1990s Bushnell and Kutzko ([1]) provided a construction of all supercuspidal representations of $\mathrm{GL}_n(k)$. These results lie at the heart of many results in the representation theory of GL_n. For example, they play a key role in the construction of an explicit local Langlands correspondence for GL_n as well as in the study of the correspondence's fine structure. Using similar methods, Stevens ([23]) constructed all supercuspidal representations of classical groups for $p \neq 2$, and Sécherre and Stevens ([22]) constructed all supercuspidal representations of inner forms of GL_n. Their work was proceeded by a series of partial results ([2,11,14,15,18,19,26]). The construction of supercuspidal representations for inner forms of GL_n plays a crucial role in the explicit description of the local Jacquet–Langlands correspondence, which is an instance of Langlands functoriality.

Shortly after Bushnell and Kutzko's work was published, Vigneras ([24]) showed that a similar construction yields all irreducible, cuspidal, mod-ℓ representations of $\mathrm{GL}_n(F)$ for $\ell \neq p$, and recently Kurinczuk and Stevens ([13]) proved the analogous result for classical groups ($p \neq 2$).

Unfortunately, the picture is much less complete for arbitrary reductive groups. An important first step was achieved by Moy and Prasad ([20, 21]) in the 1990s. They introduced the notion of *depth*, which is a non-negative rational number that measures the first occurrence of a fixed vector in a given representation. They then classified the depth-zero representations by relating them to representations of finite groups of Lie type. A similar result was obtained around the same time by Morris ([16]). The first construction of positive-depth supercuspidal representations for general reductive groups (that split over a tamely ramified extension) was given by Adler ([1]) in 1998 and generalized by Yu in his seminal work [14] in 2001. All these representations are constructed via compact induction from an

[1] The statement of [9, Theorem 2.4.(b)] contains a typo. From the proof provided by [9] one can deduce that the stated representation of $P(E_+, j)H(E_+^\perp, j)$ (i.e. the pull-back to $P(E_+, j)H(E_+^\perp, j)$ of a representation of $SH(E_0, j_0)$ as in part (a')) should be tensored with $\chi^{E_+} \ltimes 1$ before inducing it to $P(E_+, j)H(E, j)$ in order to define π_+ (using the notation of [9]).

irreducible representation $\widetilde{\rho}$ of a compact-mod-center, open subgroup $\widetilde{K}$ of $G(F)$. Since then, Yu's construction has been widely used, e.g. to understand distinction of representations of p-adic groups, to study the Howe correspondence, to obtain character formulas and to construct an explicit local Langlands. In 2007 Kim ([6]) showed that Yu's construction yields all supercuspidal representations if F has characteristic zero and p is very large. Using a different approach, I proved recently ([7]) that Yu's construction yields all supercuspidal representations without any assumption on the characteristic of F and only assuming that p does not divide the order of the (absolute) Weyl group of G (and G splits over a tamely ramified extension as assumed by Yu). A motivation for why this assumption is necessary in general when also considering types can be found in [8]. Moreover, I have shown that a construction analogous to Yu's construction yields all irreducible, cuspidal, mod-ℓ representations ([6]). In conclusion, we seem to have reached a good understanding of all (super)cuspidal representations of $G(F)$ in the tame setting based on Yu's construction.

However, it was recently noticed by Spice that Yu's proof that his representations are irreducible and supercuspidal ([14]) relied on a misprinted (and therefore false) statement in [9]. Thus we no longer knew if the representations constructed by Yu were irreducible and supercuspidal. In my recent preprint [5] I illustrate the impact of this false statement on Yu's proof by providing a counterexample to Proposition 14.1 and Theorem 14.2 of [14]. Proposition 14.1 and Theorem 14.2 are the main intertwining results in [14] and form the heart of the proof. The counterexample that I provide arises in the case where the group G is Sp_{10} by considering a twisted Levi subgroup G' of shape $\mathrm{U}(1) \times \mathrm{Sp}_8$ and a well chosen point in the Bruhat–Tits building of G'.

In my talk, I outlined Yu's construction based on my viewpoint presented in [5], discussed the above issue, and conveyed the crucial, reassuring message: Yu's construction yields nevertheless irreducible, supercuspidal representations. In other words: No worries! My proof ([5]) that Yu's construction provides irreducible, supercuspidal representations relies on the first part of Yu's proof and provides a shorter, alternative second part that does not rely on [14, Proposition 14.1 and Theorem 14.2] and the misprinted version of [9, Theorem 2.4(b)]. Yu's approach consists of following a strategy already employed by Bushnell–Kutzko that required to show that a certain space of intertwining operators has dimension precisely one, i.e., in particular, is non-trivial. My approach does not require such a result. Instead I use the structure of the constructed representation including the structure of Weil–Heisenberg representations, and the Bruhat–Tits building to show more directly that every element that intertwines $\widetilde{\rho}$ is contained in $\widetilde{K}$ (where $\text{c-ind}_{\widetilde{K}}^{G(F)}\widetilde{\rho}$ is the representation constructed by Yu), which implies the desired result.

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Parameterizing Unramified Tori

Stephen DeBacker

Suppose k is a nonarchimedian local field and $\mathbf{G}$ is a connected reductive k-group. A natural question one may ask is: How to describe the embeddings of maximal k-tori into $\mathbf{G}$? For example, for $p \neq 2$ and up to rational conjugacy there are seven embeddings of maximal k-tori into SL_2. At the level of k-points these are, up to rational conjugacy, the embedding of $k^\times$ (as a split torus) and for each of the three (up to isomorphism) quadratic extensions of k there are two embeddings of the norm one elements (as elliptic tori). While Galois-cohomology provides an answer to this question, one may hope for something more concrete via Bruhat-Tits theory. This note is about a strategy to answer this question. We also discuss how to parameterize unramified tori; this joint work with Jeff Adler extends the known parameterization of maximal unramified tori [1].

1. Notation

Let $\mathfrak{f}$ denote the residue field of k. Let $\bar{k}$ denote an algebraic closure of k, and let $K \leq \bar{k}$ be the maximal unramified extension of k. Let $\mathfrak{F}$ denote the residue field of K, it is an algebraic closure of $\mathfrak{f}$. If $\mathbf{H}$ is an algebraic K-group, then we denote its group of K-points by H.

We identify $\mathrm{Gal}(K/k)$ with $\mathrm{Gal}(\mathfrak{F}/\mathfrak{f})$ and fix a topological generator Fr for $\mathrm{Gal}(K/k)$.

Let $\mathbf{A}$ denote a maximal K-split k-torus in $\mathbf{G}$ that contains a maximal k-split torus; such a torus is unique up to G^{Fr}-conjugacy [3]. To ease the exposition, we assume $\mathbf{G}$ is K-split. That is, we assume that $\mathbf{A}$ is a maximal k-torus.

1.1. **Unramified twisted Levis and tori.** A subgroup $\mathbf{M}$ of $\mathbf{G}$ is called an *unramified twisted Levi* provided that $\mathbf{M}$ is defined over k and there exists a parabolic K-subgroup for which $\mathbf{M}$ is a Levi component.

Suppose $\mathbf{S}$ is a k-torus. We shall call $\mathbf{S}$ an *unramified torus* provided that $\mathbf{S}$ is the K-split component of the center of an unramified twisted Levi.

1.2. **Buildings.** We denote by $\mathcal{B}(G)$ the building of G. If F is a facet in $\mathcal{B}(G)$, then we denote the corresponding parahoric by G_F and its prounipotent radical by $G_{F,0^+}$. The quotient $G_F/G_{F,0^+}$ is the group of $\mathfrak{F}$-points of a connected reductive group G_F. If F if Fr-stable, then all of the above objects are defined over k or $\mathfrak{f}$, as appropriate. The group G^{Fr} acts simplicially on $\mathcal{B}(G)^{\mathrm{Fr}}$, and a fundamental domain for this action is called an alcove.

Example. For the group $\mathrm{Sp}_4(k)$, each alcove is a right isocesles triangle. For every facet in the alcove pictured in Figure 1 we describe the algebraic $\mathfrak{f}$-group G_F.

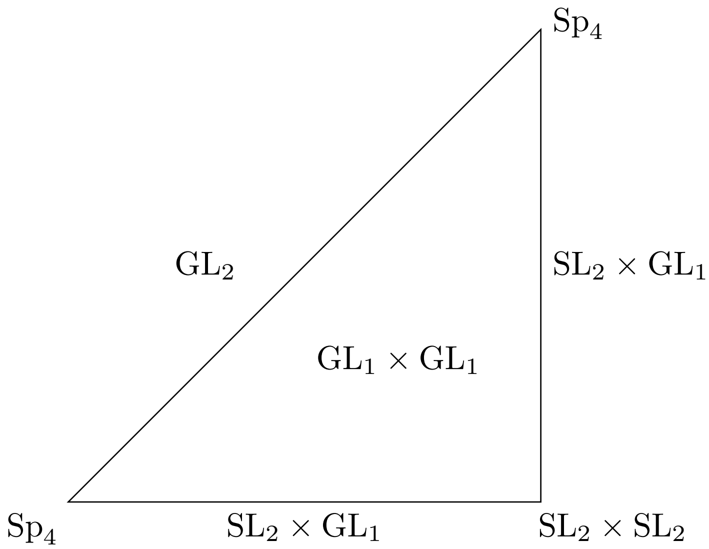

FIGURE 1. The reductive quotients for Sp_4

2. Motivation

Suppose $\mathbf{T}$ is a maximal k-torus in $\mathbf{G}$. Let $\mathbf{T}_u$ be a maximal K-split torus in $\mathbf{T}$. Since the set of K-split tori in $\mathbf{T}$ form a single $T = \mathbf{T}(K)$ conjugacy class and $\mathbf{T}$ is abelian, $\mathbf{T}_u$ is the unique maximal K-torus in $\mathbf{T}$. Since $\mathrm{Fr}(\mathbf{T}_u)$ is also a maximal K-split torus in $\mathbf{T}$, we conclude, by uniqueness, that $\mathbf{T}_u$ is defined over k.

We shall call $\mathbf{T}$ a *K-minisotropic torus in* $\mathbf{G}$ provided that $\mathbf{T}_u$ is the K-split component of the center of $\mathbf{G}$.

Set $\mathbf{M} = C_{\mathbf{G}}(\mathbf{T}_u)$. Then $\mathbf{M}$ is an unramified twisted Levi. Note that (a) $\mathbf{T}_u$ is an unramified torus, and (b) $\mathbf{T}$ is a maximal K-minisotropic torus in $\mathbf{M}$.

Based on this idea, a possible strategy for identifying all tori (up to some natural relation) has the following two natural parts:

(1) understand unramified tori and
(2) understand maximal K-ministropic tori.

This note focuses on understanding unramfied tori. In the tame case (e.g., when the characteristic of $\mathfrak{f}$ does not divide the order of the Weyl group) there is also a good way to understand the maximal K-ministropic tori via Bruhat-Tits theory; we will not discuss this here.

It turns out that describing tori up to G^{Fr}-conjugacy is not always the right thing to do, especially for questions in harmonic analysis. Instead, one would like to describe *embeddings* of maximal tori up to G^{Fr}-conjugacy.

Definition. Suppose $\mathbf{S}$ is a k-torus in $\mathbf{G}$. If $\mathbf{S}$ is either an unramified torus or a maximal torus, then an *embedding* of $\mathbf{S}$ in $\mathbf{G}$ is a k-morphism of the form $\mathrm{Int}(g)\colon \mathbf{S} \to \mathbf{G}$ for $g \in G$.

3. On unramified tori

We want to understand unramified tori, that is, those tori $\mathbf{S}$ in $\mathbf{G}$ for which $\mathbf{S}$ is the K-split component of $C_{\mathbf{G}}(\mathbf{S})$.

Remark. This does not include tori such as $\{\mathrm{diag}(t, t^2, t^{-3})\}$ in SL_3.

We already know (see, for example, [1]) how to describe the set of G^{Fr}-conjugacy classes of maximal unramified tori (the situation where $\mathbf{S} = C_{\mathbf{G}}(\mathbf{S})$) in terms of Bruhat-Tits theory. In this case, there is a bijective correspondence between the set of G^{Fr}-conjugacy classes of maximal unramified tori in $\mathbf{G}$ and the set of equivalence classes[1] of pairs (F, S) where F is a facet in $\mathcal{B}(G)^{\mathrm{Fr}}$ and S is a maximal $\mathfrak{f}$-minisotropic torus[2] in G_F:

$$\{\text{maximal unramified tori}\}/G^{\mathrm{Fr}}\text{-conjugacy} \longleftrightarrow \{(F, \mathsf{S})\}/\text{equivalence}.$$

The basic idea of the correspondence is that given a pair (F, S) there is a lift of S to a maximal K-split k-torus $\mathbf{S}$ in $\mathbf{G}$ and any two lifts of S are conjugate by an element of $G^{\mathrm{Fr}}_{F,0^+}$. In the other direct, given a maximal K-split k-torus $\mathbf{S}$ in $\mathbf{G}$, the building of $S = \mathbf{S}(K)$ embeds into that of G. We let F be a maximal G^{Fr}-facet in the building of S and we let S be a maximal $\mathfrak{f}$ torus in G whose group of $\mathfrak{F}$-points equals the image of $S \cap G_F$ in G_F.

Example. Assuming I have calculated correctly, for Sp_4 there are nine G^{Fr}-conjugacy classes of maximal unramified tori. Since G^{Fr} acts transitively on the alcoves in $\mathcal{B}(G)^{\mathrm{Fr}}$, to see how the above correspondence works it is enough to restrict our attention to a single alcove. For a finite group of Lie type, the conjugacy clases of maximal tori are in bijective correspondence with the Fr-conjugacy classes in the Weyl group. In Figure 2 for each conjugacy class of $\mathfrak{f}$-minosotrpic torus in G_F we list one element in the corresponding conjugacy class in the Weyl group of G_F; we have chosen a set of simple roots α and β with α short and β long so that in Figure 2 the hypotenuse lies on a hyperplane defined by an affine root with gradient α and the horizontal edge lies on a hyperplane defined by an affine root with gradient β; the label w_α is the simple reflection in W corresponding to α, etc. In this way we enumerate the nine pairs (F, S) that occur, up to equivalence.

[1]Two pairs (F, S) and (F', S') are equivalent provided that there exists $g \in G^{\mathrm{Fr}}$ and an apartment $\mathcal{A}'$ in $\mathcal{B}(G)^{\mathrm{Fr}}$ for which (a) the smallest affine subspace of $\mathcal{A}'$ that contains F is both nonempty and equals the smallest affine subspace of $\mathcal{A}'$ that contains gF' and (b) under the identification of G_F with $\mathsf{G}_{gF'}$ resulting from (a) we have that S is identified with S'.

[2]S is said to be $\mathfrak{f}$-minisotropic in G_F provided that the $\mathfrak{f}$-split component of S is equal to the $\mathfrak{f}$-split component of the center of G_F.

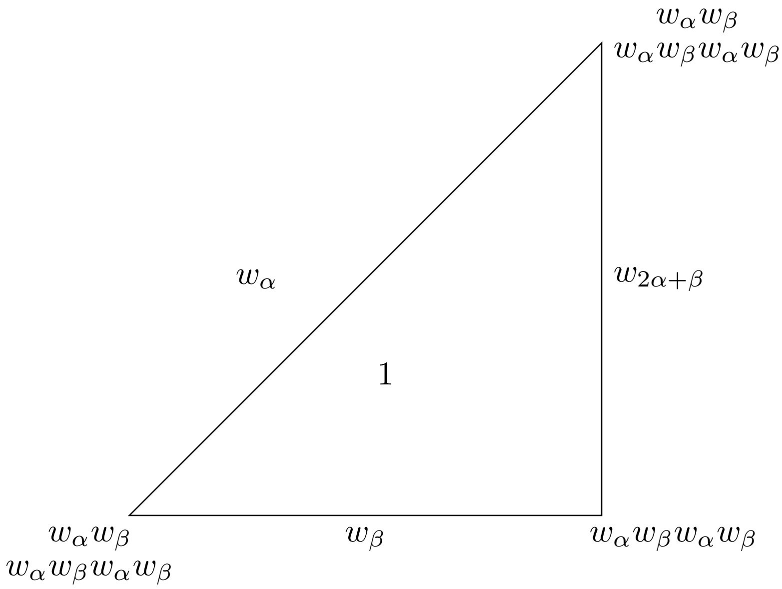

FIGURE 2. A labeling of representatives for the pairs (F, S) for Sp_4

In order to understand all unramified tori, not just the maximal ones, we should consider pairs (F, S') where F is a facet in $\mathcal{B}(G)^{\mathrm{Fr}}$ and S' lifts to an unramified torus in $\mathbf{G}$. The problem with this approach is that F cannot "see" which S' are important. A solution is to introduce some additional structure.

Let $\Phi = \Phi(\mathbf{G}, \mathbf{A})$ denote the roots of $\mathbf{G}$ with respect to $\mathbf{A}$ and let $W = N_G(\mathbf{A})/A$. Fix a set of simple roots $\Delta \subset \Phi$ and set

$$\Theta = \{w\rho \,|\, w \in W \text{ and } \rho \subset \Delta\}.$$

Note that Fr acts on Φ, W, and Θ. If F is a facet in $\mathcal{A}(A)^{\mathrm{Fr}}$, then the image of $A \cap G_F$ in G_F is a maximally $\mathfrak{f}$-split maximal $\mathfrak{f}$-torus, which we call A_F. We denote by W_F the Weyl group $N_{\mathsf{G}_F}(\mathsf{A}_F)/\mathsf{A}_F$. Note that we may and do naturally identify W_F with a subgroup of W. If $\theta \in \Theta$, then we let $\Phi_\theta \subset \Phi$ denote the root subsystem generated by θ, we let $W_\theta \leq W$ denote the corresponding Weyl group, and we let

$$\mathbf{A}_\theta = \Big(\bigcap_{\alpha \in \theta} \ker(\alpha)\Big)^\circ.$$

Suppose that (F, S') is a pair with F a G^{Fr}-facet in $\mathcal{A}(A)^{\mathrm{Fr}}$ and S' an $\mathfrak{f}$-torus in G_F. Let S be a maximal $\mathfrak{f}$-torus in G_F that contains S'. We can lift S to a maximal unramified torus $\mathbf{S}$ in $\mathbf{G}$, and we let $\mathbf{S}'$ be the subtorus of $\mathbf{S}$ corresponding to S'. We can choose $g \in G_F$ so that $\mathbf{S} = {}^g\mathbf{A}$. Since $\mathbf{S}$ is defined over k, we have that $\mathrm{Fr}(g)^{-1}g$ belongs to the normalizer of $\mathbf{A}$ in G, and we let $w \in W_F \leq W$ denote its image in the Weyl group. If $\mathbf{S}'$ is going to be unramified, that is, if $\mathbf{S}'$ is going to be the K-split component of $C_{\mathbf{G}}(\mathbf{S}')$, then one checks that there exists $\theta \in \Theta$ such that $\mathbf{S}' = {}^g\mathbf{A}_\theta$ and $\mathrm{Fr}(\Phi_\theta) = w\Phi_\theta$. Thus, our pair (F, S') corresponds to a triple (F, θ, w) where F is a facet in $\mathcal{A}(A)^{\mathrm{Fr}}$, $\theta \in \Theta$, $w \in W_F$, and $\mathrm{Fr}(\Phi_\theta) = w\Phi_\theta$.

This approach leads to a bijective correspondence between the set of G^{Fr}-conjugacy classes of unramified tori in $\mathbf{G}$ and the set of equivalence[3] classes of elliptic[4] triples (F, θ, w):

$$\{\text{unramified tori}\}/G^{\mathrm{Fr}}\text{-conjugacy} \longleftrightarrow \{\text{elliptic } (F, \theta, w)\}/\text{equivalence}.$$

When we restrict this correspondence to maximal unramified tori, the triples under consideration are of the form $(F, \emptyset, w)$.

Example. Assuming I have calculated correctly, for Sp_4 there are sixteen G^{Fr}-conjugacy classes of unramified tori. Since G^{Fr} acts transitively on the alcoves in $\mathcal{B}(G)^{\mathrm{Fr}}$, to see how the above correspondence works it is enough to restrict our attention to a single alcove. In Figure 3 we enumerate the sixteen triples (F, θ, w) that occur, up to equivalence. The centralizer of the unramified torus

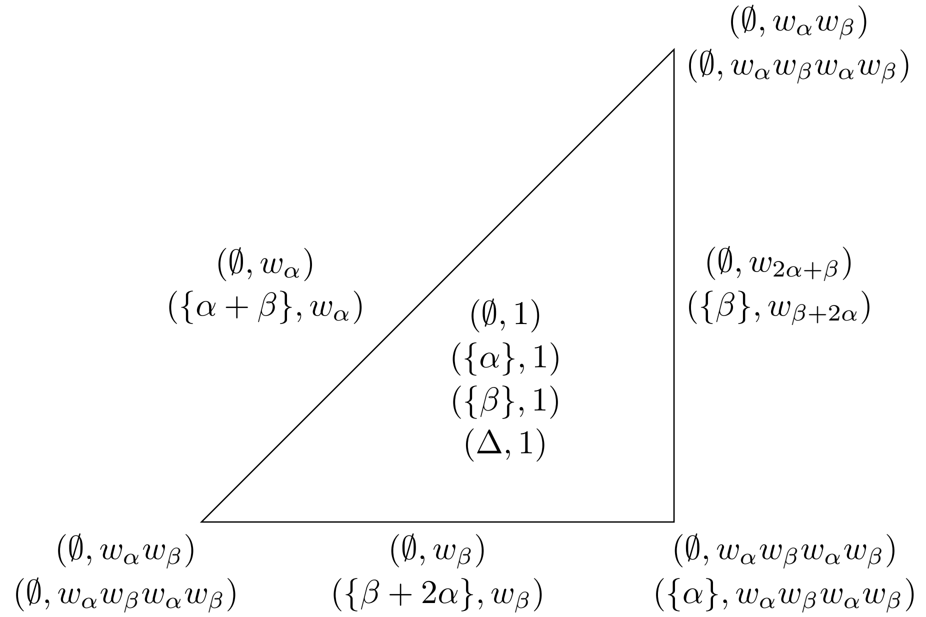

FIGURE 3. A parameterization of the rational conjugacy classes of unramified tori for Sp_4

corresponding to the pair $(\{\alpha+\beta\}, w_\alpha)$ is unramified $U(1,1)$ while the centralizer of the unramified torus corresponding to the pair $(\{\alpha\}, w_\alpha w_\beta w_\alpha w_\beta)$ is unramified $U(2)$ (using Jabon's notation [2]). The centralizers of the tori corresponding to the pairs $(\{\beta\}, w_{\beta+2\alpha})$ and $(\{\beta+2\alpha\}, w_\beta)$ are of the form $\mathrm{SL}_2 \times \mathbf{S}$ where $\mathbf{S}(k)$ is the norm one elements of an unramified quadratic extension of k. The four unramified tori with labels of the form $(\theta, 1)$ are the k-split components of the centers of the

[3]We say that (F, θ, w) is equivalent to (F', θ', w') provided that there exists an $n \in N_G(\mathbf{A})$ such that (a) the smallest affine subspace in $\mathcal{A}(A)^{\mathrm{Fr}}$ that contains nF is equal to the smallest affine subspace in $\mathcal{A}(A)^{\mathrm{Fr}}$ that contains F', (b) $\theta' = n\theta$, and (c) we have $\mathrm{Fr}(n)wn^{-1} \in w'(W_{nF} \cap W_{\theta'})$.

[4]A triple (F, θ, w) is said to be elliptic provide that the set $\{\mathrm{Fr}(w')^{-1}ww' \mid w' \in W_F\}$ doesn't intersect a proper parabolic subgroup of W_F.

four (up to conjugacy) distinct k-subgroups of Sp_4 that occur as a Levi factor for a parabolic k-subgroup of Sp_4.

What about the embeddings of unramified tori up to G^{Fr}-conjugacy? If $\mathbf{S}$ corresponds to (F, θ, w) and $\mathbf{S}'$ correspond to (F', θ', w'), we say that the unramified tori $\mathbf{S}$ and $\mathbf{S}'$ are *stably conjugate* provided that there is an embedding of $\mathbf{S}$ into $\mathbf{G}$ with image $\mathbf{S}'$. One can show that $\mathbf{S}'$ is stably conjugate to $\mathbf{S}$ if and only if there exists a $n \in W$ for which $\theta' = n\theta$ and $w' \in \mathrm{Fr}(n)wn^{-1}W_{\theta'}$. Moreover, the non-$G^{\mathrm{Fr}}$-conjugate embeddings of $\mathbf{S}$ into $\mathbf{G}$ with image $\mathbf{S}$ are naturally indexed by $W_{w\circ\mathrm{Fr},\theta}/W(F)_{w\circ\mathrm{Fr},\theta}$. Here $W_{w\circ\mathrm{Fr},\theta} = \{w' \in N_W(W_\theta)\colon w^{-1}\,\mathrm{Fr}(w')^{-1}ww' \in W_\theta\}$ and $W(F)_{w\circ\mathrm{Fr},\theta} = W(F) \cap W_{w\circ\mathrm{Fr},\theta}$ where $W(F)$ denotes the image of the stabilizer (not fixator) in $N_G(\mathbf{A})$ of the smallest affine subspace of $\mathcal{A}(\mathbf{A})$ containing F.

Example. In Figure 4 we have added a number and a letter to each datum of Sp_4. Modulo calculation errors, the number records the number of embeddings, up to G^{Fr}-conjugacy, of the unramified torus corresponding to the datum into itself. If two data have the same letter, then their corresponding unramified tori are stably conjugate. So, for example, the number of embeddings of an unramified torus corresponding to the datum $(\{\alpha+\beta\}, w_\alpha)$ is three, up to G^{Fr}-conjugacy.

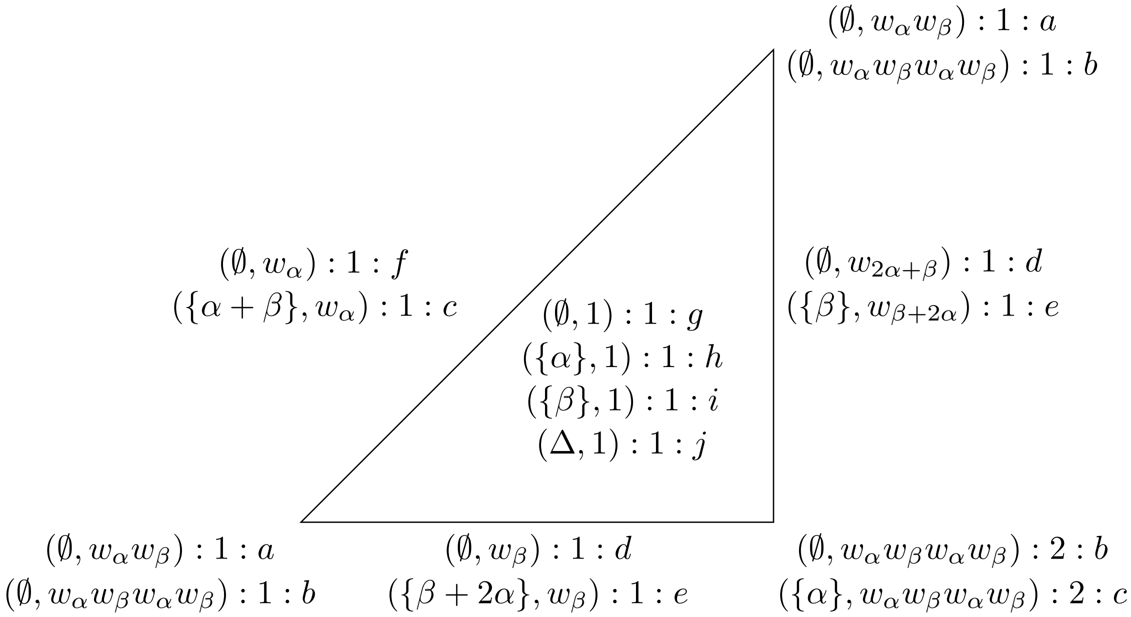

FIGURE 4. A parameterization of the embeddings of unramified tori for Sp_4

4. Conclusion

The above gives a reasonably concrete method for understanding the embeddings of unramifed tori into $\mathbf{G}$. Together with an understanding of the maximal K-ministropic tori in all unramified twisted Levis of $\mathbf{G}$ (in the tame setting), this provides a good understanding of all tori (in the tame setting). Beyond the cleanup work that needs to be done to explicate these results in their most general form, one would also like to understand how this parameterization via Bruhat-Tits theory looks on the dual side and, in particular, how the parameterization interacts with endoscopy.

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On the unicity of types

Peter Latham

(joint work with Monica Nevins)

This report is on a talk presented by the first named author at the conference *New developments in the representation theory of p-adic groups* at Mathematisches Forschunginstitut Oberwolfach in October 2019. It is our pleasure to thank MFO and the organizers for providing us with the opportunity to attend such a wonderful conference.

An effective means of studying the (smooth, complex) irreducible representations of a connected, reductive p-adic group $G = \mathbf{G}(F)$ is by the Bushnell–Kutzko *theory of types*, as developed in [1]. Our focus will be on the case of types for supercuspidal representations: a type (J, λ) for a supercuspidal representation π of G consists of an irreducible representation λ of a compact open subgroup J of G, such that any irreducible subquotient of the compactly induced representation $\text{c-Ind}_J^G \lambda$ is isomorphic to the twist of π by some unramified character of G; equivalently, $\text{c-Ind}_J^G \lambda$ is a projective generator for the Bernstein block of π.

Under various hypotheses on either the group G or the supercuspidal representation π, types are now known to exist in many cases, arising via explicit constructions which have many applications, especially within the broader context of the local Langlands programme. In each case, these explicit constructions of types provide the major step in establishing the long-standing folklore conjecture that any supercuspidal representation should be compactly induced from an open, compact-modulo-centre subgroup.

One expects that types should be rather hard to come by, and that the known constructions should in fact give *all* possible types, up to minor representation-theoretic renormalizations such as composing with the adjoint action of G, and inducing to a larger subgroup. We refer to the conjecture that this expectation is true as *the unicity of types*; this conjecture is known in several cases, under extremely restrictive hypotheses on the structure of the group G and/or the representation π. The goal of this report is to discuss recent reports by the authors towards establishing the unicity of types in the most general setting currently available.

Specifically, J.K. Yu has constructed types for many supercuspidal representations π of G, under no hypotheses on G [10]; we refer to the supercuspidal representations and types arising from this construction as *essentially tame*. Recently, J. Fintzen [2], extending previous results of J.L. Kim [3], proved that Yu's construction gives all supercuspidal representations under mild hypotheses on the residual characteristic of F.

In order to state our results, we require some details regarding J.K. Yu's construction of types, which in turn requires some groups associated to the reduced Bruhat–Tits building $\mathscr{B}(G)$ of G; this is a polysimplicial complex equipped with a natural action of G. For $x \in \mathscr{B}(G)$, we denote by G_x the (unique) maximal compact subgroup of $\mathrm{Stab}_G(x)$. A. Moy and G. Prasad have defined a filtration $G_{x,r}$, $r \geq 0$, of G_x by open normal subgroups. Writing $G_{x,r+} = \bigcup_{r>s} G_{x,s}$, the subgroup $G_{x,0}$ of G_x is a parahoric subgroup of G, and $G_{x,0+}$ is the pro-p radical of G_x. The quotient $G_{x,0:0+} := G_{x,0}/G_{x,0+}$ isomorphic to a finite group of Lie type.

J.K. Yu's essentially tame types (J, λ) then have the following properties:

(1) The quotient of J by its pro-p radical J^+ is isomorphic to a group of the form $G_x^0/G_{x,0+}^0$, where $G^0 \subset G$ is a twisted Levi subgroup, split over a tamely ramified extension and with a centre which is anisotropic modulo the centre of G, and x is a vertex of $\mathscr{B}(G^0)$, viewed as a subspace of $\mathscr{B}(G)$.

(2) The representation λ factors as $\lambda = \sigma \otimes \kappa$, where:
   (a) the representation σ is inflated from one of $J/J^+ \simeq G_x^0/G_{x,0+}^0$ which, upon restriction to $G_{x,0}^0$, is isomorphic to a sum of cuspidal representations; and
   (b) there exists a pro-p subgroup H^+ of J^+, and a character θ of H^+, such that $\kappa|_{H^+}$ is a sum of copies of θ.

In fact, since the representation κ is uniquely determined by θ, writing $H = G_{x,0}^0 H_+$, one sees that the pair $(H, \sigma \otimes \theta)$ is also a type; while the type (J, λ) often has preferable representation-theoretic properties, it will be more convenient for our purposes to work with $(H, \sigma \otimes \theta)$.

In this setting, we wish to prove the following:

Conjecture 1 (The unicity of types)**.** Let π be an essentially tame supercuspidal representation of G, let K be a maximal compact subgroup of G, and suppose that (K, τ) is a type for π. Then there exists an essentially tame type $(H, \sigma \otimes \theta)$ such that $H \subset K$ and $\tau \simeq \text{c-Ind}_H^K \sigma \otimes \theta$.

The unicity of types is known in several cases, under strong hypotheses on either the group G or the representation π [5–9]. The proofs of these results follow variations on the same basic strategy; in this report we describe recent work which generalizes this strategy as far as possible in order to reduce the unicity of types to an explicit conjecture regarding the structure of the Bruhat–Tits building of G [8] as a G-space.

We may now describe our results towards the unicity of types. Fix a maximal compact subgroup of G, which is necessarily of the form G_y, for some $y \in \mathscr{B}(G)$. One has a Mackey decomposition

$$\mathrm{Res}_{G_y}^G \text{c-Ind}_H^G \lambda = \bigoplus_{g : G_y \backslash G / H} {}^g\tau(y, g),$$

where $\tau(y, g) = \text{c-Ind}_{J \cap G_{g^{-1}y}}^{G_{g^{-1}y}} \sigma \otimes \theta$. This representation is isomorphic to a sum of copies of $\pi|_{G_y}$. Thus, we wish to show that, for each point $g^{-1}y$ in the G-orbit of $y \in \mathscr{B}(G)$, either $H \subset G_{g^{-1}y}$, in which case $\tau(y, g)$ is a sum of types of the expected form, or that for each component τ of $\tau(y, g)$, there exists an irreducible representation π' of G which contains τ but which is not isomorphic to an unramified twist of π. Our results provide two sufficient conditions for $\tau(y, g)$ to satisfy the latter of these two properties, in terms of the geometry of the geodesic $[x, g^{-1}y]$ joining x and $g^{-1}y$.

One may define a projection map $\mathrm{proj}_0 : \mathscr{B}(G) \to \mathscr{B}(G^0)$ by mapping a point to its closest neighbour in $\mathscr{B}(G^0)$. We then prove the following:

Theorem 1. *Suppose that the image under proj_0 of the simplicial closure of $[x, g^{-1}y]$ is not equal to x. Then, for any irreducible component τ of $\tau(y, g)$, there exists a (explicitly constructed) non-cuspidal irreducible representation π' of G such that τ is contained in $\pi'|_{G_{g^{-1}y}}$.*

The key observation is that the hypothesis that $\mathrm{proj}_0(g^{-1}y) \neq x$ guarantees that $J \cap G_{g^{-1}y}/J^+$ is contained in a proper parabolic subgroup of J/J^+. This allows us to modify the data underlying the type $(H, \sigma \otimes \theta)$ to produce a type for a non-cuspidal Bernstein block of G using the construction of [4] with which τ must intertwine. When this hypothesis fails, which is to say that $g^{-1}y$ is contained in the fibre $\mathrm{proj}_0^{-1}(x)$, one must follow a different approach.

Let Z^0 denote the centre of G^0, equipped with the filtration $Z^0_t = Z^0 \cap G_{x,t}$. Then $\Xi_t := \mathscr{B}(G)^{Z^0_t}$ is a subset of $\mathscr{B}(G)$ which is of bounded distance from $\mathscr{B}(G^0)$. On the other hand, one considers the filtration $H_t = H \cap G_{x,t}$ of H, and the sets $\Omega_t := \mathscr{B}(G)^{H_t} \subset \Xi_t$, which form an increasing sequence of bounded neighbourhoods of x in $\mathscr{B}(G)$. When t is a jump point in the filtration of Z^0, in the sense that $Z^0_{t+} \subsetneq Z^0_t$, it is possible that one has $\Omega_{t+} \not\subset \Xi_t$. In this situation, we prove the following:

Theorem 2. *There exists a constant $s_0 > 0$ such that, if there exists some $0 < t < s_0$ for which the geodesic $[x, g^{-1}y]$ has non-empty intersection with $\Omega_{t+} \backslash (\Omega_{t+} \cap \Xi_t)$, then no irreducible component of $\tau(y, g)$ is a type for π.*

Compared with Theorem 1 has two downsides. The first is that it does not explicit construct representations demonstrating that no component of $\tau(y, g)$ is a type. The second is that the hypotheses are much more difficult to check.

We conjecture, however, that Theorems 1 and 2 are together sufficient to show that $\tau(y,g)$ contains no types, for $g^{-1}y$ outside of some small neighbourhood of x. Note that $g^{-1}y \in \mathscr{B}(G)^H$ if and only if $H \subset G_{g^{-1}y}$; thus the unicity of types is equivalent to the assertion that $\tau(y,g)$ contains a type if and only if $g^{-1}y \in \mathscr{B}(G)^H$.

Conjecture 2. There exists a bounded convex neighbourhood Γ of x, containing $\mathscr{B}(G)^H$, such that, for every $z \in \mathscr{B}(G)\backslash\Gamma$, the geodesic $[x,z]$ has non-empty intersection with either $\mathrm{proj}_0^{-1}(\mathscr{B}(G^0)\backslash\{x\})$, or with $\bigcup_{0<t<s_0} \Omega_{t+}\backslash(\Omega_{t+} \cap \Xi_t)$.

In particular, this conjecture would imply that any type contained in $\pi|_{G_y}$ must be contained in ${}^g\tau(y,g)$ for some $g^{-1}y \in \Gamma$. As the representations $\tau(y,g)$ are of finite length, a consequence is that $\pi|_{G_y}$ contains a (small) finite number of types.

While Conjecture 2 appears to be a rather reasonable expectation, and is one which one may verify by hand in particularly simple examples, in general the actions of Moy–Prasad filtration subgroups of anisotropic tori of G on $\mathscr{B}(G)$ are poorly understood. Our arguments actually suffice to show that any counterexample to the unicity of types must necessarily arise from either a counterexample to the above conjecture, or from an example of some unexpected behaviour of a finite group of Lie type—specifically, a cuspidal representation of $G^0_{x,0}$ which is uniquely determined by its restriction to $G^0_{x,0} \cap G_{g^{-1}y}$, for some $g^{-1}y \in \Gamma$. While such examples are not expected to exist, to the authors' knowledge little is known beyond the case of $\mathbf{GL}_n(F)$ studied in [9]; it would be interesting to explore this problem for more general finite groups of Lie type.

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Comparison of Bushnell-Kutzko and Yu's constructions of supercuspidal representations

Arnaud Mayeux

At each step of Bushnell-Kutzko's construction (BK), we associate a part of a J.-K. Yu datum, at the end we obtain a full Yu datum and the supercuspidal representations associated to both data are the same. We use Bushnell-Kutzko [3] and Yu [4] 's original notations. Let F be a non archimedean local field and ord the valuation on algebraic extensions such that $\mathrm{ord}(\pi_F) = 1$ (π_F is a uniformizer of F). Let V be an F-vector space of dimension N, let $A = \mathrm{End}_F(V)$ and $G = A^\times$. Fix an additive character ψ of F with conductor the maximal ideal of F.

Theorem. Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum in A such that the attached field extension $F[\beta]/F$ is tame and such that $r = -k_0(\beta, \mathfrak{A})$.

(1) (Bushnell-Henniart [2, 3.1 Corollary]) There exists an element $\gamma \in F[\beta]$ such that $[\mathfrak{A}, n, r, \gamma]$ is a simple stratum and such that

$$[\mathfrak{A}, n, r, \gamma] \sim [\mathfrak{A}, n, r, \beta].$$

(2) (M) For all γ as in (i), the stratum $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ is simple.

Let us fix a maximal simple stratum $[\mathfrak{A}, n, 0, \beta]$ such that the attached field $E = F[\beta]$ is a tame extension of F. The previous theorem is related to BK's defining sequences and implies the following corollary.

Corollary. (M) There exists a defining sequence $[\mathfrak{A}, n, r_i, \beta_i], 0 \leq i \leq s$ of $[\mathfrak{A}, n, 0, \beta]$ ($r_0 = 0, \beta_0 = \beta$) such that for all $0 \leq i \leq s-1$,

(1) $F[\beta_{i+1}] \underset{\neq}{\subset} F[\beta_i]$

(2) $[\mathfrak{B}_{\beta_{i+1}}, r_{i+1}, r_{i+1}-1, \beta_i - \beta_{i+1}]$ is a simple stratum.

Let us fix such a defining sequence. We assume here[1] that $\beta_s \notin F$. We put

$$\begin{aligned}
&d := s+1\\
&E_i := F[\beta_i] \ 0 \leq i \leq s, E_d := F\\
&B_i := \mathrm{End}_{E_i}(V) \ 0 \leq i \leq d\\
&\mathfrak{B}_{\beta_i} := \mathfrak{A} \cap B_i\\
&U^0(\mathfrak{B}_{\beta_i}) := \mathfrak{B}_{\beta_i}^\times, U^k(\mathfrak{B}_{\beta_i}) := 1 + (\mathrm{rad}(\mathfrak{B}_{\beta_i}))^k \ k \geq 1\\
&c_i := \beta_i - \beta_{i+1} \ \ 0 \leq i \leq s-1, \ c_s := \beta_s
\end{aligned}$$

We can now describe BK's group $H^1(\beta, \mathfrak{A})$ associated to $[\mathfrak{A}, n, 0, \beta]$.

[1] The other case can be done in a similar way and can also be deduced from the case done here.

Fact. We have

$$H^1(\beta,\mathfrak{A}) = U^1(\mathfrak{B}_{\beta_0})U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1})\dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s})U^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1}(\mathfrak{A}).$$

We now fix a simple character $\theta \in \mathcal{C}(\mathfrak{A},0,\beta)$ of $H^1(\beta,\mathfrak{A})$. The fact that the sequence of fields $E_0, E_1, \dots, E_d$ is decreasing for $\subset$ implies that we can factorize the character θ as follows.

Theorem. (M) There exists $\phi_0,\dots,\phi_s$, smooth characters of $E_0^\times,\dots,E_s^\times$ such that

$$\theta = \prod_{i=0}^{s} \theta^i$$

where θ^i is determined by

- $\theta^i \mid_{H^1(\beta,\mathfrak{A})\cap B_i} = \phi_i \circ \det_{B_i}$
- $\theta^i \mid_{H^1(\beta,\mathfrak{A})\cap U^{[-\frac{\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\mathfrak{A})} = \psi_{c_i}$ where $\psi_{c_i}(x) = \psi \circ \mathrm{Tr}_{A/F}(c_i(x-1))$.

This allows us to introduce parts of a Yu datum.

Definition. (M) We put

$$G^i := \mathrm{Res}_{E_i/F}\underline{\mathrm{Aut}}_{E_i}(V)\ 0 \le i \le d,\ \overrightarrow{G} := G^0,\dots,G^d$$
$$R_i := -\mathrm{ord}(c_i)\ \ 0 \le i \le s,\ R_d := R_s, \overrightarrow{R} := R_0,\dots,R_d$$
$$\mathbf{\Phi}_i := \phi_i \circ \det_{B_i}\ 0 \le i \le s, \mathbf{\Phi}_d := 1,\ \overrightarrow{\mathbf{\Phi}} = \mathbf{\Phi}_0,\dots,\mathbf{\Phi}_d.$$

We have $G^0 \subset \dots \subset G^d$, and it forms a Yu tame twisted Levi sequence.

Fact about $H^1(\beta,\mathfrak{A})$ and similar considerations for $J^0(\beta,\mathfrak{A})$ allow us, using Broussous-Lemaire [1], to obtain the following result.

Proposition. (M) There exists a point y in the Bruhat-Tits building of G^0 such that the following equalities hold

$$H^1(\beta,\mathfrak{A}) = K_+^d(\overrightarrow{G},y,\overrightarrow{R})$$
$$J^0(\beta,\mathfrak{A}) = {}^\circ K^d(\overrightarrow{G},y,\overrightarrow{R})$$
$$E_0^\times J^0(\beta,\mathfrak{A}) = K^d(\overrightarrow{G},y,\overrightarrow{R}).$$

On the left: BK's groups - On the right: Yu's groups

Using the second assertion of Corollary, we are able to prove the following result.

Theorem. (M) The character $\mathbf{\Phi}_i$ is G^{i+1}-generic of depth R_i for $0 \le i \le s$.

Let us now fix a β-extension κ of θ and an irreducible cuspidal representation σ of $J^0(\beta,\mathfrak{A})/J^1(\beta,\mathfrak{A})$. Let us also fix an extension Λ (BK last step) of $\kappa\otimes\sigma$ to $E^\times J^0(\beta,\mathfrak{A})$. The final theorem of our comparison is as follows.

Theorem. (M) There exists ρ such that $(\overrightarrow{G}, y, \overrightarrow{R}, \rho, \overrightarrow{\mathbf{\Phi}})$ is a Yu datum and such that

$$\Lambda \simeq \rho_d(\overrightarrow{G}, y, \overrightarrow{R}, \rho, \overrightarrow{\mathbf{\Phi}}),$$

here $\rho_d(\overrightarrow{G}, y, \overrightarrow{R}, \rho, \overrightarrow{\mathbf{\Phi}})$ is Yu's representation. In particular the supercuspidal representations obtained by compact induction from ρ_d and Λ are isomorphic.

Remark/Conclusion. (M)

(1) If we remove the assumption that $F[\beta]/F$ is tame in our fixed simple stratum $[\mathfrak{A}, n, 0, \beta]$, then the sequence of fields $E_0, \ldots, E_s$ attached to a defining sequence is not decreasing for $\subset$ in general. It always decreases for $[\bullet : F]$.

(2) In a certain sense, we have explained that BK and Yu's constructions are compatible. Does there exist a construction Ⓢ generalizing both of them ?

(3) If one tries to obtain Ⓢ generalizing Yu's approach, one has to remove (among other things) the axiom of inclusions in the twisted Levi sequence by (1) of this remark and by definition of $\overrightarrow{G}$. This implies that one can not expect a factorable construction « $\rho_d = \otimes \kappa^i$ »as Yu's one.

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A compare of two versions of constructions of supercuspidal representations and its application to local theta correspondence

Jia-Jun Ma

(joint work with Hung Yean Loke)

Supercuspidal representations can be parameterized and constructed from certain arithmetic data in many cases. I am interested in the application of these constructions. My current experiences are on the explicit descriptions of local theta correspondence using these data.

There are roughly two versions of supercuspidal data generalizing Howe's construction [4] of tamely ramified supercuspidal representations for the general linear groups. One version is developed by Bushnell-Kutzko [1], Stevens [13], Sécherre [9] etc. This version constructs all supercuspidal representations for general linear groups and classical groups. Another version is developed by Yu [14], Hakim-Murnahghan [3], Kim [6] etc. This version constructs tamely ramified supercuspidal representations for arbitrary p-adic reductive groups.

As a user of the type theory, it is natural to ask whether these two versions are the same in the "tame case". The explicit translation between the Bushnell-Kutzko and Yu's constructions was written down by Mayeux [8]. In fact, the relevant supercuspidal representations are called "essentially tame" by Bushnell-Henniart [1] and are called regular supercuspidal by Kaletha [5].

In a joint project with Loke, we describe the translation for classical groups defined over a p-adic field F_0 with $p \neq 2$. From now on, we use notation in [13] and [14]: Let G be a classical group viewed as an symmetric subgroup in $\mathrm{GL}_F(V)$, $F = F_0$ if G is a symplectic group or an orthogonal group and F is a quadratic extention of F if G is unitary group, and V is the standard representation of G. For any semisimple element $\beta \in \mathfrak{g}$, let G^β denote the centralizer of β in G.

A Bushnell-Kutzko-Stevens' datum consists of a tuple (x, β, ρ, ϕ). Here x is a vertex in the building $\mathcal{B}(G^\beta) \subset \mathcal{B}(G)$ (a rational point in the building naturally corresponds to a lattice chain, which is denoted by Λ in [13]), $[x, 0, \beta]$ is a self-dual semisimple stratum (note that β is a semisimple element in $\mathfrak{g}$), ρ is a cuspidal representation of $G^\beta_{x,0:0^+}$ and ϕ is a self-dual semisimple character in $\mathcal{C}_-(x, 0, \beta) \subset \mathcal{C}(x, 0, \beta)$.

A Yu's datum consists of a tuple $(x, \overrightarrow{G}, \rho, \overrightarrow{\phi})$. Here $\overrightarrow{G} = (G^0, G^1, \cdots, G^d = G)$ is a twisted Levi sequence (a chain of nesting subgroups) of G, $x \in \mathcal{B}(G^0) \subset \mathcal{B}(G)$ is a vertex, ρ is a cuspidal representation of $G^0_{x,0:0^+}$ and $\overrightarrow{\phi} = (\phi_0, \phi_1, \cdots, \phi_d)$ is a chain of quasi-characters of $\{\, G^i \,\}$. Let $\mathcal{D}^{\mathrm{Y}}(G) = \{\, (x, \overrightarrow{G}, \rho, \overrightarrow{\phi}) \,\}$ denote the set of Yu's data, π_Σ denote the supercuspidal representation attached to $\Sigma \in \mathcal{D}^{\mathrm{Y}}(G)$ by Yu's construction. Let $\mathrm{Scusp}_{\mathrm{tame}}(G) = \{\, \pi_\Sigma \mid \Sigma \in \mathcal{D}^{\mathrm{Y}}(G) \,\}$.

There are obvious similarities of the two versions: x and ρ should be "the same". On the other hand, a factorization of ϕ is built in the definition of semisimple character. More precisely, there is a chain of semisimple elements $\beta = \gamma_0, \gamma_1, \cdots, \gamma_d$ called a "defining sequence" of β (see [10, Definition 4.56]). To compare with Yu's notation, we assume that γ_d is in the center $Z(\mathfrak{g})$ of $\mathfrak{g}$. Now the semisimple character ϕ is determined by a chain of characters $(\chi_0, \cdots, \chi_d)$ of $F[\gamma_i]^\times$ (see the new definition of semisimple character in [11, Definition 9.5]). We will call the chain $(\chi_0, \cdots, \chi_d)$ a defining sequence of ϕ. Let $G^i := G^{\gamma_i}$ and now the setting becomes very similar to Yu's. In fact, the relevant compact subgroups in Bushnell-Kutzko-Stevens' setting can be defined using Yu's formula, despite G^{i+1} may not contain G^i in general. Let $k_x(\beta)$ denote the critical exponent of β (see [12, (3.6)]) and let $s_i = -k_x(\gamma_i)/2$. Then the following groups are crucial in the construction of supercuspidal representations:

$$\begin{aligned} K^+ &:= G^0_{x,0^+} G^1_{x,s_0^+} \cdots G^d_{x,s_{d-1}^+} \\ K_{0^+} &:= G^0_{x,0^+} G^1_{x,s_0} \cdots G^d_{x,s_{d-1}} \\ K &:= G^0_x G^1_{x,s_0} \cdots G^d_{x,s_{d-1}}. \end{aligned}$$

Now we consider the case that β is tamely ramified, i.e. β is split over a tamely ramified field extension of F. Let $\mathcal{D}^{\mathrm{BKS}}_{\mathrm{tame}}(G) = \{\, (x, \beta, \rho, \phi) \mid \beta \text{ is tamely ramified} \,\}$

be the set of "essentially tame" data of Bushnell-Kutzko-Stevens. For $(x, \beta, \rho, \phi) \in \mathcal{D}^{\mathrm{BKS}}_{\mathrm{tame}}(G)$, we define a data $(x, \overrightarrow{G}, \rho, \overrightarrow{\phi}) \in \mathcal{D}^{\mathrm{Y}}(G)$ from defining sequences. The key technical part is a computation of the critical exponent $k_x(\beta)$ using a good factorization/Howe factorization:

Lemma. Let $[x, 0, \beta]$ be a semisimple stratum and $\beta \neq 0$. Then there is a good factorization

$$(\star) \qquad \beta = \Gamma_d + \Gamma_{d-1} + \cdots + \Gamma_0$$

such that Γ_i satisfies condition in [6, §7.1]. Moreover

$$k_x(\beta) = \mathrm{depth}(\Gamma_0).$$

It becomes clear that the sequence $\gamma_i := \sum_{j=i}^{d} \Gamma_i$ gives a defining sequence of β, $G^i := G^{\gamma_i}$ form a twisted Levi sequence[1]. Let $\det_{\mathrm{GL}(V)^{\gamma_i}} : \mathrm{GL}(V)^{\gamma_i} \to F[\gamma_i]^{\times}$ be the map which is the determinant map on each factors of $\mathrm{GL}(V)^{\gamma_i}$. We define a sequence of characters by

$$\phi_d := \chi_d \circ \det_{\mathrm{GL}(V)}|_G \quad \text{, and}$$

$$\phi_i := \chi_i \circ \det_{\mathrm{GL}(V)^{\gamma_i}}|_{G^i} \otimes (\chi_{i+1}^{-1} \circ \det_{\mathrm{GL}(V)^{\gamma_{i+1}}})|_{G^i} \quad \forall i = 0, \cdots, d-1$$

from a defining sequence $(\chi_0, \cdots, \chi_d)$ of ϕ. By definition, θ_d (in [14, Proposition 4.1]) is exactly the semisimple character ϕ. This procedure yields a map $\iota : \mathcal{D}^{\mathrm{BKS}}_{\mathrm{tame}}(G) \to \mathcal{D}^{\mathrm{Y}}(G)$. Up to a twist by a depth zero character[2], the Heisenberg-Weil representation defined by Yu's recipe (see [7, A.2]) is a β-extension. Overall, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{BKS}}_{\mathrm{tame}}(G) & \overset{\iota}{\dashrightarrow} & \mathcal{D}^{\mathrm{Y}}(G) \\ & \searrow \quad \swarrow & \\ & \mathrm{Scusp}_{\mathrm{tame}}(G) & \end{array}$$

We use dashed line to emphasize that ι depends on the choice of a defining sequence of ϕ.

In a previous joint work with Loke, we described local theta correspondence between supercuspidal representations using Kim's variation of Yu's data (this variation resembles that of Bushnell-Kutzko-Stevens'). In the current project, we reformulate Bushnell-Kutzko-Stevens' notation in Yu's fashion. In this way, we can reuse most part of our previous work to extend the description to all supercuspidal representations ($p \neq 2$).

Note that β plays the role of "Γ" in [7], the final result is the same as [7, §1.3], except a replacement of the definition of the positive depth data lift map ϑ^+. The new definition is as the following: For $(x, \beta, \rho, \phi) \in \mathcal{D}^{\mathrm{BKS}}_{\mathrm{tame}}(G)$ and all eigenvalues of β has valuation $-r$, we define $V' := V_\beta, x', w, \beta' := -M(w), \rho'$ by

[1] By [6, Proposition 4.7], $\{ G^i \}$ is independent of the choice of the factorization $(\star)$.

[2] This ambiguity could be removed by making certain canonical choices in the classical group case.

[7, Definition 5.8,(5.7), (5.8)], and define ϕ' to be the contragredient of the image $\tau_{x,x',\beta}(\phi)$ of ϕ under the "change of ring map" $\mathrm{End}_F(V) \to \mathrm{End}_F(V') : X \mapsto wXw^{-1}$ ([12, Proposition 3.26]).

In the end, I would like to thank Hung Yean Loke for comments and proofreading.

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Local Langlands in families in depth zero

Robert Kurinczuk

(joint work with Jean-Francois Dat, David Helm, Gil Moss)

Local Langlands. Let F be a non-archimedean local field whose residue field is finite of cardinality q, and let G be the F-points of an F-quasi-split reductive group defined over F. Fix a prime ℓ not dividing q. In this setting, it is expected that there is a natural surjective map LL

$$\{\text{Irreducible smooth representations of G}\}/\simeq \xrightarrow{\text{LL}} \{\text{Langlands parameters for G}\}/\text{"conjugacy"},$$

with finite fibres called *L-packets*, with the representations typically considered on $\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$ vector spaces depending on taste. Many cases are now known, which over p-adic fields include: $\mathrm{GL}_n(\mathrm{F})$ [7, 11, 14], and for tempered representations of certain classical groups [1, 13].

Goal. To equip both sides of LL with integral geometric structures, and enhance LL to a map of these structures. For $\mathrm{GL}_n(\mathrm{F})$ with $\mathrm{F}/\mathbb{Q}_p$ over $\mathrm{W}(\overline{\mathbb{F}}_\ell)$, this was established in the pioneering work of Emerton, Helm, and Moss [5, 8–10].

What is a Langlands parameter? Let ${}^L\mathrm{G} = \widehat{\mathrm{G}} \rtimes \mathrm{W}$ denote the L-group of G; where we consider the dual group $\widehat{\mathrm{G}}$ as a $\mathbb{Z}[1/p]$-group scheme, and W is a finite quotient of the Weil group W_F of F through which the action of W_F on $\widehat{\mathrm{G}}$ factors.

There are three forms of Langlands parameters for G often considered:

(1) (SL_2-parameters): morphisms $\rho : \mathrm{W}_\mathrm{F} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{G}(\mathbb{C})$ with continuous restriction to W_F, algebraic restriction to $\mathrm{SL}_2(\mathbb{C})$, such that

$$(\star) \qquad \mathrm{W}_\mathrm{F} \xrightarrow{\rho} {}^L\mathrm{G} \to \mathrm{W} \text{ is the natural projection}$$

(2) (Weil-Deligne parameters): pairs (ρ, N) with $\rho : \mathrm{W}_\mathrm{F} \to {}^L\mathrm{G}(\mathbb{C})$ continuous, $N \in \mathrm{Lie}(\widehat{\mathrm{G}}(\mathbb{C}))$ such that $\mathrm{Ad}(\rho)(w)N = |w|N$, satisfying $(\star)$.

(3) (ℓ-adically continuous parameters): ℓ-adically continuous morphisms $\rho : \mathrm{W}_\mathrm{F} \to {}^L\mathrm{G}(\overline{\mathbb{Q}}_\ell)$, satisfying $(\star)$.

One usually also requires Frobenius semisimplicity, however a posteriori we find it is harmless to ignore this for our purposes. There are bijections between certain conjugacy classes of the different forms of parameters. However their moduli spaces exhibit strikingly different structures. For studying congruences modulo ℓ, it is more natural to consider ℓ-adically continuous parameters.

Moduli of Langlands parameters. To define a moduli space of ℓ-adically continuous parameters over $\mathbb{Z}[1/p]$ we must extend the notion of ℓ-adically continuous to representations over arbitrary $\mathbb{Z}[1/p]$-algebras.

In [8], Helm introduced the following notion: Let R be a $\mathbb{Z}[1/p]$-algebra. A morphism $\rho : \mathrm{W}_\mathrm{F} \to {}^L\mathrm{G}(\mathrm{R})$ is *ℓ-adically continuous* if there exist a morphism of $\mathbb{Z}[1/p]$-algebras $f : \mathrm{R}' \to \mathrm{R}$ with R' ℓ-adically separated; and a morphism $\rho' : \mathrm{W}_\mathrm{F} \to {}^L\mathrm{G}(\mathrm{R}')$ satisfying the following two conditions:

(1) $f \circ \rho' = \rho$;
(2) for all n, $(\rho')^{-1}(\ker({}^L\mathrm{G}(\mathrm{R}') \to {}^L\mathrm{G}(\mathrm{R}'/\ell^n\mathrm{R}')))$ is open in W_F.

However, it is not a priori clear that the functor which associates to a $\mathbb{Z}[1/p]$-algebra the set of ℓ-adically continuous parameters valued in ${}^L\mathrm{G}(\mathrm{R})$ is representable; or even that it is a sheaf for the Zariski topology. We adopt a serpentine route, following Helm [8], the key idea being to *discretize* tame inertia: Let $\mathrm{I}_\mathrm{F}, \mathrm{P}_\mathrm{F}$ denote the inertia and wild inertia subgroups of W_F, and fix a topological generator σ of $\mathrm{I}_\mathrm{F}/\mathrm{P}_\mathrm{F}$ and a geometric Frobenius Fr. Let W_F^0 be the preimage of the subgroup $\langle \mathrm{Fr}, \sigma \rangle$ under the projection $\mathrm{W}_\mathrm{F} \to \mathrm{W}_\mathrm{F}/\mathrm{P}_\mathrm{F}$.

Let $\mathrm{P}_{\mathrm{F}}^{e}$ be a decreasing exhaustive filtration of open normal subgroups of P_{F} with $\mathrm{P}_{\mathrm{F}}^{0} = \mathrm{P}_{\mathrm{F}}$, and define a functor

$$\underline{\mathrm{Z}}^1(\mathrm{W}_{\mathrm{F}}^0/\mathrm{P}_{\mathrm{F}}^e, \widehat{\mathrm{G}}) : \{\mathbb{Z}[1/p]\text{-algebras}\} \to \mathcal{S}et$$
$$\underline{\mathrm{Z}}^1(\mathrm{W}_{\mathrm{F}}^0/\mathrm{P}_{\mathrm{F}}^e, \widehat{\mathrm{G}})(\mathrm{R}) = \underline{\mathrm{Z}}^1(\mathrm{W}_{\mathrm{F}}^0/\mathrm{P}_{\mathrm{F}}^e, \widehat{\mathrm{G}}(\mathrm{R})),$$

with cocycles continuous with respect to the discrete topology on the coefficients. It is representable by a finite type affine scheme $\mathrm{Spec}(\mathcal{R}^e_{{}^L\mathrm{G}})$ and we prove:

Theorem. [3] The ring $\mathcal{R}^e_{{}^L\mathrm{G}}$ is ℓ-adically separated and the universal representation $\rho^e_{{}^L\mathrm{G}} : \mathrm{W}_{\mathrm{F}}^0 \to {}^L\mathrm{G}(\mathcal{R}^e_{{}^L\mathrm{G}})$ extends uniquely to an ℓ-adically continuous representation $\rho^{e,\ell}_{{}^L\mathrm{G}} : \mathrm{W}_{\mathrm{F}} \to {}^L\mathrm{G}(\mathcal{R}^e_{{}^L\mathrm{G}} \otimes \mathbb{Z}_\ell)$. The pair $(\mathcal{R}^e_{{}^L\mathrm{G}} \otimes \mathbb{Z}_\ell, \rho^{e,\ell}_{{}^L\mathrm{G}})$ is universal for all ℓ-adically continuous Langlands parameters whose kernel contains $\mathrm{P}_{\mathrm{F}}^e$.

The integral Bernstein centre. Let R be a $\mathbb{Z}[1/p]$-algebra. Let $\mathrm{Rep}_{\mathrm{R}}(\mathrm{G})$ denote the abelian category of smooth R-representations of G. The *Bernstein centre* $\mathfrak{Z}_{\mathrm{R}}(\mathrm{G})$ of $\mathrm{Rep}_{\mathrm{R}}(\mathrm{G})$ is the endomorphism ring of natural transformations of the identity functor from $\mathrm{Rep}_{\mathrm{R}}(\mathrm{G})$ to itself. It is a commutative ring which acts naturally on all smooth R-representations. If e is an idempotent of $\mathfrak{Z}_{\mathrm{R}}(\mathrm{G})$ then we obtain a decomposition of the category $\mathrm{Rep}_{\mathrm{R}}(\mathrm{G}) = e\mathrm{Rep}_{\mathrm{R}}(\mathrm{G}) \times (1-e)\mathrm{Rep}_{\mathrm{R}}(\mathrm{G})$. If e is a primitive idempotent then $e\mathrm{Rep}_{\mathrm{R}}(\mathrm{G})$ is called a *block* of $\mathrm{Rep}_{\mathrm{R}}(\mathrm{G})$. In this way one decomposes $\mathrm{Rep}_{\mathrm{R}}(\mathrm{G}) = \prod e\mathrm{Rep}_{\mathrm{R}}(\mathrm{G})$.

There is an explicit description of $\mathfrak{Z}_{\overline{\mathbb{Q}}_\ell}(\mathrm{G})$ [2], which includes, of course, the block decomposition of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{G})$, but it is much more intricate integrally. For example, a description of the block decomposition of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathrm{G})$ is currently open; although there has been recent progress by Lanard in depth zero [12] and by Dat on reducing to depth zero (see his report in this volume).

Local langlands in families in depth zero. Following the approach of [9], we can now state our conjectures; these statements are adapted from [4]. We suppose R is contained in $\overline{\mathbb{Q}}_\ell$.

The mild form of the conjecture *local Langlands in families* predicts an intimate connection between moduli of Langlands parameters and the Bernstein centre:

Conjecture. [4] For all $e \geqslant 0$, there exists a unique morphism $(\mathcal{R}^e_{{}^L\mathrm{G}})^{\widehat{\mathrm{G}}} \to \mathfrak{Z}_{\mathrm{R}}(\mathrm{G})$ compatible with the ℓ-adic local Langlands correspondence LL.

Without introducing extra structure, Langlands parameters do not see inside L-packets. Therefore, following Haines in the complex case [6], to refine our conjecture we consider the *stable Bernstein centre* $\mathfrak{Z}^{\mathrm{St}}_{\mathrm{R}}(\mathrm{G})$; that is the subring of $\mathfrak{Z}_{\mathrm{R}}(\mathrm{G})$ of all elements which act uniformly (by the same scalar) on all irreducible $\overline{\mathbb{Q}}_\ell$-representations in the same L-packet. We can now state a refined version of *local Langlands in families*, which for this note we restrict to depth zero. Let $e_0 \in \mathfrak{Z}_{\mathrm{R}}(\mathrm{G})$ be the idempotent corresponding to the depth zero subcategory.

Conjecture. [4] There exists a unique morphism $(\mathcal{R}^0_{{}^L\mathrm{G}})^{\widehat{\mathrm{G}}} \to \mathfrak{Z}_{\mathrm{R}}(\mathrm{G})$ compatible with LL. Moreover, the image identifies with $e_0\mathfrak{Z}^{\mathrm{St}}_{\mathrm{R}}(\mathrm{G})$.

In current work in progress, we establish a refined form of LLIF for split classical p-adic groups (and more generally under assumptions on LL), in arbitrary depth, after inverting a finite set of primes (the *banal* case). However, inverting this set of primes destroys the most interesting congruences and the general case, which we have begun to study in depth zero, is more elusive.

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On reduction to depth 0

Jean-François Dat
(joint work with D. Helm, G. Moss, R. Kurinczuk)

Let F be a non archimedean local field of residue characteristic p and G a reductive group over F. In joint work with Helm, Moss and Kurinczuk, we have introduced a moduli space of "Langlands parameters" over $\bar{\mathbb{Z}}[\frac{1}{p}]$ for G and F, and we have studied it in two steps. The first step is to decompose this space as a disjoint union of subspaces, each of which is induced from the space of *tame* parameters of some auxiliary tamely ramified reductive group. The second step is to study directly the space of tame parameters, which is more explicit. This suggests a parallel framework to study the category $\mathrm{Rep}_{\mathbb{Z}[\frac{1}{p}]}(G(F))$. Namely, it should have a decomposition parametrized similarly as before, and such that each factor can be understood from the depth 0 category of some auxiliary tamely ramified reductive group. The desired decomposition has been constructed when p is large enough, using Yu's types, a strategy of Kaletha, and an exhaustion result of Fintzen. The corresponding auxiliary groups have been identified, but the construction of the desired equivalences of categories remains a challenge.

The Langlands parameters side

Denote by P_F, resp. I_F, the wild inertia subgroup, resp. the inertia subgroup, of the Weil group W_F. For technical reasons having to do with continuity conditions of parameters, we work with a subgroup W_F^0 of W_F in which the tame inertia group has been "discretized". Namely, we fix a topological generator σ of I_F/P_F and define W_F^0 as the inverse image in W_F of the discrete subgroup $\langle\sigma, \mathrm{Fr}\rangle$ of W_F/P_F. Then there is a scheme $\underline{Z}^1(W_F^0, \hat{G})$ over $\mathbb{Z}[\frac{1}{p}]$ that represents the functor $R \mapsto Z^1(W_F^0, \hat{G}(R))$. Here $\hat{G}$ can be any split reductive group over $\mathbb{Z}[\frac{1}{p}]$ equiped with a finite action of W_F^0. When this action is trivial on P_F, the "tame summand" $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is simply given by the following "toy model"

$$Z^1(W_F^0/P_F, \hat{G})(R) = \{(s, F) \in \hat{G}(R)\sigma \times \hat{G}(R)\mathrm{Fr},\ FsF^{-1} = s^q\}.$$

When the action of W_F^0 moreover normalizes a Borel pair of $\hat{G}$, then a direct study shows that this tame summand enjoys very nice properties : it is flat over $\mathbb{Z}[\frac{1}{p}]$, local complete intersection of relative dimension $\dim(\hat{G})$, and reduced.

In order to extend these properties to the whole of $\underline{Z}^1(W_F^0, \hat{G})$, we use the restriction map $\underline{Z}^1(W_F^0, \hat{G}) \longrightarrow \underline{Z}^1(P_F, \hat{G})$ and the remarkable simple structure of the target of this map given as follows.

Proposition. [3] *There is a set* $\Phi \subset \underline{Z}^1(P_F, \hat{G})(\bar{\mathbb{Z}}[\frac{1}{p}])$ *such that*

(i) $\underline{Z}^1(P_F, \hat{G}) = \coprod_{\phi\in\Phi} \hat{G}\cdot\phi$ *with* $\hat{G}\cdot\phi$ *the étale sheaf theoretic orbit of* ϕ.

(ii) $\forall\phi \in \Phi$, *the centralizer* $C_{\hat{G}}(\phi)$ *is smooth over* $\bar{\mathbb{Z}}[1/p]$ *with reductive neutral component and constant* π_0.

Via the restriction map, we thus get a decomposition

$$\underline{Z}^1(W_F^0, \hat{G}) = \coprod_{\phi\in\Phi} \hat{G} \times^{C_{\hat{G}}(\phi)} \underline{Z}^1(W_F^0, \hat{G})_\phi$$

where $\underline{Z}^1(W_F^0, \hat{G})_\phi = \{\varphi \in \underline{Z}^1(W_F^0, \hat{G}),\ \varphi_{|P_F} = \phi\}$ is a closed $C_{\hat{G}}(\phi)$-invariant subscheme of $\underline{Z}^1(W_F^0, \hat{G})$. We say that ϕ is admissible if this scheme is non-empty. Here is the main technical result, in a particular case, that provides a handle on this closed subscheme.

Proposition. [3] *Assume ϕ is admissible and $C_{\hat{G}}(\phi)$ is connected. Then there exists an extension $\varphi \in \underline{Z}^1(W_F^0, \hat{G})_\phi(\bar{\mathbb{Z}}[\frac{1}{p}])$ of ϕ such that the subgroup $\varphi(W_F^0) \subset {}^LG(\bar{\mathbb{Z}}[\frac{1}{p}])$ normalizes a Borel pair of $C_{\hat{G}}(\phi)$.*

Using conjugation inside ${}^LG$, we get from φ as in this proposition a finite action Ad_φ of W_F^0/P_F on $C_{\hat{G}}(\phi)$. Then one easily checks that the map $\eta \mapsto \eta \cdot \varphi$ defines an isomorphism of schemes

$$\underline{Z}^1_{\mathrm{Ad}_\varphi}(W_F^0/P_F^0, C_{\hat{G}}(\phi)) \xrightarrow{\sim} \underline{Z}^1(W_F^0, \hat{G})_\phi,$$

and since Ad_φ preserves a Borel pair of $C_{\hat{G}}(\phi)$, the LHS is an instance of our toy model.

When all centralizers $C_{\hat{G}}(\phi)$ for $\phi \in \Phi$ are connected, we thus have a complete reduction process to toy models. Note that this happens e.g. if $\hat{G}$ is a classical group and $p > 2$, or if p does not divide the order of the Weyl group of $\hat{G}$, in which case $C_{\hat{G}}(\phi)$ is even a Levi subgroup of $\hat{G}$.

In general we can prove the existence of finite sets $\tilde{\Phi}(\phi)$ for each $\phi \in \Phi$ consisting of extensions of ϕ that preserve a Borel pair of $C_{\hat{G}}(\phi)^\circ$ under conjugation inside ${}^LG$, and such that we have a decomposition

$$\underline{Z}^1(W_F^0, \hat{G}) \simeq \coprod_{\phi \in \Phi} \coprod_{\varphi \in \tilde{\Phi}(\phi)} \hat{G} \times^{C_{\hat{G}}(\phi)^\circ} \underline{Z}^1_{\mathrm{Ad}_\varphi}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ).$$

Moreover, we can prove :

Theorem. [3] *It is the decomposition of $\underline{Z}^1(W_F^0, \hat{G})$ into connected components.*

In other words, our toy model is connected. The proof of this result uses similar decompositions as above for each base change $\underline{Z}^1(W_F^0, \hat{G})_{\bar{\mathbb{Z}}_\ell}$ where $\ell \neq p$ is a prime, and where restriction to the prime-to-ℓ inertia subgroup I_F^ℓ is used in place of restriction to P_F. We like to consider it as a Galois analogue of a theorem of Sécherre and Stevens that two irreductible complex representations of $GL_m(D)$ have the same endoclass iff they can be "connected" by a chain of congruences modulo various primes.

The Representation theoretic side

Suppose now that G is reductive over F and denote by $\hat{G}$ its Langlands dual. For simplicity, let us assume that all centralizers $C_{\hat{G}}(\phi)$ are connected, for $\phi \in \Phi$. The above discussion suggests the following expectations :

(1) To each ϕ should correspond an idempotent e^ϕ in the Bernstein center $\mathfrak{Z}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F))$, or equivalently a direct factor subcategory $\mathrm{Rep}^\phi_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F))$ of the category of all smooth $\bar{\mathbb{Z}}[\frac{1}{p}]G(F)$-modules. It should be uniquely determined by its compatibility with LLC whenever the latter is known : the action of e^ϕ on an irreducible $\pi \in \mathrm{Irr}_{\mathbb{C}}(G(F))$ should be 1 if $(\varphi_\pi)_{|P_F} \sim \phi$ and 0 otherwise.

(2) e^{ϕ} should belong to the "stable" Bernstein center $\mathfrak{Z}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F))^{\mathrm{st}}$ and be primitive in this ring. Here, an element of $\mathfrak{Z}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F))$ is called "stable" if its action is constant accross an L-packet of complex representations.

(3) There should be a decomposition $e^{\phi} = \sum_{\alpha} e^{\phi,\alpha}$ as a sum of pairwise orthogonal primitive idempotents in $\mathfrak{Z}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F))$, with α running over characters of the group $\pi_0(Z(C_{\hat{G}}(\phi))^{\varphi(W_F)}/Z(\hat{G})^{W_F})$ (here φ is any extension of ϕ.)

(4) Fix $\phi \in \Phi$ and denote by G_{ϕ} the reductive quasi-split group over F that is dual to $C_{\hat{G}}(\phi)$ endowed with the outer action of $\mathrm{Ad}_{\varphi}(W_F)$ (with φ any extension of ϕ to W_F). Further, let α be as in (3) and use Kottwitz' isomorphims to see α as an element of $H^1(F, G_{\phi})$, whence a pure inner form $G_{\phi,\alpha}$ of G_{ϕ}. Then, there should be an equivalence of categories

$$\mathrm{Rep}^{0}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G_{\phi,\alpha}(F)) \xrightarrow{\sim} \mathrm{Rep}^{\phi,\alpha}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F)).$$

Note that the combination of (3) and (4) entails that the depth 0 idempotent should be primitive in $\mathfrak{Z}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F))$ whenever G is tamely ramified. For $G = GL_n$, this follows from the result of Sécherre and Stevens that we alluded to above. In general, this seems quite accessible.

Here is the main piece of evidence towards the above expectations.

Theorem. [2] *Suppose that p does not divide the order of the Weyl group of G. Then, there is a decomposition*

$$\mathrm{Rep}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F)) = \prod_{\phi,\alpha} \mathrm{Rep}^{\phi,\alpha}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F))$$

which is compatible with parabolic induction and Kaletha's construction of regular supercuspidal L-packets [5].

Actually, our construction of $\mathrm{Rep}^{\phi,\alpha}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F))$ works under the only assumption that $C_{\hat{G}}(\phi)$ is a Levi subgroup plus some technical hypothesis. In this setting, it provides a Serre subcategory but only under the stronger assumption of the theorem can we prove that this Serre subcategory is a direct factor, thanks to a result of Fintzen [4].

Here is a sketch of the construction. Let us assume G quasi-split for simplicity, and fix ϕ such that $C_{\hat{G}}(\phi)$ is a Levi subgroup. We first show that for each α attached to ϕ, there is a canonical $G(F)$-conjugacy class of F-embeddings of $G_{\phi,\alpha}$ as a twisted Levi subgroups of G. Then, fixing such a ι, we use ideas of Kaletha in [5] to define a twisted Levi sequence starting with $\iota(G_{\phi,\alpha})$ and ending with G, together with a sequence of generic characters in Yu's sense. The construction of Yu [7] then associates to any point x in the image of the building $B(G_{\phi,\alpha}, F))$ in $B(G, F)$ a character of some open pro-p-subgroup of the isotropy group $G(F)_x$ of this point. We then use these characters and all the nice intertwining results of Yu to associate to *any* point of the building an idempotent $e_x^{\phi,\alpha}$ of the smooth distribution algebra of $G(F)$ with support in $G(F)_x$, independent of any choice of embedding. The desired category is then defined as

$$\mathrm{Rep}^{\phi,\alpha}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F)) = \left\{ V \in \mathrm{Rep}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G(F)),\ V = \sum_{x\in B(G,F)} e_x^{\phi,\alpha} V \right\}.$$

It turns out that the system of idempotents $(e_x^{\phi,\alpha})_x$ satisfies the Meyer and Solleveld axioms [6] for some finer polysimplicial structure on $B(G,F)$, and this suffices to show that the above category is a Serre subcategory. Moreover, Yu's intertwining result imply that we get pairwise orthogonal subcategories when we vary ϕ and α.

In order to constuct the desired equivalence of categories, we intend to use coefficient systems on the building, following a strategy already used in [1].

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Supercuspidal representations of $\mathbf{GL}_n(F)$ distinguished by a Galois involution

Vincent Sécherre

Let E/F be a separable quadratic extension of non-Archimedean locally compact fields of residue characteristic p, and σ denote its non-trivial automorphism. It is known that any irreducible, smooth complex representation π of $G = \mathrm{GL}_n(E)$ which is G^σ-distinguished, that is, which carries a non-zero G^σ-invariant linear form, satisfies the two following conditions:

(1) its central character $c_\pi : E^\times \to \mathbb{C}^\times$ is trivial on $F^\times$;

(2) its contragredient $\pi^\vee$ is isomorphic to its $\mathrm{Gal}(E/F)$-conjugate π^σ;

the latter point is due to Flicker and Prasad ([6, 13]). It is natural to ask whether or not these two necessary conditions of distinction are sufficient as well.

Note that, when E/F is replaced by a quadratic extension of finite fields, Condition (2) is known to be a necessary and sufficient condition of distinction ([8]) and it implies Condition (1).

In the non-Archimedean case, the situation is more complicated since the norm map:

$$\mathrm{N}_{E/F} : E^\times \to F^\times$$

is not surjective. Let ω denote the unique non-trivial character of $F^\times$ with kernel $\mathrm{N}_{E/F}(E^\times)$.

According to Kable and Anandavardhanan-Kable-Tandon ([1, 11]), if F has characteristic different from 2 and π is a discrete series representation of G such that $\pi^\vee \simeq \pi^\sigma$, then π is either distinguished or ω-distinguished (the latter meaning that the vector space $\mathrm{Hom}_{G^\sigma}(\pi, \omega \circ \det)$ is non-zero), but not both. It follows that, when n is odd, a discrete series representation π is distinguished if and only if Conditions (1) and (2) above are both satisfied.

Assume from now on that π is a supercuspidal representation of G such that $\pi^\vee \simeq \pi^\sigma$ and $p \neq 2$. Thanks to the equality between the Asai L-factors of π and its Langlands parameter ([3, 10]), one gets a necessary and sufficient condition of distinction of π in terms of its Langlands parameter (see [7]).

On the other hand, π can be described by compact induction from an irreducible representation of a compact mod centre subgroup of G, and it is natural to ask for a distinction criterion in terms of the inducing datum. This has been done for tame supercuspidal representations (Hakim-Murnaghan [9], Coniglio [5]) via their parametrization in terms of Howe's admissible pairs.

In the present work ([15]) we obtain a necessary and sufficient condition of distinction for *any* supercuspidal representation of G, via Bushnell-Kutzko's theory of types ([1, 12]). Our method is purely local, and it also works for supercuspidal $\overline{\mathbb{F}}_\ell$-representations with $\ell \neq p$.

Our method is not well-suited for the case where $p = 2$, since we extensively use the fact that the first cohomology set of a group of order 2 in a pro-p-group is trivial.

The case of other involutions is currently investigated by Jiandi Zou: unitary involutions ([16]) and orthogonal involutions.

When π is cuspidal but not supercuspidal, dichotomy fails, that is, a cuspidal $\overline{\mathbb{F}}_\ell$-representation π such that $\pi^\vee \simeq \pi^\sigma$ may be nor distinguished nor ω-distinguished. For instance, this happens when $G = \mathrm{GL}_2(E)$, E/F is unramified and $\ell \neq 2$ divides $q_E + 1$, where q_E is the cardinality of the residue field of E. The cuspidal non-supercuspidal case is currently investigated in a joint work with Kurinczuk and Matringe.

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Epsilon dichotomy for linear models for tempered representations

Miyu Suzuki

Background. Let E/F be a quadratic extension of number fields and $\mathbb{A}_F, \mathbb{A}_E$ be their rings of adeles. Let $X(E:F)$ denote the set of quaternion algebras over F which has an F-subalgebra isomorphic to E. For an element D in $X(E:F)$, we fix an embedding $E \hookrightarrow D$ and regard $H_D = \mathrm{GL}_n(E)$ as a subgroup of $G_D = \mathrm{GL}_n(D)$. Let $G' = \mathrm{GL}_{2n,F}$ be the quasi-split inner form of G_D and π' be the Jacquet-Langlands transfer of π to $G'(\mathbb{A}_F)$. Let π'_E be its base change lift to $G'_E(\mathbb{A}_F) = \mathrm{GL}_{2n}(\mathbb{A}_E)$. In this setting, there is a following conjecture:

Conjecture 1 (Guo-Jacquet conjecture [3])**.** Let $D \in X(E:F)$ and π be an irreducible cuspidal automorphic representation of $G_D(\mathbb{A}_F)$. Assume that π' is cuspidal. If π is H_D-distinguished, then $L(s, \pi, \wedge^2)$ has a pole at $s = 1$ and $L(1/2, \pi'_E)$ is non-zero. If n is odd, the converse also holds.

The $n = 1$ case of this conjecture is the well-known result of Waldspurger [8]. For general n, Feigon, Martin and Whitehouse [2] proved this under some local assumptions by using a relative trace formula.

Local conjecture. Now we switch to the local situation. Let E/F be a quadratic extension of non-archimedean local fields of characteristic 0 and D be a quaternion algebra over F. Fix an embedding of E to D. The groups H_D and G_D are defined as in the global setting. We set

$$(A, d) = \begin{cases} (F, 2) & \text{if } D \text{ is split} \\ (D, 1) & \text{if } D \text{ is division.} \end{cases}$$

so that G_D is isomorphic to $\mathrm{GL}_{nd}(A)$.

In the $n = 1$ case, Saito and Tunnell [7] proved a local analogue of the Guo-Jacquet conjecture. Generalizing their result, Prasad and Takloo-Bighash [6] formulated the following conjecture, which is called ε-dichotomy.

Conjecture 2 ([6] Conjecture 1)**.** Let π be an irreducible tempered representation of G_D. If π is H_D-distinguished, then we have $\varepsilon(\pi'_E) = \varepsilon(D)^n \eta(-1)^n$. Here,

$$\varepsilon(D) = \begin{cases} 1 & \text{if } D \text{ is split} \\ -1 & \text{if } D \text{ is division} \end{cases}$$

is the Hasse invariant of D and η denotes the quadratic character on $F^\times$ associated with E/F via the local class field theory.

The original conjecture is stated for arbitrary inner forms of GL but we only consider quaternionic inner forms.

There are several known cases. Feigon, Martin and Whitehouse [2] proved this when D is split and π is supercuspidal. Chommaux [C19] proved this for Steinberg representations. For discrete series representations, Xue [9] showed this conjecture under the following two working hypotheses (for detail, see [9] section 1.1): full fundamental lemma for Guo's relative trace formula (FL), a relative analogue of Howe's finiteness theorem for p-adic symmetric spaces (H).

Our main result is a reduction of Conjecture 2 to the case of discrete series representations.

Theorem 3. If Conjecture 2 holds for discrete series representations, then it holds for tempered representations.

Classification. Theorem 3 immediately follows from the classification of H_D-distinguished tempered representations.

Proposition 4. Let $\lambda = (n_1, \ldots, n_r)$ be a partition of nd and Δ_i be an irreducible discrete series representation of $\mathrm{GL}_{n_i}(A)$. Suppose that the representation $\pi = \Delta_1 \times \cdots \times \Delta_r$ of G_D is irreducible and tempered. Then π is H_D-distinguished if and only if there is an permutation σ of $\Delta_1, \ldots, \Delta_r$ which satisfies the following conditions:

(i) Δ_i is isomorphic to the contragredient of $\Delta_{\sigma(i)}$;
(ii) if i is a fixed point of σ, n_i is a multiple of d and Δ_i is $\mathrm{GL}_{n_i/d}(E)$-distinguished.

We sketch the proof of this classification. The method we use is developed by Matringe [4]. The main tool is the Mackey theory. Let $\lambda = (n_1, \ldots, n_r)$ be a partition of nd and $P = P_\lambda = M_\lambda N_\lambda$ be the corresponding standard parabolic subgroup of $\mathrm{GL}_{nd}(A) = G_D$. First, we describe the double cosets $P_\lambda \backslash G_D / H_D$ explicitly. Set

$$\mathcal{S}(\lambda) = \left\{ S = (s_{i,j}) \in \mathrm{M}_r(\mathbb{Z}_{\geq 0}) \;\middle|\; \begin{array}{ll} \text{(i)} & S = {}^tS \\ \text{(ii)} & s_{i,i} \in d\mathbb{Z} \\ \text{(iii)} & \sum_{j=1}^r s_{i,j} = n_i \end{array} \right\}.$$

Each element of this set can be regarded as a subpartition of λ. The double cosets $P_\lambda \backslash G_D / H_D$ is parametrized by this set, *i.e.* for $S \in \mathcal{S}(\lambda)$, there is an explicitly given element w_S of G_D so that we have a disjoint decomposition

$$G_D = \coprod_{S \in \mathcal{S}(\lambda)} P_\lambda w_S H_D.$$

Let Δ_i be an irreducible discrete series representation of $\mathrm{GL}_{n_i}(A)$. Suppose that the representation $\pi = \Delta_1 \times \cdots \times \Delta_r$ of G_D is irreducible and tempered. The Mackey theory for p-adic groups provides us:

- a total order $\geq$ on the set $\{w_S\}_{S \in \mathcal{S}(\lambda)}$;
- a filtration $\mathcal{F}_w$ indexed by $\{w_S\}_{S \in \mathcal{S}(\lambda)}$ of π as an H_D-module which satisfies
$$\mathcal{F}_w / \sum_{w' > w} \mathcal{F}_{w'} \cong \mathrm{ind}_{w \cdot P \cap H_D}^{H_D} (w^{-1}(\delta_P^{-1/2} \Delta)).$$

Here, ind denotes the unnormalized compact induction, δ_P is the modulus character of P and $\Delta = \Delta_1 \boxtimes \cdots \boxtimes \Delta_r$ is a representation of M_λ. Suppose that π is H_D-distinguished. Then, there is at least one $w = w_S$ so that

$$\mathrm{Hom}_{H_D}\left(\mathcal{F}_w / \sum_{w' > w} \mathcal{F}_{w'}, \mathbb{C}\right) \neq 0.$$

By Frobenius reciprocity and a computation of modulus characters, we obtain

$$\mathrm{Hom}_{H_D}\left(\mathcal{F}_w / \sum_{w' > w} \mathcal{F}_{w'}, \mathbb{C}\right) \cong \mathrm{Hom}_{M_S^{\theta_S}}(\Delta_{P'_S}, \mathbb{C}).$$

Here,

- θ is an involution of G_D such that the group of its fixed points is H_D;
- $\theta_S = \mathrm{Int}(w_S) \circ \theta \circ \mathrm{Int}(w_S)^{-1}$ is an involution of M_S;
- $P'_S = M_\lambda \cap P_S$ is a parabolic subgroup of M_λ with a Levi subgroup M_S;
- $\Delta_{P'_S}$ is the normalized Jacquet module of Δ with respect to P'_S. This is a representation of M_S.

One implication of Proposition 4 follows from the next lemma.

Lemma 5. In the above situation, S is a monomial matrix. The corresponding permutation σ satisfies the conditions (i) and (ii) of Proposition 4.

The converse implication follows from the open and closed orbits theory of Offen [5].

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Towards the twisted endoscopic character relation for regular supercuspidal *L*-packets

Masao Oi

One fundamental goal in representation theory of p-adic groups is to establish the conjectural local Langlands correspondence, which predicts a natural connection between irreducible smooth representations and L-parameters. Roughly speaking, we can say that there are two big approaches to this problem. First one is an attempt by specifying a p-adic group. Typical successful examples in this direction are represented by Harris–Taylor's construction of the local Langlands correspondence for GL_n ([HT01]) and Arthur's one for quasi-split classical groups ([Art13]). Second one is an approach by restricting a class of representations. Since such an attempt was started by DeBacker–Reeder (depth zero case, [DR09]), a lot of works have been done so far. The latest and biggest result in this direction is Kaletha's construction of the local Langlands correspondence for regular supercuspidal representations ([Kal]). By focusing on so-called *regular supercuspidal representations*, he constructed L-packets consisting of them and their corresponding L-parameters, under an assumption that a p-adic group is tamely-ramified.

The objective of my project is to unify these two different approaches to the local Langlands correspondence. To be more precise, we let F be a p-adic field and $\mathbf{G}$ a connected reductive group over F. We put $\Pi(\mathbf{G})$ (resp. $\Phi(\mathbf{G})$) to be the set of equivalence classes of irreducible smooth representations of $\mathbf{G}(F)$ (resp. $\widehat{\mathbf{G}}$-conjugacy classes of L-parameters of $\mathbf{G}$). We write

$$\mathrm{LLC}^{\mathrm{HT}}_{\mathrm{GL}_n}\colon \Pi(\mathrm{GL}_n) \to \Phi(\mathrm{GL}_n)$$

for Harris–Taylor's local Langlands correspondence for GL_n. Similarly, when $\mathbf{G}$ is a quasi-split classical group, we write

$$\mathrm{LLC}^{\mathrm{Art}}_{\mathbf{G}}\colon \Pi(\mathbf{G}) \to \Phi(\mathbf{G})$$

for Arthur's one. On the other hand, under a small restriction on the residual characteristic p, every classical group can be treated in Kaletha's framework. Let $\mathrm{LLC}^{\mathrm{Kal}}_{\mathbf{G}}$ denote Kaletha's local Langlands correspondence for $\mathbf{G}$ (note that this is defined only on the subset $\Pi^{\mathrm{reg}}_{\mathrm{sc}}(\mathbf{G})$ of $\Pi(\mathbf{G})$ consisting of regular supercuspidal representations). Then we can ask whether Kaletha's local Langlands correspondence coincides with Harris–Taylor's one (resp. Arthur's one) for regular supercuspidal representations of GL_n (resp. quasi-split classical groups).

Let us first introduce our result on GL_n-case:

Theorem 1 (joint work with Kazuki Tokimoto)**.** Every $\pi \in \Pi^{\mathrm{reg}}_{\mathrm{sc}}(\mathrm{GL}_n)$ satisfies

$$\mathrm{LLC}^{\mathrm{Kal}}_{\mathrm{GL}_n}(\pi) = \mathrm{LLC}^{\mathrm{HT}}_{\mathrm{GL}_n}(\pi).$$

Let me explain a very rough outline of the proof of this result. The key is Tam's result on Bushnell–Henniart's rectifier ([Tam16]). In GL_n-case, regular supercuspidal representations are nothing but essentially tame supercuspidal representations in the sense of Bushnell–Henniart. Such representations can be parametrized by pairs (E, η) consisting of degree n tamely-ramified extension of F and an admissible character η of $E^\times$. Then Bushnell–Henniart showed that the essentially tame supercuspidal representation $\pi_{(E,\eta)}$ $\mathrm{LLC}^{\mathrm{HT}}_{\mathrm{GL}_n}$ corresponding to a pair (E, η) maps to $\mathrm{Ind}^F_E(\eta\mu^{-1}_{\mathrm{rec}})$ under $\mathrm{LLC}^{\mathrm{HT}}_{\mathrm{GL}_n}$, where μ_{rec} is a tamely ramified character of $E^\times$ called the *rectifier* of (E, η). On the other hand, Kaletha's construction of $\mathrm{LLC}^{\mathrm{Kal}}_{\mathrm{GL}_n}$ is given as follows. First, we regard $E^\times$ as the set of F-valued points of an elliptic maximal torus $\mathbf{S}$ of GL_n and consider its L-parameter $\mathrm{LLC}_{\mathbf{S}}(\eta)$ determined by the local Langlands correspondence for tori. Second, we take an L-embedding ι from the L-group of $\mathbf{S}$ to that of GL_n according to Langlands–Shelstad's construction. Then we can regard $\iota \circ \mathrm{LLC}_{\mathbf{S}}(\eta)$ as an L-parameter of GL_n. What Tam did in [Tam16] is rewriting the character μ_{rec} in terms of Langlands–Shelstad's construction of L-embedding. Thus we can say that the basic philosophy of Tam's earlier result is the same as Kaletha's one. However, we have to be careful about the following points:

- Kaletha's construction of $\mathrm{LLC}^{\mathrm{Kal}}_{\mathrm{GL}_n}$ requires an additional twist. It maps $\pi_{(E,\eta\epsilon)}$ to $\iota \circ \mathrm{LLC}_{\mathbf{S}}(\eta)$ for certain character ϵ.
- Tam used a different set of χ-data, which is an auxiliary choice needed for Langlands–Shelstad's construction, to Kaletha's set of χ-data.

If we understand these points, then our task is reduced to checking that the "difference" between Tam's χ-data and Kaletha's one is given by ϵ. It follows from a little complicated, but not theoretically difficult calculation on relative root systems.

Next I would like to explain the present status of my project for the case of quasi-split classical groups. Let $\mathbf{G}$ be a quasi-split classical group over F in the following. Recall that $\mathrm{LLC}^{\mathrm{Art}}_{\mathbf{G}}$ is characterized by the *endoscopic character relation.* More precisely, $\mathbf{G}$ is a twisted endoscopic group for GL_n for a suitable n. Thus, for $\pi \in \Pi(\mathbf{G})$, its corresponding L-parameter $\phi := \mathrm{LLC}^{\mathrm{Art}}_{\mathbf{G}}(\pi)$ can be regarded as an L-parameter of GL_n. Thus we get $\pi^{\mathrm{HT}}_{\mathrm{GL}_n}(\phi) := \mathrm{LLC}^{\mathrm{HT},-1}_{\mathrm{GL}_n}(\phi) \in \Pi(\mathrm{GL}_n)$. In this situation, the endoscopic character relation is formulated as an equality between the twisted character of $\pi^{\mathrm{HT}}_{\mathrm{GL}_n}(\phi)$ and the sum of characters of representations belonging to the L-packet $\Pi^{\mathrm{Art}}_{\mathbf{G}}(\phi) := \mathrm{LLC}^{\mathrm{Art},-1}_{\mathbf{G}}(\phi)$.

Keeping this in our mind, our problem can be reformulated in the following way. Let $\phi \in \Phi(\mathbf{G})$. Then our final goal is to show the coincidence of L-packets $\Pi_{\mathbf{G}}^{\mathrm{Art}}(\phi)$ and $\Pi_{\mathbf{G}}^{\mathrm{Kal}}(\phi) := \mathrm{LLC}_{\mathbf{G}}^{\mathrm{Kal},-1}(\phi)$. To show this, by considering the characterization of $\mathrm{LLC}_{\mathbf{G}}^{\mathrm{Art}}$, it suffices to prove the endoscopic character relation for $\Pi_{\mathbf{G}}^{\mathrm{Kal}}(\phi)$ and $\pi_{\mathrm{GL}_n}^{\mathrm{Kal}}(\phi)$ (note that we already proved that $\mathrm{LLC}_{\mathrm{GL}_n}^{\mathrm{HT}} = \mathrm{LLC}_{\mathrm{GL}_n}^{\mathrm{Kal}}$).

In fact, in [Kal], he proved the character relation in the case of standard endoscopy. Thus what I am trying to do now is to imitate his strategy in the twisted setting. Kaletha's proof can be summarized (very roughly) as follows:

- by using Adler–DeBacker–Spice's character formula for supercuspidal representations ([AS08, AS09, DS18]), express the value of the character at a given element in terms of
  - a root of unity defined by DeBacker–Spice and
  - the Fourier transform of an orbital integral on the Lie algebra,
- rewrite the DeBacker–Spice's root of unity by using the transfer factor, and
- apply the transfer of the Fourier transforms of orbital integrals, which was established by Ngô and Waldspurger.

Up to now, by focusing on elements which are p-adically near to the unit element, I checked that a twisted version of Adler–DeBacker–Spice's formula holds. In this case, the character formula does not contain a root of unity part. Moreover, if we furthermore specify a group $\mathbf{G}$ and supercuspidal representations, we can carry out the above strategy by using Waldspurgers transfer for non-standard endoscopy ([Wal08]) instead of Ngô–Waldspurger's standard endoscopic transfer. In other words, at least in an extremely specialized setting, I checked that the same strategy as in Kaletha's proof works in the twisted setting. Although it is too specialized, my result so far can be written as follows:

Theorem 2. Let $\mathbf{G}$ be SO_{2n+1} over F. We assume that $\pi_{\mathrm{GL}_n}^{\mathrm{Kal}}(\phi)$ and every member of $\Pi_{\mathbf{G}}^{\mathrm{Kal}}(\phi)$ are toral of depth $r > 0$. Moreover, we assume that the tori associated to these toral supercuspidal representations are totally tamely-ramified. Then, for every $\tilde{g} \in \mathrm{GL}_{2n}(F)$ and $g \in \mathbf{G}(F)$ such that

- g is a norm of $\tilde{g}$,
- g and h have "no head" in the sense of Adler–Spice's r-approximation.

Then we have

$$\widetilde{\Theta}_{\pi_{\mathrm{GL}_n}^{\mathrm{Kal}}(\phi)}(\tilde{g}) = \sum_{\pi \in \Pi_{\mathbf{G}}^{\mathrm{Kal}}(\phi)} \Theta_{\pi}(g),$$

where the left-hand side is the twisted character of $\pi_{\mathrm{GL}_n}^{\mathrm{Kal}}(\phi)$ and the right-hand side is the sum of the characters of representations belonging to $\Pi_{\mathbf{G}}^{\mathrm{Kal}}(\phi)$.

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The local Langlands correspondence and ramification

Colin J. Bushnell

(joint work with Guy Henniart)

This is part of a project aimed at rendering the local Langlands correspondence into an effective and explicit tool. Here, we introduce the ideas of [bh17] and [bh19]. See [b19] for a broad account of the background.

Notation. Let F be a non-Archimedean, locally compact field of residual characteristic p and $\mathcal{W}_F$ the Weil group of F, relative to a chosen separable algebraic closure of F. Let $\widehat{\mathcal{W}}_F$ be the set of equivalence classes of irreducible, smooth, complex representations of $\mathcal{W}_F$. For an integer $n \geq 1$, let $\widehat{\mathrm{GL}}_F(n)$ be the set of equivalence classes irreducible, cuspidal, complex representations of the general linear group $\mathrm{GL}_n(F)$ and write $\widehat{\mathrm{GL}}_F = \bigcup_{n\geq 1} \widehat{\mathrm{GL}}_F(n)$. The Langlands correspondence gives a bijection $\widehat{\mathrm{GL}}_F \to \widehat{\mathcal{W}}_F$ that we denote by $\pi \mapsto {}^L\pi$.

Endo-classes and wild ramification. A representation $\pi \in \widehat{\mathrm{GL}}_F(n)$ contains a *simple character* θ_π, in the sense of [bk93]. The conjugacy class of θ_π is uniquely determined by π [bh13]. The class of all simple characters in all groups $\mathrm{GL}_n(F)$, $n \geq 1$, is subject to an equivalence relation called *endo-equivalence* [bh96]. Let $\mathcal{E}(F)$ be the set of these equivalence classes (called "endo-classes"). If $\varTheta(\pi)$ denotes the endo-class of θ_π, then $\pi \mapsto \varTheta(\pi)$ is a canonical surjection $\widehat{\mathrm{GL}}_F \to \mathcal{E}(F)$. The endo-class $\varTheta(\pi)$ and the integer n, such that $\pi \in \widehat{\mathrm{GL}}_F(n)$, together determine the conjugacy class of θ_π in $\mathrm{GL}_n(F)$.

Let $\mathcal{P}_F$ be the wild inertia subgroup of $\mathcal{W}_F$ and let $\widehat{\mathcal{P}}_F$ be the set of classes of irreducible smooth representations of $\mathcal{P}_F$. If $\sigma \in \widehat{\mathcal{W}}_F$, let $[\sigma]_0^+ \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$ be the conjugacy class of an irreducible component of $\sigma \,\big|\, \mathcal{P}_F$. The map $\sigma \mapsto [\sigma]_0^+$ provides a canonical surjection $\widehat{\mathcal{W}}_F \to \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$. The Langlands correspondence induces a unique bijection $\mathcal{E}(F) \to \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, denoted $\Theta \mapsto {}^L\Theta$, such that

$$\left[{}^L\pi\right]_0^+ = {}^L\Theta(\pi), \quad \pi \in \widehat{\mathrm{GL}}_F,$$

[bh03]. As in [bh14], the Langlands correspondence can be (almost) completely re-constructed from the map $\Theta \mapsto {}^L\Theta$.

Ultrametrics and conductors. The set $\mathcal{E}(F)$ carries a canonical *ultrametric,* denoted by $(\Theta_1,\Theta_2) \mapsto \mathbb{A}(\Theta_1,\Theta_2)$ [bh17]. Thus $\mathbb{A}$ is a symmetric pairing, with non-negative real values, such that

$$\mathbb{A}(\Theta_1,\Theta_3) \le \max\{\mathbb{A}(\Theta_1,\Theta_2),\mathbb{A}(\Theta_2,\Theta_3)\}, \quad \Theta_j \in \mathcal{E}(F),$$

and $\mathbb{A}(\Theta_1,\Theta_2) = 0$ if and only if $\Theta_1 = \Theta_2$. Given $\Theta \in \mathcal{E}(F)$, there is a unique continuous function Φ_Θ such that

$$\frac{\mathrm{sw}(\check\pi \times \rho)}{nm} = \Phi_\Theta\Big(\mathbb{A}\big(\Theta,\Theta(\rho)\big)\Big),$$

for any $\pi \in \widehat{\mathrm{GL}}_F(n)$ with $\Theta(\pi) = \Theta$ and any $\rho \in \widehat{\mathrm{GL}}_F(m)$. Here, $\mathrm{sw}(\check\pi \times \rho)$ is the Swan exponent defined via the local constant $\varepsilon(\check\pi \times \rho, s, \psi)$ of [jpss], [s].

The function Φ_Θ is increasing, convex and piecewise linear. Its derivative has only finitely many discontinuities. The conductor formula of [bhk98] yields an explicit formula for Φ_Θ [bh17]. This is in terms of the basic invariants of a simple stratum underlying a simple character of endo-class Θ, in the manner of [bk93].

There are parallel structures on the Galois side. For $x \ge 0$, let $\mathcal{R}_F(x)$ be the ramification subgroup of $\mathcal{W}_F$ at x: that is, $\mathcal{R}_F(x) = \mathcal{W}_F^x$ in traditional notation. Let $\mathcal{R}_F^+(x)$ be the closure of $\bigcup_{y>x}\mathcal{R}_F(y)$. Thus $\mathcal{R}_F(0)$ is the inertia subgroup of $\mathcal{W}_F$ and $\mathcal{R}_F^+(0) = \mathcal{P}_F$. For $\sigma,\tau \in \widehat{\mathcal{W}}_F$, set

$$\Delta(\sigma,\tau) = \inf\{x \ge 0 : \mathrm{Hom}_{\mathcal{R}_F(x)}(\sigma,\tau) \ne 0\}.$$

The value of $\Delta(\sigma,\tau)$ depends only on the classes $[\sigma]_0^+, [\tau]_0^+ \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, and Δ induces an *ultrametric* on this orbit space.

For $\sigma \in \widehat{\mathcal{W}}_F$, there is a unique continuous function Σ_σ with the following property. If $\tau \in \widehat{\mathcal{W}}_F$, then

$$\frac{\mathrm{sw}(\check\sigma \otimes \tau)}{\dim(\sigma)\,\dim(\tau)} = \Sigma_\sigma\Big(\Delta\big(\sigma,\tau\big)\Big).$$

Here, sw denotes the usual Swan exponent of a representation of $\mathcal{W}_F$. The function Σ_σ depends only on $[\sigma]_0^+$. It is increasing, convex and piecewise linear. Its derivative has only finitely many discontinuities, called the *jumps* of Σ_σ. If x is not a jump of Σ_σ, the derivative satisfies

$$\Sigma_\sigma'(x) = \dim\mathrm{End}_{\mathcal{R}_F(x)}(\sigma)\big/(\dim\sigma)^2$$

In particular, j is a jump if and only if the algebras $\mathrm{End}_{\mathcal{R}_F(j)}(\sigma)$, $\mathrm{End}_{\mathcal{R}_F^+(j)}(\sigma)$ have different dimensions.

The function Σ_σ may be given by an explicit formula [bh17]. It is a fundamental, and informative, invariant of σ, encapsulating a *canonical* presentation of σ in group-theoretic terms: see 3.4 Theorem of [bh19] for the first steps. (Remark here that the ramification sequence $\{\mathcal{R}_F(x)\}_{x\geq 0}$ is an *arithmetic* construction, not a purely group-theoretic one.)

The Herbrand function. For $\Theta \in \mathcal{E}(F)$, set $\xi = {}^L\Theta \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$ and define

$$\Psi_\Theta = \Phi_\Theta^{-1} \circ \Sigma_\xi.$$

The Langlands correspondence preserves Swan exponents of pairs, so we have the relation

$$\mathbb{A}(\Theta, \Upsilon) = \Psi_\Theta\big(\Delta(\xi, {}^L\Upsilon)\big), \quad \Upsilon \in \mathcal{E}(F).$$

If $\Theta = \Theta(\pi)$, where $\pi \in \widehat{\mathrm{GL}}_F$, and $\sigma = {}^L\pi$, we can unambiguously label any of these functions by π or σ, for example, $\Psi_\Theta = \Psi_\pi = \Psi_\sigma$. The definitions readily imply that $\Phi_\sigma(x) = \Sigma_\sigma(x) = \Psi_\sigma(x) = x$ in the range $x \geq \mathrm{sw}(\sigma)/\dim\sigma$.

The factor Φ_Θ is easily described in terms of the structure theory of [bk93]. The critical point is that *the function Ψ_Θ can be calculated directly in terms of Θ, without recourse to any knowledge of the Langlands correspondence.* The calculation is not straightforward but, when feasible, it determines the function Σ_ξ explicitly.

We outline the method. First, if K/F is a finite, tamely ramified field extension, there is a canonical surjection $\mathfrak{i}_{K/F} : \mathcal{E}(K) \to \mathcal{E}(F)$ [bh96]. If $\Theta \in \mathcal{E}(F)$, the fibre $\mathfrak{i}_{K/F}^{-1}(\Theta)$ is finite. Its elements are the *K/F-lifts of Θ.*

Lifting property. Let K/F be a finite tame extension with $e = e(K|F)$, and let $\Theta \in \mathcal{E}(F)$. If $\Theta^K \in \mathcal{E}(K)$ is a K/F-lift of Θ, then

$$\Psi_\Theta(x) = e^{-1}\Psi_{\Theta^K}(ex), \quad x \geq 0.$$

(Note that the factors Φ_Θ and $\Sigma_{{}^L\Theta}$ do not, individually, have a tidy property of this kind.) The lifting property reduces the problem of computing Ψ_Θ to the case where Θ is *totally wild,* in the sense that Θ has a *unique* K/F-lift for any finite tame extension K/F.

The group of characters χ of $F^\times$ acts on $\mathcal{E}(F)$ in a natural way: if $\Theta = \Theta(\pi)$, for some $\pi \in \widehat{\mathrm{GL}}_F$, then $\chi\Theta = \Theta(\chi\pi)$, where $\chi\pi$ is the usual twist of π by χ. We remark in passing that $\Psi_{\chi\Theta} = \Psi_\Theta$.

From [bh17], we have:

Interpolation Theorem. Let $\Theta \in \mathcal{E}(F)$ be totally wild. There exists a finite set D_Θ of positive real numbers with the following property. Let K/F be a finite tame extension, and put $e = e(K|F)$. If χ is a character of $K^\times$ such that $e^{-1}\mathrm{sw}(\chi) \notin D_\Theta$, then

$$\mathbb{A}_K(\Theta^K, \chi\Theta^K) = e\Psi_\Theta(e^{-1}\mathrm{sw}(\chi)),$$

where $\mathbb{A}_K$ is the canonical ultrametric on $\mathcal{E}(K)$.

The result gives $\Psi_\sigma(x)$ at a set of points x that is dense on the positive real line. Since Ψ_σ is continuous, this is enough to determine it completely.

Carayol representations. Let $\sigma \in \widehat{\mathcal{W}}_F$. Say that σ is *totally wild* (and write $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$) if $\sigma\,\big|\,\mathcal{P}_F$ is irreducible. In particular, if $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$, then $[\sigma]_0^+ = {}^L\Theta$, where $\Theta \in \mathcal{E}(F)$ is totally wild in the above sense.

Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ and put $\mathrm{sw}(\sigma) = m$. In particular, $\dim\sigma = p^r$, for a non-negative integer r. Say that σ is *of Carayol type* if $r \geq 1$ and the positive integer m is not divisible by p. We follow [bh19] to compute Ψ_σ.

There are some preliminary steps. If $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ is of Carayol type and dimension p^r, then

$$\begin{aligned} \Phi_\sigma(0) &= \Sigma_\sigma(0) = m(p^r-1)/p^{2r}; \\ \Phi_\sigma(x) &= \Sigma_\sigma(x) = x, \quad x \geq m/p^r; \\ \Phi'_\sigma(x) &= p^{-r}, \qquad 0 < x < m/p^r. \end{aligned}$$

In particular, it is easy to recover Σ_σ from Ψ_σ. The graph $y = \Psi_\sigma(x)$ is interesting only in the range $0 < x < m/p^r$. There, it is a convex increasing function.

Symmetry Theorem. The graph $y = \Psi_\sigma(x)$, $0 \leq x \leq m/p^r$, is symmetric with respect to the line $x+y = m/p^r$.

This result, taken from [bh19], implies a full description of the ramification structure of σ. It exhibits striking degrees of rigidity and symmetry, as described in section 9 of [bh19].

To determine Ψ_σ, we return to the endo-class $\Theta_\sigma \in \mathcal{E}(F)$, such that $[\sigma]_0^+ = {}^L\Theta_\sigma$, and describe it following the scheme of [bk93]. Fix a character μ_F of F inflated from a non-trivial character of the residue field of F. Set $M = \mathrm{M}_{p^r}(F)$ and $\mu_M = \mu_F \circ \mathrm{tr}_M$. Consider a simple stratum $[\mathfrak{a}, m, 0, \alpha]$ in M, such that the field extension $E = F[\alpha]/F$ is totally ramified of degree p^r and p does not divide m. Set $H^1(\alpha, \mathfrak{a}) = U_E^1 U_{\mathfrak{a}}^{1+[m/2]}$. The set $\mathcal{C}(\mathfrak{a}, \alpha)$ of simple characters attached to $[\mathfrak{a}, m, 0, \alpha]$ then consists of all characters ϕ of $H^1(\alpha, \mathfrak{a})$ such that

$$\phi(x) = \mu_M(\alpha(x-1)), \quad x \in U_{\mathfrak{a}}^{1+[m/2]}.$$

We define a certain non-empty subset of $\mathcal{C}(\mathfrak{a}, \alpha)$. Let $w_{E/F}$ be the *wild exponent* of E/F: by definition, $w_{E/F} = d_{E/F}-p^r+1$, where $d_{E/F}$ is the differental exponent of E/F. Write $l_\alpha = \max\{0, m-w_{E/F}\}$. Let $\mathcal{C}^\star(\mathfrak{a}, \alpha)$ be the set of those $\phi \in \mathcal{C}(\mathfrak{a}, \alpha)$ such that $\phi(1+y) = \mu_M(\alpha y)$, for all $y \in E$ satisfying $\upsilon_E(y) \geq 1+[l_\alpha/2]$.

Continuing with the same notation, let $\psi_{E/F}$ be the classical (convex) Herbrand function of the (not necessarily Galois) field extension E/F. We state the main result of [bh19].

Explicit formula. There exists a simple stratum $[\mathfrak{a}, m, 0, \alpha]$ in M such that $E = F[\alpha]/F$ is totally ramified of degree p^r and Θ_σ is the endo-class of some $\theta \in \mathcal{C}^\star(\mathfrak{a}, \alpha)$. For any such α, we have $m = \mathrm{sw}(\sigma)$ and

$$\Psi_\sigma(x) = p^{-r}\psi_{E/F}(x)$$

in the region $0 \leq x \leq m/p^r$, $x+\Psi_\sigma(x) \leq m/p^r$.

Taken together with the Symmetry Theorem, this result determines Ψ_σ completely.

Concluding remarks. The rigidly inductive structures of [bk93] demand that, in the problem to hand, the special case of Carayol representations be treated first.

The symmetry property is essentially characteristic of Carayol representations. Let $\tau \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ and suppose that $m = \mathrm{sw}(\tau) \le \mathrm{sw}(\chi \otimes \tau)$, for all characters χ of $\mathcal{W}_F$. The graph $y = \Psi_\tau(x)$, $0 \le x \le m/p^r$, is symmetric relative to $x+y = m/p^r$ if and only if τ is of Carayol type.

Consider the class of representations $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ of dimension p^2, for which $\mathrm{sw}(\sigma)$ is divisible by p but not by p^2. These provide the first examples of non-Carayol representations. Their Herbrand functions are *never* convex but, given the detailed results on Carayol representations, they are often easy to compute (details to appear).

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Langlands functoriality in the geometrization of the local Langlands correspondence

Naoki Imai

Recently, L. Fargues formulates a conjecture on the geometrization of the local Langlands correspondence with P. Scholze (*cf.* [1], [2]).

Let F be a p-adic number field with residue field $\mathbb{F}_q$. Let G be a quasi-split reductive group over F. Then we can define a moduli stack Bun_G of G-bundle on the Fargues–Fontaine curve attached to F. For a discrete Langlands parameter

$\varphi\colon W_F \to {}^LG$, the geometrization conjecture predicts the existence of a conjectural "perverse Weil sheaf" $\mathscr{F}_\varphi$ on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ satisfying the Hecke eigensheaf property for any cocharacter μ of G. Assume that φ is cuspidal in the sequel. Then we can construct $\mathscr{F}_\varphi$ in the case where the local Langlands correspondence is constructed. The conjecture is known if G is torus by [1], if $G = \mathrm{GL}_2$ and μ is minuscule by [3].

In this talk, we show the following theorem:

Theorem ([4]). Assume that $G = \mathrm{GL}_3$ and μ is minuscule. Then $\mathscr{F}_\varphi$ satisfies the Hecke eigensheaf property on the semi-stable locus.

We explain a consequence of this theorem. Let ϖ be a uniformizer of F. We put

$$b = MAT\begin{pmatrix} 0 & 0 & 1 \\ \varpi & 0 & 0 \\ 0 & \varpi & 0 \end{pmatrix}, \quad b' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varpi & 0 & 0 \end{pmatrix}.$$

Let J_b and $J_{b'}$ denote the inner form of GL_3 attached to b and b' respectively. Assume that

$$\mu(z) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathcal{M}_{b,b'}^{\leq\mu}$ be the moduli space of modifications between $\mathscr{E}_b$ and $\mathscr{E}_{b'}$ bounded by μ, where $\mathscr{E}_b$ and $\mathscr{E}_{b'}$ denote the vector bundle of rank 3 on the Fargues–Fontaine curve attached to b and b' respectively. Let ℓ be a prime number different from p. Then the above theorem implies that the ℓ-adic etale cohomology of $\mathcal{M}_{b,b'}^{\leq\mu}$ realizes the local Jacquet–Langlands transfer between $J_b(F)$ and $J_{b'}(F)$, and the local Langlands correspondences for them.

We explain also about a joint work in progress with L. Fargues and I. Gaisin on the formulation of a conjecture realizing an automorphic induction from an unramified maximal torus in the setting of the geometrization of the local Langlands correspondence. We will construct a p-adic analogues of Deligne–Lusztig variety by imitating the construction of Deligne–Lusztig variety and using the Fargues–Fontaine curve.

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Cohomological representations of parahoric subgroups

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(joint work with Alexander Ivanov)

Deligne–Lusztig varieties [DL76] are subvarieties of flag varieties whose cohomology encodes the representations of reductive groups $\mathbb{G}_0$ over a finite field $\mathbb{F}_q$. In a remarkable series of works culminating in [L84], Lusztig wrote down the character tables for $\mathbb{G}_0(\mathbb{F}_q)$. Today, this remains the only known construction of all irreducible representations of $\mathbb{G}_0(\mathbb{F}_q)$.

For a reductive group G over a non-archimedean local field k with residue field $\mathbb{F}_q$, the *depth-zero* supercuspidal representations of $G(k)$ are inductions from $Z_G(k)G_{x,0}(\mathcal{O}_k)$ of representations π of $\mathbb{G}_0(\mathbb{F}_q) := G_{x,0}(\mathcal{O}_k)/G_{x,0+}(\mathcal{O}_k)$:

$$\mathrm{Ind}_{Z_G(k)G_{x,0}(\mathcal{O}_k)}^{G(k)}(\pi).$$

Hence depth-zero supercuspidal representations of $G(k)$ come from the *geometric* input of Deligne–Lusztig varieties. Building on the work of many people [G75, MP94, A98], Yu [Y01] gave a very general construction of supercuspidal representations of $G(k)$ by combining the geometric depth-zero input with an *algebraic* recipe to obtain positive-depth representations. In this talk, I explained recent progress in constructing positive-depth supercuspidal representations *geometrically* by considering an analogue of Deligne–Lusztig theory for $\mathbb{G}_r(\mathbb{F}_q) := G_{x,0}(\mathcal{O}_k)/G_{x,r+}(\mathcal{O}_k)$ where r is a non-negative integer and $G_{x,r+}$ comes from the Moy–Prasad filtration [MP94].

Let G be a reductive group over k, let $G \subset T$ be a maximal k-rational torus, and assume that T (and therefore G) splits over the maximal unramified extension $\breve{k}$. Let σ be a q-Frobenius associated to the k-rational structure on G and pick any σ-fixed point x in the apartment of T. Let U be the unipotent radical of a $\breve{k}$-rational Borel subgroup of G containing T. We write $\mathbb{T}_r := (T \cap G_{x,0})/(T \cap G_{x,r+})$ and $\mathbb{U}_r := (U \cap G_{x,0})/(U \cap G_{x,r+})$. In joint work with A. Ivanov [CI19], we consider the varieties

$$X_r := \{g \in \mathbb{G}_r : g^{-1}\sigma(g) \in \mathbb{U}_r\}$$

which are endowed with a natural action of $\mathbb{G}_r(\mathbb{F}_q) \times \mathbb{T}_r(\mathbb{F}_q)$. Here, the $r = 0$ case is the setting of classical Deligne–Lusztig theory. The central results of this talk were the following two:

Theorem 1 (Chan–Ivanov)**.** Let $\theta : \mathbb{T}_r(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ be sufficiently generic. Then

$$H_c^*(X_r, \overline{\mathbb{Q}}_\ell)[\theta] \text{ is an irreducible } \mathbb{G}_r(\mathbb{F}_q)\text{-representation.}$$

Theorem 2 (Chan–Ivanov)**.** Let $\theta : \mathbb{T}_r(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ be any character. If $g \in \mathbb{T}_r(\mathbb{F}_q)$ is very regular, then

$$\mathrm{Tr}\Big(g; H_c^*(X_r, \overline{\mathbb{Q}}_\ell)[\theta]\Big) = \sum_{w \in W} (\theta \circ \mathrm{Ad}(w))(g),$$

where W is the Weyl group for $G_{x,0}(\mathcal{O}_k)$.

In the setting that $G_{x,0}$ is reductive (i.e. x is hyperspecial), Theorem 1 was proved by Lusztig [L04], and our *sufficiently generic* assumption is the same as in *op. cit.* From experience, establishing the character formula on very regular elements (Theorem 2) is often enough to determine the representation. I'll now discuss an example of this by explaining a related geometric construction for p-adic groups called *loop Deligne–Lusztig varieties* (Lusztig [L79], Boyarchenko [B12], Chan [C19], Chan–Ivanov [CI18]).

In 1979, Lusztig conjectured [L79] that the loop Deligne–Lusztig set

$$X := \{g \in G : g^{-1}\sigma(g) \in U\}/(U \cap \sigma^1(U)).$$

should have the structure of an infinite-dimensional variety over $\overline{\mathbb{F}}_q$. Furthermore, he conjectured that for a fixed character $\theta\colon G(k) \to \overline{\mathbb{Q}}_\ell^\times$, the subspace $H_i(X, \overline{\mathbb{Q}}_\ell)[\theta]$ wherein $G(k)$ acts by θ should vanish for large i and should be concentrated in a single cohomological degree if θ is in general position. Today, these conjectures remain almost completely open, primarily because it is not known how to give X an algebro-geometric structure. In the setting when G is a division algebra and T is elliptic, Lusztig gave an *ad hoc* solution to this issue (later formalized by Boyarchenko). When G is any inner form of GL_n and T is again elliptic, Ivanov and I resolved this in [CI18], studied its relation to affine Deligne–Lusztig varieties at infinite level, and proved that it is an infinite disjoint union (indexed by $G(k)/G_{x,0}(\mathcal{O}_k)$) of copies of $\varprojlim_r X_r$. The representation-theoretic consequence of this is that for depth-r characters θ,

$$H_*(X, \overline{\mathbb{Q}}_\ell)[\theta] = \mathrm{Ind}_{Z_G(k)G_{x,0}(\mathcal{O}_k)}^{G(k)}(H_c^*(X_r, \overline{\mathbb{Q}}_\ell)[\theta]).$$

Using purely geometric and purely local methods, we prove in *op. cit.* that for *sufficiently generic* θ, the representation $H_*(X, \overline{\mathbb{Q}}_\ell)[\theta]$ is irreducible and therefore supercuspidal. It is at this point that Theorem 2 assumes its crucial role: the compatibility of $\theta \mapsto H_*(X, \overline{\mathbb{Q}}_\ell)[\theta]$ with local Langlands and Jacquet–Langlands now follows after invoking Henniart's characterization [H92, H93] of the local Langlands correspondence for GL_n using very regular elements.

I ended the talk with a discussion of recent and ongoing work with Masao Oi studying a comparison of Yu's algebraic construction with a variant of X_r. At this point, we can prove that the cohomology of this variant realizes the L-packets of *unramified toral representations* for general $G(k)$ appearing in [DS18].

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Representations of depth 0 and endoscopy

J.-L. Waldspurger

Let F be a non-archimedean local field of characteristic 0. Denote p the residual characteristic and val_F the usual valuation. Let G be a connected reductive group over F. I assume

$$p \geq c(G)(1 + val_F(p)),$$

where $c(G)$ is some constant. We denote $\mathfrak{g}$ the Lie algebra of G and we fix an algebraic closure $\bar{F}$ of F.

Definition. $\epsilon \in G(F)$ is a p'-element if and only if ϵ is semi-simple and, for each eigenvalue α of $ad(\epsilon)$ in $\mathfrak{g}(\bar{F})$ such that $val_{\bar{F}}(\alpha) = 0$, α is a root of unity of order prime to p.

For such ϵ, denote G_ϵ the identity component of the centralizer of ϵ in G, $\mathfrak{g}_\epsilon$ its Lie algebra, $\mathfrak{g}_{\epsilon,tn}$ the set of topologically nilpotent elements of $\mathfrak{g}_\epsilon$, $Nil(\mathfrak{g}_\epsilon(F))$ the set of nilpotent conjugacy classes in $\mathfrak{g}_\epsilon(F)$ and, for each $\mathcal{O} \in Nil(\mathfrak{g}_\epsilon(F))$, $\hat{j}(\mathcal{O}, .)$ the function giving the Fourier transform of the orbital integral over $\mathcal{O}$.

Denote $I(G)^*$ the space of invariant distributions on $G(F)$.

Definition. $D \in I(G)^*$ is a quasi-character of depth 0 if and only if the two following properties are satisfied:
(a) D is locally integrable, so given by a function θ_D;
(b) for each p'-element ϵ, there exists a family of complex numbers $(c_{\mathcal{O}})_{\mathcal{O}\in Nil(\mathfrak{g}_\epsilon(F))}$ such that, for almost all $X \in \mathfrak{g}_{\epsilon,tn}(F)$, we have

$$\theta_D(\epsilon exp(X)) = \sum_{\mathcal{O}\in Nil(\mathfrak{g}_\epsilon(F))} c_{\mathcal{O}}\hat{j}(\mathcal{O}, X).$$

Let $Irr(G)$ be the set of isomorphism classes of irreductible admissible representations of $G(F)$ in complex spaces and let $Irr^0(G)$ be the subset of representations of depth 0. For any set E, we denote $\mathbb{C}[E]$ the complex space with basis E. Let $p^0 : \mathbb{C}[Irr(G)] \to \mathbb{C}[Irr^0(G)]$ be the obvious projection and let $\Theta : \mathbb{C}[Irr(G)] \to I(G)^*$ be the map which, to a representation π, associates its character Θ_π.

Theoreme. For $\pi \in \mathbb{C}[Irr(G)]$, we have $\pi \in \mathbb{C}[Irr^0(G)]$ if and only if Θ_π is a quasi-character of depth 0.

Assume G is quasi-split. Denote $\mathbb{C}[Irr(G)]^{st}$ the subspace of the $\pi \in \mathbb{C}[Irr(G)]$ for which Θ_π is a stable distribution. Arthur has defined a projection $p^{st} : \mathbb{C}[Irr(G)] \to \mathbb{C}[Irr(G)]^{st}$.

Theoreme. We have $p^0 \circ p^{st} = p^{st} \circ p^0$.

For general G, let $\mathbf{G}' = (G', s, \mathcal{G}')$ be an elliptic endoscopic datum of G. Assume to simplify no auxiliary data are needed, that is we can defined a transfer factor Δ on $G'(F) \times G(F)$. Then Arthur has defined the spectral transfer

$$transfer : \mathbb{C}[Irr(G')]^{st} \to \mathbb{C}[Irr(G)].$$

Theoreme. We have $p^{0,G} \circ transfer = transfer \circ p^{0,G'}$, with an obvious notation.

To each facet of the reduced Bruhat-Tits' building of G, we can associate a connected reductive group $G_{\mathcal{F}}$ over the residual field $\mathbb{F}_q$ and, more generally, some twisted spaces $G^\nu_{\mathcal{F}}$ under this group. The space of quasi-characters of depth 0 is strongly related to the functions on the spaces $G^\nu_{\mathcal{F}}(\mathbb{F}_q)$ which are the characteristic functions of character-sheaves on $G^\nu_{\mathcal{F}}$. Then it seems natural to consider the following space. Denote $S(G)$ the set of vertices of the reduced Bruhat-Tits' building of G. For each $s \in S(G)$, we define the parahoric lattice $\mathfrak{k}_s \subset \mathfrak{g}(F)$ and a sublattice $\mathfrak{k}_s^+$ such that $\mathfrak{k}_s/\mathfrak{k}_s^+ \simeq \mathfrak{g}_s(\mathbb{F}_q)$ (here G_s is $G_{\mathcal{F}}$ for $\mathcal{F} = \{s\}$ and $\mathfrak{g}_s$ is its Lie algebra). Let $FC(\mathfrak{g}_s(\mathbb{F}_q))$ be the space of functions on $\mathfrak{g}_s(\mathbb{F}_q)$ generated by the characteristic functions of character-sheaves on $\mathfrak{g}_s$ which are cuspidal, with nilpotent support and fixed by the Galois-action. The space $FC(\mathfrak{g}_s(\mathbb{F}_q))$ can be considered as a subspace of $C_c^\infty(\mathfrak{g}(F))$. We define

$$FC(\mathfrak{g}(F)) = \sum_{s\in S(G)} FC(\mathfrak{g}_s(\mathbb{F}_q)).$$

I have presented some results similar to the preceedings theorems saying more or less that the spaces $FC(\mathfrak{g}(F))$ behave well under endoscopy.

Theta correspondence for symplectic–orthogonal and metaplectic–orthogonal p-adic dual pairs

Marcela Hanzer
(joint work with Petar Bakić)

One of the very active areas of research in the local representation theory, as well as in the global setting, is the classical theta correspondence. Globally, theta correspondence gives us, by the integration against the theta kernel, one of the few direct ways to construct new automorphic forms. We will dedicate ourselves here to the local situation. Theta correspondence has its origins in the work of Howe on dual reductive pairs ([6, 7]) and of Weil ([25]). Namely, Howe was considering dual reductive pairs, i.e., the pairs of subgroups which are mutual commutants inside a symplectic group. On the other side, Weil constructed, motivated by the aim to reformulate Siegel's analytic theory of quadratic forms in the group theoretic setting, the metaplectic group, i.e. a unique non-trivial two-fold cover of an symplectic group and a distinguished representation of it, the Weil representation. We are now turning our attention to reductive (mostly classical or their covers) groups defined over p-adic fields, and all the representation we are considering are over the complex numbers. The Weil representation of a metaplectic group is infinite-dimensional, but small (actually, more precisely, minimal representation) in terms of Gelfand-Kirillov dimension; so it also allows that line of treatment, but we are not going to pursue that here; we just note that its "minimality" dictates that the branching on the specific subgroups does not behave wildly. The dual reductive pairs in a symplectic group can split in the metaplectic cover or they might not. We are interested in the most studied case of dual reductive pairs: namely (G, H) where G is a symplectic group $Sp(W)$, H is $O(V)$–an orthogonal group (or vice versa), and they form a dual reductive pair in the symplectic group $Sp(W \otimes V)$. If the dimension of V is even, both $O(V)$ and $Sp(W)$ split in the metaplectic cover $Mp(W \otimes V)$ of $Sp(W \otimes V)$ so that their lifts commute in $Mp(W \otimes V)$. We can then view the metaplectic representation ω_ψ of $Mp(W \otimes V)$ (it depends on the choice of an additive character ψ) as a representation of $Sp(W) \times O(V)$. If the dimension of V is odd, we view it as a representation of $Mp(W) \times O(V)$. Having this in mind, we denote this (restricted) representation of the dual reductive pair as $\omega_{W,V,\psi}$.

Let (G, H) denote one of these pairs (note that we allow G to be either metaplectic/symplectic or orthogonal). Now we explain in more detail what kind of branching we consider by theta correspondence.

For any irreducible admissible representation π of G we may look at the maximal π-isotypic quotient of $\omega_{W,V,\psi}$. We denote it $\Theta(\pi, V)$ and call it the full theta lift of π to V. This representation, when non-zero, has a unique irreducible quotient, denoted $\theta(\pi, V)$—the small theta lift of π. This basic fact, called the Howe duality conjecture, was first formulated by Howe [6], proven by Waldspurger [24] (for odd residue characteristic) and by Gan and Takeda [5] in general.

The Howe duality establishes a map $\pi \mapsto \theta(\pi, V)$ which is called the theta correspondence. The study of theta correspondence was further developed by Kudla [8, 9], Rallis [22], Kudla-Rallis [10], Mœglin-Vigneras-Waldspurger [18], Waldspurger [24] and many others. In recent years, there were many significants developments: e.g., the conservation conjecture is proved by Sun and Zhu [23] (we extensively use it in our proofs: namely, say π is an irreducible representation of a symplectic group $Sp(W)$. Then we can consider lifts of π on the two orthogonal towers simultaneously-these two towers are comprised of the quadratic spaces which have even dimension, and the same discriminant, and are distinguished by their Hasse invariant. Then, if π occurs "early" in one tower (i.e., has a non-zero theta lift early), then it occurs "late" in the other tower. This property is made explicit by the conservation conjecture). The research then moved to the determination of the theta lifts explicitly, firstly if the lifted representation is of a certain sort (e.g. cuspidal, discrete series, etc. [20], [21]). Of course, the question relayed also on the lack of the classification data for the discrete series representations for classical groups themselves. This is resolved by [17], but up to some assumptions, which are proved to be true by the seminal work of Arthur [2] (up to some questions about the stabilization of twisted trace formulas). Building on the Arthur's work, and previous work of Gan and Ichino [4], Atobe and Gan ([3]) obtained full description of the theta lifts of tempered representations of the dual reductive pairs of our sort (they also cover the unitary dual reductive pairs). The papers of Loke and Ma, Savin [12],[13], have a bit different flavor, as they use Yu's classification of (parts of) supercuspidal representations to describe their (supercuspidal) lifts.

Relying heavily on [3], in this work we describe the theta lifts (in terms of their Langlands parameters) of any irreducible admissible representation π (also given by its Langlands parameter) of a group in our dual pair and, in this way, we completely describe the Howe correspondence using the Langlands parametrization. We formulate our results using the concept of "ladder representations" of general linear groups ([11]) to describe important parts of the Langlands parameter of the representation π which we are lifting and which affect the first occurrence index of this representation and the form of the lifts. The proof has a bit of an algorithmic flavor and parts of it resemble the Mœglin-Waldspurger algorithm for finding Aubert dual of an irreducible representation of a general linear group [19]. Now, with the explicit Howe's correspondence at hand on one side, and Mœglin's parametrization of the local Arthur's packets on the other ([15],[16]) one might ask what is the current state of affairs of the Adams conjecture (cf. [1],[14]) which aims to describe the relation between the Arthur packets of two representations which are each other's theta lifts?

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A homological version of the restriction problem from $\mathbf{GL}_{n+1}$ to $\mathbf{GL}_n$.

Gordan Savin
(joint work with Kei Yuen Chan)

Let F be a p-adic field, and O its ring of integers. If π is an irreducible representation of $\mathrm{GL}_n(F)$, let $\mathrm{Wh}(\pi)$ be the space of Whittaker functionals on π. It is well known that the dimension of $\mathrm{Wh}(\pi)$ is 1 or 0, and we say that π is generic or degenerate, respectively. Let π_1 and π_2 be a pair of irreducible representations of $\mathrm{GL}_{n+1}(F)$ and $\mathrm{GL}_n(F)$, respectively. It is also a well known but difficult result that the space $\mathrm{Hom}_{\mathrm{GL}_n}(\pi_1, \pi_2)$ is at most one-dimensional. Moreover, it is one dimensional if π_1 and π_2 are both generic. On the other hand, Dipendra Prasad has proved the following beautiful formula for the Euler-Poincaré characteristic

$$\mathrm{EP}(\pi_1, \pi_2) := \sum_{i=0}^{n} (-1)^i \dim(\mathrm{Ext}^i_{\mathrm{GL}_n(F)}(\pi_1, \pi_2)) = \dim(\mathrm{Wh}(\pi_1)) \cdot \dim(\mathrm{Wh}(\pi_2))$$

without relaying on the multiplicity one results. Based on this formula, he conjectured that $\mathrm{Ext}^i(\pi_1, \pi_2) = 0$ for $i > 0$, if π_1 and π_2 are generic, since in this case $\mathrm{Hom}_{\mathrm{GL}_n}(\pi_1, \pi_2)$ is one-dimensional. Chan and I have proved this conjecture, but there is more to it, that I would like to discuss here. One possible explanation for vanishing of higher extension spaces would be projectivity of π_1, when restricted to $\mathrm{GL}_n(F)$. For example, cuspidal π_1 are projective when restricted to $\mathrm{GL}_n(F)$. However, this is too much to hope for in general. If π_2 is a degenerate quotient of π_1 then $\mathrm{EP}(\pi_1, \pi_2) = 0$, so there must be $i > 0$ such that $\mathrm{Ext}^i_{\mathrm{GL}_n(F)}(\pi_1, \pi_2) \neq 0$ to offset the contribution of $\mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi_1, \pi_2)$. In particular, if π_1 is projective then only generic π_2 may appear as quotients. We proved that the converse is true, thus having only generic quotients is also a sufficient condition for π_1 to be projective. Moreover, we proved that this condition is true for square integrable (modulo center) π_1, so these representations are projective when restricted to $\mathrm{GL}_n(F)$.

Let us explain how some of this work is done. Let $\omega = \mathrm{ind}_U^{\mathrm{GL}_n}(\psi)$ be the Gelfand-Graev representation of $\mathrm{GL}_n(F)$, where U is the subgroup of unipotent, upper-triangular matrices, and ψ a Whittaker character of U. Our first result is the description of Bernstein components of ω in terms of the corresponding Hecke algebras. For simplicity of exposition, let's work here with the Bernstein component of ω generated by ω^I, the subspace of I-fixed vectors, where I is an Iwahori subgroup of $\mathrm{GL}_n(F)$, which we may assume to be contained in $\mathrm{GL}_n(O)$, as is customary. Let H the Hecke algebra of compactly supported I-biinvaraint functions on $\mathrm{GL}_n(F)$. The algebra H acts naturally on ω^I, and describing the Iwahori component of ω is equivalent to describing the H-module ω^I. The Hecke algebra H admits a Bernstein decomposition

$$H \cong A \otimes H_0$$

where $A = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $H_0 \cong \mathbb{C}[S_n]$ is the subalgebra of functions supported on $\mathrm{GL}_n(O)$. The center Z of H is contained in A as the subalgebra of

S_n-invariant polynomials. It is known that $H_0 \cong \mathbb{C}[S_n]$, where S_n is the symmetric group of permutations. Hence H_0 has two one-dimensional representations, 1 and ϵ, corresponding to the trivial and the sign representations of S_n, respectively. The following hold for ω:

- Bernstein components of ω are finitely generated.
- ω is projective.
- Irreducible quotients of ω appear with multiplicity one.

The three bullets is all one needs to describe the H-module ω^I. Indeed, the first two bullets imply that $\omega^I \cong A^r$, as an A-module, and the third, with some work, that $r = 1$. A bit more work shows that any H-module isomorphic to A, as an A-module, must be isomorphic to $H \otimes_{H_0} 1$ or $H \otimes_{H_0} \epsilon$. Since the Steinberg representation is a quotient of ω and it does not contain the trivial H_0-type, it follows that $\omega^I \cong H \otimes_{H_0} \epsilon$. These soft arguments extend to all Hecke algebras corresponding to Bushnell-Kutzko types, giving a similar description of all Bernstein components of ω.

Now let's go back to the restriction problem. Let π_1 be an irreducible generic representation of $\mathrm{GL}_{n+1}(F)$. Let π_2 be a generic representation of $\mathrm{GL}_n(F)$, generated by π_2^I. Then π_2^I is an irreducible H-module and let J be the annihilator of π_2^I in the center $Z \subseteq H$. By the Schur lemma, J is a maximal ideal in Z. Since π_1 is generic, the restriction of π_1 to $\mathrm{GL}_n(F)$ contains the Gelfand-Graev representation ω of $\mathrm{GL}_n(F)$. If π_2^I is the unique irreducible quotient of π_1^I annihilated by J then, as a consequence of the Nakayama lemma, π_1 is locally isomorphic to ω, that is, we have an isomorphism of J-adic completions $\hat{\pi}_1^I \cong \hat{\omega}^I$ of H-modules. In particular,

$$\hat{\pi}_1^I \cong \hat{H} \otimes_{H_0} \epsilon,$$

and we have local projectivity. If this is true for every J, then by abstract nonsense π_1^I is a projective H-module, in fact, it is isomorphic to $H \otimes_{H_0} \epsilon$. This completes (a sketch of) the proof that having only generic quotients is also a sufficient condition on π_1 to be projective.

Multiplicity formula for spherical varieties

Chen Wan

Let F be a local field of characteristic zero, G be a connected reductive group defined over F, and H be a closed subgroup of G (not necessary reductive). Assume that H is a spherical subgroup of G (i.e. the Borel subgroup B of G acts with a Zariski open orbit on $X = G/H$). Let π be an irreducible smooth representation of $G(F)$ and χ be a character of $H(F)$, we want to study the multiplicity

$$m(\pi, \chi) = dim(Hom_{H(F)}(\pi, \chi)).$$

More specifically, we want to prove a multiplicity formula

$$m(\pi, \chi) = m_{geom}(\pi, \chi)$$

where $m_{geom}(\pi, \chi)$ is defined via the Harish-Chandra character θ_π of π and it is called the *geometric multiplicity.*

The first known case for this problem is the Whittaker model case proved by Rodier [Rod] for split groups, by Moeglin-Waldspurger ([MW]) for quasi-split groups, and by Matumoto ([Mat]) for the archimedean case. 30 years after Rodier's result, Waldspurger developed a new method to study this problem. His idea is to prove a local trace formula $I_{geom}(f) = I(f) = I_{spec}(f)$ for the model (G, H), which would imply the multiplicity formula $m(\pi, \chi) = m_{geom}(\pi, \chi)$. In his pioneering works [Wal1] and [Wal2], Waldspurger applied this method to the orthogonal Gan-Gross-Prasad models. By proving the local trace formula and the multiplicity formula, he was able to show that for the orthogonal Gan-Gross-Prasad model, the summation of the multiplicities is always equal to 1 for all tempered local Vogan L-packets. Later his idea was adapted by Beuzart-Plessis [Beu1], [Beu2] for the unitary Gan-Gross-Prasad model, and by myself [Wan1], [Wan2] for the Ginzburg-Rallis model. Subsequently, in [Beu3], Beuzart-Plessis applied this method to the Galois model; in a joint work with Beuzart-Plessis [BW], we applied this method to the generalized Shalika model; and in a joint work with Zhang [WZ], we applied this method to the unitary Ginzburg-Rallis model.

Remark 1. Although all the cases above were proved by the trace formula method, the proofs of the trace formula, especially the geometric side, have each time been done in some ad hoc way pertaining to the particular features of the case at hand.

In [Wan3], based on the previous works mentioned above, I defined the geometric multiplicity $m_{geom}(\pi, \chi)$ (as well as the geometric side of the local trace formula) for general spherical varieties. As a result, I formulated the conjectural multiplicity formula and trace formula for general spherical varieties. For the rest of this report, I will briefly explain the definition of the geometric multiplicity.

Let (G, H) be a spherical pair and $\chi : H(F) \to \mathbb{C}^\times$ be a character. For simplicity, we assume that G has trivial center. Let π be an irreducible smooth representation of $G(F)$. We want to define the geometric multiplicity $m_{geom}(\pi, \chi)$. Let us first recall the finite group case. If both G and H are finite groups, we have

$$m(\pi, \chi) = m_{geom}(\pi, \chi) := \frac{1}{|H|} \sum_{h \in H} \theta_\pi(h)\chi^{-1}(h) = \sum_x \frac{1}{|Z_H(x)|}\theta_\pi(x)\chi^{-1}(x).$$

Here the second summation is over a set of representatives of conjugacy classes of H and $Z_H(x)$ is the centralizer of x in H.

Guided by the finite group case and all the known cases, it is natural to expect that for general spherical pair (G, H), $m_{geom}(\pi, \chi)$ should be an integral over certain semisimple conjugacy classes of $H(F)$ of the product of the Harish-Chandra character θ_π and the character χ^{-1}. However, compared with the finite group case, there are three difficulties in the definition of $m_{geom}(\pi, \chi)$ for general spherical varieties over local field.

First, we need to define the support (i.e. a subset of semisimple conjugacy classes of $H(F)$) of the geometric multiplicity. In the finite group case, the support of geometric multiplicity contains all the conjugacy classes of H. But this will not be the case for spherical varieties over local field. For general spherical varieties, the geometric multiplicity is only supported on those "elliptic conjugacy classes" whose centralizers in $G(F)$ and $H(F)$ form a *minimal spherical variety* and whose centralizer in $G(F)$ is quasi-split. Here we say a spherical variety $X = G/H$ is minimal if the stabilizer of the open Borel orbit is finite modulo the center. The quasi-split condition provides the existence of the regular germs of the Harish-Chandra character, while the minimal spherical variety condition ensures that the "homogeneous degree" of the spherical variety is equal to the homogeneous degree of the regular germs of the Harish-Chandra character. We refer the readers to Section 3 of [Wan3] for details. We use $\mathcal{T}(G,H)$ to denote the set of these conjugacy classes.

Secondly, unlike the finite group case, the Harish-Chandra character θ_π is only defined on the set of regular semisimple elements of $G(F)$. On the other hand, many semisimple conjugacy classes in $\mathcal{T}(G,H)$ are not regular in $G(F)$ which means that θ_π is not defined in those conjugacy classes. In order to solve this issue, we need to use the germ expansion for θ_π. Roughly speaking, near every semisimple element (not necessarily regular) of $G(F)$, θ_π can be written as a linear combination of the Fourier transform of the nilpotent orbital integrals. The coefficients associated to regular nilpotent orbits in this linear combination are called the regular germs of θ_π. In order to define θ_π at non-regular semisimple conjugacy classes, we need to use the regular germs of θ_π. This creates the second difficulty: in general when $F \neq \mathbb{C}$, we may have more than one F-rational regular nilpotent orbits (an easy example of this would be the Whittaker model for $\mathrm{SL}_2(\mathbb{R})$ where there are two regular nilpotent orbits). Hence for each $t \in \mathcal{T}(G,H)$, we need to define a subset of regular nilpotent orbits of $\mathfrak{g}_t(F)$ (G_t is the neutral component of the centralizer of t in G and $\mathfrak{g}_t$ is the Lie algebra of G_t) whose regular germs appear in the geometric multiplicity. In [Wan3], we solved this issue by studying the conjugacy classes of the tangent space of the spherical varieties G_t/H_t. We refer the readers to Section 5 of [Wan3] for more details. We use $\mathcal{N}(G_t, H_t, \chi)$ to denote this subset of regular nilpotent orbits.

Thirdly, in the finite group case, we normalize the character θ_π by the number $\frac{1}{|Z_H(x)|}$. For general spherical varieties, we would need an extra number $d(G_t, H_t, F)$ ($t \in \mathcal{T}(G,H)$) which characterizes how the $G_t(\bar{F})$-conjugacy class (i.e. stable conjugacy class) in the tangent space of G_t/H_t decomposes into $H_t(F)$-conjugacy classes. To be specific, we define

$$d(G_t, H_t, F) = |ker(H^1(F, H_t) \to H^1(F, G_t))| \times \frac{|W_{G_t}|}{|W(X_t)|}$$

where $W(X_t)$ is the little Weyl group of the spherical variety $X_t = G_t/H_t$ and W_{G_t} is the Weyl group of $G_t(\bar{F})$. By Proposition 4.11 of [Wan3], every quasi-split stable conjugacy class in the tangent space of G_t/H_t decomposes into $d(G_t, H_t, F)$ many $H_t(F)$-conjugacy classes.

After we solved the three difficulties above, we can define the geometric multiplicity to be

$$m_{geom}(\pi,\chi)=\int_{\mathcal{T}(G,H)}\chi^{-1}(t)D^{H}(t)\frac{d(G_t,H_t,F)}{|Z_H(t)(F):H_t(F)|\times c(G_t,H_t,F)}$$
$$\times\frac{1}{|\mathcal{N}(G_t,H_t,\chi)|}\sum_{\mathcal{O}\in\mathcal{N}(G_t,H_t,\chi)}c_{\theta_\pi,\mathcal{O}}(t)dt.$$

Here dt is some measure on $\mathcal{T}(G,H)$, and the number $\frac{1}{|Z_H(t)(F):H_t(F)|\times c(G_t,H_t,F)}$ is an analogue of the number $\frac{1}{Z_H(x)}$ in the finite group case. We refer the readers to Section 6 of [Wan3] for more details. In Section 7 of [Wan3], I proved that for all the cases where the multiplicity formula has been proved, including the Whittaker model, Gan-Gross-Prasad model, Ginzburg-Rallis model, Galois model and Shalika model, my definition of the geometric multiplicity recovers the one in the known multiplicity formula. In particular, the conjectural multiplicity formula and trace formula hold for all these models.

Remark 2. Unlike the finite group case, the multiplicity formula $m(\pi,\chi)=m_{geom}(\pi,\chi)$ will not hold for all irreducible smooth representations of $G(F)$. For example, in the Shalika model case, the multiplicity formula only holds for all the discrete series and for almost all tempered representations (Remark 3.4 of [BW]).

In general, the multiplicity formula should always hold for all supercuspdial representations. When the spherical pair is tempered, it should hold for all discrete series and for almost all tempered representations. When the spherical pair is strongly tempered, it should hold for all tempered representations. Moreover, if we want to make the multiplicity formula holds for all irreducible smooth representations of $G(F)$, we need to replace the multiplicity $m(\pi,\chi)$ by the Euler-Poincaré pairing $EP(\pi,\chi)$. We refer the readers to Section 6 of [Wan3] for more details.

Remark 3. Both the definitions of multiplicity and geometric multiplicity also make sense when χ is a finite dimensional representation of $H(F)$ (we just need to replace χ^{-1} in the definition of $m_{geom}(\pi,\chi)$ by the distribution character $\theta_{\chi^\vee}$ of the dual representation of χ).

When F is p-adic, this is not interesting since finite dimensional representations of $H(F)$ are essentially characters. The case we are interested in is when $F=\mathbb{R}$ and $H(\mathbb{R})=K$ is a maximal connected compact subgroup of $G(\mathbb{R})$. In this case, $m(\pi,\chi)=m_{geom}(\pi,\chi)$ gives a multiplicity formula of K-types for all the irreducible smooth representations of $G(\mathbb{R})$ (note that since $H(\mathbb{R})$ is compact, we have $m(\pi,\chi)=EP(\pi,\chi)$ for all π). In [Wan3], I proved this multiplicity formula of K-types for the general linear groups and for all the complex reductive groups.

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Explicit Plancherel formula for the space of alternating forms

Erez Lapid
(joint work with Omer Offen)

We give an explicit Plancherel formula for the space of non-degenerate alternating bilinear forms on a $2n$-dimensional vector space over a local field F, in terms of the Plancherel formula for $G' = \mathrm{GL}_n(F)$.

The Plancherel decomposition for general real reductive symmetric spaces is one of the greatest achievement of harmonic analysis in the post Harish-Chandra era. It is the combined effort of many mathematicians which was completed around the turn of the century. (See [3] and the references therein.) In the non-archimedean case, a completely explicit Plancherel decomposition (even in the group case) is unwieldy at best. However, several years ago, Sakellaridis–Venkatesh made remarkable conjectures on the L^2-decomposition of p-adic symmetric spaces, and more generally, spherical varieties [9]. Under these conjectures, the Plancherel measure is expected to be supported on representations of Arthur's type pertaining to the L-group of the spherical variety, originally defined by Gaitsgory–Nadler [4]. Sakellaridis–Venkatesh essentially reduced their conjectures to the discrete part. However, so far, there has been little progress as far as the discrete part is concerned.

Recently, Beuzart-Plessis embarked on a different approach to study the spectrum of certain symmetric spaces [2].

We follow his footsteps in obtaining the Plancherel decomposition of $L^2(H\backslash G)$ where $G = \mathrm{GL}_{2n}(F)$ and $H = \mathrm{Sp}_n(F)$ is the symplectic group, both over a local field F [8]. It will be in terms of the Plancherel formula for $G' = \mathrm{GL}_n(F)$, as explained below. This is arguably the easiest, non-trivial higher rank case of the conjectures of [9].

Denote by $\mathrm{Irr}_{\mathrm{temp}}G'$ the set of irreducible tempered representations of G', up to equivalence. Fix a Haar measure dg for G' and let μ_{pl} be the Plancherel measure on $\mathrm{Irr}_{\mathrm{temp}}G'$ [11], characterized by the relation

$$f(e) = \int_{\mathrm{Irr}_{\mathrm{temp}}G'} \mathrm{tr}\,\pi(f)\ d\mu_{\mathrm{pl}}(\pi), \quad f \in \mathcal{S}(G').$$

For any $\pi \in \mathrm{Irr}_{\mathrm{temp}}G'$ let $\sigma = \mathfrak{S}(\pi)$ be the Langlands quotient of the representation $I_P(\pi)$ parabolically induced from $\pi\,|\det|^{\frac{1}{2}} \otimes \pi\,|\det|^{-\frac{1}{2}}$ with respect to the standard maximal parabolic subgroup $P = M \ltimes U$ of type (n, n). The representation σ is irreducible and unitarizable.

We realize σ on its Zelevinsky model $\mathfrak{M}_{\psi_N}(\sigma)$ ([12, §8]). Namely, let N be the maximal unipotent subgroup of G consisting of upper unitriangular matrices. Let ψ_N be a character on N that is trivial on U and whose restriction to $N \cap M$ is non-degenerate. Then, up to a constant there exists a unique (N, ψ_N)-equivariant functional on σ, and this gives rise to a unique realization $\mathfrak{M}_{\psi_N}(\sigma)$ of σ in the space of left (N, ψ_N)-equivariant functions on G.

In [7], a G-invariant inner product for general Speh representations was defined, generalizing the classical Bernstein inner product for the Whittaker model of tempered representations [1]. In the case at hand it is given by the convergent integral

$$[W_1, W_2] = \int_{D\cap N\backslash D} W_1(g)\overline{W_2(g)}\ dg, \quad W_1, W_2 \in \mathfrak{M}_{\psi_N}(\sigma)$$

where D be the subgroup of matrices in G whose n-th and $2n$-th rows are those of the identity matrix.

We also define an H-invariant functional on $\mathfrak{M}_{\psi_N}(\sigma)$ by the convergent integral

$$\ell_H(W) = \int_{N\cap H\backslash Q\cap H} W(h)\ dh, \quad W \in \mathfrak{M}_{\psi_N}(\sigma)$$

where Q be the mirabolic subgroup of G, whose elements are those whose last row is $(0, \dots, 0, 1)$, and we assume, as we may, that ψ_N is trivial on $N \cap H$. Moreover, any other H-invariant functional is proportional to ℓ_H.

The inner product $[\cdot, \cdot]$ and the H-invariant functional ℓ_H are the data used to normalize the H-spherical trace of σ. The latter is defined by

$$(f_1, f_2)_\sigma = \sum_v \ell_H(\sigma(f_1)v)\overline{\ell_H(\sigma(f_2)v)}$$

for any $f_1, f_2 \in \mathcal{S}(G)$ where v ranges over a suitable orthonormal basis of $\mathfrak{M}_{\psi_N}(\sigma)$. Since ℓ_H is H-invariant, the positive semi-definite hermitian form $(f_1, f_2)_\sigma$ factors through the canonical map $\mathcal{S}(G) \to \mathcal{S}(H\backslash G)$. We continue to denote the resulting form on $\mathcal{S}(H\backslash G)$ by $(\cdot,\cdot)_\sigma$.

Our main result is the following.

Theorem. For a suitable choice of Haar measures and for any $\varphi_1, \varphi_2 \in \mathcal{S}(H\backslash G)$ we have

$$(\varphi_1,\varphi_2)_{L^2(H\backslash G)} = \int_{\mathrm{Irr}_{\mathrm{temp}}(G')} (\varphi_1,\varphi_2)_{\mathfrak{S}(\pi)}\ d\mu_{\mathrm{pl}}(\pi) \tag{1}$$

where the right-hand side is an absolutely convergent integral.

Consequently, we have the following decomposition of unitary representations of G:

$$L^2(H\backslash G) \simeq \int_{\mathrm{Irr}_{\mathrm{temp}}(G')} \mathfrak{S}(\pi)\ d\mu_{\mathrm{pl}}(\pi).$$

In particular, an irreducible representation σ of G is relatively discrete series with respect to $H\backslash G$ if and only if $\sigma = \mathfrak{S}(\pi)$ for a square-integrable π.

We remark that the fact that $\mathfrak{S}(\pi)$ is relatively discrete series for any square-integrable π had been proved by Jacquet (unpublished) and independently by Smith [10].

One can give an alternative non-trivial G-invariant pairing on $\mathfrak{M}_{\psi_N}(\sigma)$ by realizing it as the image of the intertwining operator on $I_P(\pi)$. It turns out that up to an explicit sign, this coincides with the inner product $[\cdot,\cdot]$ defined above [7, Appendix A]. An analogous phenomenon occurs for ℓ_H. Namely, we can relate it to a an H-invariant functional on $I_P(\pi)$, via the intertwining operator (cf. [6]). These two facts are at the heart of the proof of Theorem . A related argument was given in [5].

Let $\tilde{H}$ be the symplectic similitude group and let χ be a unitary character of F^*, viewed as a character of $\tilde{H}$ via the similitude factor. Theorem can also be formulated for the space $L^2(\tilde{H}\backslash G;\chi)$ of $(\tilde{H},\chi)$-equivariant functions on G that are square-integrable modulo $\tilde{H}$. The Plancherel formula will be in terms of the Plancherel formula for the space $L^2(Z\backslash G';\chi)$ of (Z,χ)-equivariant functions on G' that are square-integrable modulo the center Z of G'.

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L-functions and Hamiltonian duality

Yiannis Sakellaridis
(joint work with David Ben-Zvi, Akshay Venkatesh)

In this talk, I outlined ongoing joint work with David Ben-Zvi and Akshay Venkatesh, aiming to interpret automorphic L-functions in the setting of the geometric Langlands program.

Automorphic L-functions appear as the output of automorphic period integrals and related analytic constructions. A basic example is provided by a spherical variety X, satisfying certain conditions, under the action of a reductive group G. Over a local field F, the unitary representation $L^2(X(F))$ is conjectured to admit a Plancherel decomposition in terms of Langlands parameters into the dual group $\check{G}_X$ of X (assuming, for simplicity, that G is split), introduced in work of Gaitsgory–Nadler [4], Sakellaridis–Venkatesh [9], and Knop–Schalke [6]. The Plancherel decomposition of the "basic function" $\Phi_0 \in L^2(X(F))$ (which coincides with the characteristic function of $X(\mathfrak{o})$ when X is smooth and F is non-Archimedean with ring of integers $\mathfrak{o}$) coincides with a standard measure on the space of Langlands parameters, times an L-value, computed in some generality in [7]. This L-value is determined by a graded representation $\check{G}_X \to \mathrm{GL}(\check{V}_X)$. Globally, period integrals related to X are conjectured to be Euler products, whose local factors coincide with this L-value at almost every place. A similar phenomenon appears in the theta correspondence, when the theta series of an element in the Weil representation is integrated against automorphic forms for a dual pair [8].

Our current work aims to provide a conceptual, geometric interpretation of the appearance of L-values, generalizing the derived geometric Satake equivalence of Bezrukavnikov and Finkelberg [1]. Roughly speaking, we conjecture that, when $\mathfrak{o} = k[[t]]$ where k is a finite field or the field of complex numbers, the derived category of $G(\mathfrak{o})$-equivariant constructible sheaves on $X(F)$ is equivalent to a differential graded category of $\check{G}$-equivariant coherent sheaves on a certain Hamiltonian space $\check{M}$, which is isomorphic to $\check{V}_X \times^{\check{G}_X} \check{G}$. When $X = H$ under the $G = H \times H$-action, we have $\check{M} = T^*\check{H}$, and $\check{M}/\check{G} = \check{\mathfrak{h}}^*/H$ (as stacks), recovering the derived Satake category as a dg-category of $\check{H}$-equivariant coherent sheaves on $\check{\mathfrak{h}}^*$, which is the result of Bezrukavnikov and Finkelberg. Under reasonable assumptions on the action of Frobenius, our conjecture implies the Plancherel decomposition of the basic function in terms of L-values.

We also conjecture that if the derived category of constructible sheaves on $X(F)$ is viewed as a quantization of the Hamiltonian space $M = T^*X$, the assignment $M \to \check{M}$ is involutive. This point of view allows us to incorporate the theta correspondence, as well, as the quantization of $M =$ a symplectic space under the action of a dual pair $G = G_1 \times G_2$. This seems to correspond to a duality conjectured by Gaiotto and Witten [3] in the context of quantum field theory. Assuming our conjecture, we have verified various examples of this duality: for example, the trivial space $X = G\backslash G$ is dual to the Whittaker model for $\check{G}$, and the theta correspondence for the dual pair $\mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}$ is dual to the Gross–Prasad variety $\mathrm{SO}_{2n}^{\mathrm{diag}}\backslash\mathrm{SO}_{2n} \times \mathrm{SO}_{2n+1}$ [5].

Globally, the geometric incarnation of the "theta series" for the quantization of M is a distinguished D-module on Bun_G, and we conjecture that its dual under the geometric Langlands correspondence is a sheaf on the space of $\check{G}$-local systems which arises from a suitable "coherent quantization" of $\check{M}$. For example, when $\check{M} = T^*\check{X}$ for a $\check{G}$-space $\check{X}$, this is the push-forward of the structure sheaf on the (derived) stack of sections of the $\check{X}$-local system induced from any $\check{G}$-local system. This is a geometric incarnation of the conjectural relations between automorphic periods and L-functions. In some cases, we are able to verify a semiclassical limit of this conjecture, in the context of the semiclassical geometric Langlands correspondence of Donagi and Pantev [2].

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Enhanced Langlands parameters and the ABPS Conjecture

Anne-Marie Aubert
(joint work with Paul Baum, Ahmed Moussaoui, Roger Plymen, Maarten Solleveld)

Let G be the group of the F-rational points of a connected reductive algebraic group defined over a local non archimedean field F. We denote by $G^\vee$ the Langlands dual group of G (the complex Lie group with root system dual to that of G) and by $\Phi(G)$ the set of $G^\vee$-conjugacy classes of Langlands parameters for G.

Let $\mathfrak{s} = [L, \sigma]_G$ be a given inertial class in G (here L is a Levi subgroup of G and σ a supercuspidal irreducible smooth representation of L). We denote by $T_\mathfrak{s}$ and $W_\mathfrak{s}$ the complex torus and finite group assigned by Bernstein to $\mathfrak{s}$ and by $\mathrm{Irr}(G)_\mathfrak{s}$ the Bernstein component of G attached to $\mathfrak{s}$. An important feature of the ABPS Conjecture, stated with P. Baum, R. Plymen and M. Solleveld, is the construction of a *geometrical model of the cuspidal support map* $\pi_\mathfrak{s}\colon \mathrm{Irr}(G)_\mathfrak{s} \to T_\mathfrak{s}/W_\mathfrak{s}$.

When the group G is quasi-split or is an inner form of $\mathrm{GL}_n(F)$, this construction involves the *extended quotient* $T_\mathfrak{s}/\!/W_\mathfrak{s}$ of $T_\mathfrak{s}$ by $W_\mathfrak{s}$, which is defined to be the quotient $\widetilde{T}_\mathfrak{s}/W_\mathfrak{s}$, where $\widetilde{T}_\mathfrak{s} := \{(t,w) : w \in W_{\mathfrak{s},t}\}$, with $W_{\mathfrak{s},t}$ denoting the isotropy subgroup of t in $W_\mathfrak{s}$. While $\mathrm{Irr}(G)_\mathfrak{s}$ and $\pi_\mathfrak{s}$ are often very difficult to describe and calculate, the extended quotient $T_\mathfrak{s}/\!/W_\mathfrak{s}$ and its projection $\rho_\mathfrak{s}\colon T_\mathfrak{s}/\!/W_\mathfrak{s} \to T_\mathfrak{s}/W_\mathfrak{s}$ onto the ordinary quotient are usually quite easily calculated.

The *spectral extended quotient* (also called *extended quotient of the second kind*) denoted by $T_\mathfrak{s}/\!/\widehat{W}_\mathfrak{s}$, is constructed by replacing the orbit of $t \in T_\mathfrak{s}$ (for the action of $W_\mathfrak{s}$ on $T_\mathfrak{s}$) by $\mathrm{Irr}(W_{\mathfrak{s},t})$, the set of isomorphism classes of irreducible representations of $W_{\mathfrak{s},t}$. It is in bijection (in a non-canonical way, in general) with $T_\mathfrak{s}/\!/W_\mathfrak{s}$.

When G is arbitrary, we need to use a twisted version $(T_\mathfrak{s}/\!/\widehat{W}_\mathfrak{s})_\natural$ of $T_\mathfrak{s}/\!/\widehat{W}_\mathfrak{s}$ (see [ABPS1]). The ABPS Conjecture, asserts that, for each inertial class $\mathfrak{s}$, there exists a family of 2-cocycles

$$\natural_t\colon W_{\mathfrak{s},t} \times W_{\mathfrak{s},t} \to \mathbb{C}^\times \qquad t \in T_\mathfrak{s},$$

and a bijection

$$\mu_\mathfrak{s}\colon \mathrm{Irr}(G)_\mathfrak{s} \longrightarrow (T_\mathfrak{s}/\!/\widehat{W}_\mathfrak{s})_\natural$$

which

- restricts to a bijection between the tempered representations and the unitary part of the twisted extended quotient;
- is canonical up to permutations within L-packets, that is, for any $\phi \in \Phi(G)$, the image of the intersection of $\mathrm{Irr}_\mathfrak{s}(G)$ with the L-packet defined by ϕ is canonically defined (assuming the existence of a LLC for G).

The validity of the conjecture was proved in several cases, notably for the principal series of split connected groups (in [ABPS2]), and for the inner forms of $\mathrm{GL}_n(F)$ and of $\mathrm{SL}_n(F)$ (in [ABPS3]).

Let $\mathcal{O}(T_\mathfrak{s})$ be the coordinate algebra of the complex affine variety $T_\mathfrak{s}$ and let $\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s}$ be the crossed-product algebra for the action of $W_\mathfrak{s}$ on $\mathcal{O}(T_\mathfrak{s})$. We conjectured that the bijection $\mu_\mathfrak{s}$ comes from a *stratified equivalence* (as defined in [ABPS4]) of the two unital finite-type $\mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}})$-algebras $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$ and the Bernstein algebra $\mathcal{H}(G,\mathfrak{s})$ associated to $\mathfrak{s}$. It is known to be the case for G an inner form of $\mathrm{GL}_n(F)$ and for principal series of the exceptional group G_2 (see [ABP]).

In [1], for an arbitrary p-adic reductive group G, notions of *cuspidality* and *cuspidal support* for enhanced Langlands parameters were introduced and a partition of the set $\Phi_{\mathrm{e}}(G)$ of $G^\vee$-conjugacy classes of enhanced Langlands parameters for G into *Bernstein series* $\Phi_{\mathrm{e}}(G)_{\mathfrak{s}^\vee}$ was constructed. Here $\mathfrak{s}^\vee = [{}^L L, (\varphi,\varepsilon)]_{G^\vee}$ is the inertial class of $({}^L L, (\varphi,\varepsilon))$, where ${}^L L$ is the L-group of a Levi subgroup L of G and the pair (φ,ε) is a cuspidal enhanced Langlands parameter for L. For a survey of the construction, see [Au]. The partition has several nice properties.

1. Let ϕ be a Langlands parameter for G. We denote by $\mu_\phi \colon I_F \to G^\vee$ the restriction of ϕ to to the absolute inertia group I_F of F. We will denote by $\Phi(I_F, G^\vee)$ the set of $G^\vee$ conjugacy classes of continuous morphisms $\mu\colon I_F \to G^\vee$ such that $\mu = \mu_\phi$ for some $\phi \in \Phi(G)$. For $\mu \in \Phi(I_F, G^\vee)$, we define the *Lusztig series of enhanced Langlands parameters* attached to μ to be the set

$$\Phi_{\mathrm{e}}(G)_\mu := \{(\phi,\eta) \in \Phi_{\mathrm{e}}(G) \ : \ \mu_\phi = \mu\}\,.$$

The Lusztig series of enhanced Langlands parameters for G provide a partition of $\Phi_{\mathrm{e}}(G)$:

$$\Phi_{\mathrm{e}}(G) = \bigsqcup_{\mu \in \Phi(I_F, G^\vee)} \Phi_{\mathrm{e}}(G)_\mu. \tag{1}$$

The partition of $\Phi_{\mathrm{e}}(G)$ in (1) is compatible with its partition into Bernstein series in the sense that every Lusztig series is a finite disjoint union of Bernstein series. More precisely, for L a Levi subgroup of a parabolic subgroup of G, we can write similarly

$$\Phi_{\mathrm{e}}(L) = \bigsqcup_{\nu \in \Phi(I_F, L^\vee)} \Phi_{\mathrm{e}}(L)_\nu.$$

Let ι_L denote the canonical inclusion $\iota_L \colon L^\vee \hookrightarrow G^\vee$. Then for each $\mu \in \Phi(I_F, G^\vee)$, we obtain that

$$\Phi_{\mathrm{e}}(G)_\mu = \bigsqcup_{\substack{\mathfrak{s}^\vee = [L^\vee, (\varphi,\varepsilon)]_{G^\vee} \\ (\varphi,\varepsilon) \in \Phi_{\mathrm{e}}(L)_\nu \text{ with } \iota_L \circ \nu = \mu}} \Phi_{\mathrm{e}}(G)_{\mathfrak{s}^\vee}.$$

2. For each $\mathfrak{s}^\vee$, the series $\Phi_{\mathrm{e}}(G)_{\mathfrak{s}^\vee}$ satisfies a Galois analogue of the ABPS Conjecture (see [1, Theorem 9.3]). In the cases when the LLC Conjectures for G and L are established, and the latter known to send the inertial class $\mathfrak{s}$ to an inertial class $\mathfrak{s}^\vee$, and when the groups $W_{\mathfrak{s}}$ and $W_{\mathfrak{s}^\vee}$ are isomorphic (properties that we expect to always hold), then it implies the validity of the ABPS Conjecture for $\mathrm{Irr}(G)_{\mathfrak{s}}$.

3. In [2], we attached to each $\mathfrak{s}^\vee$ a twisted graded Hecke algebra, that we used in [3] to attach to $\mathfrak{s}^\vee$ a *twisted affine Hecke algebra with parameters* $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$, which have the property that its simple modules are in bijection with the Bernstein series $\Phi_{\mathrm{e}}(G)_{\mathfrak{s}^\vee}$.

We will recall below the construction of $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$. We write $\mathfrak{s}_L{}^\vee := [L^\vee, (\varphi, \varrho)]_{L^\vee}$. We have $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to L^\vee$, and we define the groups

$$W_{\mathfrak{s}^\vee} = \mathrm{N}_{G^\vee}(\mathfrak{s}_L^\vee)/L^\vee \quad \text{and} \quad J := \mathrm{Z}_{G^\vee}(\varphi(I_F)).$$

The group J is a complex (possibly disconnected) reductive group. We have $J = \mathcal{G}_{\mu_\phi}$, for any $(\phi, \rho) \in \Phi_{\mathrm{e}}(G)_{\mathfrak{s}^\vee}$. We set $\mathcal{T} := \mathrm{Z}(L^\vee)^\circ$, and define $R(J^\circ, \mathcal{T})$ as the set of $\alpha \in X_*(\mathcal{T}) \setminus \{0\}$ which appear in the adjoint action of $\mathcal{T}$ on the Lie algebra of J°. It is a root system.

We set $W_{\mathfrak{s}^\vee}^\circ = \mathrm{N}_{J^\circ}(\mathcal{T})/\mathrm{Z}_{J^\circ}(\mathcal{T})$, where $W_{\mathfrak{s}^\vee}^\circ$ is the Weyl group of $R(J^\circ, \mathcal{T})$. Let $R_+(J^\circ, \mathcal{T})$ be the positive system defined by a parabolic subgroup $P^\circ \subset J^\circ$ with Levi factor $(L^\vee)^\circ$. Two such P° are J°-conjugate, so the choice is inessential. Since $W_{\mathfrak{s}^\vee}$ acts simply transitively on the collection of positive systems for $R(J^\circ, \mathcal{T})$, we obtain a semi-direct factorization

$$W_{\mathfrak{s}^\vee} = W_{\mathfrak{s}^\vee}^\circ \rtimes \mathfrak{R}_{\mathfrak{s}^\vee},$$
$$\text{where } \mathfrak{R}_{\mathfrak{s}^\vee} = \{w \in W_{\mathfrak{s}^\vee} \,:\, w \cdot R_+(J^\circ, \mathrm{Z}(L^\vee)^\circ) = R_+(J^\circ, \mathrm{Z}(L^\vee)^\circ)\}.$$

We fix the following data:

- an inertial class $\mathfrak{s}^\vee = [L^\vee, (\varphi, \varrho)]_{G^\vee}$;
- a 2-cocycle $\kappa_{\mathfrak{s}^\vee}\colon (W_{\mathfrak{s}^\vee}/W_{\mathfrak{s}^\vee}^\circ)^2 \to \mathbb{C}^\times$;
- two $W_{\mathfrak{s}^\vee}$-invariant functions
$$\lambda\colon R(J^\circ, \mathcal{T})_{\mathrm{red}} \to \mathbb{Z}_{\geq 0},$$
$$\lambda_*\colon \{\alpha \in R(J^\circ, \mathcal{T})_{\mathrm{red}} \,:\, \alpha^\vee \in 2X_*(\mathcal{T})\} \to \mathbb{Z}_{\geq 0};$$
- an array of invertible variables $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots, \mathbf{z}_d)$.

The group $W_{\mathfrak{s}^\vee}^\circ$ is a finite Weyl group. Let ℓ be the lenght function on $W_{\mathfrak{s}^\vee}^\circ$ and let $\mathcal{H}(W_{\mathfrak{s}^\vee}^\circ, \vec{\mathbf{z}}^{2\lambda})$ denote the Iwahori-Hecke algebra of $W_{\mathfrak{s}^\vee}^\circ$ with parameters $\vec{\mathbf{z}}^{2\lambda(\alpha)}$. We proved in [3] that the vector space $V := \mathcal{O}(\mathfrak{s}_{L^\vee}^\vee) \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] \otimes \mathbb{C}[W_{\mathfrak{s}^\vee}^\circ] \otimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \kappa_{\mathfrak{s}^\vee}]$ admits a unique algebra structure such that:

- $\mathcal{O}(\mathfrak{s}_{L^\vee}^\vee)$, $\mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}]$ and $\mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \kappa_{\mathfrak{s}^\vee}]$ are embedded as subalgebras;
- $\mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] = \mathbb{C}[\mathbf{z}_1, \mathbf{z}_1^{-1}, \ldots, \mathbf{z}_d, \mathbf{z}_d^{-1}]$ is central;
- the span of $W_{\mathfrak{s}^\vee}^\circ$ is the Iwahori-Hecke algebra $\mathcal{H}(W_{\mathfrak{s}^\vee}^\circ, \vec{\mathbf{z}}^{2\lambda})$;
- for $\gamma \in \mathfrak{R}_{\mathfrak{s}^\vee}$, $w \in W_{\mathfrak{s}^\vee}^\circ$ and $x \in X_*(\mathrm{Z}(L^\vee)^\circ)$:
$$N_\gamma N_w \cdot \theta_x \cdot N_\gamma^{-1} = N_{\gamma w \gamma - 1} \cdot \theta_{\gamma(x)},$$
- for simple root $\alpha \in R(J^\circ, \mathcal{T})$ and $x \in X_*(\mathcal{T})$, corresponding to $\theta_x \in \mathcal{O}(\mathfrak{s}_{L^\vee}^\vee)$, the expression $\theta_x \cdot N_{s_\alpha} - N_{s_\alpha} \cdot \theta_{s_\alpha(x)}$ equals
$$(\mathbf{z}_j^{\lambda(\alpha)} - \mathbf{z}_j^{-\lambda(\alpha)})(\theta_x - \theta_{s_\alpha(x)})/(\theta_0 - \theta_{-\alpha}), \quad \text{if } \alpha^\vee \notin 2X_*(T),$$
$$(\mathbf{z}_j^{\lambda(\alpha)} - \mathbf{z}_j^{-\lambda(\alpha)} + \theta_{-\alpha}(\mathbf{z}_j^{\lambda_*(\alpha)} - \mathbf{z}_j^{-\lambda_*(\alpha)}))(\theta_x - \theta_{s_\alpha(x)})/(\theta_0 - \theta_{-2\alpha}),$$
otherwise. Let $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ denote this algebra.

We proved in [3, Theorem 2.15] that, for each $\vec{r} \in \mathbb{R}_{>0}^d$ and each $\mathfrak{s}^\vee$, there exists a canonical bijection

$$\mu_{\mathfrak{s}^\vee}\colon \mathrm{Irr}_{\vec{r}}(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})) \longrightarrow (T_{\mathfrak{s}^\vee} /\!/ \widehat{W}_{\mathfrak{s}^\vee})_\natural,$$

which restricts to a bijection between the tempered modules and the unitary part of the twisted extended quotient, where $\mathrm{Irr}_{\vec{r}}(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}))$ is the subset of the equivalence classes of simple $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$-modules on which $\vec{\mathbf{z}}$ acts as $\vec{r}$, and $T_{\mathfrak{s}^\vee} := \Phi_{\mathrm{e}}(L)_{\mathfrak{s}_L^\vee}$ with $\mathfrak{s}_L^\vee := [{}^L L, (\varphi, \varepsilon)]_{L^\vee}$.

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A local Langlands correspondence for unipotent representations

Maarten Solleveld

Unipotent representations of reductive p-adic groups G were first studied by Lusztig, in the nineties. They generalize Iwahori-spherical representations, with which they have a lot in common. In this talk we present of a proof of a suitable version of the conjectures of Langlands, for the set of irreducible unipotent representations $\mathrm{Irr}_{\mathrm{unip}}(G)$.

Let G be a connected reductive group over a non-archimedean local field F, which splits over an unramified extension of F. Let $\Phi_{nr,e}(G)$ be the set of those enhanced L-parameters for G which are trivial on the inertia subgroup of the Weil group of F.

Theorem. *There exists a bijection*

$$\mathrm{Irr}_{\mathrm{unip}}(G) \to \Phi_{nr,e}(G) : \pi \mapsto (\phi_\pi, \rho_\pi)$$

It has nice properties with respect to: direct products, central characters, temperedness, square-integrability, twisting by unramified characters, cuspidality, parabolic induction, uniqueness and formal degrees.

For adjoint groups this correspondence is due to Lusztig [7, 8]. The general case was established in [9], although the proof of a few properties was only concluded in [6]. We will discuss some aspects of these correspondences:

- the case of supercuspidal unipotent representations, as worked out together with Feng and Opdam [4, 5];
- the parametrization of the set of unipotent Bernstein components;
- the associated affine Hecke algebras, which stem from joint work with Aubert and Moussaoui [1–3];
- the behaviour with respect to central isogenies [10].

It would be interesting to extend the methods employed in these references to all irreducible depth zero G-representations. This is work in progress, although at the moment there seem to be many obstacles.

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Fourier transforms on the basic affine space of a quasi-split group

Nadya Gurevich

(joint work with David Kazhdan)

1.1. **Gelfand-Graev intertwining operators for quasi-split groups.** Let G be a simple quasi-split group defined over a local non-archimedean field F. We fix a Borel subgroup B of G and a decomposition $B = T \cdot U$. The basic affine space

$X = G/U$ admits unique (up to a scalar) G-invariant measure ω_X. We define a unitary representation θ of the group $G \times T$ on $L^2(X, \omega_X)$ by:

$$\theta(g,t)f(hU) = \delta_B^{1/2}(t)f(g^{-1}htU)$$

for the modular character δ_B. Let $W := N_G(T)/T$ be the Weyl group.

For split groups Gelfand and Graev in [2], see also [3], extended the action of $G \times T$ to the representation of $G \times (T \rtimes W)$, so that elements w of W act on $L^2(X)$ by generalized Fourier transforms Φ_w. Our first goal is to extend this result for quasi-split groups.

More precisely, for a quasi-split simply-connected group G with a maximal torus T and a relative Weyl group W, we construct a family of unitary operators $\Phi_w, w \in W$, on $L^2(X)$ such that

$$(1) \qquad \begin{cases} \Phi_w \theta(g,t) = \theta(g, w(t))\Phi_w & \forall w \in W, t \in T, g \in G \\ \Phi_{w_1}\Phi_{w_2} = \Phi_{w_1 w_2} & \forall w_1, w_2 \in W \end{cases}$$

Since W is generated by simple reflections, it is sufficient to define the operators Φ_s for simple reflections $s \in W$ and to check the braid relations. Moreover, it is easy to reduce the definition of the operators Φ_s to the case of simple groups of rank one.

Any simple simply connected group of rank one is a restriction of scalars of one of the following groups:

- The split group $SL(V_2)$ for a symplectic two-dimensional space V_2 over F. In this case $X = V_2 \backslash \{0\}$ and $W = \{e, w_0\}$. The operator Φ_{w_0} is given by the usual Fourier transform.
- The quasi-split group $SU(W_3^K)$, where K/F is a quadratic field extension and W_K^3 is a Hermitian three-dimensional space. In this case $X = W_3^0 \backslash \{0\}$ is the cone of non-zero isotropic vectors, and the Weyl group consists of two elements $W_K = \{e, w_K\}$. We shall construct an operator Φ_K on $L^2(X)$, which is an involution satisfying 1 and put $\Phi_{w_K} = \Phi_K$. Our construction can also be applied to the case $K = F \oplus F$, so that $SU(W_3^K) \simeq SL_3$, and w_K is the longest element in the Weyl group of SL_3.

Our construction of the operator $\Phi_w, w \in W$ is based on the observation that for quasi-split groups G of rank one, there is a map

$$j : G \times (T' \rtimes W) \to H,$$

where T' is a subgroup of finite index in T and H is a reductive group, such that $(j(G), j(T' \rtimes W))$ is a commuting pair in H. The unitary minimal representation $\hat{\Pi}$ of H has realization on $L^2(X)$, so that the restriction of $\hat{\Pi}$ to $G \times T'$ is θ. We define the operator Φ_w to be $\hat{\Pi}(j(w))$.

For the case $G = SL_2$, one has $T' = T$, the group H is the metaplectic group of rank 2 and $\hat{\Pi}$ is the Weil representation realized in the Schrodinger model.

For the group $G = SU(W_3^K)$ the group T' can be specified, the group H is $O(V_8^K)$, for a quadratic space V_8^K of dimension 8, Witt index 3 and discriminant

K. The minimal representation $\hat{\Pi}_K$ of $O(V_8^K)$ has been constructed by Howe and realized on $L^2(X)$ by Savin (in the split case $K = F \oplus F$). We define $\Phi_K = \hat{\Pi}_K(j(w_K))$. By definition Φ_K is an involution and satisfies 1. We shall obtain an explicit formula for this operator.

In fact, we work in more general setting. For a quadratic algebra K we consider a non-degenerate quadratic space V_{2n+2}^K of discriminant K and Witt index n, if K is a field, or $n+1$ if $K = F \oplus F$. The unitary minimal representation $\hat{\Pi}_K$ of the group $O(V_{2n+2}^K)$ can be realized on the space $L^2(C_{2n}^K)$, where C_{2n}^K is the cone of isotropic vectors in V_{2n}^K. This model has been considered by Savin for the split algebra K. The action of the parabolic subgroup P_1 with the Levi subgroup $GL_1 \times O(V_{2n}^K)$ in this model is very explicit. We write a formula for the action $\hat{\Pi}_K(s_1)$, where s_1 is an involution, not contained in P_1. This result has an independent interest.

For a general group G, having defined the operators Φ_s for all simple reflections, we define by a closed formula the operators Φ_w for any $w \in W$ and prove that $\Phi_{w_1 w_2} = \Phi_{w_1} \circ \Phi_{w_2}$ whenever $l(w_1) + l(w_2) = l(w_1 w_2)$. Here $l(w)$ is the length of the element w. This implies the braid relations between the operators Φ_w for any simple quasi-split group, and therefore the action of $G \times (T \rtimes W)$ on $L^2(X)$.

1.2. **Schwarz space of a quasi-split group.** Let $S_c(X)$ denote the space of smooth functions on X of compact support. The Schwarz space

$$S(X) = \sum_{w \in W} \Phi_w(S_c(X)),$$

has been defined and studied in [1] for split groups. We propose the following characterization of this space:

Conjecture. The space $S(X)$ coincides with the space of functions $\tilde{S}(X) = \{f \in L^2(X)$, such that $\Phi_w(f)$ has bounded support for all $w \in W\}$ For split groups the inclusion $S(X) \subset \tilde{S}(X)$ was proven in [1]. We show that the equality holds for SL_3 and $SU(W_K^3)$ using the interpretation of $S(X)$ as the space of smooth vectors Π_K of the minimal representation of $O(V_8^K)$.

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Invariant hermitian forms, Bernstein projectives, and Jacquet modules

Dan Ciubotaru
(joint work with Dan Barbasch)

This work is motivated by the goal of devising an algorithm for computing signatures of invariant hermitian forms for representations of reductive p-adic groups analogous to the signature algorithm of [1]. The first question is the definition of an appropriate analogue for c-invariant hermitian forms and their normalisations.

1. Let k denote a p-adic field with the ring of integers $\mathcal{O}$ and finite residue field $\mathbb{F}_q$. Let G be the k-points of a connected reductive k-split algebraic group. Fix a Borel subgroup $B = AN$ in G (A the maximal split torus) and $K = G(\mathcal{O})$ the maximal hyperspecial compact open subgroup such that $G = KB$. Let $I \subset K$ be an Iwahori subgroup and $\mathcal{H}(G,I) = C_c(I\backslash G/I)$ be the Iwahori-Hecke algebra. The Borel equivalence of categories states that the Bernstein component $\mathcal{C}_I(G)$ of unramified smooth representations of G is equivalent to the category of $\mathcal{H}(G,I)$-modules via the functor $V \mapsto V^I$.

Via the Bruhat decomposition $G = \sqcup_{w\in\widetilde{W}} IwI$, where $\widetilde{W}$ is the Iwahori-Weyl group, it is apparent that a $\mathbb{C}$-basis of $\mathcal{H}(G,I)$ is given by the functions $T_w = \delta_{IwI}$. The algebra $\mathcal{H}(G,I)$ admits the well-known Iwahori presentation in terms of the T_w's, $w \in \widetilde{W}$, and with the braid and quadratic relations. The natural $*$-operation in $\mathcal{H}(G,I)$ (a conjugate-linear anti-automorphism) is given by

$$T_w^* = T_{w^{-1}}, \quad w \in \widetilde{W}.$$

Barbasch and Moy showed that the Borel functor induces a bijection between irreducible unitary representations in $\mathcal{C}_I(G)$ and simple $*$-unitary modules for $\mathcal{H}(G,I)$.

The Iwahori-Weyl group can be identified with $\widetilde{W} = X_* \rtimes W$, where W is the finite Weyl group and X_* is the coweight lattice of A. Bernstein and Lusztig gave a different presentation of $\mathcal{H}(G,I)$, where the generators are $\{T_w : w \in W\}$ and $\{\theta_x : x \in X_*\}$ such that $\mathcal{A} = \mathbb{C}[\theta_x : x \in X_*]$ is an abelian subalgebra. The natural $*$-operation takes the form

$$\theta_x^* = T_{w_0} \cdot \theta_{-w_0(x)} \cdot T_{w_0}^{-1}, \quad w_0 \text{ is the longest element in } W.$$

In [2], we considered a different operation c for $\mathcal{H}(G,I)$:

$$T_w^{\mathsf{c}} = T_{w^{-1}},\ \theta_x^{\mathsf{c}} = \theta_x, \quad w \in W,\ x \in X_*. \tag{1}$$

We believe that this is the analogue of the c-form introduced for $(\mathfrak{g}, K)$-modules of real reductive groups in [1]. We offer some supporting evidence for this claim.

The centre of $\mathcal{H}(G,I)$ is $\mathcal{Z} = \mathcal{A}^W$ which implies that the central characters of $\mathcal{H}(G,I)$ are parameterised by W-orbits in $A^\vee = X^* \otimes_{\mathbb{Z}} \mathcal{C}^\times$, where X^* is the weight lattice of A. Let $A^\vee = A_u^\vee A_{\mathbb{R}}^\vee$ be the polar decomposition induced by $\mathcal{C}^\times = S^1 \times \mathbb{R}_{>0}$. Let V denote an irreducible G-representation in $\mathcal{C}_I(G)$. Denote by $\mathsf{cc}(V)$ the central character of the simple $\mathcal{H}(G,I)$-module V^I and by $\Omega(V^I)$ the set of $\mathcal{A}$-weights of V^I. It is easy to see that $\Omega(V^I) \subseteq W \cdot \mathsf{cc}(V)$.

We say that V has *positive real* central character if $\mathsf{cc}(V) \in A_{\mathbb{R}}^\vee/W$.

Proposition 1 ([2])**.** Every irreducible representation $V \in \mathcal{C}_I(G)$ with positive real central character admits a c-invariant hermitian form on V^I.

This result follows easily from the known fact (Evens-Mirković) that if V and V' have positive real central characters then $V \cong V'$ if and only if $\Omega(V^I) = \Omega((V')^I)$, together with the easy observation that $\Omega((V^I)^{\mathsf{c}}) = \overline{\Omega(V^I)}$, where $(V^I)^{\mathsf{c}}$ denotes the c-hermitian dual of V^I.

To discuss the normalisation of the c-invariant hermitian forms, restrict V to K. Then V^I contains the K-types σ such that $\sigma^I \neq 0$. These are in one-to-one correspondence with the simple $\mathcal{H}(K,I)$-modules. The finite Hecke algebra $\mathcal{H}(K,I)$ is isomorphic to the finite Hecke algebra $\mathbb{C}[W]$ (by Tits' deformation theorem). Denote by $\mathcal{W}(\sigma)$ the W-type corresponding to σ^I. By the Springer correspondence, $\mathcal{W}(\sigma)$ is parameterised by a local system on an adjoint nilpotent orbit in the Langlands dual Lie algebra $\mathfrak{g}^\vee$. Denote this orbit by $\mathcal{N}(\sigma)$. From the work of Kazhdan and Lusztig, one can deduce the following statement.

Theorem 1 (Kazhdan-Lusztig, see [4])**.** Suppose V is an irreducible representation in $\mathcal{C}_I(G)$ with positive real central character. There exists a unique adjoint nilpotent orbit $\mathcal{O}^\vee_{\min}$ in $\mathfrak{g}^\vee$ such that for every $\sigma \in \widehat{K}$ with $\mathrm{Hom}_K[\sigma, V] \neq 0$ and $\sigma^I \neq 0$, $\mathcal{O}^\vee_{\min} \subseteq \overline{\mathcal{N}(\sigma)}$.

We call $\sigma \in \widehat{K}$ such that $\mathcal{N}(\sigma) = \mathcal{O}^\vee_{\min}$ a *lowest K-type* of V. An irreducible V can have more than one lowest K-type and these could appear with multiplicity greater than 1. By computing the signature of the c-invariant hermitian form at ∞ and using a deformation argument, we prove the following:

Theorem 2 ([2])**.** Under the assumptions in Theorem 1, there exists a normalisation of the c-invariant hermitian form on V^I such that on the isotypic component of every lowest K-type of V, the form is the identity (hence positive definite).

In [2] and subsequent works, we prove several other interesting facts about c-invariant forms, including:

(1) If V^I is c-unitary, then all Jacquet modules of V are *semisimple*.
(2) If V^I is both $*$-unitary and c-unitary, then V^I is *irreducible* when restricted to $\mathcal{H}(K,I)$.
(3) Suppose $G = GL(n, \mathsf{k})$ and V has integral real central character. Then V^I is c-unitary if and only if it is a *ladder* representation in the sense of [6]. Moreover, V^I is both $*$-unitary and c-unitary if and only if V is a Speh representation.
(4) The c-unitary dual of $\mathcal{H}(G,I)$ is unbounded (in fact, more precise results are available).

2. A natural question is how to see the c-operation appearing for the group, rather than the Iwahori-Hecke algebra. We believe that this is via the Bernstein projective modules. We illustrate this idea in the Iwahori case. In the category $\mathcal{C}_G(I)$, there are two types of projective generators. The first we have already seen:

$$\mathcal{Q} = \mathrm{ind}_I^G(1).$$

The second, defined by Bernstein, is

$$\mathcal{P} = i_{B^-}^G(\mathrm{ind}_{A(\mathcal{O})}^A(1)), \tag{2}$$

where B^- is the opposite Borel subgroup. The fact that $\mathcal{P}$ is a projective generator implies that the functor $V \mapsto \mathrm{Hom}_G[\mathcal{P}, V]$ is an equivalence of categories between $\mathcal{C}_I(G)$ and right $\mathrm{End}_G(\mathcal{P})$-modules. If V_N denotes the Jacquet module of V, it is well known and easy to see that Bernstein's second adjointness theorem implies that

$$\mathrm{Hom}_G[\mathcal{P}, V] \cong \mathrm{Hom}_A[\mathrm{ind}_{A(\mathcal{O})}^A(1), V_N] \cong V_N^{A(\mathcal{O})} \cong V^I, \tag{3}$$

the last step being Jacquet's lemma. Now $V^I = \mathrm{Hom}_G[\mathcal{Q}, V]$, hence Yoneda's lemma implies that $Q \cong \mathcal{P}$. In particular,

$$\mathrm{End}_G(\mathcal{P}) \cong \mathrm{End}_G(Q) \cong \mathcal{H}(G, I)^{\mathsf{opp}} \cong \mathcal{H}(G, I).$$

We refer to [5] for a much more general and direct proof of the isomorphism $\mathcal{P} \cong \mathcal{Q}$.

Let C denote a Chevalley involution of G such that $C(a) = a^{-1}$ for all $a \in A$ and $C(B) = B^-$. For example, if $G = GL(n, \mathsf{k})$, then $C(g) = (g^t)^{-1}$. If G is a classical group, this is the involution considered by Mœglin-Vigneras-Waldspurger. Define the c-star operation on the Hecke algebra $\mathcal{H}(G)$ by

$$f \mapsto f^{\mathsf{c}}(g) = \overline{f(C(g)^{-1})}. \tag{4}$$

Suppose a G-representation V admits a nondegenerate c-invariant hermitian form. An easy variant of Casselman's result about the contragredient implies:

Proposition 2. If $P = MU$ is a standard parabolic subgroup such that $C(M) = M$, then the Jacquet module V_U acquires a nondegenerate (M, c)-invariant hermitian form $\langle\ ,\ \rangle_U$.

Finally, we have:

Theorem 3 ([3])**.** The form $\langle\ ,\ \rangle_N$ on $V_N^{A(\mathcal{O})} \cong V^I$ is c-invariant for the right action of $\mathrm{End}_G(\mathcal{P}) \cong \mathcal{H}(G, I)^{\mathsf{opp}}$.

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Quantitative results in relative harmonic analysis

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(joint work with Eitan Sayag, Sandeep Varma)

We prove some homogeneity results for spherical characters of p-adic symmetric spaces.

Let F denote a nonarchimedean local field, and for simplicity of exposition assume that F has characteristic zero and large residual characteristic. Let $\underline{G}$ denote a connected reductive F-group, σ an F-involution of $\underline{G}$, and $\underline{H}$ a reductive F-group that lies between the group of σ-fixed points in $\underline{G}$ and the connected part of this group: $\underline{G}^\sigma \supseteq \underline{H} \supseteq (\underline{G}^\sigma)^\circ$. Let $G = \underline{G}(F)$, $H = \underline{H}(F)$, etc.

Say that a representation π of G is H-*distinguished* provided that $\dim \operatorname{Hom}_H(\pi, 1) \neq 0$. Suppose that π and its smooth dual $\pi^\vee$ are both H-distinguished. Then we may choose nonzero vectors

$$\lambda \in \operatorname{Hom}_H(\pi, 1), \qquad \lambda^\vee \in \operatorname{Hom}_H(\pi^\vee, 1),$$

and from these choices we obtain a *spherical character* $\Theta_{\pi,\lambda,\lambda^\vee}$, which can be thought of as an H-invariant distribution on G/H, or as an H-bi-invariant distribution on G. That is, for a smooth function $f \in C_c^\infty(G)$,

$$\Theta_{\pi,\lambda,\lambda^\vee}(f) := \langle \pi(f)\lambda^\vee, \lambda \rangle.$$

The spherical character has many properties analogous to those of the Harish-Chandra character Θ_π, but we will only consider one of them.

Local character expansion. Let $\mathfrak{g} = \operatorname{Lie}(G)$, and let $\mathcal{N}$ denote the nilpotent set in $\mathfrak{g}$. We can use the exponential map to transfer the character Θ_π from the group G to a neighborhood of 0 in $\mathfrak{g}$. From a theorem of Howe [5] and Harish-Chandra [4], $\Theta_\pi \circ \exp$ should, when restricted to a small enough neighborhood of 0, coincide with a linear combination of Fourier transforms of nilpotent orbital integrals. For many applications, it is important to know on what neighborhood such an identity is valid. Hales and Moy-Prasad conjectured that the right neighborhood is $\mathfrak{g}_{r+} := \cup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r+}$, where r is the depth of π. Here, $\mathcal{B}(G)$ is the (enlarged) Bruhat-Tits building of G, and $\mathfrak{g}_{x,r+}$ is a Moy-Prasad filtration lattice in $\mathfrak{g}$ associated to $x \in \mathcal{B}(G)$ and $r \in \mathbb{R}$. Note that $\{\mathfrak{g}_{r+}\}_{r \in \mathbb{R}}$ is a system of open, closed, G-invariant neighborhoods of $\mathcal{N}$. Thus,

$$\text{(LCE)} \qquad \operatorname{res}|_{\mathfrak{g}_{r+}} \Theta_\pi \circ \exp = \operatorname{res}|_{\mathfrak{g}_{r+}} \sum c_{\mathcal{O}}(\pi) \hat{\mu}_{\mathcal{O}}.$$

Taking Fourier transforms of both sides, we get

$$\operatorname{res}|_{D_{r+}} (\Theta_\pi \circ \exp)^\wedge = \operatorname{res}|_{D_{r+}} \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi) \mu_{\mathcal{O}},$$

where $D_{r+} = \sum_x \mathfrak{g}/\mathfrak{g}_{x,r+}$, the sum is over the set of nilpotent orbits, and $\mu_{\mathcal{O}}$ is the orbital integral corresponding to $\mathcal{O}$. That is, the Hales-Moy-Prasad conjecture [6, §1] asserts that two distributions coincide when restricted to the functions in D_{r+}. See DeBacker [1] for a definition of a space $\tilde{J}_{r+}$ of distributions that includes $(\Theta_\pi \circ \exp)^\wedge$ for every admissible representation π of depth $\leq r$. To prove the conjecture, it would thus be enough to show that for all $r \in \mathbb{R}$,

$$\text{(LCE}^\wedge\text{)} \qquad \operatorname{res}|_{D_{r+}} \tilde{J}_{r+} = \operatorname{res}|_{D_{r+}} J(\mathcal{N}),$$

where $J(\mathcal{N})$ denotes the space of G-invariant distributions on $\mathcal{N}$. Waldspurger [9, 10] proves this statement for many groups when $r \in \mathbb{Z}$, and DeBacker [1] proves it in general, apart from some hypotheses on the residual characteristic p and assumptions about the tameness of G.

Relative local character expansion. Let V be the -1-eigenspace of σ in $\mathfrak{g}$, i.e., the tangent space at the identity to G/H, so that $\mathfrak{g} = \mathfrak{h} \oplus V$, where $\mathfrak{h} = \operatorname{Lie}(H)$. Then H acts on V, which thus contains a nilpotent cone $\mathcal{N}_V$. The following result is due to Hakim [3] and Rader-Rallis [8].

Theorem 1. Let π be an irreducible, H-distinguished representation of G, and let λ and $\lambda^\vee$ be as before.

$$\text{(RLCE)} \qquad \Theta_{\pi,\lambda,\lambda^\vee} \circ \exp \in J(\mathcal{N}_V)^\wedge$$

on some neighborhood of $\mathcal{N}_V$ in V.

Here we haven't expressed our distribution as a linear combination of Fourier transforms of explicitly given distributions that form a a basis for $J(\mathcal{N}_V)$, because it is not always clear what a basis should look like. (More on this below.)

We would like to know on what neighborhood (RLCE) is valid.

Like $\mathfrak{g}$, the space V comes equipped with a collection of Moy-Prasad filtrations: For every $x \in \mathcal{B}(H)$ and $r \in \mathbb{R}$, we have that $V_{x,r+} = V \cap \mathfrak{g}_{x,r+}$. (Here we are identifying $\mathcal{B}(H)$ with $\mathcal{B}(G)^\sigma$.) Thus, as in $\mathfrak{g}$, we can form a collection of closed, open H-invariant neighborhoods of $\mathcal{N}_V$ by setting $V_{r+} = \cup_x V_{x,r+}$.

Conjecture 2. The equation (RLCE) is valid on V_{r+}, where r is the depth of π.

By analogy with the classical case, we can form a space $\tilde{J}(V)_{r+}$ of H-invariant distributions on V that contains the Fourier transform of $\Theta_{\pi,\lambda,\lambda^\vee}$ for all H-distinguished π of depth $\leq r$; and the space of functions $D(V)_{r+} = \sum_x V/V_{x,r+}$. The Conjecture 2 would follow from

Conjecture 3. For all $r \in \mathbb{R}$,

$$\text{(RLCE}^\wedge\text{)} \qquad \operatorname{res}|_{D(V)_{r+}} \tilde{J}(V)_{r+} = \operatorname{res}|_{D(V)_{r+}} J(\mathcal{N}_V).$$

A more general statement, and a theorem. Now let $\underline{H}$ be a reductive F-group, and V a finite-dimensional representation of H. Under favorable conditions, V comes equipped with a nilpotent cone $\mathcal{N}_V$; a family of Moy-Prasad lattices $\{V_{x,r+}\}$, indexed by points $x \in \mathcal{B}(H)$ and numbers $r \in \mathbb{R}$; a system of neighborhoods $\{V_{r+}\}$ of $\mathcal{N}_V$; a space of functions $D(V)_{r+} = \sum_x V/V_{x,r+}$; etc. One can make the following statement about the pair (H, V).

Homog(H, V). For all $r \in \mathbb{R}$, (RLCE$^\wedge$) is valid for the action of H on V.

While this statement is not a theorem, nor even a conjecture, we know it to be true in some cases.

Theorem 4 (ASV). Suppose G/H is a symmetric space, and V as before.

(1) $\mathrm{Homog}(H, V)$ is true in the "Galois" case, i.e., where $\underline{G} = R_{E/F}\underline{H}$ for some separable quadratic extension E/F.
(2) $\mathrm{Homog}(H, V)$ is true if it is true in the special case where $\underline{G}$ is absolutely almost simple and simply connected.
(3) $\mathrm{Homog}(H, V)$ is true if G/H has rank one.

$\mathrm{Homog}(H, V)$ is also true if E/F is a cyclic extension of degree n coprime to p, $\underline{G} = R_{E/F}\underline{H}$, and V is the μ-eigenspace for the action of a generator of $\mathrm{Gal}(E/F)$ on $\mathrm{Lie}(G)$, where μ is a primitive nth root of unity.

Problems overcome. As pointed out by Rader-Rallis, in the symmetric-space case, a nilpotent orbit $\mathcal{O}$ for (H, V) need not carry an H-invariant measure, and thus need not give rise, via an orbital integral, to an H-invariant distribution on $\mathcal{O}$. Even if it does, such a distribution need not extend to give a distribution on V. That is, the set of nilpotent orbital integrals $\{\mu_{\mathcal{O}}\}$ need not be a basis for $J(\mathcal{N}_V)$.

However, this problem doesn't arise in the Galois case, and we can find an alternative basis in the rank-one cases.

Problems remaining. We don't yet know how to find a basis for $J(\mathcal{N}_V)$ in the higher-rank symmetric space case.

In order to adapt DeBacker's arguments, we used Portilla's adaption [7] of DeBacker's parametrization [2] of nilpotent orbits. But this work is not available to us in the case of, say, general affine spherical varieties. Coming up with an alternative parametrization will require us to deal with the fact that in such cases V can have infinitely many nilpotent H-orbits.

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