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Partial Differential Equations (hybrid meeting)

Organized by
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Abstract. The workshop covered topics in nonlinear elliptic and parabolic Partial Differential Equations as well as topics in Geometric Measure Theory, touching topics such as geometric variational problems and minimal surfaces, geometric flows, free boundaries and the structure of nodal sets of eigenfunctions as well as real and complex Monge-Ampère equations.

Mathematics Subject Classification (2010): 35J, 35K, 53C44, 49Q20, 49R, 35R35, 35P, 32W20.

Introduction by the Organizers

The workshop *Partial Differential Equations*, organized by Guido De Philippis (Courant), Richard Schoen (Irvine) and Felix Schulze (Warwick) was held July 26 - July 30, 2021. The meeting was held in hybrid format, with 13 in person and 22 virtual participants. Despite the pandemic the in person participants still had a wide geographic representation. The program consisted of 21 talks, with an evening talk adjusting to the majority of the time zones of the virtual participants and gave sufficient time for discussions among the in person participants. There were also several informal chats sessions including the in person and virtual participants.

Following the workshop tradition, a variety of results concerning the interaction of nonlinear PDE with Geometry have been presented. An application of minimal surface theory in the classification of sufficiently connected manifolds with positive scalar curvature was presented, as well as a result showing that the p-widths of a surface are attained by smooth, immersed geodesics. Furthermore, an approach via Morse theory of moduli spaces of Yang-Mills gauge theories to extend the Bogomolov-Miyaoka-Yau inequality was presented.

Several talks announced advances in the regularity theory for minimal surfaces. This included a talk on boundary regularity for area minimizing hypersurfaces mod(p), the possible degeneration of 7-dimensional minimal hypersurfaces with bounded index as well as an existence theory for hypersurfaces with prescribed mean curvature using the approach via the Allen-Cahn functional. For the anisotropic area functional, a proof of the anisotropic Michael-Simon inequality without using a suitable monotonicity formula was given, as well as a local regularity result for Lipschitz graphs with L^p-bounded anisotropic mean curvature. Furthermore, an extension of classical symmetry results for smooth minimal surfaces to the class of Plateau's surfaces was introduced.

A well established application of non-linear PDE to geometric problems concerns geometric flows. In this context, a construction of a Ricci flow with initial data only in $W^{2,2}$ in four dimensions was presented, as well as an existence theory for Ricci flow on noncompact surfaces with rough initial data together with applications to the existence of special solutions, including expanding solitons and breather solutions. In mean curvature flow, advances in the classification of noncollapsed translators was presented, as well as applications to the structure of asymptotically conical self-shrinkers and an approximation result for weak solutions with only generic singularities by flows with surgery. For higher non-linear flows, a classification of translating flows of power-of-Gauss curvature was given.

In the setting of the Monge-Ampère equation, there were two presentations, one concerning possible singular structure in exterior solutions for the real case, and for complex Monge-Ampère equations higher order estimates for solutions with small fibre size.

Other interactions of PDEs with Geometry included a presentation on an ε-regularity theorem for solutions to the vectorial free boundary problem and the regularity and structure of bilpschitz, quasiconformal, and Sobolev mappings, in the sub-Riemannian setting.

A result in the more classical PDE setting demonstrated the approach using quasiconformal mappings in understanding nodal sets on closed two-dimensional surfaces and a successful application to a solution to Landis' conjecture.

Workshop (hybrid meeting): Partial Differential Equations

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Abstracts

Ricci flow of $W^{2,2}$ metrics in four dimensions

Miles Simon
(joint work with Tobias Lamm)

In this talk we explain how to construct solutions to Ricci flow and Ricci DeTurck flow which are instantaneously smooth but whose initial values are (possibly) non-smooth Riemannian metrics whose components, in smooth coordinates, belong to certain Sobolev spaces.

For a given smooth Riemannian manifold (M, h), and an interval $I \subseteq \mathbb{R}$, a smooth family $g(t)_{t\in I}$ of Riemannian metrics on M is a solution to Ricci DeTurck h Flow if

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} =& g^{ab}({}^h\nabla_a {}^h\nabla_b g_{ij}) - g^{kl} g_{ip} h^{pq} R_{jkql}(h) - g^{kl} g_{jp} h^{pq} R_{ikql}(h) \\ &+ \tfrac{1}{2} g^{ab} g^{pq} \left({}^h\nabla_i g_{pa} {}^h\nabla_j g_{qb} + 2 {}^h\nabla_a g_{jp} {}^h\nabla_q g_{ib} - 2 {}^h\nabla_a g_{jp} {}^h\nabla_b g_{iq} \right. \\ &\left. -2 {}^h\nabla_j g_{pa} {}^h\nabla_b g_{iq} - 2 {}^h\nabla_i g_{pa} {}^h\nabla_b g_{jq}\right), \end{aligned} \tag{1}$$

in the smooth sense on $M \times I$, where here, and in the rest of the paper, ${}^h\nabla$ refers to the covariant derivative with respect to h. A smooth family $\ell(t)_{t\in I}$ of Riemannian metrics on M is a solution to Ricci flow if

$$\frac{\partial \ell}{\partial t} = -2\mathrm{Rc}(\ell) \tag{2}$$

in the smooth sense on $M \times I$. Ricci DeTurck flow and Ricci flow in the smooth setting are closely related : given a Ricci DeTurck flow $g(t)_{t\in I}$ on a compact manifold and an $S \in I$ there is a smooth family of diffeomorphisms $\Phi(t) : M \to M$, $t \in I$ with $\Phi(S) = Id$ such that $\ell(t) = (\Phi(t))^* g(t)$ is a smooth solution to Ricci flow. The diffeomorphisms $\Phi(t)$ solve the following ordinary differential equation:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi^\alpha(x,t) =& V^\alpha(\Phi(x,t),t), \quad \text{for all } (x,t) \in M^n \times I, \\ \Phi(x,S) =& x. \end{aligned} \tag{3}$$

where $V^\alpha(y,t) := -g^{\beta\gamma}\left({}^g\Gamma^\alpha_{\beta\gamma} - {}^h\Gamma^\alpha_{\beta\gamma}\right)(y,t)$

There are a number of papers on solutions to Ricci DeTurck flow and Ricci flow starting from non-smooth Riemannian metric/distance spaces: Given a non-smooth starting space (M, g_0) or (M, d_0), it is possible in some settings, to find smooth solutions $g(t)_{t\in(0,T)}$ to (1), respectively $\ell(t)_{t\in(0,T)}$ to (2) defined for some $T > 0$, where the initial values are achieved in some weak sense. Here is a non-exhaustive list of papers, where examples of this type are constructed : [5, 3, 6, 7, 1, 2, 4]. The initial non-smooth data considered in these papers has certain structure, which when assumed in the smooth setting, leads to a priori estimates for solutions, which are then used to construct solutions in the class being considered. In some papers this initial structure comes from geometric conditions, in others from regularity conditions on the initial function space of the metric components in smooth coordinates. In the second instance, this is usually in the setting, that one has some C^0 control of the metric. That is, the metric is close in the L^∞ sense to the standard euclidean metric in smooth coordinates: $(1-\varepsilon)\delta \leq g(0) \leq (1+\varepsilon)\delta$ for a sufficiently small ε. In this talk, the structure of the initial metric $g(0)$ comes from the assumption, in the four dimensional compact setting, that the components in coordinates are in $W^{2,2}$, and uniformly bounded from above and below : $\frac{1}{a}\delta \leq g(0) \leq a\delta$ for some constant c. Closeness of the metric to δ is not assumed. With this initial structure, we show that a solution to Ricci DeTurck flow exists. In the non-compact setting, we further require that the $W^{2,2}$ norm on balls of radius one is uniformly small and a uniform bound from above and below in the L^∞ sense, both with respect to a geometrically controlled background metric. We also investigate the question of how the initial values are achieved, in the metric and distance sense, as time goes back to zero.

Using this solution $g(t)_{t\in[0,T)}$ to Ricci DeTurck flow, we consider the Ricci flow realted solution $(\Phi(t))^*(g(t))_{t\in(0,T)}$ as defined above, where $\Phi(S) = Id$ for some $S > 0$. The convergence as time goes back to zero in the distance and metric sense is investigated for this Ricci Flow solution. We require some new estimates on convergence in the L^p sense for solutions to Ricci flow, in order to show that there is indeed a limiting weak Riemannian metric, as time approaches to zero. We also show that the initial metric value of the Ricci flow that is achieved is isometric, in a weak sense, to the initial value $g(0)$ of the Ricci DeTurck flow solution.

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Translating flow of power-of-Gauss curvature

Kyeongsu Choi

(joint work with Beomjun Choi, Soojung Kim)

Closed solutions to the flow by super-affine-critical powers of Gauss curvature converge to round points, and those solutions to the flow by the affine-critical power converges to ellipsoids after normalization. However, the flow by sub-affine-critical powers of Gauss curvature generically develops Type II singularities, and the translating flows are the Type II singularity models. Thus, the classification of translators in the sub-affine-critical case is important for the singularity analysis.

Since the translators are the complete graphs of solutions to the Monge-Ampère type equation

$$\det D^2 u = (1+|Du|^2)^{\frac{n+2}{2}-\frac{1}{2\alpha}}, \tag{1}$$

we can classify the translators by establishing a Liouville type theorem for the Monge-Ampère equation.

Entire solutions to (1) diverge at infinity and thus its blow-down $\bar{u}$ satisfy

$$\det D^2 \bar{u} = |D\bar{u}|^{n+2-\frac{1}{\alpha}}. \tag{2}$$

Therefore, its convex dual $\bar{v}$ satisfies

$$\det D^2 \bar{v}(p) = |p|^{\frac{1}{\alpha}-n-2}. \tag{3}$$

Since the sub-affine-critical power α is less than $\frac{1}{n+2}$, $\bar{v}$ solves

$$\det D^2 \bar{v}(p) = |p|^{\beta}, \tag{4}$$

for some β.

In [3], Daskalopoulos and Savin studied asymptotic behaviour of a solution to (4) near the origin in $\mathbb{R}^2$. We can modify their method so that we can show that an entire solution $\bar{v}$ in $\mathbb{R}^2 \setminus \{0\}$ must be a homogeneous function. This implies that the level curves of a translating soliton converges to a closed shrinker to $\frac{\alpha}{1-\alpha}$-curve shortening flow after normalization.

On the other hand, Andrews [1] classified every closed shrinker to the α-curve shortening flow. Therefore, we can find every potential shape of the translator at infinity. Thus, by using the unstable eigenfunctions of the Jacobi operator of the shrinkers, we can construct translators slowly converging to the homogeneous functions with shrinker-level-sets.

Finally, we need to prove that they are the only possible translators to complete the classification. To this end, we first show that the neutral eigenfunctions and the unstable eigenfunctions dominate as in [2]. Then, we exclude the neutral eigenfunctions dominance case by explicit calculations. The unstable eigenfunction dominance case can be done by the classical stable manifold theory.

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An epsilon-regularity theorem for the solutions of a vectorial free boundary problem

Bozhidar Velichkov

(joint work with Francesco Paolo Maiale and Giorgio Tortone)

In this talk we consider a free boundary system arising in the study of a class of shape optimization problems. The problem involves three variables: an open set $\Omega \subset B_1$ and two continuous non-negative functions

$$u : B_1 \to \mathbb{R} \qquad \text{and} \qquad v : B_1 \to \mathbb{R}\,,$$

The functions u and v are harmonic in Ω:

$$\Delta u = \Delta v = 0 \quad \text{in} \quad \Omega, \tag{1}$$

and satisfy the following overdetermined boundary condition on $\partial\Omega$:

$$u = v = 0 \quad \text{and} \quad \frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = 1 \quad \text{on} \quad \partial\Omega \cap B_1. \tag{2}$$

The domain Ω is also part of the system as we assume that:

$$\Omega := \{u > 0\} = \{v > 0\}. \tag{3}$$

Our main result is an ε-regularity theorem for solutions of the system (1)-(2)-(3).

1. Main theorem, viscosity solutions and flatness

1.1. **Viscosity solutions.** We assume that the equations (1) holds in the classical sense, while the boundary condition (2), since we do not assume any a priori regularity of $\partial\Omega$, holds in the viscosity sense proposed by Caffarelli in [1, 2] for the classical two-phase problem. Precisely, we say that u and v satisfy (2) if

at any point $x_0 \in \partial\Omega \cap B_1$, at which $\partial\Omega$ admits a one-sided tangent ball,

the functions u and v can be expanded as

$$u(x) = \alpha\big((x-x_0)\cdot\nu\big)_+ + o\big(|x-x_0|\big) \quad \text{and} \quad v(x) = \beta\big((x-x_0)\cdot\nu\big)_+ + o\big(|x-x_0|\big),$$

where ν is a unit vector and α and β are positive real numbers such that $\alpha\beta = 1$.

1.2. **Definition of flatness.** We adapt the geometric notion of flatness proposed by De Silva in [4] in the context of the one phase problem:

a function u is flat if its graph is flat.

Precisely, we say that u and v are ε-flat in B_1, if there is a unit vector $\nu \in \partial B_1$ and positive constants α and β such that $\alpha\beta = 1$ and for every $x \in B_1$

$$\alpha\left(x\cdot\nu-\varepsilon\right)_+ \le u(x) \le \alpha\left(x\cdot\nu+\varepsilon\right)_+ \qquad \text{and} \qquad \beta\left(x\cdot\nu-\varepsilon\right)_+ \le v(x) \le \beta\left(x\cdot\nu+\varepsilon\right)_+.$$

1.3. **Main theorem.** In [6], we prove the following theorem.

Theorem 1 ([6]). *There is a constant $\varepsilon_0 > 0$ such that the following holds. Let u and v be non-negative continuous functions on B_1 and $\Omega \subset B_1$ be an open set. If u, v and Ω are solutions of (1)-(2)-(3) and if u and v are ε-flat in B_1, for some $\varepsilon \in (0, \varepsilon_0]$, then the free boundary $\partial\Omega$ is a $C^{1,\alpha}$-regular manifold in $B_{1/2}$.*

2. Shape optimization problems

A shape optimization problem is a variational problem of the form

$$\min\Big\{J(\Omega)\ :\ \Omega\in\mathcal{A}\Big\},$$

where $\mathcal{A}$ is an admissible class of subsets of $\mathbb{R}^d$ and J is a given function on $\mathcal{A}$. The shape functionals are often related to models in Engineering, Mechanics and Material Sciences; the most studied ones fall in one of the following classes:

spectral functionals and *integral functionals.*

2.1. **Spectral functionals.** The spectral functionals are functionals of the form

$$J(\Omega)=F(\lambda_1(\Omega),\dots,\lambda_k(\Omega)).$$

where $F:\mathbb{R}^k\to\mathbb{R}$ is a given function and $\lambda_1(\Omega),\dots,\lambda_k(\Omega)$ are the eigenvalues of the Dirichlet Laplacian on Ω. The regularity of the optimal sets for spectral functionals is the question that inspired and motivated most of the research on another free boundary system, the so-called vectorial problem (see [3, 5, 7]), which involves vector-valued functions $U=(u_1,\dots,u_k):B_1\to\mathbb{R}^k$ satisfying

$$\Delta U=0\quad\text{in}\quad\Omega:=\{|U|>0\}\ ,\qquad\sum_{j=1}^k|\nabla u_j|^2=1\quad\text{on}\quad\partial\Omega\cap B_1. \tag{4}$$

2.2. **Integral functionals.** The integral functionals can be written in the form

$$J(\Omega)=\int_D j(u_\Omega,x)\,dx\,, \tag{5}$$

where $j:\mathbb{R}\times D\to\mathbb{R}$ is a given function and u_Ω is the solution of

$$-\Delta u=f\quad\text{in}\quad\Omega\ ,\qquad u\in H^1_0(\Omega),$$

where the right-hand side $f:D\to\mathbb{R}$ is a given function.

The system (1)-(2)-(3) *corresponds to the equation "first variation of J"=0.* Let us formally compute the first variation of J for smooth sets Ω in the case

$$j(x,u)=-g(x)u.$$

Given a smooth a compactly supported vector field $\xi:\mathbb{R}^d\to\mathbb{R}^d$, we consder the sets $\Omega_t:=(Id+t\xi)(\Omega)$ and the solutions $u_t:=u_{\Omega_t}$. The first variation of J is

$$\delta J(\Omega)[\xi]:=\frac{d}{dt}\Big|_{t=0}J(\Omega_t),$$

where u' is the derivative of u_t at $t=0$ and solves the PDE

$$\Delta u'=0\quad\text{in}\quad\Omega\,,\qquad u'=-\xi\cdot\nabla u_\Omega\quad\text{on}\quad\partial\Omega.$$

Let now v_Ω be the solution of the problem

$$-\Delta v_\Omega=g\quad\text{in}\quad\Omega\ ,\qquad v_\Omega\in H^1_0(\Omega).$$

We now integrate by parts in Ω, obtaining

$$-\int_\Omega u' g(x)\,dx = \int_\Omega u'\Delta v_\Omega\,dx = -\int_\Omega \nabla u'\cdot\nabla v_\Omega\,dx + \int_{\partial\Omega} u'\frac{\partial u_\Omega}{\partial n} = \int_{\partial\Omega} u'\frac{\partial u_\Omega}{\partial n}\ .$$

Since the gradient $u_\Omega = 0$ on $\partial\Omega$, we have $\xi\cdot\nabla u_\Omega = (\xi\cdot n)(n\cdot\nabla u_\Omega)$. Thus,

$$\delta J(\Omega)[\xi] = -\int_{\partial\Omega} -\frac{\partial u_\Omega}{\partial n}\frac{\partial v_\Omega}{\partial n}\,(n\cdot\xi)\ .$$

Now, if Ω is a (local) minimizer of J among all sets of fixed Lebesgue measure, then there is a positive constant c such that the state functions u_Ω and v_Ω satisfy

$$\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = c \quad \text{on} \quad \partial\Omega.$$

3. Final remarks and open questions

3.1. **Regularity of the optimal sets.** A regularity theorem for the optimal sets Ω, that minimize the functional (5) among all sets of fixed measure, will be proved in a subsequent paper. The proof strongly relies on Theorem 1 for viscosity solution. The main difficulty is in proving that the functions have the same behavior close to the free boundary (that is, that the optimal sets satisfy a Boundary Harnack Principle); for the vectorial problem, for instance, this was done by showing that the optimal domain is NTA (see [7, 3]).

3.2. **Free boundary systems.** We can write a general free boundary system as

$$\Delta u = \Delta v = 0 \quad \text{in} \quad \Omega = \{u > 0\} = \{v > 0\}\,; \quad G\Big(\frac{\partial u}{\partial n}, \frac{\partial u}{\partial n}\Big) = 0 \quad \text{on} \quad \partial\Omega\cap B_1.$$

In the vectorial case $G(u,v) = u^2 + v^2 - 1$, while for our system, $G(u,v) = uv - 1$.

It is not known at the moment if one can prove an epsilon-regularity theorem for a general function G. For sure, the proofs of Theorem 1 and the epsilon-regularity theorems for the vectorial case strongly rely on the invariance of the function G. In the vectorial case the invariance is rotational, while in our case G is invariant with respect to dilations of the form $(u,v)\mapsto(tu,\frac{1}{t}v)$.

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Approximation of mean curvature flow with generic singularities by smooth flows with surgery

Joshua Daniels-Holgate

Mean curvature flow is the L^2-gradient flow for the area functional. In general, the flow from a hypersurface can develop singularities and there are multiple notions of weak flow that allow for the continuation of the flow past such singularities. An alternate approach is to approximate the flow by a piece-wise smooth flow, known as a mean curvature flow with surgery. The surgery procedure for mean curvature flow from a 2-convex hypersurface of dimension $n \geq 3$ was introduced by Huisken–Sinestrari in [12], and extended to $n = 2$ by Huisken–Brendle [1]. Independently, Haslhofer–Kleiner [10] established a surgery procedure that works for all dimensions $n \geq 2$. It shown independently by Lauer [13] and Head [11] that 2-convex flows with surgery converge in the Hausdorff sense to the weak flow as the surgery parameters are taken to infinity.

The essence of surgery is to 'cut out' regions of high curvature that form as the flow evolves. To be precise, we consider three curvature scales: $H_{\mathrm{trig}} > H_{\mathrm{neck}} > H_{\mathrm{th}}$. The flow is stopped when a point in the flow achieves $H(x,t) = H_{\mathrm{trig}}$. Necks of curvature H_{neck} are identified, cut and replaced with caps, and connected components with $H > H_{\mathrm{th}}$ are then dropped. The flow is then continued from this new hypersurface. In addition to extending the existence of the flow, this provides a simple way for topological information to be tracked, as only finitely many surgeries can occur.

In both methodologies, existence of 2-convex surgery boils down to the classification of regions of high curvature that develop: a canonical neighbourhood theorem for 2-convex flow. Such a theorem classifies regions of high curvature, showing there are always necks on which to perform surgery. Canonical neighbourhoods of neck-pinch singularities for mean curvature flows of dimension $n = 2$ were established in [4] and for $n \geq 3$ in [5], as a corollary to their resolution of the mean convex neighbourhood conjecture. It is from this result that we can extend the notion of a flow with surgery to flows with spherical and neck-pinch singularities.

Spherical and generalised cylindrical singularities were conjectured by Huisken to be 'generic'. The pioneering work of Colding–Minicozzi, [6, 7, 8] showed spherical and generalised cylindrical singularities are the only linearly stable singularity models. Recently, Chodosh–Choi–Mantoulidis–Schulze, [2, 3], showed that flows with spherical and neck-pinch singularities occur generically in $\mathbb{R}^4$ amoungst initial conditions with entropy less than that of $\mathbb{S}^1 \times \mathbb{R}^2$. Such results provide a strong motivation for establishing a flow with surgery for such flows.

In [9], we detail the construction of a flow with surgery for unit-regular cyclic (mod 2) Brakke flows with only spherical and neck-pinch singularities. Further, we establish a convergence result that relate the behaviour of the flows with surgery back to the original flow in a smooth sense.

Theorem 1 (Existence)**.** *For any unit-regular, cyclic (mod 2) Brakke flow with only spherical and neck-pinch singularities starting from a smooth closed hypersurface $M^n \subset \mathbb{R}^{n+1}$, there are parameters $H_{\min}, \Theta < \infty$ such that if $H_{\text{th}} > H_{\min}$ and $H_{\text{trig}}/H_{\text{neck}}, H_{\text{neck}}/H_{\text{th}} > \Theta$, then there is a smooth flow with surgery with these parameters that exists until it vanishes completely.*

The following result is analogous to that of Lauer and Head [13, 11]

Theorem 2 (Hausdorff Convergence)**.** *Taking the limit as $H_{\text{th}} \to \infty$, the surgical flows converge in the Hausdorff sense to the level set flow from M.*

Finally, we improve the convergence result. Such a convergence result shows makes rigorous the notion of the surgery flows approximating the weak flow.

Theorem 3 (Smooth Convergence)**.** *Away from the singular set the convergence is smooth.*

To highlight why existence of a surgical flow is non-trivial consider a hypersurface, M, whose mean curvature flow has only spherical and neck-pinch singularities, and a single neck-pinch singularity at the first singular time. With the canonical neighbourhood theorems of [4, 5] in mind, one can follow the arguments of [10] to pick surgery parameters suitable for surgical modifications to be made at some time before the flow become singular. Such a process would construct a new hypersurface M'. One immediately runs into a problem: without assuming global 2-convexity, we do not have any knowledge of how the flow from M' will proceed. In the worst case, it may run into non-generic singularities.

To overcome these difficulties, we develop a technical framework that allows us to pass to limits locally. In fact, this approach to the problem is such that once we show existence of a flow with surgery, the convergence results are achieved for free.

The framework boils down to two ingredients:

(1) Construction of barriers. We show that for any equidistant hypersurfaces $M_{\pm\varepsilon}$ to the initial condition M, the 'thick scale' H_{th} can be chosen, depending on $\varepsilon > 0$, such that the flow with surgery avoids the flow from these hypersurfaces. At smooth times, this follows from the standard avoidance principle for mean curvature flow. At surgery times, we use the geometric observation that connected components with $H > H_{\text{th}}$, and necks with $H > H_{\text{th}}$ are 'too small' for the barrier flows to be present in the interior, and thus surgeries don't decrease the distance of the flow with surgery from the barrier flows. This is sufficient to get Hausdorff convergence. See Lauer [13].

(2) Following paths in the regular set. The connectedness of the regular set for flows with spherical and neck-pinch singularities was established in [2]. Following paths in regular set contained in a canonical neighbourhood, we use standard tools of mean curvature flow (namely, psuedolocality, Ecker–Huisken's graphical estimate and Haslhofer–Kleiner's gradient estimate) to establish convergence of the surgery flows back to the original weak flow

in a small neighbourhood of the path. Indeed, we show no surgeries can occur in a neighbourhood of a given regular point provided the surgeries are done at a sufficiently large scale.

These tools allow us to establish a stability result, showing that the flows with surgical modifications cannot 'stray' to far from the flow we wish to approximate, allowing for further surgical modifications.

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Some new applications of the MCF to asymptotically conical self shrinkers

Alexander Mramor

The mean curvature flow (abbreviated MCF) is where one deforms a submanifold by its mean curvature vector. It is a natural flow to consider applying to problems in topology concerning spaces of submanifolds (such as the Schoenflies and Smale conjectures and generalizations thereof) and on the applied side has relations to material science and computer imaging. As can be seen a number of ways, given sufficiently regular initial submanifold $M_0 \subset N$ of an ambient space N the existence of such a deformation M_t, $t > 0$ is equivalent to finding a solution to a nonlinear heat equation; there are many general results which give at least the short time existence/uniqueness/regularity of the mean curvature flow. However, the maximum principle can be used to show that the mean curvature flow in many cases will develop a singularity by some time $T < \infty$.

There are a number of interrelated notions of weak solutions to the MCF which can be used to continue the flow through singularities. However, if one wishes to use the flow to study the initial data (for instance in applications of the flow to topology) a detailed understanding of the singularities which occur is necessary because otherwise some relevant information could be "lost" in the singular set. For example, there are known potential singularity models of nontrivial topology. To study the singularities, a rescaling procedure is used and when one rescales about a fixed point one finds a so-called self shrinker $\Sigma \subset \mathbb{R}^N$ after a suitable normalization by Husken's monotonicty formula [7] – these correspond to flows which move by contractions given by $t \to \sqrt{-t}\Sigma$, where $t \in (-\infty, 0)$.

Outside of some convexity conditions or entropy bounds (in the sense of Colding and Minicozzi, see [5]), the space of self shrinkers is generally poorly understood and there are many examples showing the space is complicated. As an aside we note that a way to avoid dealing with this overabundance of singularity models to develop the theory of generic mean curvature flow initiated in [5], which roughly speaking is a program to show that for generic initial data the only shrinkers which occur are modeled on shrinkers of the form $S^k_{round} \times \mathbb{R}^{n-k}_{flat}$ (naturally, typically called generic shrinkers); in [3, 4] this has been developed within the dimensions, entropy assumptions, and conjectural framework we consider below however there are cases where one might imaginably need to deal with nongeneric shrinkers as well, besides the intrinsic interest in their study.

Equivalent to the definition above, they are minimal surfaces in the Gaussian metric $G_{ij} = e^{\frac{-|x|^2}{2n}}\delta_{ij}$, which (when they are smooth) implies they satisfy in the hypersurface case $H - \frac{x^\perp}{2} = 0$. It turns out that under very broad conditions, self shrinkers are unstable. This can be seen as a manifestation of the fact that the Gaussian metric is f-Ricci positive in the sense of Bakry and Emery – such metrics generally satisfy many of the same properties as true Ricci positive metrics. Hence in principle they can be perturbed to find a nearby shrinker mean convex surface (that is, one which satisfies $H - \frac{x^\perp}{2} > 0$) by the first eigenfunction of the Jacobi operator. Shrinker mean convexity under some decay assumptions is preserved under the renormalized MCF, which is related to the regular mean curvature flow by a coordinate change and is given by deforming the surface by the mean curvature vector plus the position vector. Mean convex MCFs satisfy very good properties and these carry over to shrinker mean convex flows (in fact they even satisfy more), which suggests that it could be profitable to study a self shrinker by considering the flow of shrinker mean convex perturbations of it. Indeed this strategy has been applied before in [6, 1, 2] amongst other works and is used to show the results below, hence the title of this extended abstract.

A (relatively minor) analytic novelty of the flows we use is that singularity formation is not ruled out by, say, an entropy assumption; we must check a number of technical properties hold through singular times and because we are most often in a noncompact setting below these doesn't immediately follow from off the shelf results. The key fact about shrinker mean convex MCFs M_t we leverage (under some very mild assumptions) is that they must clear out in that in any bounded region B there is some time T for which $M_t \cap B = \emptyset$ for $t > T$. To see this, White in [16] showed that the nonempty limit of such a flow (in dimensions less than 7) must be a smooth stable minimal surface, which in our case violates the aforementioned instability of self shrinkers (it would also violate the Frankel property, another Ricci positive type property enjoyed by the Gaussian metric). We now state our first result, shown in [10] which partially extends a previous joint work with S. Wang [13] to the noncompact setting:

Theorem 1. *Let $M^2 \subset \mathbb{R}^3$ be a two-sided, possibly noncompact, self shrinker with finite topology and no more than one end. Then if M has an asymptotically conical end or is compact, it is topologically standard.*

The idea to show this result is to show that M must be a Heegaard splitting. Then one may apply Waldhausen's theorem [15] to see that M is topologically standard which essentially says that M is embedded in the simplest possible way (for instance, a standardly embedded torus is isotopic to a tubular neighborhood of an unknotted circle). To show that M is a Heegaard splitting we appeal to a "π_1 surjectivity criterion" taken from [8, 9], which says essentially that if the universal cover of either bounded component of $\mathbb{R}^3$ by M has connected boundary then it is a Heegaard splitting. We suppose this doesn't hold true for some bounded component and perturb in a shrinker mean convex way and flow into it. By an intersection number argument we can then show the flow doesn't clear out, which gives a contradiction to the previous paragraph.

In the conjectural picture all one ended self shrinkers of finite topology are asymptotically conical. It is a useful assumption for instance in actually constructing the first eigenfunction to perturb by – see [1], proposition 4.1. In fact this is part of the reason why we only consider asymptotically conical shrinkers in these results. Geometrically speaking, the picture in the sketch above is that the flow of (the perturbation of) M gets "snagged" in case M is knotted/topologically nonstandard. In [11] we used this rough idea along with the eventual star-shapedness of shrinker mean convex flows to show some unknottedness results in situations where Waldhausen's theorem doesn't apply. Now we state our next result, shown in [12], where Λ_k is the Colding-Minicozzi entropy of the round k-sphere:

Theorem 2. *Suppose $M^3 \subset \mathbb{R}^4$ is a smooth 2-sided asymptotically conical self shrinker with entropy less than Λ_1 and k ends. Then it is diffeomorphic to S^3 with k 3-balls removed and replaced with k copies of $S^2 \times \mathbb{R}_+$ attached along their respective boundaries. If $k = 1$ then $M \simeq \mathbb{R}^3$ and in particular this is the case when $\lambda(M) \leq \Lambda_2$.*

This generalizes a previous joint work again with S. Wang in [14], where we considered closed self shrinkers (in any dimension) under essentially this entropy bound as well as a result of Bernstein and L. Wang shown in [2] for noncompact shrinkers in $\mathbb{R}^4$ where they used a stronger bound. The primary difficulty here is that the link might have nontrivial topology, which because of the noncompactness its topology is less amenable to being "detected" by the flow. The most difficult case is if there is a homotopically nontrivial curve γ in the link which is in fact homotopically trivial in M. In this case, one can use Dehn's lemma along with the entropy assumption to basically show that γ stays homotopically trivial in M_t, which eventually leads to a contradiction to the clearing out property of the flow for a properly chosen perturbation and subsequent flow.

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Existence of hypersurfaces with prescribed mean curvature

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(joint work with Neshan Wickramasekera)

We report on [4], where (largely by PDE methods) we establish:

Theorem 1. *Let (N, h) be a compact Riemannian manifold of dimension $n+1$, $n \geq 2$, and let $g \geq 0$ be a Lipschitz function on N. There exists an immersed hypersurface $M \subset N$ of class C^2 such that (i) M is two sided, i.e. there is a global choice of unit normal ν, (ii) $dim_{\mathcal{H}}(\overline{M} \setminus M) \leq n-7$, where $dim_{\mathcal{H}}$ denotes the Hausdorff dimension, (iii) the mean curvature vector of M is given by $g\nu$. More* *precisely, M is quasi-embedded, i.e. for every $p \in M$ around which M fails to be embedded, there exists a neighbourhood $\mathcal{N}_\rho(p) \subset N$ such that $M \cap \mathcal{N}_\rho(p) = D_1 \cup D_2$, where D_1 and D_2 are embedded C^2 disks lying on one side of each other and intersecting tangentially (p is contained in the intersection).*

Remark 1. M locally satisfies a quasi-linear elliptic PDE, the prescribed mean curvature (PMC) equation. By Schauder theory, M is then $C^{2,\alpha}$, for every $\alpha \in (0,1)$; moreover, if $g \in C^{k,\alpha}(N)$ for $k \geq 1$, then M is of class $C^{k+2,\alpha}$.

Remark 2. When $n \leq 6$, M is closed. The failure of M to be closed can only arise for $n \geq 7$ and is caused by the possible presence of a set of 'singular points'. A point p is singular if $\overline{M}$ fails to be immersed in any neighbourhood of p. (It follows from the proof that $\overline{M}$ admits only non-planar tangent cones at singular points.)

The possible presence of a "small" singular set for $n \geq 7$ is expected, since such set appears already for the long-studied class of area-minimisers, as well as in the previously known case $g \equiv 0$ (minimal hypersurfaces) of Theorem 1. The possible singular set (and the closely related lack of curvature estimates) introduces an extra level of difficulty in the construction of the PMC hypersurface, specifically in the (essential) development of a suitable regularity and compactness theory.

The case $g \equiv 0$ of Theorem 1 has a long and fruitful history. It was proved for $n \leq 5$ in the combined works of Almgren [1] and Pitts [8], relying on the curvature estimates of Schoen–Simon–Yau [11], and in arbitrary dimension thanks to the regularity and compactness theory developed by Schoen–Simon [10]. The construction was carried out by means of what is nowadays called the 'Almgren–Pitts minmax method'. The latter finds the minimal hypersurface as a stationary point for the area functional (area refers to the n-dimensional measure), through a mountain pass construction in a suitable space of 'integral varifolds' (weak notion of submanfiolds). This method has been further extended and refined in the last decade (notably starting with the proof of the Willmore conjecture by Marques–Neves) and has led, among other results, to the establishment of Yau's conjecture, stating that a compact Riemannian manifold of dimension between 3 and 7 admits infinitely many closed minimal hypersurfaces (Marques–Neves, Song). A drawback of the Almgren–Pitts approach is that the relevant functional, area, is defined on a non-linear space and does not satisfy a Palais–Smale condition; substantial technical machinery is required to compensate for that.

A new, more direct proof for the case $g \equiv 0$ of Theorem 1 has been given in recent years, by means of what we will refer to as 'Allen–Cahn minmax'. This was carried out by Guaraco [6] relying on works by Hutchinson, Tonegawa, Wickramasekera [7] [12] [13]. The idea is to replace the area functional with a regularised version $\mathcal{E}_\epsilon$ of it, defined for $\epsilon > 0$ on the Hilbert space $W^{1,2}(N)$: $\mathcal{E}_\epsilon(u) = \frac{1}{2\sigma}\left(\int_N \epsilon\frac{|\nabla u|^2}{2} + \int_N \frac{W(u)}{\epsilon}\right)$, where $W : \mathbb{R} \to [0,\infty)$ is a C^2 'double well' potential, i.e. with two nondegenerate global minima at -1 and $+1$, with $W(\pm 1) = 0$, and $\sigma > 0$ is a normalising constant (determined by W). Employing this energy goes back to a deep insight of De Giorgi, developed by Modica–Mortola and others, who realised that "minimisers of $\mathcal{E}_\epsilon$ converge to area-minimisers as $\epsilon \to 0$"

and pioneered the framework known as Γ-convergence. In [7] it is shown that, more generally, "critical points of $\mathcal{E}_\epsilon$ converge to critical points of area (stationary integral varifolds) as $\epsilon \to 0$". In [6] a minmax construction is carried out for $\mathcal{E}_\epsilon$, for each $\epsilon > 0$, in $W^{1,2}(N)$, capitalising on the validity of the Palais–Smale condition (so a standard mountain pass lemma from classical PDE theory can be brought to bear); then a suitable limit as $\epsilon \to 0$ yields a stationary integral varifold ([7]); its smooth embeddedness away from a set of dimension $\leq n-7$ is obtained thanks to the regularity theory in [12], [13].

In [4] we prove Theorem 1, at first for $g > 0$, $g \in C^{1,1}(N)$, by means of an Allen–Cahn minmax. The (straightforward) starting point is to carry out a classical mountain pass construction in $W^{1,2}(N)$, for each $\epsilon > 0$, for the energy $\mathcal{F}_{\epsilon,g}(u) = \mathcal{E}_\epsilon(u) - \frac{1}{2}\int gu$. Unlike in the case $g \equiv 0$, however, it is not necessarily true that "critical points of $\mathcal{F}_{\epsilon,g}$ converge to hypersurfaces with mean curvature g as $\epsilon \to 0$". The latter statement is true only if the limit (which is an integral varifold, still by [7]) appears with multiplicity 1 (a.e.). On the other hand, simple examples show that critical points of $\mathcal{F}_{\epsilon,g}$ may converge, as $\epsilon \to 0$, to hypersurfaces with (even) multiplicity higher than 1, and with vanishing mean curvature (i.e. minimal). This threatens the success of the Allen–Cahn minmax strategy.

We successfully prove the theorem by first showing a *regularity result*, that applies to any integral varifold (in any Riemannian manifold of dimension 3 or higher) that arises as limit (subsequential, as $\epsilon \to 0$) of critical points of $\mathcal{F}_{\epsilon,g}$ with $g > 0$, $g \in C^{1,1}(N)$, with (locally) equibounded Morse index. This result proves that the varifold in question is either (i) a minimal hypersurface M_0 with locally constant even multiplicity and with a singular set $\overline{M_0} \setminus M_0$ of dimension $\leq n-7$; or (ii) a quasi-embedded two-sided hypersurface M_g with multiplicity 1 and mean curvature $g\nu$ (where ν is a choice of unit normal) and with singular set $\overline{M_g} \setminus M_g$ of dimension $\leq n-7$; or (iii) a union of a minimal hypersurface as in (i) and a prescribed-mean-curvature hypersurface as in (ii). This regularity result is based on the regularity and compactness framework that we developed in [2], [3], combined with Röger–Tonegawa's work [9] and standard tools from GMT and from quasi-linear elliptic PDEs. As a consequence of our regularity result, in the case $g > 0$, $g \in C^{1,1}(N)$, we may carry out the minmax (leading for each ϵ to a critical point of $\mathcal{F}_{\epsilon,g}$ with Morse index ≤ 1), send (subsequentially) $\epsilon \to 0$, and pick M_g to be the desired hypersurface (M in Theorem 1) *unless* we are in case (i), that is, unless the integral varifold obtained is a minimal hypersurface M_0 with even multiplicity. If the latter happens, we suitably perturb M_0 and use the resulting hypersurface to build (for each ϵ) a function that we use as initial data for a negative gradient flow for $\mathcal{F}_{\epsilon,g}$. The minmax characterisation of M_0 and the specific choice of its perturbation guarantee that (a) the flow is mean convex and thus converges to a stable critical point v_ϵ of $\mathcal{F}_{\epsilon,g}$; (b) any integral varifold obtained (subsequentially) from v_ϵ as $\epsilon \to 0$ cannot be completely minimal, hence (we can apply the regularity result discussed above to this varifold) is of the type (ii) or (iii) above. We can then select the prescribed-mean-curvature component of this varifold. This establishes Theorem 1 in the case $g > 0$, $g \in C^{1,1}(N)$.

Then a (fairly simple) approximation argument (based on $C^{2,\alpha}$ estimates that appear in the regularity result for $g > 0$, $g \in C^{1,1}(N)$) leads to Theorem 1 in full.

When $g \equiv$ cnst, Theorem 1 gives the existence of ("closed") constant-mean-curvature hypersurfaces for a given value of the mean curvature. When $2 \leq n \leq 6$ and $g \equiv$ cnst, and when $2 \leq n \leq 6$ and $g : N \to \mathbb{R}$ is C^∞ and satisfies a constraint on the set $\{g = 0\}$, the Almgren–Pitts minmax method has been employed in [14], [15] to reach the existence result. The dimensional restriction permits a short-cut in the relevant regularity/compactness arguments, as curvature estimates ([11]) are available. The extension of the Almgren–Pitts approach of [14] to $n \geq 7$ has been carried out in [5], employing the regularity/compactness theory of [2], [3].

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Morse theory and the Bogomolov–Miyaoka–Yau inequality

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(joint work with Thomas G. Leness)

1. Abstract

We review the classical Bogomolov–Miyaoka–Yau inequality for complex surfaces, summarize prior attempts to prove the inequality for a broader class of four-dimensional smooth manifolds, and outline our own approach to prove the inequality for four-dimensional smooth manifolds of Seiberg–Witten simple type, which includes symplectic manifolds.

1.1. Bogomolov–Miyaoka–Yau inequality for complex surfaces.

We begin by recalling the well-known

Theorem 1 (Bogomolov–Miyaoka–Yau inequality for complex surfaces of general type)**.** *(See Miyaoka* [11, Therem 4] *and Yau* [17, Theorem 4]*.) If X is a compact, complex surface of general type, then*

$$c_1(X)^2 \leq 3c_2(X). \tag{1}$$

Theorem 1 was proved by Miyaoka [11] using methods of algebraic geometry and a weaker version was proved by Bogomolov [1]. Yau proved inequality (1) in a slightly more restricted setting, using methods of non-linear partial differential equations to solve an equation of complex Monge–Ampére type.

1.2. Bogomolov–Miyaoka–Yau inequality for four-manifolds.

Our submitted monograph [2] contains first steps towards a proof of the

Conjecture 2 (Bogomolov–Miyaoka–Yau inequality for smooth four-dimensional manifolds with non-zero Seiberg–Witten invariants)**.** *If X is a closed, four-dimensional, oriented, smooth manifold with $b_1(X) = 0$, odd $b^+(X) \geq 3$, and Seiberg–Witten simple type with a non-zero Seiberg–Witten invariant, then* (1) *holds.*

If X obeys the hypotheses of Conjecture 2, then it has an almost complex structure J [10] and in the inequality (1), the Chern classes are those of the complex vector bundle (TX, J). Conjecture 2 is based on [12, Problem 4], though often stated for four-dimensional, simply connected symplectic manifolds — see Gompf and Stipsicz [7, Remark 10.2.16 (c)] or Stern [12, Problem 2]. Taubes [14, 15] proved that four-dimensional, symplectic manifolds have non-zero Seiberg–Witten invariants, generalizing Witten's result for Kähler surfaces [16], while Kotschick, Morgan, and Taubes [9] and Szabó [13] proved existence of four-dimensional, non-symplectic, smooth manifolds with non-zero Seiberg–Witten invariants.

1.3. **SO(3) monopoles and the Bogomolov–Miyaoka–Yau conjecture.** We shall summarize the approach described in our submitted monograph [2] with Leness to prove Conjecture 2 and describe some of our results proved in that monograph.

Our strategy to prove Conjecture 2 uses a new approach to Morse–Bott theory, which we call *virtual Morse–Bott theory*, that applies to singular analytic spaces that typically arise in gauge theory — including moduli spaces of SO(3) monopoles over closed smooth four-manifolds, stable holomorphic pairs of bundles and sections over closed complex Kähler surfaces, and moduli spaces of Higgs pairs over closed Riemann surfaces. Such moduli spaces over complex Kähler surfaces or Riemann surfaces are *complex analytic spaces*, equipped with Kähler metrics and Hamiltonian functions for circle actions. When the smooth four-manifold is almost Hermitian (as are four-manifolds of Seiberg–Witten simple type) — where the almost complex structure is not necessarily integrable and the fundamental two-form defined by the almost complex structure and Riemannian metric is not necessarily closed — one can still show that the moduli space of SO(3) monopoles is almost Hermitian [4]. Such almost Hermitian moduli spaces are real analytic spaces and carry a circle action compatible with the almost complex structure and Riemannian metric and a corresponding Hamiltonian function to which our virtual Morse–Bott theory applies. Our development of Morse theory extends one due to Hitchin in his study of the moduli space of Higgs pairs over Riemann surfaces [8].

In [2, Theorem 1], we give an extension of Frankel's Theorem [6, Section 3] for Hamiltonian functions for circle actions on complex Kähler manifolds by allowing smooth almost Hermitian manifolds; this allows us to compute eigenvalues of the Hessian of the Hamiltonian function in terms of weights of circle actions. We introduce the concept of a *virtual Morse–Bott index* for the Hamiltonian function of a circle action on a complex analytic space. We prove in [3] that positivity of the virtual Morse–Bott index of a critical point implies that it cannot be a local minimum.

In [2, Section 1.3], we explain our strategy to use virtual Morse–Bott theory on moduli spaces of SO(3) monopoles to prove Conjecture 2 by establishing existence of *anti-self-dual Yang–Mills connections* on certain complex rank two, Hermitian vector bundles E over X and which attained (by gradient flow, for example) as absolute minima of the Hamiltonian function.

Our monograph [2] is a first step towards a proof of Conjecture 2, where we begin by validating our strategy in the case of complex Kähler surfaces where Conjecture 2 holds. Our [2, Theorem 5] shows that points in these moduli spaces are *critical points* of the Hamiltonian function (the square of the L^2 norm of the spinor section, by analogy with Hitchin's definition in terms of the Higgs field in [8, Section 7]) if and only if they represent *Seiberg–Witten monopoles* or *anti-self-dual connections*. Our [2, Theorem 4] shows that one can always choose moduli spaces of SO(3) monopoles to implement our strategy to prove Conjecture 2. By analogy with results of Hitchin [8, Proposition 7.1], our [2, Theorem 4] indicates that the virtual Morse–Bott indices of points representing Seiberg–Witten monopoles are *positive* and thus cannot be local minima while points representing anti-self-dual connections are absolute minima, where the Hamiltonian function is zero and their existence is detectable via Morse theory. Our [2, Theorem 6] gives the virtual Morse–Bott index of the Hamiltonian function for a point represented by a Seiberg–Witten monopole, computed using the Hirzebruch–Riemann–Roch Index Theorem.

In our article [4] in preparation, we extend our results in [2] from complex Kähler surfaces to smooth four-manifolds that have $b_1 = 0$, odd $b_+ \geq 3$, and Seiberg–Witten simple type. Additional articles towards a proof of Conjecture 2 are in preparation with Leness and Richard Wentworth.

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Surviving without monotonicity: anisotropic Michael-Simon inequality

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(joint work with Frederick J. Almgren and Guido De Philippis)

While the variational theory for the area functional has by now a huge literature, with several cornerstone results, much less is known for a simple generalization of it, called *anisotropic energy*. Given a Riemannian ambient (M^{n+k}, g) and a fixed smooth positive function F on the Grassmannian bundle of n-planes, we define the anisotropic energy $F(\Sigma)$ to be

$$F(\Sigma) := \int_\Sigma F(T_x\Sigma)\, d\mathcal{H}^n(x),$$

for any compact n-submanifold $\Sigma \subset M$. Note that, for $F = 1$, we recover the usual n-dimensional area. In the sequel, we deal with codimension $k = 1$ and choose $M = \mathbb{R}^{n+1}$, for simplicity. Writing the Grassmannian bundle as $\mathbb{R}^{n+1} \times \mathrm{Gr}_n(\mathbb{R}^{n+1})$, we also assume that F depends only on the second variable.

The *monotonicity formula* for the area functional states that, if Σ is minimal, i.e., critical for the area in $\mathbb{R}^{n+1} \setminus \partial\Sigma$ (with respect to ambient deformations), then

$$r \mapsto \frac{\mathcal{H}^n(\Sigma \cap B_r(x))}{r^n}$$

is increasing for any $x \notin \partial\Sigma$, on the range $r \in (0, \mathrm{dist}(x, \partial\Sigma))$. More precise versions allow for an arbitrary mean curvature H, which represents precisely the *first variation* of the area ($H = 0$ in the minimal case), and hold also across the boundary $\partial\Sigma$ (which can be regarded as a singular part of the so-called *generalized mean curvature* in geometric measure theory).

The monotonicity formula, which also holds for other important functionals enjoying enough symmetry, such as the Dirichlet energy for maps between Riemannian manifolds, the Yang–Mills energy for connections, etc., is a very basic tool which is used crucially in the proof of a number of fundamental facts. For instance, it implies the *upper semicontinuity of the support* in the limit, for a sequence of stationary varifolds (a weak notion of minimal submanifold) assuming a lower density bound. It also implies the *compactness* of stationary rectifiable and integral varifolds (under local bounds on mass and first variation) and the existence of tangent cones. The first two consequences are particularly important in soft arguments by compactness and contradiction.

However, for anisotropic energies, it was shown by Allard [1] that we cannot hope for anything resembling too closely the monotonicity formula for the area. Given that, we have to look for completely different ways to obtain the aforementioned facts in the anisotropic setting.

In this talk we discuss the anisotropic version of the Michael–Simon inequality, which is a more robust fact, usually stated for functions defined on a submanifold. Namely, given $\Sigma^n \subset \mathbb{R}^{n+1}$ and a nonnegative function $f : \Sigma \to \mathbb{R}$, it states that

$$\Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \le C(n) \int_\Sigma (|df| + f|H|) + C(n) \int_{\partial\Sigma} f$$

(for $n \geq 2$). It resembles the classical Sobolev inequality in the Euclidean space, except for the extra term involving the mean curvature H of Σ (and the boundary of Σ, which can be regarded as a singular part of H). Thus, Sobolev's inequality holds with a constant independent of Σ, provided we add the term $\int_N f|H|$ on the right-hand side.

As in the case of Sobolev's inequality (for the exponent $p = 1$), this inequality is equivalent to its validity for the special case where $f = 1$, in which case the statement becomes

$$\mathcal{H}^n(\Sigma)^{\frac{n-1}{n}} \leq C(n) \int_\Sigma |H| + C(n)\mathcal{H}^{n-1}(\partial\Sigma). \tag{1}$$

While the initial proof of the Michael–Simon inequality deduced it from the monotonicity formula [5], other proofs are available nowadays: we quote, for instance, the work [3], using methods which are connected to optimal transport ideas.

In turn, using (1), we can recover immediately a lower bound of the form

$$\mathcal{H}^n(\Sigma \cap B_r(x)) \geq c(n)r^n \tag{2}$$

for a minimal hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ and a point $x \in \Sigma$ (assuming $B_r(x) \cap \partial\Sigma = \emptyset$), by applying (1) to all the truncations $\Sigma \cap B_s(x)$ (for $0 < s < r$) and integrating the resulting differential inequality. This argument immediately gives the upper semicontinuity of the support for sequences of stationary varifolds in the anisotropic setting.

With slightly more work, using the anisotropic counterpart of the rectifiability theorem, proved by De Rosa–De Philippis–Ghiraldin [4], and earlier work by Allard [2], we also recover the compactness of rectifiable and integral varifolds.

Finally, let us mention that the lower area bound (2) is the main missing ingredient in order to extend Allard's small-excess-regularity result for stationary integral varifolds to the anisotropic situation. It should be noted that the quoted papers require an ellipticity condition on F, namely (qualitative or quantitative) *strict convexity* of F once we identify it with an even function on the sphere S^n and extend it in a 1-homogeneous way to $\mathbb{R}^{n+1}$.

Our main result is the following.

Theorem. *For $n = 2$ the Michael–Simon inequality* (1) *holds, provided that (the 1-homogeneous extension of) F is convex and that F is close to 1 (the area functional) in the C^1 topology. More precisely, we have*

$$|V|(\mathbb{R}^3) \leq C(F)\sqrt{\mathcal{H}^2\{\Theta > 0\}}|\delta_F V|(\mathbb{R}^3) \tag{3}$$

for any rectifiable 2-varifold V in $\mathbb{R}^3$ with finite total mass $|V|(\mathbb{R}^3)$ and finite total first variation $|\delta_F V|(\mathbb{R}^3)$ with respect to F. Here Θ denotes the density, which exists $|V|$-a.e.; thus, $\mathcal{H}^2\{\Theta > 0\}$ is the area of the rectifiable set supporting V.

Note that when $\Theta \geq 1$ a.e. we can bound $\mathcal{H}^2\{\Theta > 0\} \leq |V|(\mathbb{R}^3)$ and deduce that

$$|V|(\mathbb{R}^3)^{1/2} \leq C(F)|\delta_F V|(\mathbb{R}^3), \tag{4}$$

which is the same as (1). Compared to (4), inequality (3) is both scale invariant and homogeneous in V.

This result (actually, the slightly weaker version (4)) was initially proved in a long set of posthumous notes by Frederick J. Almgren. We are grateful to his wife, Jean Taylor, for sharing them with me and Guido De Philippis. In collaboration with him, we greatly simplified his proof (although retaining some important ideas).

The initial step is to project the situation onto a plane, e.g. $\mathrm{span}\{e_1, e_2\}$. On this plane, identified with $\mathbb{R}^2$, the boundedness of the first variation becomes the fact that

$$\operatorname{div} A \in L^1, \quad \|\operatorname{div} A\|_{L^1} \leq C(F)|\delta_F V|(\mathbb{R}^3)$$

for a suitable matrix $A \in \mathbb{R}^{2\times 2}$ (varying with the point in $\mathbb{R}^2$ and supported on the projection of $|V|$) which can be assumed to have nonnegative diagonal entries, after a special change of coordinates (this is where convexity plays a role).

The main difference with respect to Almgren's presentation is the use of the following inequality, which is possibly new.

Theorem. *Given two vector fields* S, $T \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ *on the plane, with* S^x, $T^y \geq 0$ *and* $\det(S,T) \geq 0$*, we have*

$$\int_{\mathbb{R}^2} \det(S,T) \leq \frac{1}{4} \int_{\mathbb{R}^2} |\operatorname{div} S| \int_{\mathbb{R}^2} |\operatorname{div} T|. \tag{5}$$

It is important to note that the nonlinear conditions on S and T cannot be dropped. Essentially, S^x, $T^y \geq 0$ guarantees that the integral curves of the vector fields do not form loops, while $\det(S,T) \geq 0$ guarantees that they intersect at most once. Inequalities like (5) are closely related to the so-called *multilinear Kakeya inequality* in harmonic analysis, which is an open problem for non-straight tubes in dimension higher than two.

The inequality (5) immediately implies the classical isotropic Michael–Simon inequality for 2-varifolds (in the version (4)), once applied to the rows of A, while for anisotropic energies some additional ideas are required.

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Singular structures in exterior solutions to the Monge-Ampère equation

Connor Mooney

The Monge-Ampère equation

$$\det D^2 u = 1 \text{ in } \Omega \subset \mathbb{R}^n,\ u \text{ convex} \tag{1}$$

arises in problems involving prescribed Gauss curvature, Kähler geometry, and optimal transport. Global solutions of (1) are necessarily quadratic polynomials. This was proven by Jörgens [9] in dimension $n = 2$, Calabi [4] in dimensions $n \leq 5$, and Pogorelov [16] in all dimensions. The *local* behavior of solutions to (1) is delicate. In dimension $n = 2$ solutions to (1) are smooth and enjoy pure interior C^2 estimates, as in the case of the Laplace equation [7]. This is not true in higher dimensions. Pogorelov constructed singular solutions to (1) in dimension $n \geq 3$ of the form $u(x',\, x_n) = |x'|^{2-\frac{2}{n}} f(x_n)$, $|x_n| < 1$, where f is positive, analytic and uniformly convex [17]. The key feature of this example is the presence of a line segment in the graph of u. This is in fact the only obstruction to the interior smoothness of solutions to (1): strictly convex solutions are smooth. The complete proof of the latter fact has a long history that includes important contributions from Calabi [4], Pogorelov [17], Cheng-Yau [5], Lions [11], Caffarelli-Nirenberg-Spruck [3], Evans [6] and Krylov [10].

In applications, the intermediate situation of "exterior solutions" arises, namely:

$$\det D^2 u = 1 \text{ in } \mathbb{R}^n \backslash \Sigma,\ \Sigma \text{ compact.} \tag{2}$$

For example, the potential u of the optimal transport map which takes the uniform measure plus a sum of M Dirac masses of sizes $\{a_i\}_{i=1}^M$ centered at points $\{p_i\}_{i=1}^M$ to the uniform measure (that is, ∇u "spreads" the Dirac masses out in a cost-minimizing fashion) solves $\det D^2 u = 1 + \sum_{i=1}^M a_i \delta_{p_i}$. In this case $\Sigma = \{p_i\}_{i=1}^M$. Another example is motivated by mirror symmetry. The Strominger-Yau-Zaslow conjecture predicts that certain families of degenerating Calabi-Yau metrics converge to a space whose metric is given by the Hessian of a convex function u that solves (2), with nonempty Σ. Of particular interest is the case that Σ is a "Y" shape, see for example the works of Loftin [11] and Loftin-Yau-Zaslow [13].

Central questions about solutions to (2) are: (a) what is the asymptotic behavior of u at infinity?, and (b) what is the local behavior of u away from Σ? Caffarelli-Li showed that any solution to (2) is smooth outside the convex hull of Σ, and is asymptotic to a quadratic polynomial at infinity [2]. This result can be viewed as a generalization of the Jörgens-Calabi-Pogorelov theorem. The local behavior remained unclear. For example: Can the optimal transport maps of Dirac masses have discontinuities away from the masses? Or: Is there a robust method to generate Monge-Ampère metrics with Y-shaped singular structures? In the talk we discussed work which answered these questions in the positive [14]:

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a compact convex polytope, and let Γ_k denote its k-skeleton (that is, the collection of faces of dimension at most k). Assume further* *that $n = 3$ or $n = 4$. Then there exists a convex function $u : \mathbb{R}^n \to \mathbb{R}$ such that $\Gamma_1 \subset \{u = 0\}$, $u \in C^\infty(\mathbb{R}^n \backslash \Gamma_1)$, and*

$$\det D^2 u = 1 + \sum_{q \in \Gamma_0} a_q \delta_q \tag{3}$$

in the Alexandrov sense, for some coefficients $a_q > 0$. Furthermore, ∇u is discontinuous on the set of edges Γ_1.

Such examples do not exist in dimension $n = 2$ in view of the purely local regularity theory [7]. Also, the polytope Ω from Theorem 1 is allowed to be degenerate (e.g. Γ_0 may consist of two points, in which case $\Omega = \Gamma_1$ is a line segment). The examples are built by solving an obstacle problem for the Monge-Ampère equation with an obstacle whose graph is a convex polytope, and then taking the Legendre transform. This approach required developing new tools related to the propagation of singularities in solutions to the Monge-Ampère equation, as well as the use of sophisticated regularity results for the case of linear obstacle due to Savin [15].

Theorem 1 opens up several interesting research directions. The first one is:

Problem 1. *What happens in dimension $n \geq 5$?*

In [1], Caffarelli constructed generalizations of the Pogorelov example which solve (1) near the origin, but whose graphs have flat regions of dimension up to (but not including) $n/2$. In view of these examples, we expect that our approach produces solutions to (2) with Σ again finite and consisting of the vertices of a convex polytope, such that the solutions are singular on higher-dimensional faces. Another interesting question is that of stability:

Problem 2. *Do the singularities in the examples from Theorem (1) remain after making small perturbations of the locations and sizes of the Dirac masses?*

There are several ways to interpret this problem. One way is to choose a bounded domain and its image under ∇u, fix these as the initial and target domains of an optimal transport problem, and then vary the mass sizes and locations. In [14] the locations of the masses are prescribed, and the mass sizes are determined by the method of construction. Alternatively, one can remain in the setting of global exterior solutions. Jin and Xiong have characterized the space of global solutions to $\det D^2 u = 1 + \sum_{i=1}^{M} a_i \delta_{p_i}$ as an explicit orbifold of a certain dimension, parametrized by the mass locations and sizes [8]. It would be interesting to investigate the geometry and topology of the set of points that correspond to solutions that have singular structures:

Problem 3. *Determine the geometric properties of the set on the solution orbifold that corresponds to solutions with singular structures. For example: What is its dimension? What is its regularity? What are its connectivity properties? Likewise, find algebro-geometric conditions on the locations and the sizes of the masses that rule out the existence of singular structures.*

In connection with the Strominger-Yau-Zaslow conjecture, it would be very interesting to understand the asymptotic behavior of the Hessians of the examples from Theorem 1 near the masses:

Problem 4. *Find precise descriptions of the tangent cones to the examples from* [14] *at the Dirac masses.*

A starting point will be to study the case of two masses (that is, Γ_0 consists of two points) and axisymmetry. The behavior near Y-shaped singular structures is a more delicate problem.

Finally, the approach in [14], which involves solving an obstacle problem and taking the Legendre transform, is quite robust. It is natural to ask what happens when we change the types of obstacle:

Problem 5. *Investigate similar questions to those above, but choose obstacles whose graphs are not necessarily convex polytopes.*

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Higher order estimates for complex Monge-Ampère equations with small fiber size

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(joint work with Hans-Joachim Hein)

Let (X,ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$. A fundamental theorem of Yau [10] guarantees the existence of a smooth solution of the elliptic complex Monge-Ampère equation on X, as conjectured by Calabi who had also proved uniqueness. More precisely, we have:

Theorem 1 (Yau [10])**.** *Let (X^n,ω) be a compact Kähler manifold, and $F \in C^\infty(X)$ be normalized by $\int_X (e^F - 1)\omega^n = 0$. Then there is a unique $\varphi \in C^\infty(X)$ normalized by $\int_X \varphi\omega^n = 0$ such that $\omega + i\partial\overline{\partial}\varphi > 0$ (i.e. this defines a new Kähler metric) and*

$$(\omega + i\partial\overline{\partial}\varphi)^n = e^F\omega^n. \tag{1}$$

In local holomorphic coordinates, (1) is the complex Monge-Ampère equation

$$\det\left(g_{j\overline{k}} + \frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}\right) = e^F \det(g_{j\overline{k}}).$$

When X is *Calabi-Yau*, in the sense that $c_1(X) = 0$ in $H^2(X,\mathbb{R})$, then we can choose F so that $\mathrm{Ric}(\omega) = i\partial\overline{\partial}F$, and in this case the Kähler metric $\omega + i\partial\overline{\partial}\varphi$ provided by Theorem 1 has vanishing Ricci curvature (i.e. it is Ricci-flat).

A natural question, which arose from consideration coming from mirror symmetry [3], is to study the behavior of these Ricci-flat metrics in families, when the total volume of the manifold is shrinking to zero. The following is the precise setup, which was first studied by Gross-Wilson [3] for elliptically fibered K3 surfaces and by the author [7] in general.

We suppose that X^{m+n} is a compact Calabi-Yau manifold which admits a surjective holomorphic map $f : X \to B$ with connected fibers onto a compact Kähler manifold B^m (which may also be allowed to be singular, but we will not stress this point here), with $m, n > 0$, and we fix a Ricci-flat Kähler metric ω_X on X, and a Kähler metric ω_B on B. Such maps can be thought of as fibrations with possibly singular fibers, in the sense that if we denote by $D \subset B$ the critical values of f and $S = f^{-1}(D)$, then D, S are proper closed analytic subsets and $f : X\backslash S \to B\backslash D$ is a proper holomorphic submersion, in particular a C^∞ fiber bundle. The smooth fibers $X_z = f^{-1}(z), z \in B\backslash D$, are then Calabi-Yau n-folds, pairwise diffeomorphic, and S is referred to as the singular fibers of f. Many such fibrations can be constructed using tools from algebraic geometry.

Given such a fiber structure on X, we can consider Ricci-flat Kähler metrics on X with small fiber volume. More precisely, for all $t \geq 0$ we let ω_t be the unique Ricci-flat Kähler metric on X of the form

$$\omega_t = f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t, \quad \int_X \varphi_t\omega_X^{m+n} = 0,$$

provided by Theorem 1, which therefore solve the family of complex Monge-Ampère equations

$$\omega_t^{m+n} = (f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t)^{m+n} = c_t e^{-nt}\omega_X^{m+n},$$

where c_t are explicit positive constants that are bounded away from 0 and ∞ for all t. The RHS of the PDE is thus of the order of e^{-nt}, and furthermore the reference metrics $f^*\omega_B + e^{-t}\omega_X$ are very small in the fiber directions. The ellipticity of the equation is then degenerating along the fibers, as $t \to \infty$, and one would like to know how the solutions φ_t (or equivalently the metrics ω_t) behave as $t \to \infty$.

The first result in this direction was:

Theorem 2 (T. [7]). *There is $\varphi_\infty \in C^\infty(B\backslash D)$ with $\omega_\infty = \omega_B + i\partial\overline{\partial}\varphi_\infty > 0$ solving the Monge-Ampère equation*

$$(\omega_B + i\partial\overline{\partial}\varphi_\infty)^m = c_\infty f_*(\omega_X^{m+n})$$

(for suitable $c_\infty \in \mathbb{R}_{>0}$) such that $\varphi_t \to f^*\varphi_\infty$ in $C^{1,\alpha}_{\mathrm{loc}}(X\backslash S)$ for $\alpha < 1$. Furthermore, given any $K \Subset X\backslash S$ there is C such that on K for all $t \geq 0$

$$C^{-1}(f^*\omega_B + e^{-t}\omega_X) \leq \omega_t \leq C(f^*\omega_B + e^{-t}\omega_X).$$

The main conjecture was then that the convergence in this result can be improved to the smooth topology:

Conjecture 3. *In the above setting, as $t \to \infty$ we have that $\varphi_t \to f^*\varphi_\infty$ locally smoothly on $X\backslash S$. Equivalently, there are uniform a priori estimates*

$$\|\omega_t\|_{C^k(K,\omega_X)} \leq C, \tag{2}$$

independent of t, for each $K \Subset X\backslash S$ and $k \geq 0$, where C depends on k, K and the background data.

This conjecture was proved in [3] for elliptic K3 surfaces when f is generic, by constructing ω_t via a gluing procedure. From the work of [1] one can see that conjecture 3 holds when $S = \emptyset$, i.e. f is a submersion everywhere. This assumption is however very restrictive, as it implies that f is a holomorphic fiber bundle [9]. In [2, 4] it was shown that the conjecture holds if the smooth fibers X_z are tori (or finite quotients of tori). However, none of these methods can be applied to solve the conjecture in general.

In [8] we showed that $\omega_t \to f^*\omega_\infty$ locally uniformly on $X\backslash S$, and this was improved to C^α_{loc} convergence in [5], where it was also shown that the conjecture holds whenever all smooth fibers X_z are pairwise biholomorphic. Finally, we have:

Theorem 4 (Hein-T. [6]). *Conjecture 3 is true. More precisely, given $0 \leq j \leq k, 0 < \alpha < 1$ and $z \in B\backslash D$ there is a coordinate ball $B' = B_r(z) \subset B$ and smooth functions $G_{i,p,k}, 2 \leq i \leq j, 1 \leq p \leq N_{i,k}$ on $f^{-1}(B')$ so that there we can write*

$$\omega_t = f^*\omega_\infty + e^{-t}\omega_F + \gamma_{t,0} + \gamma_{t,2,k} + \cdots + \gamma_{t,j,k} + \eta_{t,j,k},$$

where $\omega_F = \omega_X + i\partial\overline{\partial}\rho$ is defined so that its restriction to any smooth fiber is Ricci-flat, and

$$\gamma_{t,0} = i\partial\overline{\partial}\underline{\psi_t} \to 0 \text{ in } C^j(B'),$$

where $\psi_t = \varphi_t - f^*\varphi_\infty - e^{-t}\rho$ *and* $\underline{\psi_t}$ *is its fiberwise average w.r.t.* ω_X^n,

$$\gamma_{t,i,k} = i\partial\overline{\partial}\sum_{p=1}^{N_{i,k}} \mathfrak{G}_{t,k}(A_{t,i,p,k}, G_{i,p,k}) \to 0 \text{ in } C^j(f^{-1}(B'), \omega_X),$$

where $\mathfrak{G}_{t,k}$ *is a certain approximate Green operator and* $A_{t,i,p,k}$ *are smooth functions on* B' *which go to zero in* $C^{j+2}(B')$*, and the remainder* $\eta_{t,j,k}$ *goes to zero in a "shrinking"* C^j *norm on* $f^{-1}(B')$.

All objects that appear in this decomposition are also bounded in their corresponding Hölder norms $C^{\bullet,\alpha}$, $\alpha < 1$. One can think of this result as an asymptotic expansion for the Ricci-flat metrics ω_t, which however stops at some arbitrarily chosen level k, and where all the pieces of the expansion have explicit estimates in Hölder norms. The remainder $\eta_{t,j,k}$ is bounded in a much stronger "shrinking" $C^{j,\alpha}$ norm on $f^{-1}(B')$, where the length of derivatives is measured with respect to the shrinking reference metrics $\omega_t^{\mathrm{ref}} = f^*\omega_\infty + e^{-t}\omega_F$ instead of the fixed metric ω_X (there is a technicality as to which covariant derivatives and parallel transport are used, that we gloss over here). The "obstruction functions" $G_{i,p,k}$ arise as obstructions to the remainder $\eta_{t,i-1,k}$ at the previous step (which by induction is bounded in shrinking $C^{i-1,\alpha}$) being bounded in shrinking $C^{i,\alpha}$. Once these are identified, the functions $A_{t,i,p,k}$ on the base can be thought of as the fiberwise L^2 component of $\mathrm{tr}^{g_t^{\mathrm{ref}}}\eta_{t,i-1,k}$ onto $\mathbb{R}.G_{i,p,k}$, while the Green operator $\mathfrak{G}_{t,k}$ satisfies $\Delta^{g_t^{\mathrm{ref}}}\mathfrak{G}_{t,k}(A,G) \approx AG$ for A any polynomial on the base of degree $< 2k+2$.

When the smooth fibers X_z are tori or are pairwise biholomorphic, the terms $\gamma_{t,i,k}$ vanish and the remainder decays faster then e^{-Nt} for all N. In general however, the terms $\gamma_{t,i,k}$ are not zero.

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Non collapsed translators in $\mathbb{R}^4$

Or Hershkovits
(joint work with Kyeongsu Choi, Robert Haslhofer)

A hypersurface $M^n \subset \mathbb{R}^{n+1}$ is called a *translator* if its mean curvature vector satisfies

$$\mathbf{H} = v^{\perp} \tag{1}$$

for some $0 \neq v \in \mathbb{R}^{n+1}$. Solutions of (1) correspond to selfsimilarly translating solutions $\{M_t = M + tv\}_{t\in\mathbb{R}}$ of the mean curvature flow,

$$(\partial_t x)^{\perp} = \mathbf{H}(x). \tag{2}$$

Translators model the formation of type II singularities under mean curvature flow, see e.g. [Ham95, HS99, Whi03]. We recall that Huisken and Hamilton grouped singularities of the mean curvature flow at some time T into type I and II, depending on whether $(T-t)|A|^2$ stays bounded or not [Hui90, Ham95]. Type I singularities are modelled on *shrinkers*, and are easier to analyze than type II singularities. For example it is known in any dimension that the round cylinders $\mathbb{R}^k \times S^{n-k}$ are the only mean-convex shrinkers [Hui93, Whi03], and also the only stable shrinkers [CM12]. In an attempt to get a grasp on type II singularities, translators have received a lot of attention over the last 25 years, but despite these efforts no general classification result has been obtained for $n \geq 3$, not even for convex graphs.

Whenever a translator appears as a blow-up limit of a compact mean convex mean curvature flow, it has to be convex and noncollapsed, in the sense of [SW09, And12, HK17]. More generally, by Ilmanen's mean-convex neighborhood conjecture [Ilm03], which has been proved recently in the case of neck-singularities in [CHH18, CHHW19], it is expected even without mean-convexity assumption that all blowup limits near any cylindrical singularity are ancient noncollapsed flows.

When $n = 2$, it was shown in [Has15] that Bowl_2 - the unique rotationally symmetric translating graph (which was constructed by Altschuler-Wu [AW94]) - is the unique non collapsed translator in $\mathbb{R}^3$.

When $n = 3$, there are two examples of non-collapsed translators that have been known for quite a while: $\mathbb{R} \times \mathrm{Bowl}_2$ - the product of the line with the 2-dimensional bowl from from Altschuler-Wu [AW94], and Bowl_3 - the 3d *round bowl* constructed by Clutterbuck-Schnürer-Schulze [CSS07]. More recently, Hoffman-Ilmanen-Martin-White [HIMW19a] constructed examples that are not rotationally symmetric. Specifically, for every triple (k_1, k_2, k_3) of nonnegative numbers with $k_1 + k_2 + k_3 = 1$ they proved that there exists at least one unit-speed graphical translator with tip principal curvatures (k_1, k_2, k_3). Moreover, they showed that when one takes $k_1 \leq k_2 = k_3$ then one always gets a translator that is an entire graph and has circular symmetry in the last two variables. It is not hard to show that these entire graphical translators are in fact noncollapsed. Hence, for every $k \in (0, \frac{1}{3})$ there exists at least one noncollapsed translators $M_k \subset \mathbb{R}^4$

that is noncollapsed and circular symmetric and whose principal curvatures at the tip are $(k, \frac{1-k}{2}, \frac{1-k}{2})$. The HIMW-translators $\{M_k\}_{k\in(0,1/3)}$ interpolate between $M_0 = \mathbb{R} \times \mathrm{Bowl}_2$ and $M_{1/3} = \mathrm{Bowl}_3$.

In the talk, I described the proof of the following result, which was obtained jointly with Kyeongsu Choi and Robert Haslhofer

Theorem 1 (classification of noncollapsed translators [CHH21]). *Every noncollapsed translator in $\mathbb{R}^4$ is, up to rigid motion and scaling,*

- *either* $\mathbb{R} \times \mathrm{Bowl}_2$*,*
- *or the 3d round bowl* Bowl_3*,*
- *or belongs to the one-parameter family $\{M_k\}_{k\in(0,1/3)}$ constructed by Hoffman-Ilmanen-Martin-White.*

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Ricci flow on noncompact surfaces with rough initial data

Peter Topping
(joint work with Hao Yin)

In previous work [1, 5, 6, 4, 7] it has been shown that given any smooth connected Riemannian surface, there exists a unique Ricci flow evolution for a specific time interval $[0, T)$, that is complete for all $t \in (0, T)$.

In this talk we discussed to what extent we could broaden this theory to initial data that was rougher than a smooth surface, insisting always that we are working on a possibly noncompact underlying space.

Our starting proposal was to consider a Radon measure on a Riemann surface. The measure can be thought of as a very rough conformal factor on the Riemann surface. In the talk we showed that this was unreasonably general, by showing that uniqueness and even existence would fail in this case. In contrast, if we consider nonatomic measures then a theory can be developed.

Theorem 1 (Main existence theorem, [11])**.** *Suppose M is any (connected, possibly noncompact) Riemann surface and μ is any nonnegative nontrivial Radon measure on M that is nonatomic in the sense that*

$$\mu(\{x\}) = 0 \text{ for all } x \in M.$$

Define

$$T := \begin{cases} \frac{\mu(M)}{4\pi} & \text{if } M = \mathbb{C} \simeq \mathbb{R}^2 \\ \frac{\mu(M)}{8\pi} & \text{if } M = S^2 \\ \infty & \text{otherwise.} \end{cases}$$

Then there exists a smooth complete conformal Ricci flow $g(t)$ on M, for $t \in (0, T)$, such that the Riemannian volume measure $\mu_{g(t)}$ converges weakly to μ as $t \searrow 0$.

In the cases that $T < \infty$, as $t \nearrow T$ we have

$$\mathrm{Vol}_{g(t)}(M) = (1 - \tfrac{t}{T})\mu(M) \to 0.$$

Moreover, if μ has no singular part then $\mu_{g(t)} \to \mu$ in $L^1_{loc}(M)$. More generally, if Ω is the complement of the support of the singular part of μ, then $\mu_{g(t)} \llcorner \Omega \to \mu \llcorner \Omega$ in $L^1_{loc}(\Omega)$.

The proof of the theorem involves an $L^1 - L^\infty$ smoothing estimate proved in [10] and a Harnack estimate that was inspired by [2].

Several applications of this new theory were presented. First, we demonstrated how it leads to a large new class of expanding Ricci solitons that provide an answer to a number of open problems concerning Ricci solitons in two dimensions. Second, we used the theory to resolve the problem of whether a smooth Ricci flow that attains smooth initial data only in the sense of local uniform convergence of the Riemannian distance must in fact be smooth all the way down to time zero. See, for example, the discussion in [8]. A counterexample was constructed by evolving the Radon measure on the plane that is the sum of Lebesgue measure and the Hausdorff $\mathcal{H}^1$ measure restricted to a line.

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Rigidity, flexibility, and regularity of Sobolev mappings in sub-Riemannian geometry

Bruce Kleiner

(joint work with Stefan Müller, László Székelyhidi, Xiangdong Xie)

The lecture covered results from a series of recent papers on geometric mapping theory in Euclidean space and Carnot groups; these were motivated by geometric group theory, analysis on metric spaces, differential geometry, and PDEs.

The first results concern rigidity of product structure in $\mathbb{R}^n$. A mapping $f : X_1 \times X_2 \to Y_1 \times Y_2$ between product sets **splits** (or preserves product structure) if it is of the form $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for some mappings $f_i : X_i \to Y_i$, after possibly reindexing the factors Y_1, Y_2.

Theorem 1 (K-Müller-Szekelyhidi-Xie). *If $n \geq 2$ and $f : \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ is a $W^{1,2}_{loc}$-mapping such that $Df(x)$ is split and bijective for a.e. $x \in \Omega$, then f is split.*

The exponent 2 is sharp:

Theorem 2 (KMSX). *For every $1 \leq p < 2$ there is a $W^{1,p}_{loc}$-mapping*

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$$

such that $Df(x)$ is split and bijective for a.e. $x \in \Omega$, but f is not split.

The assertion in Theorem 1 is false when $n = 2$ because of the map $\mathbb{R}^2 \to \mathbb{R}^2$ which folds along the diagonal line $\{x_1 = x_2\}$. In fact, there are even bilipschitz counterexamples:

Theorem 3 (KMSX). *There is a bilipschitz homeomorphism $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ such that*

- *$Df(x)$ is split and bijective for a.e. x.*
- *f is not split.*
- *f is area preserving: $detDf(x) = 1$ for a.e. x.*
- *There is a null set N such that $Df(x)$ takes only five values for $x \notin N$.*

The remaining results are joint with Stefan Müller and Xiangdong Xie, and are concerned with bilipschitz, quasiconformal, and Sobolev mappings in the Carnot group setting. Such mappings have been studied since the 1970s, and arise, for example, in geometric group theory as boundary homeomorphisms associated with bilipschitz mappings $X \to X'$ between negatively curved manifolds, or as blow-downs of quasi-isometries between finitely generated nilpotent groups. Since the work of Pansu on rigidity in 1989, they have been of interest to a broader community of differential geometers, people working on analysis in metric spaces, and on PDEs.

In what follows, all Carnot groups will be equipped with their Carnot-Caratheodory (i.e. sub-Riemannian) distance.

Theorem 4. *Let $\mathbb{H}$ be the Heisenberg group and $U_1, U_2 \subset \mathbb{H}$ be connected open subsets. Suppose*

$$f : U_1 \times U_2 \to U' \subset \mathbb{H} \times \mathbb{H}$$

is a quasisymmetric homeomorphism. Then f is split.

The same holds more generally for a $W^{1,3}_{loc}$-mapping provided the horizontal differential $d_H f$ is nonsingular almost everywhere. We do not know the optimal Sobolev exponent for rigidity. This is a special case of a general result for maps between products of Carnot groups.

Theorem 5. *Let $\mathbb{H}^{\mathbb{C}}_n$ be the n^{th} complex Heisenberg group.*

- *(Hypoellipticity) Any quasisymmetric homeomorphism $\mathbb{H}^{\mathbb{C}}_n \supset U \to U' \subset \mathbb{H}^{\mathbb{C}}_n$ is locally holomorphic or antiholomorphic.*
- *Any globally defined quasisymmetric homeomorphism $\mathbb{H}^{\mathbb{C}}_n \to \mathbb{H}^{\mathbb{C}}_n$ is affine.*

Reimann-Ricci proved the first assertion above, assuming f is C^2. The regularity result only requires that $f \in W^{1,2n+1}_{loc}$, and nondegeneracy of the horizontal differential $d_H f$ almost everywhere.

Theorem 6. *If G is a nonrigid Carnot group (in the sense of Ottazzi-Warhurst) other than $\mathbb{R}^n$ or a Heisenberg group, and $f : G \supset U \to U' \subset G$ is quasisymmetric, then $f \in W^{1,\infty}_{loc}$, and is locally bilipschitz. If $U = G$, then f is bilipschitz.*

The starting point for our results is theorem about Pansu pullback – the pullback of differential forms using the Pansu differential. Although Pansu pullback need not commute with the exterior derivative even for smooth contact diffeomorphisms, a partial analog does hold.

The lecture also mentioned:

- Applications to quasiregular mappings in Carnot groups.
- Sobolev mappings induce a chain mapping on Rumin complexes, in the setting of contact manifolds.
- A rigidity result characterizing Sobolev mappings $N \supset U \to N$ where $N \subset GL(n, \mathbb{R})$ is the Carnot group of upper triangular matrices with 1s on the diagonal.

Degeneration of 7-dimensional minimal hypersurfaces with bounded index

Nick Edelen

An n-dimensional area-minimizing hypersurface M^n will in general have a $(n-7)$-dimensional singular set $\mathrm{sing}M \equiv \overline{M} \setminus M$ ($\overline{M}$ denoting set-theoretic closure), which is countably $(n-7)$-rectifiable with locally-finite $(n-7)$-area [10], [7]. The same regularity holds if M is locally stable for the area-functional [12], [14], provided one knows a priori that $\mathcal{H}^{n-1}(\overline{M} \setminus M) = 0$ (often satisfied for reasons of orientability or minimization). Any M with finite index, such as those arising from minimization or min-max procedures, is locally stable.

When $n = 7$ the singular set is discrete, with a priori bounds on the number of singular points in terms of the area of M [7]. Every tangent cone to M is stable and smooth (away from 0), and near each singular point deep work of [1], [9] showed that M is a C^1 perturbation of its tangent cone. The quadratic "Simons' cones[1]" $\mathbf{C}^{3,3}$, $\mathbf{C}^{2,4}$, $\mathbf{C}^{1,5} \subset \mathbb{R}^8$ are all examples of smooth, stable cones. No other such cones in $\mathbb{R}^8$ are known. Except for the question of classifying the tangent cones themselves, the aforementioned results give us a very precise picture of the small-scale behavior of any particular M^7.

In this talk we are interested in *moduli spaces* of 7D stable minimal hypersurfaces, which requires knowledge of how uniform the regular and singular structure are. For example, can arbitrarily bad topology, geometry, or singular set collapse along a sequence into a singular limit? Precisely, we are led to the following question:

Question 1. *Let $g_i \to g$ be a sequence of metrics on $B_1(0) \subset \mathbb{R}^8$ converging in C^3, and M_i^7 be a sequence of stable embedded minimal hypersurfaces in (B_1, g_i) with discrete singular set, which converge as varifolds in B_1 to some minimal cone $\mathbf{C} \subset (B_1, g)$. What can be said about the M_i in $B_{1/2}$? What if instead $\sup_i index(M_i) < \infty$?*

---

[1]Given integers $p, q \in \mathbb{N}$, the quadratic hypercone $\mathbf{C}^{p,q} := \{(x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : p|x|^2 = q|y|^2\}$ is always stationary and smooth.

An instructive example: [3], [6] showed that one can foliate $\mathbb{R}^8 \setminus \mathbf{C}^{3,3}$ by a family of smooth, area-minimizing hypersurfaces $\cup_{\lambda\neq 0} S_\lambda$ asymptotic to $\mathbf{C}^{3,3}$, which in fact are all dilations of each other. If one takes $\lambda \to 0$, the S_λ converges to $\mathbf{C}^{3,3}$, but the geometry/topology/singular set changes in the limit. Therefore, even in the nicest possible singular setting, different regular structures can "collapse" into singularities of the limit. In general, by blowing-up along the sequence M_i, one would expect the geometry of M_i to be modelled on entire, locally-stable minimal hypersurfaces in $\mathbb{R}^8$ asymptotic to $\mathbf{C}$.

When $n \geq 8$, there is essentially no hope to answer Question 1 in the class of smooth metrics. [11] has constructed remarkable examples of stable minimal hypersurfaces $M^8 \subset (\mathbb{R}^9, g)$, such that g is smoothly close to g_{eucl}, M is varifold close to $\mathbf{C}^{3,3} \times \mathbb{R}$, and sing$M$ is any preassigned closed subset of $\{0\} \times \mathbb{R}$. On the other hand, when $n \leq 6$, then $\mathbf{C}$ is necessarily a plane-with-multiplicity, and the sheeting theorem of [12] imply M_i are all graphs (for $i >> 1$). Even when the M_i are only locally stable, but with a priori bounded index, then [4] have shown M_i have controlled geometry/topology

We prove the following answer to Question 1.

Theorem 1. *In the setup of Question 1, then after passing to a subsequence, all the M_i admit a "cone decomposition," and we can a radius $r \in (3/4, 1)$ and bi-Lipschitz maps $\phi_i : B_r \to B_r$, so that $\phi_i(M_1 \cap B_r) = M_i \cap B_r$, $\phi_i : B_r \setminus \text{sing}M_1 \to B_r \setminus \text{sing}M_i$ are C^2 diffeomorphisms, and $Lip(\phi_i) \leq C$ independent of i.*

A "cone decomposition," analogous to the "bubble trees" of [13], [8], is a quantitative decomposition of M_i into a collection of annular cone regions, wherein M_i is a small perturbation of some smooth stable hypercone, and a collection of balls, wherein the curvature of M_i is controlled. The main consequence of Theorem 1 is the following finiteness theorem.

Corollary 2. *Let (N^8, g) be a closed, 8-dimensional Riemannian manifold with C^3 metric g, and take $\Lambda, I \geq 0$. There is a number $K(N, g, I, \Lambda)$ so that every C^2, closed, embedded minimal hypersurface $M \subset (N, g)$ having $\mathcal{H}^7(M) \leq \Lambda$ and index$(M) \leq I$ fits into one of at most K diffeomorphism classes. If M instead $\overline{M} \setminus M$ is discrete, then $\overline{M}$ fits into one of at most K bi-Lipschitz equivalence classes.*

The main difficulty in proving Theorem 1, and the key difference from other finitness results like [5], [4], is that we do not have any classification of tangent cones $\mathbf{C}$ beyond being smooth and stable. In particular, we cannot assume $\mathbf{C}$ is "integrable through rotations," and have to allow for the possibility that $\mathbf{C}$ could "rotate" through a pathological family of smooth, stable minimal hypercones. In higher codimension there are examples of this behavior. We must use very heavily the analytic nature of $\mathbf{C}$, in form of the Lojasiewicz-Simon inequality [9].

We use analyticity in two ways. First, the Lojasiewicz-Simon inequality combined with standard compactness/regularity theory implies that the set of densities

$$\left\{ \theta_{\mathbf{C}}(0,1) \equiv \frac{\mathcal{H}^7(\mathbf{C} \cap B_1)}{\omega_7} : \mathbf{C} \subset \mathbb{R}^8 \text{ smooth, stable} \right\}$$

forms a discrete set $1 = \theta_1 < \theta_2 < \dots$. We can therefore induct on the density θ_k as follows: If one performs a blow-up procedure on the M_i as indicated above, then one obtains an entire stable minimal surface $M' \subset \mathbb{R}^8$ asymptotic to some cone $\mathbf{C}$ with $\theta_{\mathbf{C}}(0) = \theta_k$. By choosing the blow-up scale well, one can assume M' is not itself a cone, and therefore by monotonicity any singular point of M' has density $\leq \theta_{k-1} < \theta_k$. One can then perform a secondary blow-up at each singular point of M', obtaining surfaces M'' asymptotic to a cone with density $\leq \theta_{k-1}$. After a finite number of iterative blow-ups, one will exhaust all possible densities, and get a *smooth* surface asymptotic to a non-flat cone, much like the foliation S_λ.

The second point we use analyticity is in controlling the annular "cone regions," where each M_i look close to a fixed cone. In the blow-up argument above, one can only pass information between finitely-many scales, and so one is left with the possibility that as the radius r decreases M looks like a "rotating" family of cones, e.g. $M_i \cap \partial B_r \approx \mathbf{C}_{\log r/\sigma} \cap \partial B_r$ for some family of smooth cones $\{\mathbf{C}_t\}_{t\in\mathbb{R}}$, $\sigma \in \mathbb{R}$ small, and $r \in [\rho_i, R]$. This becomes a problem because, although the $M_i \cap B_R \setminus B_{\rho_i}$ are all diffeomorphic, we lose control over how these annular pieces are glued into other parts of M_i. Imagine if $\mathbf{C}_t \cap B_1$ where a genus two surface which as t increased rotated one handle, and kept the other fixed. By rotating more and more, one obtains gluing maps which are not isotopic to each other, and have uncontrolled C^1 norm. To deal with this, we adapt the decay-growth estimates of [9] to show that, a posteriori, the cones cannot rotate in cone regions. We must relax the original assumptions of [9], which required a sharp lower bound $\theta_{M_i}(0, \rho_i) \geq \theta_{\mathbf{C}}(0)$, to allow instead for $\theta_{M_i}(0, \rho_i) \geq \theta_{\mathbf{C}}(0) - \delta$.

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Classifying sufficiently connected PSC manifolds in 4 and 5 dimensions

Chao Li
(joint work with Otis Chodosh, Yevgeny Liokumovich)

We are concerned here with the problem of classification of manifolds admitting positive scalar curvature (PSC). For closed (compact, no boundary) 2- and 3-manifolds this problem is completely resolved, namely the sphere and projective plane are the only closed surfaces admitting positive scalar curvature and a 3-manifold admits positive scalar curvature if and only if it has no aspherical factors in its prime decomposition. In particular, a 3-manifold admitting positive scalar curvature has a finite cover diffeomorphic to S^3 or to a connected sum of finitely many $S^2 \times S^1$.

It is a long standing challenge to seek analogous classification results in higher dimensions. In particular, Schoen-Yau in [8] made the following conjecture:

Conjecture 1. *Let $n \geq 4$, M^n be a closed smooth aspherical manifold. Then M does not admit any Riemannian metric of positive scalar curvature.*

A stronger formulation in terms of metric geometry was also conjectured by Gromov in [5]. As pointed out by Rosenberg in [6, Theorem 3.5], Conjecture 1 is also closely related to a (still widely open) strong form of the Novikov conjecture: if a certain form of the Novikov conjecture holds, then Conjecture 1 holds.

In a joint work with Chodosh, the author proved Conjecture 1 in 4 and 5 dimensions.

Theorem 2 ([2])**.** *Let $n \in \{4, 5\}$, and M^n a closed smooth aspherical manifold of dimension n. Then any Riemannian metric on M with nonnegative scalar curvature is flat.*

A key step in the proof of Theorem 2 is a certain homological filling estimate, which we describe here. (This is inspired by a paper of Schoen-Yau [7].) Given (M^n, g) as in Theorem 2 and suppose, for the sake of contradiction, that $R_g \geq 1$, then there exists a large constant $L = L(M, g)$, such that the following holds: any $(n-2)$-cycle Γ in the universal cover $\tilde{M}$ can be realized as $\partial\Sigma$, where Σ is an $(n-1)$-chain such that it is contained in an L tubular neighborhood of Γ. It is worth noting that Schoen-Yau [7] proved this estimate when $n = 3$ with $L = 2\pi$. Any further extension of such an estimate to high dimensions will deduce Conjecture 1 in such dimensions.

Inspired by an ongoing project of Alpert-Balitsky-Guth [1], Chodosh, Liokumovich and the author extended Theorem 2 to a positive result.

Theorem 3 ([3]). *Suppose that N is a closed smooth n-manifold admitting a metric of positive scalar curvature and*

- $n = 4$ *and* $\pi_2(N) = 0$, *or*
- $n = 5$ *and* $\pi_2(N) = \pi_3(N) = 0$.

Then a finite cover $\hat{N}$ of N is homotopy equivalent to S^n or connected sums of $S^{n-1} \times S^1$.

Some remarks are in order. First, recall that a closed N^n is aspherical if $\pi_j(N) = 0$ for all $j \geq 2$. Thus Theorem 3 is a generalization of Theorem 2. Second, the conditions that $\pi_2(N) = 0$ when $n = 4$ ($\pi_3(N) = 0$ when $n = 5$) are necessary: compare to the product metric on $T^2 \times S^{n-2}$. Third, if $n = 4$ and $\hat{N}$ is homotopy equivalent to S^4 or $S^3 \times S^1$, or if $n = 5$ (with no further restriction on the homotopy type), then homotopy equivalence in the conclusion can be upgraded to homeomorphism [4]. However, the approach does not seem possible to conclude diffeomorphism types (in contrast to the case of dimension three).

There is also a more general mapping version of Theorem 3.

Theorem 4 ([3]). *Suppose that N is a closed smooth orientable n-manifold with a metric of positive scalar curvature and there exists a non-zero degree map $f : N \to X$, to a manifold X satisfying*

- $n = 4$ *and* $\pi_2(X) = 0$, *or*
- $n = 5$ *and* $\pi_2(X) = \pi_3(X) = 0$.

Then a finite cover $\hat{X}$ of X is homotopy equivalent to S^n or connected sums of $S^{n-1} \times S^1$.

In particular, this implies that if $n \in \{4, 5\}$, $f : N \to X$ is a non-zero degree map to an aspherical manifold X, then N does not admit any Riemannian metric with positive scalar curvature.

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The p-widths of a surface

Christos Mantoulidis
(joint work with Otis Chodosh)

Fix a closed Riemannian manifold (M^{n+1}, g). The *p-widths* of (M, g), denoted $\omega_p(M, g) \in (0, \infty)$ for $p \in \mathbb{N}^*$, are a geometric nonlinear analogue of the spectrum of its Laplace–Beltrami operator. They are obtained by replacing the Rayleigh quotient of the Laplace–Beltrami operator along families of scalar-valued functions on M with the n-dimensional area along sweepouts of M of (possibly singular) hypersurfaces. They were introduced by Gromov [Gro88, Gro03, Gro09], studied further by Guth [Gut09], and have played a central and exciting role in minimal surface theory when combined with the Almgren–Pitts–Marques–Neves Morse theory program for the area functional. We invite the reader to refer to [Gro88] for the analogy between the Laplace spectrum and the volume spectrum, and to [MN21] for a thorough overview of the importance of this analogy in minimal surface theory.

Let us recall the main existence theorem for p-widths. By the combined work of Almgren–Pitts, Schoen–Simon, Marques–Neves, and Li, it is known that in ambient dimensions $n + 1 \geq 3$ every p-width is attained as the area of a smoothly embedded minimal hypersurface Σ_p whose singular set $\bar{\Sigma}_p \setminus \Sigma_p$ has dimension $\leq n - 7$, whose connected components may contribute to area with different multiplicities, and whose total Morse index (discounting multiplicities) is bounded by p. That is:

Theorem 1 ([Pit81, SS81, MN16, Li20])**.** *Let (M^{n+1}, g) be a closed Riemannian manifold with $n + 1 \geq 3$. For every $p \in \mathbb{N}^*$, there exists a smoothly embedded minimal hypersurface $\Sigma_p \subset M$, with $\bar{\Sigma}_p \setminus \Sigma_p$ of Hausdorff dimension $\leq n - 7$ and components $\Sigma_{p,1}, \ldots, \Sigma_{p,N(p)} \subset \Sigma_p$, such that*

$$\omega_p(M, g) = \sum_{j=1}^{N(p)} m_j \cdot \mathrm{area}_g(\Sigma_{p,j}),$$

where $m_j \in \mathbb{N}^*$ for all $j \in \{1, \ldots, N(p)\}$ and $\mathrm{ind}(\Sigma_p) \leq p$.

Note that, when $3 \leq n + 1 \leq 7$, Σ_p is necessarily smoothly embedded. On the other hand, in the case of a two-dimensional Riemannian manifold ($n + 1 = 2$), min-max methods not only need not produce *embedded* geodesics (see [Aie19] for examples of immersed geodesics being produced), but in full generality they could *a priori* produce *geodesic nets* as opposed to (immersed) geodesics (see [MN16, Remark 1.1]).

Our first main result shows that the min-max methods described above can be guaranteed to produce (immersed) geodesics regardless of the number of parameters. Throughout the paper, a geodesic is said to be primitive if it is connected and traversed with multiplicity one.

Theorem 2. *Let (M^2, g) be a closed Riemannian manifold. For every $p \in \mathbb{N}^*$, there exists a $\sigma_p \subset M$ consisting of primitive closed geodesics $\sigma_{p,1}, \ldots, \sigma_{p,N(p)} \subset \sigma_p$*

such that

$$\omega_p(M,g) = \sum_{j=1}^{N(p)} m_j \cdot \mathrm{length}_g(\sigma_{p,j}),$$

where $m_j \in \mathbb{N}^*$ *for all* $j \in \{1,\dots,N(p)\}$.

The existence of immersed geodesics representing the p-widths was previously known for $p = 1$ by Calabi–Cao [CC92] and for $p \in \{1,\dots,8\}$ and nearly round metrics on $\mathbf{S}^2$ by Aiex [Aie19].

Our second main result is a computation of the full p-width spectrum of the round two-sphere. (To this point there had not been a single (M^n, g), $n \geq 2$, for which the areas $\omega_p(M,g)$ (let alone the surfaces Σ_p) are known for all $p \in \mathbb{N}^*$, not even in the two-dimensional case. For comparison, the spectrum of the Laplacian is completely determined for a large class of Riemannian manifolds.)

Theorem 3. *Let* g_0 *denote the unit round metric on* $\mathbf{S}^2$. *For every* $p \in \mathbb{N}^*$,

$$\omega_p(\mathbf{S}^2, g_0) = 2\pi\lfloor\sqrt{p}\rfloor,$$

and is attained by a sweepout constructed out of homogeneous polynomials. The corresponding σ_p *is a union of* $\lfloor\sqrt{p}\rfloor$ *great circles (repetitions allowed).*

One application of Theorem 3 concerns Weyl law for the p-widths. Recall that the Laplacian spectrum (denoted by $\lambda_p(M,g)$) of a closed Riemannian $(n+1)$-manifiold satisfies the celebrated Weyl law

$$\lim_{p\to\infty} \lambda_p(M,g)p^{-\frac{2}{n+1}} = 4\pi^2 \operatorname{vol}(B)^{-\frac{2}{n+1}} \operatorname{vol}(M,g)^{-\frac{2}{n+1}}$$

showing that the high-frequency behavior of the spectrum is universal in a certain sense. Liokumovich–Marques–Neves have recently proven [LMN18] that the p-widths satisfy the following Weyl-type law

$$\lim_{p\to\infty} \omega_p(M,g)p^{-\frac{1}{n+1}} = a(n) \operatorname{vol}(M,g)^{\frac{n}{n+1}} \tag{1}$$

for some constant $a(n) > 0$. This result has had important implications for existence of minimal hypersurfaces, cf. [IMN18]. However, the constant $a(n)$ has not been determined for any dimension n (see [LMN18, §1.5]). This is in contrast with the classical Weyl law, where one can use e.g. the (explicitly known) spectrum of a cube to compute the constant in a straightforward manner. Our full computation of the p-widths of the round two-sphere in Theorem 3 readily implies:

Corollary 4. *When* $n = 1$, *the constant in* (1) *satisfies* $a(1) = \sqrt{\pi}$.

This settles the "simplest case" of the first question in [LMN18, §1.5].

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Regularity of anisotropic minimal surfaces

Antonio De Rosa
(joint work with Riccardo Tione)

A celebrated theorem of W. Allard [1] states that, given a rectifiable m-varifold V in $\mathbb{R}^N$ with density greater or equal than 1 and generalized mean curvature bounded in $L^p(\|V\|)$ with $p > m$, then V is regular around $x \in \mathbb{R}^N$ provided x has density ratio sufficiently close to 1. The proof deeply relies on the monotonicity formula of the density ratio, which is strictly related to the special symmetries of the area functional, [2]. Hence, it is a hard and widely open question whether this result holds for anisotropic energies, [7, Question 1], i.e. assuming an L^p bound on the anisotropic mean curvature with respect to functionals of the form

$$\Sigma_\Psi(V) := \int_\Gamma \Psi(T_z\Gamma)\theta(z)d\mathcal{H}^m(z), \quad \text{where } V = (\Gamma, \theta) \text{ is a rectifiable } m\text{-varifold.}$$

To the best of our knowledge, the only available result is the regularity for codimension one varifolds with bounded generalized Ψ-mean curvature [3], under a *density lower bound assumption.*

In joint work with R. Tione [14], we provide an affirmative answer to the question above in any dimension and codimension in the case the varifold V is associated to a Lipschitz graph, solving the open question [7, Question 5] for graphs:

Theorem 1 [14]**.** Let $\Psi \in C^2$ be a functional satisfying (USAC), let $p > m$ and consider an open, bounded set $\Omega \subset \mathbb{R}^m$. Let $u \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$ be a map whose graph Γ_u induces a varifold with Ψ-mean curvature in L^p in $\Omega \times \mathbb{R}^n$. Then there exists $\alpha > 0$ and an open set Ω_0 of full measure in Ω such that $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^n)$.

As mentioned above, our proof cannot rely on the monotonicity formula. Hence, we are not able to extend it to general rectifiable varifolds. Instead, we introduce a novel ellipticity condition (USAC), which allows us to obtain a Caccioppoli inequality, giving an answer to [7, Question 6]. (USAC) reveals to be useful to tackle another open problem in the literature: providing non-trivial examples of anisotropic energies in general codimension satisfying the atomic condition (AC). The classical Almgren ellipticity (AE), ([5, IV.1(7)] or [4, 1.6(2)]), allowed F. J. Almgren to prove regularity for minimizers of anisotropic energies, [4]. Very recently, an ongoing interest on the anisotropic Plateau problem has lead to a series of results [8, 9, 11, 12, 16, 17]. In particular, in joint work with G. De Philippis and F. Ghiraldin [10], we introduced (AC) and proved it to be necessary and sufficient for the validity of the rectifiability of varifolds whose anisotropic first variation is a Radon measure. In codimension one and in dimension one, we proved that (AC) is equivalent to the strict convexity of the integrand, [10]. However, in general codimension, characterizing (AC) in terms of more classical conditions (such as (AE), policonvexity, or others) remains an open problem, [10, Page 2]. In joint work with S. Kolasiński [13], we have recently obtained one implication, proving that (AC) implies (AE). However, in general codimension there were no examples of anisotropic energies (besides the area functional) satisfying (AC). We address this question with R. Tione [14] proving the following:

Theorem 2 [14]**.** Integrands Ψ in a C^2 neighborhood of the area functional satisfy (USAC); (USAC) implies (AC).

Hence, the anisotropic energies in a C^2-neighborhood of the area are the first functionals in the literature in general codimension to justify the regularity theory developed in [10]. In particular, we deduce the rectifiability of varifolds with locally bounded anisotropic first variation for a C^2 neighborhood of the area functional.

(AC) can be relaxed to a condition (AC1), which is equivalent to the rectifiability of the mass of varifolds whose anisotropic first variation is a Radon measure, [6]. In codimension one, the convexity of the integrand implies (AC1), [6]. However, in general codimension, there are no non-trivial examples of anisotropic energies satisfying (AC1). We address this problem with R. Tione [14] by proving:

Theorem 3 [14]**.** Integrands Ψ in a C^1 neighborhood of the area functional satisfy (AC1).

Theorem 3 implies that, in codimension one, (AC1) is a strictly weaker notion than convexity of the integrand. This shows that the result of [6, Page 656, point (b)] is indeed optimal. We also find a class of integrands on $\mathbb{G}(4, 2)$ (l^p-norms) satisfying (AC1), which are not C^1-close to the area.

There are profound connections between anisotropic geometric variational problems and questions arising in the study of polyconvex energies, see [15, Page 229]. This link was investigated in [7, 18]. In particular, there is a *canonical* way to associate a function $F_\Psi : \mathbb{R}^{n\times m} \to \mathbb{R}$ to an integrand Ψ defined on $\mathbb{G}(N, m)$, in such a way that a Lipschitz graph $[\Gamma_u]$ is stationary for Σ_Ψ if and only if u is critical for outer and inner variations for the energy

$$\mathbb{E}_{F_\Psi}(u) = \int_\Omega F_\Psi(Du(x))dx.$$

In [7, 18], C. De Lellis, G. De Philippis, B. Kirchheim, J. Hirsch and R. Tione investigated the possibility of constructing a nowhere regular stationary graph for Σ_Ψ, exploiting the convex integration techniques introduced by S. Müller and V. Šverák and L. Székelyhidi in [21, 22]. However they proved that it is impossible to complete this task *using the same strategy* of [21, 22]. In particular, they proved that if the polyconvexity of F_Ψ complies with the stationarity of u, then one can exclude a certain type of Young measures, referred to as T'_N *configurations.* With R. Tione [14], we show a much more systematic result in this direction:

Theorem 4 [14]**.** (AC) excludes non-trivial Young measures in the case of Ψ-stationary graphs.

Theorem 4 provides an answer to [7, Question 4]. In [19, Question 1], B. Kirchheim, S. Müller and V. Šverák leave as an open question to find rank-one convex functions whose differential inclusion associated to critical points (for outer variations only) supports only trivial Young measures. Theorem 4 provides an answer in a neighborhood of the area (in arbitrary dimension and codimension), adding the hypothesis of criticality for inner variations. To conclude, we remark that Theorem 1 provides partial answers to questions that naturally arose in the context of quasiconvex energies, [20, Page 65], and [19, Question 2].

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Geometry of nodal sets of Laplace eigenfunctions.

Aleksandr Logunov

We will discuss geometrical and analytic properties of zero sets of harmonic functions and eigenfunctions of the Laplace operator. For harmonic functions on the plane there is an interesting relation between local length of the zero set and the growth of harmonic functions. The larger the zero set is, the faster the growth of harmonic function should be and vice versa. Zero sets of Laplace eigenfunctions on surfaces are unions of smooth curves with equiangular intersections. Topology of the zero set could be quite complicated, but Yau conjectured that the total length of the zero set is comparable to the square root of the eigenvalue for all eigenfunctions. We will start with open questions about spherical harmonics and explain some methods to study nodal sets, which are zero sets of solutions of elliptic PDE.

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Area minimizing hypersurfaces mod(p): A geometric free boundary problem

Jonas Hirsch

(joint work with C. De Lellis, A Marches, S. Stuvard and L. Spolaor)

In this talk I would like to give an idea of our resent result on the structure of area minimizing hypersurfaces mod(p), [2].

Motivation: If one considers real soap films one notice that from time to time one can find configurations where different soap films join on a common piece. One possibility to allow this kind of phenomenon is to consider flat chains with coefficients in $\mathbb{Z}_p$. For instance for $p = 2$ one can deal with unoriented surfaces, for $p = 3$ one allows triple junctions. Using known results it can be shown that for $p = 3$ this common piece is itself nicely regular. It was our aim to investigate the situation for higher p.

We consider area minimizing m-dimensional currents mod(p) in complete C^2 Riemannian manifolds Σ of dimension $m + 1$. For odd moduli we prove that, away from a closed rectifiable set of codimension 2, the current in question is, locally, the union of finitely many smooth minimal hypersurfaces coming together at a common $C^{1,\alpha}$ boundary of dimension $m-1$, and the result is optimal. For even p such structure holds in a neighborhood of any point where at least one tangent cone has $(m-1)$-dimensional spine. These structural results are indeed the byproduct of a theorem that proves (for any modulus) uniqueness and decay towards such tangent cones. The underlying strategy of the proof is inspired by the techniques developed by Simon, [3], in a class of *multiplicity one* stationary varifolds. The major difficulty in our setting is produced by the fact that the cones and surfaces under investigation have arbitrary multiplicities ranging from 1 to $\lfloor \frac{p}{2} \rfloor$.

Some ideas to the proof. The first step is an analysis of possible tangent cones $\mathbf{C}$ with $(m-1)$-dimensional spine. Two elementary facts will play an important role. First of all, any such $\mathbf{C}$ can be described as the union of finitely many, but at least 3, half-hyperplanes $\mathbf{H}_i$ meeting at a common $(m-1)$-dimensional subspace V and counted with appropriate multiplicities κ_i. Secondly, if the modulus p is odd, then the angle formed by a pair $(\mathbf{H}_i, \mathbf{H}_j)$ of consecutive half-hyperplanes is necessarily smaller than $\pi - \vartheta_0(p)$, where $\vartheta_0(p)$ is a positive geometric constant depending only on p. This is effectively the reason why for odd moduli our conclusion is stronger.

Now we are able to formulate the most important result of the paper, an Decay Theorem: It states, roughly speaking, that if the current T is sufficiently close, at a given scale ρ, to a cone $\mathbf{C}$ as above around a point q where T has density at least $\frac{p}{2}$, at a scale $\delta\rho$ the distance to a suitable cone $\mathbf{C}'$ with the same structure will decay by a constant factor. This is the counterpart of a similar decay theorem proved by Simon in his pioneering work on cylindrical tangent cones [3] of multiplicity 1 under the assumption that the cross section satisfies a suitable integrability condition, which in turn is a far-reaching generalization of the work of Taylor in

[4] for the specific case of 2-dimensional area-minimizing cones $\mod 3$ in $\mathbb{R}^3$ with 1-dimensional spines (to our knowledge, the first theorem of the kind ever proved in the literature for a "singular cone").

In order to deal with the fact that the multiplicities are allowed to be larger than 1 substantial work is needed. To perform our analysis, the theorem is proved for cones $\mathbf{C}$ which in turn are sufficiently close to a fixed reference cone $\mathbf{C}_0$. While $\mathbf{C}_0$ is assumed to be area-minimizing mod(p), both $\mathbf{C}$ and $\mathbf{C}'$ are not. The Decay Theorem can be iterate and we obtain as a consequence the mentioned uniqueness of the tangent cone and that locally we are in the situation of a "classical" geometric free boundary. Hence the main part of our work is devoted to the proof of the Decay Theorem.

As in many similar regularity proofs (starting from the pioneering work of De Giorgi [1]) the main argument is a "blow-up" procedure: after scaling, we focus on a sequence of area-minimizing currents T_k which are close at scale 1 to cones $\mathbf{C}_k$, which in turn converge to a reference cone $\mathbf{C}_0$. $\mathbf{C}_k$ and $\mathbf{C}_0$ are assumed to share the same spine V. The distance between T_k and $\mathbf{C}_k$ (which is measured in an L^2 sense) is the relevant parameter and will be called *excess* and denoted by $\mathbf{E}_k$. The distance between $\mathbf{C}_k$ and $\mathbf{C}_0$ is not assumed to be related to $\mathbf{E}_k$. The overall idea is then to approximate the currents T_k and $\mathbf{C}_k$ with Lipschitz graphs over the halfplanes $\mathbf{H}_{0,i}$ forming $\mathbf{C}_0$, consider the differences between these graphs, renormalize them by $\mathbf{E}_k^{-\frac{1}{2}}$, and study their limits. These are proved to be harmonic (an idea that dates back to De Giorgi), while the remarkable insight of Simon's work [3] is used to prove suitable estimates (and compatibility relations) at the spine V. The novelty this time is that we need an suitable decay for "multivalued" harmonic functions. A fundamental realization of Simon is that, in order to accomplish the above program, one needs to introduce an additional object, for which we propose the term *binding function*, and whose role will be explained in a moment.

As already mentioned, the biggest source of complication is that the multiplicities $\kappa_{0,i}$ of the halfplanes $\mathbf{H}_{0,i}$ forming the support of $\mathbf{C}_0$ are typically larger than 1. In particular it is necessary to use $\kappa_{0,i}$ (not necessarily all distinct) functions to approximate the portions of the current T_k which are close to $\mathbf{H}_{0,i}$. Likewise, it is necessary to use $\kappa_{0,i}$ functions to describe the portions of $\mathbf{C}_k$ which are close to $\mathbf{H}_{0,i}$. Notice that while we know that the number N_i of *distinct* functions needed in the representation ranges between 1 and $\kappa_{0,i}$ and that the multiplicities of the corresponding graphs are positive integers $\kappa_{i,j}$ which sum up to $\kappa_{0,i}$, any choice of coefficients respecting these conditions is possible, and moreover the choice might be different for T_k and $\mathbf{C}_k$ and depend on k.

In order to produce graphical parametrizations of the current T_k at appropriate scales, we take advantage of the ϵ-regularity result proved by White in [5], but we also need to show that each such parametrization is close to one of the linear functions describing the cone $\mathbf{C}_k$. This major issue is absent in Simon's work [3] thanks to the multiplicity one assumption. A large part of our work address it leading to a the relevant "graphical approximation theorems".

First we show how to use [5] to gain a graphical parametrization of $T = T_k$. Inspired by [3] we subdivide the support of the current in regions Q of size comparable to their distance d_Q from the spine of $\mathbf{C}_0$. For practical reasons, d_Q will range in a dyadic scale and we will put an order relation on all the regions according to whether a region Q' is lying "above" the region Q. We then apply the regularity theorem of [5] on any "good" region, i.e. any Q with the property that at Q and at every region above Q the current T is sufficiently close to $\mathbf{C} = \mathbf{C}_k$. A simple argument (which uses heavily the fact that the codimension of T in Σ is 1), allows to "patch" together the graphical approximations across different regions to achieve $p = \sum_i \kappa_{0,i}$ "sheets" which approximate efficiently the current.

In fact we show that on each region Q each "graphical sheet" of T is close to some sheet of $\mathbf{C}$: the main ingredient is an appropriate Harnack-type estimate for solutions of the minimal surface equation. While at this stage the choice might depend on the region Q, an appropriate selection algorithm allows to bridge across different regions and show that there is a single sheet of $\mathbf{C}$ to which each single graphical sheet of T is close on every region Q. The latter selection algorithm will in fact be used again twice later on. An important thing to be noticed is that, since we use a one-sided excess, there might be some sheets of $\mathbf{C}$ which are not close to any of the graphical sheets of T: this phenomenon, which is not present in [3], is due to the fact that the multiplicities $\kappa_{0,i}$ might be higher than 1, and forces us to introduce an intermediate cone $\tilde{\mathbf{C}}$ which consists of those sheets of $\mathbf{C}$ which are close to at least one graphical sheet of T.

We next appropriately modify the key idea of [3] that the remainder in the classical monotonicity formula can be used to improve the estimates near the spine of the cone $\mathbf{C}_0$. First this is done to estimate the distance of T to suitable shifted copies of $\tilde{\mathbf{C}}$, centered at points of high density of T. It is in this section that we exploit crucially a reparametrization of the graphical sheets of T over $\tilde{\mathbf{C}}$ and, in particular, the fact that $\tilde{\mathbf{C}}$ does not contain any "halfspace of $\mathbf{C}$ far from T". The mod(p) structure allows us to prove the so-called "no-hole condition", namely some point of high density of T must be located close to any point of the spine of $\tilde{\mathbf{C}}$ (which, we recall, is the same as the spine of $\mathbf{C}$ and $\mathbf{C}_0$). The latter is combined with the previously established estimates, inspired by Simon, to prove that, upon subtracting some suitable piecewise constant functions with a particular cylindrical structure (the *binding functions*), the graphical sheets enjoy good estimates close to the spine. However, again caused by multiplicities $\kappa_{0,i}$, unlike in [3], we need to introduce a suitable correction to the binding functions, and a crucial point is that the size of the latter can be estimated by the product of the excess $\mathbf{E}^{\frac{1}{2}} = \mathbf{E}_k^{\frac{1}{2}}$ and the distance of $\mathbf{C} = \mathbf{C}_k$ to $\mathbf{C}_0$.

With the sketched modifications we are able to perform the intended "blow-up" procedure.

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Symmetry results for Plateau's surfaces

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(joint work with Jacob Bernstein)

A **minimal Plateau's surface** Σ is defined here as a closed subset of $\mathbb{R}^3$ such that, for each point $p \in \Sigma$, one can find $r > 0$, $\alpha \in (0, 1)$, and a $C^{1,\alpha}$-diffeomorphism $f_p : B_r(p) \to B_r(p)$, with $Df_p(p)$ a linear isometry, such that $f_p(\Sigma \cap B_r(p)) = K \cap B_r(p)$, where K is either a plane P, a half-plane H, a union Y of three half-planes meeting along a common line at 120-degrees, or a regular tetrahedral cone T. Moreover, the interior set Σ of the points p with $K = P$ is assumed to have vanishing mean curvature – given the $C^{1,\alpha}$-regularity, at first in distributional sense, and thus, by elliptic regularity, in the classical, smooth sense.

The above definition captures, in elementary mathematical terms, the content of the experimental laws of Plateau for soap films at equilibrium. It should be noted that Plateau also experimented with soap bubbles, where the mean curvature of the interior set may take different constant values on different connected components. Also, the above definition does not include the possibility of "singular boundary points", which are indeed physically possible, although (apparently) not exhaustively described in the physical and mathematical literature.

As shown in the works of Almgren [2] and Taylor [8], minimal Plateau's surfaces arise as *Almgren minimal sets*, i.e., as closed sets locally minimizing the two-dimensional Hausdorff measure $\mathcal{H}^2$ in $\mathbb{R}^3$ with respect to local Lipschitz deformations. Variational characterization of Almgren minimal sets as global minimizers in suitable variational problems have been first proposed, and then obtained, by several authors in recent years. Limiting ourselves to the first results concerning area minimization in codimension one we mention here [4, 5, 6] as entry points in a vaster literature.

Classical minimal surfaces are often motivated in terms of their application to the description of soap films. From this viewpoint, given the ubiquity of Y-type and T-type singularity, we consider the fascinating idea of reviewing classical results for smooth minimal surfaces in the physically more relevant context of minimal Plateau's surfaces.

In this direction, we consider as a case study a rigidity theorem of Schoen [7] for catenoids: given two co-axial circles in $\mathbb{R}^3$, the only minimal surfaces bounded by those circles are either catenoids or disks. The expected result in the context of minimal Plateau's surfaces should include an additional rigidity case, which will be present depending on the metric data of the problem (radii of the circles and their distance), and consists of singular catenoids, i.e. union of two catenoidal necks and a disk meeting along a common boundary circle of Y-points.

In [3] we obtain the expected extension of Schoen's rigidity theorem to minimal Plateau's surfaces. For reasons whose nature is likely just technical, this is done under the assumptions that the two circles have the same radii, and under a global to local topological assumption called "cellular structure" (for each $p \in \Sigma$ there exists $r_p > 0$ such that $\mathbb{R}^3 \setminus \Sigma$ and $B_r(p) \setminus \Sigma$ have the same finite number of connected components if $r < r_p$).

The result is obtained by an original application of the classical moving planes method introduced by Alexandrov in [1]. Interestingly, Alexandrov's method has been concurrently applied in the non–smooth setting of varifold solutions to the mean curvature flow in [9], and, independently from our work, to the study of Schoen's rigidity theorem in the varifold setting, but assuming *a priori* the absence of Y-singularities, in [10]. A novel contribution we can offer is the insight that the application of the moving planes can be compatible with the actual presence of singularities, while still working as a tool to obtain additional regularity (we exclude T-type singularities in a situation where they could indeed be possible).

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