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Algebraic Groups (hybrid meeting)

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Abstract. Linear algebraic groups is an active research area in contemporary mathematics. It has rich connections to algebraic geometry, representation theory, algebraic combinatorics, number theory, algebraic topology, and differential equations. The foundations of this theory were laid by A. Borel, C. Chevalley, J.-P. Serre, T. A. Springer and J. Tits in the second half of the 20th century. The Oberwolfach workshops on algebraic groups, led by Springer and Tits, played an important role in this effort as a forum for researchers, meeting at approximately 3 year intervals since the 1960s. The present workshop continued this tradition, covering a range of topics, with an emphasis on recent developments in the subject.

Mathematics Subject Classification (2010): 14Lxx, 17Bxx, 20Gxx, 14Mxx.

Introduction by the Organizers

The theory of linear algebraic groups originated in the work of E. Picard in the mid-19th century. Picard assigned a “Galois group” to an ordinary differential equation. This construction was developed into what is now known as “differential Galois theory” by J. F. Ritt and E. R. Kolchin in the 1930s and 40s. Another source of inspiration comes from Invariant theory and the representation theory of classical groups.

Their work was a precursor to the modern theory of algebraic groups, founded by A. Borel, C. Chevalley, J. P. Serre, T. A. Springer and J. Tits in the second half of the 20th century. The Oberwolfach workshops on algebraic groups, originated by Springer and Tits, played an important role in this effort as a forum for researchers, meeting at regular intervals since the 1960’s.

The present workshop continued this tradition despite of the pandemy. There were 50 participants (2 in Oberwolfach, 48 online) from 13 countries: Australia, Canada, China, France, Germany, Great Britain, India, Israel, Italy, the Netherlands, Russia, Switzerland and the United States. The scientific program consisted of 21 lectures. The lectures covered a broad range of topics of current interest, including

- Geometric invariant theory;
- Spherical varieties;
- Algebraic groups over imperfect fields;
- Algebraic groups and opers;
- Geometric Satake correspondence and its relation to Modular representation theory;
- Hessenberg varieties and character sheaves;
- Quantum groups and cluster algebras;
- Cluster varieties and potential functions;
- Modular representation theory;
- Bruhat-Tits group schemes.

Due to the special format, the organization of recreational activities like the traditional Wednesday afternoon hike and the Thursday night concert had to be cancelled. But during the discussions, again and again, people regretted not to be in Oberwolfach. They even missed the standard interchange train stop at Offenburg, not to speak about the cake. More seriously: despite the perfect organization of the meeting and the excellent online talks, people missed the possibility for a personal exchange. And this is a point, for which the meetings at the MFO are usually well known for.

Workshop (hybrid meeting): Algebraic Groups

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Abstracts

Semisimplification for subgroups of reductive algebraic groups

Gerhard Röhrle
(joint work with Michael Bate and Benjamin Martin)

This is a report on the joint work [3]. There we present a construction of the *semisimplification* of a subgroup H of a reductive linear algebraic group G over an arbitrary field k. This unifies and generalizes many concepts already in the literature within a single framework. It is also an ingredient in recent work of Lawrence-Sawin on the Shafarevich Conjecture for abelian varieties [8], which involves Galois representations taking values in non-connected reductive p-adic groups. Our definition of a k-semisimplification is new and generalizes the one given by Serre in [9, §3.2.4]. Our aim is a Jordan–Hölder Theorem: the k-semisimplification of a subgroup H of G is unique up to $G(k)$-conjugacy, generalizing [9, Prop. 3.3(b)].

1. Cocharacter-closed orbits

Following [6], [5], and [1], we regard an affine variety over a field k as a variety X over the algebraic closure $\overline{k}$ together with a choice of k-structure. We write $X(k)$ for the set of k-points of X and $X(\overline{k})$ (or just X) for the set of $\overline{k}$-points of X. By a subvariety of X we mean a closed $\overline{k}$-subvariety of X; a k-subvariety is a subvariety that is defined over k. We denote by M_n the associative algebra of $n \times n$ matrices over k. Below G denotes a possibly non-connected reductive linear algebraic group over k. By a subgroup of G we mean a closed $\overline{k}$-subgroup and by a k-subgroup we mean a subgroup that is defined over k. By G^0 we denote the identity component of G, and likewise for subgroups of G.

We define $Y_k(G)$ to be the set of k-defined cocharacters of G and $Y(G) := Y_{\overline{k}}(G)$ to be the set of all cocharacters of G. Let H be a subgroup of G. Even if H is k-defined, the (set-theoretic) centralizer $C_G(H)$ need not be k-defined in general. For instance, if k is perfect and H is k-defined then $C_G(H)$ is k-defined.

Next we recall some basic notation concerning parabolic subgroups in (non-connected) reductive groups G from [2, §6] and [5]. Given $\lambda \in Y(G)$, we define

$$P_\lambda = \{g \in G \mid \lim_{a \to 0} \lambda(a) g \lambda(a)^{-1} \text{ exists}\}$$

and $L_\lambda = C_G(\mathrm{Im}(\lambda))$. We call P_λ an *R-parabolic subgroup* of G and L_λ an *R-Levi subgroup* of P_λ. We have $P_\lambda = L_\lambda = G$ if $\mathrm{Im}(\lambda)$ belongs to the centre of G.

We denote the canonical projection from P to L by c_L; this is k-defined if P and L are. If we are given $\lambda \in Y(G)$ such that $P = P_\lambda$ and $L = L_\lambda$ then we often write c_λ instead of c_L. We have $c_\lambda(g) = \lim_{a \to 0} \lambda(a) g \lambda(a)^{-1}$ for $g \in P_\lambda$; the kernel of c_λ is the unipotent radical $R_u(P_\lambda)$ and the set of fixed points of c_λ is L_λ.

Let $m \in \mathbb{N}$. Below we consider the action of G on G^m by simultaneous conjugation: $g \cdot (g_1, \ldots, g_m) = (g g_1 g^{-1}, \ldots, g g_m g^{-1})$. Given $\lambda \in Y(G)$, we have a map $P_\lambda^m \to L_\lambda^m$ given by $\mathbf{g} \mapsto \lim_{a \to 0} \lambda(a) \cdot \mathbf{g}$; we abuse notation slightly and also call this map c_λ. For any $\mathbf{g} \in P_\lambda^m$, there exists an R-Levi k-subgroup L of P_λ with $\mathbf{g} \in L^n$ if and only if $c_\lambda(\mathbf{g}) = u \cdot \mathbf{g}$ for some $u \in R_u(P_\lambda)(k)$.

Our main tool from GIT is the notion of cocharacter-closure from [5] and [1].

Definition 1. *Let X be an affine G-variety and let $x \in X$ (we do not require x to be a k-point). We say that the orbit $G(k) \cdot x$ is* cocharacter-closed over k *if for all $\lambda \in Y_k(G)$ such that $x' := \lim_{a \to 0} \lambda(a) \cdot x$ exists, x' belongs to $G(k) \cdot x$. If $k = \overline{k}$ then it follows from the Hilbert-Mumford Theorem that $G(k) \cdot x$ is cocharacter-closed over k if and only if $G(k) \cdot x$ is closed* [7, Thm. 1.4]. *If $\mathcal{O}$ is a $G(k)$-orbit in X then we say that $\mathcal{O}$ is* accessible from x over k *if there exists $\lambda \in Y_k(G)$ such that $x' := \lim_{a \to 0} \lambda(a) \cdot x$ belongs to $\mathcal{O}$.*

Theorem 2 (Rational Hilbert-Mumford Theorem ([1, Thm. 1.3]))**.** *Let G, X, x be as above. Then there is a unique $G(k)$-orbit $\mathcal{O}$ such that $\mathcal{O}$ is cocharacter-closed over k and accessible from x over k.*

2. G-complete reducibility

Definition 3. *Let H be a subgroup of G. We say that H is* G-completely reducible over k *(G-cr over k) if for any R-parabolic k-subgroup P of G such that P contains H, there is an R-Levi k-subgroup L of P such that L contains H. We say that H is* G-irreducible over k *(G-ir over k) if H is not contained in any proper R-parabolic k-subgroup of G at all. We say that H is G-cr if H is G-cr over $\overline{k}$.*

For more on G-complete reducibility, see [9] and [2]. Note that the definition make sense even if H is not k-defined. It is immediate that G-irreducibility over k implies G-complete reducibility over k. We have $P_{g\cdot\lambda} = gP_\lambda g^{-1}$ and $L_{g\cdot\lambda} = gL_\lambda g^{-1}$ for any $\lambda \in Y(G)$ and any $g \in G$ (cf. [2, §6]). It follows that if H is G-cr over k (resp., G-ir over k) then so is any $G(k)$-conjugate of H.

Fix a k-embedding $G \to \mathrm{GL}_n$ for some $n \in \mathbb{N}$. Let H be a subgroup of G. Let $m \in \mathbb{N}$ and let $\mathbf{h} = (h_1, \ldots, h_m) \in H^m$. We call $\mathbf{h}$ a *generic tuple for H* if $h_1, \ldots, h_m$ generates the subalgebra of M_n generated by H [5, Def. 5.4]. Note that we don't insist that $\mathbf{h}$ is a k-point.

Theorem 4 ([1, Thm. 9.3])**.** *Let H be a subgroup of G and let $\mathbf{h} \in H^m$ be a generic tuple for H. Then H is G-completely reducible over k if and only if $G(k) \cdot \mathbf{h}$ is cocharacter-closed over k.*

Using this result one can derive many results on G-complete reducibility: for instance, see [2] for the algebraically closed case and [5], [1] for arbitrary k.

3. k-semisimplification

Definition 5. *Let H be a subgroup of G. We say that a subgroup H' of G is a* k-semisimplification of H *if there exist an R-parabolic k-subgroup P of G and an R-Levi k-subgroup L of P such that $H \subseteq P$, $H' = c_L(H)$ and H' is G-completely reducible (or equivalently, L-completely reducible, by* [3, Prop. 3.6(b)]*) over k. We say* the pair (P, L) yields H'.

Remark 6. *(a). Let H be a subgroup of G. If H is G-cr over k then clearly H is a k-semisimplification of itself, yielded by the pair (G, G).*

(b). Suppose (P, L) yields a k-semisimplification H' of H. Let L_1 be another R-Levi k-subgroup of P. Then $L_1 = uLu^{-1}$ for some $u \in R_u(P)(k)$, so $c_{L_1}(H) = uc_L(H)u^{-1}$. Hence (P, L_1) also yields a k-semisimplification of H. We say that P yields a k-semisimplification of H.

(c). For G connected and H a subgroup of $G(k)$, Definition 5 generalizes Serre's "G-analogue" of a semisimplification from [9, §3.2.4].

Remark 7. *Let $\mathbf{h} = (h_1, \ldots, h_m) \in H^m$ be a generic tuple for H. It is easy to see that $c_\lambda(\mathbf{h}) = (c_\lambda(h_1), \ldots, c_\lambda(h_m))$ is a generic tuple for $c_\lambda(H)$. Hence by Theorem 4, $c_\lambda(H)$ is a k-semisimplification of H if and only if $G(k) \cdot c_\lambda(\mathbf{h})$ is cocharacter-closed over k. It follows from Theorem 2 that H admits at least one k-semisimplification: for we can choose $\lambda \in Y_k(G)$ such that $G(k) \cdot c_\lambda(\mathbf{h})$ is cocharacter-closed over k, so $c_\lambda(H)$ is a k-semisimplification of H, yielded by (P_λ, L_λ).*

Here is our main result, which was proved in the special case $k = \overline{k}$ in [5, Prop. 5.14(i)], cf. [9, Prop. 3.3(b)]. The uniqueness statement is akin to the theorem of Jordan–Hölder.

Theorem 8 ([3, Thm. 4.5])**.** *Let H be a subgroup of G. Then any two k-semisimplifications of H are $G(k)$-conjugate.*

Remark 9. *Given a reductive k-group G and a subgroup H of G, we may regard G as a $\overline{k}$-group by forgetting the k-structure, so it makes sense to consider the $\overline{k}$-semisimplification of H. It can happen that H is G-cr over k but not G-cr, or vice versa: see* [2, Ex. 5.11] *and* [4, Ex. 7.22]. *So there is no direct relation between the notions of k-semisimplification and $\overline{k}$-semisimplification in general.*

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Overgroups of regular unipotent elements in simple algebraic groups

Donna M. Testerman
(joint work with Gunter Malle)

Let G be a simple linear algebraic group defined over an algebraically closed field. The regular unipotent elements of G are those whose centraliser has minimal possible dimension (the rank of G) and these form a single conjugacy class which is dense in the variety of unipotent elements of G. In [3], we make a contribution to the study of positive-dimensional subgroups of G which meet the class of regular unipotent elements. Since any parabolic subgroup must contain representatives from every unipotent conjugacy class, the question is tractable only for reductive, not necessarily connected, subgroups. Our main result is:

Theorem 1. *Let G be a simple linear algebraic group over an algebraically closed field, $X \leq G$ a closed reductive subgroup containing a regular unipotent element of G. If $[X^\circ, X^\circ] \neq 1$, then X lies in no proper parabolic subgroup of G.*

On the other hand, we show that for many simple groups G, there exists a closed positive-dimensional reductive subgroup $X \leq G$ with X° a torus, such that X meets the class of regular unipotent elements of G and lies in a proper parabolic subgroup of G. Note that a direct corollary of the theorem concerns representations of positive-dimensional reductive groups:

Corollary 2. *Let $X < \mathrm{SL}(V)$ be a closed reductive subgroup containing a unipotent element acting as one Jordan block on V. If $[X^\circ, X^\circ] \neq 1$, then X acts irreducibly on V.*

The investigation of the possible overgroups of regular unipotent elements in simple linear algebraic groups has a long history. The *maximal* closed positive-dimensional reductive subgroups of G which meet the class of regular unipotent elements were classified by Saxl and Seitz in 1997; see [4]. In earlier work, see [5, Thm 1.9], Suprunenko obtained a particular case of their result. In order to inductively derive from the Saxl–Seitz classification a description of all closed positive-dimensional reductive subgroups $X \leq G$ containing regular unipotent elements, one needs to exclude that any of these can lie in proper parabolic subgroups. For connected X this was shown by Testerman and Zalesski in [6, Thm 1.2] in 2013. They then went on to determine all connected reductive subgroups of simple algebraic groups which meet the class of regular unipotent elements. Our result generalises [6, Thm 1.2] to the disconnected case and thus makes the inductive approach possible. It is worth pointing out that the analogous result is no longer true even for simple subgroups once one relaxes the condition of positive-dimensionality. For example, there exist reducible indecomposable representations of the group $\mathrm{PSL}_2(p)$ whose image in the corresponding $\mathrm{SL}(V)$ contains a matrix with a single Jordan block, i.e., the image meets the class of regular unipotent elements in $\mathrm{SL}(V)$.

Our proof of the main theorem relies on the result of Testerman–Zalesski [6] in the connected case, which actually implies our theorem in characteristic 0, as well as on results of Saxl–Seitz [4] classifying almost simple irreducible and tensor indecomposable subgroups of classical groups containing regular unipotent elements and maximal reductive subgroups in exceptional groups with this property. For the exceptional groups we also use information on centralisers of unipotent elements and detailed knowledge of Jordan block sizes of unipotent elements acting on small modules, as found in Lawther [2]. For establishing the existence of positive-dimensional reductive subgroups $X \leq G$, with X° a torus, and X meeting the class of regular unipotent elements, we produce subgroups which centralise a non-trivial unipotent element and hence necessarily lie in a proper parabolic subgroup of G.

The Testerman–Zalesski classification can be seen as being a "recognition" result: identify a reductive subgroup of the simple algebraic group G simply by knowing that the subgroup meets a certain conjugacy class. In this vein, and in the case of disconnected subgroups, the following natural questions remain open:

1. Determine (up to conjugacy) all positive-dimensional disconnected reductive subgroups $X < G$, where X° is not a torus and X contains a regular unipotent element of G.
2. Determine all disconnected reductive subgroups of G containing a regular unipotent element of G and lying in a proper parabolic subgroup of G.

Finally, it is also natural to ask to whether one can extend the main theorem to reductive subgroups of G meeting a conjugacy class of non-regular distinguished unipotent elements. (A unipotent element is *distinguished* if its connected centralizer contains no nontrivial semisimple elements.) In [1], Korhonen considers this question and gives examples of connected reductive subgroups X of simple classical algebraic groups such that X meets a class of distinguished unipotent elements of G and lies in a proper parabolic subgroup of G; see for example [1, 7.1].

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Langlands duality, shifted quantum groups and cluster algebras

David Hernandez

In the framework of the study of K-theoretical Coulomb branches, Finkelberg-Tsymbaliuk introduced remarkable new algebras, the shifted quantum affine algebras and their truncations. We establish that the Grothendieck ring of the category of their finite-dimensional representations has a natural cluster algebra structure. We propose a conjectural parameterization of simple modules of a non simply-laced truncation in terms of the Langlands dual quantum affine Lie algebra. We have several evidences, including a general result for finite-dimensional representations.

Shifted quantum affine algebras and their truncations arose [FT] in the study of quantized K-theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories in the sense of Braverman-Finkelberg-Nakajima which are at the center of current important developments. A presentation of (truncated) shifted quantum affine algebras by generators and relations was given by Finkelberg-Tsymbaliuk. Their rational analogs, the shifted Yangians, and their truncations, appeared in type A in the context of the representation theory of finite W-algebras by Brundan-Kleshchev, then in the study of quantized affine Grassmannian slices [KTWWY] for general types and in the study of quantized Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories by Braverman-Finkelberg-Nakajima for simply-laced types and [NW] for non simply-laced types.

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra, and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Kac-Moody algebra, central extension of the loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$. Drinfeld and Jimbo associated to each complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ called a quantum affine algebra. Shifted quantum affine algebras $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ can be seen as variations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, but depending on two coweights μ_+, μ_- of the underlying simple Lie algebra $\mathfrak{g}$. These coweights corresponding to shifts of formal power series in the Cartan-Drinfeld elements (that is quantum analogs of the $t^r h \in \mathcal{L}\mathfrak{g}$, with $r \in \mathbb{Z}$ and $h \in \mathfrak{h}$ in the Cartan subalgebra of $\mathfrak{g}$). In particular $\mathcal{U}_q^{0,0}(\hat{\mathfrak{g}})$ is a central extension of the ordinary quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. Up to isomorphism, $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ only depends on $\mu = \mu_+ + \mu_-$ and will be denoted simply by $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$.

The truncations are quotients of $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$ and depend on additional parameters, including a dominant coweight λ. In this context, these parameters λ and μ can be interpreted as parameters for generalized slices of the affine Grassmannian $\overline{\mathcal{W}}_{\mu}^{\lambda}$ (usual slices when μ is dominant).

For simply-laced types, representations of shifted Yangians and related Coulomb branches have been intensively studied, see [KTWWY] and references therein. For non simply-laced types, representations of quantizations of Coulomb branches have been studied by Nakajima and Weekes [NW].

In [H], we develop the representation theory of shifted quantum affine algebras. We establish several analogies with the representation theory of ordinary quantum affine algebras. But our approach is also based on several techniques, which are new for ordinary as well as for shifted quantum affine algebras : induction and restriction functors to the category $\mathcal{O}$ of representations of the Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$, relations of truncations with Baxter polynomiality in quantum integrable models, and parametrization of simple modules via Langlands dual interpolating (q,t)-characters.

We first we relate these representations to the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$. For general untwisted types, the category $\mathcal{O}$ of representations of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ was introduced and studied in [HJ]. Some representations in this category extend to a representation of the whole quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$, but many do not, including the prefundamental representations constructed in [HJ] and whose transfer-matrices have remarkable properties for the corresponding quantum integrable systems [FH2].

Consider a category $\mathcal{O}_\mu$ of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ which is an analog of the ordinary category $\mathcal{O}$. We obtain induction/restriction functors to the category $\mathcal{O}$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules and we establish the following. Let us denote by α_i the simple roots of $\mathfrak{g}$ and let n be the rank of $\mathfrak{g}$.

Theorem 1 ([H]). *The simple representations in $\mathcal{O}_\mu$ are parametrized by n-tuples $(\Psi_i(z))$ of rational fractions regular at 0 with $deg(\Psi_i(z)) = \alpha_i(\mu)$.*

We define a ring structure on the sum of Grothendieck groups $K_0(\mathcal{O}_\mu)$ from fusion products. It contains the Grothendieck ring $K_0(\mathcal{C}^{sh})$ of finite-dimensional representations as a subring. Recall that the cluster algebra $\mathcal{A}(Q)$ attached to a quiver Q is a commutative ring with a distinguished set of generators called cluster variables and obtained inductively by a procedure called mutation. Using induction/restriction functors, as well as results in [HL], we obtain the following (the last part of the Theorem relies on recent results in [KKOP]).

Theorem 2 ([H]). *The Grothendieck ring $K_0(\mathcal{C}^{sh})$ has a structure of a cluster algebra with an initial cluster variables which are classes of prefundamental representations. Moreover, all cluster monomials are classes simple objects.*

Let us now discuss truncated shifted quantum affine algebras, quotients of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. For simply-laced types, simple representations of truncated shifted Yangians have been parametrized in terms of Nakajima monomial crystals [KTWWY]. See the Introduction of [H] for comments on related results in [NW].

We will use Baxter polynomiality in quantum integrable systems. Let us recall that to each representation V of $\mathcal{U}_q(\hat{\mathfrak{b}})$ in the category $\mathcal{O}$, is attached a transfer-matrix $t_V(z)$ which is a formal power series in a formal parameter z with coefficients in $\mathcal{U}_q(\hat{\mathfrak{g}})$. Given another simple finite-dimensional representation W of $\mathcal{U}_q(\hat{\mathfrak{g}})$, we get a family of commuting operators on $W[[z]]$. This is a quantum integrable model generalizing the XXZ-model. It is established in [FH2], the spectrum of this system, that is the eigenvalues of the transfer-matrices, can be described in terms of certain polynomials, generalizing Baxter's polynomials associated to the XXZ-model. These Baxter's polynomials are obtained from the eigenvalues of transfer-matrices associated to prefundamental representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$. Moreover, this Baxter polynomiality implies the polynomiality of certain series of

Cartan-Drinfeld elements acting on finite-dimensional representations [FH2]. We relate this result to the structures of representations of truncated shifted quantum affine algebras. In particular, we give in [H] a uniform proof of the finiteness of the number of simple isomorphism classes for truncations.

In non-simply-laced types, we propose a parametrization of these simple representations. We use a limit obtained from interpolating (q, t)-characters. The latter were defined by Frenkel and the author as an incarnation of Frenkel-Reshetikhin deformed W-algebras interpolating between q-characters of a non simply-laced quantum affine algebra and its Langlands dual. They lead to the definition of an interpolation between the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_t(\hat{\mathfrak{g}}^L))$ of finite-dimensional representations of the Langlands dual algebra quantum affine algebra $\mathcal{U}_t(\hat{\mathfrak{g}}^L)$. We found it is relevant for our purposes to introduce a different specialization of interpolating (q, t)-characters that we call Langlands dual q-characters.

Conjecture 3 ([H]). *The simple modules of a truncation are explicitly parametrized by monomials in the Langlands dual q-character of a finite-dimensional representation of the Langlands dual quantum affine algebra.*

Recall that the deformed W-algebras were introduced by Frenkel-Reshetikhin in the context of the geometric Langlands program. The parametrization in [KTWWY] for simply-laced types can be understood in the context of symplectic duality. Hence the statement of our conjecture can be seen as motivated by symplectic and Langlands duality. Our main evidence for the Conjecture is the following, obtained as a consequence of the Baxter polynomiality.

Theorem 4 ([H]). *A finite-dimensional simple representation in $\mathcal{O}_\mu$ descends to a certain explicit truncation as predicted by Conjecture 3.*

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Fixed points on toroidal compactifications and essential dimension of congruence covers

Patrick Brosnan
(joint work with Najmuddin Fakhruddin)

1. **Introduction.** Let $f : X \to Y$ be a generically finite morphism of quasi-projective varieties over $\mathbb{C}$. (We work over $\mathbb{C}$ in this extended abstract mostly for simplicity.) The *pullback dimension* of $f : X \to Y$ is the minimum dimension $\operatorname{pbd} f$ of a quasi-projective variety Z such that $f : X \to Y$ is isomorphic to the projection on the second factor $W \times_Z Y \to Y$ from a fiber product, where $g : Y \to Z$ and $h : W \to Z$ are morphisms of quasi-projective complex schemes. The *essential dimension* $\operatorname{ed} f$ of f is the minimum of the pullback dimensions of the morphisms $X \times_Y U \to U$ obtained from f via pullback from an open immersion $U \to Y$. Similarly, if p is a prime number, then the essential dimension $\operatorname{ed}(f; p)$ of f at p is the minimum of the pullback dimensions of the morphisms $X \times_Y V \to V$, where $V \to Y$ ranges over all generically finite morphisms of degree prime to p. We say that f is *incompressible* (resp. *p-incompressible*) if $\operatorname{ed} f = \dim Y$ (resp. $\operatorname{ed}(f; p) = \dim Y$).

If G is a finite group (viewed as a constant group scheme), then the essential dimension $\operatorname{ed} G$ of G (resp. the essential dimension $\operatorname{ed}(G; p)$ of G at p) can be defined as the essential dimension $\operatorname{ed}(\mathbf{GL}_n \to \mathbf{GL}_n/G)$ (resp. $\operatorname{ed}(\mathbf{GL}_n \to \mathbf{GL}_n/G; p)$) where $G \to \mathbf{GL}_n$ is any faithful representation of G. It turns out that the essential dimension of G is the maximum of the essential dimensions of all G-torsors $E \to B$ (and similarly for the essential dimension at p).

2. **Work of Farb, Kisin and Wolfson.** Essential dimension was invented by J. Buhler and Z. Reichstein in the paper [4]. Since then, there has been a large amount of work done on computing the essential dimension of groups. Moreover, the notion of essential dimension was very quickly generalized to algebraic groups [8] (and later to stacks in [3]). However, there has been less work on computing the essential dimension of morphisms in general. One notable recent exception can be found in a preprint by B. Farb, M. Kisin and J. Wolfson [5].

Theorem 1 (Farb-Kisin-Wolfson)**.** *Let g and N be integers greater than 2, and let p be a prime not dividing N. Let $A_g[N]$ denote the moduli space of principally polarized abelian varieties of dimension g equipped with a full level N structure. Then the congruence cover $A_g[pN] \to A_g[N]$ (obtained by the forgetful functor) is p-incompressible.*

The above theorem was proved by arithmetic means involving the reduction modulo p of the scheme of principally polarized abelian varieties with full level N structure. Owing to the flexibility of their method, Farb, Kisin and Wolfson were able to prove several generalizations involving varieties closely related to $A_g[N]$. For example, in [5] they prove generalizations where $A_g[N]$ is replaced with the moduli space $M_g[N]$ of genus g curves with full level N structure, and they also prove generalizations where A_g is replaced with certain (connected) Shimura varieties of Hodge type. (These are, roughly speaking, the Shimura varieties which can be embedded in A_g.)

Note that the morphism $A_g[pN] \to A_g[N]$ can be viewed as an $\mathbf{Sp}_{2g}(\mathbb{Z}/p)$-torsor. So Theorem 1 proves that $\mathrm{ed}(\mathbf{Sp}_{2g}(\mathbb{Z}/p); p) \geq \dim A_g = \binom{g+1}{2}$. On the other hand, using the theorem of Karpenko and Merkurjev on essential dimension of p-groups [7], D. Benson has shown [2, Theorem 79] that, in fact, $\mathrm{ed}(\mathbf{Sp}_{2g}(\mathbb{Z}/p); p) = p^{g-1}$ when p is odd.

3. **Summary of work with Fakhruddin.** The work described here was motivated by a suggestion of Zinovy Reichstein to recover Theorem 1 using the fixed-point method, a geometric method for proving lower bounds on essential dimension. Using it we are able to recover not only Theorem 1 but also its generalization to M_g. Moreover, our methods apply to some Shimura varieties, which are not of Hodge type. For example, we can prove p-incompressibility for certain congruence covers of (connected) Shimura varieties of type E_7 [2, Corollary 76]. We also prove the p-incompressibility of certain "quantum" congruence covers of the moduli space of curves, and we generalize Theorem 1 to fields k of finite characteristic ℓ as long as $\ell \nmid pN$. On the other hand, while the arithmetic methods of Theorem 1 apply to many compact Shimura varieties, for reasons which will hopefully become obvious below, our fixed point methods do not prove nontrivial lower bounds on essential dimension for étale morphisms $f : X \to Y$ between proper varieties.

4. **Fixed point theorems.** Our results are based on the following fixed point theorem of Gille and Reichstein [6].

Theorem 2. *Suppose G is a finite group and $f : X \to Y$ is a G-torsor with X and Y quasi-projective varieties over $\mathbb{C}$. Let p be a prime and suppose $H \leq G$ is a subgroup isomorphic to $(\mathbb{Z}/p)^r$ for some integer r. Suppose further that X admits a G-equivariant partial compactification $\bar{X}$ on which H has a smooth fixed point. Then* $\mathrm{ed}(f; p) \geq r$.

This reduces the problem of proving lower bounds on essential dimension to the problem of finding fixed points. For this we use the following new result, which is [2, Proposition 16].

Theorem 3. *Let $\bar{S}$ be a toroidal singularity, that is, $\bar{S}$ is the completion of the local ring of a toric variety at a toric fixed point. Let S denote the complement of the boundary divisors in $\bar{S}$, and suppose that*

$$\begin{array}{ccccc} S & \xrightarrow{g} & Y & \xleftarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ \bar{S} & \xrightarrow{\bar{g}} & \bar{Y} & \xleftarrow{\bar{f}} & \bar{X} \end{array}$$

is a commutative diagram with $\bar{X}$ and $\bar{Y}$ quasi-projective complex varieties. Assume that all vertical arrows are open immersions and assume that f and $\bar{f}$ are proper. Suppose further that the left and right squares are both pullback diagrams *and that the entire diagram is equivariant for the action of a finite group G which acts trivially on all schemes except (possibly) on X and $\bar{X}$, and with $f : X \to Y$ a G-torsor.*

Write H for the conjugacy class of the image of $\pi_1(S)$ in G arising from the Galois correspondence, and suppose that H is a abelian. Then H has a fixed point on $\bar{X}$.

The proof of Theorem 3 is in some ways similar to the proof of the fixed point theorem itself, especially to the proof given by Kollár and Szabó in [8]. If we pull back the right square to the left, we get an H-torsor X_S over S and an extension of the map $X_S \to S$ to $\bar{S}$. Roughly speaking, the point is to show that H has a fixed point on $\bar{X}_{\bar{S}}$. The main tool which allows us to prove this is Abhyankar's lemma.

In the applications of Theorem 3 to the Shimura variety case, we can take $X \to Y$ to be a congruence cover, $\bar{Y}$ to be the Baily-Borel compactification of Y and $\bar{X}$ to be any smooth G-equivariant compactification of X, where G is the group of the congruence cover. We then take $\bar{S}$ to be the formal neighborhood of a point in one of the toroidal compactifications of Y constructed in [1]. The image H of $\pi_1(S)$ in G depends on the neighborhood chosen, but, in the Shimura variety case, the neighborhoods have a group theoretic interpretation as does the Baily-Borel compactification $\bar{Y}$. Applying these ideas give us the following result, which follows directly from [2, Lemma 47].

Theorem 4. *Suppose $Y = D/\Gamma$ is the quotient of a tube domain by a neat arithmetic subgroup, and assume that D has a zero dimensional rational boundary component. Then, for every prime p, there is a finite index normal subgroup Γ' of Γ such that the congruence cover $D/\Gamma' \to D/\Gamma$ is p-incompressible.*

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On a "Wonderful" Bruhat-Tits group scheme

Vikraman Balaji
(joint work with Y. Pandey)

Let G be an almost simple, simply-connected group over an algebraically closed field k and let $G_{ad} := G/Z(G)$. We construct certain *universal group schemes* on

- the De Concini-Procesi wonderful compactification [7] $\mathbf{X}$ of G_{ad},
- the Mumford embeddings $\bar{G}_{ad,A}$ of the relative group scheme $G_{ad,A}$ [12],
- the loop "wonderful embedding" $\mathbf{X}^{aff}$ of the adjoint affine Kac-Moody group G_{ad}^{aff}, constructed by Solis [15].

We term these group schemes "wonderful" Bruhat-Tits group schemes (see `http://arxiv.org/abs/2101.09212`).

Group embeddings and buildings. Let us recall that Tits' buildings are basically of two types. The first one is the "absolute" Tits building or spherical building which is attached to a semi-simple group over a general field. This simplicial complex is built from simplices which correspond to parabolic subgroups. The apartments of the building correspond to parabolic subgroups containing a fixed maximal torus. This is built up out of Euclidean spaces decomposed by the usual Weyl chambers. The second one is the Bruhat and Tits building which is the "relative" building attached to a semi-simple group over a complete-valued field. This is based on its *parahoric subgroups* and built up out of Euclidean spaces decomposed into affine Weyl chambers.

The two types of Tits' buildings can also be seen from an algebro-geometric perspective. In the absolute case, we work with a semisimple group G_{ad} of adjoint type. In this setting one has the wonderful embedding

$$G_{ad} \subset \mathbf{X} \tag{1}$$

where G_{ad} sits as an open dense subset of $\mathbf{X}$. The complement $\mathbf{X} \setminus G_{ad}$ is stratified by subsets Y and there is a bijection $Y \mapsto P_Y$ from these strata to parabolic subgroups $P_Y \subset G$. Furthermore, this bijection extends to an isomorphism between the Tits building and the canonical complex associated with the toroidal embedding (see Mumford [12, Page 178]).

A second perspective is when the ground field is endowed with a complete non-archimedean discrete valuation. Let $A = k[[z]]$ be a complete discrete valuation ring, $K = k((z))$ its quotient field. In this setting our basic model is the one constructed by Mumford [12]. He constructs a toroidal embedding $G_{ad,A} \subset \bar{G}_{ad,A}$ of the split group scheme $G_{ad,A} = G_{ad} \times \operatorname{Spec} A$. The strata of $\bar{G}_{ad,A} \setminus G_{ad,A}$ correspond bijectively to parahoric subgroups of $G(K)$ in a way that naturally extends to an isomorphism of the graph of the embedding $G_{ad,A} \subset \bar{G}_{ad,A}$ with the Bruhat-Tits building of $G \times \operatorname{Spec} A$ over A.

Statement of main results. Classical Bruhat-Tits theory associates, to each facet σ of the Bruhat-Tits building, a smooth group scheme $\mathcal{G}_\sigma$ on Spec A with connected fibres whose generic fibre is $G \times_{\operatorname{Spec} k} \operatorname{Spec} K$. We call $\mathcal{G}_\sigma$ a Bruhat-Tits group scheme on Spec A. The A-valued points $\mathcal{G}_{\sigma}(A) \subset G(K)$ are precisely the parahoric subgroups of $G(K)$. In this paper we construct universal analogues of the Bruhat-Tits group scheme.

A new point of view which plays a central role in this paper is that the notion of a *parabolic vector bundle* on *logarithmic schemes* can be used as a tool to make geometric constructions. Parabolic bundles have been encountered hitherto in the literature as objects in certain moduli spaces of bundles.

The case of Tits building. In the first setting, namely in the case of the Tits building, we construct an affine group scheme $\mathcal{G}_{\mathbf{X}}$ over $\mathbf{X}$ whose restriction along each curve transversal to a strata of $\mathbf{X} \setminus G_{ad}$ corresponds to the Bruhat-Tits group scheme associated to the parabolic subgroup under the bijection $Y \mapsto P_{Y}$ mentioned above.

To state our theorem, let us introduce some relevant notations and notions. Let $\mathbf{X} := \overline{G_{ad}}$ be the wonderful compactification of G_{ad}. We firstly construct a "parahoric" Lie-algebra bundle R on X.

We fix data (T, B, G) of G. Let S denote the set of simple roots of G and $\mathbb{S} = S \cup \{\alpha_0\}$ denote the set of affine simple roots. For $I \subset \mathbb{S}$ let $\mathcal{G}_I$ denote the associated Bruhat-Tits group scheme on a dvr and for $I' \subset S$ let $\mathcal{G}^{st}_{I'} = \mathcal{G}_I$ where $I = I' \cup \{\alpha_0\}$. For $I' \subset S$, let $Z_{I'}$ denote the corresponding strata of $\mathbf{X}$.

Theorem 1. *There exists an affine "wonderful" Bruhat-Tits group scheme $\mathcal{G}^{\varpi}_{\mathbf{X}}$ on $\mathbf{X}$ together with a canonical isomorphism Lie$(\mathcal{G}^{\varpi}_{\mathbf{X}}) \simeq \mathcal{R}$. It further satisfies the following classifying property:*

For any proper $I' \subset S$ and any point $z_{I'} \in Z_{I'}$, let $C_{I'} \subset \mathbf{X}$ be a smooth curve with generic point in G_{ad} and closed point $z_{I'}$ and let $U_z \subset C_{I'}$ be a formal neighbourhood of the closed point $z_{I'}$. Then, the restriction $\mathcal{G}^{\varpi}_{\mathbf{X}}|_{U_z}$ is isomorphic to the standard Bruhat-Tits group scheme $\mathcal{G}^{st}_{I'}$ on Spec (A).

The relative case. In the second setting we construct an affine group scheme over $\bar{G}_{ad,A}$ whose restriction along each curve transversal to a strata satisfies properties similar to $\mathcal{G}_{\mathbf{X}}$. We work in the setting of loop groups and construct an affine group scheme over a "wonderful" embedding $\mathbf{X}^{aff}$ constructed by Solis [15]. In the relative case, we construct a a Lie-algebra bundle $\mathbf{R}$ on $\mathbf{X}^{aff}$ and the group scheme is obtained by integrating this bundle. Its construction is achieved by constructing a Lie-algebra bundle J on a finite dimensional scheme $\mathbf{Y}$ which is the closure of a torus-embedding and whose translates build up the ind-scheme $\mathbf{X}^{aff}$.

The ind-scheme $\mathbf{X}^{aff}$ has divisors D_α for $\alpha \in \mathbb{S}$ such that the complement of their union is $\mathbf{X}^{aff} \setminus G^{aff}_{ad}$. Our second theorem is the following:

Theorem 2. *There exists an affine "wonderful" Bruhat-Tits group scheme $\mathcal{G}^{\varpi}_{\mathbf{X}^{aff}}$ on $\mathbf{X}^{aff}$ together with a canonical isomorphism Lie$(\mathcal{G}^{\varpi}_{\mathbf{X}^{aff}}) \simeq \mathbf{R}$. It further satisfies the following classifying property:*

For any non-empty subset $I \subset \mathbb{S}$ and any point $h \in \cap_{\alpha \in I} D_\alpha$, let $C_I \subset \mathbf{X}^{aff}$ be a smooth curve with generic point in G^{aff}_{ad} and closed point h. Let $U_h \subset C_I$

be a formal neighbourhood of the closed point h. Then, the restriction $\mathcal{G}^{\varpi}_{\mathbf{X}^{aff}}|_{U_h}$ *is isomorphic to the Bruhat-Tits group scheme* $\mathcal{G}_I$ *on* $\mathrm{Spec}\,(A)$.

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Complexity of actions over perfect fields

Friedrich Knop
(joint work with Vladimir S. Zhgoon)

Let G be a connected reductive group defined over a ground field K. Let $P \subseteq G$ be a minimal parabolic K-subgroup.

Definition 1. *A G-variety X is* K-spherical *if it is normal and there is a K-rational point $x_0 \in X(K)$ such that the orbit Px_0 is open in X.*

Consider first the case that K is algebraically closed. Then $P = B$ be a Borel subgroup of G and K-spherical is spherical in the usual sense. It is a basic theorem of Brion [3] and, independently, Vinberg [8] that the number of B-orbits in a spherical variety is finite. The method of both proofs is the same namely a deformation argument to horospherical varieties.

Later Matsuki [7] gave a simpler proof by using reduction to $\operatorname{rk} G = 1$. In the same paper he discussed the possibility of generalizations to non-closed ground fields K. He observed in particular that naïvely replacing the Borel subgroup B by a minimal parabolic P does not work. On the other hand, he conjectured in the case $K = \mathbb{R}$ that for an $\mathbb{R}$-spherical X, the number of $G(\mathbb{R})$-orbits in $X(\mathbb{R})$ is finite. This conjecture was proved by Bien [1] and, independently by Krötz-Schlichtkrull [6].

In the talk we presented the following generalization of the Brion-Vinberg theorem.

Theorem 2. *Let K be a perfect field, let G be a connected reductive K-group and let $P \subseteq G$ be a minimal parabolic K-subgroup. Let X be a K-spherical G-variety. Then the number of P-orbits $Px \subseteq X$ with $x \in X(K)$ is finite. Equivalently, the image of*

$$X(K)/P(K) \to X/P$$

is finite.

By a theorem of Borel-Serre [2] we get

Corollary 3. *Let K be a local field of characteristic zero and X a K-spherical G-variety. Then $X(K)/P(K)$ is finite.*

In the proof, we first reduce (using [5]) to the case that X is homogeneous. Then we use the idea of Matsuki, i.e., it suffices to consider only groups with $\operatorname{rk}_K G = 1$. But even then, a classification of all K-spherical X does not seem feasible. Instead, we have to resort to some general arguments when $X = G/H$ with H semisimple. Here, a crucial ingredient is Kempf's instability theorem, whence the restriction to perfect fields.

As a matter of fact, Vinberg proved a more general theorem which is valid for all normal G-varieties X. For this, define for a P-stable subvariety $Y \subseteq X$ its *complexity* as

$$c(Y/P) := \operatorname{trdeg}_K K(Y)^P.$$

We say that Y is K*-dense* if $Y(K)$ is Zariski-dense in Y. Observe that X is K-spherical iff X is K-dense with $c(X/P) = 0$. Hence, the first theorem follows easily from:

Theorem 4. *Let K be a perfect field, let G be a connected reductive K-group and let $P \subseteq G$ be a minimal parabolic K-subgroup. Let X be a normal K-dense G-variety. Then*

$$c(Y/P) \leq c(X/P)$$

for all K-dense P-stable subvarieties $Y \subseteq X$.

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Rigidity and Unirational Groups

Zev Rosengarten

A basic result in the theory of abelian varieties is the rigidity lemma, whose statement we recall.

Lemma 1. [4, §4, Rigidity Lemma (Form I)] *Let X and Y be geometrically integral schemes of finite type over a field k, and let Z be a separated k-scheme. Let $f\colon X \times Y \to Z$ be a k-morphism, and assume that X is proper and that, for some $y \in Y$, the restriction $f_y\colon X_{k(y)} \to Z_{k(y)}$ is constant. Then f depends only on the Y coordinate. That is, there is a map $g\colon Y \to Z$ such that f is the composition of the projection $X \times Y \to Y$ with g.*

This is applied in particular when $X = Y$ and Z are abelian varieties to show that any k-morphism of abelian varieties as *schemes* which preserves identities is a k-group homomorphism. The proof of the rigidity lemma above depends crucially upon the properness of X. Indeed, it is easy to construct examples of morphisms of affine group schemes that are not k-group homomorphisms. For example, the endomorphism of $\mathbf{G}_a$ given by the polynomial $f(X) = X^6$ is a k-scheme endomorphism preserving identities that is not a homomorphism. Nevertheless, the purpose of the talk discussed in this abstract is to observe that for maps from open subschemes of $\mathbf{P}^1$ into wound unipotent groups (and, slightly more generally, into solvable groups not containing a copy of $\mathbf{G}_a$, though the real meat lies in the wound unipotent case), there is (or should be) a certain rigidity principle that has several important applications.

We begin with the simplest interesting case, which was the subject of the talk – namely, fields of degree of imperfection 1. The main result in that context is the following.

Theorem 2. *Let k be a field of degree of imperfection 1, and let G and H be finite type k-group schemes with G unirational and H not containing a k-subgroup scheme k-isomorphic to $\mathbf{G}_a$. Then any k-scheme morphism $f\colon G \to H$ such that $f(1_G) = 1_H$ is a homomorphism.*

Among the corollaries of this result, we obtain that, over fields of degree of imperfection 1, every unirational wound unipotent k-group scheme is commutative. Indeed, this follows by applying the above result to the inversion morphism of such a group. Using this commutativity, we may also show that, again in the degree of imperfection 1 context, unirationality of group schemes descends through separable extensions. More precisely, we have the following result.

Theorem 3. *Let k be a field of degree of imperfection 1, and let K/k be a (not necessarily algebraic) separable field extension. Then, for a finite type k-group scheme G, one has that G is unirational over k if and only if G_K is over K.*

We remark that the analogous statement for *commutative* group schemes over arbitrary fields is a theorem of Achet [1, Thm. 2.3].

The above results are nice, and already sufficient for certain applications to arithmetic (global function fields have degree of imperfection 1), but we would like to understand the situation over fields of higher degree of imperfection as well. Unfortunately, all of the above results fail over *every* field of degree of imperfection > 1. Nevertheless, the degree of imperfection 1 case, as well as the counterexamples given in the talk, point the way to suitable positive results over general fields. One considers maps from open subschemes of $\mathbf{P}^1$, and the crucial observation/conjcture is then the following.

Conjecture 4. *Let U be a wound unipotent group over the field k, let $x \in \mathbf{P}^1_k$ be a closed point, and let $X := \mathbf{P}^1 \backslash x$. Finally, let $y_1, y_2 \in X(k)$. Then the only k-morphism $f\colon X \times X \to U$ such that $f|\{y_1\} \times X = f|X \times \{y_2\} = 1$ is $f = 1$.*

I have phrased the above statement as a conjecture, but in fact I have a proof for $p \neq 2$ (where $p := \mathrm{char}(k)$). This rigidity result, in conjunction with additional arguments, already has important implications. One, which was the motivation for these rigidity results in the first place, is that Theorem 3 holds over every field k, not just those of degree of imperfection 1. Another application is a fundamental fact about the structure of unirational wound unipotent groups – namely, that they are generated by their unirational commutative k-subgroups, and in particular by their smooth connected commutative k-subgroups. This is false in general without the assumption of unirationality: Over every imperfect field k, Gabber has produced examples of non-commutative two-dimensional wound unipotent k-groups [3, Example 2.10]. One may show that any smooth, connected, non-commutative, two-dimensional unipotent k-group U admits as its only smooth connected k-subgroups $1, \mathscr{D}U$, and U, so in particular it is not generated by its smooth connected commutative k-subgroups.

But there is in fact an even stronger rigidity result than Conjecture 4 that should hold. Stating it, however, requires introducing certain additional definitions. Given a finite extension L/K of fields of characteristic p, we define the *degree of primitivity* of the extension to be the quantity $[L : KL^p]$, where KL^p denotes the compositum inside L of the fields K and L^p. This quantity roughly measures the minimum number of elements required to generate L from K. In fact, this is literally true if L/K is purely inseparable, in the sense that the minimum number of generators is r, where $[L : KL^p] = p^r$ [2, Thm. 6]. It turns out that one may naturally extend this definition from the case of a finite extension of fields to any finite reduced algebra over a field. The idea is that, for a finite reduced K-algebra $A = \prod_{i=1}^n L_i$, where each L_i/K is a finite field extension, one may choose K-embeddings $j_i \colon L_i \hookrightarrow F$ of the L_i into some fixed extension field F/K. Then one defines the degree of imprimitivity of A over K to be that of the compositum $j_1(L_1)j_2(L_2)\dots j_n(L_n)$ inside F. Of course, one must show that this is independent of the choice of embeddings. Having done this, one may then define, for any divisor $D \subset \mathbf{P}^1_k$, the degree of primitivity of D to be that of the finite reduced k-algebra $\mathcal{O}(D)_{\mathrm{red}}$. Then we hope that the following rigidity result, which generalizes Conjecture 4 (because closed points have degree of primitivity 1), holds.

Conjecture 5. *Let U be a wound unipotent group over the field k, and let $X_1, \dots, X_n \subset \mathbf{P}^1_k$ be nonempty open subschemes and choose $x_i \in X_i(k)$. For each $i = 1, \dots, n$, let $D_i := \mathbf{P}^1 \backslash X_i$ be the complementary divisor, and let $D := \cup_{i=1}^n D_i$. Suppose that $n > r$, the degree of primitivity of D. Then the only k-scheme morphism $f \colon \prod_{i=1}^n X_i \to U$ such that $f|X_1 \times \dots \{x_i\} \times \dots X_n = 1$ for all i is the constant map $f = 1$.*

One may show that any finite reduced algebra over a field of degree of imperfection r has degree of primitivity $\leq r$. Using this fact and the above conjecture, one could then show that, for any unirational wound unipotent group over a field of degree of imperfection r, the rth descending central subgroup is trivial. This generalizes the statement that every unirational wound unipotent group over a field of degree of imperfection 1 is commutative.

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Local-Global Principles for Homogeneous spaces and Berkovich Analytic Curves

Vlerë Mehmeti

Let k be a complete ultrametric field. Let C/k be a normal irreducible projective algebraic curve with function field F. We study local-global principles for certain homogeneous spaces defined over F by using an analytic point of view. We work in the setting of Berkovich's theory, which is one of the several possible approaches to non-archimedean analytic geometry. The results we obtain generalize those of the literature and provide geometric insight into the strategy. We will use the notations introduced in this paragraph throughout the text.

1. Berkovich analytic curves

Analytification. There are many positive aspects to working with Berkovich spaces: geometric intuition, a good topology, an analogy with complex analytic spaces, etc. There exists an analytification functor, associating to a f.t. scheme X/k a k-analytic space X^{an}, and with respect to which we have GAGA-type theorems. For instance, as in the complex case, a proper k-analytic curve is algebraic.

Meromorphic functions ($\mathscr{M}$). We can construct over these spaces the sheaf of meromorphic functions $\mathscr{M}$ in a natural way. This object plays a very important role in the study of the local–global principle through analytic curves, in part due to the fact that $F = \mathscr{M}(C^{\mathrm{an}})$.

The sheaf $\mathscr{M}$ does not behave locally as in the complex case, but its stalks still satisfy some nice properties: 1) $\mathscr{M}_x$ is naturally endowed with a valuation; we denote by $\widehat{\mathscr{M}_x}$ its completion; 2) $\mathscr{M}_x$ is Henselian with respect to this valuation.

Valuations. To conclude, we mention a fact that is quite useful when studying more "classical" local–global principles: the points of C^{an} are valuations of F.

Proposition 1. *Let M_F denote the set of non-trivial rank 1 valuations v over F such that $v_{|k}$ is either trivial or induces the norm on k. Then, there is a bijection $C^{\mathrm{an}} \leftrightarrow M_F$ such that if $x \mapsto v_x$, $\widehat{\mathscr{M}_x} = F_{v_x}$, where F_{v_x} is the completion of F with respect to v_x.*

2. History and a local–global principle

We call local–global principle the following kind of statement:

Statement 2. *Let K be a field and $(K_i)_i$ a family of* overfields, *meaning $K \subsetneq K_i$ for all i. Let V/K be a variety. Then, $V(K) \neq \emptyset \iff V(K_i) \neq \emptyset\ \forall i$.*

In the last few decades, a lot of focus has been put on studying geometric variants of this statement, especially when $K = F$. In [1], Harbater, Hartmann and Krashen proved such a local–global principle by constructing the overfields over a model of the curve C. To do so, HHK used *patching techniques*, whose application to the study of the local–global principle was a novelty.

I adapted the patching of HHK, which is quite algebraic, to the setting of Berkovich analytic curves (see [3]). It yielded the following:

Theorem 3. *If V is a projective homogeneous variety or a torsor over a rational linear algebraic group G/F, then $V(F) \neq \emptyset \iff V(\mathscr{M}_x) \neq \emptyset \ \forall x \in C^{\text{an}}$.*

The above result generalizes those of [1]. If char $k \neq 2$, then Theorem 3 can be applied to quadratic forms. Thanks to the Henselianity of the fields $\mathscr{M}_x$, as a consequence we obtain a generalization of the following result, originally shown by Parimala and Suresh in [6] and then by HHK in [1].

Corollary 4 (Parimala–Suresh '09, HHK '09, M. '19)**.** *Any quadratic form over $\mathbb{Q}_p(T), p \neq 2$, of dimension $\geqslant 9$, has a non-trivial zero over $\mathbb{Q}_p(T)$.*

3. Other local-global principles

3.1. **All valuations.** As the $\mathscr{M}_x$ are Henselian, they satisfy a classical approximation property. Then, by applying Proposition 1 and Theorem 3, we obtain:

Theorem 5. *If V/F is a projective homogeneous space or a torsor over a rational linear algebraic group G/F, then $V(F) \neq \emptyset \iff V(F_v) \neq \emptyset \ \forall v \in M_F$.*

The above statement also applies to quadratic forms provided char $k \neq 2$.

3.2. **Discrete valuations.** Classically, we are interested in local–global principles where the overfields are *discrete* completions of F.

Conjecture 6 (Colliot-Thélène, Parimala, Suresh)**.** *Suppose k is discretely valued. If V/F is a projective homogeneous space over a connected linear algebraic group G/F, then $V(F) \neq \emptyset \iff V(F_v) \neq \emptyset$ for all discrete valuations v on F.*

The conjecture above was shown to be true for quadratic forms by the same authors (see [2]). Other advances include [7].

Let $S \subset C^{\text{an}}$ denote the set of points corresponding to the discrete valuations of F. We remark that S is dense in C^{an}. Then, thanks to Proposition 1, using the same notation, Conjecture 6 can be paraphrased as follows:

Conjecture 7. $V(F) \neq \emptyset \iff V(\mathscr{M}_x) \neq \emptyset \ \forall x \in S$.

In recent work in progress ([5]), by using Theorem 3, I have showed the following:

Theorem 8. *Conjecture 7 is true for certain varieties V that satisfy a "strong smoothness" condition.*

Intuitively speaking, this "strong smoothness" condition aims to prevent a uniformizer of k from being an obstruction to the smoothness of V/F. As an application, we find another proof of Conjecture 6 in the case of quadratic forms.

3.3. **Higher dimension.** By using the geometric nature of patching over Berkovich spaces, we can devise a strategy to generalize the results above from curves to higher dimensional analytic spaces. This strategy consists of starting from the study of neighbourhoods of fibers in the case of relative analytic curves. In [4], I show that this is true for certain fibers, thus generalizing the results of [3].

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Character Sheaves for graded Lie algebras

Kari Vilonen

(joint work with Ting Xue)

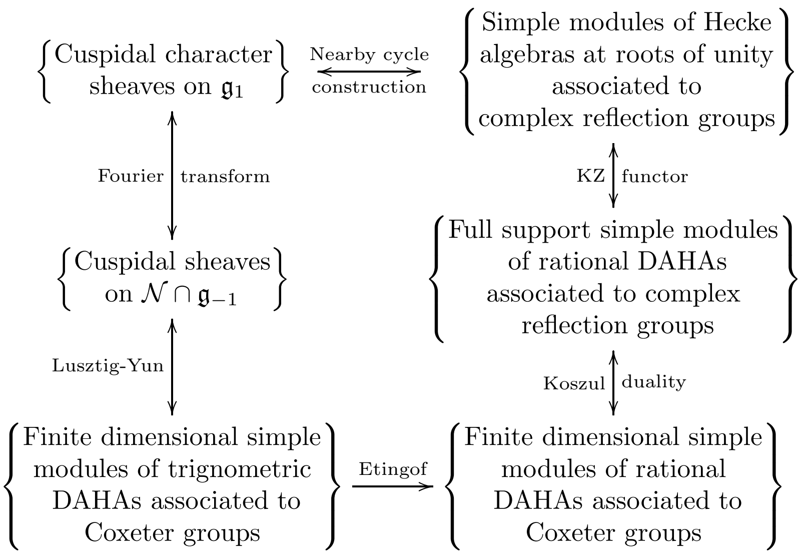
We consider character sheaves for graded Lie algebras. To that end let G be a complex reductive group with a finite order automorphism θ. The automorphism θ induces a grading on $\mathfrak{g}$. We write $\mathfrak{g}_1$ for the first graded piece and set $K = G^\theta$. The character sheaves are irreducible K-equivariant perverse sheaves on $\mathfrak{g}_1$ whose singular support is nilpotent. The goal of our program is to classify all graded characters sheaves or at least find all the cuspidals. This program is a significant extension of Lusztig's generalized Springer correspondence.

One way to construct character sheaves is to proceed analogously to the original Springer correspondence making use of functoriality of the Fourier transform. This results in a family $\check{\pi} : \mathfrak{X} \to \mathfrak{g}_1$ whose generic fibers are smooth Hessenberg varieties. By construction any direct summand in the push-forward $R\check{\pi}_*\mathbb{C}$ is a character sheaf. This turns out to be a rather difficult way to construct character sheaves, but it can be turned around to conversely compute cohomology of Hessenberg varieties. These ideas are explored in papers [1, 2, 3] in the special case of a symmetric pair $(SL(n), SO(n))$.

In his thesis Misha Grinberg replaced the Springer resolution with a nearby cycle construction in the context of polar representations. He then showed that the Fourier transform of the nearby cycle sheaf is an intersection cohomology sheaf of a local system on the generic locus of $\mathfrak{g}_1$. It is a priori a representation of the braid group, but one can show that it factors through a Hecke algebra with unequal parameters. These ideas are worked out in the joint papers [5, 6] where we show how to reduce the calculation of the Hecke algebra to rank one. This work forms a basis of the papers [4, 7, 8]. In [4] we work out the classification of character sheaves for the symmetric pair $(SL(n), SO(n))$ and in [7] we classify character sheaves for arbitrary classical symmetric pairs.

In [8] we, conjecturally, produce all cuspidal character sheaves in the case of arbitrary stable gradings. This work relies on [6]. The cuspidal character sheaves for unstable gradings are more difficult to analyze. A new ingredient is to introduce a third way of constructing cuspidal character sheaves by explicitly writing down a D-module with nilpotent characteristic variety in analogy with the Harish-Chandra system in the non-graded case. From this point of view the Hecke relations are given by b-functions.

Finally, to show that we have constructed sufficiently many cuspidal character sheaves we propose to fit our construction in the following conjectural diagram containing other interesting nodes and arrows where all the arrows are supposed to be bijections.



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Perverse sheaves on affine flag varieties and geometry of the Langlands dual group

Simon Riche
(joint work with R. Bezrukavnikov and L. Rider)

This abstract reports on a joint project with Roman Bezrukavnikov (and initially Laura Rider) aiming at constructing a version for positive characteristic coefficients of Bezrukavnikov's equivalence relating the two geometric incarnations of the affine Hecke algebra attached to a reductive group [Be2].

1. The affine Hecke algebra and its geometric incarnations

Let us start by recalling what the affine Hecke algebra is, and where it comes from. Let $\mathcal{F}$ be a nonarchimedian local field, $\mathcal{O}$ its ring of integers, and k the residue field. Let also G be a split reductive group scheme over $\mathcal{O}$ with connected center, and $B \subset G$ a Borel subgroup. Then we can consider the groups

$$G(\mathcal{F}) \supset G(\mathcal{O}) \twoheadrightarrow G(k),$$

and also the Iwahori subgroup $I \subset G(\mathcal{O})$ consisting of the elements whose image in $G(k)$ belongs to $B(k)$ (a compact open subgroup in $G(\mathcal{F})$). To these data one can attach an *affine Hecke algebra* $\mathscr{C}_c(I\backslash G(\mathcal{F})/I)$, consisting of compactly supported locally constant functions from $G(\mathcal{F})$ to $\mathbb{C}$ which are I-biinvariant (for an appropriate convolution product). A classical result of Borel [Bo] provides an equivalence of categories between the category of admissible representations of $G(\mathcal{F})$ (on complex vector spaces) spanned by their I-fixed vectors and the category of finite-dimensional modules for $\mathscr{C}_c(I\backslash G(\mathcal{F})/I)$.

To proceed further one needs to understand the algebra $\mathscr{C}_c(I\backslash G(\mathcal{F})/I)$ better. This is exactly what is provided by results of Iwahori–Matsumoto [IM], with later contributions of Bernstein and Lusztig. To explain this one chooses a maximal torus $T \subset B$, and denotes by $\mathbf{X}$ the associated cocharacter lattice and by W the Weyl group of (G,T). The choice of B determines a subset $S \subset W$ of Coxeter generators, and for $s \neq t$ in S we will denote by $m_{s,t}$ the order of st in W. One then considers the algebra $\mathscr{H}$ over $\mathbb{Z}[v^{\pm 1}]$ generated by $\{T_s : s \in S\} \cup \{\theta_\lambda : \lambda \in \mathbf{X}\}$, and with relations given by

- $(T_s + 1)(T_s - v^{-2}) = 0$ for $s \in S$;
- $T_s T_t \cdots = T_t T_s \cdots$ (with $m_{s,t}$ terms) for $s \neq t \in S$;

- $\theta_\lambda \theta_\mu = \theta_{\lambda+\mu}$ for $\lambda, \mu \in \mathbf{X}$, $\theta_0 = 1$;
- $T_s \theta_\lambda - \theta_{s(\lambda)} T_s = (1 - v^{-2}) \frac{\theta_{s(\lambda)} - \theta_\lambda}{1 - \theta_{-\alpha_s}}$ for $s \in S$ and $\lambda \in \mathbf{X}$.

It is known that this algebra is free over $\mathbb{Z}[v^{\pm 1}]$, and that we have isomorphisms

$$\mathscr{H} \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbb{Z}_{v=1} \cong \mathbb{Z}[W \ltimes \mathbf{X}], \quad \mathscr{H} \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbb{C}_{v=\frac{1}{\sqrt{q}}} \cong \mathscr{C}_c(I \backslash G(\mathcal{F})/I).$$

The algebra $\mathscr{H}$ has another geometric incarnation thanks to work of Kazhdan–Lusztig [KL] and Ginzburg. For this one considers an algebraically closed field $\Bbbk$, the Langlands dual group $G^\vee_\Bbbk$ over $\Bbbk$ and a Borel subgroup $B^\vee_\Bbbk$, denotes by $U^\vee_\Bbbk$ the unipotent radical of $B^\vee_\Bbbk$, and considers the Springer variety

$$\widetilde{\mathcal{N}} = \{(x, gB^\vee_\Bbbk) \in \mathrm{Lie}(G^\vee_\Bbbk) \times G^\vee_\Bbbk / B^\vee_\Bbbk \mid x \in g \cdot \mathrm{Lie}(U^\vee_\Bbbk)\}$$

(a vector bundle over the flag variety $G^\vee_\Bbbk / B^\vee_\Bbbk$) with the natural action of $G^\vee_\Bbbk \times \Bbbk^\times$. The *Steinberg variety* is the fiber product $\mathrm{St} := \widetilde{\mathcal{N}} \times_{\mathrm{Lie}(G^\vee_\Bbbk)} \widetilde{\mathcal{N}}$, and the results of Kazhdan–Lusztig and Ginzburg provide an algebra isomorphism

$$\mathsf{K}^{G^\vee_\Bbbk \times \Bbbk^\times}(\mathrm{St}) \cong \mathscr{H}$$

where the left-hand side is the equivariant K-theory of St. This isomorphism allows to classify the simple finite-dimensional $\mathscr{C}_c(I \backslash G(\mathcal{F})/I)$-modules, and therefore the simple admissible $G(\mathcal{F})$-modules with nonzero I-fixed vectors, and hence prove the so-called *Deligne–Langlands conjecture*.

2. Geometric Satake equivalence and Gaitsgory's central functor

For various reasons (coming from the Geometric Langlands Program but also from Representation Theory) it is desirable to obtain a "categorical upgrade" of the relation between $\mathsf{K}^{G^\vee_\Bbbk \times \Bbbk^\times}(\mathrm{St})$ and $\mathscr{C}_c(I \backslash G(\mathcal{F})/I)$. For this one fixes an algebraically closed field $\mathbb{F}$ of characteristic p, and assumes that $\Bbbk$ is an algebraic closure either of $\mathbb{Q}_\ell$ or of $\mathbb{F}_\ell$, for some prime $\ell \neq p$. One also replaces $\mathcal{F}$ by the field $\mathscr{K} := \mathbb{F}((z))$, and $\mathcal{O}$ by $\mathscr{O} := \mathbb{F}[[z]]$. One chooses a connected reductive group over $\mathbb{F}$ (still denoted by G), and associate to it the loop group $\mathcal{L}G$ and the arc group $\mathcal{L}^+G$. The affine Grassmannian Gr is the ind-projective ind-scheme of ind-finite type over $\mathbb{F}$ defined as the fppf quotient $\mathcal{L}G/\mathcal{L}^+G$; the celebrated *Geometric Satake Equivalence* provides an equivalence of monoidal categories between the category $\mathsf{Perv}_{\mathcal{L}^+G}(\mathrm{Gr}, \Bbbk)$ of $\mathcal{L}^+G$-equivariant perverse sheaves on Gr (for an appropriate convolution product) and the category $\mathsf{Rep}(G^\vee_\Bbbk)$ of representations of the Langlands dual group over $\Bbbk$. (This equivalence is due to Mirković–Vilonen [MV], after earlier contributions of Lusztig and Ginzburg.) This can be thought of as a categorical enhancement of the Satake isomorphism [Sa] describing the spherical affine Hecke algebra, i.e. the algebra defined as for $\mathscr{C}_c(I \backslash G(\mathcal{F})/I)$, but for I replaced by $G(\mathcal{O})$.

A result of Bernstein provides an isomorphism between the spherical affine Hecke algebra and the center of $\mathscr{C}_c(I \backslash G(\mathcal{F})/I)$. At the categorical level, this relation is provided by Gaitsgory's central functor

$$\mathsf{Z} : \mathsf{Perv}_{\mathcal{L}^+G}(\mathrm{Gr}, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{Iw}}(\mathrm{Fl}, \Bbbk),$$

where $\mathrm{Iw} \subset \mathcal{L}^+G$ is an Iwahori subgroup (determined by a choice of Borel subgroup in G) and Fl is the affine flag variety, i.e. the fppf quotient $\mathcal{L}G/\mathrm{Iw}$, see [Ga]. One can therefore expect an equivalence of categories relating appropriate variants of the equivariant derived category $D^{\mathrm{b}}_{\mathrm{Iw}}(\mathrm{Fl}, \Bbbk)$ and the derived category of equivariant coherent sheaves on St, and which sends the object $\mathsf{Z}(\mathscr{F})$ (with $\mathscr{F}$ in $\mathsf{Perv}_{\mathcal{L}^+G}(\mathrm{Gr}, \Bbbk)$) to $V \otimes \mathscr{O}_{\Delta\widetilde{\mathcal{N}}}$, where V is the $G^\vee_\Bbbk$-module corresponding to $\mathscr{F}$ under the geometric Satake equivalence. In the case when $\Bbbk$ is an algebraic closure of $\mathbb{Q}_\ell$ this is exactly what is provided by [Be2]; the case when $\Bbbk$ is an algebraic closure of $\mathbb{F}_\ell$ is the subject of the project presented here.

3. Regular quotients

As a first step towards this goal, in [BRR] we describe (under mild assumptions on ℓ) the category $\mathsf{P}^0_{\mathrm{Iw}}$ obtained as the Serre quotient of the category P_{Iw} of Iw-equivariant perverse sheaves on Fl by the subcategory generated by simple objects with positive-dimensional support. More precisely we show that this category is equivalent (as a monoidal category, for the monoidal structure naturally induced by the convolution product on $D^{\mathrm{b}}_{\mathrm{Iw}}(\mathrm{Fl}, \Bbbk)$) to the category of representations of the centralizer $Z_{G^\vee_\Bbbk}(u)$ of a regular unipotent element $u \in G^\vee_\Bbbk$ (or in other words of $G^\vee_\Bbbk$-equivariant coherent sheaves on the regular part of St), in such a way that the functor from $\mathsf{Perv}_{\mathcal{L}^+G}(\mathrm{Gr}, \Bbbk)$ induced by Z corresponds to the restriction functor $\mathsf{Rep}(G^\vee_\Bbbk) \to \mathsf{Rep}(Z_{G^\vee_\Bbbk}(u))$. One crucial ingredient in our approach is a general lemma on central functors due to Bezrukavnikov in [Be1].

In a paper in preparation [BR] we construct a version of this equivalence for a similar quotient of the category of perverse sheaves on the natural T-torsor $\widetilde{\mathrm{Fl}}$ over Fl generated by objects obtained by pullback from P_{Iw}. On the dual side one obtains in this case the category of representations of the pullback of the universal centralizer group scheme (of $G^\vee_\Bbbk$) to $T^\vee_\Bbbk \times_{T^\vee_\Bbbk/W} T^\vee_\Bbbk$ supported set-theoretically on the base point. In later work we will show that one can reconstruct appropriate derived categories of constructible sheaves on $\widetilde{\mathrm{Fl}}$ and coherent sheaves on (a variant of) St from the categories considered in [BR], and therefore obtain an equivalence of categories as alluded to in Section 2.

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R-equivalence on reductive group schemes

Anastasia Stavrova

(joint work with Philippe Gille)

The following definition of R-equivalence goes back to Yu. Manin [7].

Definition 1. Let X be an algebraic variety over a field k. Denote by $k[t]_{(t),(t-1)}$ the semilocal ring of the affine line $\mathbf{A}^1_k$ over k at the points 0 and 1. Two points $x_0, x_1 \in X(k)$ are called *directly R-equivalent*, if there is $x(t) \in X(k[x]_{(x),(x-1)})$ such that $x(0) = x_0$ and $x(1) = x_1$. The *R-equivalence relation* on $X(k)$ is the equivalence relation generated by direct R-equivalence. The *R-equivalence class group* $G(k)/R$ of an algebraic k-group G is the quotient of $G(k)$ by the R-equivalence class of the neutral element $1 \in G(k)$.

It is easy to see that the R-equivalence class of the neutral element $1 \in G(k)$ is a normal subgroup of $G(k)$, so $G(k)/R$ is indeed a group. If k is infinite and G is reductive, then $G(k)/R$ is a birational invariant of G [1]. If $G = T$ is a k-torus and

$$1 \to F \to P \to T \to 1$$

is a flasque resilution of T, then $T(k)/R \cong H^1(k, F)$ and T is a retract rational variety if and only if $T(K)/R = 1$ for any field extension K of k [2]. If G is a simply connected absolutely almost simple isotropic k-group, then $G(k)/R$ coincides with the Whitehead group of G, which is the subject of the Kneser–Tits problem [5].

We propose the following more general definition of R-equivalence that allows to extend to group schemes over rings the above-mentioned and other properties of R-equivalence of algebraic groups.

Definition 2. Let B be a unital commutative ring. We denote by Σ the multiplicative subset of polynomials $P \in B[T]$ satisfying $P(0), P(1) \in B^\times$. Let F be a B-functor in sets. We say that two points $x_0, x_1 \in F(B)$ are *directly R–equivalent* if there exists $x \in F\big(B[t]_\Sigma\big)$ such that $x_0 = x(0)$ and $x_1 = x(1)$. The *R-equivalence relation* on $F(B)$ is the equivalence relation generated by this elementary relation.

The following properties of the generalized R-equivalence are immediate.

- $X(B)/R$ is functorial in B;
- $(X \times_B Y)(B)/R \cong X(B)/R \times Y(B)/R$;
- $X(B[t])/R \cong X(B)/R$;
- If U is an open subscheme of a finitely generated B-vector bundle, and B is semilocal or U is principal, then $U(B)/R = *$;

- If G is a B-group scheme, then $G(B)/R$ is the group $G(B)/RG(B)$, where $RG(B)$ is the R-equivalence class of $1 \in G(B)$.

The following properties are specific for reductive B-group schemes. They are proved in [6].

Proposition 3. Assume that B be a regular domain, and let $1 \to S \to E \xrightarrow{\pi} T \to 1$ be a flasque resolution of a B-torus T. Then

$$T(B)/R \cong \ker\bigl(H^1(B,S) \to H^1(B,E)\bigr).$$

Proof. We have Since E is a quasi-trivial B-torus, we have $E(B)/R = 1$. Hence the inclusion $\pi(E(B)) \subseteq RT(B)$. For the converse, it is enough to show that a point $x \in T(B)$ which is directly R–equivalent to 1 belongs to $\pi(E(R))$. By definition, there exists a polynomial $P \in B[t]$ such that $P(0), P(1) \in B^\times$ and $x(t) \in T\bigl(B[t,1/P]\bigr)$ satisfying $x(0) = 1$ and $x(1) = x$. We consider $\delta(x(t)) \in H^1(B[t,1/P],S)$. Since S is flasque, the map

$$H^1(B,S) \to H^1(B[t,1/P],S)$$

is onto [2, cor. 2.6]. It follows that $\delta(x(t)) = \delta(x)(0) = \delta(x(0)) = 1$ so that $x(t)$ belongs to the image of $\pi : E\bigl(B[t,1/P]\bigr) \to T\bigl(B[t,1/P]\bigr)$. Then $x = x(1) \in \pi\bigl(E(R)\bigr)$. □

Definition 4. A reductive B-scheme G is called *strictly isotropic*, if G contains a non-trivial parabolic B-subgroup scheme that intersects properly every semisimple normal B-subgroup of G.

Theorem 5. Let B be a semilocal regular domain containing an infinite field, and let K be the fraction field of B. Let G be a reductive B-scheme that is either a torus or simply connected and strictly isotropic. Then G is retract rational over B if and only if G_K is retract rational over K if and only if $G(C)/R = 1$ for each semilocal B-ring C.

Definition 6. Assume that G is strictly isotropic, and let P be a strictly proper parabolic B-subgroup of G. The *non-stable K_1-functor* $K_1^{G,P}$ is the functor $K_1^{G,P}(A) = G(A)/\langle R_u(P)(A), R_u(P^-)(A)\rangle$ on the category of B-algebras A, where P^- is any opposite parabolic B-subgroup and $R_u(P)$, $R_u(P^-)$ are the unipotent radicals of P and P^-. (We omit P^- in the notation since $K_1^{G,P}$ is independent of P^- by [3, Exp. XXVI Cor. 1.8].) There are canonical surjections

$$G(B) \to K_1^{G,P}(B) \to G(B)/R.$$

Proposition 7. Let G be a simply connected strictly isotropic semisimple group scheme over a semilocal ring B. If B is henselian local or if B is an equicharacteristic regular ring and G has isotropic rank ≥ 2, then $G(B)/R \cong K_1^{G,P}(B)$.

Let A be a henselian local domain with residue field k and fraction field K. Let G be a reductive group scheme over A. One may ask if there exists a specializaion homomorphism $G(K)/R \to G(k)/R$ and what are the conditions for it to be an isomorphism. It is productive to address this question by comparing these R-equivalence class groups with $G(A)/R$.

Proposition 8. Let G be a reductive A-group scheme. If G is a torus or simply connected semisimple and strictly isotropic, then $G(A)/R \cong G(k)/R$.

Corollary 9. If T is an A-torus and A is regular, then $T(k)/R \cong T(A)/R \cong T(K)/R$.

It has been previously known that $G(k)/R \cong G\big(k((t))\big)/R$ for any field k and any reductive k-group G [4]. The following result is a generalization to the case of several variables.

Theorem 10. Let A be a complete equicharacteristic regular local ring. Then for any reductive A-group scheme G one has $G(k)/R \cong G(A)/R \cong G(K)/R$.

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Reduction for branching multiplicities

Nicolas Ressayre

(joint work with Pierre-Emmanuel Chaput)

Fix an n-dimensional vector space V. Let $\Lambda_n^+ = \{(\lambda_1 \geq \cdots \geq \lambda_n \geq 0 \,:\, \lambda_i \in \mathbb{N}\}$ denote the set of partitions. For $\lambda \in \Lambda_n^+$, let $S^\lambda V$ be the corresponding Schur module, that is the irreducible $\mathrm{GL}(V)$-module of highest weight $\sum \lambda_i \epsilon_i$ (notation as in Bourbaki). The Littlewood-Richardson coefficients (or LR-coefficients for short) $c_{\lambda,\mu}^\nu$ are defined by

$$S^\lambda V \otimes S^\mu V \simeq \bigoplus_{\nu \in \Lambda_n^+} \mathbb{C}^{c_{\lambda,\mu}^\nu} \otimes S^\nu V \tag{1}$$

(here $\mathbb{C}^{c_{\lambda,\mu}^\nu}$ is a multiplicity space).

Let $1 \leq r \leq n$ and $\mathrm{Gr}(r, n)$ be the Grassmannian of r-dimensional linear subspaces of V. Recall that the Schubert basis (σ^I) of the cohomology ring

$H^*(\mathrm{Gr}(r,n),\mathbb{Z})$ is parametrized by the subsets I of $\{1,\ldots,n\}$ with r elements. The Schubert constants $c_{I,J}^K$ are defined by

$$\sigma^I\sigma^J = \sum_K c_{I,J}^K \sigma^K. \tag{2}$$

Actually, $c_{I,J}^K$ is also a LR-coefficient.

Given a partition λ and a subset I, let λ_I be the partition whose parts are λ_i with $i \in I$. Let also $\overline{I}$ denote the complementary subset $\{1,\cdots,n\}\setminus I$. By [4, 2], we have:

Theorem 1. *(1) Let r and $I,J,K \subset \{1,\ldots,n\}$ be subsets with r elements such that $c_{I,J}^K \neq 0$. For any $(\lambda,\mu,\nu) \in (\Lambda_n^+)^3$, if $c_{\lambda,\mu}^\nu \neq 0$ then*

$$|\lambda_I| + |\mu_J| \geq |\nu_K|. \tag{3}$$

(2) Assume that $c_{I,J}^K = 1$. Let λ,μ,ν be partitions such that

$$|\lambda_I| + |\mu_J| = |\nu_K|. \tag{4}$$

Then

$$c_{\lambda,\mu}^\nu = c_{\lambda_I,\mu_J}^{\nu_K} \cdot c_{\lambda_{\overline{I}},\mu_{\overline{J}}}^{\nu_{\overline{K}}}. \tag{5}$$

Formula (5) is a multiplicativity property. Let us report on a similar property for Belkale-Kumar [1] coefficients (BK-coefficients for short). These numbers are all intersection numbers of Schubert classes or zero. Consider an inclusion $P \subset Q$ of parabolic subgroups of a reductive algebraic group G, and the corresponding fibration $G/P \to G/Q$. Richmond [5] proves that any BK-coefficient d of G/P is the product of two BK-coefficients in G/Q and Q/P. In type A, this implies that a non zero BK-coefficient of any two steps flag manifold is a product of two LR-coefficients: $d = c_1c_2$.

If moreover $c_1 = 1$, Theorem 1 implies that c_2 itself is the product of two LR-coefficients: $c_2 = c_2'c_2''$. Thus $d = c_1c_2'c_2''$ is the product of *three* LR-coefficients. This is the content of [3, Theorem 3], which even more generally states that on a k-step flag variety, a BK-coefficient can be factorized as a product of $\frac{k(k-1)}{2}$ LR-coefficients. Unfortunately, this assertion needs $c_1 = 1$ and is not correct in general. Our original motivation was to correct this result. We get such a correction if $c_1 = 2$.

Fix now r and $I,J,K \subset \{1,\ldots,n\}$ of cardinal r such that $c_{I,J}^K = 2$, and consider the multiplicities associated to the triples of partitions satisfying equation (4). We prove that the set of triples $(\lambda,\mu,\nu) \in (\Lambda_n^+)^3$ such that

$$|\lambda_I| + |\mu_J| = |\nu_K| \text{ and } 0 \neq c_{\lambda,\mu}^\nu < c_{\lambda_I,\mu_J}^{\nu_K} \cdot c_{\lambda_{\overline{I}},\mu_{\overline{J}}}^{\nu_{\overline{K}}}$$

contains a unique minimal element (α,β,γ). Observe that in the paper, we obtain an explicit expression for this triple.

Theorem 2. *Assume* $c_{I,J}^K = 2$*. Let* λ, μ, ν *be partitions such that* $|\lambda_I| + |\mu_J| = |\nu_K|$*. Then,*

$$c_{\lambda,\mu}^{\nu} = \sum_{k\geq 0} (-1)^k c_{\lambda_I - k\alpha_I, \mu_J - k\beta_J}^{\nu_K - k\gamma_K} \cdot c_{\lambda_{\overline{I}} - k\alpha_{\overline{I}}, \mu_{\overline{J}} - k\beta_{\overline{J}}}^{\nu_{\overline{K}} - k\gamma_{\overline{K}}}. \tag{6}$$

We also compute the LR coefficients corresponding to (α, β, γ):

Theorem 3. *For all* $k \geq 0$*, we have*

$$c_{k\alpha,k\beta}^{k\gamma} = \frac{(k+1)(k+2)}{2},$$

and

$$c_{k\alpha_I,k\beta_J}^{k\gamma_K} = c_{k\alpha_{\overline{I}},k\beta_{\overline{J}}}^{k\gamma_{\overline{K}}} = k+1.$$

For example, Theorem 2 with $(\lambda, \mu, \nu) = (k\alpha, k\beta, k\gamma)$ is true since $\frac{(k+1)(k+2)}{2} + \frac{k(k+1)}{2} = (k+1)^2$.

We also explained why the assumption $c_{I,J}^K = 2$. Indeed, it implies that some morphism from an incidence variety is of degree 2. A crucial point in our proof is that such a finite morphism is cyclic.

Finally, note that Theorem 2 admits a general extension to any branching problem (the case of LR-coefficients corresponding to the tensor product decomposition for the linear groups). At the opposite, Theorem 3 is specific to the type A.

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Homomorphisms of algebraic groups: representability and rigidity

Michel Brion

This talk is based on the preprint [1] with the same title. A motivation comes from the problem of classifying algebraic group actions on (say) projective varieties, that we briefly present.

We work over a field k of characteristic $p \geq 0$. Given a projective variety X, there exists a group scheme Aut_X such that $\mathrm{Aut}_X(R) = \mathrm{Aut}_{R-\mathrm{sch}}(X_R)$ for any algebra R, where X_R denotes the R-scheme obtained from X by base change. Moreover, the automorphism group scheme Aut_X is locally of finite type. As a consequence, the connected component of the identity in Aut_X is an algebraic group (i.e., a group scheme of finite type) that we denote by Aut_X^0.

An action of an algebraic group G on X is given by a homomorphism (i.e., a morphism of group schemes) $f : G \to \mathrm{Aut}_X$. Two actions are equivalent if the corresponding homomorphisms are conjugate by an element of $\mathrm{Aut}(X) = \mathrm{Aut}_X(k)$. So classifying G-actions on X amounts to describing the orbit space $\mathrm{Hom}_{\mathrm{gp}}(G, \mathrm{Aut}_X)/\,\mathrm{Aut}(X)$ for the conjugation action. If G is connected, then we may replace Aut_X with Aut^0_X.

This motivates the following:

Questions 1. *Let G and H be algebraic groups.*
(i) Is there a scheme M such that $\mathrm{Hom}_{R-\mathrm{gp}}(G_R, H_R) = M(R)$ for any algebra R?
(ii) If so, H acts on M via its action on itself by conjugation. How to describe the orbits?

(In fact, one should allow H to be locally of finite type for applications to G-actions as above. We refer to [1] for this technically more complicated setting).

Question (i) is equivalent to the representability of the functor of homomorphisms $\mathbf{Hom}_{\mathrm{gp}}(G, H)$. For example, $\mathbf{Hom}_{\mathrm{gp}}(\mathbb{G}_m, \mathbb{G}_m)$ is represented by the constant scheme $\mathbb{Z}_k$ by rigidity of tori. In particular, the scheme M is not necessarily affine, nor of finite type (if it exists). But M is locally of finite type, since $R \mapsto \mathrm{Hom}_{R-\mathrm{gp}}(G_R, H_R)$ commutes with direct limits.

Also, M has an action of $\mathrm{End}_{\mathrm{gp}}(G) \times \mathrm{End}_{\mathrm{gp}}(H)$ by composition. We will mostly use the action of H by inner automorphisms. If H is commutative, then M is a group scheme via pointwise multiplication.

A related parameter space exists when $p = 0$ and G, H are linear: then the set $\mathrm{Hom}_{\mathrm{gp}}(G, H)(\bar{k})$ has the structure of an affine finite-dimensional ind-variety, see [5, Part 8]. Yet the functor $\mathbf{Hom}_{\mathrm{gp}}(G, H)$ contains more information than its sets of K-points for all field extensions K/k, and is not necessarily representable. For example, every homomorphism $\mathbb{G}_{a,K} \to \mathbb{G}_{m,K}$ is trivial, but $\mathbf{Hom}_{\mathrm{gp}}(\mathbb{G}_a, \mathbb{G}_m)$ is not representable (see e.g. [8, Ex. 1.1]). More generally, given a connected linear algebraic group H, one can show that $\mathbf{Hom}_{\mathrm{gp}}(\mathbb{G}_a, H)$ is representable if and only if H is unipotent (when $p = 0$), resp. trivial (when $p > 0$).

Next, we introduce a class of algebraic groups for which Questions (i) and (ii) have positive answers. We say that an algebraic group G (possibly non-affine) is *linearly reductive* if every finite-dimensional representation of G is semi-simple.

Examples of such groups include tori, abelian varieties (since their representations are trivial), and reductive groups in characteristic 0. The affine linearly reductive groups have a well-known structure, due to Nagata and Demazure-Gabriel (see [2, IV.3.3]); the general case is analyzed in [1].

We may now state our main result:

Theorem 2. *Let G be a linearly reductive group, and H a smooth algebraic group. Then $\mathbf{Hom}_{\mathrm{gp}}(G, H)$ is represented by a smooth scheme M. Moreover, every H-orbit in M is open and the stabilizer is smooth.*

The openness of orbits is a rigidity result: the only way to deform a homomorphism is via conjugation on the target.

The above theorem is due to Grothendieck when G is of multiplicative type and H is affine over an arbitrary base scheme. See [3, Exp. XI] and a recent generalization by Romagny in [9]. Grothendieck's proof is based on "local" methods (from what is now known as deformation theory) together with the restriction of homomorphisms to n-torsion subgroups of G for all positive integers n. By contrast, our proof uses many structure results for algebraic groups over a field, together with some "global" arguments such as a rigidity lemma adapted from the theory of abelian varieties.

Here is a first application of the main result:

Proposition 3. *Assume k algebraically closed. Let G be a linearly reductive group, and H a smooth algebraic group. Then the natural map*

$$\mathrm{Hom}_{\mathrm{gp}}(G,H)/H(k) \longrightarrow \mathrm{Hom}_{K-\mathrm{gp}}(G_K,H_K)/H(K)$$

is a bijection for any algebraically closed field extension K/k.

This is due to Vinberg (see [10]) and Margaux (see [7]) for G affine.

Proof sketch. The scheme M representing $\mathbf{Hom}_{\mathrm{gp}}(G,H)$ is a disjoint union of open orbits of k-rational points. So the connected components of M are the orbits of the neutral component H^0. Thus, we obtain

$$\mathrm{Hom}_{\mathrm{gp}}(G,H)/H(k) = M(k)/H(k) = (M/H^0)/(H/H^0) = \pi_0(M)/\pi_0(H),$$

where $\pi_0(M)$ denotes the set of connected components of M.

A further application is the following:

Proposition 4. *Assume k algebraically closed. Let G be a finite group of order prime to p, and H an algebraic group. Then $\mathrm{Hom}_{\mathrm{gp}}(G,H)/H(k)$ is finite.*

Proof sketch. Given a finite scheme X, the functor $R \mapsto \mathrm{Hom}_{R-\mathrm{sch}}(X_R,H_R)$ is represented by an algebraic group: the Weil restriction $\mathrm{R}_{X/k}(H)$. As a consequence, the functor $\mathbf{Hom}_{\mathrm{gp}}(G,H)$ is represented by a closed subscheme $M \subset \mathrm{R}_{G/k}(H)$. In particular, M is of finite type. As G is linearly reductive, we conclude by the main result.

In particular, for any finite group G of order prime to p, there are finitely many equivalence classes of G-actions on a projective variety X, if Aut_X is algebraic. But this fails in general, since there exist smooth projective complex varieties X such that Aut_X is discrete and has infinitely many conjugacy classes of involutions. Examples of such varieties were first constructed by Lesieutre in dimension 6 (see [6]) and then by Dinh and Oguiso in any dimension ≥ 2 (see [4]).

The main result is close to optimal, as shown by the following:

Proposition 5. *Let G be an algebraic group. Assume that the main result holds for any smooth affine algebraic group H. Then G is linearly reductive.*

This leaves open the following question:

Characterize the algebraic groups G such that $\mathbf{Hom}_{\mathrm{gp}}(G,H)$ is representable for any algebraic group H.

One can show that this holds if G is an iterated extension of finite groups, reductive groups and anti-affine groups. If $p > 0$, this is a weaker condition than G being linearly reductive. A further natural question asks when the representing scheme M is of finite type. This is very useful for applications, and holds e.g. if G is an extension of a finite group by a semi-simple one.

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Optimality of string cone inequalities and potential functions

Bea Schumann
(joint work with Gleb Koshevoy)

Let G be a simple, simply connected, simply-laced algebraic group over $\mathbb{C}$ and U be the unipotent radical of a Borel subgroup $B \subset G$. Let W be the Weyl group of G with longest element w_0 of length N.

For every reduced expression $w_0 = s_{i_1} s_{i_2} \dots s_{i_N}$ there exists a polyhedral parametrization of Lusztig's canonical basis of $\mathbb{C}[U]$ by the integer points of a rational polyhedral cone $\mathcal{S}_{\mathbf{i}} \subset \mathbb{R}^N$. Here $\mathbf{i} = (i_1, i_2, \dots i_N)$ is the word of the reduced expression $s_{i_1} s_{i_2} \dots s_{i_N}$.

Let for example $G = \mathrm{SL}_3$, then $W \cong S_3$ and $s_1 s_2 s_1 = w_0$ is a reduced expressions $\mathbf{i} = (1,2,1)$ of w_0. We have

$$\mathcal{S}_{\mathbf{i}} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq 0,\ x_1 \geq 0,\ x_2 - x_3 \geq 0\}.$$

If we consider two reduced words $\mathbf{i_1}$ and $\mathbf{i_2}$, then there is a piecewise linear bijection $\Psi^{\mathbf{i_1}}_{\mathbf{i_2}} : \mathbb{R}^N \to \mathbb{R}^N$ such that $\Psi(\mathcal{S}_{\mathbf{i}_1}) = \mathcal{S}_{\mathbf{i}_2}$. In our example:

$$\begin{aligned} \Psi^{1,2,1}_{2,1,2}: \quad & S_{(1,2,1)} \to S_{(2,1,2)} \\ (x_1,x_2,x_3) & \mapsto (\max(x_3, x_2-x_1), x_1+x_2, \min(x_1, x_2-x_3)). \end{aligned} \tag{1}$$

The piecewise linear bijection $\Psi : \mathcal{S}_{\mathbf{i}_1} \to \mathcal{S}_{\mathbf{i}_2}$ can be used to compute the inequalities of all string cones. However, in general many recursive steps are needed to compute the inequalities and the number of inequalities grows in general exponentially with the rank of G. The questions we want to study in this talk are:

For an arbitrary reduced word $\mathbf{i}$ *how far are the inequalities from being optimal? How many redundancies appear and can we classify them?*

Let us specify what we mean by redundancies. We call an inequality $\sum_{k=1}^{N} a_k x_k \geq 0$ redundant if omitting this inequality gives the same cone. This is equivalent to the fact that this inequality is a positive sum of the others. The reader might guess that this problem is hard if we compute inequalities recursively using compositions of piecewise linear maps as in (1). We hence use another approach to study the string cone inequalities.

Since U is a so-called partial compactification of a cluster variety, we can apply the machinery of Gross-Hacking-Keel-Kontsevich [1] to U (up to some technical conditions) giving a basis for $\mathbb{C}[U]$ together with many parametrizations of this basis by rational polyhedral cones $\mathcal{C}_\Sigma$ (Σ a possibly infinite index set).

The string cones appear as a subset of these polyhedral cones, as the next theorem shows.

Theorem 1 ([2]). *The string cones appear as a subset of the parametrizations* $\mathcal{C}_\Sigma$*, i.e. for any reduced expression* $\mathbf{i}$ *there exists an index* $\Sigma_\mathbf{i}$ *and a unimodular bijection*

$$\mathcal{C}_{\Sigma_\mathbf{i}} \to \mathcal{S}_\mathbf{i}$$

(and the technical conditions are satisfied here).

Let us look as the $G = \mathrm{SL}_3$ example again, with reduced expression $s_1 s_2 s_1 = w_0$ (i.e. $\mathbf{i} = (1,2,1)$). In this example we have

$$\mathcal{C}_{\Sigma_\mathbf{i}} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid -x_2 \geq 0,\ -x_3 \geq 0,\ -x_1 - x_3 \geq 0\}.$$

The inequalities are grouped into disjoint sets of inequalities, one for every simple root of G, i.e. $\mathcal{C}_{\Sigma_\mathbf{i}} = \bigcap_{\alpha_j \text{ simple}} \mathcal{C}_{\Sigma_{\mathbf{i},j}}$ Here $\mathcal{C}_{\Sigma_{\mathbf{i},2}}$ is determined by the inequalities $-x_3 \geq 0,\ -x_1 - x_3 \geq 0$ and $\mathcal{C}_{\Sigma_{\mathbf{i},1}}$ is determined by the inequality $-x_2 \geq 0$. This reduces our problem as follows:

Proposition 2 ([3]). *If a redundancy occurs it involves only inequalities of* $\mathcal{C}_{\Sigma_{\mathbf{i},j}}$ *for a fixed* j*.*

Let us look closer at the definition of the cones $\mathcal{C}_\Sigma$. To each index Σ is associated an open torus $T^\vee_\Sigma = (\mathbb{C}^*)^N$ in the dual cluster variety to U, denoted by $\mathcal{X}$. There exists a regular function $W : \mathcal{X} \to \mathbb{C}$ (called potential) such that Gross-Hacking-Keel-Kontsevich's basis for $\mathbb{C}[U]$ is parametrized by the integer points of

$$\mathcal{C}_\Sigma = \{x \in \mathbb{R}^N \mid [W|_{T_\Sigma^\vee}]_{trop}(x) \geq 0 \text{ for a } \Sigma \ (\iff \text{ for any } \Sigma)\}.$$

Here $W|_{T_\Sigma^\vee} \in \mathbb{C}[T_\Sigma^\vee]$, hence $W|_{T_\Sigma^\vee} \in \mathbb{C}[x_k^{\pm 1} \mid 1 \leq k \leq N]$ is a Laurent polynomial. The function $[W|_{T_\Sigma^\vee}]_{trop} : \mathbb{R}^N \to \mathbb{R}$ is the piecewise linear map we get when we replace multiplication by addition and addition by taking the minimum.

This function gives us a necessary condition for the appearance of redundancies, as the following theorem shows.

Theorem 3 ([3]). *If the absolute value of the exponent of any variable in $W|_{T_{\mathbf{i}}^\vee}$ is less or equal to 1, then the inequalities of $\mathcal{C}_{\Sigma_{\mathbf{i}}}$ (and hence $\mathcal{S}_{\mathbf{i}}$) are non-redundant.*

The theorem is applicable in many situations:

- The inequalities of $\mathcal{C}_{\Sigma_{\mathbf{i}},j}$ are non-redundant if ω_j is minuscule. In particular this holds for $\mathcal{C}_{\Sigma_{\mathbf{i}}}$ and $G = \mathrm{SL}_n$.
- The inequalities of $\mathcal{C}_{\Sigma_{\mathbf{i}}}$ if $\mathbf{i}$ is a reduced word adapted to a Dynkin quiver of the same type as G which has only sinks at vertices j such that ω_j is minuscule.
- The inequalities of $\mathcal{C}_{\Sigma_{\mathbf{i}}}$ if $\mathbf{i}$ is a "nice" word in the sense of Littelmann ([4]).

We conjecture that Theorem 3 is indeed an equivalence, i.e. the inequalities of the string cone $\mathcal{S}_{\mathbf{i}}$ are non-redundant if and only if the absolute value of the exponent of any variable in $W|_{T_{\mathbf{i}}^\vee}$ is less or equal to 1.

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Non-normality of Schubert varieties

Timo Richarz

(joint work with Thomas J. Haines, João N. P. Lourenço)

Schubert varieties in finite dimensional flag varieties are normal with mild singularities regardless of the characteristic of the ground field. More generally, similar results hold true in the Kac–Moody setting by the works of Kumar [3], Mathieu [6] and Littelmann [4]. The ambient affine flag varieties were later reinterpreted via the theory of affine Grassmannians as parametrizing torsors under parahoric group schemes over the formal disk equipped with a trivialization over the punctured one. In this setting, Faltings [2] and Pappas–Rapoport [7] proved the normality of Schubert varieties whenever the characteristic of the ground field is either zero or big enough. In my talk, I reported on joint work [5] with Haines and Lourenço where we show that such normality fails in small positive characteristic.

Main result. Let $F = k(\!(t)\!)$ be the Laurent series field in the formal variable t over an algebraically closed field k of characteristic $p > 0$. Let G be an absolutely almost simple, semi-simple, tamely ramified, reductive F-group, $\mathbf{f}$ a facet of its Bruhat-Tits building and $\mathbf{a}$ an alcove containing $\mathbf{f}$ in its closure. For each class $w \in W_{\text{aff}}$ in the affine Weyl group, let $S_w = S_w(\mathbf{a}, \mathbf{f})$ be the associated Schubert variety in the neutral component $\mathrm{Fl}^{\circ}_{G,\mathbf{f}}$ of the partial affine flag variety. Note that $\mathrm{Fl}_{G,\mathbf{f}}$ is possibly non-connected, but that every Schubert variety is isomorphic to one of the S_w after translation to the neutral component.

Theorem 1. *If p divides $|\pi_1(G)|$, then only finitely many Schubert varieties S_w, $w \in W_{\text{aff}}$ are normal. The non-normal Schubert varieties are geometrically unibranch and regular in codimension* 1 *(=R1), but not S2 (hence, not Cohen-Macaulay), not weakly normal and not Frobenius-split.*

Schubert varieties in Kac–Moody flag varieties of affine type are isomorphic to Schubert varieties attached to suitable *simply connected* groups G so that $|\pi_1(G)| = 1$ in these cases. Hence, the above result is consistent with the normality results in the Kac–Moody setting.

Example. Assume S_w is the quasi-minuscule Schubert variety inside the affine Grassmannian for $G = \mathrm{PGL}_2$ in characteristic $p = 2$. Then the completed local ring $\hat{\mathcal{O}}_{S_w,e}$ at the base point is isomorphic to the k-algebra

$$k[\![x, y, v, w]\!]/(vw + x^2y^2, v^2 + x^3y, w^2 + xy^3, xw + yv).$$

This is a surface singularity which is not weakly normal. Its (weak) normalization morphism identifies with the inclusion map of the subalgebra of $k[\![x, y, z]\!]/(z^2+xy)$ generated by $x, y, v = xz, w = yz$.

Heuristic. The reason why non-normal Schubert varieties must exist can be summarized in a few lines: One has a map

$$S_{\text{sc},w} = S_{\text{sc},w}(\mathbf{a}, \mathbf{f}) \longrightarrow S_w(\mathbf{a}, \mathbf{f}) = S_w$$

where $S_{\text{sc},w}$ is the Schubert variety for w inside $\mathrm{Fl}_{G_{\text{sc}},\mathbf{f}}$ and $G_{\text{sc}} \to G$ is the simply connected cover. The Schubert variety $S_{\text{sc},w}$ is normal [7, Thm. 8.4], and the map $S_{\text{sc},w} \to S_w$ is a birational homeomorphism by using Demazure resolutions. In other words, $S_{\text{sc},w} \to S_w$ is the (weak) normalization morphism of S_w, just as in the example above. On the other hand, the affine flag variety $\mathrm{Fl}_{G_{\text{sc}},\mathbf{f}}$ is reduced [7, Thm. 6.1], that is, equals the union of its Schubert varieties. If all Schubert varieties in $\mathrm{Fl}_{G,\mathbf{f}}$ were normal, then these two facts would imply the map $\mathrm{Fl}_{G_{\text{sc}},\mathbf{f}} \to \mathrm{Fl}_{G,\mathbf{f}}$ is a monomorphism. By looking at tangent spaces, this fails as soon as the kernel of $G_{\text{sc}} \to G$ is non-étale, or equivalently, as soon as p divides $|\pi_1(G)|$. Exploiting tangent spaces a bit further, we show that the normality of S_w is equivalent to the injectivity of the induced map $T_eS_{\text{sc},w} \to T_eS_w$ on tangent spaces at base points. This yields the following observation:

Key Lemma 2. *Let $w \in W_{\text{aff}}$.*

(1) *If S_w is normal, then S_v is normal for all $v \leq w$.*
(2) *If S_w is not normal, then S_v is not normal for all $v \geq w$.*

The above reasoning only shows that there are infinitely many non-normal Schubert varieties in $\mathrm{Fl}_{G,\mathbf{f}}$. In order to give an effective normality criterion, we are led to a deeper study of tangent spaces of Schubert varieties, see [5] for details.

Classification. Assume that G is absolutely simple. Examining the tables in [1, Ch. VI, Planche IX], here is the list of all pairs (G,p) such that $p \mid |\pi_1(G)|$. Split groups:

- type A_n for $p \mid n+1$;
- types B_n, C_n, D_n, E_7 for $p = 2$;
- type E_6 for $p = 3$.

The split groups E_8, F_4 and G_2 have connection index 1, and hence are excluded from the list. For the list of twisted groups, the reader is referred to [5, §6].

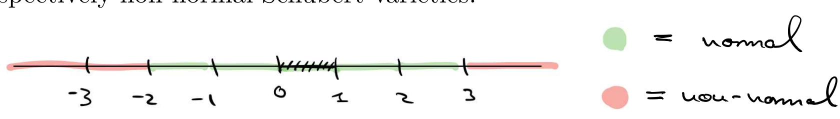
Problem 3. *Classify all finitely many Schubert varieties inside the full affine flag variety $\mathrm{Fl}^{\circ}_{G,\mathbf{a}}$ for the pairs (G,p) as above.*

Fix a maximally split torus $S \subset G$ such that the base alcove $\mathbf{a}$ is contained in the apartment $\mathcal{A} = \mathcal{A}(G,S)$. Using the bijections

$$\{\text{Schubert varieties in } \mathrm{Fl}^{\circ}_{G,\mathbf{a}}\} \leftrightarrow W_{\text{aff}} \leftrightarrow \{\text{alcoves in } \mathcal{A}\},$$

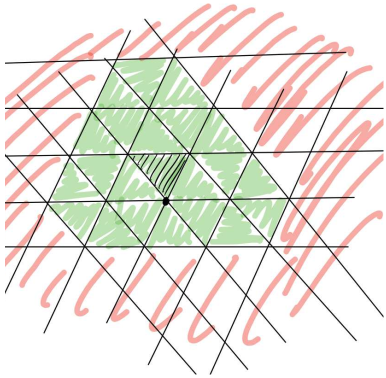
one can visualize the normal Schubert varieties.

Case $G = \mathrm{PGL}_2$, $p = 2$. The picture shows the alcoves corresponding to normal, respectively non-normal Schubert varieties:



There are exactly 5 normal Schubert varieties. Further, a Schubert variety S_w is normal if and only if $\dim(S_w) \leq 2$ if and only if S_w is smooth.

Case $G = \mathrm{PGL}_3$, $p = 3$. Our calculations indicate that there are 22 normal Schubert varieties:



If $\dim(S_w) \leq 3$, then S_w is normal. If $\dim(S_w) \geq 5$, then S_w is non-normal. In dimension 4, some S_w are normal and others are non-normal.

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Infinite dimensional geometric invariant theory and the moduli of gauged maps

Daniel Halpern-Leistner

(joint work with Andres Ferrero Hernandez, Eduardo Gonzalez)

Let C be a smooth curve over a field. The moduli stack of principal G-bundles $\mathrm{Bun}_G(C)$ plays a central role in geometric representation theory, mathematical physics, and arithmetic. $\mathrm{Bun}_G(C)$ also poses an interesting challenge from a foundational perspective: it is unbounded, i.e., there is no finite type scheme parameterizing all G-bundles, and it is highly non-separated, which is related to the fact that points have large automorphism groups.

One thing that makes $\mathrm{Bun}_G(C)$ manageable is a special stratification by locally closed substacks due to Harder-Narasimhan and Shatz [6, 9]. For simplicity, let us consider the moduli of vector bundles, i.e., $G = \mathrm{GL}_r$, of a fixed degree. There is an open substack $\mathrm{Bun}_G(C)^{\mathrm{ss}}$ that parameterizes "semistable" G-bundles, and this stack admits a projective good moduli space, which can be constructed using geometric invariant theory (GIT). The other strata parameterize "unstable" bundles along with a canonical filtration whose associated graded pieces are semistable.

It is natural to search for this structure on other moduli problems. Recently, a general theory has developed for describing this structure, which is now called a Θ-stratification, and giving necessary and sufficient criteria for constructing Θ-stratifications [4]. This report explains work in progress applying this theory to a particular class of moduli problems.

1. The moduli problem

Viewing $\mathrm{Bun}_G(C)$ as the moduli stack of maps $C \to BG$, one natural generalization is the stack of maps to the quotient stack $\mathrm{Map}(C, X/G)$. In order to get a Θ-stratification, we must first enlarge the moduli problem. We consider the moduli stack of *gauged maps* from C to X (with marked points):

$$\mathcal{M}^G_{C,n}(X) := \left\{ \begin{array}{c} G\text{-bundle } P \text{ over } C \\ u : \tilde{C} \to P \times^G X \end{array} \middle| \begin{array}{c} \tilde{C} \text{ is nodal with } n \text{ marked points,} \\ u \text{ is Kontsevich stable, and} \\ \text{the composition } \tilde{C} \to C \text{ has degree 1} \end{array} \right\}.$$

Note that $\mathrm{Map}(C, X/G) \subset \mathcal{M}^G_C(X)$ is the open substack of points for which $\tilde{C} \to C$ is an isomorphism, and when X is affine the two stacks agree.

This is not merely a toy problem. $\mathcal{M}^G_{C,n}(X)$ has been studied in the context of gauged Gromov-Witten (GW) theory (see [2] for a survey), and is part of a program to use $\mathrm{Bun}_G(C)$ to study the Gromov-Witten invariants of varieties and orbifolds arising as GIT quotients. For technical reasons, it is easier to work with K-theoretic invariants than cohomological ones. Our interpretation of this program is as follows:

(1) For any $\delta > 0$ there is an open substack of δ-semistable points $\mathcal{M}^G_{C,n}(X)^{\delta-\mathrm{ss}} \subset \mathcal{M}^G_{C,n}(X)$ that is a projective DM stack [7, 8].[1] One can define gauged GW invariants as the holomorphic Euler characteristic $\chi(\mathcal{M}^G_{C,n}(X)^{\delta-\mathrm{ss}}, P^\bullet)$ of certain tautological K-theory classes $[P^\bullet] \in K_0(\mathcal{M}^G_{C,n}(X))$.

(2) When $\delta \gg 0$, these invariants are related to the usual K-theoretic GW invariants of the GIT quotient $X/\!/G$ by certain wall-crossing formulas [3, 11].

(3) Forgetting the map u defines a morphism $\mathcal{M}^G_{C,n}(X) \to \mathrm{Bun}_G(X)$ which is proper – its fibers are certain moduli spaces of Kontsevich stable maps. One can obtain explicit formulas for $\chi(\mathcal{M}^G_{C,n}(X), P^\bullet)$ by pushing forward to $\mathrm{Bun}_G(C)$ and using the index formulas of [10].

[1] There is actually a stability condition for any G-ample line bundle L on X, and δ-stability is δL stability for a fixed choice of G-ample bundle L that we have suppressed from the notation.

(4) Compare $\chi(\mathcal{M}^G_{C,n}(X), P^\bullet)$ with $\chi(\mathcal{M}^G_{C,n}(X)^{\delta-\mathrm{ss}}, P^\bullet)$ using the "non-abelian" virtual localization formula [5].

Steps (1)-(3) are basically covered by previous work. The last step requires that $\mathcal{M}^G_{C,n}(X)^{\delta-\mathrm{ss}}$ is the open piece of a Θ-stratification.

2. Construction of the Θ-stratification

For simplicity, we restrict our discussion to the stack $\mathcal{M} = \mathcal{M}^{\mathrm{GL}_r}_{C,n}(X)$ where X is projective. The general case can be reduced to this one via a somewhat elaborate argument.

The stack $\Theta := \mathbb{A}^1/\mathbb{G}_m$ plays a central role in the general theory. For a field k, we regard a map $\Theta_k \to \mathfrak{X}$ as a "filtration" of the point $f(1) \in \mathfrak{X}(k)$. It is an interesting exercise to verify that a map $\Theta_k \to B\,\mathrm{GL}_n$ is the same thing as a $\mathbb{Z}$-indexed filtered k-vector space. More generally a map $\Theta_k \to \mathrm{Bun}_{\mathrm{GL}_r}(C)$ is a diagram of inclusions of vector bundles on C,

$$\cdots \subset E_{w+1} \subset E_w \subset E_{w-1} \subset \cdots$$

such that $E_w = 0$ for $w \gg 0$, $E_w = E_{w-1}$ for $w \ll 0$, and $\mathrm{gr}_w(E_\bullet) := E_w/E_{w+1}$ is locally free. It turns out that because $\mathcal{M} \to \mathrm{Bun}_G(C)$ is proper, a filtration in $\mathcal{M}$ is the same as a filtration in $\mathrm{Bun}_G(C)$. So the question is: **does every δ-unstable point of $\mathcal{M}$ come with a canonical filtration of the underlying vector bundle?**

In order to define canonical filtrations on a general stack $\mathfrak{X}$, the theory of Θ-stability takes as input a *numerical invariant* μ, which can be used to define a real-valued function on the set of all filtrations in $\mathfrak{X}$. In our example, for any filtration $f : \Theta_k \to \mathcal{M}^G_C(X)$, corresponding to a filtration $\cdots E_{w+1} \subset E_w \subset \cdots$ of the underlying vector bundle on C, the numerical invariant corresponding to δ-stability assigns $\mu_\delta(f) = (\ell_{\mathrm{Bun}_G} + \delta \ell_X)/\sqrt{b}$, where ℓ_X is the Hilbert-Mumford weight in the stack X/G at the generic point of C, $b = \sum_{w\in\mathbb{Z}} w^2\, \mathrm{rank}(\mathrm{gr}_w(E_\bullet))$, and

$$\ell_{\mathrm{Bun}_G} = \sum_{w\in\mathbb{Z}} w \left(\deg(\mathrm{gr}_w(E_\bullet)) - \frac{\deg(E_{-\infty})}{r}\, \mathrm{rank}(\mathrm{gr}_w(E_\bullet)) \right).$$

Theorem 1. *For every unstable point $p \in \mathcal{M}(k)$, there is a unique maximizer of the function $\mu_\delta(f)$ among all filtrations $f : \Theta_k \to \mathcal{M}$ equipped with an isomorphism $f(1) \cong p$. This defines a Θ-stratification of $\mathcal{M}$, i.e., a stratification by locally closed substacks that parameterize points along with the filtration maximizing μ_δ.*[2]

As mentioned above, this is an application of a general criterion for a numerical invariant on an algebraic stack to define a Θ-stratification [4, Thm. B]. The geometric insight behind Theorem 1 is that one of these conditions, called "strict Θ-monotonicity," can be verified using an "infinite dimensional" analogue of GIT. It is an interesting and (perhaps) unexpected application of the theory of Beilinson-Drinfeld Grassmannians [1]:

[2]Technically, the canonical filtration f is only unique up to composition with ramified covering maps $\Theta_k \to \Theta_k$, and this only gives a "weak" Θ-stratification in positive or mixed characteristic.

In classical GIT, one considers a numerical invariant on a stack of the form X/G where X is projective and G is a reductive group. Another way to think of this is that GIT applies to an algebraic stack $\mathfrak{X}$ equipped with a morphism $\mathfrak{X} \to BG$ that is relatively representable by projective schemes, and the numerical invariant is determined by a relatively ample line bundle. It turns out that it is easy to verify monotonicity from this perspective.

In our setting, $\mathcal{M}$ is not a global quotient stack. Instead, one can consider a stack $(B\,\mathrm{GL}_r)_{\mathrm{rat}}$ parameterizing a vector bundle of rank r on an open subset of C. This is not an algebraic stack, but it shares some key properties with the stack BG for a reductive group G. The canonical morphism $\mathcal{M} \to (B\,\mathrm{GL}_r)_{\mathrm{rat}}$ is not representable, but its fibers are representable by colimits of ind-projective ind-Deligne-Mumford stacks. This picture is close enough to the classical GIT set up to carry over the argument for monotonicity.

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Sugawara operators and local opers with two singularities in the case of $\mathfrak{sl}(2)$

Andrea Maffei

(joint work with Giorgia Fortuna, Davide Lombardo and Valerio Melani)

Proofs, and more complete definitions, of the results explained in the seminar are contained in [2].

Let $\mathfrak{g}$ be a complex simple Lie algebra, G be a complex algebraic group with Lie algebra equal to $\mathfrak{g}$, and G^L be the Langlands dual of G. Frenkel and Gaitsgory have put forward a relationship between, on one side, the geometry of the "space" of G^L-local systems on the formal disc with a possible singularity in the origin, and on the other, certain categories of representations of the affine Lie algebra $\hat{\mathfrak{g}}$ equipped with an action of the loop group $G(\mathbb{C}((t)))$ [4, 6, 5, 9, 7, 10, 8]. The starting point of this connection is the description of the center of the enveloping algebra of an affine Lie algebra at the critical level proved by Feigin and Frenkel [1, 3]. More recently, Dennis Gaitsgory pointed Giorgia Fortuna in the direction of investigating the situation in which the relevant G^L-local systems are allowed to have more than one singular point. We begun to develop this suggestion for the case of the Lie algebra $\mathfrak{sl}(2)$.

Let us fix a coordinate t around the origin of the formal disc. We wish to consider functions in the variable t, parametrised by a second variable a, having poles only at $t = 0$ and $t = a$. Formally, we set $A = \mathbb{C}[[a]]$ and introduce the A-algebra

$$K_2 = \mathbb{C}[[t, a]][\frac{1}{t(t-a)}],$$

whose elements are the functions we are interested in. The variables t and a are of very different nature here: t is a local coordinate for the geometric object of interest, namely the formal disc, while a should be considered as a parameter.

The ring K_2 can be equipped with a 'residue' map, defined as the sum of the residues around $t = 0$ and $t = a$. We can use this map to define a structure of A-Lie algebra on the space

$$\mathfrak{g} \otimes_{\mathbb{C}} K_2 \oplus A\, C_2$$

in a way that closely mimics the construction of the usual affine Lie algebra. We can then proceed exactly as in the case of one singularity: we first construct a suitable completion of the enveloping algebra, and then specialise C_2 to $-1/2$ to obtain a certain 'critical level' enveloping algebra $\hat{U}_2$. Just as in the case of the usual affine Lie algebra, the center of $\hat{U}_2$ turns out to be nontrivial, and we show that it is generated by certain 2-variables analogues of the classical Sugawara operators.

To introduce these generalised Sugawara elements, let J^α be a basis of $\mathfrak{g}$ and let J_α be the dual basis with respect to the Killing form of $\mathfrak{g}$. For every integer k we can then define

$$S_k^{(2)} = \sum_{n\in\mathbb{Z},\alpha} :(J^\alpha t^n s^n)(J_\alpha t^{k-n} s^{k-n-1}): + :(J^\alpha t^n s^{n+1})(J_\alpha t^{k-n-1} s^{k-n-1}):$$

$$S_{k+\frac{1}{2}}^{(2)} = \sum_{n\in\mathbb{Z},\alpha} :(J^\alpha t^n s^n)(J_\alpha t^{k-n} s^{k-n}): + :(J^\alpha t^n s^{n+1})(J_\alpha t^{k-n-1} s^{k-n}):$$

where $s = t - a$ and the colons denote a suitable (two-variables) normal ordered product.

In order to describe the geometric side of the correspondence, one should consider G^L-connections in the case of G^L being an adjoint group. Hence, in our case, we take $G = \mathrm{SL}(2)$ and $G^L = \mathrm{PSL}(2)$. We consider G^L-connections on the formal disc, parametrised by a, with possible singularities at $t = 0$ and $t = a$. We define 2-opers in this context in complete analogy to local opers with one singularity, in particular the space of 2-opers is represented by a smooth ind-scheme Op_2^* over Spec A. Concretely the A points of this ind-scheme can be seen as the set of connections of the form:

$$d + \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} dt$$

with $f \in K_2$. We prove the following result, which is a two-singularities analogue of the Feigin-Frenkel theorem [1] for $\mathfrak{g} = \mathfrak{sl}(2)$.

Theorem 1. *The operators $S_k^{(2)}$ are algebraically independent and topologically generate the centre of the algebra $\hat{U}_2$. Moreover, there is an isomorphism*

$$\mathcal{F}_2 : \mathrm{Funct}(\mathrm{Op}_2^*) \xrightarrow{\sim} Z(\hat{U}_2).$$

In their study of the "spherical case" [10], Frenkel and Gaitsgory describe the endomorphism ring of what they call a Weyl module $\mathbb{V}^\lambda$ of $\hat{\mathfrak{g}}$, where λ is an integral dominant weight of $\mathfrak{g}$. The Weyl modules have been recognised as the fundamental objects in the category of *spherical modules*, that is, those continuous representations of the affine Lie algebra $\hat{\mathfrak{g}}$ at the critical level on which the action of $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ integrates to an action of $G(\mathbb{C}[[t]])$. An important step in understanding the category of spherical modules is the determination of the endomorphism rings of the Weyl modules. These rings have been shown [10] to admit a very nice description in terms of the geometry of the space of opers. Indeed there exist disjoint subschemes Op_1^λ of the space of opers parametrised by the integral dominants weights of $\mathfrak{g}$ such that their union is the space of the unramified opers, that is, those that are trivial as G^L-connections. The connection between the Weyl modules $\mathbb{V}^\lambda$ and the schemes Op_1^λ is provided by [10], where the authors show that, for every λ, the Feigin-Frenkel isomorphism induces an isomorphism $\mathrm{Funct}(\mathrm{Op}_1^\lambda) \simeq \mathrm{End}(\mathbb{V}_1^\lambda)$.

This result generalises to our setting in the following way. Given two integral dominant weights λ and μ of $\mathfrak{g}$, we construct, by analogy to the 1-singularity case, a corresponding Weyl module $\mathbb{V}_2^{\lambda,\mu}$. The idea to define a corresponding space of opers $\mathrm{Op}_2^{\lambda,\mu}$ is to consider opers with two singularities such that around $t = 0$ are connections in Op_1^λ and around $t = a$ are connections in Op_1^μ. This idea can be used to define a closed subscheme of $\mathrm{Op}_2 |_{a \neq 0}$ and we define $\mathrm{Op}_2^{\lambda,\mu}$ as its closure in Op_2. In the case of $\mathfrak{sl}(2)$, we extend the main result of [10] to this context as follows:

Theorem 2. *The action of $Z(\hat{U}_2)$ on $\mathbb{V}_2^{\lambda,\mu}$ and the isomorphism $\mathcal{F}_2$ of the previous theorem induce an isomorphism*

$$\mathcal{G}_2 : \mathrm{Funct}(\mathrm{Op}_2^{\lambda,\mu}) \xrightarrow{\sim} \mathrm{End}(\mathbb{V}_2^{\lambda,\mu}).$$

The main ingredient to prove this theorem is the description of the schemes $\mathrm{Op}_2^{\lambda,\mu} |_{a=0}$. By definition, the scheme $\mathrm{Op}_2 |_{a=0}$ is isomorphic to the usual space of opers. Hence $\mathrm{Op}_2^{\lambda,\mu} |_{a=0}$ can be seen as a subscheme of the usual space of opers and we prove that

$$\mathrm{Op}_2^{\lambda,\mu}|_{a=0} \simeq \coprod_{\substack{|\mu-\lambda|\leq\nu\leq\lambda+\mu \\ \nu\equiv\lambda+\mu \bmod 2}} \mathrm{Op}_1^{\nu}$$

While some of the ingredients in the proof of this isomorphism apply to general Lie algebras $\mathfrak{g}$, we make use of the hypergeometric series to construct some specific elements of $\mathrm{Op}_2^{\lambda,\mu}|_{a=0}(\mathbb{C})$, which restricts some of our arguments to the case $\mathfrak{g} = \mathfrak{sl}_2$.

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Smith-Treumann theory and modular representation theory

Geordie Williamson

(joint work with Simon Riche)

Smith theory relates the mod p cohomology of the space to that of its fixed points under the action of a cyclic p group. It is a precursor to equivariant localization for torus action. The archetypal example of localization for a circle action is the equality

$$\chi(X) = \chi(X^{S^1})$$

where χ denotes Euler characteristic, and X is a reasonable topological space (e.g. compact manifold or algebraic variety). The analogue for Smith theory is the equality

$$\chi(X) = \chi(X^{\mathbb{Z}/p\mathbb{Z}}) \mod p$$

for the action of a $\mathbb{Z}/p\mathbb{Z}$ on a reasonable topological space.

Smith-Treumann theory is a version for sheaves. Suppose that $\varpi = \mathbb{Z}/p\mathbb{Z}$ acts on a complex variety X, and consider sheaves with $\mathbb{F}_p$-coefficients. We first consider the Smith quotient category on the fixed points,

$$\mathrm{Sm}(X^\varpi) = D^b_\varpi(X^\varpi)/\langle \mathrm{Perf}\rangle$$

where Perf denotes the full subcategory of sheaves on the fixed points whose stalks are free $\mathbb{F}_p[\varpi]$-modules. A key lemma of Treumann asserts if $i : X^\varpi \hookrightarrow X$ denotes the inclusion of the fixed points, then the cone over the canonical arrow $i^! \to i^*$ belongs to $\langle \mathrm{Perf}\rangle$ and hence one has a self-dual restriction functor

$$i^{!*} : D^b_\varpi(X) \to \mathrm{Sm}(X^\varpi).$$

For more on these constructions, see Treumann's paper [1]. In applications, an equivariant version discussed at length in [3] is indispensible.

In this talk I explained a recent application of these ideas to modular representation theory [2]. The punchline is the following: if one considers the loop rotation $\mathbb{G}_m$-action on the affine Grassmannian, then its fixed points under the subgroup $\varpi \subset \mathbb{G}_m$ of p^{th}-roots of unity have a beautiful description:

$$Gr^\varpi = \bigsqcup_{\lambda \in X/W_p} Gr^\lambda.$$

This decomposition closely resembles the decomposition of the category or rational representations into blocks governed by the affine Weyl group W_p. Our main theorem asserts that Smith restriction provides an equivalence in a suitable Iwahori-Whittaker version of geometric Satake. From this we recover a geometric proof of the linkage principle, as well as a proof (for all p!) of conjectures on tilting module characters that Simon and I made eight years ago.

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Equivariant multiplicities of Mirković–Vilonen cycles

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(joint work with Joel Kamnitzer, Allen Knutson)

Let G be a connected reductive group over $\mathbb{C}$, equipped with a Borel subgroup B and a maximal torus $T \subset B$. Let N^- be the unipotent radical of the opposite Borel, let $X_*(T)$ and $X^*(T)$ be the cocharacter and character lattices, let $\Phi = \Phi_+ \sqcup \Phi_-$ be the root system, let ρ be the half-sum of the positive roots, and set $\mathfrak{t}_{\mathbb{Q}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $\mathrm{Gr} = G(\mathbb{C}(\!(z)\!))/G(\mathbb{C}[\![z]\!])$ be the affine Grassmannian of G. The torus T acts on Gr, and the fixed points for this action are the elements $L_\mu = [\mu(z)]$ for $\mu \in X_*(T)$. Given a dominant $\lambda \in X_*(T)$, define $\mathrm{Gr}^\lambda = G(\mathbb{C}[\![z]\!])L_\lambda$; given $\mu \in X_*(T)$, define $S_\mu^- = N^-(\mathbb{C}(\!(z)\!))L_\mu$. The intersection $\overline{\mathrm{Gr}^\lambda} \cap S_\mu^-$ has pure dimension $d = \rho(\lambda - \mu)$; its irreducible components are called Mirković–Vilonen (MV) cycles.

Let $G^\vee$ be the Langlands dual of G (over $\mathbb{Q}$). The geometric Satake equivalence [7, 8] identifies the intersection homology $\mathrm{IH}\left(\overline{\mathrm{Gr}^\lambda}, \underline{\mathbb{Q}}\right)$ with the irreducible $G^\vee$-module of highest weight λ, denoted by $V(\lambda)$. Mirković and Vilonen further identify the weight space $V(\lambda)_\mu$ with the top-dimensional Borel–Moore homology $H_{2d}\left(\overline{\mathrm{Gr}^\lambda} \cap S_\mu^-, \mathbb{Q}\right)$. An MV cycle Z therefore yields a vector $v_Z \in V(\lambda)_\mu$.

Let Z be an MV cycle, irreducible component of $\overline{\mathrm{Gr}^\lambda} \cap S_\mu^-$. Then Z is an affine T-variety and has a unique T-fixed point, namely L_μ. The torus T acts on $\Gamma(Z, \mathcal{O}_Z)$ with weights in $\mathbb{N}\Phi_+$, and acts on the tangent space of Z at L_μ with weights in Φ_-. Therefore L_μ is a nondegenerate T-fixed point of Z in the sense of Brion [2], and we can consider the equivariant multiplicity $e_{L_\mu}Z$ of Z at L_μ, a rational function on $\mathfrak{t}_{\mathbb{Q}}$. If $\theta \in X_*(T)$ is regular antidominant, then $(e_{L_\mu}Z)(\theta)$ is the multiplicity of the finitely generated graded algebra

$$\Gamma(Z, \mathcal{O}_Z) = \bigoplus_{n \geq 0} \left(\bigoplus_{\substack{\chi \in X^*(T) \\ \langle -\theta, \chi \rangle = n}} \Gamma(Z, \mathcal{O}_Z)_\chi \right).$$

On the other side of the geometric Satake equivalence, the group $G^\vee$ comes with a Borel subgroup $B^\vee$ and a maximal torus $T^\vee \subset B^\vee$. Let $N^\vee$ be the unipotent radical of $B^\vee$, and let $\mathfrak{t}^\vee = X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathfrak{n}^\vee$ be the Lie algebras of $T^\vee$ and $N^\vee$. Fix a principal nilpotent element $e \in \mathfrak{n}^\vee$. For each regular element x in $\mathfrak{t}^\vee$, there exists a unique element $n_x \in N^\vee$ such that $\mathrm{Ad}_{n_x}(x) = x + e$.

Let v_λ^* be the linear form on $V(\lambda)$ dual to the highest weight vector $v_{\{L_\lambda\}}$. We can evaluate the matrix coefficient $\langle v_\lambda^*, ?\, v_Z \rangle$ at any element in $N^\vee$. Let $q : X_*(T) \to \mathbb{Z}$ be a quadratic form invariant under the Weyl group and let $\iota : X_*(T) \to \mathbb{X}^*(T)$ be the polar form of q. The main result of this talk is that for any regular element $\theta \in X_*(T)$,

$$(\dagger) \qquad (e_{L_\mu} Z)(\theta) = \langle v_\lambda^*, n_{\iota(\theta)} v_Z \rangle.$$

The right-hand side of this equation seemingly depends on the choice of q and e (via $n_{\iota(\theta)}$), while the left-hand side does not. To overcome this contradiction, we note that q and e are used to normalize the action of $G^\vee$ on $\mathrm{IH}\Big(\overline{\mathrm{Gr}^\lambda}, \underline{\mathbb{Q}}\Big)$. Explicitly, q determines a projective embedding of Gr, whence a very ample line bundle $\mathcal{L}$ on Gr, and Ginzburg shows in [5] that e acts as the cup-product with $c_1(\mathcal{L})$. The contributions compensate, and $n_{\iota(\theta)} v_Z$ does not actually depend on the choice of q or e.

The formula ($\dagger$) leads to a short proof of the following conjecture by Muthiah: For any dominant $\lambda \in X_*(T)$, the linear map $V(\lambda)_0 \to \mathbb{Q}(\mathfrak{t}_\mathbb{Q})$ that extends the assignment $v_Z \mapsto e_{L_0} Z$ is equivariant under the action of the Weyl group. Muthiah's paper [9] contains a beautiful proof of this result in the case where $G^\vee = \mathrm{SL}_d$ and $\lambda \leq d\varpi_1$, which translates the problem through the Schur–Weyl duality to the classical results of Joseph and Hotta on Springer representations.

The proof presented during the talk follows the line of sects. 8 and 9 in [1]. The equivariant multiplicity of Z at L_μ is obtained as a coefficient in the Fourier transform of the Duistermaat–Heckman (DH) measure of $\overline{Z}$ (defined in the algebro-geometric setup by Brion and Procesi [3]). This measure is computed by a method due to Knutson [6]: it is equal to the sum, over the set of all maximal chains of closures of MV cycles $\{L_\lambda\} \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_{d-1} \subsetneq \overline{Z}$, of DH measures of projective spaces. The sum is weighted by intersection multiplicities, which are matched to the action of Chevalley monomials on v_Z by means of Ginzburg's result.

Several questions have been asked at the end of the talk.

Can ($\dagger$) be proved without Duistermaat–Heckman measures?

In [10], Yun and Zhu construct an isomorphism of group schemes over $\mathfrak{t}_\mathbb{Q}$ from the spectrum of the T-equivariant homology of Gr (a Hopf algebra for the Pontryagin product) to the universal centralizer

$$C = \{(b, x) \in B^\vee \times \mathfrak{t}^\vee \mid b \text{ centralizes } e + x\}$$

(C is viewed as a scheme over $\mathfrak{t}_\mathbb{Q}$ thanks to $\iota^{-1} : \mathfrak{t}^\vee \to \mathfrak{t}_\mathbb{Q}$). The space C comes with a map $C \to B^\vee$, and by composition we get a homomorphism of algebras $\mathbb{Q}[B^\vee] \to \mathbb{Q}[C] \cong H_*^T(\mathrm{Gr}, \mathbb{Q})$. Let $Z \subset \overline{\mathrm{Gr}^\lambda}$ be an MV cycle; then the matrix coefficient $\langle v_\lambda^*, ?\, v_Z \rangle$, viewed as an element in $\mathbb{Q}[B^\vee]$, is mapped to the T-equivariant fundamental class $[Z] \in H_*^T(\mathrm{Gr}, \mathbb{Q})$. After localization in equivariant homology, this observation (due to Kamnitzer) provides another proof of ($\dagger$).

Can one actually compute examples?

Computing the equivariant multiplicity $e_{L_\mu} Z$ of an MV cycle Z seems almost as hard as computing the corresponding basis element v_Z. Methods currently known to perform this task are not completely algorithmic.

Suppose that G has semisimple rank one, write α for the positive root, and set $\mu = \lambda - n\alpha$ with $0 \leq n \leq \langle \alpha^\vee, \lambda \rangle$. The intersection $\overline{\mathrm{Gr}^\lambda} \cap S_\mu^-$ is irreducible, hence is an MV cycle. It is smooth and its equivariant multiplicity at L_μ is $(-1/\alpha)^n$.

Take now G of type A_3, take for λ the highest positive root (so that $V(\lambda)$ is the adjoint representation), and set $\mu = 0$. Here there are three MV cycles, whose equivariant multiplicities at L_0 are

$$\frac{-1}{\alpha_1(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)}, \quad \frac{-(\alpha_1+2\alpha_2+\alpha_3)}{\alpha_2(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}, \quad \frac{-1}{\alpha_3(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}.$$

In particular, we see that the second MV cycle is not smooth at L_0.

One can also use (†) the other way around, to acquire information on v_Z from the geometry of Z; see the appendix of [1] for an application. Recently, Casbi undertook a more systematic study of these equivariant multiplicities [4].

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Verlinde rings, eigenfunctions and DAHA actions

Catharina Stroppel

This talk will be about Verlinde rings (or also called fusion rings) which are certain Grothendieck rings attached to a semisimple tensor category constructed from representations of quantum groups at roots of unities. These fusion rings were already studied from many different aspects. For instance using categories of representations of affine Kac-Moody algebras at a fixed level (by Finkelberg and Kazhdan-Lusztig) or using K-theory (by Freedman-Hopkins-Teleman) or using generalisd Θ-functions (by Beauville). These rings come with an integral basis and integral structure constants which (originally also by Verlinde) can be interpreted as dimensions of conformal blocks. This integral structure comes from Jordan-Hoelder multiplicities of tensor product in the underlying category. The category itself is a crucial player in Chern-Simons theory and can also be used to construct invariants of knots and 3-manifolds (for instance by Reshethikin-Turaev).

The setup of the talk was as follows: Let $\mathfrak{g}$ be a semisimple complex Lie algebra or $\mathfrak{g} = \mathfrak{gl})_m$ with Cartan matrix $C = (c_{i,j})$. An important number is $D \in \{1, 2, 3\}$, which denotes the maximum of the $c_{i,j}$ for $i/not = j$. Let now $q \in \mathbb{C}$ be a fixed primitive ℓth root of unity. We set $\ell = \ell'$ if ℓ is odd and $\ell = \frac{1}{2}\ell'$ if ℓ is even. Finally we will distinguish two cases

$$\text{Case I)}: D \text{ does not divide } \ell', \quad \text{and} \quad \text{Case II)} : D \mid \ell'.$$

Consider the category Rep(U) of finite dimensional complex representations of Lusztig's divided power quantum group $U = U_q(\mathfrak{g})$ with its Weyl and dual Weyl modules $\Delta(la)$, $\nabla(\lambda)$ for $\lambda \in^+$, the set of dominant integral weights. An object $T' \in \text{Rep}(U)$ is called tilting if it has both, a Verma and also a dual Verma flag. By general theory of highest weight categories the indecomposable tiltings are classified and in natural bijection to X^*. We denote them by $T(\lambda)$. By a theorem of Paradowski, the tiltings form an additive monoidal ribbon category $\mathcal{T}$. Since Paradowski's proof is quite involved we provide an easier proof for type A, in fact for all types where the fundamental weights are miniscule.

We now take $\mathcal{T}$ modulo the tensor ideal given by all negligible objects, i.e. whose quantum dimension vanishes. This quotient is a semisimple tensor category. Its Grothendieck ring is the Verlinde or fusion ring, which we call A. We denote

$$\mathcal{A}_\ell = \{\lambda \in X^+ \mid (\lambda + \rho, \Theta_0) < \ell'\} \tag{1}$$

where Θ_0 is the longest short (resp. maximal) root if we are in Case I (resp. II).

Proposition 1. *Assume ℓ is larger than the Coxeter number. Then*

(1) $\lambda \notin \mathcal{A}_\ell$ implies that $T(\lambda)$ is negligible.
(2) The classes of the $T(\lambda)$ with $\lambda \in \mathcal{A}_\ell$ form a basis of A. In particular is A finite dimensional except in case $\mathfrak{g} = \mathfrak{gl}_n$.
(3) $\mathcal{A}_\ell$ is a fundamental domain for the action of the affine Weyl group which is the semiproduct of the Weyl group with ℓ' times the (co)root lattice in Case I (respectively in Case II).

We like to understand now the ring A. Observe that A is a quotient of the character ring. In [1], an explicit description in terms of generators and relations is given. The following result goes back to [6] and then reproved in [5].

Theorem 2. *In case of $\mathfrak{g} = \mathfrak{gl}_n$, the fusion ring A is isomorphic to the quantum cohomology of the Grassmannian of n dimensional subspaces in ℓ'-dimensional space. Under this isomorphism, the basis vector given by $T(\lambda)$ is mapped to the quantum Schubert class labeled by the partition attached to λ and the quantum parameter corresponds to the class of the determinant representation in A.*

The proof was given in terms of an integrable systems model. More precisely we constructed a certain Bethe eigenbasis for certain symmetrised hopping operators and used this to match the multiplication in this eigenbasis with the well-known Verlinde product formula. The defining ideal $I(n, \ell')$ in the Siebert-Tian presentation of A arises than nicely as a solution to the Bethe Ansatz equations. This approach has more interesting side-effects. For instance, the hopping operators from the intergable systems model give rise to a crystal action of the affine Kac-Moody Lie algebra of $\hat{\mathfrak{sl}}_n$ on the fusion ring, turning it into certain KR-module, and the fusion product can be completely described via an action of symmetric functions in affine plactic generators. Moreover, the base change matrix from the standard basis to the Bethe basis can be matched with the S matrix of the underlying tensor category, in case the category is modular. In that case the Verlinde ring inherits an $\mathrm{SL}_2(\mathbb{Z})$-action and should be seen as an example of a Verlinde ring in the sense of Cherednik. Cherednik's philosophy, from [2] says that any Verlinde algebra should be a module for a double affine Hecke algebra (DAHA). We make this claim precise.

For that, let $\mathcal{H}$ be the DAHA in the sense of [2] corresponding to $\mathfrak{g}$ with its weight lattice. We specialise both parameters to q. Let Pol be the polynomial representation. Now to connect with the character ring and fusion ring we consider the spherical DAHA $e\mathcal{H}e$. Then $e\,\mathrm{Pol}$ becomes an $e\mathcal{H}e$-module.

Proposition 3. *(1) Via the triangular decomposition of $\mathcal{H}$, $e\,\mathrm{Pol}$ can be identified with the symmetrised Laurent polynomials identified with the symmetrised negative part of $\mathcal{H}$.*

(2) Factoring out the radical for a certain bilinear form defines a quotient Q of $e\,\mathrm{Pol}$ which is an irreducible representation of $e\mathcal{H}e$ which is semisimple for the action of the positive part of $e\mathcal{H}e$ with 1-dimensional eigenspaces.

We can now consider A as a quotient of the character ring $R(\mathfrak{g})$ which we embed into $e\,\mathrm{Pol}$. The following is shown in [3], [4].

Theorem 4. *The following holds*

(1) The surjection onto Q factors through A and defines a surjection $A \mapsto Q$.

(2) This surjection is an isomorphism in case our tensor category is modular.

(3) In case of $\mathfrak{g} = \mathfrak{gl}_n$ we have a commutative diagram with exact row:

$$\begin{array}{ccccccccc} \{0\} & \longrightarrow & I(n,\ell') & \longrightarrow & R(\mathfrak{g}) & \longrightarrow & A & \longrightarrow & \{0\} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \{0\} & \longrightarrow & \mathrm{Rad} & \longrightarrow & e\,\mathrm{Pol} & \longrightarrow & Q & \longrightarrow & \{0\} \end{array}$$

Here, the obvious middle map restricts to a map on the left and then induces a map on the quotient which is in fact an isomorphism. Moreover, the eigenvectors in A constructed in [5] coincide with the eigenvectors for the positive part of $e\mathcal{H}e$ under this isomorphism.

We like to note that the spherical DAHA action can be used to detect modularity, even more the generalised eigenspace decomposition for the action of the positive part of $e\mathcal{H}e$ gives a precise measure for the failure of modularity and allows to define a further quotient of our tensor category which is modular or at least spin modular. These categories appear in work of Brugieres. As an upshot:

The spherical DAHA acts on fusion rings and MacDonald operators can be used to verify, measure the failure and analyse modularity.

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Characteristic cycles, categories of singularities and Hall algebras

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(joint work with Michela Varagnolo)

Let Q be a quiver of Kac-Moody type. Let X be the variety of all finite dimensional representations of Q and $\mathcal{X}$ its moduli stack. Lusztig has introduced a graded additive monoidal subcategory of the bounded constructible derived category $D^{\mathrm{b}}_c(\mathcal{X})$ of the stack $\mathcal{X}$ whose split Grothendieck group is isomorphic to the integral form of the quantum unipotent enveloping algebra of type Q. This construction can be viewed as a categorification of Ringel-Hall algebras. There are no known analogues of Ringel-Hall algebras for super quantum groups.

K-theoretic Hall algebras yield another geometric construction of quantum groups. In this theory quantum groups are categorified by the derived category $D^{\mathrm{b}}(T^*\mathcal{X})$ of coherent sheaves on the cotangent dg-stack of $\mathcal{X}$. The dg-stack $T^*\mathcal{X}$ is identified with the derived moduli stack of finite dimensional representations of the preprojective algebra Π_Q, which is a Calabi Yau algebra of dimension 2. In an informal way, the relation between Ringel-Hall algebras and K-theoretic Hall algebras is given by characteristic cycles of constructible sheaves. Our goal is to prove that K-theoretic Hall algebras categorify both quantum groups and super quantum groups.

We first consider the K-theoretic Hall algebra of the preprojective algebra Π_Q of the quiver Q following Schiffmann-Vasserot's approach. For each dimension vector β, let $X_\beta \subset X$ be the variety of all representations of dimension β. A linear group $G_\beta \times \mathbb{C}^\times$ acts on the cotangent T^*X_β. The $\mathbb{C}^\times$-action is prescribed by a weight function. The G_β-action is Hamiltonian. The moment map $\mu_\beta : T^*X_\beta \to \mathfrak{g}_\beta$ is $G_\beta \times \mathbb{C}^\times$-equivariant, with $\mathbb{C}^\times$ acting with the weight 2 on $\mathfrak{g}_\beta$. The moduli stack $\mathcal{X}_\beta$ is the quotient of X_β by G_β. The cotangent dg-stack of $\mathcal{X}_\beta$ is the derived fiber product

$$T^*\mathcal{X}_\beta = [T^*X_\beta \times^R_{\mathfrak{g}_\beta} \{0\} \,/\, G_\beta \times \mathbb{C}^\times].$$

We define $T^*\mathcal{X}$ to be the sum of all dg-stacks $T^*\mathcal{X}_\beta$. The category $D^{\mathrm{b}}(T^*\mathcal{X})$ is a graded triangulated category with a graded triangulated monoidal structure. The internal grading is given by the $\mathbb{C}^\times$-action. Thus the rational Grothendieck group $\mathbf{K}(\Pi_Q) = \mathrm{G}_0(T^*\mathcal{X})$ of $D^{\mathrm{b}}(T^*\mathcal{X})$ has a $\mathbb{Q}[q, q^{-1}]$-algebra structure. We call it the K-theoretic Hall algebra of Π_Q. We'll mostly consider another similar algebra $\mathbf{NK}(\Pi_Q)$, which is the Grothendieck group of the derived category of all coherent sheaves of dg-modules over the dg-stack $T^*\mathcal{X}$ which are supported on a closed substack of nilpotent elements.

Let $\mathbf{U}^+$ be the Lusztig integral form of the quantum enveloping algebra of the loop algebra of the positive part of the Kac-Moody algebra of type Q, i.e., a Drinfeld half of a toroidal quantum group. Our first goal is to prove the following.

Theorem 1. Let Q be a quiver of Kac-Moody type. Fix a normal weight function.
(a) There is a surjective $\mathbb{N}^{Q_0}$-graded algebra homomorphism $\phi : \mathbf{U}^+ \to \mathbf{NK}(\Pi_Q)$.
(b) If Q is of finite or affine type but not of type $A_1^{(1)}$, then the map ϕ is injective.

This work is motivated by [4], where the graded triangulated category $D^{\mathrm{b}}(T^*\mathcal{X})$ for a quiver Q of type A_1 is compared with a derived category of graded modules over the quiver-Hecke algebra of affine type $A_1^{(1)}$. The method of the proof goes back to some previous work of Schiffmann-Vasserot which considers the case of Borel-Moore homology. There, it was observed that an essential step to prove a similar isomorphism in Borel-Moore homology was to deform the algebra $\mathbf{NK}(\Pi_Q)$ by allowing a big torus action, and to prove torsion freeness of this deformation relatively to a commutative symmetric algebra generated by the classes of some universal bundles. We prove a similar torsion freeness in K-theory. We also discuss briefly the case of quivers which are not of Kac-Moody type.

Next, we consider the Drinfeld half $\boldsymbol{\mathcal{U}}^+$ of the super toroidal quantum groups of type A introduced in [1]. In this case, the relevant geometric algebra is the deformed K-theoretic Hall algebra $\mathbf{K}(Q, W)$ of a quiver with potential. We'll use the approach of Padurariu [3], which is inspired by the cohomological Hall algebras of Kontsevich-Soibelmann. The algebra $\mathbf{K}(Q, W)$ is defined as the Grothendieck group of the category of singularities of some Landau-Ginzburg model attached to the pair (Q, W). The quiver Q and the potential W are described as follow. The set of vertices is $Q_0 = \mathbb{Z}/N\mathbb{Z}$ and is partitioned into even and odd vertices. The set of arrows is

$$Q_1 = Q_1^+ \sqcup Q_1^0 \sqcup Q_1^-$$

with

$$Q_1^+ = \{x_i : i \to i+1\,;\, i \in Q_0\},$$
$$Q_1^- = \{y_i : i+1 \to i\,;\, i \in Q_0\},$$
$$Q_1^0 = \{\omega_i : i \to i\,;\, i \in Q_0^{ev}\}.$$

The potential is

$$W = \sum_{i \in Q_0^{ev}} (\omega_i y_i x_i - \omega_i x_{i-1} y_{i-1}) + \sum_{i \in Q_0^{odd}} y_i x_i x_{i-1} y_{i-1}.$$

This category is Calabi Yau of dimension 3. The relation with the first part is given by dimensional reduction [2].

Theorem 2. There is an $\mathbb{N}^{Q_0}$-graded algebra homomorphism $\psi : \boldsymbol{\mathcal{U}}^+ \to \mathbf{K}(Q, W)$.

We do not have a torsion freeness statement similar to the one of toroidal quantum groups.

Conjecture 3. The map ψ is injective.

Both theorems have analogues in Borel-Moore homology involving Yangians.

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