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Moduli spaces and Modular forms (hybrid meeting)

Organized by
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31 January – 6 February 2021

Abstract. The relation between moduli spaces and modular forms goes back to the theory of elliptic curves. On the one hand both topics experience their own growth and development, but from time to time new unexpected links show up and usually these lead to progress on both sides. One subject where there has been a lot of progress concerns the moduli of abelian varieties and K3 surfaces and especially on the Kodaira dimension of these spaces. The idea of the workshop was to bring together the experts of the two areas in the hope that discussion, interaction and lectures would spur the development of new ideas. The lectures of the workshop gave ample evidence of the interaction and provided opportunities for further interaction. Besides the lectures participants interacted via zoom in smaller groups.

Mathematics Subject Classification (2010): 11xx, 14xx.

Introduction by the Organizers

The workshop "Moduli Spaces and Modular Forms", organized by Jan Bruinier (Darmstadt), Gerard van der Geer (Amsterdam) and Valéry Gritsenko (Lille) was held 31 January-5 February 2021 in hybrid format because of the COVID-19 pandemic. There were only 8 participants at MFO in Oberwolfach; all other participants 48 were present only via zoom link. This restricted the interaction a lot. Nevertheless participation was very high and despite time differences up to 12 hours. Participants told the organizers afterwards that they were very happy with the quality of the lectures. Almost all lectures were given by the lecturers from their home, but a few talks were given in the lecture hall in Oberwolfach.

We had 18 talks of different lengths ranging from 30 minutes to one hour. Live attendance of the talks was very high. Since the talks were recorded it was possible for participants to view the talks at more suitable times.

Gaëtan Chenevier gave a beautiful talk on his search for unimodular integral lattices of rank 26, 27 and 28.

The Kodaira dimension of moduli spaces figured prominently in the talks of Ma, Farkas, Salvati Manni and Möller.

Shouhei Ma discussed construction of pluri-canonical forms on moduli spaces $F_{g,n}$ of n-pointed K3 surfaces of genus g and holomorphic symplectic varieties and obtained results on the Kodaira dimension. Riccardo Salvati Manni explained the construction of a cusp form of weight 14 on the moduli space A_6 that shows that the Kodaira dimension of A_6 is not unirational. Gavril Farkas explained in his lecture his new results on the Kodaira dimension of the moduli spaces M_g of curves of genus g for the cases where this is still not known. His main result is that M_{22} and M_{23} are of general type. The Kodaira dimension of moduli spaces of abelian differentials was the topic of a talk by Martin Möller. He showed that certain components of the strata of such moduli spaces have maximal Kodaira dimension and discussed the singularities of such strata.

Klaus Hulek discussed the cone of effective surfaces of the compactified moduli space of abelian threefolds and described extremal rays and a conjecture on the generators of this cone.

Jürg Kramer's talk also dealt with modular forms on the moduli of abelian varieties and the question when formal Fourier-Jacobi expansions give Siegel modular forms. He presented an alternative approach to the results of Bruinier and Raum.

Teichmüller modular forms live on the moduli spaces M_g and Giulio Codogni associated such modular forms to a holomorphic vertex algebra.

Orthogonal and unitary Shimura varieties featured prominently in the talks of Stephen Kudla and Ben Howard. Kudla showed that the products in the subring of cohomology generated by the special cycles are controlled by the Fourier coefficients of triple pullbacks of certain Siegel-Eisenstein series. Howard considered classes in the arithmetic Chow groups of unitary Shimura varieties and defined the arithmetic volume as an iterated intersection and showed that this volume can be expressed in terms of logarithmic derivatives of Dirchlet L-functions at integer points.

Rings of modular forms were the topic of talks by Haowu Wang and Christophe Ritzenthaler. Wang considered the cases where these rings are polynomial and Ritzenthaler discussed the cases of Siegel modular forms where the degree is small.

Harmonic Maass forms were the topic of a talk by Claudi Alfes. She described how to generalize results of Zagier and Bruinier and Funke on traces on singular moduli. Kathrin Bringmann gave a talk on recent work on false theta functions, that is, theta functions that are not modular due to sign factors and are examples of quantum modular forms.

Another topic during the workshop was that of paramodular forms and modularity. It occurred in the talks of Poor and Yuan and in the one of van Straten.

The two nicely coordinated talks of Cris Poor and David Yuan centered around the paramodular conjecture of Brumer and Kramer who conjectured a bijection between isogeny classes of abelian surfaces defined over $\mathbb{Q}$ with conductor N and (certain) paramodular forms of level N. The two talks discussed the history of the problem and recent advances on this modularity problem. In particular it discussed recent progress using restriction to Humbert surfaces.

Van Straten in a lively talk discussed the modularity of Calabi-Yau manifolds in relation with the paramodular conjecture. He discussed an example of candidates for paramodular forms of level 79 and 61 as part of an ongoing project.

The final talk of Don Zagier was given on the blackboards of the MPIM Bonn and discussed certain invariants that generalize Witten's r-spin intersection numbers and are associated to simply laced simple Lie algebras. The talk concentrated on the A_4 case and discussed differential equations associated to the generating functions of these invariants.

The organizers and participants of the workshop thank the Mathematisches Forschungsinstitut Oberfwolfach for making this hybrid workshop possible and for providing a great environment for it. Despite the restrictions it was a very stimulating event.

Workshop (hybrid meeting): Moduli spaces and Modular forms

Table of Contents

Abstracts

Unimodular Hunting

Gaëtan Chenevier

1. Introduction

Consider the standard Euclidean space $\mathbb{R}^n$, with inner product $x.y$. A lattice $L \subset \mathbb{R}^n$ is called *integral* if we have $x.y \in \mathbb{Z}$ for all $x, y \in L$, and *unimodular* if its covolume is 1. We denote by $\mathcal{L}_n$ the set of all integral unimodular lattices in $\mathbb{R}^n$, and by X_n the (finite) set of isometry classes of such lattices. Due to the works of many authors, including Gauss, Witt, Kneser, Niemeier, Conway-Sloane and Borcherds, representatives of X_n are known up to $n \leq 25$ (see [CS99]).

Theorem A. *We have* $|\mathrm{X}_{26}| = 2566$*,* $|\mathrm{X}_{27}| = 17059$ *and* $|\mathrm{X}_{28}| = 374062$*. In each cases, representatives for the isometry classes are listed in* [Lists].

The most difficult case of X_{28} is a joint work with Bill Allombert, in which we also classified the class of L in X_{29} with no vector $v \in L$ with $v.v \leq 2$ (there are 10092 such classes). Some of the lattices we found in Theorem A had already been discovered by Bacher and Venkov [BV01].

Of course, the proof of Theorem A involved computer calculations (we used PARI/GP). They took about 1 month of CPU time for X_{26}, 1 year for X_{27}, and 72 years for X_{28}. Fortunately, it can be checked directly that the lists given in [Lists] are complete by verifying that: (i) all given lattices L_i are non isometric, (ii) $\sum_i 1/|\mathrm{O}(L_i)|$ fits the Minkowski-Siegel-Smith mass formula. For step (i), it turns out that all the given L_i have a different configuration of vectors v with $v.v \leq 3$ (we use ad hoc invariants to distinguish them). For step (ii), we relied on the Plesken-Souvignier algorithm. In the end, checking steps (i) and (ii), hence proving Theorem A, only takes respectively 5 hours, 40 hours and 27 days in dimensions $26, 27$ and 28.

2. Kneser neighbors of I_n

A key tool in our proof is the notion of Kneser neighbor, a familiar construction in lattice theory. For $d \geq 1$ an integer and $L, N \in \mathcal{L}_n$, we say that N is a (cyclic) d-neighbor of L if we have $L/(L \cap N) \simeq \mathbb{Z}/d$. The d-neighbors of L are naturally parameterized by the finite quadric $\mathrm{C}_L(\mathbb{Z}/d)$ of *isotropic lines* ℓ in $L \otimes \mathbb{Z}/d$, which are the subgroups $\ell \subset L \otimes \mathbb{Z}/d$ satisfying $\ell \simeq \mathbb{Z}/d$ and $\ell.\ell \equiv 0 \bmod d$ (case d odd) or $\ell.\ell \equiv 0 \bmod 2d$ (case d even). To each $\ell \in \mathrm{C}_L(\mathbb{Z}/d)$ is associated a sublattice of L defined by $\mathrm{M}_d(\ell) = \{v \in L \,|\, v.\ell \equiv 0 \bmod d\}$. Also, we may always choose $w \in L$ generating ℓ with $w.w \equiv 0 \bmod d^2$ and set

$$\mathrm{N}_d(\ell) = \mathrm{M}_d(\ell) + \mathbb{Z}\frac{w}{d}.$$

It is easy to check that $\mathrm{N}_d(\ell)$ is in $\mathcal{L}_n$, and a d-neighbor of L satisfying $\mathrm{N}_d(\ell) \cap L = \mathrm{M}_d(\ell)$. When d is odd, it does not depend on the choice of w. When d is even, it takes exactly two possible values (not necessarily isometric) but to simplify the notations we denote any of them here by $\mathrm{N}_d(\ell)$.

Fact. *The d-neighbors of $L \in \mathcal{L}_n$ are exactly the $\mathrm{N}_d(\ell)$ for $\ell \in \mathrm{C}_L(\mathbb{Z}/d)$.*

We apply these constructions to the special case $L = \mathrm{I}_n = \mathbb{Z}^n$ (standard unimodular lattice). It is equivalent to give an $\ell \in \mathrm{C}_{\mathrm{I}_n}(\mathbb{Z}/d)$ and a vector $(x_i) \in \mathbb{Z}^n$ with coordinates prime to d and with $\sum_i x_i^2 \equiv 0 \bmod d$ (case d odd) or $\equiv 0 \bmod 2d$ (case d even). This provides both beautiful and compact definitions of unimodular lattices. For instance we have

$$\mathrm{N}_2(1^8) \simeq \mathrm{E}_8 \text{ and } \mathrm{N}_{94}(1,3,5,7,...,47) \simeq \text{Leech},$$

a construction of the Leech lattice attributed to Thompson by Conway and Sloane (note $2 \cdot 94 = 4 \cdot 47$ and $1^2 + 2^2 + \cdots + 23^2 \equiv 0 \bmod 47$).

This work emerged from the desire of the author to find such descriptions for all known unimodular lattices. After doing so, we learnt that it had already been done by Bacher years ago in dimension ≤ 25, see [Bac97]. All the representatives mentioned in Theorem A are given in [Lists] in the form $\mathrm{N}_d(x_1, \ldots, x_n)$.

3. Theoretical justification: statistics for p-neighbors

Consider more generally any genus $\mathcal{G}$ of integral lattices in $\mathbb{R}^n$. To simplify, we assume $n > 2$ and $\mathcal{G}$ is a single spinor genus in the sense of Eichler. For instance, $\mathcal{G}$ can be $\mathcal{L}_n^{\mathrm{odd}}$ or $\mathcal{L}_n^{\mathrm{even}}$ (odd or even lattices). For $L, L' \in \mathcal{G}$, and p an odd prime not dividing $\det L = \det L'$, we denote by $\mathrm{N}_p(L, L')$ the number of p-neighbors of L which are isometric to L'. Our second main result is

Theorem B. *We have* $\frac{\mathrm{N}_p(L,L')}{|\mathrm{C}_L(\mathbb{Z}/p)|} = \frac{1/|\mathrm{O}(L')|}{\mathrm{mass}\,\mathcal{G}} + \mathrm{O}(\frac{1}{\sqrt{p}})$ *for $p \to \infty$.*

We can even replace the $1/\sqrt{p}$ above by $1/p$ for $n > 4$. Note $|\mathrm{C}_L(\mathbb{Z}/p)| \sim p^{n-2}$ for $p \to \infty$. The theorem asserts that the probability to find L' as a p-neighbor of L tends to be proportional to the mass $1/|\mathrm{O}(L')|$ of L' (independently of L). In particular, this explains why the L' with large isometry groups are usually harder to construct this way, a striking fact in numerical applications.

The fact that $\mathrm{N}_p(L, L')$ is nonzero for p big enough had already been proved by Hsia and Jöchner in [HJ97]. Our proof of Theorem B uses quite deep results from the theory of automorphic forms, such as Arthur's theory [Art13] and the Jacquet-Shalika estimates towards the general Ramanujan conjecture. We also prove a number of variants (several spinor genera, level structures, biaised statistics).

4. Some ideas to prove Theorem A

If L is an integral lattice and $j \in \mathbb{Z}_{\geq 0}$, we denote by $\mathrm{R}_j(L)$ the finite set of vectors $v \in L$ with $v.v = j$. Arguing by induction on n, we easily reduce to classify the L in $\mathcal{L}_n$ with $\mathrm{R}_1(L) = \emptyset$. Also, $\mathrm{R}_2(L)$ has a natural structure of **ADE** root system. Using a computation of King [Kin03] we know, for any $n \leq 30$ and any **ADE** root system R in $\mathbb{R}^n$, the mass $\mathrm{m}_n(R)$ of all $L \in \mathcal{L}_n$ with $\mathrm{R}_1(L) = \emptyset$ and $\mathrm{R}_2(L) \simeq R$. This allows to split the classification "root system by root system".

For $n = 26$, 27 and 28, there are respectively 1086, 2797 and 4722 root systems R with $\mathrm{m}_n(R) \neq 0$. We do not know how to efficiently determine the isometry class of $\mathrm{R}_{\leq 3}(L) = \cup_{j\leq 3}\mathrm{R}_i(L)$, but use some ad hoc invariants, denoted here by $\mathrm{inv}_3(L)$.

The basic method is to enumerate, for $d = 2, 3, 4, \ldots$, all isotropic vectors $x = (x_i)$ in $\mathrm{I}_n \otimes \mathbb{Z}/d$, and to study the isometry classes of the associated neighbors $\mathrm{N}_d(x)$. As the isometry group $\mathrm{O}(\mathrm{I}_n)$ is the quite big $\{\pm 1\}^n \times \mathrm{S}_n$, and since two isotropic lines in the same orbit have isometric associated neighbors, we may restrict to elements x with $x_1 \leq x_2 \leq \cdots \leq x_n \leq d/2$. For a given x, we compute $|\mathrm{R}_1(\mathrm{N}_d(x))|$ (fast), and if it vanishes we compute the isomorphism class of $R = \mathrm{R}_2(\mathrm{N}_d(x))$ (a few `ms`), and if we still have to find lattices with root system R, we compute $\mathrm{inv}_3(\mathrm{N}_d(x))$ (may take a few sec). If we find a new invariant (hence a new lattice), we compute $1/|\mathrm{O}(\mathrm{N}_d(x))|$ using the Plesken-Souvignier algorithm (takes a few sec. using some tricks), and check using $\mathrm{m}_n(R)$ if we have found all lattices with root system R.

This method allows to find most lattices in X_n but not all, and other ideas have to be used to find the remaining ones (typically, lattices with very small mass). In the end, many root systems have to be studied case by case. When searching for lattices with given root system R, a simple but important constraint is the *visible root system*. By this we mean the root system $\mathrm{R}_2(\mathrm{M}_d(x))$, which is both trivial to compute and a subroot system of $\mathrm{R}_2(\mathrm{N}_d(x))$. By imposing the visible root system, which amounts to impose simple conditions on the x_i, we efficiently biaise the statistics in Theorem B and reach much higher d's. As an illustration, I discussed the case $R \simeq 10\,\mathbf{A}_1$ for $n = 26$ during the talk. We have $2^{11}\mathrm{m}_{26}(R) = 4424507/58060800$ and eventually find 7 classes $[L]$ in X_{26}, with $2^{11}/|\mathrm{O}(L)| = 1/32, 1/48, 1/48, 1/320, 1/6144, 1/3686400$ and $1/46448640$ respectively. The first 5 classes quickly appear by imposing a visible root system $8\,\mathbf{A}_1$ (4 for $d = 36$, and 1 for $d = 39$), but not the last 2 up to $d = 50$: they require more clever constructions (and appear for $d = 70$).

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Universal holomorphic symplectic varieties and Borcherds products

Shouhei Ma

In my talk I reported on my study [13], [14] of the Kodaira dimension of universal families of $K3$ surfaces and more generally holomorphic symplectic varieties using orthogonal modular forms. This is based on the following general correspondence. Let $X \to M$ be a smooth algebraic family of (lattice-)polarized holomorphic symplectic manifolds of dimension $2d$ with polarized Beauville lattice L of signature $(2, b)$. If $\Gamma < \mathrm{O}^+(L)$ contains the monodromy group, we have the period map $M \to \Gamma\backslash\mathcal{D}_L$. We take the n-fold fiber product $X \times_M \cdots \times_M X$ and let X_n be its smooth projective model. Let $S_k(\Gamma, \det)$ be the space of Γ-cusp forms on $\mathcal{D}_L$ of weight k and character det. Then, if the period map is dominant and generically finite, we have an injective map

$$S_{b+dn}(\Gamma, \det) \hookrightarrow H^0(K_{X_n}). \tag{1}$$

For the moduli space $\mathcal{F}_{g,n}$ of n-pointed $K3$ surfaces of genus g with at worst rational double points, we have a more refined correspondence between pluricanonical forms on the regular locus of $\mathcal{F}_{g,n}$ and certain modular forms.

We apply (1) to the study of the Kodaira dimension of X_n. We are mainly interested in the case when we have an explicit construction of the family, because $\kappa(X_n) \equiv b$ when the base modular variety $\Gamma\backslash\mathcal{D}_L$ is of general type. As a consequence of (1), we find that we always have $\kappa(X_n) = b$ in $n \gg 0$. Then we use quasi-pullback of the Borcherds Φ_{12} form ([4]) to produce an explicit cusp form, which itself is again a Borcherds product, and use it to study the explicit transition of $\kappa(X_n)$ from $-\infty$ to ≥ 0.

In the $K3$ case, if $F(g)$ denotes the quasi-pullback of Φ_{12}, then $\kappa(\mathcal{F}_{g,n}) \geq 0$ for $n \geq \mathrm{wt}(F(g)) - 19$. Thus $\mathrm{wt}(F(g)) - 19$ gives an "arithmetic" bound for $\kappa \geq 0$, which can be calculated explicitly by root number computation. When $g \gg 0$ so that $\mathrm{wt}(F(g)) \leq 19$, $F(g)$ was used to study $\kappa(\mathcal{F}_g)$ ([11], [8]). In this way $F(g)$ plays a crucial geometric role in *every* genus g.

In the opposite direction, we have a "geometric" bound for $\kappa = -\infty$ calculated from the Mukai models of $K3$ surfaces of genus $g \leq 20$ (see, e.g., [15]). We compare these two bounds of different nature, "arithmetic" for $\kappa \geq 0$ and "geometric" for $\kappa = -\infty$. The result is as in Table 1. In particular, we find the exact transition point at $g = 3, 4, 6, 12, 20$, and also nearly exact in some other g. Here the bounds for $\kappa = -\infty$ in $g = 8, 9, 10, 14, 22$ are due to Farkas-Verra [6], [7], and in $g = 11$ due to Barros [1]. Our bound for $\kappa \geq 0$ in $g = 11$ agrees with the geometric result of Barros-Mullane [2]. In this way $F(g)$ meets explicit geometry in small g.

As a byproduct we observe the following strange coincidence. In $3 \leq g \leq 10$, the classical and the Mukai models show that general $K3$ surfaces are linear sections of an (almost) homogeneous space embedded in a representation space $\mathbb{P}V_g$ of an algebraic group. Then we always have the equality

$$\mathrm{weight}(F(g)) - 19 = \dim(V_g). \tag{2}$$

Is this accidental?

TABLE 1. $\kappa(\mathcal{F}_{g,n})$

g	2	3	4	5	6	7	8	9	10	11	12
$\kappa \geq 0$	56	35	30	21	23	16	15	14	14	9	14
$\kappa = -\infty$	38	34	29	18	22	14	9	10	11	7	13

g	13	14	15	16	17	18	19	20	21	22
$\kappa \geq 0$	9	9	8	7	6	8	5	6	5	6
$\kappa = -\infty$	7	1		4		5		5		1

TABLE 2. Symplectic varieties of dimension > 2

	BD	DV	LLSS	IR	OG	IKKR
$\kappa \geq 0$	14	6	7	6	11	16
$\kappa = -\infty$	13	5	5	1	0	0

Our result for holomorphic symplectic varieties of dimension > 2 is as in Table 2. Here BD, DV, LLSS, IR, OG, IKKR mean the families of polarized symplectic varieties discovered by Beauville-Donagi [3], Debarre-Voisin [5], Lehn-Lehn-Sorger-van Straten [12], Iliev-Ranestad [10], O'Grady [16] and Iliev-Kapustka-Kapustka-Ranestad [9] respectively. In particular, we find the exact transition point in the Beauville-Donagi and Debarre-Voisin cases.

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On the Kodaira Dimension of $\mathcal{A}_6$

Riccardo Salvati Manni

In the theory of the classification of algebraic varieties a fundamental birational invariant is the Kodaira dimension. Let X be smooth projective defined over $\mathbb{C}$, we can consider the space of sections of the canonical bundle K_X and of its powers, mK_X. When X is not smooth, neither projective, we can consider a smooth completion of X, in fact these spaces of sections are birational invariants. Thus we can consider rational maps (i.e. they are not necessarily definied everywhere)

$$\Psi_m : X \to \mathbb{P}^n(\mathbb{C})$$

We can say that the variety X is of general type if , for $m \gg 0$ the map Ψ_m is birational onto its image. More generally, the Kodaira dimension of is defined as the dimension of the image. The Kodaira dimension is a birational invariant, that is, it does not depend on the representative in the birational equivalence class.

To the opposite of the varieties of general type we have the case in which $|mK_X| = \emptyset$, i.e the Kodaira dimension is $-\infty$. Among these varieties, an important role is played by the unirational varieties, i.e. those for which there exists a dominant rational morphism $\mathbb{P}^n \to X$.

We will report on the case of the moduli space of principally polarized abelian varieties $\mathcal{A}_n$,i.e of the pairs (A, Θ) with A an abelian variety and Θ a principal polarization. In the complex case we have an explicit realization of these moduli spaces as $\mathcal{A}_n = \mathcal{H}_n/\operatorname{Sp}(2n, \mathbb{Z})$, here $\mathcal{H}_n$ is the Siegel upper half space of degree n.

The problem of unirationality and Kodaira dimension of $\mathcal{A}_n$ is related to modular forms of weight k, i.e to holomorphic functions defined on $\mathcal{H}_n$ such that

$$f(M \cdot \tau) = \det(c\tau + d)^k f(\tau) \text{ for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Sp}(2n, \mathbb{Z}).$$

In fact, sections of $mK_{\mathcal{A}_n})$ are of the form

$$\omega = f(\tau)(d\tau)^{\otimes m},\ d\tau = d\tau_{11} \wedge d\tau_{12} \wedge \cdots \wedge d\tau_{nn}$$

and f a modular form of weight $m(n+1)$.

Let $\mathcal{A}_n^0$ be the set of smooth points of $\mathcal{A}_n$, thus the main problem is the extension of these forms to a smooth compactification. There are two kind s of obstructions produced by resolution of singularities and compactification. As first result at the begin of seventies Freitag in a sequence of papers proved that a canonical differential forms extends providing that the modular form f is a cusp form. Hence he was able to prove that $\mathcal{A}_n$ is not unirational for $n \equiv 0 \mod 24$,[4] . Moreover, at the same time, he proved the not unirationality of $\mathcal{A}_n$ for $n \equiv 1 \mod 8, n \geq 17$, [5], constructing explicitely sections of the sheaf of holomorphic differential forms, not of maximal degree. These results induced to the development of the study of modular forms with pluriharmonic coefficients and of vector valued modular forms.

At the end of seventies in a milestone paper, Tai, in [9], gave an criterion on the extensibility of pluri-canonical differential forms and proved that there are sufficiently many pluri-canonical forms which extend for $n \geq 9$, so that in these cases A_n is of general type. After that there have been extension of this result to the case $n \geq 8$, [6] and to $n \geq 7$, [8]. Meanwhile several authors gave proofs about the unirationality of A_n for $n \leq 5$, [1],[3],[7],[10]. Thus the only case that remained open whose the case $n = 6$. Recently, in [2], inspired by Freitag's method, Dittmann, Salvati Manni and Scheithauer gave explicitely the existence of a bicanonical differential forms on $\mathcal{A}_6$ proving the not unirationality of this space.

I shortly recall the construction of the cusp form of weight 14 related to the bicanonical differential forms. We used theta series with pluriharmonic coefficients. Let V be an euclidean vector space of dimension m, $\langle , \rangle$ is the scalar product, $L \subset V$ is an even unimodular lattice, $h = (h_1, \dots, h_g)$ a sequence of g vectors in $V_{\mathbb{C}}$ such that

$$Q(h) := Q(h,h) = \langle h_i, h_j \rangle = 0, \quad Q(h, \overline{h}) = \langle h_i, \overline{h_j} \rangle > 0,$$

then

$$\Theta_{L,h,k} = \sum_{x \in L^g} \det((Q(x,h))^k \exp(\pi i Tr(Q(x)\tau)$$

is a Siegel cusp form of weight $m/2 + k$ with Fourier coefficients

$$a(T) = \sum_{x \in L^g / Q(x) = T} \det(Q(x,h)^k$$

Now we take $\Lambda =$ Leech's lattice, hence $m = 24$, $k = 2$, thus $\Theta_{\Lambda,h,2}$ has weight 14 and vanishing at the boundary at least 2 since Leech's lattice is even and has no roots.

We are able to show that $\Theta_{\Lambda,h,2} \neq 0$. This is obtained considering a suitable sequence h, namely

$$\begin{aligned}
h_1 &= (i\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\
h_2 &= (0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ i) \\
h_3 &= (0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ i\ 0\ 0\ 0) \\
h_4 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ i\ 0\ 0\ 0\ 0\ 0) \\
h_5 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ i\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\
h_6 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ i\ 0\ 0\ 0\ 0)
\end{aligned}$$

Hence, summing up all previous results, we got

Theorem 1. *$\mathcal{A}_g$ is unirational* $\iff$ $g \leq 5$

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On the subring of special cycles

Stephen Kudla

For a totally real field F with $d = |F:\mathbb{Q}|$, suppose that V, $(\ ,\ )$ is an inner product space with

$$\mathrm{sig}(V_\sigma) = \begin{cases} (m,2) & \text{for } \sigma \in \Sigma_+(V), \\ (m+2,0) & \text{otherwise.} \end{cases}$$

Here $\Sigma_+(V)$ is a set of real embeddings $\sigma : F \to \mathbb{R}$ with $|\Sigma_+(V)| = d_+$. and $V_\sigma = V \otimes_{F,\sigma} \mathbb{R}$. Assume that $0 < d_+$ and that V is anisotropic. Let $G = R_{F/\mathbb{Q}}\mathrm{GSpin}(V)$ and let

$$D = \prod_{\sigma \in \Sigma_+(V)} D_\sigma,$$

where D_σ is the space of oriented negative 2-planes in V_σ. For a neat open compact subgroup $K \subset G(\mathbb{A}_f)$ of the finite adèles of G, the coset space

$$S_K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$$

is the complex points of a smooth projective variety of dimension md_+, a Shimura variety of orthogonal type. The Betti cohomology groups of S_K with complex coefficients define a graded ring with an inner product defined by taking the normalized degree of the cup product

$$\langle z, z' \rangle = \deg_K(z \cup z') = \mathrm{vol}(K)\ \deg_K^\natural(z \cup z').$$

For $1 \le n \le m$, $T \in \mathrm{Sym}_n(F)_{\ge 0}$, and a weight function $\varphi \in S(V(\mathbb{A}_f)^n)^K$, there is a special cycle class

$$Z(T, \varphi, K) \in H^{2nd_+}(S_K),$$

defined as a weighted linear combination of the cycle classes of Shimura subvarieties of orthogonal type. Note that these classes occur in degrees that are multiples of d_+. These classes behave well under pullbacks. More precisely, for an open compact subgroup $K' \subset K$,

$$\mathrm{pr}^*(Z(T,\varphi,K)) = Z(T,\varphi,K'), \qquad \mathrm{pr} : S_{K'} \longrightarrow S_K,$$

so that there are well defined classes $Z(T,\varphi)$ in the limit

$$H^\bullet(S) = \varinjlim_K H^\bullet(S_K).$$

Theorem A. *For $T_i \in \mathrm{Sym}_{n_i}(F)$ and $\varphi_i \in S(V(\mathbb{A}_f)^{n_i})$, there is a product formula*

$$Z(T_1,\varphi_1)\cdot Z(T_2,\varphi_2) = \sum_{\substack{T\in \mathrm{Sym}_{n_1+n_2}(F)_{\geq 0}\\ T=\begin{pmatrix} T_1 & * \\ {}^t* & T_2\end{pmatrix}}} Z(T,\varphi_1\otimes\varphi_2).$$

Thus the special cycle classes form a subring of the cohomology

$$\mathrm{SC}(V)^\natural = \bigoplus_{n=0}^{m} \mathrm{SC}^n(V)^\natural \ \subset\ \bigoplus_{n=0}^{m} H^{2nd_+}(Sh(V)).$$

Note the shift in the grading, eliminating $2d_+$. Since our information about special cycle classes is based on their inner products, we pass to a quotient. By associativity of the cup product, the radical of the restriction of this pairing to $\mathrm{SC}^\bullet(V)^\natural$ is an ideal, and we define the **reduced ring of special cycles** is the subquotient of the cohomology ring

$$\mathrm{SC}(V) := \mathrm{SC}(V)^\natural/\mathrm{Rad}.$$

The aim of this talk was to explain how the structure of this subring is controlled by the Fourier coefficients of pullbacks of Hilbert-Siegel Eisenstein series.

For a Schwartz function $\varphi \in S(V^m(\mathbb{A}_f))$, there is a Hilbert-Siegel Eisenstein series of genus m and parallel weight $(\kappa,\dots,\kappa)$,

$$E(\tau,s_0,\varphi), \qquad \boldsymbol{\tau} = (\tau_1,\dots,\tau_d)\in \mathfrak{H}_m^d, \quad s_0 = \frac{1}{2},$$

where $\kappa = \frac{1}{2}m+1$, arising from Siegel-Weil data $\varphi_\infty\otimes\varphi$ for a certain archimedean Schwartz function $\varphi_\infty \in \mathcal{S}(V^m\otimes_{\mathbb{Q}}\mathbb{R})$. It is obtained by analytic continuation from a convergent Eisenstein series $E(\tau,s,\varphi)$ defined in a half-plane $\Re(s) > \frac{1}{2}(m+1)$, [7].

For $n_1+n_2=m$ and for $\varphi=\varphi_1\otimes\varphi_2$, consider the pullback of this Eisenstein series under the diagonal map

$$\mathfrak{H}_{n_1}^d \times \mathfrak{H}_{n_2}^d \longrightarrow \mathfrak{H}_m^d.$$

Write the Fourier expansion of this function as

$$E(\begin{pmatrix}\boldsymbol{\tau}_1 & \\ & \boldsymbol{\tau}_2\end{pmatrix}, \frac{1}{2}, \varphi_1\otimes\varphi_2) = \sum_{T_1\in \mathrm{Sym}_{n_1}(F)_{\geq 0}} \sum_{T_2\in \mathrm{Sym}_{n_2}(F)_{\geq 0}} A(T_1,T_2;\varphi_1\otimes\varphi_2)\, \mathbf{q}_1^{T_1}\,\mathbf{q}_2^{T_2}.$$

Theorem B. *For $n_1+n_2=m$, $T_i\in \mathrm{Sym}_{n_i}(F)_{\geq 0}$ and $\varphi_i \in \mathcal{S}(V^{n_i}(\mathbb{A}_f))$, the inner product of the special cycles $z(T_1,\varphi_1)$ and $z(T_2,\varphi_2)$ is given by*

$$\langle z(T_1,\varphi_1), z(T_2,\varphi_2)\rangle = A(T_1,T_2;\varphi_1\otimes\varphi_2).$$

In short, the inner products of special cycle classes are given by Fourier coefficients of pullbacks of Hilbert-Siegel Eisenstein series.

But one can get more!
For $n_1 + n_2 + n_3 = m$, and for weight functions $\varphi_i \in S(V(\mathbb{A}_f)^{n_i})$, write

$$E(\begin{pmatrix} \boldsymbol{\tau}_1 & & \\ & \boldsymbol{\tau}_2 & \\ & & \boldsymbol{\tau}_3 \end{pmatrix}, \frac{1}{2}, \varphi_1 \otimes \varphi_2 \otimes \varphi_3)$$
$$= \sum_{T_1 \in \mathrm{Sym}_{n_1}(F)_{\geq 0}} \sum_{T_2 \in \mathrm{Sym}_{n_2}(F)_{\geq 0}} \sum_{T_3 \in \mathrm{Sym}_{n_3}(F)_{\geq 0}} A(T_1, T_2, T_3; \varphi_1 \otimes \varphi_2 \otimes \varphi_3)\, \mathbf{q}_1^{T_1}\, \mathbf{q}_2^{T_2}\, \mathbf{q}_3^{T_3}$$

for the Fourier expansion of the pullback of the Eisenstein series under the map

$$\mathfrak{H}_{n_1}^d \times \mathfrak{H}_{n_2}^d \times \mathfrak{H}_{n_3}^d \longrightarrow \mathfrak{H}_m^d.$$

Theorem C. *For $n_1 + n_2 + n_3 = m$,*

$$\langle z(T_1, \varphi_1) \cdot z(T_2, \varphi_2), z(T_3, \varphi_3) \rangle = A(T_1, T_2, T_3; \varphi_1 \otimes \varphi_2 \otimes \varphi_3).$$

Since the pairing $\langle\ ,\ \rangle$ is non-degenerate pairing on $\mathrm{SC}(V)$, this last formula uniquely determines the product

$$z(T_1, \varphi_1) \cdot z(T_2, \varphi_2) \ \in\ \mathrm{SC}^{n_1+n_2}(V).$$

Corollary. *The structure of the ring $\mathrm{SC}(V)$ is determined by the Fourier coefficients of triple pullbacks of Hilbert-Siegel Eisenstein series.*

The results described here are consequences of the old joint work of the author with John Millson on cohomological theta series and special cycles, [4], [5], [6], and with Steve Rallis on the extended Siegel-Weil formula, [7]. Full proofs are given in the second of the preprints [2] and [3]. The case $d_+ = 1$ was considered in earlier work [1].

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The Kodaira dimension of the moduli space of curves: new progress on a century-old problem

Gavril Farkas

(joint work with Dave Jensen, Sam Payne)

Following a principle due to Mumford, most moduli spaces that appear in algebraic geometry (classifying curves, abelian varieties, $K3$ surfaces) are of general type, with a finite number of exceptions, which are unirational, or at least uniruled. Understanding the transition from negative Kodaira dimension to being of general type is usually quite difficult. With one exception (the moduli space of spin curves), for all these moduli spaces there are notorious open cases, when the Kodaira dimension is not known. The aim of the talk was to shed some light on this change of the birational nature of the moduli space of curves.

In a series of landmark papers [8], [7], [2] published in the 1980s, Harris, Mumford and Eisenbud proved that $\overline{\mathcal{M}}_g$ is a variety of general type for $g > 23$. This contrasts with the classical result of Severi [10] that $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$ (see [1] for a beautiful modern treatment) and with the more recent results of Chang-Ran, Sernesi, Verra [11] and Schreyer, which summarized, amount to the statement that $\overline{\mathcal{M}}_g$ is rationally connected or unirational for $g \leq 15$. The Slope Conjecture of Harris and Morrison predicted that the Brill-Noether divisors are the effective divisors on $\overline{\mathcal{M}}_g$ having minimal slope $6 + \frac{12}{g+1}$. This led people to expect that the moduli space changes from uniruledness to being of general type precisely at genus $g = 23$. However the Slope Conjecture turned out to be false and there are instances of effective divisors on $\overline{\mathcal{M}}_g$ for infinitely many genera $g \geq 10$ having slope less than $6 + \frac{12}{g+1}$, see [4], [5]. In view of these examples it is to be expected that there should be an effective divisor of slope less than $\frac{13}{2} = 6 + \frac{12}{24}$ on $\overline{\mathcal{M}}_{23}$ as well, which would imply that $\overline{\mathcal{M}}_{23}$ is of general type. The best known result on $\overline{\mathcal{M}}_{23}$ is the statement $\kappa(\overline{\mathcal{M}}_{23}) \geq 2$, proven in [3] via a study of the relative position of the three Brill-Noether divisors.

The main aim of the talk was to discuss the recent breakthrough result in [6]:

Theorem 1. *Both moduli spaces $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type.*

This result is obtained by constructing an effective divisor of slope less than $\frac{13}{2}$ on the corresponding moduli space. In this abstract, we shall restrict to the case of $\mathcal{M}_{23}$. By Brill-Noether theory, a general curve C of genus 23 carries a two-dimensional family of linear series $L \in W^6_{26}(C)$, all satisfying $h^1(C,L) = 3$. Each of these linear series is complete and very ample. Consider the multiplication map

$$\phi_L : \mathrm{Sym}^2 H^0(C,L) \to H^0(C, L^{\otimes 2}).$$

By Riemann-Roch $h^0(C, L^{\otimes 2}) = 30$, whereas $\dim \mathrm{Sym}^2 H^0(C,L) = 28$. Imposing the condition that ϕ_L be non-injective, one expects a codimension 3 locus inside the parameter space of pairs $[C,L]$. Since this parameter space has 2-dimensional fibres over $\mathcal{M}_{23}$, by projection, one expects a divisor inside the moduli space $\mathcal{M}_{23}$.

Theorem 2. *The following locus consisting of curves of genus* 23

$$\mathfrak{D}:=\Big\{[C]\in\mathcal{M}_{23}:\exists L\in W^6_{26}(C)\ \textit{with}\ \mathrm{Sym}^2H^0(C,L)\xrightarrow{\phi_L}H^0(C,L^{\otimes 2})\ \textit{not injective}\Big\}$$

is an effective divisor on $\mathcal{M}_{23}$. *The class of its compactification inside* $\overline{\mathcal{M}}_{23}$ *equals*

$$[\widetilde{\mathfrak{D}}]=\frac{4}{9}\binom{19}{8}\Big(470749\lambda-72725\ \delta_0-401951\ \delta_1-\sum_{j=2}^{11}b_j\ \delta_j\Big)\in CH^1(\overline{\mathcal{M}}_{23}),$$

where $b_j\geq b_1$ *for* $j\geq 2$. *In particular,* $s\big([\widetilde{\mathfrak{D}}]^{\mathrm{virt}}\big)=\frac{470749}{72725}=6.473\ldots<\frac{13}{2}$.

The question whether the virtual divisor $\mathfrak{D}$ is an actual divisor is very much related to the Maximal Rank Conjecture, originally due to Eisenbud and Harris and predicting that for a pair $[C,L]$, where C is a general curve of genus g and $L\in W^r_d(C)$ is a general linear system, the multiplication of global sections $\phi_L:\mathrm{Sym}^2H^0(C,L)\to H^0(C,L^{\otimes 2})$ is of maximal rank. The conjecture has been the focus of much attention, both a couple of decades ago using embedded degenerations in projective space, as well as recently using tropical geometry, or limit linear series.

A refined version of the Maximal Rank Conjecture, taking into account *every* linear series $L\in W^r_d(C)$ on a general curve (rather than the general one), has been put forward by myself. The *Strong Maximal Rank Conjecture*, motivated by applications to the birational geometry of the moduli space of curves, predicts that for a general curve C of genus g and for positive integers r,d such that $0\leq\rho(g,r,d)\leq r-2$, the determinantal variety

$$\Big\{L\in W^r_d(C):\phi_L:\mathrm{Sym}^2H^0(C,L)\to H^0(C,L^{\otimes 2})\text{ is not of maximal rank}\Big\}$$

has the expected dimension. The Strong Maximal Rank Conjecture in the case $g=23$, $d=26$ and $r=6$ amounts to the statement that the virtual divisor $\mathfrak{D}$ on $\mathcal{M}_{23}$ is a genuine divisor. Using novel tropical methods these cases of the Strong Maximal Conjecture are established in [6].

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Vertex algebras and Teichmüller modular forms

Giulio Codogni

Vertex algebras have been introduced in the 80's by physicists working in conformal field theory and string theory; the formal mathematical definition was given by the Fields medallist Richard Borcherds in 1986. Many problems about vertex algebras are still open, in particular there is still a lot of ongoing research on their classification. In this note we focus on holomorphic vertex algebras, also known as holomorphic vertex operator algebras.

Physicists, when studying Conformal Field Theory and String Theory, have always combined vertex algebras with Riemann surfaces and their moduli spaces; following our work [2], we would like to explain how to make this connection rigorous, and how it can provide new insights about both mathematical objects.

Our work mirrors the relation between quadratic forms, Siegel modular forms, and the moduli space of principally polarized abelian varieties, so let us describe it before getting into the actual matter.

The Smith-Minkowski-Siegel mass formula shows that there exists only finitely many even, unimodular, positive definite quadratic forms of a given rank, and it moreover gives an effective bound on their number. It is classically known how to associate to such a quadratic form Q of rank c, a section $\Theta_{Q,g}$ of the $c/2$-th power of the Hodge bundle L_g over the Satake compactification $\mathcal{A}_g^S$ of the moduli space of g-dimensional princiaplly polarized abelian varietiies; in other words, $\Theta_{Q,g}$ is a Siegel modular forms of degree g and weight $c/2$. Fixed two quadratic forms P and Q, it is possible to find a g such that the two theta series $\Theta_{P,g}$ and $\Theta_{Q,g}$ are distinct. A classical result of Freitag [5], see also [1, Section 3], states that, fixed c, there exists a value $g_c > 0$ such that, for all $g \geq g_c$, $h^0(\mathcal{A}_g^S, L_g^{\otimes c/2})$ is equal to the number of even, unimodular positive definite quadratic forms of rank c (so, in particular, it does not depend on g), and theta series provide a basis of $H^0(\mathcal{A}_g^S, L_g^{\otimes c/2})$.

We now consider a holomorphic vertex algebra V of central charge c, and the moduli space of curves $\overline{\mathcal{M}}_g$ (when $g = 1$, we rather consider $\overline{\mathcal{M}}_{1,1}$) endowed with the Hodge line bundle λ_g. Sections of the k-th power of λ_g are called Teichmüller modular forms of degree g and weight k.

In [2], for every $g \geq 1$, we associate to V a Teichmüller modular form $1_g(V)$ of degree g and weight $c/2$, which is invariant under the natural gluing morphisms. When V is a lattice vertex algbera, which means that it is constructed out of a rank c even, positive definite, unimodular quadratic form, this section is the pull-back of the theta series $\Theta_{Q,g}$ via the jacobian morphism $j\colon \overline{\mathcal{M}}_g \to \mathcal{A}_g^S$. For vertex algebras which do not come from quadratic forms, we conjecture that, for g big enough, the partition functions are not pull-backs of Siegel modular forms.

We call $1_g(V)$ the genus g *partition function* of the vertex algebra V (or, from physicists' shoes, the partition function of the conformal field theory defined by V). The existence of this section is hinted in many works such as [7, 8], and constructed for low values of g and special kind of vertex algebras in [9, 11]. Our apraoch is however totally different. In all previous works, authors define a function of the Schottky coordinates, which are geometrically meaningful coordinates near the boundary of $\overline{\mathcal{M}}_g$, and then the they try to extend this function to a section of $\lambda_g^{\otimes c/2}$ on the entire moduli space. In [2], we directly construct the section, and then we show that it has the predicted expansion near the boundary of the moduli space.

Conjecturally, the vertex algebra is uniquely determined by the collection of partition functions $\{1_g(V)\}_{g\geq 1}$, but this is a widely open problem.

The newly established relation between vertex algebras and Teichmüller modular forms permits to combine the two theories and discover new results. Let us give a first example. In [3], it is shown that, given two even, positive definite, unimodular quadratic forms P and Q, there exists a g such that the pull-back of the two theta series $\Theta_{P,g}$ and $\Theta_{Q,g}$ to $\overline{\mathcal{M}}_g$ are different. This, combined with the results from [2] that we have just described, gives a new proof that the vertex algebras associated to P and Q are not isomorphic, see [2, Corollary 6.4].

Let us now focus on a way more ambitious problem. One of the most prominent examples of vertex algebras is the moonshine vertex algebra. It was conjectured in 1993 (cf. [6]) that this is the unique holomorphic vertex algebra of central charge 24 such that the weight one space is trivial. Let us also recall that the slope s_g of $\overline{\mathcal{M}}_g$ is the smallest number such that if a degree g and weight k Teichmüller modular form f vanishes along the boundary with multiplicity at least ks_g, then f itself is identically zero. A long standing conjecture, proven just for low value of g, claims that $s_g > 6$ for all g, see [4]. In [2, Corollary 1.5], we show that if this conjecture about s_g is true, and if the partition functions uniquely determinate the vertex algebra, then the moonshine vertex algebra is the unique holomorphic vertex algebra of central charge 24 such that the weight one space is trivial.

We finish this short note mentioning the problem of finding a mass formula for vertex algebras. It is not known, though expected because of the analogy with quadratic forms, if there exists only finitely many holomorphic vertex algebras of a given central charge c. In [2, Corollary 1.6], we relate this problem to the stabilization of $h^0(\overline{\mathcal{M}}_g, \lambda_g^{\otimes c/2})$ when c is fixed and g grows. More generally, we expect that the partition functions of holomorphic vertex algebras of central charge c provide a basis of the space of degree g and weight $c/2$ Teichmüller modular forms for all g big enough. In [6], it is conjectured that there are exactly 71 holomorphic vertex algbras with $c = 24$; the author also propose a complete set of invariants, now known as Schellekens' list. One of the last work on this problem is the very recent [10], where the conjecture is almost completely settled: the only missing part is to show that the moonshine vertex algebra is the unique holomorphic vertex algebra of central charge 24 such that the weight one space is trivial. In view of these results, of the analogy with quadratic forms, and of [2], we speculate that

$$\dim H^0(\overline{\mathcal{M}}_g, \lambda_g^{\otimes 12}) = 71 \qquad \text{for all } g \gg 0\,.$$

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Arithmetic volumes of unitary Shimura varieties

Ben Howard
(joint work with Jan Bruinier)

On the integral model of the usual modular curve parametrizing elliptic curves, there is a metrized line bundle of weight one modular forms. One can view this metrized line bundle as an element in the codimension arithmetic Chow group of Gillet-Soulé, and compute its self-intersection. It is a theorem of Bost and Kühn [4] that this self-intersection is essentially the logarithmic derivative of the Riemann zeta function at $s = -1$. A similar result with the modular curve replaced by a quaternionic Shimura curve was subsequently proved by Kudla-Rapoport-Yang [3].

The main result of the talk is a generalization of this to $\mathrm{GU}(n-1,1)$ Shimura varieties. The integral model of such a Shimura variety carries a natural metrized line bundle, whose arithmetic volume is defined as its iterated self-intersection of top degree in the arithmetic Chow ring.

When $n = 2$ the Shimura variety is essentially a modular curve or quaternionic Shimura curve, and the arithmetic volume in question can be computed using the results mentioned above [2]. When $n > 2$, the arithmetic volume can be computed inductively, by using the theory of Borcherds products to express the line bundle in question as a linear combination of Kudla-Rapoport divisors, each of which is essentially a unitary Shimura variety in one dimension lower. In all cases, the volume is given as a sum of logarithmic derivatives of Dirichlet L-functions at integer points.

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Free algebras of modular forms on type IV symmetric domains

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(joint work with Brandon Williams)

Let Γ be an arithmetic group on a complex symmetric domain $\mathcal{D}$. Automorphic forms on $\mathcal{D}$ for Γ are very interesting objects in mathematics. All such modular forms of integral weight form a graded algebra $M_*(\Gamma)$ over $\mathbb{C}$. By the theory of Baily–Borel [1], $M_*(\Gamma)$ is finitely generated over $\mathbb{C}$ and the Satake–Baily–Borel compactification of the modular variety $\mathcal{D}/\Gamma$ is a projective variety determined by $\mathrm{Proj}(M_*(\Gamma))$. But it is very difficult to find the generators because this is equivalent to find a projective model of the modular variety. When $M_*(\Gamma)$ is freely generated, it is isomorphic to a polynomial algebra and thus has the simplest structure. However, free algebras of modular forms are very rare and there is no general approach to construct such algebras. If $M_*(\Gamma)$ is free, then Γ must be generated by reflections. It is known that reflections exist only in two infinite families of symmetric domains: complex balls and symmetric domains of type IV in Cartan's classification. In this report, we introduce our recent work on the classification and construction of free algebras of modular forms on type IV symmetric domains in which case the groups Γ are orthogonal groups of signature $(n, 2)$. We assume that $n \geq 3$. A famous theorem of Igusa asserts that the algebra of even-weight Siegel modular forms of genus 2 is freely generated by forms of weights 4, 6, 10, 12. This is the first free algebra of orthogonal modular forms in high dimension. It was proved in [3] that $M_*(\Gamma)$ is never free when the dimension

n is larger than 10. We next establish an automorphic approach to classify and construct free algebras of orthogonal modular forms.

As an analogue of Rankin-Cohen-Ibukiyama differential operators for Siegel modular forms, we define the Jacobian of orthogonal modular forms. Let M be even lattice of signature $(n, 2)$ and Γ be a finite index subgroup of $\mathrm{O}^+(M)$. We take $n+1$ modular forms for Γ of weights $k_1, ..., k_{n+1}$. If they are algebraically independent over $\mathbb{C}$, then their Jacobian is a cusp form of weight $n + \sum_{t=1}^{n+1} k_t$ for Γ with the determinant character. Following Vinberg's insights [4], we are able to prove the following result which gives a necessary and sufficient condition for the graded algebra of modular forms for Γ being free.

Theorem 1 (see [6])**.**

(1) *If $M_*(\Gamma)$ is free, then the Jacobian of the $n+1$ free generators defines a cusp form for Γ with the determinant character which vanishes exactly on all mirrors of reflections in Γ with multiplicity one.*

(2) *If there is a Jacobian of $n+1$ modular forms for Γ which vanishes precisely on all mirrors of reflections in Γ with multiplicity one, then $M_*(\Gamma)$ is freely generated by the $n+1$ forms and Γ is generated by all reflections whose mirrors are contained in the divisor of the Jacobian.*

We first derive an explicit classification from the necessary condition. The modular form with special divisor in the above theorem is called reflective in the literature. Reflective modular forms have many applications in generalized Kac–Moody algebras, reflection groups and algebraic geometry, and the number of such modular forms is finite. In [5] we established an approach to classify reflective modular forms based on the theory of Jacobi forms of lattice index. Applying this approach to the case below, we find that if $M_*(\Gamma)$ is a free algebra then Γ corresponds to a root system of the same rank as L and the Coxeter numbers of the irreducible components of the root system satisfy some conditions. It is enough to deduce the following theorem from these conditions.

Theorem 2 (see [6])**.** *Let $M = 2U \oplus L$ be an even lattice of signature $(n, 2)$ splitting two hyperbolic planes. Suppose $\Gamma < \mathrm{O}^+(M)$ is a subgroup containing the discriminant kernel $\widetilde{\mathrm{O}}^+(M)$. If $M_*(\Gamma)$ is a free algebra, then (L, Γ) can only take $(E_8, \mathrm{O}^+(2U \oplus E_8))$ and one of the 25 pairs described in the next theorem.*

Theorem 3 (see [8])**.** *Let R be a root system of type $A_r (1 \le r \le 7)$, $B_r (2 \le r \le 4)$, $D_r (4 \le r \le 8)$, $C_r (3 \le r \le 8)$, G_2, F_4, E_6, or E_7. We define $\Gamma_R < \mathrm{O}^+(2U \oplus L_R)$ as the subgroup generated by $\widetilde{\mathrm{O}}^+(2U \oplus L_R)$ and $W(R)$, where $W(R)$ is the Weyl group of R, and L_R is the root lattice generated by R (we rescale its bilinear form by 2 such that it is even when L_R is an odd lattice). Then $M_*(\Gamma_R)$ is freely generated by $r+3$ forms of weights 4, 6, and $k_j + 12m_j$, $1 \le j \le r+1$.*

The case of E_8 was proved by Hashimoto and Ueda in 2014 using the property of moduli space of lattice-polarized K3 surfaces. It is well-known that Jacobi forms appear in the Fourier–Jacobi expansions of orthogonal modular forms at an 1-dimensional cusp. The structure result is proved using this connection. The above parameters k_j and m_j are weights and indices of generators of the bigraded ring of Weyl invariant weak Jacobi forms associated to the root system R (see [7]). Besides, we construct the generators as additive lifts of some particular Jacobi forms, e.g. Jacobi Eisenstein series.

Our sufficient condition is very useful to construct free algebras of modular forms. We present two applications. We first prove the following theorem which provides some smooth modular varieties of orthogonal type isomorphic to projective spaces.

Theorem 4 (see [9]). *There are* 16 *reflection groups* Γ *acting on type IV symmetric domains* $\mathcal{D}$*, for which the graded algebras of modular forms are freely generated by forms of the same weight, and in particular the Satake–Baily–Borel compactification of* $\mathcal{D}/\Gamma$ *is isomorphic to the projective space of dimension 3 or 4.*

Four of these are previously known results of Runge (1993), Matsumoto (1993), Freitag–Salvati Manni (2006) and Perna (2016). In order to prove these results using our criterion, we only need to choose a suitable lattice model and construct the potential Jacobian as a reflective modular form using the Borcherds automorphic product.

As the second application, we construct free algebras of modular forms for simple lattices. An even integral lattice M of signature $(n, 2)$ is called *simple* if the dual Weil representation attached to M admits no cusp forms of weight $1 + n/2$. In other words, M is simple if and only if every Heegner divisor on the modular variety attached to M occurs as the divisor of a Borcherds product. In 2016, Bruinier, Ehlen and Freitag [2] proved that there are finitely many simple lattices and gave a full classification of simple lattices. For such lattices, we prove the following theorem.

Theorem 5 (see [10]). *Let* M *be a simple lattice of signature* $(n, 2)$*, with* $3 \leq n \leq 10$*. (37 such lattices)*

1. *Let* $\mathrm{O}_r(M)$ *denote the subgroup generated by all reflections in* $\mathrm{O}^+(M)$*. Then the graded ring of modular forms* $M_*(\mathrm{O}_r(M))$ *is freely generated.*
2. *Let* $\widetilde{\mathrm{O}}_r(M)$ *denote the subgroup generated by all reflections in* $\widetilde{\mathrm{O}}^+(M)$*. With five exceptions, the ring* $M_*(\widetilde{\mathrm{O}}_r(M))$ *is freely generated.*

In the end, we propose an interesting conjecture.

Conjecture 6 (see [6]). *Let* $\Gamma < \mathrm{O}^+(M)$ *be a finite index subgroup generated by reflections. Let* Γ' *be a finite index subgroup of* Γ*. If* $M_*(\Gamma')$ *is a free algebra, then the smaller algebra* $M_*(\Gamma)$ *is also free.*

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On formal Fourier–Jacobi expansions

Jürg Kramer

In our talk, we would like to revisit the problem of formal Fourier–Jacobi expansions giving rise to Siegel modular forms investigated by J. Bruinier and M. Raum in [1]. This problem arises from the modularity conjecture for Shimura varieties associated to orthogonal groups of signature $(n, 2)$ stated by S. Kudla in [5] and from its reformulation by W. Zhang in [6]. In particular, we will give an alternative approach to the one pursued in [1], which allows to transfer the result from the complex category to the arithmetic setting.

Let us now be more specific and fix some notation. We let $\mathbb{H}_g$ denote the Siegel upper half-space of degree g, which allows an action of the Siegel modular group $\Gamma_g := \mathrm{Sp}_g(\mathbb{Z})$ by fractional linear transformations. It is known that the quotient space $\mathscr{A}_g(\mathbb{C}) := \Gamma_g\backslash\mathbb{H}_g$ equals the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$. Denoting by $\pi\colon \mathscr{B}_g(\mathbb{C}) \to \mathscr{A}_g(\mathbb{C})$ the universal abelian variety and by e the zero section, the Hodge bundle $\omega_\mathbb{C}$ is given by $e^* \det(\Omega^1_{\mathscr{B}_g(\mathbb{C})/\mathscr{A}_g(\mathbb{C})})$. The $\mathbb{C}$-vector space $M_k(\Gamma_g)$ of modular forms of weight k for Γ_g is then obtained as the space of global sections $\Gamma(\mathscr{A}_g(\mathbb{C}), \omega_\mathbb{C}^{\otimes k})$.

A Siegel modular form $f \in M_k(\Gamma_g)$ is now known to have various expansions. For example, writing $\mathbb{H}_g \ni Z = \left(\begin{smallmatrix} \tau & z \\ z^t & \tau' \end{smallmatrix}\right)$ with $\tau \in \mathbb{H}_{g-1}, \tau' \in \mathbb{H}_1$, and $z \in \mathbb{C}^{g-1}$, then f has a so-called Fourier–Jacobi expansion (in codimension 1), which is of the form

$$f(Z) = \sum_{m\in\mathbb{N}} f_m(\tau, z)\, q'^m \quad (q' := \exp(2\pi i\tau')), \tag{1}$$

where the functions $f_m(\tau, z)$ are Jacobi forms of weight k and index m for the Jacobi group $\Gamma_{g-1} \ltimes \mathbb{Z}^{g-1}$, in other words $f_m \in \Gamma(\mathscr{B}_{g-1}(\mathbb{C}), \mathscr{L}_\mathbb{C}^{\otimes m} \otimes \pi^*\omega_\mathbb{C}^{\otimes k})$, where $\mathscr{L}_\mathbb{C}$ is the pull-back of the Poincaré bundle on $\mathscr{B}_{g-1}(\mathbb{C}) \times \mathscr{B}_{g-1}(\mathbb{C})$ by the diagonal morphism Δ.

Substituting the Fourier expansions of the Jacobi forms f_m, i.e.,

$$f_m(\tau,z)=\sum_{\substack{n\in\mathrm{Sym}_{g-1}(\mathbb{Q}),\,\text{half-integral}\\ r\in\mathbb{Z}^{g-1},\,4mn-rr^t\geq 0}} c_m(n,r)\exp(2\pi i(\mathrm{tr}(n\tau)+r^tz)), \tag{2}$$

into (1), we obtain the Fourier expansion of the Siegel modular form under consideration

$$f(Z)=\sum_{N\in\mathrm{Sym}_g(\mathbb{Q}),\,N\geq 0,\,\text{half-integral}} \exp(2\pi i\mathrm{tr}(NZ)); \tag{3}$$

here we have set $c(N):=c_m(n,r)$ with $N:=\begin{pmatrix} n & r/2\\ r^t/2 & m\end{pmatrix}$. As a consequence of the modularity of f, we note that the Fourier coefficients in (3) satisfy the symmetry condition

$$c(u^tNu)=c(N) \tag{4}$$

for all $u\in\mathrm{GL}_g(\mathbb{Z})$.

The main result of the paper [1] now states that given a *formal* Fourier–Jacobi expansion $\sum_{m\in\mathbb{N}} f_m(\tau,z)\,q'^m$ with $f_m\in\Gamma(\mathscr{B}_{g-1}(\mathbb{C}),\mathscr{L}_{\mathbb{C}}^{\otimes m}\otimes\pi^*\omega_{\mathbb{C}}^{\otimes k})$ and such that the Fourier coefficients $c_m(n,r)$ of the Fourier expansions (3) satisfy the symmetry condition (4), then the given formal expansion is the Fourier–Jacobi expansion of a Siegel modular form f as described above. The proof proceeds in the analytic category and starts with the observation that the series in question transforms like a Siegel modular form of weight k for Γ_g. In order to prove the convergence of the formal Fourier–Jacobi expansion, the authors first establish its local convergence and subsequently its analytic continuation to $\mathscr{A}_g(\mathbb{C})$ up to a divisor. The convergence is finally extended to the whole of $\mathscr{A}_g(\mathbb{C})$ using that the Picard group of the minimal compactification $\mathscr{A}_g^*(\mathbb{C})$ of $\mathscr{A}_g(\mathbb{C})$ is, up to torsion, generated by $\omega_{\mathbb{C}}$. Our main result now states (see [4])

Theorem. *The main result of* [1] *continues to hold in the arithmetic setting over the ring of integers* $\mathbb{Z}$.

The remaining part of this abstract is devoted to a sketch of the ideas of proof of our approach. We start with a brief digression on arithmetic compactifications based on [2]. We let $\mathscr{A}_g/\mathbb{Z}$ denote the moduli stack of principally polarized abelian schemes of dimension g over $\mathbb{Z}$ and by $\overline{\mathscr{A}}_g/\mathbb{Z}$ a smooth toroidal compactification of $\mathscr{A}_g/\mathbb{Z}$. Such a compactification depends on the choice of a smooth rational polyhedral cone decomposition $\mathfrak{C}=\{\sigma\}$ of the cone $C(X)$ of positive semi-definite quadratic forms on $X\otimes_{\mathbb{Z}}\mathbb{R}$, which is in addition invariant under the natural $\mathrm{GL}(X)$-action (here $X:=\mathbb{Z}^g$). To obtain a more precise description of the 1-codimensional boundary components of $\overline{\mathscr{A}}_g$, we let $X_\xi:=\mathbb{Z}$ and consider the subcone $C(X_\xi)$ of $C(X)$ consisting of those positive semi-definite quadratic forms that factor through $X_\xi\otimes_{\mathbb{Z}}\mathbb{R}$ with induced cone decomposition $\mathfrak{C}_\xi:=\mathfrak{C}\cap C(X_\xi)$. By means of the 1-dimensional torus $E_\xi:=\mathbb{G}_{\mathrm{m}}$, we obtain the torus embedding $\overline{E}_\xi:=\bigcup_{\sigma\in\mathfrak{C}_\xi}E(\sigma)$ with $E(\sigma)$ being the E_ξ-invariant torus embedding defined by

σ, and the locally closed subscheme $Z_\xi := \bigcup_{\sigma \in \mathfrak{C}_\xi} Z(\sigma)$ given by the E_ξ-orbits $Z(\sigma)$ determined by σ. In a next step, we use these data to construct an E_ξ-bundle $\mathscr{E}_\xi$ over the universal abelian scheme $\mathscr{B}_{g-1}/\mathbb{Z}$, and put

$$\overline{\mathscr{E}}_\xi := \mathscr{E}_\xi \times^{E_\xi} \overline{E}_\xi = \bigcup_{\sigma \in \mathfrak{C}_\xi} \mathscr{E}(\sigma) \quad \text{and} \quad \overline{\mathscr{Z}}_\xi := \mathscr{E}_\xi \times^{E_\xi} Z_\xi = \bigcup_{\sigma \in \mathfrak{C}_\xi} \mathscr{Z}(\sigma).$$

We now recognize that the 1-codimensional boundary components of $\overline{\mathscr{A}}_g$ are given by the $\mathscr{E}(\sigma)$'s with $\sigma \neq \{0\}$ running through a complete set of representatives of the $\mathrm{GL}(X)$-orbits of $\mathfrak{C}_\xi$.

With these notations at hand, we are now able to give a reinterpretation of the Fourier–Jacobi expansion. For this, we let $\widehat{\overline{\mathscr{E}}}_\xi$ denote the completion of $\overline{\mathscr{E}}_\xi$ along the locally closed subscheme $\mathscr{Z}_\xi$. By pull-back to the completion, we obtain a map

$$\mathrm{FJ}_\xi \colon \Gamma(\overline{\mathscr{A}}_g, \omega^{\otimes k}) \longrightarrow \Gamma(\widehat{\overline{\mathscr{E}}}_\xi, \widehat{\omega}_\xi^{\otimes k}),$$

which assigns to a Siegel modular form f its expansion $\sum_{m \in \mathbb{N}} f_m \chi^m$ according to the action of the characters χ^m of the torus E_ξ. The a priori formal functions f_m are algebraizable and we have $f_m \in \Gamma(\mathscr{B}_{g-1}, \mathscr{L}^{\otimes m} \otimes \pi^* \omega^{\otimes k})$; see [3] for an arithmetic theory of Jacobi forms in higher dimensions. This gives the desired reinterpretation of the Fourier–Jacobi expansion. By the $\mathrm{GL}(X)$-invariance of the construction, we eventually arrive at the injective map

$$\mathrm{FJ}_\xi \colon \Gamma(\overline{\mathscr{A}}_g, \omega^{\otimes k}) \hookrightarrow \Gamma(\widehat{\overline{\mathscr{E}}}_\xi, \widehat{\omega}_\xi^{\otimes k})^{\mathrm{GL}(X)}. \tag{5}$$

We are thus left with the task to show that the map FJ_ξ is surjective. To see this, we use Grothendieck's existence theorem on formal functions, which shows that the right-hand side of (5) is isomorphic to the inverse limit of the $\mathrm{GL}(X)$-invariants of the spaces of global sections $\Gamma(\overline{\mathscr{E}}_\xi, \omega_\xi^{\otimes k} / \mathscr{J}_\xi^{\otimes m} \otimes \omega_\xi^{\otimes k})$, where $\mathscr{J}_\xi$ is the ideal sheaf of the locally closed subscheme $\mathscr{Z}_\xi$. Since $\bigcup_\sigma \mathscr{E}(\sigma)$, with the union being taken over all $\sigma \in \mathfrak{C}_\xi$ modulo $\mathrm{GL}(X)$, describes $\overline{\mathscr{A}}_g$ up to codimension 2, we find affine open subsets $U \subseteq \overline{\mathscr{A}}_g$ such that a formal expansion $\sum_{m \in \mathbb{N}} f_m \chi^m \in \Gamma(\widehat{\overline{\mathscr{E}}}_\xi, \widehat{\omega}_\xi^{\otimes k})^{\mathrm{GL}(X)}$ defines a Siegel modular form $G_{m_0} \in \Gamma(U, \omega^{\otimes k})$ on U, which, in fact, extends as a holomorphic Siegel modular form, again denoted by G_{m_0}, to the whole of $\overline{\mathscr{A}}_g$ (again, using that the minimal compactification $\mathscr{A}_g^*$ of $\mathscr{A}_g$ is, up to torsion, generated by ω) having the property that its Fourier–Jacobi expansion

$$\mathrm{FJ}_\xi(G_{m_0}) = \sum_{m \in \mathbb{N}} g_m \chi^m$$

satisfies the relation $g_m = f_m$ for $m = 0, \ldots, m_0$. Choosing m_0 large enough, the finite dimensionality of the right-hand side of (5) gives that the two expansions have to coincide, which proves that G_{m_0} with $m_0 \gg 0$ gives a preimage of the formal expansion $\sum_{m \in \mathbb{N}} f_m \chi^m$ in $\Gamma(\overline{\mathscr{A}}_g, \omega^{\otimes k})$. This leads to the surjectivity of the map FJ_ξ and hence completes the sketch of proof of the theorem.

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Harmonic Maass forms and periods

Claudia Alfes-Neumann

(joint work with Jan H. Bruinier, Markus Schwagenscheidt)

We present work in progress on the relation of coefficients of harmonic weak Maass forms of half-integral weight and periods of associated differentials. This generalizes work of Bruinier [3] and Bruinier and Ono [6] who investigated the situation in the case that the harmonic weak Maass forms have weight 1/2.

1. Preliminaries

We start with a definition of harmonic weak Maass forms. These functions were introduced by Bruinier and Funke in [4]. A function $f : \mathbb{H} \to \mathbb{C}$ is called a *harmonic weak Maass form of weight $k \in \mathbb{Z}$ for the group* $\mathrm{SL}_2(\mathbb{Z})$ if the following conditions hold:

- For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$.
- The function f is smooth on the upper half-plane $\mathbb{H}$ and $\Delta_k f = 0$, where $\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, $z = x + iy$.
- There exists a Fourier polynomial $P_f(z) = \sum_{n \leq 0} c_f^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) - P_f(z) = O(e^{-Cy})$ as $y \to \infty$ for a $C > 0$.

By $H_k(\mathrm{SL}_2(\mathbb{Z}))$ we denote the space of harmonic weak Maass forms of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$. As in the case of usual (holomorphic) modular forms the definition can be generalized to include half-integral weights, congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and characters.

The Fourier expansion of a function $f \in H_k(\mathrm{SL}_2(\mathbb{Z}))$ naturally splits into a holomorphic part f^+ and a non-holomorphic part f^-

$$f(z) = f^+(z) + f^-(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(1-k, 4\pi|n|y) q^n, \tag{1}$$

where $q = e^{2\pi i z}$ and $\Gamma(\alpha, x)$ denotes the incomplete Γ-function.

In [4] Bruinier and Funke introduced a certain differential operator that relates harmonic weak Maass forms to cusp forms. By $S_k(\mathrm{SL}_2(\mathbb{Z}))$ we denote the space of weight k cusp forms for the full modular group. Then we have that

$$\xi_k := 2iy^k \overline{\frac{\partial}{\partial \bar{z}}} : H_k(\mathrm{SL}_2(\mathbb{Z})) \to S_{2-k}(\mathrm{SL}_2(\mathbb{Z})).$$

Moreover, ξ_k is surjective [4].

2. Fourier coefficients of harmonic weak Maass forms and the vanishing of central L-derivatives

To ease notation we let p be prime (the results hold for arbitrary level). We let $G \in S_2(\Gamma_0(p))$ be a newform with rational Fourier coefficients. By the modularity theorem we know that $L(G,s) = L(E,s)$ for a rational elliptic curve E. We let $g \in S_{3/2}(\Gamma_0(4p))$ be the half-integral weight cusp form that corresponds to G under the Shimura correspondence. Moreover, we let $f \in H_{1/2}(\Gamma_0(4p))$ with Fourier expansion as in (1) be a preimage of $\xi_{1/2}$ of the cusp form g that satisfies $\xi_{1/2} f = \frac{g}{||g||^2}$, where $||g||$ denotes the Petersson norm of g. Moreover, we normalize f such that its principal part has rational coefficients.

Bruinier and Ono then proved the following theorem.

Theorem 2.1 (Theorem 1.1,[6]). *Let the notation be as above. For a fundamental discriminant $\Delta > 0$ with $\left(\frac{\Delta}{p}\right) = 1$ we have*

$$L'(G, \chi_\Delta, 1) = 0 \Leftrightarrow c_f^+(\Delta) \in \mathbb{Q},$$

where $L(G, \chi_\Delta, s)$ denotes the usual twisted L-function.

Bruinier and Ono also give an interpretation of the coefficients of the non-holomorphic part of f in terms of the central L-value of G via Waldspurger's theorem [9, 8].

Obviously, it would be interesting to obtain a more direct relation between the Fourier coefficient $c_f^+(\Delta)$ and $L'(G, \chi_\Delta, 1)$. The first approach towards this problem is to derive a formula for the coefficient $c_f^+(\Delta)$. We present two different formulas here.

In the following, we let $\mathcal{Q}_d$ denote the set of all integral binary quadratic forms $Q = [a, b, c]$ of discriminant $d = b^2 - 4ac$. Note that the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_d$ with finitely many orbits if $d \neq 0$.

For $d < 0$ one is led to study CM points z_Q, i.e. the zeros of $Q(z_Q, 1)$ which lie in the upper half-plane. For $d > 0$ we obtain a geodesic

$$C_Q := \left\{ z \in \mathbb{H} : a|z|^2 + bx + c = 0 \right\} \qquad (z = x + iy).$$

By $c_Q := \Gamma_Q \backslash C_Q$ we denote the image in $\Gamma \backslash \mathbb{H}$ of the geodesic C_Q.

As before, we let $\Delta > 0$ be a fundamental discriminant. We define a twisted Heegner divisor for $d < 0$ as

$$Z_\Delta(d) = \sum_{Q \in \mathcal{Q}_{d\Delta}/\Gamma_0(p)} \frac{\chi_\Delta(Q)}{\omega_Q} z_Q \in X_0(N),$$

where χ_Δ denotes the generalized genus character as in [7] and ω_Q is the order of the stabilizer of Q in $\Gamma_0(p)$.

We can then describe the Fourier coefficient $c_f^+(\Delta)$ as a trace over the CM values of a related weight 0 harmonic weak Maass form. (This is a special case of a generalization of the results of Zagier [10] and Bruinier and Funke [5] on traces of singular moduli).

Theorem 2.2 (Theorem 4.5, [1]). *With the notation as above we have*

$$c_f^+(\Delta) = \sum_{z\in \tilde{Z}_\Delta(d)} F(z),$$

where F maps to f under the so-called (twisted) Millson theta lift. (Also known as the Zagier lift.) By $\tilde{Z}_\Delta(d)$ we denote a slightly modified version of the Heegner divisor defined above.

The right hand side can be interpreted as a sum of a period of a differential of the first (holomorphic) and second kind (meromorphic) via Stokes' theorem. This description can also be used to give a different proof of Theorem 2.1.

A different formula for $c_f^+(\Delta)$ in terms of periods of differentials of the third kind was given by Bruinier in [3].

Theorem 2.3 (Theorem 1.1, [3]). *Let the notation be as above. There is a unique differential $\zeta_\Delta(f)$ of the third kind (poles of order one with integral residues) with residue divisor $\sum_{n<0} c_f^+(n) Z_\Delta(n)$ that satisfies:*

- *Its first Fourier coefficient vanishes.*
- *We have that $T\zeta_\Delta(f) - \lambda_G(T)\zeta_\Delta(f) = \frac{dF}{F}$, for a function $F \in \mathbb{C}(X)^\times$ and all Hecke operators T. Here, $\lambda_G(T)$ denotes the eigenvalue of G under the Hecke operator T.*
- *Denote by C_G a generator of the G-isotypical component of the part of $H_1(X_0(p), \mathbb{R})$ that is invariant under the involution induced by complex conjugation. Then*

$$c_f^+(\Delta) = \frac{\Re\left(\int_{C_G} \zeta_\Delta(f)\right)}{\sqrt{\Delta}\int_{C_G} \omega_G},$$

where $\omega_G = 2\pi i\ G(z)dz$.

3. The higher weight case

We now describe an extension of these results to the case that the newform G has even weight $2k+2>2$. The situation is summarized in the following diagram

$$\begin{array}{ccc} F \in H_{-2k}(\Gamma_0(p)) & \xrightarrow{\xi_{-2k}} & G \in S_{2k+2}(\Gamma_0(p)) \\ \Big\downarrow \text{Millson-lift} & & \Big\downarrow \text{Shintani-lift} \\ f \in H_{\frac{1}{2}-k}(\Gamma_0(4p)) & \xrightarrow{\xi_{1/2-k}} & g \in S_{\frac{3}{2}+k}(\Gamma_0(4p)). \end{array}$$

Again, we can describe the Fourier coefficient of the half-integral weight harmonic weak Maass form f as a CM trace (now of a weight 0 function $R^k_{-2k}F$, where R^k_{-2k} denotes the Maass raising operator), see [2].

Moreover, we can again relate the coefficient to periods of differentials as follows.

Theorem 3.1. *Let the notation be as above. We have*

$$c_f^+(\Delta) = \frac{\Re\left(\int_{C_G} \zeta_\Delta(f,z) Q_G(z,1)^k dz\right)}{||G||^2},$$

where we chose C_G and Q_G such that $\int_{C_G} G(z)Q_G(z,1)^k dz = ||G||^2$ and $\zeta_\Delta(f,z)$ is given as a certain linear combination of the functions $f_{k,d,\Delta}(z) = \sum_{Q\in\mathcal{Q}_{\Delta d}} \frac{\chi_\Delta(Q)}{Q(z,1)^k}$ and $G(z)$.

We can interpret the numerator as the pairing of homology and cohomology with values in certain local systems which then mimics the description given in Theorem 2.3.

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Modular forms in small dimension: geometry and arithmetic

Christophe Ritzenthaler

In this talk, we start by recalling the definition of analytic Siegel modular forms of degree g and weight h. Using theta constants with integral characteristics, it is then possible to construct interesting such forms. We introduce in particular the forms χ_h which are the products of all theta constants of a given degree. We also introduce the form $F_{4,8}$ which is of degree 4 and weight 8 (called Schottky form).

These forms cut interesting loci in the moduli spaces of principally polarized abelian varieties of dimension g over $\mathbb{C}$. For instance for $g = 3$, $\chi_{18}(\tau)$ is not zero if and only if $A_\tau = \mathbb{C}^3/\tau\mathbb{Z}^3 + \mathbb{Z}^3$ is the Jacobian of a non-hyperelliptic curve. Similarly $F_{4,8}(\tau) = 0$ if and only if $A_\tau = \mathbb{C}^4/\tau\mathbb{Z}^4 + \mathbb{Z}^4$ is the Jacobian of a (possibly reducible) curve of genus 4.

This geometric information can be made more precise and also available over more general fields, using the definition of geometric Siegel modular forms as sections of power of the Hodge bundle. When starting with an analytic Siegel modular form f, there is a recipe (à la Katz) to get such a form by multiplying f by an appropriate normalization factor. In particular, if f is a polynomial over $\mathbb{Z}$ in the theta constants, then the value of the associated geometric modular form at a principally polarized abelian variety defined over a field k (along a basis of regular differentials defined over k) is in k.

As a first illustration, I then speak about results with Reynald Lercier. Tsuyumine in 1986 showed that the ring $R_3$of modular forms in degree 3 is generated by 34 polynomials in the theta constants. Performing an evaluation/interpolation strategy using rational expressions of quotients of the theta constants in terms of coefficients of well-chosen plane quartics over $\mathbb{Q}$, we could show that R_3 is actually generated by a minimal set of 19 generators. Using then the interpretation of geometric modular forms as section of the Hodge bundle, we can pull them back to the space of invariants of ternary quartics under the action of $\mathrm{SL}_3(\mathbb{C})$ and give an explicit dictionary between the two worlds.

As a second illustration, I speak of results obtained with Markus Kirschmer, Fabien Narbonne and Damien Robert. Starting with a principally polarized abelian variety (A, a) in the isogeny class of E^g, where E is an ordinary elliptic curve over a finite field k, we study when we can descent the product polarization a_0 on E^g to a. This is done using an equivalence of categories between certain hermitian R-lattices and these abelian varieties (here $R = \mathrm{End}(E)$ which we assume to be generated by the Frobenius). When this is possible, a `Magma` package called avisogenies designed by Gaetan Bisson, Romain Cosset and Damien Robert allows to compute the theta nullpoint (a generalization of theta constants over any field) from the theta nullpoint on (E^g, a_0). We can then compute for instance the form $F_{4,8}$ and show that some classes of E^4 do not contain Jacobians. Using geometric modular forms, and showing that the avisogenies package behaves well with respect to the normalization of analytic modular forms, we can even compute the so-called Serre obstruction in dimension 3 by testing when χ_{18} is a square in k. Thanks to this, one can find for instance a class E^3 which contains a unique maximal curve of genus 3 (i.e. a curve which number of rational points is as large as possible) which automorphism group is trivial. This means that the existence of this curve would have been hard to prove by any other existing means.

False theta functions and their modularity properties

Kathrin Bringmann

In my talk I reported about recent work on false theta functions.

False theta functions have wrong sign factors which prevent them from being modular. One such example is given by

$$\sum_{n\in\mathbb{Z}}(-1)^n \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{\left(n+\frac{1}{2}\right)^2},$$

where $\operatorname{sgn}(x) := \frac{x}{|x|}$ for $x \in \mathbb{R}\setminus\{0\}$, $\operatorname{sgn}(0) := 0$.

False theta functions are also examples of so-called quantum modular forms. Following Zagier [8], $f : \mathcal{Q} \to \mathbb{C}$ ($\mathcal{Q} \subset \mathbb{Q}$) is a *quantum modular form of weight k* if for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\tau) - (c\tau + d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

is "nice". False theta functions occur for example in vertex algebras and in combinatorics.

To describe modularity properties of false theta functions, define

$$\psi(z;\tau) := i\sum_{n\in\mathbb{Z}}(-1)^n \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\zeta^{n+\frac{1}{2}}$$

throughout $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi i z}$, $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} : \tau_2 := \mathrm{Im}(\tau) > 0\}$, $z \in \mathbb{C}$. Removing the sign yields a Jacobi form, namely the Jacobi theta function

$$\vartheta(z;\tau) := i\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\zeta^{n+\frac{1}{2}}.$$

The idea now is to complete $\psi(z;\tau)$. To be more precise, define for $w \in \mathbb{H}$, $z_2 := \mathrm{Im}(z)$

$$\widehat{\psi}(z;\tau,w) := i\sum_{n\in\mathbb{Z}} \operatorname{erf}\left(-i\sqrt{\pi i(w-\tau)}\left(n+\frac{1}{2}+\frac{z_2}{\tau_2}\right)\right)(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\zeta^{n+\frac{1}{2}},$$

where erf denotes the error function. Note that, for $-\frac{1}{2} < \frac{z_2}{\tau_2} < \frac{1}{2}$ and $\varepsilon > 0$,

$$\lim_{t\to\infty} \widehat{\psi}(z;\tau,\tau+it+\varepsilon) = \psi(z;\tau).$$

Using Poisson summation we showed in [5] the following.

Theorem 1. *The function $\widehat{\psi}$ transforms like a Jacobi form.*

Theorem 1 has applications to asymptotics for so-called unimodal sequences and quantum modularity of functions occuring in W-algebras.

Let me next describe higher-dimensional false theta functions. A typical example is given by (see [1, 4])

$$F(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\mathbb{N}_0^2+\boldsymbol{\alpha}} q^{Q(\boldsymbol{n})} + \frac{1}{2}\sum_{n\in\mathbb{Z}} \operatorname{sgn}\left(n+\frac{1}{N}\right) q^{\left(n+\frac{1}{N}\right)^2},$$

where $\boldsymbol{n} = (n_1, n_2)$, $\quad Q(\boldsymbol{n}) := 3n_1^2 + 3n_1n_2 + n_2^2$,

$$\mathscr{S} := \left\{\left(1-\tfrac{1}{N},\tfrac{2}{N}\right),\left(\tfrac{1}{N},1-\tfrac{2}{N}\right),\left(1,\tfrac{1}{N}\right),\left(0,1-\tfrac{1}{N}\right),\ \left(\tfrac{1}{N},1-\tfrac{1}{N}\right),\left(1-\tfrac{1}{N},\tfrac{1}{N}\right)\right\},$$

$$\varepsilon(\boldsymbol{\alpha}) := \begin{cases} -2 & \text{if } \boldsymbol{\alpha} \in \left\{\left(1-\frac{1}{N},\frac{2}{N}\right),\left(\frac{1}{N},1-\frac{2}{N}\right)\right\}, \\ 1 & \text{otherwise.}\end{cases}$$

Note that

$$F(q) = \tfrac{1}{2} \sum_{\boldsymbol{\alpha}\in\mathcal{S}^*} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\mathbb{Z}^2+\boldsymbol{\alpha}} \operatorname{sgn}(n_2)\left(\operatorname{sgn}(n_1)+\operatorname{sgn}(n_2)\right) q^{Q(\boldsymbol{n})}, \tag{1}$$

where

$$\mathcal{S}^* := \left\{\left(\tfrac{1}{N},1-\tfrac{2}{N}\right),\left(0,1-\tfrac{1}{N}\right),\left(\tfrac{1}{N},1-\tfrac{1}{N}\right)\right\}.$$

The first step for seeing modularity is to view F as coefficient of a two-dimensional (meromorphic) Jacobi form. Note the classical fact that Fourier coefficients of holomorphic Jacobi forms are modular forms [7]. Moreover, we showed previously [6] that one-dimensional false theta functions occur as coefficients of one-dimensional meromorphic Jacobi forms. In the case of F, we have [3] the following theorem.

Theorem 2. *We have*

$$F(q) = \frac{\eta(\tau)^5}{\eta(2\tau)} \mathrm{CT}_{[\zeta_1,\,\zeta_2]} \frac{\vartheta(z_1;2\tau)\vartheta(z_2;2\tau)\vartheta(z_1+z_2;2\tau)}{\vartheta(z_1;\tau)\vartheta(z_2;\tau)\vartheta(z_1+z_2;\tau)},$$

where $\zeta_j := e^{2\pi i z_j}$ satisfy $|q| < |\zeta_j| < 1$, $|q| < |\zeta_1\zeta_2| < 1$. Here CT *denotes the constant term.*

To fully understand modularity of functions like (1) one writes these as iterated integrals [2]. A key ingredient is the following lemma.

Lemma 3. *Let* $\ell_1, \ell_2 \in \mathbb{R}$, $\kappa \in \mathbb{R}$, *with* $(\ell_1, \ell_2 + \kappa\ell_1) \neq (0,0)$. *Then*

$$\mathrm{sgn}(\ell_1)\mathrm{sgn}(\ell_2+\kappa\ell_1)q^{\frac{\ell_1^2}{2}+\frac{\ell_2^2}{2}} = \int_\tau^{\tau+i\infty} \frac{\ell_1 e^{\pi i \ell_1^2 w_1}}{\sqrt{i(w_1-\tau)}} \int_\tau^{w_1} \frac{\ell_2 e^{\pi i \ell_2^2 w_2}}{\sqrt{i(w_2-\tau)}} dw_2 dw_1$$
$$+ \int_\tau^{\tau+i\infty} \frac{m_1 e^{\pi i m_1^2 w_1}}{\sqrt{i(w_1-\tau)}} \int_\tau^{w_1} \frac{m_2 e^{\pi i m_2^2 w_2}}{\sqrt{i(w_2-\tau)}} dw_2 dw_1 + \frac{2}{\pi}\arctan(\kappa) q^{\frac{\ell_1^2}{2}+\frac{\ell_2^2}{2}},$$

where $m_1 := \frac{\ell_2+\kappa\ell_1}{\sqrt{1+\kappa^2}}$ $m_2 := \frac{\ell_1-\kappa\ell_2}{\sqrt{1+\kappa^2}}$.

Summing now over the (shifted) lattices in the summations of F yields iterated generalized Eichler integrals. Moreover, we also considered generic characters of vertex algebras of type A_2 and B_2 [2].

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Rank four Calabi-Yau motives of low conductor

Duco van Straten
(joint work with Vasily Golyshev and many others)

1. Machine calculations by Birch and Swinnerton-Dyer [6],[7],[8] in the 1960s led to important conjectures on the arithmetic of elliptic curves and tables of elliptic curves and modular forms that supported them. Apparently the first such list appeared in [9] and became referred to as the *Antwerp tables*, which contained a list of elliptic curves over $\mathbb{Q}$ with conductor ≤ 200 and corresponding modular forms. Later, the Antwerp tables were extended to higher conductor by J. Cremona, which by the hard work of many people, culminated in the electronic database LMFDB [16] that is used on a daily basis by workers in the field.

The tables were lending strong support to the conjecture, made by Shimura, Taniyama and Weil [43], that any elliptic curve E over $\mathbb{Q}$ of conductor N admits a modular parametrisation $\phi : X_0(N) \longrightarrow E$, which then leads via the Eichler-Shimura relation to the equality of L-functions

$$L(s, H^1E) = L(s, f), \quad f \in S_2(\Gamma_0(N)).$$

The Hasse-Weil L-function of E is defined by an Euler product

$$L(s, H^1E) = \prod_p E_p(T),$$

where the Euler factors for primes of good reduction are given by

$$\det(1 - T.Frob_p : H^1E) = 1 - a_pT + p^2T^2.$$

The trace of Frobenius a_p relates directly to the number of points of E over $\mathbb{F}_p$:

$$\#E(\mathbb{F}_p) = 1 - a_p(E) + p.$$

The proof of the Taniyama-Shimura-Weil conjecture by Wiles and Taylor [42], [38] and completed in [10] leads to a bijection between isogeny classes of elliptic curves over $\mathbb{Q}$ and $\mathbb{Q}$-Hecke newforms in $S_2(\Gamma_0(N))$: *"All elliptic curves are modular"*.

The first entry in the Antwerp table is the curve $11A$, the first elliptic curve in nature. It recieved the code $11.a3$ in the LMFDB and is given by the equation

$$y^2 + y = x^3 + x^2,$$

which corresponds to the η-product $f_{11} = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2$:

$$\begin{aligned} f_{11} &= \textstyle\sum_{n=1}^{\infty} a_n(f_{11})a^n \\ &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + \ldots, \end{aligned}$$

The curve $11A$ above is nothing else then the modular curve $X_1(11)$. The isogeny $X_1(11) \longrightarrow X_0(11)$ has $\mathbb{Z}/5$ (the full Mordell-Weil group) as kernel. This 5-isogeny leads to the fact that $\#E(\mathbb{F}_p)$ is divisible by 5 so that

$$a_p(f_{11}) = 1 + p \mod 5, \quad (p \neq 11)$$

We say that the modular form f_{11} *has a 5-congruence.* (The equation for the modular curve $X_0(11)$ appears in the work of Fricke and Klein, [32], p.436; for the elliptic curves with good reduction outside 11 and their isogenies, see [41].)

2. The so-called *small Apéry numbers*

$$A_0 = 1,\;\; A_2 = 3,\;\; A_2 = 19,\;\; A_3 = 147, \ldots, A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

played a key role in Apéry's irrationality proof of $\zeta(2)$, [2]. These numbers satisfy the recursion relation

$$(n+1)^2 A_{n+1} - (11n^2 + 11n + 3)A_n - n^2 A_{n-1} = 0, \quad n \geq 1,$$

so that the generating series $\phi(t) := \sum_{n=0}^{\infty} A_n t^n$ satisfies the second order differential equation $\mathcal{L}\phi = 0$, where

$$\mathcal{L} := \theta^2 - t(11\theta^2 + 11\theta + 3) - t^2(\theta+1)^2, \quad \theta = t\frac{d}{dt},$$

which has singular points at 0, ∞ and the two roots $\frac{-11\pm 5\sqrt{5}}{2}$ of the polynomial

$$\Delta(t) = 1 - 11t - t^2.$$

It was discovered by F. Beukers [4] that this operator is the Picard-Fuchs operator of a rational elliptic surface $\pi : \mathcal{E} \longrightarrow \mathbb{P}^1$, which can be identified with the universal elliptic curve over the modular curve $X_1(5) = \mathbb{P}^1$. The fibres of π are elliptic curves, with a cyclic $\mathbb{Z}/5$ isogeny; over the four cusps of $X_1(5)$ we find generalised elliptic curves of Kodaira type I_5, I_1, I_1, I_5. The fact that $\Delta(1) = -11$ suggests that the fibre at $t = 1$ has reduction only for $p = 11$ so that might be $X_1(11)$. The family is of toric nature: the Laurent polynomial

$$f(x,y) = \frac{(1+x)(1+y)(1+x+y)}{xy} \in \mathbb{Z}[x, x^{-1}, y, y^{-1}]$$

has the property of having the constant term of its n-th power being A_n:

$$A_n = [f(x,y)^n]_0.$$

Consequently, the period function ϕ can be represented as

$$\phi(t) = \frac{1}{(2\pi i)^2} \oint \oint \frac{1}{1 - tf(x,y)} \frac{dxdy}{xy}$$

The polar locus defines a family of open elliptic curves

$$E_t^\circ = \{(x,y) \in \mathbb{C}^* \times \mathbb{C}^* \mid 1 - tf(x,y) = 0\}$$

and it is an amusing exercise to show that for $t = 1$ we get indeed the curve $11A$. It follows from the relation between the period function $\phi(t)$ and the Hasse-invariant of E_t that

$$\phi(1) \mod p = \sum_{n=0}^{p-1} A_n = a_p(f_{11}) \mod p,$$

which relates the Apéry numbers to the Fourier-coefficients of f_{11} in an unexpected way.

3. By a *Calabi-Yau manifold* we will understand a smooth projective variety with trivial canonical bundle, $\mathcal{O}_X = \omega_X$ and we require $h^i(\mathcal{O}_X) = 0$ for $i \neq 0, \dim(X)$ to exclude the case of abelian varieties. Even for Calabi-Yau threefolds there is a bewildering zoo of examples, and contrary to the case of elliptic curves or abelian varieties, no uniform construction is known. For a Calabi-Yau threefold X, Poincaré-duality defines a non-degenerate alternating form on the third cohomology H^3X of a Calabi-Yau threefold, making it a so-called *symplectic motive* of rank $2 + 2a$, as the Hodge numbers of H^3X are

$$1 \quad a \quad a \quad 1,$$

where $a = h^{12}$ has the interpretation as dimension for its local moduli space.

A lot of work has been done on the arithmetic of *rigid Calabi-Yau threefolds* (i.e. those for which $a = 0$) defined over $\mathbb{Q}$. As then H^3X is a rank 2 motive of weight 3, its L-function can be expected to be described in terms of classical modular forms. The Euler factor now has the form

$$\det(1 - F_pT : H^3X) = 1 - a_pT + p^3T^2$$

and the a_ps can be expected to be the p-th Fourier coefficient of a $\mathbb{Q}$-new eigenform $\in S_4(\Gamma_0(N))$. In fact, it was shown by F. Gouvêa and N. Yui [21] and L. Dieulefait [18] that *All rigid Calabi-Yau threefolds are modular!* But the correspondence is not as straightforward as in the elliptic curve case: we do not know which weight 4 modular forms can be realised by a Calabi-Yau threefold and furthermore, examples show that there are topologically distinct rigid Calabi-Yau threefolds with the same modular form. Many examples can be found in the book by C. Meyer [34]; much remains to be done.

There are nice examples of Siegel modular varieties $\mathbb{H}_2/\Gamma$ which compactify to Calabi-Yau threefolds described by B. van Geemen, N. Nygaard, E. Freitag, R. Salvati-Manni, S. Cynk ([28], [29], [20], [17]) as coverings of a specific complete intersection of four quadrics in $\mathbb{P}^7$, defined by relations between genus two theta-functions. However, in all these cases described one obtained motives that were *split*: $1\ a\ a\ 1 = 1\ 0\ 0\ 1 + 0\ a\ a\ 0$. The sub-motive 1 0 0 1 corresponds to a classical cusp form from $S_4(\Gamma_0(N)$.

The next case is $a = 1$. In this case H^3X is a symplectic motive of rank four, with Hodge numbers 1 1 1 1. These naturally appear as the fibres of one-parameter families of Calabi-Yau threefolds, parametrised in the simplest cases by $\mathbb{P}^1$:

$$\mathcal{X} \longrightarrow \mathbb{P}^1$$

Many such families are known explicitly, and studied in various levels of detail. Simplest are the 14 hypergeometric families, which directly generalise the famous example described by P. Candelas, X. de la Ossa, P. Green and L. Parkes and are known from the earliest days of mirror symmetry. For general values of t one obtains irreducible 1 1 1 1-motives, whose automorphic origin one would like to find. Candidates of moderate low conductor 525 an 257 were identified by H. Cohen and D. Roberts, [15].

4. Paramodular levels. Apart from the congruence subgroups in $\mathrm{Sp}_4(\mathbb{Z})$, the *paramodular groups* $K(N)$ of level N have great relevance. By definition, the Siegel modular threefolds

$$Y(N) := \mathbb{H}_2/K(N)$$

have the natural interpretation as moduli spaces for abelian surfaces with $(1 : N)$-polarisation. The algebraic geometric study of projective models has a long history. Abelian surfaces with $(1 : 5)$ polarisation appear in the study of the Horrocks-Mumford bundle on $\mathbb{P}^5$; a model of $Y(11)$ as a cubic hypersurface in $\mathbb{P}^4$ appears already in Klein's paper [31], (see [25]), the question of unirationality of $Y(N)$ is also adressed by Gritsenko [22] and [26],[27].

In [37], a local new-form theory for the paramodular levels in GSp(4) was developed by B. Roberts and R. Schmidt. This theory shows the complete analogy of

$K(N)$ with the classical theory for $\Gamma_0(N)$ and suggests that the threefolds $Y(N)$ are the most natural generalisations of the modular curves $X_0(N)$. In analogy of the Shimura-Taniyama-Weil conjecture, Brumer-Kramer [11] conjectured: *All abelian surfaces defined over* $\mathbb{Q}$ *are paramodular!*:

$$L(s, H^1 A) = L(s, F), \quad F \in S_2(K(N))$$

Several refinements must be taken into account, but by now there is overwhelming evidence for this conjecture, [12].

The Langlands correspondence predicts that all symplectic motives of rank $2n$ come from automorphic forms for the split orthogonal group $SO(n, n+1)$. (In [24] and [40] the local new-form theory was developed for paramodular levels, generalising the work B. Roberts and R. Schmidt had done for $\mathrm{Sp}_4 = SO(2,3)$). In particular, one may conjecture in a similar vein that *All* $(1,1,1,1)$*-Calabi-Yau motives are paramodular!* More precisely, if M is such a motive with conductor N, then there should exist a weight 3 paramodular new-form $F \in S_3(K(N))$ so that

$$L(s, M) = L(s, F), \quad F \in S_3(K(N)).$$

One may furthermore ask how such a motive may be realised geometrically, inside the cohomology of a nice variety, of course preferably a smooth projective Calabi-Yau threefold with $b_3 = 4$. Undoubtedly, many refinements will have to be made, but one would like to start collecting evidence.

6. The low levels 61, 73, 79,... In [3], Ash-Gunnels-McConnel found indications for the existence of cusp forms in H^5 for congruence subgroups $\Gamma_0(N) \subset SL_4(\mathbb{Z})$ for $N = 61, 73, 79$ and conjectured their Siegel modular origin. Using rational combinations of Gritsenko lifts, C. Poor and D. Yuen [35] constructed paramodular cusp forms $F_N \in S^3(K(N))$ for $N = 61, 73, 79$ and computed Euler-factors for $p = 2, 3, 5$. By forcing the functional equation $\Lambda(s) = \Lambda(4-s)$ for a putative completed L-function

$$\Lambda(s) = \left(\frac{N}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L(s, F_N)$$

numerically, A. Mellit and V. Golyshev were able to guess the first few hundred coefficients of the Dirchilet series for F_N, $N = 61, 73, 79, \ldots$. Subsequent computations of algebraic modular forms by J. Hein (2016) [30], W. Ladd (2018) [33] and G. Tornaria (2019) produced Euler factors up to $p = 31$ and Dirichlet series up to more than 1000 terms, confirming these finds. For further information we refer to [36].

The AESZ-list [1] contains about 500 so-called *Calabi-Yau differential equations*. These are self-dual differential equations of order four of the sort that arise as Picard-Fuchs equation for 1-parameter families of Calabi-Yau varieties. As such, they can be considered as analogues of the Apéry second order equation $\mathcal{L}$ discussed in **2**. The list starts with the 14 hypergeometric operators, but in general, it appears that the conductors N never get very low. Running through the list, and evaluating the discriminant $\Delta(t)$ at 1 and -1 we encounter several remarkable cases.

Candidate for the realisation of F_{79}: Operator AESZ # 25

$$\theta^4 - 4t(2\theta+1)^2(11\theta^2+11\theta+3) - 16t^2(2\theta+1)^2(2\theta+3)^2$$

has its origin in the mirror symmetry for Calabi-Yau manifold that is the (2,2,1) complete intersection in the Grassmanian $G(2,5)$ [5]. Its holomorphic solution is the period of the mirror pencil $\pi : \mathcal{X} \to \mathbb{P}^1$ and expands as

$$\phi(t) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 A_n t^n = 1 + 12t + 684t^2 + \dots$$

where A_n are the small Apéry numbers as in **2**. Apart from 0 and ∞, the singularities of the operator are the roots of the polynomial

$$\Delta(t) = 1 - 176t - 256t^2.$$

We observe that

$$\Delta(-1) = -79$$

suggesting that the fibre motive at $t = -1$ has good reduction for all primes $p \neq 79$. The Dirichlet series L_{79} starts as

$$L_{79} = \sum_{n=1}^{\infty} a_n(F_{79}) \frac{1}{n^s} = 1 - 5\frac{1}{2^s} - 5\frac{1}{3^s} + 11\frac{1}{4^s} + 3\frac{1}{5^3} + 25\frac{1}{6^7} + 15\frac{1}{7^s} + \dots$$

It can be observed that for $p = 2, 3, 5, 7, \dots$ we have

$$\sum_{n=0}^{p-1} \binom{2n}{n}^2 A_n(-1)^n = a_p(F_{79}) \mod p.$$

As the operator # 25 is a Hadamard product of two second order operators, the geometry X_t of this Calabi-Yau threefold is best understood as a fibre product of the Legendre family and the Apéry family of elliptic curves, so we are dealing here with one of the simplest types of Calabi-Yau varieties.
From this description corresponding equations can be written down. A convenient way is to use the Laurent polynomial

$$\lambda(x, y, u, v) := \frac{(1+x)^2(1+y)^2(1+u)(1+v)(1+u+v)}{xyuv}$$

One has

$$[\lambda^n]_0 = \binom{2n}{n}^2 A_n$$

The Calabi-Yau variety X_{-1} is a compactification of the affine sub-variety of the torus

$$U_{79} := \{f(x, y, u, v) + 1 = 0\} \subset \mathbb{G}_m^4.$$

For the number of points in $\mathbb{F}_p$

$$\#U_{79}(\mathbb{F}_p) = \#\{(x, y, u, v) \in (\mathbb{F}_p^*)^4 \mid f(x, y, u, v) + 1 = 0 \mod p\},$$

we find the following relation to the Dirichlet series of F_{79}:

$$\#U_{79}(\mathbb{F}_p) = Corr_{79}(p) - a_p(F_{79}), \quad Corr_{79}(p) := p^3 - 8p^2 + 21p - 23.$$

as one would expect. So this is very good evidence that the Galois representation on $Gr_3^W H^3(U_{79})$ realises the Galois representation attached to F_{79}; we can extend this to higher values of p and the table below can be extended to a count of points in $\mathbb{F}_p^2$ to obtain the Euler factors at p. Rather than point counting, there are much more efficient methods to obtain the complete degree four Euler-factors directly from the differential equation, [13]. One finds a perfect match with the available data.

Candidate for the realisation of F_{61}: Operator # 195 is more complicated:

$$(29)^2\theta^4 + t(-6728 - 479370\theta - 1327330\theta^2 - 1695920\theta^3 - 877540\theta^4)+$$

$$+t^2(5568 + 577680\theta + 2391590\theta^2 + 4242200\theta^3 + 2586470\theta^4)+$$

$$+t^3(76560 - 3368640\theta - 5816470\theta^2 - 5326140\theta^3 - 2727430\theta^3)+$$

$$+t^4(75616{+}3327920\theta{+}5522280\theta^2{+}4211240\theta^3{+}1306960\theta^4){-}3468t^5(\theta{+}1)^2(3\theta{+}2)(3\theta{+}4)$$

Its holomorphic solution expands at 0 as

$$\phi(t) = \sum_{n=0}^{\infty} B_n t^n = 1 + 8t + 264t^2 + 13040t^3 + \dots$$

where

$$B_n := \sum_{i=0}^{n}\sum_{j=0}^{n} \binom{n}{i}\binom{n}{j}\binom{n+i}{i}\binom{i+j}{j}.$$

These numbers appeared first in [44], but the geometry of the associated pencel has apparently not been studied before. The operator has singularities at 0, ∞ and the roots of the polynomial

$$\Delta(t) = -\Delta_1(t)(34t-29)^2, \quad \Delta_1(t) = 27t^3 - 67t^2 + 102t - 1.$$

One has $\Delta_1(1) = 61$) and the factor $(34t - 29)$ is an *apparent singularity* of the operator, for which the geometry remains smooth, but introduces a spurious 5 in the model. From the beginning of the Dirichlet series

$$L_{61} := \sum_{n=1}^{\infty} a_n(F_{61})\frac{1}{n^s} = 1 - 7\frac{1}{2^s} - 3\frac{1}{3^s} + 25\frac{1}{4^s} + 3\frac{1}{5^s} + 21\frac{1}{6^s} - 9\frac{1}{7^s} + \dots$$

it can be observed that for $p = 2, 3, 5, \dots$ we have

$$\sum_{n=0}^{p-1} B_n = a_p(F_{61}) \mod p$$

The coefficients $B_n = [\lambda^n]_0$ can be obtained from the Laurent polynomial

$$\lambda(x,y,z,t) = \frac{(1+x)(1+y)(1+t)(1+x+z)((1+t)(1+z)+yz)}{xyzt}$$

The affine threefold

$$U_{61} := \{(x,y,z,t) \in (\mathbb{C}^*)^4 \mid \lambda(x,y,z,t) = 1\}$$

has modulo 61 a single ordinary double point at $x = 28, y = -9, z = -10, t = 28$. For $p = 2, 3, 5, 7, \ldots$ we find

$$\#U_{61}(\mathbb{F}_p) = Corr_{61}(p) - a_p(F_{61}), \quad Corr_{61}(p) := p^3 - 8p^2 + 25p - 33$$

p	$\#U_{61}(\mathbb{F}_p)$	$Corr_{61}(p)$	$-a_p(F_{61})$	$\#U_{79}(\mathbb{F}_p)$	$Corr_{79}(p)$	$-a_p(F_{79})$
2	0	−7	7	0	−5	5
3	0	−3	3	0	−5	5
5	14	17	−3	4	7	−3
7	102	93	9	60	75	−15
11	609	605	4	545	571	−26
13	1140	1137	3	8345	8395	−50
17	2956	2993	−37	2995	2935	60
19	448	4413	75	4315	4347	−32
23	8467	8477	−10	8345	8395	−50
29	18141	18353	−212	18223	18247	−24
31	22851	22845	6	22589	22731	−142
37	40681	40593	88	40955	40455	500
41	56468	56465	3	56071	56311	−240
43	64210	65757	−547	65915	65595	320
47	87440	87293	147	87220	87115	105
53	127805	127697	108	126865	127495	−630
59	179018	178973	45	178722	178747	−25
61	198560	198705	−145	198277	198471	−194

We are confident that one will be able to rigorously prove that these varieties realise the Galois representations attached to the paramodular forms F_{61} and F_{79}, using the extension of the Serre-Faltings-Livne method to Galois representations in Sp_4 described in the paper [12] of A. Brumer, A. Pacetti, C. Poor, G. Tornaría, J. Voight and D. Yuen, whom I would like to thank here for interest and support in this ongoing project. Obvious next steps are, besides climbing up in level, the systematic study of congruences and central L-values (Deligne-conjecture, Bloch-Kato conjectures) for these low conductor examples.

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Modularity of Abelian Surfaces I

Cris Poor
(joint work with David S. Yuen)

We outline theoretical evidence for the paramodular conjecture (see the presentation "Modularity of Abelian Surfaces II" for computational evidence [10, 8, 9]). The model for modularity results is the Modularity Theorem.

Theorem (Wiles; Wiles & Taylor; Breuil, Conrad, Diamond & Taylor)**.** *Let* $N \in \mathbb{N}$*. There is a bijection between*

1. *isogeny classes of elliptic curves* $E/\mathbb{Q}$ *with conductor* N*, and*
2. *normalized Hecke eigenforms* $f \in S_2(\Gamma_0(N))^{\text{new}}$ *with rational eigenvalues.*

In this correspondence we have $L(E, s, \text{Hasse}) = L(f, s, \text{Hecke})$.

It was M. Eichler who gave the first instances of modularity, making possible appealing examples such as $L(X_0(11), s, \text{Hasse}) = L(\eta(\tau)^2\eta(11\tau)^2, s, \text{Hecke})$. Although specific examples of the modularity of abelian surfaces defined over $\mathbb{Q}$ have been recently proven [3, 2], we still have no result parallel to that of G. Shimura, who gave a construction from item 2 to item 1.

More generally, we may associate Hilbert $\mathbb{Q}$-newforms of parallel weight two over totally real extensions $K/\mathbb{Q}$ to elliptic curves defined over K, and the converse association is also expected. To elliptic curves over imaginary quadratic fields K, we may associate weight two Bianchi $\mathbb{Q}$-newforms; however, there is an issue, pointed out by John Cremona, with the converse association. There are Bianchi $\mathbb{Q}$-newforms f_o whose L-function is not the Hasse L-function of any elliptic curve over an imaginary quadratic field. Rather, $L(f_o, s)^2$ is the Hasse-Weil L-function of an abelian surface A_o/K, whose endomorphisms defined over K include an order of a quaternion algebra. A theorem of G. Faltings proves that the Hasse-Weil L-function of an abelian variety determines its isogeny class, and this precludes the existence of an elliptic curve E/K whose L-function is $L(f_o, s)$ since $E \oplus E$ would have the same L-function as A_o but a different ring of endomorphisms. Examples of this phenomenon have been given by Ciaran Schembri. As pointed out by Frank Calegari, these same examples can be lifted to the context of the paramodular conjecture using Weil restriction and the lifting of Bianchi forms due to Tobias Berger, and Lassina Dembélé, Ariel Pacetti, and Mehmet Haluk Sengun [1].

Armand Brumer and Ken Kramer have incorporated this insight into the current version of the conjecture. An abelian fourfold $B/\mathbb{Q}$ has *quaternionic multiplication* (QM) if $\mathrm{End}_{\mathbb{Q}}(B)$ is an order in a non-split quaternion algebra over $\mathbb{Q}$. A cuspidal, nonlift Siegel paramodular newform $f \in S_2(K(N))$ with rational Hecke eigenvalues will be called a *suitable* paramodular form of level N.

Paramodular Conjecture. *Let $N \in \mathbb{N}$. Let $\mathcal{A}_N$ be the set of isogeny classes of abelian surfaces $A/\mathbb{Q}$ of conductor N with $\mathrm{End}_{\mathbb{Q}} A = \mathbb{Z}$. Let $\mathcal{B}_N$ be the set of isogeny classes of QM abelian fourfolds $B/\mathbb{Q}$ of conductor N^2. Let $\mathcal{P}_N$ be the set of suitable paramodular forms of level N, up to nonzero scaling. There is a bijection $\mathcal{A}_N \cup \mathcal{B}_N \leftrightarrow \mathcal{P}_N$ such that*

$$L(C, s, \text{H-W}) = \begin{cases} L(f, s, \text{spin}), & \text{if } C \in \mathcal{A}_N, \\ L(f, s, \text{spin})^2, & \text{if } C \in \mathcal{B}_N. \end{cases}$$

The statement of the paramodular conjecture uses the theory of paramodular newforms, begun by Tomoyoshi Ibukiyama [5], and systematically developed and completed by Brooks Roberts and Ralf Schmidt [11, 12]. It also requires excluding the paramodular Gritsenko lifts, which are the arithmetically uninteresting paramodular forms. The most subtle condition is that the endomorphisms defined over $\mathbb{Q}$ are minimal; when this is not the case, the abelian surface A is of GL(2)-type and the modularity is known. The QM case was not seen in levels $N < 1000$ and Brumer and Kramer have shown that QM implies $N = M^2 s$ with $s \mid \gcd(30, M)$.

Although the paramodular conjecture is open, there are consistency results that may be viewed as theoretical evidence for the conjecture. For example, there is a way to twist an abelian surface A by a character χ so that the Hasse-Weil L-function of the twist A^χ is the twist of the Hasse-Weil L-function of A. When A has conductor N and χ is the nontrivial quadratic Dirichlet character modulo an odd prime p prime to N, the conductor of the twist A^χ is Np^4. Drawing an inference from the paramodular conjecture, there should be a compatible way to twist paramodular forms (at least weight two nonlift newforms). Johnson-Leung and Roberts have such a theory of twisting paramodular forms, [7].

Theorem (Johnson-Leung and Roberts)**.** *Let χ be the nontrivial quadratic Dirichlet character modulo an odd prime p prime to N. There exists a linear twisting map*

$$\mathcal{T}_\chi : S_k\left(K(N)\right) \to S_k\left(K(Np^4)\right)$$

such that if f is a new eigenform and $\mathcal{T}_\chi(f) \neq 0$ then

$$L\left(\mathcal{T}_\chi(f), s, \mathrm{spin}\right) = L^\chi\left(f, s, \mathrm{spin}\right).$$

As a consequence of this theorem, and the modularity of A_{277}, we deduce the modularity of A_{277}^χ whenever $\mathcal{T}_\chi(f_{277}) \neq 0$.

Further consistency with the paramodular conjecture lies in an automorphic counterpart to Weil restriction. Given an elliptic curve E/K over a real quadratic field K, the Weil-restriction is an abelian surface $A/\mathbb{Q}$ whose $\mathbb{Q}$-rational points are in bijection with the K-rational points of E. When E is not isogenous to its conjugate, we have $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. An inference from the paramodular conjecture is that there should be a way to lift the Hilbert modular form that shows the modularity of E/K to the paramodular form that shows the modularity of the Weil restriction A. Again, Johnson-Leung and Roberts have such a theory [6], which provides an infinite set of modularity results. Over imaginary quadratic fields K, there is an analogous result [1] of Tobias Berger, and Lassina Dembélé, Ariel Pacetti, and Mehmet Haluk Sengun that lifts Bianchi forms to paramodular forms and provides modularity results for abelian surfaces that are Weil restrictions whenever the modularity of the elliptic curve over K is known. Thus the paramodular conjecture continues to inspire research.

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Modularity of Abelian Surfaces II

David S. Yuen

(joint work with Cris Poor)

We give a current history of rigorous computations that give evidence for the paramodular conjecture (see presentation "Modularity of Abelian Surfaces I" for the statement of the paramodular conjecture), and we describe the techniques used to compute paramodular forms rigorously. We present the history chronologically with publication years, although arxiv announcements often predate publication by years.

Published 2014, Brumer and Kramer [5] proved that there are no abelian surfaces for prime conductors less than 600 other than the "exceptional" conductors of $\{277, 349, 353, 389, 461, 523, 587\}$. Poor and Yuen at the same time proved [14] that there are no weight 2 nonlift paramodular newforms of prime levels less than 600 except possibly for the exceptional levels. The absence of abelian surfaces and the absence of nonlift paramodular newforms match up perfectly for these prime levels under 600 away from the interesting levels. We also proved that there were at most one (two for 587) dimension of nonlifts for the interesting levels. We proved the existence of the nonlift eigenform at level 277, but the existence of the nonlift eigenforms at the other interesting levels had to wait for techniques in subsequent papers [7] [13]. The technique used [14] was "integral closure". This technique relied on known dimensions of paramodular forms in higher weights for prime levels given by Ibukiyama [9].

Published 2016, Breeding, Poor, and Yuen used the technique of "Jacobi Restriction" and certain bounds based on the geometry of numbers to prove that there are no weight 2 nonlift paramodular newforms for levels less than or equal to 60, prime, squarefree, or not squarefree [3].

Published 2017, Poor, Shurman, and Yuen used an upgraded version of the integral closure technique combined with Jacobi restriction to prove that there were no weight 2 nonlift paramodular newforms for squarefree levels less than 300, except for the levels of $\{249, 277, 295\}$ where there was exactly one dimension of nonlift newforms [11]. The dimension formula for higher weights and squarefree levels of Ibukiyama and Kitayama [10] was crucial for this technique.

Published 2019, Brumer, Pacetti, Poor, Tonaria, Voight, Yuen proved the first examples of modularity of a typical abelian surface [4]. The technique was a generalization of Faltings-Serre.

Published 2020, Berger and Klosin prove the first example of modularity of a typical abelian surface of composite level [1]. The technique was using deformations of Galois representations. Poor, Shurman, Yuen wrote an appendix that proved the needed existence existence and congruence of certain paramodular forms.

Current work in progress, Breeding, Poor, Shurman, and Yuen are using a new technique "Humbert restriction" to prove that there are no weight 2 nonlift paramodular newforms for *a*ll levels less than or equal to 388, except for the levels $\{249, 277, 295, 349, 353, 388\}$, and furthermore there is exactly one dimension of nonlift newforms in these exceptional levels. Currently there are still sixteen more levels to be worked out. The dimensions of some Hibert modular forms over real quadratic fields are crucial to these computations [15].

We give overviews of the techniques of integral closure, Jacobi restriction, and Humbert restriction [2] [3] [11] [14].

We describe how to make nonlift newforms using techniques of integral closures, Borcherds products, and tracing down from a higher level [6] [7] [8] [12] [13] [14] .

We discuss existing and work in progress techniques for computing many eigenvalues that depend on the form of the available formula for an eigenform [4].

We look at all available current data (rigorous or heuristic) for levels less than or equal to 1000 both on the automorphic (paramodular forms) and arithmetic (abelian surfaces) sides and see that they match up perfectly, with only level 903 having a nonlift paramodular newform with no known matching abelian surface.

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Towards the Kodaira dimension of moduli spaces of Abelian differentials

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(joint work with Matteo Costantini, Dawei Chen)

The moduli spaces of Abelian differentials $\Omega\mathcal{M}_{g,n}(\mu)$ parametrize n-pointed stable curves of genus g together with an abelian differential with zeros at the marked points of orders $\mu = (m_1, \ldots, m_n)$. These strata have attracted a lot of interest from the dynamics viewpoing since they admit and action of $\mathrm{SL}_2(\mathbb{R})$ that encodes the dynamics on polygonal billiard tables. Here we report on progress towards the algebraic geometric properties of the projectivized strata $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu) = \Omega\mathcal{M}_{g,n}(\mu)/\mathbb{C}^*$.

For low genus strata, i.e. $g \leq 9$ for all n and moreover for some strata for $g \leq 11$ but large n, the strata are uniruled by results of Barros [1] and Bud [2]. For large genus the moduli space of curves $\mathcal{M}_{g,n}$ is of maximal Kodaira dimension and the moduli spaces of abelian differentials are finite of covers of such $\mathcal{M}_{g,n}$ if moreover n is large by results of [3]. This implies that $\Omega\mathcal{M}_{g,n}(\mu)$ has maximal Kodaira dimension if both g and n is large.

The main open question is thus the behaviour of the Kodaira dimension for g large in the cases with few points, including the special case of subcanonical points where $n = 1$ and thus $\mu = (2g-2)$. The connected components of strata have been classified by Kontsevich and Zorich [6]. For each μ there are up to three components, distinguished by odd or even spin parity and possibly components consisting entirely of hyperelliptic curves. The even and the hyperelliptic components are unirational for trivial reasons. For the first interesting case we are resonably confident to conjecture:

Conjecture 1. *The odd components of the minimal strata* $\mathbb{P}\Omega\mathcal{M}_{g,n}(2g-2)^{\mathrm{odd}}$ *have maximal Kodaira dimension for* $g \geq 12$.

The strategy is the classical one of Harris and Mumford [7] writing the canonical bundle of a compactification as an ample plus an effective divisor. Two prerequisits have been provided by earlier work. The compactifiction from [4] is a smooth proper Deligne-Mumford stack and [5] gives a formula the canonical class on this stack.

The first step in our program is to show that the coarse moduli space associated with the compactified stack is actually a projective variety by exhibiting an ample line bundle. We denote that coarse moduli space by $\mathbb{P}\mathrm{MS}(\mu)$, an acrynom for the multi-scale differentials whose moduli space compactifies the moduli space of Abelian differentials.

The second step is to understand the ramification divisor of the map from the compactified stack to the compact coarse moduli space. The third step is to control the singularities. Both these steps have been carried out. We give a flavour of the result in the interior. The complete result is a long case distinction.

Theorem 2. *The singularities of the coarse moduli space $\mathbb{P}\Omega M_{g,n}(\mu)$ with marked points are canonical except for the holomorphic stratum $\Omega\mathcal{M}_{1,2}(0,0)$ and the meromorphic strata $\mu = (m, 2-m)$ in genus $g = 2$ for $m \geq 4$ and $m \equiv 1 \mod 3$.*

However, the compactified coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$ has non-canonical singularities for all but finitely many μ.

The fourth step is to find an effective divisor with small slope. For this purpose we use a generalized Weierstrass divisor. At least for $\mu = (2g-2)^{\mathrm{odd}}$ and $\mu = (2g-3, 1)$ the slope beats for $g \geq 12$ the slope of the canonical class, which is the main reason for the conjecture. However, with respect to several other boundary divisors the Weierstrass divisor has pretty bad slope. So the fifth step will be to mix with known Brill-Noether divisors, pulled back from $\overline{\mathcal{M}}_{g,n}$ and estimate that the slope is good enough to control all boundary divisors, even with a compensation term stemming from non-canonical singularities.

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On the cone of effective surfaces in $\overline{\mathcal{A}_3}$

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(joint work with Samuel Grushevsky)

1. Introduction

The main purpose of our work is to start the investigation of the pseudoeffective cone of k-dimensional cycles in compactifications of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties (ppav) of dimension g. We denote by $\overline{\mathcal{A}_g}$ the perfect cone compactifiation of $\mathcal{A}_g$. Its Picard group (with rational coefficients) is generated by the Hodge line bundle L and the boundary D (which is irreducible). Setting $M := 12L - D$, an important theorem of Shepherd-Barron says

Theorem 1.1 (Shepherd-Barron 2006)**.** *The nef cone of $\overline{\mathcal{A}_g}$ is given by*

$$\mathrm{Nef}^1(\overline{\mathcal{A}_g}) = \mathbb{R}_{\geq 0}L + \mathbb{R}_{\geq 0}M.$$

This result plays a crucial role in the proof of Shepherd-Barron's theorem that $\overline{\mathcal{A}_g}$ for $g \geq 12$ is the canonical model in the sense of the minimal model program. It can also be formulated in terms of effective curves. For this let $C_A := \overline{\mathcal{A}_1} \times \{B\}$ where B is a fixed ppav of dimension $g-1$ and let C_F be an irreducible curve which is contracted under the natural morphism $\overline{\mathcal{A}_g} \to \mathcal{A}_g^{\mathrm{Sat}}$ to the Satake compactification. Since $M.C_A = L.C_F = 0$ the above theorem translates into

$$\overline{\mathrm{Eff}}_1(\overline{\mathcal{A}_g}) = \mathbb{R}_{\geq 0}C_A + \mathbb{R}_{\geq 0}C_F.$$

Here $\overline{\mathrm{Eff}}_k(\overline{\mathcal{A}_g})$ denotes the pseudoeffective cone of k-cycles, i.e. the closure of the cone of effective k-cycles.

The corresponding question about the pseudoeffective cone $\overline{\mathrm{Eff}}^1(\overline{\mathcal{A}_g})$ of divisors has also been the subject of numerous investigations. This cone is explicitly known for $g \leq 5$. The boundary D is an extremal ray for all genera g, as it is contracted under the map to $\mathcal{A}_g^{\mathrm{Sat}}$. To determine the other ray is equivalent to knowing the *slope* of $\mathcal{A}_g$ and is closely related to the question about the Kodaira dimension.

In our work we study the first case of the pseudoeffective cone of cycles which are neither in dimension nor codimension 1, namely the cone $\overline{\mathrm{Eff}}_2(\overline{\mathcal{A}_3})$ of pseudoeffective surfaces in the compactified moduli space $\overline{\mathcal{A}_3}$ of 3-dimensional ppav. We consider this cone in $H_4(\overline{\mathcal{A}_3}, \mathbb{Q}) \cong \mathbb{R}^4$. We remark that in genus 3 all three known toroidal compactifications (namely perfect cone, second Voronoi and central cone) coincide. Moreover the space $\overline{\mathcal{A}_3}$ has only finite quotient singularities and hence Poincaré duality holds. The Chow ring with rational coefficients $CH^\bullet_{\mathbb{Q}}(\overline{\mathcal{A}_3})$ was determined by van der Geer [1] and it was remarked by Tommasi and the speaker that Chow and cohomology coincide in this case [3] .

2. The main result

Our aim is to determine extremal rays of $\mathrm{Eff}_2(\overline{\mathcal{A}_g})$. For this we recall that a class $S \in \mathrm{Eff}_k(\overline{\mathcal{A}_g})$ is called *extremal effective* if $S = S_1 + S_2$ with $S_1, S_2 \in \mathrm{Eff}_k(\overline{\mathcal{A}_g})$ implies that S_1 and S_2 are non-negative multiples of S. A crucial role is played by the locus of decomposable ppav, for which we make use of the finite map $N : \overline{\mathcal{A}_1} \times \overline{\mathcal{A}_2} \to \overline{\mathcal{A}_3}$. By C_A and C_F we denote the two extremal curves in $\overline{\mathcal{A}_2}$ which we have introduced above, and set $S_{AA} := \overline{\mathcal{A}_1} \times C_A$, $S_{AF} := \overline{\mathcal{A}_1} \times C_F$, $S_{DD} := \{B\} \times \partial\overline{\mathcal{A}_2}$ and $S_{DA} := \{B\} \times (\overline{\mathcal{A}_1} \times \overline{\mathcal{A}_1})$. Here $B \in \mathcal{A}_1$ is the class of a fixed elliptic curve, and $\partial\overline{\mathcal{A}_2}$ denotes the boundary of $\overline{\mathcal{A}_2}$ parametrizing degenerate principally polarized abelian surfaces We further set $S_A := N(S_{AA}) = N(S_{DA})$ and $S_F := N(S_{AF}), S_D := N(S_{DD})$.

Being a toroidal compactification $\overline{\mathcal{A}_3}$ has a stratification into strata

$$\overline{\mathcal{A}_3} = \mathcal{A}_3 \sqcup \bigsqcup_{\Delta \in \Sigma} \sigma_\Delta,$$

where Δ runs through all positive dimensional cones in the perfect cone decomposition of the rational closure of $\mathrm{Sym}^2_{>0}(\mathbb{R}^3)$ (modulo the action of $\mathrm{GL}(3,\mathbb{Z})$). In our case this amounts to 8 different cones. The trivial cone $\{0\}$ corresponds to the open part $\mathcal{A}_3$. The dimension of Δ equals the codimension of the stratum σ_Δ in $\overline{\mathcal{A}_3}$, which is locally closed in $\overline{\mathcal{A}_3}$. Its closure $\overline{\sigma_\Delta}$ is thus an irreducible subvariety of dimension $\dim(\overline{\sigma_\Delta}) = 6 - \dim(\Delta)$. In genus 3 two orbits of 4-dimensional cones exist. These are represented by $\langle x_1^2, x_2^2, (x_1-x_2)^2, x_3^2\rangle$ and $\langle x_1^2, x_2^2, (x_1-x_3)^2, (x_2-x_3)^2\rangle$, and are called $K3+1$ and $C4$ respectively, leading to surfaces $\overline{\sigma_{K3+1}}$ and $\overline{\sigma_{C4}}$ in $\overline{\mathcal{A}_3}$. Our main result [2, Main Theorem] says:

Theorem 2.1. *The five surfaces $S_A, S_F, S_D, \overline{\sigma_{K3+1}}, \overline{\sigma_{C4}}$ are extremal effective rays of $\overline{\mathrm{Eff}}_2(\overline{\mathcal{A}_3})$.*

We further make the

Conjecture 2.2. *One has in fact equality*

$$\overline{\mathrm{Eff}}_2(\overline{\mathcal{A}_3}) = \mathbb{R}_{\geq 0}S_A + \mathbb{R}_{\geq 0}S_F + \mathbb{R}_{\geq 0}S_D + \mathbb{R}_{\geq 0}\overline{\sigma_{K3+1}} + \mathbb{R}_{\geq 0}\overline{\sigma_{C4}}.$$

This conjecture can be rephrased in cohomology in terms of the nef cone $\mathrm{Nef}^2(\overline{\mathcal{A}_3})$. The calculation of various top degree intersection numbers supports this conjecture.

3. Outlook

The cohomology in degree 4 of $\overline{\mathcal{A}_g}$ stabilizes, more precisely:

$$H_4(\overline{\mathcal{A}_g}, \mathbb{R}) \cong H^4(\overline{\mathcal{A}_g}, \mathbb{R}) \cong \mathbb{R}^4 \text{ for } g \geq 3.$$

Moreover, Poincaré duality still holds in this degree (although $\overline{\mathcal{A}_g}$ is a singular space for which Poincaré duality fails in general). By taking a product with a fixed $B \in \mathcal{A}_{g-3}$ the surfaces S_A, S_D, S_F can be viewed as surfaces in $\overline{\mathcal{A}_g}$. Similarly, we can do this for the surfaces $\overline{\sigma_{K3+1}}$ and $\overline{\sigma_{C4}}$ in $\overline{\mathcal{A}_3}$, denoting the resulting surfaces in $\overline{\mathcal{A}_g}$ by S_1, S_2. Note that this is not the same as the strata $\overline{\sigma_{K3+1}}$ and $\overline{\sigma_{C4}}$ in $\overline{\mathcal{A}_g}$

which have complex codimension 4. The surfaces just listed still span $H_4(\overline{\mathcal{A}_g}, \mathbb{R})$. A natural question is:

Question 3.1. *Is it true that*

$$\overline{\mathrm{Eff}}_2(\overline{\mathcal{A}_g}) = \mathbb{R}_{\geq 0}S_A + \mathbb{R}_{\geq 0}S_F + \mathbb{R}_{\geq 0}S_D + \mathbb{R}_{\geq 0}S_1 + \mathbb{R}_{\geq 0}S_2?$$

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Witten spin intersection numbers and the arithmetic of ordinary linear differential equations

Don Zagier

Witten's r-spin intersection numbers are integrals of certain products of the Witten classes coupling the ψ-classes on the Deligne–Mumford moduli space $\overline{\mathcal{M}_{g,n}}$ of stable algebraic curves. There are also more general invariants (Fan-Jarvis-Ruan-Witten invariants) associated to simply laced simple Lie algebras, with A_{r-1} corresponding to the r-spin case. In a recent joint paper (arXiv:2101.10924) with Boris Dubrovin (deceased) and Di Yang we gave a number of descriptions of these invariants for once-punctured surfaces (i.e., $n = 1$) and arbitrary genus g using methods from integrable systems (integrable hierarchies of Korteweg-de Vries type), wave functions, and pseudodifferential operators. In particular, we showed that the invariants in question, appropriately normalized, are the coefficients of an algebraic generating function in the A_l, D_l, and E_6 cases.

The talk concentrated on the A_4 (or 5-spin) case, for which there are particularly interesting arithmetic aspects connected with questions of integrality. The intersection numbers in this case, denoted τ_g, are rational numbers whose first few values are $\tau_0 = 1$, $\tau_1 = \frac{1}{6}$, $\tau_2 = \frac{11}{3600}$, $\tau_3 = 0$, $\tau_4 = \frac{341}{25920000}$, … and which satisfy a recursion with polynomial coefficients, meaning that their generating series $\sum_g \tau_g x^g$ satisfies an ODE with polynomial coefficients. The special feature here is that there are *three* different normalizations of τ_g, each obtained by multiplying τ_g by a suitable product of Pochhammer symbols (shifted factorials), that are simultaneously integral and of only exponential rather than factorial growth. For example, for $g = 5n$ one finds that each of the three numbers

$$a_{5n} = 2^{2n}\left(\tfrac{2}{5}\right)_n\left(\tfrac{-1}{10}\right)_n\tau_{5n}\,, \qquad b_{5n} = \left(\tfrac{4}{5}\right)_n\left(\tfrac{1}{5}\right)_n\tau_{5n}\,, \qquad c_{5n} = \left(\tfrac{4}{5}\right)_n\left(\tfrac{3}{5}\right)_n\tau_{5n}$$

(where $(x)_n := x(x+1)\cdots(x+n-1)$) belongs to $\mathbb{Z}\left[\frac{1}{30}\right]$ and has only exponential growth as n tends to infinity. This means that each of the three generating functions $\sum a_g x^g$, $\sum b_g x^g$ and $\sum c_g x^g$ is a "G-function" in the sense of Carl Ludwig Siegel and hence according to a famous conjecture should be a period function of some family of algebraic varieties (or equivalently, a solution of a Picard-Fuchs differential equation). The first of these three was shown in the paper to be not only a period function as expected, but actually an algebraic function, and as mentioned above this result was also generalized to all higher spin intersection numbers and even to the FJRW invariants for other simple Lie algebras. The other two integral versions of the intersection numbers are known so far only in the A_4 case, and at the time that the paper was uploaded (Jan. 26) the fact that the corresponding generating functions were also period functions was not known. In the course of preparing the lecture, I found that this was indeed the case, as predicted by the general conjecture, but that even more was true, namely that these generating functions were again algebraic. This is a quite puzzling phenomenon, in which three entirely different (and very complicated) algebraic functions of one variable have Taylor coefficients that are the same up to shifted factorials and hence share the same large prime factors, but have completely different small prime factors. We are currently in the process of trying to understand this better. There is also an intriguing link between the generating functions associated to the coefficients b_g and c_g and Klein's famous equations for the invariants of the icosahedron, as well as with the Hauptmodule for $\Gamma(5)$ given by the Ramanujan continued fraction.

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