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Analysis, Geometry and Topology of Singular PDE (hybrid meeting)

Organized by
Claire Debord, Paris
Rafe Mazzeo, Stanford
Paolo Piazza, Roma
Boris Vertman, Oldenburg

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Abstract. This is a report on the Oberwolfach conference "Analysis, geometry and topology of singular PDE", June 2-12, 2021. This workshop, held in an hybrid format, focused on the topology, geometry and geometric analysis of certain spaces with singularities: stratified spaces, compactifications of moduli spaces, spaces carrying a (singular) foliation, etc., and on the microlocal techniques, either in their classical forms or in more recent versions developed to handle singular PDE, or via the groupoid approach.

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Introduction by the Organizers

The Conference "Analysis, geometry and topology of singular PDE" took place at Oberwolfach from June 7-11, 2021. This workshop brought together a diverse audience of mathematicians interested in the study of stratified spaces and singular structures that appear in specific problems in topology, in various parts of geometric analysis and mathematical physics, as well as in the different techniques that are used to solve these problems. The setting of stratified geometries is a particularly convenient one, occupying a middle ground between the study of "very singular" (e.g. metric measure) spaces and the usual study of smooth manifolds. During the past decade, many important advances have been made in developing the tools of geometric analysis in these settings, and in using these tools to explore the relationships between topological, geometric and analytic concepts on these spaces. There are now a number of different, albeit related, approaches, and an increasingly wide range of problems in this area to which these techniques are being applied. During this workshop, we focused on various directions going from the more algebraic and topological ideas needed in general index theories, the very flexible use of groupoids and their applications to the deep PDE questions and geometric microlocal methods adapted to carry out analysis in these singular settings.

The workshop started with an an introductory talk by Georges Skandalis on pseudodifferential calculi associated with Lie groupoids. The aim was to provide the audience with the necessary tools to understand recent advances based on these calculi and their generalisations, and in particular the recent spectacular work of Androulidakis, Mohsen, Van Erp and Yuncken, which was presented during the week by Omar Mohsen. This work deals with inhomogeneous calculus. Mohsen explained how to define an inhomogeneous principal symbol associated to a singular filtration of the tangent bundle together with an associated deformation groupoid and a pseudo-differential calculus. He also explained how the inhomogeneous principal symbol can be used to compute the index of differential operators which are elliptic in such calculus. Another use of groupoids in the study of operators associated with singular spaces was proposed in Paulo Carrilo Rousse's lecture on Friday. He presented a collaborative work with Jean-Marie Lescure and Mario Velasquez where they study Fredholm boundary conditions on manifolds with corners. This involves the introduction of conormal homology, a very simple and computable homology group which depends on the faces, and in which the obtructions of being Fredholm live.

A number of talks were devoted to the use of suitable pseudodifferential calculi, and other methods from geometric microlocal analysis, to study singular geometric problems. Pierre Albin discussed his work with Hadrian Quan on the study of the heat kernel on a contact manifold X as the metric degenerates in the transverse direction to the corank 1 subbundle $\mathcal{H} \subset TX$ defining the contact structure. The heat kernels for this family of degenerating metrics is analyzed, leading to a formula for the limit of the analytic torsion associated to the Rumin complex. Daniel Grieser's talk described his analysis of the low energy resolvent behavior for Laplace-type operators on spaces with fibered boundary metrics. This is an analytic result, naturally within the domain of geometric scattering theory, but these types of results have been shown to be important in the analysis of long-time asymptotics of the heat kernel on these spaces, which in turn is a key tool in index theory on this class of open manifolds.

Andras Vasy explained a set of techniques he has developed over the last several years to analyze 'on-spectrum' behavior of resolvents. More specifically, if H is a quantum Hamiltonian, for example $-\Delta + V$ on an asymptotically conic space, then the resolvent $R(\lambda) = (H - \lambda)^{-1}$ is a holomorphic family of operators lying in the small scattering calculus when λ lies in the resolvent set of H. The limiting absorption theorem describes what happens as λ approaches points of the real line which lie in the smooth part of the continuous spectrum. In the limit, $R(\lambda)$ cannot be bounded on L^2, but following a very old theme in scattering theory, the goal is to find spaces on which this limit is bounded, or perhaps even better. Vasy's work, strongly motivated by his work on stability problems for cosmological spacetimes in general relativity, presents a new class of spaces for which this limit is actually Fredholm. The key point is that although $H - \lambda$ is elliptic on the asymptotically conic space itself, its scattering symbol on a compactification of this space is not elliptic, and propagation of singularities phenomena 'at infinity' become relevant. Vasy explained how initially he had defined new function spaces which allow for a microlocal drop in regularity along bicharacteristics, and on which one can show that the limit of $R(\lambda)$ is Fredholm. He concluded his talk with the description of a 2-microlocalization which makes it possible to define suitable spaces even more simply.

Other lectures focused on problems where the techniques stemmed from other parts of analysis. Gilles Carron presented his recent work with Lye and Vertman about the Yamabe flow on manifolds with iterated edge singularities. This flow and the Ricci flow had been studied in this same setting of incomplete edge metrics by several authors, including Vertman and Bahuaud, and the 'stationary' Yamabe problem itself has been analyzed in detail for iterated edge spaces by Akutagawa, Carron and Mazzeo. Analysis of this flow presents many very difficult analytic challenges. This new work shows how, under a certain condition on the integrability of the scalar curvature, the linearization of this flow may be studied using abstract methods for parabolic equations on spaces satisfying some weak global conditions, e.g. existence of a Sobolev inequality, etc. They apply this in the setting of spaces with positive Yamabe invariant to obtain long-time existence and subsequential convergence of the flow. Elmar Schrohe reported on his joint work with Thorben Krietenstein about a bounded H^∞-calculus and its use in the treatment of certain degenerate elliptic boundary value problems with non-smooth coefficients. Particular emphasis was given to the construction of a parameter dependent parametrix, using an extension of the Boutet de Monvel calculus.

Julie Rowlett's talk was about spectral geometry. She explained her recent work with Erik Nilsson and Felix Rydell concerning isospectral but non-isometric tori. Their interest is in understanding the *flat choir number* $\flat_n$, i.e., the supremum over all $k \in \mathbb{N}$ such that there is a collection $T_1, \dots, T_k$ of mutually isospectral and non-isometric flat tori.

There were also talks connecting to more topological problems. Francesco Bei reported on the use of the heat kernel in order to compare the (maximal) L^2-cohomology groups and L^p-cohomology groups for $p > 2$, on an incomplete Riemannian manifold. His main result shows that under suitable assumptions there exist homomorphisms between these groups that are injective and in some cases even isomorphisms. Applied to certain Thom-Mather spaces, this yields interesting inequalities relating the dimension of the intersection cohomology in different perversities. Chris Kottke reported on joint work with Richard Melrose on the emerging theory of bigerbes, and in particular on the so-called Brylinski-McLaughlin bigerbe. He explained why its triviality (as a bigerbe) is equivalent to the existence of a string structure on the underlying manifold X. This is also equivalent to the existence of a fusion loop-spin structure on the loop group $\mathcal{L}X$.

Yet other talks were devoted to new developments in index theory. Christian Bär reported on joint work with Alexander Strohmaier, explaining a new approach, based on Feynman propagators, to their Atiyah-Patodi-Singer index formula on Lorentzian manifolds. Matthias Lesch reported on joint work with Alexandre Baldare, Rémi Come and Victor Nistor. There, given a Γ-invariant pseudodifferential operator P on a Γ-manifold without boundary, Γ-finite, and given an irreducible representation $\alpha \in \widehat{\Gamma}$, the main question is whether the restriction of P to a α-isotypical component of a Sobolev space is Fredholm. His work gives a necessary and sufficient condition.

The other major focus of the workshop was on analysis related to moduli spaces and other 'naturally occuring' singular spaces. Richard Melrose's talk considered the problem of computing the Hodge cohomology, i.e., space of L^2 (or weighted L^2) harmonic forms on the Riemann moduli space for surfaces of genus g with respect to the Weil-Petersson metric. This moduli space is a much-studied and natural geometric object which plays a key role in many parts of mathematics. It is a 'naturally occuring' singular space, and the singular structure of this metric presents a number of new technical challenges. Melrose used this setting as motivation to describe the more general process of constructing pseudodifferential calculi adapted to certain classes of adapted metrics on manifolds with corners carrying an iterated fibration structure. This gave him an opportunity to sharpen a set of hypotheses required to carrry out this analysis.

There were also talks involving the class of gravitational instantons, i.e., four-dimensional complete hyperKaehler manifolds. These are important spaces which arise in many different contexts, for example, as the 'bubbles' in various types of degenerations of Calabi-Yau surfaces and other Einstein 4-manifolds. There is now a satisfactory theorem showing that there are six separate families of these objects, going by the monikers ALE, ALF, ALG, ALG*, ALH and ALH*. This subject has deep relationships with gauge theory, for example, these spaces may arise as gauge-theoretic moduli spaces. However, they are also natural backgrounds to study gauge theory; Two talks represented new work in this area.

The talk of Sergey Cherkis presented a set of results he has developed over the last several years, most recently with Hubach and Stern, which provides a parametrization of Yang-Mills instantons over ALF manifolds. This involves his notion of bow moduli spaces, a deep and intricate generalization of the classical ADHM construction, which provides a complete description of all Yang-Mills instantons on S^4. These bows are defined by diagrams which are partly linear algebraic and partly analytic in nature. He presented the 'up' and 'down' transforms, which carry bows to instantons and vice versa, and outlined the main result that these transforms are natural isomorphisms between the moduli space of bow representations and the moduli space of Yang-Mills instantons. Finally, Xuwen Zhu described her recent work with Mazzeo concerning the Fredholm theory of Laplace-type operators on ALH* manifolds, and some applications. After some transformations, it turns out that these operators and their inverses lie in the class of a-pseudodifferential operators, as developed initially by Grieser and Hunsicker. The analytic results that Zhu obtains sharpen results obtained earlier by Chen-Sun-Viaclovsky-Zhang using more classical methods. The main new applications include a computation of the space of L^2 harmonic forms, generalizing old work of Hausel-Hunsicker-Mazzeo, and a description of the local deformation theory of these spaces.

Overall, this workshop provided an excellent setting for specialists from different parts of this general field of research to understand the range of problems and applications being studied and to learn the variety of techniques which are being brought to bear. One of the original intents of this workshop was to help build better communication between the communities of researchers who use group techniques and those who employ geometric microlocal analysis, and in this regard, the meeting was a definite success.

Workshop (hybrid meeting): Analysis, Geometry and Topology of Singular PDE

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Abstracts

The sub-Riemannian limit of a contact manifold

Pierre Albin
(joint work with Hadrian Quan)

Let M be a contact manifold of dimension $m = 2n + 1$ endowed with a global contact form; that is, a smooth one form θ such that $\theta \wedge (d\theta)^n$ is a nowhere-vanishing form of top degree. The null space of θ is a co-rank one sub-bundle of the tangent bundle which we denote $\mathcal{H}$.

An illustrative example is $\mathbb{R}^3$ with contact form

$$\theta = dz + \tfrac{1}{2}(x\ dy - y\ dx).$$

Here $X = \partial_x + \frac{y}{2}\partial_z$ and $Y = \partial_y - \frac{x}{2}\partial_z$ make up a frame for $\mathcal{H}$. Since $[Y, X] = \partial_z$, we see that every vector field on $\mathbb{R}^3$ can be obtained as a linear combination of sections of $\mathcal{H}$ and their brackets. We can assign an $\mathcal{H}$-degree to a differential operator as the smallest degree of a polynomial in the vector fields tangent to $\mathcal{H}$ that yields that operator. Clearly X and Y have $\mathcal{H}$-degree one and ∂_z has $\mathcal{H}$-degree two. Note that the $\mathcal{H}$-degree will be well-behaved for anisotropic dilations of $\mathbb{R}^3$,

$$(x, y, z) \mapsto (\lambda x, \lambda y, \lambda^2 z).$$

Michel Rumin [Rum94] used similar anisotropic dilations on arbitrary contact manifolds to introduce a complex of differential forms, now known as the Rumin complex,

$$0 \to \Omega^0_{\mathcal{H}}M \xrightarrow{d_{\mathcal{H}}} \cdots \xrightarrow{d_{\mathcal{H}}} \Omega^n_{\mathcal{H}}M \xrightarrow{D_{\mathcal{H}}} \Omega^{n+1}_{\mathcal{H}}M \xrightarrow{d_{\mathcal{H}}} \cdots \xrightarrow{d_{\mathcal{H}}} \Omega^m_{\mathcal{H}}M \to 0.$$

This complex computes the de Rham cohomology of M and has the striking feature that the operator $D_{\mathcal{H}}$ is a differential operator of order two.

In subsequent work, Rumin [Rum00] showed that the contact complex can be derived from a spectral sequence. He also studied the behavior of the resolvents of Hodge Laplacians of Riemannian metrics g_ε equal to a fixed bundle metric $g_{\mathcal{H}}$ on $\mathcal{H}$ that are blowing-up in the directions transverse to $\mathcal{H}$ as $\varepsilon \to 0$.

In this talk I report on joint work with Hadrian Quan in which we analyze the behavior of the heat kernels of these Hodge Laplacians and particularly their traces in this limit. As a first step we carry out the Mazzeo-Melrose approach to spectral sequences via Hodge theory [MM90]. This has the advantage of making the Rumin complex appear analytically and singles out the different asymptotic regimes in which the Laplacian degenerates as $\varepsilon \to 0$.

We construct manifolds with corners on which the heat kernels of the Hodge Laplacians are polyhomogeneous down to $\varepsilon = 0$. Similarly we find manifolds with corners on which the traces of the heat kernels, initially functions of t and ε, lift to be polyhomogeneous. We apply this to analyze the behavior of global spectral invariants such as the η-invariant and the determinants of the Laplacians. In particular we show that contact versions of the relative η-invariant the relative analytic torsion are equal to their Riemannian analogues and hence topological.

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Local index theory for Lorentzian manifolds

Christian Bär

(joint work with Alexander Strohmaier)

On a closed manifold, index theory is intimately related to ellipticity. A differential operator is Fredholm on the standard Sobolev spaces if and only if it is elliptic. On compact Riemannian manifolds with boundary, the index theorem by Atiyah, Patodi, and Singer [1] says that Dirac-type operators are Fredholm if suitable global boundary conditions are imposed. This extends to general elliptic differential operators of first order [2]. Surprisingly, it turns out that there is an analog for this on Lorentzian manifolds with boundary where the Dirac operator is not elliptic but hyperbolic.

Theorem 1 (Bär-Strohmaier (2019) [4]). *Let (M, g) be an even-dimensional compact globally hyperbolic Lorentzian spin manifold with boundary $\partial M = \Sigma_- \sqcup \Sigma_+$. Here $\Sigma_\pm$ are smooth spacelike Cauchy hypersurfaces.*

Then the Dirac operator D_{APS} under Atiyah-Patodi-Singer boundary conditions is Fredholm and its index is given by

$$\text{(1)}\quad \operatorname{ind}[D_{\mathrm{APS}}] = \int_M \widehat{\mathrm{A}}(g) + \int_{\partial M} \mathrm{T}\widehat{\mathrm{A}}(g) - \frac{h(A_-) + h(A_+) + \eta(A_-) - \eta(A_+)}{2}\,.$$

The right hand side in the index formula is precisely the same as in the original Riemannian Atiyah-Patodi-Singer index theorem. Here $\widehat{\mathrm{A}}(g)$ is the $\widehat{\mathrm{A}}$-form manufactured from the curvature of the Levi-Civita connection of the Lorentzian manifold, $\mathrm{T}\widehat{\mathrm{A}}(g)$ is the corresponding transgression form which also depends on the second fundamental form of the boundary (and vanishes if the boundary is totally geodesic). Moreover, $A_\pm$ denotes the Dirac operator on $\Sigma_\pm$, h the dimension of the kernel, and η the η-invariant.

The theorem has been extended in various ways. The author and Hannes show that the Atiyah-Patodi-Singer conditions can be replaced by a large class of boundary conditions [3]. Braverman [7] drops spatial compactness and adds certain potentials to the Dirac operators so that it behaves similarly as on compact manifolds. Shen and Wrochna [8] drop temporal compactness and obtain a scattering-type result. In [5] the author and Strohmaier apply the index theorem to compute the chiral anomaly in quantum field theory on curved backgrounds.

The theorem applies also to the Dirac operator twisted with a coefficient bundle with connection ∇^E, provided ∇^E is compatible with a positive metric on E. One then has to include the Chern character form of ∇^E in the index formula as usual. The compatibility condition on the connection is necessary to ensure that the Dirac operators induced on the boundary are self-adjoint. Otherwise, it is unclear what is meant by Atiyah-Patodi-Singer boundary conditions and, more seriously, the argument in [4] involving spectral flow breaks down.

Based on [6], the talk is devoted to an alternative derivation of the index formula (1) without resorting to the Riemannian index formula using spectral flow. This direct computation of the index density uses Feynman propagators and a detailed analysis of their singular structure. The result is an analog of the *local* index theorem in the Riemannian setting. Integration then yields a generalization of (1) for general Dirac-type operators.

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L^p-cohomology, heat operator and stratified spaces

Francesco Bei

Let (M, g) be a possibly incomplete Riemannian manifold of finite volume and dimension m. For any $1 \leq p \leq q \leq \infty$ and $k \in \{0, ..., m\}$ it is well known that the identity $\Omega_c^k(M) \to \Omega_c^k(M)$ gives rise to a continuous inclusion $i : L^p\Omega^k(M, g) \hookrightarrow L^q\Omega^k(M, g)$, with $L^p\Omega^k(M, g)$ and $L^q\Omega^k(M, g)$ denoting the Banach space of L^p and L^q k-forms, respectively. Let $(L^p\Omega^*(M, g), d_{*,p,\max})$ be the L^p-maximal de Rham complex of (M, g). We recall that $d_{k,p,\max} : L^p\Omega^k(M, g) \to L^p\Omega^{k+1}(M, g)$ is defined as the distributional L^p extension of $d_k : \Omega_c^k(M) \to \Omega_c^{k+1}(M)$. Given any $q \geq p$ it is not difficult to check that the above continuous inclusion i induces a morphism of complexes $i : (L^q\Omega^*(M, g), d_{*,q,\max}) \to (L^p\Omega^*(M, g), d_{*,p,\max})$. Hence we get well-defined maps acting between the corresponding (reduced) cohomology groups:

$$i:\overline{H}^k_{q,\max}(M,g)\to\overline{H}^k_{p,\max}(M,g),\qquad i:H^k_{q,\max}(M,g)\to H^k_{p,\max}(M,g).$$

However the above maps are in general neither surjective nor injective. In this report we are going to describe some results obtained in [3] which provide a partial answer to the following question:

under what circumstances are the morphisms $i:\overline{H}^k_{q,\max}(M,g)\to\overline{H}^k_{p,\max}(M,g)$ and $i:H^k_{q,\max}(M,g)\to H^k_{p,\max}(M,g)$ injective and/or surjective?

One of the main ingredient that we used in our results is a spectral curvature condition known as "Kato class", see e.g. [4]. More precisely: let (M,g) be any Riemannian manifold. The well known Weitzenböck formula says that there exists a section $L_k\in C^\infty(M,\mathrm{End}(\Lambda^k(M)))$ such that $\Delta_k=\nabla^t\circ\nabla+L_k$. Let us define $\ell_k^-:M\to\mathbb{R}$ as

$$\ell_k^-(x):=\max\{-\lambda_k(x),0\}$$

with

$$\lambda_k(x):=\inf_{v\in\Lambda^kT_x^*M,\ g(v,v)=1}g(L_{k,x}v,v).$$

We say that the negative part of L_k lies in the the Kato class of (M,g), $\ell_k^-(x)\in\mathcal{K}(M)$, if

$$\lim_{t\to0^+}\sup_{x\in M}\int_0^t\int_M p(s,x,y)\ell^-(y)\mathrm{dvol}_g(y)ds=0$$

where $p(s,x,y)$ denotes the kernel of $e^{-t\Delta_0^{\mathcal{F}}}:L^2(M,g)\to L^2(M,g)$ and $\Delta_0^{\mathcal{F}}:L^2(M,g)\to L^2(M,g)$ is the Friedrichs extension of the Laplace-Beltrami operator $\Delta_0:C_c^\infty(M)\to C_c^\infty(M)$. We recall that for the Kato class the following properties are true: if $\ell_k^-\in L^\infty(M)$ then $\ell_k^-\in\mathcal{K}(M)$; moreover if $m>2$, (M,g) carries a Sobolev embedding $W_0^{1,2}(M,g)\hookrightarrow L^{\frac{2m}{m-2}}(M,g)$ and $\ell_k^-\in L^q(M,g)$ with $q>m/2$ then $\ell_k^-\in\mathcal{K}(M)$. We are finally in position to state our first main result:

Theorem A. *Let (M,g) be an open and incomplete Riemannian manifold of dimension $m>2$. Assume that:*

- *$vol_g(M)<\infty$;*
- *There is a Sobolev embeddings $W_0^{1,2}(M,g)\hookrightarrow L^{\frac{2m}{m-2}}(M,g)$;*
- *There exists $k\in\{0,...,m\}$ such that $\ell_k^-\in\mathcal{K}(M)$;*
- *We have the following equality $\Delta_{k,\mathrm{abs}}=\Delta_k^{\mathcal{F}}$.*

Then for any $2<p\leq\infty$ there exists an injective linear map

$$\beta_p:\overline{H}^k_{2,\max}(M,g)\to\overline{H}^k_{p,\max}(M,g).$$

Above $\Delta_{k,\mathrm{abs}}$ denotes the absolute extension of $\Delta_k:\Omega_c^k(M)\to\Omega_c^k(M)$, that is the self-adjoint extension of $\Delta_k:\Omega_c^k(M)\to\Omega_c^k(M)$ induced by the L^2-maximal de Rham complex whereas $\Delta_k^{\mathcal{F}}$ is the Friedrichs extension of $\Delta_k:\Omega_c^k(M)\to\Omega_c^k(M)$. Our second main result reads as follows:

Theorem B. *Let (M, g) be an open incomplete Riemannian manifold of dimension $m > 2$. Assume that*

- $vol_g(M) < \infty$*;*
- *There exists $k \in \{0, ..., m\}$ such that $\ell^-_{k-1}, \ell^-_k \in \mathcal{K}(M)$;*
- $\mathrm{Im}(d_{k-1,\max,q})$ *is closed in $L^q\Omega^k(M, g)$ for some $2 \leq q < \infty$;*
- *There is a Sobolev embedding $W^{1,2}_0(M, g) \hookrightarrow L^{\frac{2m}{m-2}}(M, g)$;*
- *We have the following equalities $\Delta_{k,\mathrm{abs}} = \Delta^{\mathcal{F}}_k$ and $\Delta_{k-1,\mathrm{abs}} = \Delta^{\mathcal{F}}_{k-1}$.*

Then

$$i_k : H^k_{q,\max}(M, g) \to H^k_2(M, g)$$

is injective. In addition if $\mathrm{Im}(d_{k-1,\max,2})$ *is closed in $L^2\Omega^k(M, g)$ then*

$$i_k : H^k_{q,\max}(M, g) \to H^k_2(M, g)$$

is an isomorphism.

In the proofs of both the above theorems a key ingredient is provided by the L^p-L^q mapping properties of the heat operator $e^{-t\Delta^{\mathcal{F}}_k}$. The above results are then applied in the setting of compact Thom-Mather-Witt stratified pseudomanifolds whose regular part is endowed with an iterated conic metric. Thanks to [1] and [2] we know that all the hypothesis of Theorem B except that concerning the Kato class are fulfilled. Moreover thanks to [5] we know that the L^r-cohomology of the regular part of a Thom-Mather stratified pseudomanifold X endowed with an iterated conic metric g is isomorphic to the intersection cohomology of X with respect to the perversity $q_r(j) := j - 2 - [[j/r]]$, with $[[j/r]]$ denoting the biggest integer number strictly smaller than j/r. Therefore in the presence of suitable curvature properties the above theorems can be used to show various (in)equality between the dimensions of intersection cohomology groups corresponding to different perversities. More precisely:

Theorem C. *Let X be a compact smoothly Thom-Mather-Witt stratified pseudomanifolds of dimension $m > 2$. Let g be an iterated conic metric on $reg(X)$ such that $d + d^t$ is essentially self-adjoint on $L^2\Omega^\bullet(reg(X), g)$. Then:*

(1) *If $\ell^-_k \in \mathcal{K}(reg(X))$ for some $k \in \{0, ..., m\}$ then*

$$\dim(I^{\underline{m}}H^k(X, \mathbb{R})) \leq \dim(I^{q_r}H^k(X, \mathbb{R}))$$

for any $2 \leq r < \infty$.

(2) *If $\ell^-_{k-1}, \ell^-_k \in \mathcal{K}(reg(X))$ for some $k \in \{0, ..., m\}$ then*

$$\dim(I^{\underline{m}}H^k(X, \mathbb{R})) = \dim(I^{q_r}H^k(X, \mathbb{R}))$$

for any $2 \leq r < \infty$.

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On Fredholm boundary conditions on manifolds with corners, geometric and topological obstructions

Paulo Carrillo Rouse
(joint work with Jean-Marie Lescure, Mario Velasquez)

In this talk I reported on a joint work with Jean-Marie Lescure (Paris 13) and Mario Vélasquez (Bogota), [1, 2], and ongoing work with Lescure. In a series of papers we have been studying obstructions on Fredholm boundary conditions for manifolds with corners, in particular we have showed how these obstructions live in some very simple and computable homology groups depending on the faces, conormal homology, denoted by $H^{cn}_{ev/odd}(X)$ for a manifold with corners X.

In the first part of the talk I resumed and explained the main results of these series of papers on conormal homology and Fredholm boundary conditions. Indeed, after recalling the previous work of several authors (Melrose and Piazza; Melrose and Nistor; Nazaikinskii, Savin and Sternin; Monthubert and Nistor among the principal ones) I explained how the obstruction for a given $b-$elliptic operator on a closed manifold with corners to be Fredholm (up to perturbation by a regularizing operator) is encoded in what we called the boundary analytic index of the operator (of its principal symbol class). To be more precise for D an elliptic b-pseudodifferential operator on a compact manifold with corners X it is known that D is Fredholm (up to perturbation as explain in [1, 2]) if and only if

$$Ind^{\partial}_X([\sigma_b(D)]) = 0 \text{ in } K_0(\mathcal{K}_b(X)) \cong K_0(\mathcal{K}_b(\partial X)),$$

where $K_0(\mathcal{K}_b(X))$ (or it's isomorphic restriction to the boundary version) is the K_0-group of the C^*-algebra of b-compact operators (the closure as bounded operators on some L^2-spaces of the regularizing b-operators), and where

$$Ind^{\partial}_X : K^0({}^bT^*X) \longrightarrow K_0(\mathcal{K}_b(X)) \tag{1}$$

is the index morphism associated to the principal symbol pseudodifferential extension in the calculus. This motivated in the talk the presentation of the following computation initiated in [1] and completed in [2].

Theorem [Carrillo Rouse, Lescure, Velasquez, 2019] For every connected manifold with corners X there are morphisms

$$T_{ev/odd} : K_{ev/odd}(\mathcal{K}_b(X)) \longrightarrow H^{cn}_{ev/odd}(X) \otimes \mathbb{Q} \tag{2}$$

inducing rational isomorphisms.

The computation, as explained in the talk can be made very explicit by means of the use of an appropriate embedding on some euclidean space and the construction of an associated topological space whose singular cohomology computes the conormal homology.

The previous discussion and results make clear that besides being able to compute the above groups in where the obstructions live it is important to be able to compute the conormal cycles or classes associated a b-elliptic operator, that is to be able to compute the image of the principal symbol class by the composition of the morphism (1) followed by (2) above.

In the second part I talked about how to explicitly compute, for a given elliptic pdo in the calculus, the conormal cycles that give such obstructions, it was done in low codimensions for more clarity (codimension 2 and 3). In fact, as I sketched in the talk, these cycles depend on a series of Atiyah-Singer and Atiyah-Patodi-Singer kind of indices on the (strictly positive) even codimension faces. For example for X of codimension two, for D b-elliptic and for any corner component $c \in F_2(X)$ there is an associated Fredholm operator D_c on $c \times R^2$ (translation invariant) and we have that:

In $H^{cn}_{ev/odd}(X) \cong Ker\,\delta_2 \subset \mathbb{Z}^{\#F_2}$:

$$Ind^{\partial}_X([\sigma_D]) = \sum_{c \in F_2^{cn}(X)} Ind_{AS}(D_c) \cdot c,$$

where $Ind_{AS}(D_c)$ is some type of Atiyah-Singer index (given then in particular by a classical topological index formula) associated to the Fredholm operator D_c and where $F_2^{cn}(X) \subset F_2(X)$ is the set of faces belonging to a conormal cycle (notion explained in the talk). An explicit formula for the above conormal boundary analytic index for codimension 3 manifolds was then presented, in this case the conormal class can be computed by a 2-conormal cycle whose coefficients are given this time by some Atiyah-Patodi-Singer indices on the closed manifolds with boundary obtained by closing each 2-codimensional component on the manifold. All this second part of the talk is based on work in progress with Jean-Marie Lescure.

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Yamabe flow on some singular spaces

Gilles Carron

(joint work with Jørgen Olsen Lye and Boris Vertman)

We study the convergence of the normalized Yamabe flow with positive Yamabe constant on a class of pseudo-manifolds that includes stratified spaces with iterated cone-edge metrics. We establish convergence under a low-energy condition. We also prove a concentration–compactness dichotomy, and investigate what the alternatives to convergence is.

The Yamabe flow: The Yamabe flow [7] is the gradient flow of the Hilbert functional

$$\mathcal{S} : g \in \mathcal{C}(g_0) \mapsto \int_M \mathrm{Scal}_g \mathrm{dv}_g$$

restricted to conformal metrics with fixed volume:

$$\mathcal{C}(g_0) := \left\{g = e^{2f} g_0, \text{such that } \mathrm{vol}_g(M) = 1\right\}.$$

Critical points of $\mathcal{S} : \mathcal{C}(g_0) \to \mathbb{R}$ are conformal metric with constant scalar curvature. In dimension $n \geq 3$, writing $g_t = u^{\frac{4}{n-2}}(t) g_0$, the Yamabe flow induces a non linear parabolic PDE:

$$\frac{\partial}{\partial t} u(t) = \frac{n-2}{4}\left(\sigma(t)u(t) - u^{\frac{4}{n-2}}(t) L u(t)\right),$$

where

$$L = 4\frac{n-1}{n-2}\Delta_{g_0} + \mathrm{Scal}_{g_0} = c_n \Delta_{g_0} + \mathrm{Scal}_{g_0}$$

is the Yamabe operator of the metric g_0 and

$$\begin{aligned}\sigma(t) &:= \int_M \mathrm{Scal}_{g_t} \mathrm{dv}_{g_t}\\ &= \int_M \left[c_n |du(t)|^2 + \mathrm{Scal}_{g_0} u^2(t)\right] \mathrm{dv}_{g_0},\end{aligned}$$

Recall that the Yamabe flow is the gradient flow of $\mathcal{S}$, hence $t \mapsto \sigma(t)$ is decreasing (unless the metric has constant scalar curvature).

Thanks to the works of Yamabe, Trudinger, Aubin and Schoen, we know that there is a smooth conformal metric (a Yamabe minimizer) $g \in \mathcal{C}(g_0)$ realizing

$$Y(M) = \inf_{\mathcal{C}(g_0)} \mathcal{S},$$

this metric have constant scalar curvature equals to $Y(M)$. An important tool to conclude about the existence of Yamabe minimizer is the positive mass theorem in dimension $n \in \{3, 4, 5\}$ and for locally conformally flat metrics.

Concerning the Yamabe flow on smooth closed manifold, we know that there is a global, in times, smooth solution:

$$u : [0, +\infty) \times M \longrightarrow \mathbb{R}.$$

Hence the issue is about the convergence of the flow when $t \mapsto +\infty$. It has been shown by Brendle [4, 5] and Schwetlick-Struwe [8] that when the positive mass theorem holds for g_0, then the Yamabe flow converges when $t \mapsto +\infty$ toward a smooth conformal metric with constant scalar curvature. The positive mass theorem is used to show that no bubbling phenomena occurs.

We have investigated the Yamabe flow on a singular setting, our results are parabolic counterpart of results of [1] concerning the existence of Yamabe minimizer in a singular setting. Let's first give a short and imprecise description of the geometry of these spaces.

Stratified space with iterated edge metric: If Σ is a complete metric space with distance d_Σ, the metric cone $C(\Sigma)$ over Σ is the completion of the product $(0,\infty)\times\Sigma$ with the distance for $p=(t,x),\ q=(s,y)\in(0,\infty)\times\Sigma$

$$d(p,q)=\begin{cases} t+s & \text{if} \quad d_Y(x,y)\geq\pi \\ \sqrt{t^2+s^2-2ts\cos\left(d_Y(x,y)\right)} & \text{if} \quad d_Y(x,y)\leq\pi \end{cases}$$

We have only to blown down $\{0\}\times\Sigma$ to a point (the vertex of the cone) from $[0+\infty)\times\Sigma$.

A stratified space with iterated edge metric is a compact metric space (X,d) with a stratification

$$X\supset X_{n-2}\supset\cdots\supset X_1\supset X_0$$

such that

- near each point $x\in X\setminus X_{n-2}=X_{\text{reg}}$, the geometry is Riemannian and induced by a Riemannian metric g_0.
- near each point $x\in X_k\setminus X_{k-1}$, the geometry "looks like" a product

  $$\mathbb{R}^k\times C(\Sigma_x)$$

  where Σ_x is a $(n-k-1)$- dimensional stratified space.

Scalar curvature: The regular part $X_{reg}:=X\setminus X_{n-2}$ is a smooth open Riemannian manifold and as $\operatorname{vol}X_{n-2}=0$ we can extend $\mathrm{Scal}_g\colon X\to\mathbb{R}$ to a measurable function. We can also define the Yamabe operator, for this purpose we introduce $H^1(X)$ that is the completion of $\mathrm{Lip}(X)$ with $u\mapsto\sqrt{\int_{X_{reg}}|du|^2+u^2}$. It turns out that $\mathcal{C}_0^\infty(X_{reg})$ is dense in $H^1(X)$ and the Yamabe operator L is the Friedrichs extension of the quadratic form

$$u\mapsto\int_X\left[c_n|du|^2+\mathrm{Scal}_{g_0}u^2\right]\mathrm{dv}_{g_0}.$$

It enjoys the same conformal properties as the Yamabe operator on smooth manifold. We can then define the Yamabe constant of X by

$$\begin{aligned} Y(X)&=\inf_{u\in H^1(X)}\left\{\int_X\left[c_n|du|^2+\mathrm{Scal}_{g_0}u^2\right]\mathrm{dv}_{g_0},\int_X|u|^{\frac{2n}{n-2}}\mathrm{dv}_{g_0}=1\right\}\\ &=\inf_{u\in\mathcal{C}_0^\infty(X_{reg})}\left\{\int_{X_{reg}}\left[c_n|du|^2+\mathrm{Scal}_{g_0}u^2\right]\mathrm{dv}_{g_0},\ \int_{X_{reg}}|u|^{\frac{2n}{n-2}}\mathrm{dv}_{g_0}=1\right\}.\end{aligned}$$

When $u\in\mathcal{C}(X)$ is positive and smooth on X_{reg}, one can define the conformal metric $g_u=u^{\frac{4}{n-2}}g_0$ and we will get that

$$Y(X)=\inf_{u\text{ such that }\mathrm{vol}_{g_u}(X)=1}\int_{X_{reg}}\mathrm{Scal}_{g_u}\mathrm{dv}_{g_u}.$$

Theorem A (Carron, Olsen Lye & Vertman, 2021) *Assume that X is a stratified space of dimension $n>2$ and that g_0 is iterated edge metric on X_{reg} such that*

$$Y(X)>0 \textit{ and } vol_{g_0}X_{reg}=1 \textit{ and } Scal_{g_0}\in L^{p>\frac{n}{2}}.$$

Then there is a long time solution of the Yamabe flow starting at g_0.

This solution is given by $t \mapsto u^{\frac{4}{n-2}}(t)g_0$ where for each $t > 0$, $u(t)$ is Hölder continuous and $Lu(t) \in L^p$. We also have a rough description of the convergence scenario when $t \to +\infty$:

Theorem B (Carron, Olsen Lye & Vertman, 2021) *Assume that X is a stratified space of dimension $n > 2$ and that g_0 is iterated edge metric on X_{reg} such that*

$$Y(X) > 0 \text{ and } vol_{g_0} X_{reg} = 1 \text{ and } Scal_{g_0} \in L^{p > \frac{n}{2}}.$$

and $t \mapsto u(t)$ be the solution of the Yamabe flow. There are $t_k \to \infty$ and a Hölder continuous function $u_\infty : X \to (0, \infty)$ solving the Yamabe equation

$$Lu_\infty = \sigma_\infty u_\infty^{\frac{n+2}{n-2}}$$

where $\sigma_\infty = \lim_{t \to +\infty} \sigma(t)$. Such that

$$u_k := u(t_k) \overset{H^1}{\rightharpoonup} u_\infty,$$

Moreover outside a finite set $F = \{x_1, \ldots, x_L\} \subset X$, we have strong convergence $u_k \overset{C^\alpha_{loc}(X \setminus F)}{\longrightarrow} u_\infty$.

We have also obtained a more precise description of the bubbling phenomena (bubble tree description).

Some questions

- Is it possible that concentrations occurs at smooth points or is it alway true that $F \subset X \setminus X_{reg}$?
- We cannot, in general, exclude that different scenarios occur along different sub-subsequence diverging towards infinity. For instance it could be possible that along $t_k \to +\infty$ and $s_\ell \to +\infty$ we have one of the following behavior:
  (1) $u(t_k)$ converges strongly in H^1 to some solution of the Yamabe equation and $u(s_\ell)$ converges weakly (and not strongly) to another solution of the Yamabe equation.
  (2) bubbling phenomena occurs for $\{u(t_k)\}$ in a neighborhood of a finite set F and bubbling phenomena occurs for $\{u(s_\ell)\}$ in a neighborhood of another finite set $F' \neq F$.

  It would be interesting to build such examples or to find a general criterion to exclude these kinds of behavior.
- Viaclovsky has shown that the stereographic conformal compactification of ALE gravitational instantons do not carry conformal metric with constant scalar curvature [9]. So in these case, we necessary have that $u(t)$ converges weakly in H^1 to 0 and that concentrations occurs at the unique singular point. It will be interesting to have a finer description of bubble tree. Our paper provides a general answer but in that case we do not know wether different diverging subsequences could produce different bubble trees and even more than one bubble.

- In some cases, we know that there is no Yamabe minimizers. Akutagawa and Mondello obtained this non existence for certain ramified cover of the sphere [2]. A recent result of Brendle [6] provides new examples: (M^n, g) is a complete manifold with non negative Ricci curvature with Euclidean growth that is not the Euclidean space, then we have that

$$\forall \varphi \in \mathcal{C}_0^\infty(M) : Y_\infty(M,g) \left(\int_M |u|^{\frac{2n}{n-2}} \mathrm{dv}_g\right)^{1-\frac{2}{n}} \leq \int_M c_n |du|^2 \mathrm{dv}_g$$

$$\text{with } Y_\infty(M,g) = Y(\mathbb{S}^n) \left(\lim_{R\to\infty} \frac{\operatorname{vol}(B(x,R))}{\omega_n R^n}\right)^{\frac{2}{n}}.$$

Moreover there is no u such that $du \in L^2$ and $u \in L^{2n/(n-2)}$ realizing this equality. Hence if (M, g) has a conformal compactification $(\hat{M}, \hat{g})$ that is a stratified space with an iterate edge metric (for instance if (M^n, g) is Asymptotically Locally Euclidean) then $Y(\hat{M}, \hat{g}) = Y_\infty(M, g)$ is not realized by any conformal deformation $v^{\frac{4}{n-2}}\hat{g}$ with $v \in H^1(\hat{M}, \hat{g})$.

It will be interesting to investigate the Yamabe flow on these examples and study wether we could find there other conformal metrics with constant scalar curvature.

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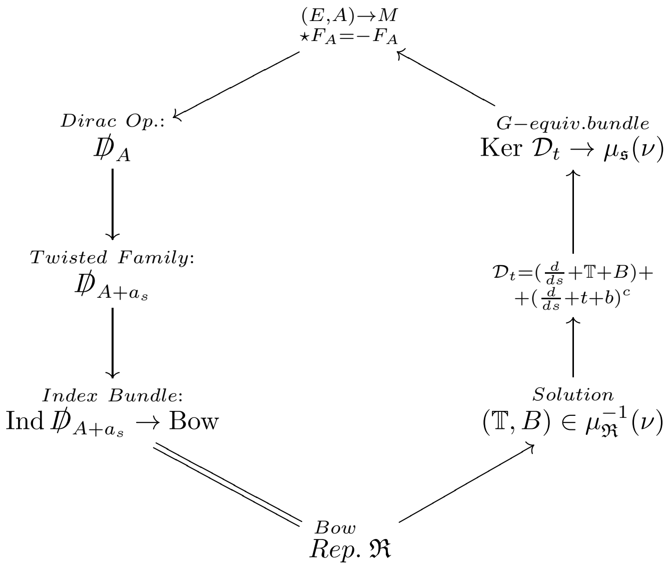
Bows to Instantons and Back

Sergey A. Cherkis
(joint work with Andrés Larraín-Hubach and Mark Stern)

We report the results of [1] relating bow data to Yang-Mills instantons on Asymptotically Locally Flat (ALF) space M. Each solution of a bow representation $\mathfrak{R}$ was expected to correspond to an instanton $(E \to M, A)$. We prove this fact (both that the resulting curvature F_A of connection A is anti-self-dual and is $L^2(M)$)

and compute all of the topological classes of the resulting instanton in terms of the bow representation ranks.

Our prior work [2] lays the groundwork for the Down transform mapping an instanton on any ALF space to a solution of the corresponding bow representation.



For brevity we focus on the A_k bow, presented as a circle of perimeter ℓ that is cut in k points, we call these p-points. Its representation $\mathfrak{R}$ consists of a Hermitian vector bundle over that circle that might change rank at p-points or at n other points, so called λ-points. The affine space $\mathrm{Dat}(\mathfrak{R})$ involves a connection and three endomorphisms T_1, T_2, T_3 of that bundle. The latter are L^2 close to prescribed poles at λ. This L^2 condition ensures that $\mathrm{Dat}(\mathfrak{R})$ is a Hilbert space and is hyperkähler. The bundle gauge group $G_{\mathfrak{R}}$ acts on it respecting its hyperkähler structure. The level set at level ν of its action is $\mu_{\mathfrak{R}}^{-1}(\nu)$. And the hyperkähler quotient is the *moduli space* of the bow representation $\mathfrak{R}$

$$\mathcal{M}_{\mathfrak{R}} = \mathrm{Dat}/\!/\!/G_{\mathfrak{R}} := \mu_{\mathfrak{R}}^{-1}(\nu)/G_{\mathfrak{R}}.$$

Any ALF space is a moduli space of some bow representation, for example a k-centered Taub-NUT space is a moduli space of a small bow representation $\mathfrak{s}$ of an A-type bow that has rank one and no λ-points. This is how the base space of the desired instanton emerges from the bow. Thus, any solution of the small bow representation $(t, b) \in \mu_{\mathfrak{s}}^{-1}(\nu)$ gives a point in M. Importantly, the level set itself $\mu_{\mathfrak{s}}^{-1}(\nu)$ can be regarded as a family (parameterized by the bow itself) of principal $U(1)$ bundles over $M = \mathcal{M}_{\mathfrak{s}}$.

Any solution $(T, B) \in \mu_{\mathfrak{R}}^{-1}(\nu)$ of some given bow representation $\mathcal{R}$, has a corresponding bow Dirac operator $\mathcal{D}_{T,B}$. Similarly, for any solution $(t, b) \in \mu_{\mathfrak{s}}^{-1}(\nu)$ there is a bow Dirac operator $\mathcal{D}_{t,b}$ and its charge conjugate $\mathcal{D}_{t,b}^c$. Combining the two, one obtains a family of bow Dirac operators

$$\mathcal{D}_t := \mathcal{D}_{T,B} \otimes 1_{\mathfrak{s}} + 1_{\mathfrak{R}} \otimes \mathcal{D}^c_{t,b},$$

parameterized by the level set $\mu_{\mathfrak{s}}^{-1}(\nu)$. Crucially, the moment map conditions and the matched levels of $\mathfrak{R}$ and $\mathfrak{s}$ ensure that $\mathcal{D}_t^*\mathcal{D}_t$ is positive and commutes with all quaternions (i.e. acts as scalar on the spin bundle). Thanks to positivity, the resulting index bundle is indeed a bundle over the level set of the small representation $\mu_{\mathfrak{s}}^{-1}(\nu)$, i.e. $\operatorname{Ker}\mathcal{D}_t = \{0\}$. Quotient by the gauge group of the small representation produces the bundle $\mathcal{E} \to \mathcal{M}_{\mathfrak{s}} = \mu_{\mathfrak{s}}^{-1}(\nu)/G_{\mathfrak{s}}$.

We prove that

(1) the connection induced on $\mathcal{E}$ is anti-self-dual (which is rather straightforward in this quaternionic setup),
(2) the asymptotic form of this connection is the direct sum of pullback of monopole connections from $\mathbb{R}^3$ (as in Theorem 1 below).

We also express all topological properties of the resulting connection in terms of the bow representation.

Technical Results. Our first regularity result is

Lemma 1:
On an ϵ-neighborhood of a λ-point for the difference $T_j(\lambda+s)+\frac{i\rho_j}{2s}$ we have $L^2 \implies s^{-\frac{1}{2}}L^\infty \implies L^\infty$.

For any representation, p- and λ-points subdivide the bow into open intervals. For each of these special points we introduce a space of solutions of the dual equation $\mathcal{D}_t\chi = 0$ over its neighborhood, satisfying appropriate matching conditions at that point. The direct sum of all such spaces is denoted by $Y = \oplus_\lambda Y_\lambda \oplus \oplus_p Y_p$. For each open interval, we introduce the space of solutions of $\mathcal{D}_t^*\psi = 0$ over it. The direct sum of all such spaces is denoted by $X = \oplus_{I\in Intervals} X_I$. Since χ and ψ satisfy dual equations, there is a good pairing $\langle\chi(s),\psi(s)\rangle$ which does not depend on s. Therefore the space $\mathcal{E} = \operatorname{Ker}\mathcal{D}_t^*$ fits into the short exact sequence

$$0 \to \mathcal{E} \to X \to Y^\vee \to 0.$$

Counting the dimensions of the spaces X and Y (and using positivity of $\mathcal{D}_t^*\mathcal{D}_t$) we find the rank of the instanton bundle expressed in

Lemma 2

$$\mathcal{E} = \dim\operatorname{Ker}\mathcal{D}_t^\dagger = -Ind\,\mathcal{D}_t = \dim X - \dim Y = |\Lambda| = n.$$

I.e. the rank of the instanton bundle equals to the number of λ-points of the bow representation $\mathfrak{R}$.

Estimating the Green's function: $(\mathcal{D}_t^*\mathcal{D}_t)^{-1} < \frac{C}{|\vec{t}|^2}$, we find approximate basis $\{f_a\}_{a=1}^n$ of $\operatorname{Ker}\mathcal{D}_t^*$ (with each element f_a of this basis concentrated near one λ-point λ_a). After establishing the accuracy of this approximation we use it to compute the induced instanton connection A with connection matrix, thus proving

Theorem 1:
There exists a local frame in which the induced connection matrix (A_{ab}) has essentially diagonal form

$$A_{ab} = \int_{Bow} f_a^* \nabla^{e_s} f_b ds + O(\frac{1}{|\vec{t}|^2}),$$

implying

$$A(\partial_\tau) = \mathrm{diag}_a \frac{-i}{\ell + \sum_p \frac{1}{2|\vec{t}-\vec{\nu}_p|}} \left(\lambda_a + \frac{m_a + |\{p_\sigma < \lambda_a\}|}{2|\vec{t}|}\right) + O(t^{-2}).. \tag{1}$$

Here, ∂_τ is the triholomorphic isometry vector field of the multi-Taub-NUT, m_a is the rank discontinuity of $\mathfrak{R}$ representation across λ_a. And the effective 'monopole charge' $\hat{m}_a$ is the sum of the rank change m_a and the number of p-points to the left of λ_a.

In order to compute the values of the Chern-Weil forms we use the Hausel-Hunsicker-Mazzeo [3] compactification $\overline{M}$ and, using the above theorem, we extend the instanton bundle $\mathcal{E} \to M$ to a bundle $\overline{\mathcal{E}} \to \overline{M}$. We judiciously extend the bundles X and Y to $\overline{X} \to \overline{M}$ and $\overline{Y} \to \overline{M}$, so that they still fit into an short exact sequence

$$0 \to \overline{\mathcal{E}} \to \overline{X} \to \overline{Y}^\vee \to 0. \tag{2}$$

The instanton connection now gives a singular connection on $\overline{\mathcal{E}}$ with logarithmic singularity at infinity. Comparing the Chern character values of $\overline{\mathcal{E}}$ computed using the short exact sequence (2) with the same Chern character values computed using the singular connection A on $\overline{\mathcal{E}}$, we obtain

Theorem 2:

$$\frac{i}{2\pi} \int_{C_p} \mathrm{tr} F_A = \Delta R_p - r_p + \sum_\lambda \frac{\lambda}{\ell},$$

$$\frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \int_{\mathcal{M}} \mathrm{tr} F_A \wedge F_A = -\frac{1}{2}\sum_\lambda \hat{m}_\lambda - R_0 + \sum_\lambda \frac{\lambda}{\ell}\hat{m}_\lambda - \frac{k}{2}\sum_\lambda \left(\frac{\lambda}{\ell}\right)^2.$$

Here r_p is the number of λ-points to the right of p, C_p is the preimage in M of a ray in $\mathbb{R}^3$ originating at ν_p, and R_0 is the rank of the representation $\mathfrak{R}$ over $s = 0$. This expresses all topological properties of the resulting instanton in terms of the bow representation $\mathfrak{R}$.

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Low energy resolvent asymptotics for fibred boundary metrics

Daniel Grieser
(joint work with Boris Vertman, Mohammad Talebi)

The resolvent of the Laplacian in $\mathbb{R}^3$ has the Schwartz kernel

$$R(z,z',k) = \text{const} \cdot \frac{e^{-k|z-z'|}}{|z-z'|}, \quad z,z' \in \mathbb{R}^3,\ k>0\,.$$

Besides the standard singularity at the diagonal $z=z'$ it exhibits rapid decay away from the diagonal, i.e. as $|z-z'|\to\infty$, uniformly as $z\to\infty$ but non-uniformly as $k\to 0$ ('low energy limit'). For $k=0$ the formula gives the Schwartz kernel of a particular inverse of Δ. The precise asymptotic behavior of the resolvent as $z\to\infty$, $z'\to\infty$ or $k\to 0$, or any combination of these, can be efficiently described in terms of blow-ups: compactify $\mathbb{R}^3$ to a manifold with boundary, X, by adding a sphere at infinity. Then the resolvent kernel R is a function on the interior of $X^2\times[0,\infty)_k$, and its pull-back to a suitable blow-up space $X^2_{\mathrm{ac},k}$ under the blow-down map $\beta : X^2_{\mathrm{ac},k}\to X^2\times[0,\infty)_k$ extends smoothly to the boundary of $X^2_{\mathrm{ac},k}$.

Such a blown-up space was constructed by Melrose and Sá Barreto, and it was shown by Guillarmou and Hassell [6] that the same asymptotic behavior occurs for the resolvent of the Laplacian on asymptotically conical spaces, which have a similar metric structure at infinity as $\mathbb{R}^n$. Guillarmou and Hassell also allow an added potential and impose certain restrictions on the null space of the Schrödinger operator. This result was generalized to weaker conditions and to the Hodge Laplacian in [7, 8].

Such results can (and have been) used for example for analyzing mapping properties of the Riesz transform or the long time behavior of the heat kernel, which in turn is needed to analyze spectral invariants like the analytic torsion.

We generalize these results further to a class of spaces, so-called ϕ-manifolds, which are modelled on products of compact Riemannian manifolds with asymptotically conical spaces and hence generalize the latter. More precisely, we consider a compact manifold with boundary, X, with interior $M=\mathrm{int}(X)$, whose boundary ∂X is equipped with a fibration

$$\phi : \partial X \to B$$

over a closed manifold B, with fibres given by copies of a closed manifold F. On M we consider a fibred boundary metric, or ϕ-metric, g. This means that for some collar neighborhood $U\cong[0,1)\times\partial X$ of the boundary ∂X it has the form

$$g_\phi = g_{\Phi,0} + h\,, \quad g_{\Phi,0} = \frac{dx^2}{x^4} + \frac{\phi^* g_B}{x^2} + g_F\,. \tag{1}$$

Here $x : U \to [0,1)$ is the first component of the collaring, so $x^{-1}(0)=\partial X$, g_B is a Riemannian metric on the base B, g_F is a symmetric bilinear form on ∂X which restricts to a Riemannian metric on each fiber F, and h is a higher order term satisfying $|h|_{g_{\Phi,0}} = O(x)$ as $x \to 0$. It is also assumed that $\phi^* g_B + g_F$ is a Riemannian submersion metric on ∂X.

An example of a ϕ-manifold is a global product $Y \times F$ where (F, g_F) is a compact Riemannian manifold and Y is a manifold with boundary B and asymptotically conic metric, i.e. a metric of the form $\frac{dx^2}{x^4} + \frac{\phi^* g_B}{x^2}$ near the boundary (for example, $Y = \overline{\mathbb{R}^n}$, where $x = \frac{1}{r}$, $B = S^{n-1}$). Non-product examples are the moduli space of non-abelian magnetic monopoles of charge 2 with its natural metric, cf. [10, 2], and gravitational instantons, i.e. complete hyperkähler 4-manifolds.

An essential complicating feature of the Hodge Laplacian on a ϕ-manifold is that it annihilates forms which at the boundary are 'fibre harmonic' (i.e. whose restriction to any fibre, in a suitable sense, is harmonic) to a higher order as $x \to 0$ than arbitrary smooth forms. This implies that it cannot be inverted in the 'small' ϕ-pseudodifferential (ΨDO) calculus introduced for such spaces by Mazzeo and Melrose [11] (it is not fully elliptic), and that it is not Fredholm on the standard Sobolev spaces associated to a ϕ-metric. A large ΨDO calculus containing the Fredholm inverse was constructed by Vaillant [14] and by Grieser and Hunsicker [3, 4]. It lives on the same blown-up space X^2_ϕ as the small ϕ-calculus, but allows more general asymptotic behavior at the boundary faces. Since this asymptotic behavior differs between fibre harmonic forms and general forms, the calculus is also called *split* calculus, and the corresponding Sobolev spaces are called split Sobolev spaces.

Our main result is as follows, see [5].

Theorem 1. *Under certain additional assumptions on the metric the Schwartz kernel of the resolvent $(\Delta_\phi + k^2)^{-1}, k > 0$, of the Hodge Laplacian lifts to a polyhomogeneous conormal distribution on an appropriate manifold with corners $X^2_{\phi,k}$, with a conormal singularity along the diagonal.*

We also determine the exponents and the leading terms in the asymptotics of the resolvent kernel, and make the different asympotic behavior with respect to the decomposition into fibrewise harmonic forms and their orthogonal complement explicit.

The additional assumptions are that $\dim B \geq 2$ (otherwise the $k \to 0$ behavior is quite different, though it may be analyzed by the same techniques), that there are no resonances (i.e. harmonic forms which are almost in L^2 are already in L^2), and a certain flatness condition on the bundle $\mathcal{H} \to B$ of fibre harmonic forms. Also, the higher order term h in (1) is required to satisfy $|h|_{g_{\Phi,0}} = O(x^3)$.

The space $X^2_{\phi,k}$ is constructed similarly as $X^2_{\mathrm{ac},k}$, with additional fibrations arising from ϕ over the boundary faces, and with an additional blow-up that accounts for the fact that the Laplacian is a ϕ-operator and its inverse is in the split calculus.

The proof proceeds by the usual techniques of geometric microlocal analysis: construction of the blown-up space, identification and solution of the model problems at the various boundary hypersurfaces, proof of a composition theorem.

Essentially it is a combination of the constructions of Guillarmou/Sher and of Grieser/Hunsicker (with some refinements of the latter even at $k = 0$).

Kottke and Rochon [9] have obtained a very similar result in parallel work. See also [12, 13] for an application to analytic torsion.

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Bigerbes and applications

Chris Kottke

(joint work with Richard Melrose)

1. Introduction and review of gerbes

Gerbes on a space X are 'geometric representatives' of the cohomology group $H^3(X;\mathbb{Z})$, in the same way that complex line bundles[1] $L \to X$ represent $H^2(X;\mathbb{Z})$ through their first Chern class $c_1(L)$, which is to say naturally with respect to

[1]Or equivalently principal U(1)-bundles.

pullbacks, products, and inverses. While various versions of 2-gerbes, representing $H^4(X;\mathbb{Z})$, have been defined, the theory [3] of *bigerbes*, summarized in this talk, affords certain advantages compared to existing theories, with an important application discussed in §3.

To review: bundle gerbes in the sense of Murray [4] may be defined equivalently (if not originally) as follows. Let $\pi : Y \to X$ be a *locally split* map of spaces, meaning a continuous surjection admitting local sections, and denote the k-fold fiber product of π by $Y^{[k]} = Y \times_X Y \times_X \cdots \times_X Y$. Then the $Y^{[\bullet]}$ along with the various projections[2] $\pi_j : Y^{[\bullet+1]} \to Y^{[\bullet]}$ form a *simplicial space* $Y \leftleftarrows Y^{[2]} \Lleftarrow Y^{[3]} \cdots$ and the alternating tensor product of pullbacks of a line bundle $L \to Y^{[\bullet]}$ defines a 'differential' $dL = \bigotimes \pi_j^* L^{(-1)^j} \to Y^{[\bullet+1]}$ such that $d^2L \to Y^{[\bullet+2]}$ is canonically trivial. A *bundle gerbe* is then a line bundle $L \to Y^{[2]}$ equipped with a trivialization[3] of $dL \to Y^{[3]}$ inducing the canonical trivialization[4] of d^2L. Such data define a characteristic *Dixmier-Douady class* $DD(L) \in H^3(X;\mathbb{Z})$ which is natural with respect to pullbacks, inverses, and tensor products, and which vanishes if and only if the gerbe admits a *trivialization*, meaning an isomorphism $L \cong dQ$ for a line bundle $Q \to Y^{[1]}$.

A primary application known as the *lifting bundle gerbe* arises in the following situation: if G is a group admitting a central U(1) extension

$$1 \to \mathrm{U}(1) \to \widehat{G} \to G \to 1$$

and $E \to X$ is a principal G-bundle with difference map $\chi : E^{[2]} \to G$, then the line bundle $L \to E^{[2]}$ associated to $\chi^*\widehat{G}$ (considered here simply as a U(1)-bundle) represents the obstruction to lifting E to a principal $\widehat{G}$-bundle, in that such a lift is equivalent to a bundle gerbe trivialization of L.

For another example, every 3-class is represented by a bundle gerbe $L \to \mathcal{P}^{[2]}X$ where $\mathcal{P}X$ is the based[5] path space mapping to X by endpoint evaluation. Identifying the space $\mathcal{P}^{[2]}X$ of endpoint-coincident pairs of paths in an obvious way with the (based) loop space $\mathcal{L}X$, the bundle gerbe structure on L makes it what has been termed a *fusion line bundle* [6, 7] on $\mathcal{L}X$, for which $c_1(L) \in H^2(\mathcal{L}X;\mathbb{Z})$ is the image of $DD(L) \in H^3(X;\mathbb{Z})$ by the *transgression map* $H^\bullet(X;\mathbb{Z}) \to H^{\bullet-1}(\mathcal{L}X;\mathbb{Z})$.

2. Bigerbes

Various higher versions of gerbes have been defined, such as *bundle 2-gerbes* [5], in which the line bundle $L \to Y^{[2]}$ in the preceding section is replaced by a bundle gerbe over $Y^{[2]}$ with a trivialization of the induced gerbe over $Y^{[3]}$. One drawback is that the associativity condition becomes more complicated: one cannot simply say that the induced trivialization over $Y^{[4]}$ agrees with the canonical one; it is

[2] Along with 'partial diagonal' inclusions going the other way, of which we do not make use.

[3] This structure is equivalent to the 'gerbe product' or groupoid structure of Murray's original definition and in other versions of gerbes.

[4] This property is equivalent to associativity of the gerbe product.

[5] It is possible to work with free path and loop spaces by incorporating an additional condition, see [3].

necessary to specify a 2-morphism relating these and then this 2-morphism must satisfy a rather complicated coherence condition when pulled back over $Y^{[5]}$. The natural extension to higher gerbes necessitates ever higher and more complicated coherence conditions over $Y^{[6]}$ and beyond.

Bigerbes provide a simpler and more symmetric definition, beginning with the prescription of a *locally split square*, meaning a commutative diagram (a) below in which all maps along with the natural map $W \to Y \times_X Z$ are locally split:

$$
\text{(a)}\quad \begin{array}{ccc} Y & \longleftarrow & W \\ \downarrow & & \downarrow \\ X & \longleftarrow & Z \end{array}
\qquad \Longrightarrow \qquad
\text{(b)}\quad \begin{array}{ccccccc}
Z^{[3]} & \longleftarrow & W^{[1,3]} & \leftleftarrows & W^{[2,3]} & \Lleftarrow & W^{[3,3]} \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
Z^{[2]} & \longleftarrow & W^{[1,2]} & \leftleftarrows & W^{[2,2]} & \Lleftarrow & W^{[3,2]} \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
Z & \longleftarrow & W & \leftleftarrows & W^{[2,1]} & \Lleftarrow & W^{[3,1]} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longleftarrow & Y & \leftleftarrows & Y^{[2]} & \Lleftarrow & Y^{[3]}
\end{array}
$$

Extending above and to the right by fiber products leads to the diagram (b) in which $W^{[\bullet,\bullet]}$ forms a *bisimplicial space* and there are two commuting differentials d_1 and d_2 taking line bundles over $W^{[\bullet,\bullet]}$ to $W^{[\bullet+1,\bullet]}$ and $W^{[\bullet,\bullet+1]}$, respectively.

By definition, a *bigerbe* is a line bundle $L \to W^{[2,2]}$ equipped with trivializations of $d_i L$ for $i = 1, 2$ which induce the canonical trivializations of $d_i^2 L$ and a coincident trivialization of $d_1 d_2 L \cong d_2 d_1 L$. Such an object has a well-defined characteristic class $G(L) \in H^4(X; \mathbb{Z})$ which is shown in [3] to be natural with respect to pullbacks, products, and inverses, and which vanishes if and only if L admits a *bigerbe trivialization*, meaning an isomorphism $L \cong d_1 Q_1 \otimes d_2 Q_2$ for two line bundles $Q_1 \to W^{[1,2]}$ and $Q_2 \to W^{[2,1]}$. A further result gives a necessary and sufficient condition for a locally split square to admit a bigerbe representing a given 4-class, and the extension to 'multigerbes' representing $H^k(X; \mathbb{Z})$ for $k \geq 5$ is straightforward.

3. The Brylinski-McLaughlin bigerbe

A major application of bigerbes concerns the transgression of *string structures* on a spin manifold X to *spin structures* (here called *loop-spin* to avoid confusion) on the loop space $\mathcal{L}X$. For context, consider the problem of lifting the structure group of a Riemannian manifold X to subsequent groups in the *Whitehead tower*

$$\mathrm{O}(n) \longleftarrow \mathrm{SO}(n) \longleftarrow \mathrm{Spin}(n) \longleftarrow \mathrm{String}(n) \;\cdots\cdots$$

of subsequently higher connected topological groups over the orthogonal group $\mathrm{O}(n)$, a reduction to $\mathrm{SO}(n)$ being an orientation, a lift to $\mathrm{Spin}(n)$ being a spin structure, and a further lift to $\mathrm{String}(n)$ (which is not a finite dimensional Lie group) being a 'string structure', all of which admit cohomological obstructions and classifications. In particular, string structures are obstructed by[6] $\frac{1}{2}p_1(X) \in H^4(X; \mathbb{Z})$ and (if they exist) are classified by $H^3(X; \mathbb{Z})$.

[6] The 1/2 comes from the fact that the Pontryagin class of the original $\mathrm{SO}(n)$ structure is twice that of the $\mathrm{Spin}(n)$ structure for a spin manifold.

Roughly speaking, the transgression of these obstruction classes to $\mathcal{L}X$ (which has structure group $\mathcal{L}G$ whenever X has the structure group G) gives the obstruction to the structure problem 'one degree lower' on $\mathcal{L}X$; for example, if X is spin, then $\mathcal{L}X$ admits an orientation, meaning a reduction to the connected structure group $\mathcal{L}_+\mathrm{SO}(n) \cong \mathcal{L}\mathrm{Spin}(n)$. However, consideration of fusion as in §1 is necessary to give a complete bijection, e.g. between spin structures on X and *fusion orientations* on $\mathcal{L}X$, as shown in [6].

A *loop-spin* structure on $\mathcal{L}X$ (for X spin) is a lift to the structure group $\widehat{\mathcal{L}\mathrm{Spin}}$, the universal central $\mathrm{U}(1)$ extension of the loop group $\mathcal{L}\mathrm{Spin}$, which is obstructed[7] by the 3-class on $\mathcal{L}X$ transgressing $\frac{1}{2}p_1(X)$. While a number of results state some form of equivalence between string structures on X and loop-spin structures on $\mathcal{L}X$ which incorporate fusion or related conditions [2, 1, 8], the *Brylinski-McLaughlin bigerbe* of [3] gives a direct accounting. This utilizes the locally split square in which $Y = E$ is the principal Spin-bundle of X, $Z = \mathcal{P}X$ is the based[8] path space of X, and $W = \mathcal{P}E$. The identification $\mathcal{P}^{[2]} \cong \mathcal{L}$ leads to the diagram

$$\begin{array}{ccccc}
E^{[2]} & \longleftarrow & \mathcal{P}E^{[2]} & \leftleftarrows & \mathcal{L}E^{[2]} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
E & \longleftarrow & \mathcal{P}E & \leftleftarrows & \mathcal{L}E \\
\downarrow & & \downarrow & & \downarrow \\
X & \longleftarrow & \mathcal{P}X & \leftleftarrows & \mathcal{L}X
\end{array}$$

and the Brylinski-McLaughlin bigerbe is defined by the line bundle over $\mathcal{L}(E^{[2]}) \cong (\mathcal{L}E)^{[2]}$ associated to the principal $\mathrm{U}(1)$-bundle $\chi^*\widehat{\mathcal{L}}\mathrm{Spin}$ pulled back by the difference map $\chi : \mathcal{L}E^{[2]} \to \mathcal{L}\mathrm{Spin}$. A main theorem in [3] states that there is a complete equivalence between (a) string structures on X, (b) fusion loop-spin structures on $\mathcal{L}X$, and (c) bigerbe trivializations of the Brylinski-McLaughlin bigerbe.

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Fredholm conditions for restrictions of invariant pseudodifferential operators to isotypical components

Matthias Lesch
(joint work with Alexandre Baldare, Rémi Come, Victor Nistor)

This is a report on the papers [1] and [2].

Let Γ be a finite group acting by diffeomorphisms on a smooth compact manifold M. For convenience we may choose a Γ-invariant Riemannian metric g on M such that Γ acts by isometries.

We need to fix some notation. As usual, $\widehat{\Gamma}$ denotes the set of equivalence classes of irreducible (unitary) representations of Γ. If $\mathcal{H}$ is a Γ–module then for $\alpha \in \widehat{\Gamma}$ we denote by $\mathcal{H}_\alpha$ the α-isotypical component of $\mathcal{H}$. Clearly, $\mathcal{H} = \bigoplus_{\alpha\in\widehat{\Gamma}} \mathcal{H}_\alpha$ decomposes into a (finite!) direct sum of such isotypical components.

If $T : \mathcal{H}_0 \to \mathcal{H}_1$ is a Γ-equivariant linear map of Γ-modules then T respects the direct sum decomposition into isotypical components, that is $T = \bigoplus_{\alpha\in\widehat{\Gamma}} \pi_\alpha(T)$, where $\pi_\alpha(T) : \mathcal{H}_{0\alpha} \to \mathcal{H}_{1\alpha}$.

Next let P be a Γ–invariant classical pseudodifferential operator of order m acting between the sections of the Γ–equivariant vector bundles E_0, E_1 over M. It is a classical result that P is a Fredholm operator $H^s(M; E_0) \to H^{s-m}(M; E_1)$ if and only if its homogeneous principal symbol

$$\sigma_m(P) \in C^\infty(S^*M; \mathrm{Hom}(E_0, E_1)) \tag{1}$$

is invertible. Here, as usual $H^s(\ldots)$ denote the respective Sobolev spaces of order s and S^*M denotes the cosphere bundle over M. By slight abuse of notation, the pullbacks of E_0, E_1 to S^*M are denoted by the same letter.

Here we are concerned with the problem of deciding whether for a fixed irreducible representation $\alpha \in \widehat{\Gamma}$ the operator

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha\,, \tag{2}$$

is Fredholm. Certainly, Fredholmness of P will be sufficient, but in general it turns out not to be necessary.

To describe our result we need to introduce some more notation. Fix $\xi \in S^*_pM$. Γ also acts on S^*M and Γ_ξ is a subgroup of Γ_p. For an irreducible representation $\rho \in \widehat{\Gamma}_\xi$ the homogeneous principal symbol induces a homomorphism

$$\sigma^\Gamma_m(P)(\xi, \rho) := \pi_\rho(\sigma_m(P)(\xi)) \in \mathrm{Hom}_{\Gamma_\xi}(E_{0x\rho}, E_{1x\rho})$$

For $g \in \Gamma$ we denote by $g \cdot \rho \in \widehat{\Gamma}_{g\xi}$ the representation $(g \cdot \rho)(a) := \rho(g^{-1}ag)$. Furthermore, let Γ_0 be a minimal isotropy group. Then let

$$X^\alpha_{M,\Gamma} := \{(\zeta, \rho) \,|\, \zeta \in S^*M, \rho \in \widehat{\Gamma}_\zeta \text{ and } \exists g \in \Gamma, \mathrm{Hom}_{\Gamma_0}(g \cdot \rho, \alpha) \neq 0\}$$

In other words the definition of $X^\alpha_{M,\Gamma}$ means $\Gamma_0 \subset \Gamma_{g\zeta}$ and $g \cdot \rho$ and α restricted to Γ_0 do have a common Γ_0–irreducible representation in their Γ_0–decomposition.

After these preparations our main result reads.

Theorem 1. *Let Γ be a finite group and let P be a Γ–invariant classical pseudo-differential operator of order m acting between sections of the Γ–equivariant vector bundles E_0, E_1 over the compact smooth manifold M. Then $\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$ is Fredholm if and only if $\sigma_m^\Gamma(P)(\zeta, p)$ is invertible for all $(\zeta, \rho) \in X^\alpha_{M,\Gamma}$.*

Remarks.

1. In [3] the result was generalized to actions of compact (non-finite) Lie groups on M.

2. If the principal orbit type Γ_0 is the trivial group or if the bundles E_0, E_1 are the trivial rank 1 bundles (i.e. P is a scalar operator) then $\pi_\alpha(P)$ is Fredholm if and only if P is elliptic in the ordinary sense and hence P itself is already Fredholm.

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Hodge theory for the Weil-Petersson metric

Richard Melrose

Using the, rather protracted, project with Jesse Gell-Redman on the Hodge theory for the Weil-Peterrson metric on the Riemann moduli spaces $\mathcal{M}_{g,n}$ as a guide I will outline a notion of 'iterated fibration' structure on a manifold with corners. The discussion here is restricted to codimension two and set in the general context of the resolution/quantization of a Lie algebroid.

Consider first the case of a manifold with boundary, the case of codimension one. Many examples have been extensively discussed in the literature – unfortunately too many to list here. The example I concentrate on comes from a 'real Weil-Petersson' metric on a compact manifold with boundary, X. This is arbitrary in the interior and near the boundary takes the form

$$g = dx^2 + h(y, dy) + x^6\alpha^2 + xe, \ x \geq 0. \tag{1}$$

Here x is a defining function for the boundary, α is a connection form on a circle bundle over (and extended off) the boundary, h is a metric on the base of the circle bundle and e is an 'error term' which is smooth and bounded by the leading part.

Appropriately scaled this corresponds to a Lie algebroid, a Lie algebra of smooth b-vector fields on X spanned locally near the boundary by

$$x^3 \times \left(x^{-3}\partial_\theta, \ x^{-1}(x\partial_x), \ \partial_{y_j}\right) \tag{2}$$

where I have multiplied the vector fields of bounded length by x^3 to make them smooth; ∂_θ is a generator of the circle action.

This is a 'geometric' Lie algebroid; in particular a $\mathcal{C}^\infty(X)$-module of smooth b-vector fields, $\mathcal{V}$, on X (that is the geometric part), and as in this case, I will assume in general that it is unrestricted in the interior (although this should be replaced by tangency to a b-fibration). By assumption (as a Lie algebroid) it has a local smooth basis near each point. The notion of a 'boundary-fibration structure' involves the boundary filtration

$$\mathcal{W}_k = \left(\mathcal{V} \cap \rho^k \mathcal{V}_{\mathrm{b}}(X)\right)/\rho^k,\ W_k = \mathcal{W}_k\big|_{\partial X} \subset \mathcal{C}^\infty(\partial X; {}^{\mathrm{b}}T_{\partial X}X). \tag{3}$$

I will demand that the W_k are subbundles. The b-tangent bundle to X has a canonical line subbundle over ∂X, spanned by $x\partial_x$, with the quotient being $T\partial X$. For each k I require that either W_k meets this b-normal bundle only at the 0 section or else contains it. It follows that for some minimal l – the boundary depth – there is a b-normal vector field (inducing the section $x\partial_x$ at the boundary) $N \in \mathcal{W}_l$.

The space $\mathcal{C}^\infty(\partial X; {}^{\mathrm{b}}T_{\partial X}X)$ is a Lie algebra. I will require four further conditions on $\mathcal{V}$ for it to be an iterated boundary fibration structure:-

$$\begin{gathered}\text{The } W_k \text{ are Lie algebras,}\\ \text{The quotients } W_k/{}^{\mathrm{b}}N \text{ define fibrations of } \partial X,\\ \text{The } W_k \text{ are exact}\\ [N, \mathcal{W}_k] \subset \mathcal{W}_k\ \forall\ k.\end{gathered} \tag{4}$$

The $W_k/{}^{\mathrm{b}}N \subset \mathcal{C}^\infty(\partial X; T\partial X)$, for $k < l$, then have local coordinate bases ∂_{y_j} and these lift to elements $\partial_{y_j} + a_j x\partial_x$ of $\mathcal{W}_k$; the exactness condition requires the closed forms $\sum_j a_j dy_j$ to be exact on the fibres. In fact the first three conditions can be combined by requiring the action of W_k on the normal bundle to the boundary to induce a fibration.

For such a Lie algebroid there is a 'Frobenius' basis analogous to (2). Most significantly such a Lie algebroid can always be resolved by the construction of a of generalized product, and in particular can be quantized to a calculus of pseudo-differential operators. As noted above many cases included here are quite familiar:

$l = 0$: b-calculus, (fibred) edge calculus
$l = 1$: scattering calculus, fibred boundary calculus, Weil-Peterrson
$l = 2$: a-calculus of Grieser and Hunsicker.

Note that, for brevity, I have excluded the 'adiabatic calculi' (where N is not in the Lie algebroid). Ideally the definition should also be broadened further to include the Θ-calculus. Such a generalization is even more relevant in higher codimension to capture the compactifications of reductive Lie groups.

The main aim of this talk is to examine appropriate conditions for an iterated boundary fibration in codimension two (and higher). So now let X be a compact manifold with corners up to codimension 2 and let $\mathcal{V} \subset \mathcal{V}_{\mathrm{b}}(X)$ be a 'geometric' Lie algebroid. I will demand conditions as in (4) at the interior of the boundary hypersurfaces. In fact, by generalizing the initial definition to allow non-trivial interior fibrations and the extra normal direction one can proceed iteratively and simply require that each of the spaces in (3), at each boundary hypersurface, define an iterated boundary fibration structure.

Still we need further restrictions at each corner, F, of codimension two; for simplicity I shall assume there is only one (connected) corner. There the boundary filtration is parameterized by a multiorider κ :

(5) $$\mathcal{U}_\kappa = \left(\mathcal{V} \cap \rho^\kappa \mathcal{V}_{\mathrm{b}}(M)\right)/\rho^\kappa,\ U_\kappa = \mathcal{U}_\kappa\big|_F \subset \mathcal{C}^\infty(F; {}^{\mathrm{b}}T_F X),\ \rho = (\rho_1, \rho_2).$$

Here the ρ_i are defining functions for the two local boundary hypersurfaces. These space are automatically decreasing under the standard partial order $\kappa' \geq \kappa$. Again we assume that

(6) $$U_\kappa = \mathcal{C}^\infty(F; {}^{\mathrm{b}}T_F M) \text{ for some } \kappa.$$

The space $\mathcal{C}^\infty(F; {}^{\mathrm{b}}T_F X)$ is again a Lie algebra (and Lie algebroid over F with two abelian 'fibre' variables) and we demand that the U_κ be Lie subalgebroids. The additional requirement I wish to emphasize – it is automatic in codimension 1 – is

'Strong iteration': There exists a sequence of distinct multiindices

(7) $$(0,0) = \kappa(0) < \kappa(1) < \cdots < \kappa(N)$$

forming a *chain* and such that for any $\kappa \in \mathbb{N}_0^2$

(8) $$U_\kappa = U_{\kappa(j)},\ j(\kappa) = \max\{k; \kappa(k) \leq \kappa\}.$$

Of course the $U_j = U_{\kappa(j)}$ then determine all the U_κ.

Beyond this a generalization of 'boundary depth' above is required. The b-normal bundle to F is a canonically trivial subbundle of ${}^{\mathrm{b}}T_F X$ with fixed basis $x_1\partial_{x_1}$, $x_2\partial_{x_2}$ corresponding to (but independent of) any local choice of defining functions.

'b-normality': For each k the intersection

(9) $${}^{\mathrm{b}}U_j = U_j \cap {}^{\mathrm{b}}NF \text{ is a subbundle with basis } p_1 x_1 \partial_{x_1} - p_2 x_2 \partial_{x_2},\ p_i \in \mathbb{N}_0.$$

By assumption ${}^{\mathrm{b}}U_N = {}^{\mathrm{b}}NF$ since we are assuming that $U_N = {}^{\mathrm{b}}T_F X$. So the ${}^{\mathrm{b}}U_j$ are decreasing starting from full rank two (so containing both generators). If the rank drops from two to one (it could drop from two to zero) then the remaining element is required to be some $p_1 x_1 \partial_{x_1} - p_2 x_2 \partial_{x_2}$. The sign condition on the integers corresponds to the fact that this should generate a b-fibration of the inward-pointing normal bundle to F.

Finally we require the fibration condition itself, that the

'Fibration': $U_j(F)/{}^{\mathrm{b}}NF$ define fibrations of F

and that the induced 1-forms on F are exact on fibres. I also require *homogeneity* with respect to the two normal vector fields.

Under these conditions an iterated fibration structure has a resolution by a generalized product and hence quantization to a calculus of pseudodifferential operators.

The Weil-Petersson case shows that the chain condition need not be trivial to arrange. Namely in codimension two the metric assumes the 'product' form

$$g = dx_1^2 + dx_2^2 + h(y, dy) + x_1^6\alpha_1^2 + x_2^6\alpha_2^2 + x_1e_1 + x_2e_2. \tag{10}$$

The conditions above, without the chain condition, are achieved on the single space defined by blow-up of the corner. The chain condition holds on the space defined by parabolic blow-up of the resulting two corners.

Is there a simpler way?

The Hodge theorem asserts that the L^2 null space of the Laplacian for a Weil-Petersson metric is isomorphic to the cohomology of the manifold without boundary obtained by collapsing the circle bundles.

Donut choirs and Schiemann's symphony

Julie Rowlett
(joint work with Erik Nilsson and Felix Rydell)

We investigate the following:

Question 1. *If two flat tori are isospectral, then are they necessarily also isometric?*

A *donut* is a common albeit inaccurate visualization of a two dimensional flat torus, that is obtained as the quotient $\mathbb{R}^2/\Gamma$ for a full rank lattice $\Gamma \subset \mathbb{R}^2$, endowed with the flat Riemannian metric inherited from the Euclidean metric on $\mathbb{R}^2$. In spite of its inaccuracy, we choose to incorporate this playful language in the spirit of Conway [1] and Kac [6]. Moreover, we identify the spectrum of a flat torus with the song the donut sings, imagining isospectral flat tori as a choir of donuts that sing in perfect unison. One such donut is shown in Figure 1.

Figure 1. Here we identify the spectrum of a flat torus with 'the song the donut sings.' This image is open source: `https://charatoon.com/?id=2325`

If the flat tori are one-dimensional, the answer to Question 1 is yes; this is a calculus exercise. On the other hand, if the flat tori are sixteen-dimensional, the answer is no. With the analogy of isospectral flat tori as a choir of donuts singing in perfect unison, we introduce

Definition 1 (Flat choir numbers $\flat_n$)**.** *In each dimension $n \in \mathbb{N}$, we define the* (flat) choir number *to be the supremum over all $k \in \mathbb{N}$ such that there is a collection $T_1, \ldots, T_k$ of mutually isospectral and non-isometric flat tori. The sequence $\flat_n$ is called the sequence of* (flat) choir numbers.

In 1978, Wolpert proved [13] that there are at most finitely many flat tori (up to isometry) that share any given spectrum. However, there could be a sequence of spectra, each of which has an increasingly larger choir of non-isometric donuts, so that à priori one cannot conclude that $\flat_n$ is finite for all n. Suwa-Bier proved in their 1984 doctoral thesis [11] that the flat choir numbers are in fact finite. In 1964, Milnor [7] proved that $\flat_{16} \geq 2$. Milnor used a construction of Witt [12] based on the root lattice, D_n, and the diamond packing, D_n^+,

$$D_n := \left\{ z = (z_1, \ldots, z_n) \in \mathbb{Z}^n : \sum_{i=1}^n z_i \in 2\mathbb{Z} \right\}, \quad D_n^+ := D_n \cup \left(\frac{1}{2}\mathbf{1} + D_n \right).$$

Theorem 1 (Milnor's duet)**.** *$\mathbb{R}^{16}/(D_8^+ \times D_8^+)$ and $\mathbb{R}^{16}/D_{16}^+$ are isospectral but not isometric.*

The flat tori in Milnor's theorem are not isometric because the first is reducible whereas the second is irreducible. They are isospectral because they have identical theta series. This follows from the fact that the lattices are even and unimodular, and consequently their theta series are modular forms for $\mathrm{PSL}_2(\mathbb{Z})$. In 16 dimensions there is only one such form (up to multiplication by scalars), hence all 16 dimensional donuts of identical volume obtained as quotients by even unimodular lattices are isospectral. What is the largest donut choir in dimension 16, or more generally we ask

Question 2. *What is the precise value of $\flat_n$ for each n?*

In soon-to-appear joint work [8], we investigate this and related questions. Two flat tori are isospectral if and only if they have identical theta series. For a lattice Γ this is

$$\theta_\Gamma(z) := \sum_{\gamma \in \Gamma} e^{i\pi z ||\gamma||^2}, \quad z \in \mathbb{C} \text{ with } \mathrm{Im} z > 0.$$

Checking isospectrality is therefore equivalent to checking whether the theta series are identical, a seemingly infinite task. However, the theta series of certain lattices are modular forms, in which case verifying isospectrality reduces to finitely many calculations thanks to the identity theorem for modular forms [4]. Another way to reduce verifying isospectrality to a finite number of calculations is to employ techniques from the theory of linear codes [3].

Question 1 can be paraphrased in entirely different language that omits the terms spectrum, flat torus, lattice, eigenvalues, and isometric. To see this, consider a full rank lattice $\Gamma = A\mathbb{Z}^n \subset \mathbb{R}^n$. Then

$$q(x) := x^T Q x, \quad Q := A^T A, \quad x \in \mathbb{R}^n$$

is a positive definite quadratic form. The *representation numbers* of the quadratic form are for $\lambda \in \mathbb{R}_{\geq 0}$

$$\mathcal{R}(q, \lambda) := \#\{x \in \mathbb{Z}^n : q(x) = \lambda\}.$$

There are many quadratic forms associated to a given lattice, because for any unimodular matrix $G \in \mathrm{GL}_n(\mathbb{Z})$, $AG\mathbb{Z}^n = A\mathbb{Z}^n$ is the same lattice. However, the quadratic form for the basis AG might not be identical to the quadratic form for the basis A. The quadratic form senses the choice of basis matrix, whereas the lattice itself is blind to this choice. This motivates the equivalence notion: two quadratic forms $q(x) = x^T Q x$, $p(x) = x^T P x$ are *integrally equivalent* if there is a unimodular matrix G with $Q = G^T P G$. Integrally equivalent quadratic forms have identical representation numbers. To each donut we therefore associate a collection of integrally equivalent quadratic forms. On the other (left) hand, for any $C \in \mathrm{O}(n)$, the lattice $CA\mathbb{Z}^n$ is congruent to $A\mathbb{Z}^n$ but is not necessarily identical. From the quadratic form's perspective $(CA)^T(CA) = A^T A$, so the quadratic form is blind to congruence. Consequently, two donuts sing the same song if and only if the representation numbers of their associated integral equivalence classes of quadratic forms are identical. The donuts are isometric if and only if these equivalence classes are in fact identical. Question 1 is therefore equivalent to: is an equivalence class of integrally equivalent quadratic forms uniquely determined by its representation numbers? The answer depends on the dimension.

Theorem 2 (Schiemann's Symphony)**.** *The flat choir numbers satisfy:* $\flat_1 = \flat_2 = \flat_3 = 1$, $\flat_n \geq 2$ *for all* $n \geq 4$.

The proof for dimensions one and two is a fairly simple exercise. In contrast, the proof for dimension three requires a tremendous amount of effort and to date has only been completed with further assistance by a computer [10]. The result for four dimensions has been proven by Schiemann [9]. In 2011, Cervino & Hein developed a method to systematically construct infinitely many isospectral non-isometric pairs of flat tori in four dimensions [5], thereby proving a conjecture of Conway & Sloane [2]. In [8], we invite readers to join us in exploring the questions posed here from different mathematical perspectives with creative descriptions of the mathematical objects intended not only to convey the concepts but also to inspire the reader's imagination.

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A pseudo-differential calculus for singular filtrations of the tangent bundle and index theorem

Omar Mohsen

(joint work with I. Androulidakis, R. Yuncken, E. van-Erp)

A linear differential operator D on a smooth manifold M is called hypoelliptic if for any distribution u on M

$$\text{singsupp}(u) \subseteq \text{singsupp}(Du)$$

where singsupp is the singular support. This is a crucial property of elliptic operators, closely related to Fredholmness when M is compact. It is well-known that hypoellipticity generalizes to other classes of linear differential operators. Two major sources of hypoelliptic operators are

- Hörmander's sums-of-squares operators [11, 18] and generalizations to other polynomials of vector fields, e.g. [12, 10].
- Rockland operators on nilpotent Lie groups [17, 9, 3], on Heisenberg manifolds [19, 2] and more generally on filtered manifolds [14].

The standard proof of hypoellipticity for elliptic and Rockland operators on manifolds is to introduce a pseudo-differential calculus in which the differential operator admits a parametrix. This was achieved for Heisenberg manifolds in [19, 2] and for general filtered manifolds in the unpublished manuscript by Melin [14].

Building on the seminal work of Connes [5] and Debord-Skandalis [6], a simple construction Melin's pseudo-differential calculus was described in [8].

The main goal of this talk is to introduce a pseudo-differential calculus appropriate to Hörmander's operators and their generalizations. This will allow a unified approach to hypoellipticity and Fredholmness results for all of the differential operators listed above. Furthermore, our approach using groupoids and pseudo-differential calculus allows one to make sense of the notion of the principal symbol for sums-of-squares operators and their generalizations. This opens up these operators to techniques such as K-theory and zeta functions which have been widely applied in the elliptic and filtered settings.

Let us describe our setting. Let $N \in \mathbb{N}$ be a natural number called the depth. And let X_i be a finite number of *real* smooth vector fields on M. To each X_i, let $1 \leq n_i \leq N$ be a natural number. One declares the order of X_i to be n_i and the order of every other vector field on M to be N (of course there are non-trivial relations forcing some vector fields to have order $< N$ as we will see shortly). Associated to this we define a pseudo-differential calculus. We prove that the expected properties of a pseudo-differential calculus are satisfied. For example

(1) Operators of order $k \in \mathbb{C}$ with $\Re(k) = 0$ are bounded operators on L^2M.
(2) Operators of order $k \in \mathbb{C}$ with $\Re(k) < 0$ are compact.
(3) All asymptotic expansions admit a limit in the calculus.
(4) Differential operators have a well defined inhomogeneous principal symbol in our calculus. To be discussed below, in more details.
(5) If the principal symbol is invertible, then the differential operator admits a parametrix in our calculus

A consequence of these properties is that if the principal symbol is invertible, then the operator is hypoelliptic. Furthermore if M is compact, then the bounded transform of the operator is Fredholm. This immediately implies Hörmander's theorem

Theorem 1 (Hörmander)**.** *Let $X_0, \cdots, X_k$ be real vector fields, $a_0, \cdots, a_k \in \mathbb{N}$ such that a_0 is odd. If the vector fields X_i together with their iterated Lie brackets span T_xM at every point $x \in M$, then the operator $X_0^{a_0} + \sum_{i=}^{k} X_i^{2a_i}$ is hypoelliptic.*

It also immediately implies and generalizes Kohn's theorem [13, Theorem A]. Some non trivial relations are forced from the choice of the orders on X_i. More precisely, if X, Y are of order k, l respectively, then $[X, Y] = XY - YX$ has to be of order at most $k + l$. To keep track of such relations, we define the $C^\infty(M, \mathbb{R})$-modules $\mathcal{D}^i$ of compactly supported vector fields of order less than or equal to i.

One has
$$0 = \mathcal{D}^0 \subseteq \mathcal{D}^1 \subseteq \cdots \subseteq \mathcal{D}^N = \mathcal{X}_c(M),$$
where $\mathcal{X}_c(M)$ is the $C^\infty(M, \mathbb{R})$-module of smooth compactly supported vector fields on M. The modules have the fundamental property
$$[\mathcal{D}^i, \mathcal{D}^j] \subseteq \mathcal{D}^{i+j}.$$

For $x \in M$, we define the Lie algebra
$$\mathfrak{gr}(\mathcal{D})_x = \oplus_{i=1}^{N} \frac{\mathcal{D}^i}{\mathcal{D}^{i-1} + I_x \mathcal{D}^i},$$
where $I_x \subseteq C^\infty(M, \mathbb{R})$ is the ideal of real valued smooth functions vanishing at x. The Lie algebra structure on $\mathfrak{gr}(\mathcal{D})_x$ comes from the Lie bracket, making it a graded nilpotent Lie algebra. Let $\mathcal{G}\mathrm{r}(\mathcal{D})_x$ be the simply connected Lie group integrating $\mathfrak{gr}(\mathcal{D})_x$. Let us make some remarks on $\mathcal{G}\mathrm{r}(\mathcal{D})_x$.

(1) The bundle $\mathcal{G}\mathrm{r}(\mathcal{D}) := \sqcup \mathcal{G}\mathrm{r}(\mathcal{D})_x$ isn't a fiber bundle in the usual topological sense, because $x \to \dim(\mathfrak{gr}(\mathcal{D})_x)$ isn't a continuous function. In fact $\dim(\mathfrak{gr}(\mathcal{D})_x) \geq \dim(T_xM)$ and the difference measures in some sense how

much $\mathcal{D}^i$ fail to be sections of a vector bundle. Melin's calculus is the case where all the modules $\mathcal{D}^i$ are sections of a subbundle of TM, in which case $\dim(\mathfrak{gr}(\mathcal{D})_x) = \dim(T_xM)$.

(2) Even though $\mathcal{G}\mathrm{r}(\mathcal{D})$ isn't a fiber bundle, we can still define the C^*-algebra of the bundle $\mathcal{G}\mathrm{r}(\mathcal{D})$, as we will see below.

Let D be a differential operator on M. We say that D is of order k if D can be written as the sum of monomials $Y_1 \cdots Y_s$ with $s \in \mathbb{N}$, $Y_i \in \mathcal{D}^{a_i}$ and $\sum_{i=1}^{s} a_i \leq k$.

The principal symbol (in our calculus) of D at x, is an unbounded multiplier of a *quotient* of $C^*\mathcal{G}\mathrm{r}(\mathcal{D})_x$. In contrast with Melin's calculus or the classical pseudo-differential calculus, our principal symbol isn't defined for all irreducible unitary representations of $\mathcal{G}\mathrm{r}(\mathcal{D})_x$, only for a closed subset $\mathcal{T}^*\mathcal{D}_x$ of $\widehat{\mathcal{G}\mathrm{r}(\mathcal{D})}_x$ which we call the characteristic set. Before giving the definition of the characteristic set, let us explain our strategy to define the pseudo-differential calculus.

The work of Debord and Skandalis and Van-Erp and Yuncken [6, 8] shows that in order to define a pseudo-differential calculus one can first define the associated deformation groupoid and then use the groupoid to define the pseudo-differential calculus. The advantage of this approach is that properties like invariance under local diffeomorphisms, closeness under composition and adjoint become geometric statements about the groupoid.

For the classical pseudo-differential calculus the associated deformation groupoid is the tangent groupoid defined by Connes [5]. For the more general calculus defined by Melin [14], the assoicated deformation groupoid was defined by Choi and Ponge [4], see also Van-Erp and Yuncken [7], and Mohsen [16]. In both cases the assoociated groupoid is a Lie groupoid. In our case this cant be true. The deformation groupoid for our calculus is algebraiclly equal to

$$M \times M \times \mathbb{R}^* \sqcup \mathcal{G}\mathrm{r}(\mathcal{D}) \times \{0\}. \tag{1}$$

Since $\mathcal{G}\mathrm{r}(\mathcal{D})$ isn't a fiber bundle, the groupoid in Eqn (1) cant be a Lie groupoid. Our approach is to define the groupoid as the holonomy groupoid of a singular foliation [1]. Recall that in Androulidakis and Skandalis terminology a singular foliation is a $C^\infty(M, \mathbb{R})$-module of vector fields that is locally finitely generated and closed under Lie bracket. On $M \times \mathbb{R}$, we define the following singular foliation

$$\mathfrak{a}\mathcal{D} = t\mathcal{D}^1 + t^2\mathcal{D}^2 + \cdots t^N\mathcal{D}^N.$$

This is a well defined singular foliation. We show that the holonomy groupoid of $\mathfrak{a}\mathcal{D}$ is equal algebraically to the one in Eqn (1). Now this groupoid is still not a Lie groupoid but it can be covered by bi-submersions (local charts introduced in [1]). These bi-submersions play exactly the role of local charts for our calculus. In fact our operators are written as standard oscillatory integrals in such charts.

Let us now introduce the characteristic set. Let $\mathrm{Diff}_{\mathcal{D}}(M)$ be the algebra of differential operators filtered as above by declaring elements of $\mathcal{D}^i$ to be of order i. To define the principal symbol, one is naturally led to consider the algebra of symbols

$$\oplus_{i=0}^{\infty} \frac{\mathrm{Diff}^i_{\mathcal{D}}(M)}{\mathrm{Diff}^{i-1}_{\mathcal{D}}(M) + I_x \mathrm{Diff}^i_{\mathcal{D}}(M)}.$$

One naturally has a Lie algebra homomorphism map

$$\mathfrak{gr}(\mathcal{D})_x \to \oplus_{i=0}^{\infty} \frac{\mathrm{Diff}^i_{\mathcal{D}}(M)}{\mathrm{Diff}^{i-1}_{\mathcal{D}}(M) + I_x \mathrm{Diff}^i_{\mathcal{D}}(M)},$$

which sends a vector field to itself as a differential operator. Hence one has an obviously surjective algebra homorphism

$$\mathcal{U}(\mathfrak{gr}(\mathcal{D})_x) \to \oplus_{i=0}^{\infty} \frac{\mathrm{Diff}^i_{\mathcal{D}}(M)}{\mathrm{Diff}^{i-1}_{\mathcal{D}}(M) + I_x \mathrm{Diff}^i_{\mathcal{D}}(M)}, \tag{2}$$

where $\mathcal{U}(\mathfrak{gr}(\mathcal{D})_x)$ is the enveloping algebra over $\mathbb{C}$. In contrast to the classical pseudo-differential calculus and Melin's calculus, the map (2) isn't injective in general. We define the algebraic characteristic set $\mathcal{T}^*\mathcal{D}_x$ to be the set of representations π of $\mathcal{G}\mathrm{r}(\mathcal{D})_x$ such that π vanishes on the kernel of the map (2).

We remark that the characteristic set $\mathcal{T}^*\mathcal{D}$ is different from $\mathfrak{gr}(\mathcal{D})^*$ even for some very simple operators like $\partial_x^2 + x^2\partial_y$.

We now come to our main theorem.

Theorem 2. *Let D be a differential operator of order k such that for every $x \in M$, $\pi \in \mathcal{T}^*\mathcal{D}_x \backslash \{1_{\mathcal{G}\mathrm{r}(\mathcal{D})_x}\}$, $\pi(\sigma_x^k(D))$ is injective. Here $\pi(\sigma_x^k(D))$ denotes the principal symbol of D evaluated at the irreducible non-trivial unitary representation π. Then D admits a left parametrix in our calculus and D is hypoelliptic.*

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Degenerate Elliptic Boundary Value Problems with Non-smooth Coefficients

Elmar Schrohe
(joint work with Thorben Krietenstein)

The first topic of this talk is the existence of a bounded H_∞-calculus in the sense of McIntosh [5] for the realization of a strongly elliptic second order differential operator A, endowed with a boundary condition T that is degenerate, so that, in general, the pair (A, T) will not be elliptic in the sense of Lopatinskij-Shapiro. To introduce the notion of H_∞-calculus, consider, for $\theta > 0$, the sector

$$\Lambda_\theta = \{\lambda = re^{i\phi} \in \mathbb{C} : r \geq 0;\ |\phi| \geq \theta\}$$

and a closed linear operator $B : \mathcal{D}(B) \to E$ in a Banach space E which is sectorial in Λ_θ in the sense that $\sup_{\lambda\in\Lambda_\theta} \|\lambda(\lambda - B)\|_{\mathcal{L}(E)} < \infty$. For a bounded holomorphic (H_∞-) function f on $\mathbb{C} \setminus \Lambda$ one defines the operator $f(B)$ by the Dunford integral

$$f(B) = \frac{1}{2\pi i}\int_{\partial\Lambda_\theta} f(\lambda)(\lambda - B)^{-1}\, d\lambda. \tag{1}$$

Then B is said to have a bounded H_∞-calculus, if there is a constant M such that

$$\|f(B)\|_{\mathcal{L}(E)} \leq M\|f\|_\infty, \quad f \in H_\infty(\mathbb{C} \setminus \Lambda).$$

The existence of an H_∞-calculus for B for some $\theta < \pi/2$ implies that the Cauchy problem $\partial_t u + Bu = f$, $u(0) = u_0$ has the property of *maximal regularity* – an important tool for the analysis of parabolic evolution equations. Indeed, we use this property to show the existence of a short time solution to the porous medium equation under the degenerate boundary condition. Apart from the existence of a bounded H_∞-calculus we obtain the solvability of the full boundary problem.

As Equation (1) suggests, the main point of the analysis is the construction of a suitable parameter-dependent parametrix. For this we use a slightly extended version of Boutet de Monvel's calculus.

Let us fix some notation: X denotes a manifold with boundary ∂X and bounded geometry of dimension n as in [1], and A a second order partial differential operator on X, written locally

$$A = \sum_{1\le k,l\le n} a^{kl}(x) D_k D_l + \sum_{1\le k\le n} b^k(x) D_k + c^0(x), \tag{2}$$

where $a^{kl} \in C^\tau(X)$ are real-valued (the results hold, with obvious modifications, for complex coefficients), the matrix $(a^{kl}(x))_{1\le k,l\le n}$ is positive definite with a uniform positive lower bound, $b^k, c^0 \in L_\infty(X)$. The boundary operator T is of the form

$$T = \varphi_0\gamma_0 + \varphi_1\gamma_1. \tag{3}$$

Here γ_0 denotes evaluation and γ_1 the evaluation of the exterior normal derivative at ∂X; $\varphi_0, \varphi_1 \in C_b^\infty(\partial X)$ are real-valued functions on the boundary with $\varphi_0, \varphi_1 \ge 0$ and $\varphi_0+\varphi_1 \ge c > 0$. We obtain the classical Dirichlet problem for $\varphi_0 = 1, \varphi_1 = 0$ and the Neumann problem for $\varphi_0 = 0, \varphi_1 = 1$. Unless $\varphi_1 \equiv 0$ or $\varphi_1(x) \ne 0$ everywhere, the problem is not elliptic.

The L_p-realization of A with the boundary condition T is the unbounded operator A_T, acting like A on the domain

$$\mathcal{D}(A_T) := \{u \in H_p^2(X) : Au \in L_p(X),\ Tu = 0 \text{ on } \partial X\}.$$

This problem has been investigated by many authors, see e.g. Egorov-Kondrat'ev, Kannai, and Taira [7], also for the case where the boundary operator T involves an additional first order tangential differential operator. The present talk is partly based on [4].

Theorem 1. *For every $0 < \theta < \pi$ a constant ν exists such that $A_T + \nu$ has a bounded H^∞-calculus in $L_p(X)$.*

Theorem 1 extends to the case where $T = \gamma_0 + \varphi^2\gamma_1$, i.e. $\varphi_0 = 1$ (not an essential restriction) and φ_1 is the square of a $C^{2+\tau}$-function φ with $\tau > 0$.

For $s \in \mathbb{R}$ and T as in (3) or $s > -2-\tau$ and T as above, we define

$$B_{p,T}^{s-1-1/p}(\partial X) = \{v = \varphi_0 v_0 + \varphi_1 v_1 : v_0 \in B_p^{s-1/p}(\partial X), v_1 \in B_p^{s-1-1/p}(\partial X)\}.$$

Clearly, this is a Banach space with the topology of the non-direct sum.

Proposition 2. *For $s > 1 + 1/p$ the mapping $T : H_p^s(X) \to B_{p,T}^{s-1-1/p}(\partial X)$ is surjective.*

Theorem 3. *For every $0 < \theta < \pi$ the operator*

$$\begin{pmatrix} A-\lambda \\ T \end{pmatrix} : H_p^2(X) \longrightarrow \begin{matrix} L_p(X) \\ \oplus \\ B_{p,T}^{1-1/p}(\partial X) \end{matrix} \tag{4}$$

is a topological isomorphism for $\lambda \in \Lambda_\theta$, $|\lambda|$ sufficiently large.

Theorem 3 is a consequence of Theorem 1 and Proposition 2.

Theorem 4. *Let $1 < p, q < \infty$, $n/p + 2/q < 1$, $m > 0$, $v_0 \in H_p^2(X)$ with $v_0 \ge c > 0$, and $\phi \in C^1([0,t_0]; B_{p,T}^{1-1/p}(\partial X))$, $t_0 > 0$, with $\phi(0) = Tv_0$. Then the* *porous medium equation*

$$\begin{cases} \dot{v} - \Delta_g v^m = 0 \\ Tv = \phi \\ v|_{t=0} = v_0 \end{cases} \tag{5}$$

has a unique short time solution

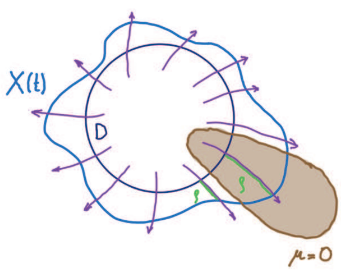
$$v \in L_q([0,t^*]; H_p^2(X)) \cap W_q^1([0,t^*]; L_p(X)), \quad t^* > 0.$$

Current work in progress is on a problem that arose in discussions with J. Escher. It concerns the melting/solidification process of ice filling a volume $X(t)$ at time $t \geq 0$. The evolution of $X(t)$ is governed by the temperature $u = u(x,t)$ in x at time t, its (outer) normal derivative $\partial_\nu u$, the normal velocity V of $\partial X(t)$, and its mean curvature κ, via the equations

$$\begin{aligned} \Delta u &= 0 \quad \text{in } X(t) \\ V + \partial_\nu u &= 0 \quad \text{on } \partial X(t) \\ \mu^2 V + \kappa &= u \quad \text{on } \partial X(t) \\ X(0) &= X_0 \quad \text{at } t = 0. \end{aligned}$$

Here, $\mu \in C_b^\infty(\mathbb{R}^n)$ is a nonnegative function with a possibly non-empty zero set (lower regularity is possible). This boundary condition lies between the Gibbs-Thomson condition $\sigma\kappa = u$, $\sigma > 0$ and thermal cooling $V + \kappa = u$.

Following partly an approach devised by Escher, Kneisel and Simonett [2, 3], we choose a smooth bounded domain D, and suppose that there is a non-tangent vector field such that $\mu =$ const. along the flow. In the flow coordinates (x, ρ), $x \in \partial D$, $\rho \geq 0$, we obtain the equations



$$\begin{aligned} A(\rho)v &= 0 \quad \text{in } J \times D && (6) \\ \partial_t \rho + L_\rho D_\partial v &= 0 \quad \text{on } J = \partial D && (7) \\ \gamma_0 v + \mu^2 D_\partial v &= H(\rho) \quad \text{on } J \times \partial D && (8) \\ \rho(0) &= \rho_0 \quad \text{on } \partial D && (9) \end{aligned}$$

with $J = [0, \tau)$, a function $L_\rho > 0$, the expression $H(\rho)$ obtained for the mean curvature, and an oblique derivative D_∂ at the boundary.

The strategy now is to first solve the degenerate boundary value problem (6)/(8) for v, given ρ. Inserting the solution v into (7) and linearizing H yields a quasilinear evolution equation for ρ with a non-elliptic generator. We intend to solve it in suitable little Hölder spaces with continuous maximal regularity techniques as developed e.g. in [6].

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Lie groupoids and pseudodifferential calculus

Georges Skandalis

(joint work with Claire Debord)

1. Groupoids

1.1. **Definitions.** A (Lie) groupoid is given by two sets (manifolds): the sets (manifolds) of *objects* denoted by $G^{(0)}$ and of *arrows* denoted by $G^{(1)}$ or just G.

We assume the following data are given.

- Every arrow has a *source* and a range – or target – which are objects. In other words, we have two maps (smooth submersions) $r, s : G \to G^{(0)}$.
- Every object x has an identity arrow with source and range x. In other words, we assume given a (smooth) map $u : G^{(0)} \to G$ such that $r \circ u = \mathrm{id}_{G^{(0)}}$ and $s \circ u = \mathrm{id}_{G^{(0)}}$. We identify $G^{(0)}$ with its image in G via u.
- Arrows x, y are *composable* if $s(x) = r(y)$. We denote by $G^{(2)}$ the set subset of $G \times G$ consisting of composable arrows. If $G^{(0)}$ and G are manifolds and r, s are submersions, $G^{(2)}$ is a submanifold of $G \times G$.

  Composition of composable arrows is a (smooth) map $(x, y) \mapsto xy$ from $G^{(2)}$ to G. It is assumed to satisfy the following relations

  Source and range: $r(xy) = r(x)$, $s(xy) = s(y)$;

  Units are units: $r(x)x = x$ and $xs(x) = x$;

  Associativity: if $s(x) = r(y)$ and $s(y) = r(z)$, then $(xy)z = x(yz)$.
- Finally, we assume that *every arrow is invertible*, *i.e.* there is a (smooth) map $x \mapsto x^{-1}$ such that, for all $x \in G$, we have $s(x^{-1}) = r(x)$, $r(x^{-1}) = s(x)$ and $xx^{-1} = r(x)$, $x^{-1}x = s(x)$.

We write $G \rightrightarrows G^{(0)}$ to mean that G is a groupoid with set of objects $G^{(0)}$.

1.2. **Elementary examples.**

(1) A manifold M is a Lie groupoid $M \rightrightarrows M$. All maps r, s, composition, inverse are the identity of M...
(2) A Lie group is a Lie groupoid with just one unit. All arrows are composable.
(3) A family $(\Gamma_x)_{x\in M}$ of groups is a groupoid $G = \sqcup_{x\in M} G_x \rightrightarrows M$ (source and range coincide). In particular, a smooth vector bundle is a Lie groupoid.
(4) *Pair groupoid:* Let M be a manifold. The set $M \times M$ is a Lie groupoid with set of objects M. We put $r(x,y) = x$, $s(x,y) = y$ and $(x,y)\cdot(y,z) = (x,z)$.

1.3. **Singularities and Lie groupoids.** Let M be an (open) manifold. Let $\overline{M}$ be a compact space containing M as a dense open subset; put $\partial M = \overline{M} \setminus M$. One may think of ∂M as the singular part (*e.g.* the boundary, or the union of all the singular strata).

We wish to study differential operators on M with given behavior near ∂M. This behavior is given by the set $\mathcal{E}$ of vector fields allowed. We assume that $\mathcal{E}$ is a module over $C^\infty(\overline{M})$ (containing all vector fields supported in M) and is closed under brackets. It is usually further assumed to be a *projective module* over $C^\infty(\overline{M})$; thus $\mathcal{E}$ is the space of sections of a vector bundle over $\overline{M}$: a *Lie algebroid* $A \to \overline{M}$. Such a situation is called a *Lie manifold* in [1]. It follows then from [4] that it is the algebroid of a natural groupoid. Note that, thanks to [2] and the recent work ([9]) one gets rid of the projectivity requirement.

Examples. For instance, let M be a manifold with boundary ∂M. We can then naturally consider various natural modules: The module of vector fields tangent to the boundary, yields the groupoid of the b-calculus of Melrose ([8], see [10]). Associated to the module of vector fields tangent to a fibration at the boundary is the groupoid of the edge-calculus: ([7]). The module of vector fields tangent to a foliation of the boundary yields the calculus in [11]...

One can generalize to more general situations allowing corners ([10]) and stratified manifolds ([5]), ...

2. Convolution on a Lie groupoid

2.1. **Convolution of functions.** Denote by $C_c^\infty(G)$ the space of smooth functions with compact support on a Lie groupoid. G. For $f_1, g_2 \in C_c(G)$, put

$$f_1 * f_2(x) = \int_{(x_1,x_2)\in G;\ x_1x_2=x} f_1(x_1)f_2(x_2)\, d\nu^x(x_1,x_2)$$

The set $\{(x_1,x_2) \in G \times G;\ x_1x_2 = x\}$ is a smooth manifold. The family $x \mapsto d\nu^x$ is a smooth "Haar system" *i.e.* a smooth choice of Lebesgue measures ν^u satisfying a *left invariance*: for every $(x,y) \in G^{(2)}$, the measure ν^{xy} is the image of ν^y by the diffeomorphism $(x_1,x_2) \mapsto (xx_1,x_2)$.

Particular cases are convolution on a group and of kernels on $M \times M$.

The *adjoint* of $f \in C_c^\infty(G)$: function $f^* : x \mapsto \overline{f(x^{-1})}$.

2.2. **Pseudodifferential operators on a Lie groupoid.** The convolution extends to many distributions. In particular to the distributions with compact support on G that are conormal to $G^{(0)} \subset G$.

If P, Q are conormal distributions, then $P * Q$ is a conormal distribution, and the principal symbol of $P * Q$ is the product of the principal symbols of P and Q.

3. Deformation to the normal cone and conormal distributions

3.1. **Deformation to the normal cone.** Let M be a manifold and let $V \subset M$ be a submanifold. Denote by N_V^M the normal bundle of V in M

The *deformation to the normal cone* of V in M is a manifold obtained by putting a very natural smooth structure on $DNC(M, V) = (M \times \mathbb{R}^*) \sqcup (N_V^M \times \{0\})$. This construction is more or less the blowup of $V \times \{0\}$ in $M \times \mathbb{R}$.

Since this construction is functorial, if M is a Lie groupoid and V is a subgroupoid, then $DNC(M, V)$ carries a groupoid structure.

Alain Connes explained how this construction naturally gives the Analytic index map of Atiyah-Singer, and used it to give a very nice proof of the Atiyah Singer Index Theorem ([3]).

3.2. **DNC and conormal distributions.** Let M be a manifold V a submanifold. Using a tubular neighbourhood, we may assume $M = E$ is a vector bundle over V. Then $N_V^M = E$ and $DNC(M, V) = E \times \mathbb{R}$. One may define $\mathcal{J} \subset C_c^\infty(DNC(M, V))$ as the set of functions f on $E \times \mathbb{R}$ with rapid decay such that $\hat{f}$ vanishes as well as all its derivatives on $V \times \{0\} \subset E^* \times \mathbb{R}$.

For $f \in \mathcal{J}$ and $t \in \mathbb{R}^*$, let $f_t \in C_c^\infty(M)$ restriction of f to $M \times \{t\}$.

Let $m \in \mathbb{C}$. Conormal distributions of order m are in fact just integrals $\int_0^{+\infty} \frac{f_t}{t^m} \frac{dt}{t}$ with $f \in \mathcal{J}$. Alternatively ([12]), one can characterize conormal distributions on $DNC(M, V)$ using the natural action of $\mathbb{R}_+^*$.

This idea can be used in order to produce pseudodifferential calculi in many more general situations.

Details and more references and examples can be found in [6].

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Fredholm theory for scattering on asymptotically conic spaces and applications

Andras Vasy

In this talk we consider generalizations of Euclidean resolvent estimates, in a Fredholm framework; these are relevant for e.g. the asymptotic behavior of wave equation solutions. Indeed, one motivation is understanding waves on Kerr spacetimes in a joint project with Häfner and Hintz [2]. However, these are already interesting in explaining the Euclidean phenomena: can one phrase the 'limiting absorption principle' as a Fredholm problem?

In the asymptotically Euclidean setting, one considers $H = \Delta_g + V$ where g_0 be the Euclidean metric, g metric on $\mathbb{R}^n$ with $g - g_0 \in S^{-\delta}$, $\delta > 0$ (i.e. $g_{ij} - (g_0)_{ij} \in S^{-\delta}$), g positive definite, $V \in S^{-\delta}$, real. ($\operatorname{Im} V \in S^{-1-\delta}$ is OK for Fredholm.) The space of symbols $S^m(\mathbb{R}^n_z)$, which is also used to capture asymptotically Euclidean behavior in geometric problems, is: $a \in S^m$ if $\forall \alpha\ |D_z^\alpha a(z)| \le C_\alpha \langle z\rangle^{m-|\alpha|}$, where $\langle z\rangle = (1+|z|^2)^{1/2}$. The talk also included an extension to asymptotically conic settings, which already arose in the work of Melrose [4]; effectively the 'sphere at infinity' in Euclidean space is replaced by another compact manifold, reflecting that Euclidean space (minus the origin) can be considered as a cone over the sphere. Then $H = \Delta_g + V$ is self-adjoint on $L^2(\mathbb{R}^n)$, so $H - \lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is invertible, e.g. as a map $H - \lambda : H^{s,l} \to H^{s-2,l}$, $s, l \in \mathbb{R}$. Moreover, the spectrum in $(-\infty, 0)$ is discrete, with 0 a possible accummulation point (e.g. Coulomb-like potentials); $[0,\infty)$ the essential spectrum. Here $H^{s,l} = \langle z\rangle^{-l} H^s$, H^s is the standard Sobolev space.

While $H - \lambda$ will no longer be invertible between the weighted Sobolev spaces when $\lambda > 0$, the limiting absorption principle states that $(H - (\lambda \pm i0))^{-1}$, understood as $\lim_{\epsilon\to 0}(H - (\lambda \pm i\epsilon))^{-1}$ exist e.g. as limits in $\mathcal{L}(H^{s-2,l}, H^{s,l'})$, $l > \frac{1}{2}$, $l' < -\frac{1}{2}$ (so $l - l' > 1$). Under stronger assumptions, $V \in S^{-2-\delta}$, $\delta > 0$, 0 is not an accummulation point of the point spectrum, and under stronger restrictions on l, l', in particular $l - l' > 2$, $(H-(\lambda \pm i0))^{-1}$ is uniformly bounded between the weighted Sobolev spaces as $\lambda \to 0$ if there are no 0-energy bound states (L^2 nullspace of H) or half-bound states (discussed in the talk). (Jensen, Kato [3],...,Fournais, Skibsted, Wang, Derezinski, Bony, Häfner, Rodnianski, Tao, Müller, Strohmaier, and N-body analogues, e.g. Wang, Skibsted, Tamura, as well as geometric microlocal analysis parametrix construction: Guillarmou, Hassell, Sikora [1])

Can one make the function spaces more precise, namely can one fit these into a Fredholm (here typically invertible) statement? Such frameworks are necessarily sharp in a sense. Here one necessarily must have different domains/target spaces for $H-\lambda$ for the cases producing the inverses $(H-(\lambda\pm i0))^{-1}$. What are these?

It is useful to write the spectral parameter as $\lambda=\sigma^2$, with $\operatorname{Im}\sigma>0$ corresponding to $\lambda\in\mathbb{C}\setminus[0,\infty)$, and let $P(\sigma)=H-\sigma^2=\Delta_g+V-\sigma^2$. We then are interested in $P(\sigma+i0)^{-1}=\lim_{\epsilon\to 0}P(\sigma+i\epsilon)^{-1}$; note that the limits with $\lambda\pm i0$ become $\pm\sigma+i0$ for $\sigma\in\mathbb{R}$, and exist e.g. as limits in $\mathcal{L}(H^{s-2,l},H^{s,l'})$, $l>\frac{1}{2}$, $l'<-\frac{1}{2}$ (so $l-l'>1$). This is *not* a sharp estimate, even though it is sharp on the standard scale of weighted Sobolev spaces; the point is that this is *not a satisfactory scale.* One way to see this is the $1+\epsilon$, $\epsilon>0$, order of loss of decay (cf. one derivative loss in hyperbolic PDE relative to elliptic ones); one expects the loss of 1 order if *done right.* One would also like to have a more precise description of the output of $P(\sigma)^{-1}$ for well-behaved inputs: should have the outgoing spherical wave form $e^{i\sigma\rho}(...)$ where $\rho=|z|$.

The reason for the non-optimality is that phase space behavior is not taken into account. To see what this looks like, consider scattering pseudodifferential operators of which $P(\sigma)$ is an example. These have a symbol calculus both in the position z and in the momentum ζ. The class $\Psi^{m,l}$ of scattering pseudodifferential operators of Melrose [4] on asymptotic cones, going back to Shubin and Parenti for $\mathbb{R}^n$, arises (in $\mathbb{R}^n$) from quantizing symbols $S^{m,l}$ satisfying the estimates $|D_z^\alpha D_\zeta^\beta a(z,\zeta)|\leq C_{\alpha\beta}\langle z\rangle^{l-|\alpha|}\langle\zeta\rangle^{m-|\beta|}$; here m is the differential and l is the decay order. One quantizes these symbols to operators via e.g. the standard left quantization to obtain $\operatorname{Op}(a)$. These pseudodifferential operators are closed under composition and adjoints, and one can compute the composition and adjoints modulo 'trivial' operators in $\Psi^{-\infty,-\infty}$, which map any weighted Sobolev space to any other. For instance, to leading order, *both* in ζ and in z decay, $\operatorname{Op}(a)\operatorname{Op}(b)$ is given by $\operatorname{Op}(ab)$! One calls $[a]$, the class of a in $S^{m,l}/S^{m-1,l-1}$, the principal symbol $\sigma_{m,l}(A)$ of $A=\operatorname{Op}(a)$. A is elliptic if $\sigma_{m,l}(A)$ is invertible. Moreover, $\Psi^{m,l}\subset\mathcal{L}(H^{s,r},H^{s-m,r-l})$ for all $s,r\in\mathbb{R}$. Indeed, one could *define* $H^{s,r}$ as consisting of tempered distributions u for which $Au\in L^2$ for some *elliptic* $A\in\Psi^{s,r}$.

Now, $\Delta-\sigma^2$ has principal symbol $|\zeta|^2-\sigma^2$, which vanishes for real σ at certain (finite) ζ; the issue is that this *persists* as $|z|\to\infty$, so in the spatial decay sense this operator is *not* elliptic. Since the principal symbol is real, within the characteristic set (where the principal symbol vanishes) we have propagation as for the wave equation, and the key question is the behavior of the Hamilton flow, i.e. the flow of H_p, or more precisely of $\mathsf{H}_p=\rho H_p$. Here the H_p-flow has a source/sink structure, with the source or sink corresponding to the phase space location of the incoming $(-)$ and outgoing $(+)$ spherical wave phase functions $e^{\mp i\sigma\rho}$. The usual propagation estimates propagate an existing estimate from one region to another, and one needs a way of starting this propagation. At the source/sink the a priori controlled term can be dropped if the decay order r is greater than the threshold value $-1/2$, while if $r<-1/2$ then one can propagate estimates from a punctured neighborhood of the source/sink to the source/sink itself; this already arose in [4].

Issue: being Fredholm needs estimates for both P and P^* on *dual* spaces, so

- for both we need high regularity (as measured by decay) at either the source or sink,
- which means low regularity at the same place for the dual,
- so we need $r > -1/2$, say, at source, $r < -1/2$ at the sink,
- so the decay order needs to be variable.

There are such variable order (or anisotropic) Sobolev spaces (going back to Unterberger, Duistermaat...), and indeed can be defined via variable order A, essentially $\mathrm{Op}(\langle\zeta\rangle^s\langle z\rangle^r)$, but $r = r(z,\zeta)$ is a homogeneous degree 0 function in z: $u \in H^{s,r}$ if for such elliptic variable order A, $Au \in L^2$. (Other uses of variable order spaces include Anosov dynamical systems: Faure, Sjöstrand, Dyatlov, Zworski, Guillarmou...) The results then extend to:

Theorem 1 ([5], see [7] for the low energy version)**.**

$$P(\sigma) : \{u \in H^{s,r} :\ P(\sigma)u \in H^{s-2,r+1}\} \to H^{s-2,r+1}$$

is invertible (in particular Fredholm), provided r is monotone along the H_p*-flow in the characteristic set,* $< -1/2$ *at one of the source/sink,* $> -1/2$ *at the other.*

We do *not* need to make sense of the limiting absorption principle resolvent as a limit; it is an honest Fredholm problem, thus sharp!

Here we can make r high everywhere except in a small neighborhood of the sink, say. But shouldn't we be able to make it high everywhere but *at* the sink? Here comes 2-microlocalization. Informally, 2-microlocalization blows up (resolves) the phase space, and a version goes back to Bony in the 80s.

- Here we blow up the outgoing source or sink manifold, which creates a new boundary hypersurface.
- Symbolic orders, as well as Sobolev space orders, arise from the order of vanishing at the boundary hypersurfaces.
- Thus, we have three orders now: differentiability, general decay (call it sc-decay) and outgoing decay (call it b-decay): $H^{s,r,l}$.
- We can have $r > -1/2$, $l < -1/2$ constant.

Theorem 2 ([6], see [8] for the low energy version)**.**

$$P(\sigma) : \{u \in H^{s,r,l} :\ P(\sigma)u \in H^{s-2,r+1,l+1}\} \to H^{s-2,r+1,l+1}$$

is invertible.

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The Fredholm theory of Tian–Yau metrics

Xuwen Zhu
(joint work with Rafe Mazzeo)

Gravitational instantons are defined as complete, noncompact four-dimensional hyperKähler manifolds with sufficient curvature decay at infinity. They play important roles in several areas of mathematics and physics. There are six known types of gravitational instantons, which go under the monikers ALE, ALF, ALG*, ALG, ALH* and ALH. Among these, ALH* spaces are total spaces of nilmanifold fibrations over a half-line, and have volume growth $r^{4/3}$ and curvature decay r^{-2}, and Tian–Yau metrics [5, 2] are a kind of ALH* spaces. Such metrics recently appear as bubbles in the degeneration theory of four-dimensional Einstein metrics [3].

The structure of Tian–Yau metrics at infinity is given by Calabi moduli. Such doubly warped structure is a kind of **a**-structures discussed in [1], and using the construction of pseudodifferential operators there we prove the Fredholm property of the Laplace operator [4].

Theorem 1. *The Laplace operator of a Tian–Yau metric (M, g) is a Fredholm operator mapping between suitable weighted Sobolev spaces*

(1)
$$\Delta : x^c H^k_{\mathbf{a}}(M) \to \Pi_\theta^\perp x^c H^{k-2}_{\mathbf{a}}(M) \oplus \Pi_\theta \Pi_y^\perp x^{c+2} H^{k-2}_{\mathbf{a}}(M) \oplus \Pi_y \Pi_\theta x^{c+4} H^{k-2}_{\mathbf{a}}(M)$$

for any c, k except when c is an indicial root for $\Pi_y \Pi_\theta \Delta \Pi_y \Pi_\theta$.

As applications we prove the following result about the L^2 cohomology of such metrics.

Theorem 2. *The L^2 harmonic forms of a Tian–Yau metric is identified with weighted cohomology and intersection cohomology with suitable perversities.*

We also show the following

Theorem 3. *Any Tian–Yau metric has a polyhomogeneous expansion at infinity.*

and

Theorem 4. *The perturbation of the Tian–Yau metrics determined by the moduli of Calabi model is unobstructed.*

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Reporter: Vito F. Zenobi

Participants

Dr. Iakovos Adroulidakis
National and Kapodistrian
University of Athens
Department of Mathematics
30 Panepistimiou St.
157-84 Athens
GREECE

Prof. Dr. Pierre Albin
Department of Mathematics
University of Illinois at
Urbana-Champaign
1409 West Green Street
Urbana IL 61801
UNITED STATES

Dr. Clara Lucia Aldana
Universidad del Norte
Km.5 Vía Puerto Colombia
081007 Barranquilla
COLOMBIA

Prof. Dr. Bernd Ammann
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Dr. Paolo Antonini
Dipartimento di Matematica e Fisica E.
de Giorgi,
Università del Salento
Piazza Tancredi n. 7
73100 Lecce
ITALY

Dr. Eric Bahuaud
Department of Mathematics
Seattle University
901 12th Ave., P.O.Box 222000
Seattle, WA 98122-1090
UNITED STATES

Prof. Dr. Christian Bär
Institut für Mathematik
Universität Potsdam
Karl-Liebknecht-Strasse 24/25
14476 Potsdam
GERMANY

Francesco Bei
Dipartimento di Matematica
Universita di Roma "La Sapienza"
Istituto "Guido Castelnuovo"
Piazzale Aldo Moro, 2
00185 Roma
ITALY

Prof. Dr. Maxim Braverman
Northeastern University
Department of Mathematics
567 Lake Hall
360 Huntington Ave.
Boston MA 02115-5000
UNITED STATES

Dr. Yaiza Canzani
Department of Mathematics
University of North Carolina
at Chapel Hill
Phillips Hall
Chapel Hill, NC 27599-3250
UNITED STATES

Dr. Paulo Carrillo-Rouse
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Gilles Carron
Département de Mathématiques Jean Leray
UMR 6629
Université de Nantes
2, rue de la Houssinière
44322 Nantes Cedex 03
FRANCE

Prof. Dr. Sergey Cherkis
Department of Mathematics
University of Arizona
617 N Santa Rita Ave
Tucson AZ 85721-0089
UNITED STATES

Clément Cren
Laboratoire d'analyse mathématique et applications (LAMA)
Faculté de Sciences et Technologie
Université Paris-Est - Creteil
61, Ave. du General de Gaulle
94010 Créteil Cedex
FRANCE

Prof. Dr. Xianzhe Dai
Department of Mathematics
University of California at Santa Barbara
South Hall 6714
Santa Barbara, CA 93106
UNITED STATES

Prof. Dr. Claire Debord
Institut de Mathématiques de Jussieu - Paris Rive Gauche
Université Paris
Campus des Grands Moulins
Bâtiment Sophie Germain
Boite Courrier 7012
8, place Aurélie Nemours
75205 Paris Cedex 13
FRANCE

Dr. Anda Degeratu
Institut für Geometrie und Topologie
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Panagiotis Dimakis
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Prof. Dr. Laura Fredrickson
Department of Mathematics
University of Oregon
97403 Eugene
UNITED STATES

Dr. Jesse Gell-Redman
Dept. of Mathematics and Statistics
University of Melbourne
Melbourne VIC 3010
AUSTRALIA

Prof. Dr. Sebastian Goette
Mathematisches Institut
Universität Freiburg
Ernst-Zermelo-Straße 1
79104 Freiburg i. Br.
GERMANY

Prof. Dr. Daniel Grieser
Institut fuer Mathematik
Carl v. Ossietzky-Universität Oldenburg
Fakultät V: Mathematik & Naturwissensch.
26111 Oldenburg
GERMANY

Prof. Dr. Batu Güneysu
Institut für Mathematik
Universität Potsdam
Karl-Liebknecht-Str. 24-25
14476 Potsdam
GERMANY

Dr. Luiz Hartmann
Av. Fimeno Rispoli, 179.
13564-200, São Carlos - SP / Brazil
Mathematics Department,
Universidade Federal de São Carlos
Rod. Washington Luís, Km 235
P.O. Box 676
13565905 Sao Carlos
BRAZIL

Prof. Dr. Chris Kottke
New College of Florida
5800 Bay Shore Road
Sarasota FL 34243-2197
UNITED STATES

Prof. Dr. Klaus Kröncke
Department Mathematik
Universität Hamburg (Geomatikum)
Bundesstraße 55
20146 Hamburg
GERMANY

Prof. Dr. Matthias Lesch
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Jean-Marie Lescure
Laboratoire d'analyse mathématique et applications (LAMA)
Faculté de Sciences et Technologie
Université Paris-Est - Creteil
61, Ave. du General de Gaulle
94010 Créteil Cedex
FRANCE

PD Dr. Ursula Ludwig
Universität Duisburg-Essen
Fakultät für Mathematik
45117 Essen
GERMANY

Prof. Dr. Xiaonan Ma
Université de Paris
Institut de Mathématiques de Jussieu - Paris Rive Gauche
Bâtiment Sophie Germain
P.O. Box Case 7012
75205 Paris Cedex 13
FRANCE

Kévin Massard
UMR 7586 CNRS
Institut de Mathématiques de Jussieu - Paris Rive Gauche, Université de Paris
Campus des Grands Moulins
Boite Courrier 7012
8 place Aurélie Nemours
75205 Paris Cedex 13
FRANCE

Prof. Dr. Rafe Mazzeo
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Prof. Dr. Richard B. Melrose
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Gerardo A. Mendoza
Department of Mathematics
Temple University
Philadelphia PA 19122
UNITED STATES

Dr. Omar Mohsen
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Dr. Jørgen Olsen Lye
Institut für Mathematik
Carl-von-Ossietzky-Universität
Oldenburg
Ammerländer Heerstrasse 114-118
26129 Oldenburg
GERMANY

Prof. Dr. Paolo Piazza
Dipartimento di Matematica
"Guido Castelnuovo"
Università di Roma "La Sapienza"
Piazzale Aldo Moro, 5
00185 Roma
ITALY

Massimiliano Puglisi
Dipartimento di Matematica
"Guido Castelnuovo"
Università di Roma "La Sapienza"
Piazzale Aldo Moro, 5
00185 Roma
ITALY

Dr. Frederic Rochon
Department of Mathematics
University of Quebec/Montreal
C.P. 8888
Montréal QC H3C 3P8
CANADA

Dr. Julie Rowlett
Department of Mathematics
Chalmers University of Technology
412 96 Göteborg
SWEDEN

Prof. Dr. Thomas Schick
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstrasse 3-5
37073 Göttingen
GERMANY

Prof. Dr. Elmar Schrohe
Institut für Analysis
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Dr. David Sher
Department of Mathematical Sciences
DePaul University
2320 N Kenmore Ave.
Chicago, IL 60614
UNITED STATES

Prof. Dr. Michael A. Singer
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
UNITED KINGDOM

Prof. Dr. Georges Skandalis
UFR de Mathématiques
Université Denis Diderot
Case 7012
5 rue Thomas Mann
75205 Paris Cedex 13
FRANCE

Prof. Dr. Alexander Strohmaier
School of Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Prof. Dr. Mathai Varghese
Department of Pure Mathematics
The University of Adelaide
North Terrace
5005 Adelaide SA 5005
AUSTRALIA

Prof. Dr. Andras Vasy
Department of Mathematics, Bldg 380
Stanford University
450 Jane Stanford Way
Stanford, CA 94305-2125
UNITED STATES

Dr. Mario Velásquez
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstrasse 3-5
37073 Göttingen
GERMANY

Prof. Dr. Boris Vertman
Institut für Mathematik
Carl von Ossietzky Universität
Oldenburg
Postfach 2503
26015 Oldenburg
GERMANY

Prof. Dr. Hartmut Weiss
Mathematisches Seminar
Christian-Albrechts-Universität Kiel
Ludewig-Meyn-Strasse 4
24098 Kiel
GERMANY

Dr. Jared Wunsch
Department of Mathematics
Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Dr. Vito Felice Zenobi
Dipartimento di Matematica Guido
Castelnuovo - Sapienza Università di
Roma
Piazzale Aldo Moro, 5
00185 Roma
ITALY

Prof. Dr. Wei-Ping Zhang
Chern Institute of Mathematics
Nankai University
Weijin Road 94
Tianjin 300071
CHINA

Prof. Dr. Xuwen Zhu
Department of Mathematics
Northeastern University
Boston MA 02115-5000
UNITED STATES