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Arbeitsgemeinschaft: Derived Galois Deformation Rings and Cohomology of Arithmetic Groups (hybrid meeting)

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Abstract. The purpose of the workshop was to study derived generalizations of Mazur's deformation ring of Galois representations, and the relationship of such a derived deformation ring to the homology of arithmetic groups.

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Introduction by the Organizers

In the 1980s, Mazur introduced the deformation ring of Galois representations in order to shed light on how Galois representations deform in p-adic families. Mazur was inspired, in part, by the construction of Hida of surjective Galois representations $\mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_p[\![x]\!])$ associated to ordinary families of modular forms. Mazur's formalism also allowed for the study of deformation rings R which capture (conjecturally) all Galois representations which come from automorphic forms of a fixed level and weight. These works were primarily concerned with representations of the Galois group of $\mathbb{Q}$ into GL_2. More recent work [CG, GV] has suggested that (in the general case) one should study instead a derived version of the ring R, and that this derived R acts on the homology of arithmetic groups (shifting homological degree).

The purpose of the workshop was to explain this picture, some parts of which remain conjectural. The overall background assumed of participants was the theory of holomorphic forms and basic properties of their Galois representations. The workshop was split into four general series of lectures, on the following themes:

- (§ A): Cohomology of arithmetic groups, and how to compute them using "Hodge theory":

  The whole picture depends in an essential way on a numerical coincidence between a dimension computation that occurs in $(\mathfrak{g}, K)$ cohomology and a dimension computation in Galois cohomology. The goal of § A was to outline the dimension computation in $(\mathfrak{g}, K)$ cohomology and how it manifests itself in the cohomology of arithmetic groups.
- (§ B): The Taylor–Wiles method (with the case of GL(1) serving as a motivating example).

  The Taylor–Wiles method is a way of analyzing the structure of the deformation space of a Galois representation by slowly enlarging the deformation space — allowing extra ramification — until it becomes formally smooth. In this process, one needs a way to bound the size of the deformation space from below, and this typically comes because one knows there are plenty of Galois representations coming from modular forms. This series of lectures introduced this method by studying the simplest (abelian) case.
- (§ C): The definition of derived deformation rings.

  The material of § B used the "usual" (i.e. underived) deformation space of a Galois representation. However, this can be enriched in a natural way to a derived space — that is to say, the ring of functions becomes a simplicial ring rather than a usual ring. To describe the usual deformation space we must describe its A-points for A a test ring; to describe the derived deformation space we must describe A-points for A a *simplicial* test ring. Many constructions with derived rings are not canonical up to unique isomorphism in the strict sense, but rather "up to contractible choice". The lectures followed the explicit construction of [GV].
- (§ D): Taylor-Wiles method in general.

  In the final series of lectures, bringing together the various strands, it was explained how to apply the Taylor–Wiles method, as explained in § B, to the cohomology of an arithmetic group in the higher rank case (focussing on GL_n). This ends up relating the derived deformation ring to the cohomology in a direct way (and in fact this is the only way of accessing the derived deformation ring that we currently know about).

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Arbeitsgemeinschaft (hybrid meeting): Derived Galois Deformation Rings and Cohomology of Arithmetic Groups

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Abstracts

A1. $(\mathfrak{g}, K)$-cohomology and Matsushima's formula

Mathilde Gerbelli-Gauthier

Let G be a semisimple Lie group, and Γ a lattice in G. We will express $H^*(\Gamma, \mathbf{C})$ in terms of $(\mathfrak{g}, K)$-cohomology of unitary representations of G.

An example. As a motivation, consider, as in [3, §3], the situation where $G = \mathrm{PSL}_2(\mathbf{C})^3$ and Γ is an irreducible torsion-free cocompact lattice. Let $\mathbf{H}^3$ be hyperbolic 3-space. Then G acts by isometries on $S = \mathbf{H}^3 \times \mathbf{H}^3 \times \mathbf{H}^3$, inducing a discrete action of Γ with quotient $\Gamma\backslash S$. Denote by $h^i(\Gamma)$ the i^{th} Betti number of $\Gamma\backslash S$. Then the vector of Betti numbers is of the form

$$(1,0,0,3,0,0,3,0,0,1) + k(0,0,0,1,3,3,1,0,0,0), \quad k \geq 1. \tag{1}$$

This restriction on the Betti numbers comes from a splitting of the tangent space at any point of $\Gamma\backslash S$. It induces a decomposition of the differential forms:

$$\Omega^q(\Gamma\backslash S) = \bigoplus_{a+b+c=q} \Omega^{a,b,c}(\Gamma\backslash S),$$

where forms of degree (a, b, c) have components of the corresponding degrees in the tangent spaces of each copy of $\mathbf{H}^3$. This decomposition is preserved by the Laplacian, and descends to cohomology. Moreover, the Hodge-$\star$ operator factors into partial $\star_i$ with for example $\star_1 : \Omega^{a,b,c}(\Gamma\backslash S) \to \Omega^{3-a,b,c}$. This implies for example $h^{3-a,b,c}(\Gamma) = h^{a,b,c}(\Gamma)$. The equality $h^{0,0,0} = h^{3,0,0} = ...$ explains the first vector in (1), and similarly $h^{1,1,1} = h^{2,1,1} = ...$ gives rise to the second, with the rest of the cohomology vanishing since Γ is irreducible.

This and more general decompositions of cohomology groups can be understood via $(\mathfrak{g}, K)$-cohomology of unitary representations of G.

$(\mathfrak{g}, K)$-cohomology. Keeping G as above, let $K \subset G$ be a compact subgroup and $\mathfrak{k} \subset \mathfrak{g}$ their complexified Lie algebras. We define $(\mathfrak{g}, K)$-cohomology following [1, §4], let V be a complex $\mathfrak{g}$-module. Define the chain complex $C^*(\mathfrak{g}, V)$ by letting $C^q(\mathfrak{g}, V) = \mathrm{Hom}_{\mathbf{C}}(\wedge^q \mathfrak{g}, V)$, with a differential given by

$$df(x_0, ..., x_q) = \sum_i (-1)^i x_i f(x_0, ..., \hat{x}_i, ..., x_q) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_q).$$

To compute $(\mathfrak{g}, K)$-cohomology we start by considering $C^*(\mathfrak{g}, \mathfrak{k}, V)$, the subcomplex of $C^*(\mathfrak{g}, V)$ realized as the common kernels of the interior products i_X and Lie derivatives θ_X for $X \in \mathfrak{k}$. If additionally V is a $(\mathfrak{g}, K)$-module, i.e. a simultaneous representation of $\mathfrak{g}$ and K, locally K-finite and satisfying compatibility conditions, then K acts on $C^*(\mathfrak{g}, \mathfrak{k}, V)$. The $(\mathfrak{g}, K)$-cohomology of V, denoted $H^*(\mathfrak{g}, K; V)$, is then defined as the cohomology of the complex

$$C^*(\mathfrak{g}, \mathfrak{k}, V)^K \simeq \mathrm{Hom}_K(\wedge^q(\mathfrak{g}/\mathfrak{k}), V).$$

In the case where G is compact and connected, the $(\mathfrak{g}, K)$-cohomology $H^*(\mathfrak{g}, K; \mathbf{C})$ computes the de Rham cohomology of the space G/K.

Note that the K-finite smooth vectors V_0 of a continuous representations (π, V) of G are a $(\mathfrak{g}, K)$-module. In fact, for π unitary, the $(\mathfrak{g}, K)$-module V_0 determines V, see [2, Corollary 9.2].

Matsushima's Formula. Let $\Gamma \subset G$ be a cocompact lattice, and let K now be a maximal compact subgroup of G. We have a decomposition

$$L^2(\Gamma\backslash G) = \hat{\bigoplus}_{\pi\in\hat{G}} m(\pi, \Gamma) H_\pi$$

for $\hat{G}$ the unitary dual of G, and $m(\pi, \Gamma)$ the multiplicity of π in $L^2(\Gamma\backslash G)$. Matsushima's formula, proved in [1, §VII], gives a decomposition of the cohomology $H^*(\Gamma, \mathbf{C})$ in terms of unitary representations:

$$H^*(\Gamma, \mathbf{C}) = \bigoplus_{\pi\in\hat{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, K; H_{\pi,0}).$$

To prove Matsushima's formula, we rewrite $H^*(\Gamma, \mathbf{C}) = H^*(\Omega(G/K, \mathbf{C})^\Gamma)$. We now want to identify the complex $\Omega^*(G/K, \mathbf{C})^\Gamma$ with $C^*(\mathbf{g}, K; C^\infty(\Gamma\backslash G/K))$. This follows since multiplication by $g \in G$ induces a canonical identification between $T_g(G)$ and $\mathfrak{g}$, resulting in an isomorphism $C^*(\mathfrak{g}, C^\infty(\Gamma\backslash G)) \simeq \Omega^*(G, \mathbf{C})^\Gamma$. Comparison of the K-actions gives $C^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G)) \simeq \Omega^*(G/K, \mathbf{C})^\Gamma$. Matsushima's formula then follows since compactness of $\Gamma\backslash G$ implies that $C^\infty(\Gamma\backslash G)$ is the space of smooth vectors in $L^2(\Gamma\backslash G)$, and only finitely many $\pi \in \hat{G}$ contribute nontrivially to $H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G))$.

Of the representations π contributing to cohomology, certain are tempered. These correspond to $h^{1,1,1} = h^{2,1,1} = ...$ in the initial example, and the contribution $h^{0,0,0} = h^{3,0,0} = ...$ comes from the trivial representation. The goal of the next three lectures will be to understand the cohomology of tempered representations. To describe it, assume now that G is the real points of a reductive group $\mathcal{G}$ defined over $\mathbf{Q}$, with K still maximal compact in G, and let A be the real points of the maximal $\mathbf{Q}$-split torus in the center of $\mathcal{G}$. Then we will show that the contribution to cohomology of tempered representations is isomorphic to some number of copies of $\wedge\mathbf{C}^{\ell_0}$ and appears in degrees $[q_0, q_0 + \ell_0]$, where

$$\ell_0 = rk(G) - rk(K) - rk(A),$$

$$q = \frac{\dim(G) - \dim(K) - \dim(A)}{2}, \quad q_0 = \frac{2q - \ell_0}{2}.$$

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A2. Cohomology of $(\mathfrak{g}, K)$-modules

XIAO LIANG

Preliminaries on Lie algebra. Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic zero. We first recall several facts about the center $Z(\mathrm{U}(\mathfrak{g}))$ of the universal enveloping algebra:

- $\mathrm{U}(\mathfrak{g})$ admits a natural filtration by degrees, and the associated algebra is isomorphic to $k[\mathfrak{g}^*]$.
- When $\mathfrak{g}$ is reductive with Cartan subalgebra $\mathfrak{t}$, the natural restriction
$$k[\mathfrak{g}^*]^{\mathrm{ad}(\mathfrak{g})} \to k[\mathfrak{t}^*]^W$$
is an isomorphism, where W is the Weyl group.
- Choose a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{t}$ with nilpotent subalgebra $\mathfrak{n}$, let $\mathfrak{n}^-$ denote the opposite nilpotent subalgebra. The composition
$$Z(\mathrm{U}(\mathfrak{g})) \subseteq \mathrm{U}(\mathfrak{g}) \twoheadrightarrow \mathrm{U}(\mathfrak{g})/\mathrm{U}(\mathfrak{g})\mathfrak{n} \cong \mathrm{U}(\mathfrak{n}^-) \otimes \mathrm{U}(\mathfrak{t})$$
is injective and induces an isomorphism $\mathrm{HC} : Z(\mathrm{U}(\mathfrak{g})) \xrightarrow{\cong} \mathrm{U}(\mathfrak{t})^{W,\bullet}$ to its image, where the superscript bullet indicates the Weyl group dotted action: $w \bullet \lambda := w(\lambda + \rho) - \rho$.

A typical example we consider is $\mathfrak{g} = \mathfrak{sl}_2$, where we take $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. The *Casimir operator* is $\Omega = \frac{1}{2}EF + \frac{1}{2}FE + \frac{1}{4}H^2 = \frac{1}{4}H^2 + \frac{1}{2}H - FE$, and $Z(\mathrm{U}(\mathfrak{sl}_2)) = k[\Omega]$.

The situation for real reductive Lie algebra $\mathfrak{g}$ is more subtle. Let θ denote the Cartan involution, which gives a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = \mathfrak{g}^{\theta=1}$ and $\mathfrak{p} = \mathfrak{g}^{\theta=-1}$. The Killing form B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. So if taking an orthogonal basis $\{X_i\}$ of $\mathfrak{p}$ and $\{X_a\}$ of $\mathfrak{k}$, the Casimir operator $C = \sum X_i^2 - \sum X_a^2$.

In the example of $\mathfrak{g} = \mathfrak{sl}_{2,\mathbb{R}}$, $\theta(X) = \mathrm{Ad}_{\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)}(X)$. Then for $\kappa = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$, $x_+ = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right)$, and $x_- = \left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$, we have
$$\mathfrak{k} = \mathbb{R}\kappa, \quad \mathfrak{p} = \mathbb{R}x_+ \oplus \mathbb{R}x_-.$$
Under the conjugation by $\left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$, κ (resp. x_+, x_-) corresponds to Hi (resp. Ei, Fi) in the above setting, and $\Omega = -\frac{1}{4}\kappa^2 + \frac{i}{2}\kappa + x_+x_-$.

Vanishing theorems of $(\mathfrak{g}, K)$-cohomology. It turns out that the $(\mathfrak{g}, K)$-cohomology are "often zero". Let G be a reductive group over $\mathbb{R}$ and $K \subseteq G(\mathbb{R})$ its maximal compact subgroup modulo center. Set $\mathfrak{g} := \mathrm{Lie}(G)$.

- Let (E, ρ) be a finite dimensional representation of G.
- Let (H, σ) be a $(\mathfrak{g}, K)$-module.

Assume that both (E, ρ) and (H, σ) admit infinitesimal characters $\chi_\rho, \chi_\rho : Z(\mathrm{U}(\mathfrak{g})) \to \mathbb{C}$.

Theorem 1. *(1)* $\mathrm{H}^q(\mathfrak{g}, K; E \otimes H) = 0$ if $\chi_\rho \neq -\chi_\sigma$.

(2) Assume that (H, σ) is a *unitary* $(\mathfrak{g}, K)$-module and that $\chi_\rho = -\chi_\sigma$. Then

$$\mathrm{H}^q(\mathfrak{g}, K; E \otimes H) \cong \mathrm{C}^q(\mathfrak{g}, K; E \otimes H) \cong \left(\wedge^q (\mathfrak{g}/\mathfrak{p})^* \otimes E \otimes H \right)^K.$$

Sketch of the proof. (1) follows from: if U and V are $(\mathfrak{g}, K)$-modules, then the actions of $z \in Z(\mathrm{U}(\mathfrak{g}))$ on $\mathrm{Ext}^q_{(\mathfrak{g},K)}(U, V)$ coming from U and from V are the same. So if the infinitesimal character of U is different from that of V, $\mathrm{Ext}^q_{(\mathfrak{g},K)}(U, V) = 0$ for any q.

(2) We use a certain variant of Hodge theory. The upshot is that one may associate each cochain $\mathrm{C}^q(\mathfrak{g}, K; V)$ an inner product structure. Then we may define $\partial : \mathrm{C}^{q+1}(\mathfrak{g}, K; V) \to \mathrm{C}^q(\mathfrak{g}, K; V)$ dual to the natural differential operator. Then the operator $\Delta := \partial d + d\partial$ behaves in a way similar to that of the Laplacian operator.

One may check that $\Delta\eta = 0 \Leftrightarrow d\eta = 0$ and $\partial\eta = 0 \Leftrightarrow (\Delta\eta, \eta) = 0$. This implies a decomposition

$$\mathrm{C}^q(\mathfrak{g}, K; V) = d\mathrm{C}^{q-1}(V) \oplus \partial\mathrm{C}^{q+1}(V) \oplus \mathcal{H}^q(\mathfrak{g}, K; V).$$

The key fact is that Δ acts on $\mathrm{C}^q(\mathfrak{g}, K; V)$ by the scalar $\rho(C) - \sigma(C)$, where C is the Casimir operator. So when $\rho(C) = \sigma(C)$, $C^q(\mathfrak{g}, K; V) \cong \mathcal{H}^q(\mathfrak{g}, K; V)$. In particular, all differentials are zero. ☐

Classification of irreducible $(\mathfrak{g}, K)$-modules for $\mathrm{SL}_2(\mathbb{R})$. We focus on the case of $\mathfrak{g} = \mathfrak{sl}_2$, $K = \mathrm{SO}(2)$. Then every $(\mathfrak{g}, K)$-module V admits a decomposition (as K-representations) $V = \bigoplus_{m \in \mathbb{Z}} V_m$, such that for $v_m \in V_m$ and $\kappa = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$, $\kappa v_m = mi \cdot v_m$. The raising and lowering operators are $x_+ = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right) : V_m \to V_{m+2}$ and $x_- = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right) : V_m \to V_{m-2}$.

We assume the Casimiar operator $\Omega = -\frac{1}{4}\kappa^2 + \frac{i}{2}\kappa + x_+x_-$ acts by γ on V.

(1) (Principal series) $P_{\gamma,n}$ with $n \in \{0, 1\}$ and $\gamma \neq \frac{m^2-1}{4}$ for any $m \in \mathbb{Z}$; or $\gamma = \frac{m^2-1}{4}$ but $m \equiv n \pmod 2$. (The following picture only gives $n = 0$ case; the case $n = 1$ is similar.)

$$\begin{tikzcd}
\cdots & v_{-4} \arrow[r, bend left, "x_+"] & v_{-2} \arrow[l, bend left, "x_-"] \arrow[r, bend left, "x_+"] & v_0 \arrow[l, bend left, "x_-"] \arrow[r, bend left, "x_+"] & v_2 \arrow[l, bend left, "x_-"] \arrow[r, bend left, "x_+"] & v_4 \arrow[l, bend left, "x_-"] & \cdots
\end{tikzcd}$$

(2) When $\gamma = \frac{m^2-1}{4}$ and $n \in \{0, 1\}$ such that $m \not\equiv n \pmod 2$. The principal series "breaks up" into three parts (again with $n = 0$ as an example)

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "x_+"] & v_{-1-m} \arrow[l, bend left, "x_-"] \arrow[r, bend left, "x_+=0"] & v_{1-m} \arrow[l, bend left, "x_-"] \arrow[r, bend left, "x_+"] & \cdots \arrow[l, bend left, "x_-"]
\end{tikzcd}$$

$$\mathrm{DS}^-_{m+1} = \bigoplus_{\substack{\ell\le -m-1\\ \ell\equiv m+1 \bmod 2}} \mathbb{C}v_\ell,$$

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "x_+"] & v_0 \arrow[l, bend left, "x_-"] \arrow[r, bend left, "x_+"] & \cdots \arrow[l, bend left, "x_-"]
\end{tikzcd}$$

$$\mathrm{FD}_{m-1} = \bigoplus_{\substack{1-m\le\ell\le m-1\\ \ell\equiv m+1 \bmod 2}} \mathbb{C}v_\ell,$$

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "x_+"] & v_{m-1} \arrow[l, bend left, "x_-"] \arrow[r, bend left, "x_+"] & v_{m+1} \arrow[l, bend left, "x_-=0"] \arrow[r, bend left, "x_+"] & \cdots \arrow[l, bend left, "x_-"]
\end{tikzcd}$$

$$\mathrm{DS}^+_{m+1} = \bigoplus_{\substack{\ell\ge m+1\\ \ell\equiv m+1 \bmod 2}} \mathbb{C}v_\ell.$$

$(\mathfrak{g},K)$-cohomology for representations of $\mathrm{SL}_2(\mathbb{R})$. As an example, we compute $\mathrm{H}^*(\mathfrak{g},K;V)$ for $V=\mathrm{PS}_{\gamma,n}$, $\mathrm{DS}^\pm_{m+1}$, and FD_{m+1}.

By Theorem 1(1), $\mathrm{H}^*(\mathfrak{g},K;V)=0$ unless C acts by zero on V, i.e. $\gamma=0$, which implies that $m=1$. So there are only four possible representations, whose $(\mathfrak{g},K)$-cohomology we list in the following table. (In fact, all the differentials are zero in all cases we consider.)

$(\mathfrak{g},K)$-module V	$\mathrm{PS}_{0,1}$	$\mathbf{1}$	DS_2^+	DS_2^-
$\mathrm{C}^0(\mathfrak{g},K;V)=\left(\wedge^0(\mathfrak{g}/\mathfrak{k})^*\otimes V\right)^K$	0	$\mathbb{C}$	0	0
$\mathrm{C}^1(\mathfrak{g},K;V)=\left(\wedge^1(\mathfrak{g}/\mathfrak{k})^*\otimes V\right)^K$	0	0	$\mathbb{C}$	$\mathbb{C}$
$\mathrm{C}^2(\mathfrak{g},K;V)=\left(\wedge^2(\mathfrak{g}/\mathfrak{k})^*\otimes V\right)^K$	0	$\mathbb{C}$	0	0

As an application, we make explicit the Matsushima formula in this case: for Γ a cocompact arithmetic lattice,

$$H^*\left(\Gamma\backslash\mathfrak{H},\underline{\mathbb{C}}\right)\cong\bigoplus_{\pi\in\widehat{\mathrm{SL}_2(\mathbb{R})}} m(\pi,\Gamma)\cdot\mathrm{H}^*(\mathfrak{g},K;H_{\pi,0})$$

$$H^0\left(\Gamma\backslash\mathfrak{H},\underline{\mathbb{C}}\right)\cong H^2\left(\Gamma\backslash\mathfrak{H},\underline{\mathbb{C}}\right)\cong\mathbb{C}^{m(\mathbf{1},\Gamma)},\qquad H^1\left(\Gamma\backslash\mathfrak{H},\underline{\mathbb{C}}\right)\cong\mathbb{C}^{m(\mathrm{DS}_2^-,\Gamma)+m(\mathrm{DS}_2^+,\Gamma)}.$$

A3. Representation theoretic background

JESSICA FINTZEN

This lecture provided some background on the representation theory of real (and complex) algebraic Lie groups with a focus on the example of $\mathrm{SL}_2(\mathbb{R})$. The following is a summary of the main topics discussed. More details can be found in [1], which is our main reference.

BASICS

We let G be a real (or complex) linear connected reductive group and K a maximal compact subgroup of G.

Definition 1. A representation of G on a complex Hilbert space $V \neq \{0\}$ is group homomorphism $\pi : G \to B(V)$ from G to the space of bounded linear operators $B(V)$ on V such that the resulting map $G \times V \to V$ given by $(g, v) \mapsto \pi(g)v$ is continuous.

Definition 2. A representation (π, V) of G is called admissible if $\pi(K)$ are unitary operators and every irreducible representation of K occurs with at most finite multiplicity in $(\pi|_K, V)$.

If (π, V) is an admissible representation of G, then every K-finite vector is a $\mathcal{C}^\infty$-vector and the space $V_{K-\mathrm{fin}}$ of K-finite vectors is stable under $\pi(\mathfrak{g})$. Hence taking K-finite vectors allows us to attach to each admissible representation of G a $(\mathfrak{g}, K)$-module. The notion of a $(\mathfrak{g}, K)$-modules was discussed in previous lectures.

PARABOLIC INDUCTION

Let $S = MAN$ be a parabolic subgroup of G with its Langlands decomposition, see Knapp [1] for details, for example. Let (σ, V_σ) be an irreducible unitary representation of M and ν an element of $\mathrm{Lie}(A)^* \otimes_{\mathbb{R}} \mathbb{C}$, which defines a character of A via the exponential map. Then Knapp defines a corresponding parabolic induction $\mathrm{ind}_{MAN}^G(\sigma \otimes \exp(\nu) \otimes 1)$ in [1, Chapter VII], which is a representation of G.

We will discuss the special case of $G = \mathrm{SL}_2(\mathbb{R})$ and $S = MAN$ with $M = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, $A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t > 0 \right\}$ and $N = \left\{ \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$. In this case we denote the parabolic induction corresponding to $\sigma_\pm : M \to \{\pm 1\}$ defined by $\sigma_\pm \left(- \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \pm 1$ and $\nu \in \mathrm{Lie}(A)^* \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}$ by $\mathcal{P}^{\pm,\nu}$. We distinguish three cases:

Case 1: $\nu \notin \mathbb{Z}$ or $\nu \in \mathbb{Z}$ and ν is even in the case of $\mathcal{P}^{+,\nu}$ and odd in the case of $\mathcal{P}^{-,\nu}$. In this situation $\mathcal{P}^{\pm,\nu}$ is irreducible and $\mathcal{P}^{\pm,\nu} \simeq \mathcal{P}^{\pm,\nu'}$ if and only if $\nu' \in \{\pm\nu\}$. The $(\mathfrak{g}, K)$-module of $\mathcal{P}^{+,\nu}$ is $P_{\frac{\nu^2-1}{4},0}$ and the $(\mathfrak{g}, K)$-module of $\mathcal{P}^{-,\nu}$ is $P_{\frac{\nu^2-1}{4},1}$ using the notation from Lecture A.2.

Case 2: $\mathcal{P}^{\pm,n-1}, n \geq 1$, with superscript "+" if n is even and "−" if n is odd. In this case $\mathcal{P}^{\pm,n-1}$ contains a direct sum $\mathcal{D}_n^+ \oplus \mathcal{D}_n^-$ consisting of two *discrete series* representations if $n \geq 2$ and of two *limit of discrete series* representations if $n = 1$. The $(\mathfrak{g}, K)$-module of $\mathcal{D}_n^\pm$ is $DS_n^\pm$ using the notation from Lecture A.2.

Case 3: $\mathcal{P}^{\pm,-(n+1)}, n \geq 0$, with superscript "+" if n is even and "−" if n is odd. In this case $\mathcal{P}^{\pm,-(n+1)}$ contains the $(n+1)$-dimensional irreducible representation Φ_n of $G = \mathrm{SL}_2(\mathbb{R})$, which can be realized as $\mathrm{SL}_2(\mathbb{R})$ acting on polynomials of degree at most n. The $(\mathfrak{g}, K)$-module of Φ_n is FD_n in the notation from Lecture A.2.

Using the classification of $(\mathfrak{g}, K)$-module of $\mathrm{SL}_2(\mathbb{R})$ from Lecture A.2., we see that every irreducible admissible representation of $\mathrm{SL}_2(\mathbb{R})$ is infinitesimally equivalent to a subrepresentation of a *nonunitary principal series representation*, i.e. a representation induced from a minimal parabolic subgroup. This statement, known as "Subrepresentation Theorem", also holds for general G.

DISCRETE SERIES AND TEMPERED REPRESENTATIONS

Definition 3. An irreducible admissible representation (π, V) of G is

(1) in the *discrete series* of G if all its K-finite matrix coefficients are in $L^2(G)$.

(2) *tempered* if all its K-finite matrix coefficients are in $L^{2+\epsilon}(G)$ for every $\epsilon > 0$.

In our example of $G = \mathrm{SL}(\mathbb{R})$, the discrete series consists of the representations $\mathcal{D}_n^{\pm}$ for $n \geq 2$, and the tempered representations are $\mathcal{D}_n^{\pm}$ for $n \geq 1$, $\mathcal{P}^{+,i\nu}$ for $\nu \in \mathbb{R}$ and $\mathcal{P}^{-,i\nu}$ for $\nu \in \mathbb{R} - \{0\}$.

For general G, with appropriate assumptions on $\Gamma \subset G$ and in a sense made precise in [2], "most" representations occurring in $L^2(\Gamma \backslash G)$ are tempered ([2]).

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A4. $(\mathfrak{g}, K)$-cohomology of tempered representations

SUG WOO SHIN

Let G be a real reductive Lie group. (Often G is the group of $\mathbb{R}$-points of a reductive algebraic group, but we also allow groups like $\mathbb{G}_m(\mathbb{R})^0 = \mathbb{R}^{\times}_{>0}$.) Write $\mathfrak{g}$ for the Lie algebra of G. Fix a maximal compact subgroup $K \subset G$.

Fix a minimal parabolic subgroup $P_0 \subset G$ and a maximal (connected) split torus $A_0 \subset P_0$ in the radical of P_0. Let (P, A) be a *standard p-pair*, that is a parabolic subgroup P and a (connected) split torus A in the radical of P such that $P \supset P_0$ and $A \subset A_0$. This gives rise to a Levi decomposition $P = M \rtimes N$ with $M := Z_G(A)$, and a direct decomposition $M = {}^0M \times A$. When $G = GL_n$, it is common to take P_0 (resp. A_0) to be the subgroup of upper triangular (resp. diagonal) matrices, where the diagonal entries of A_0 are required to be positive (so that A_0 is connected). If $\vec{n} = (n_1, ..., n_r)$ is an ordered partition of n (with $n_i \mathbb{G}e1$) then there is a corresponding standard p-pair $(P_{\vec{n}}, A_{\vec{n}})$, where $P_{\vec{n}}$ consists of block upper triangular matrices whose block sizes are $n_1, ..., n_r$ from the left top to the right bottom. In this case, ${}^0M$ is the group of block diagonal matrices with $n_i \times n_i$-matrices M_i on the block diagonal such that $\det M_i \in \{\pm 1\}$.

A Cartan subgroup $T \subset G$ is *fundamental* if its real rank is minimal, that is, equal to $\ell_0(G) = \mathrm{rk}G - \mathrm{rk}K$. There is a unique G-conjugacy class of fundamental

Cartan subgroups. This is analogous to the fact that split maximal tori form a single conjugacy class in a connected reductive group. A parabolic subgroup is *fundamental* if it is minimal among parabolic subgroups containing fundamental Cartan subgroups. As a related notion, a standard p-pair (P, A) (or simply P) is cuspidal if ${}^0M$ contains a compact Cartan subgroup. A couple of important facts are as follows.

- P is fundamental $\Leftrightarrow$ P is cuspidal and ${}^0M$ contains a regular element of G.
- P is cuspidal $\Leftrightarrow$ ${}^0M$ has discrete series representations.

In the above example, $P_{\vec{n}} \subset GL_n$ is fundamental if and only if $\{n_1, ..., n_r\} = \{2, 2, ..., 2, 1 \text{ or } 2\}$ as a multi-set, and cuspidal if and only if all $n_i \leq 2$. If G is a unitary group then $P = G$ itself is fundamental, but not every parabolic subgroup is cuspidal.

To state the main theorem, we recall the three important numerical invariants.

$$q(G) = \frac{1}{2}(\dim G - \dim K), \quad \ell_0(G) = \mathrm{rk}G - \mathrm{rk}K, \quad q_0(G) = q(G) - \frac{1}{2}\ell_0(G).$$

It is easy to see that the numbers do not depend on the choice of K and that $\ell_0(G), q_0(G) \in \mathbb{Z}_{\mathbb{G}e0}$. We omit G from the notation if it is clear from the context. When $G = GL_n$, we have $q = \frac{1}{4}n(n+1)$ and $\ell_0 = n - \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n+1}{2} \rfloor$. We have $q_0 = 0$ if G is a split torus, whereas $\ell_0 = 0$ if G is itself a fundamentla parabolic.

Write I_P for the parabolic induction as $(\mathfrak{g}, K)$-modules, ρ for the half sum of positive roots relative to (P_0, A_0), and W^P for W_G/W_M (or its set of representatives in W_G) where W_G (resp. W_M) is the the Weyl group of G (resp. M).

Now we state the main theorem, namely Theorem III.5.1 of Borel-Wallach.

Theorem 1. Consider the following data.

- (P, A) be a standard cuspidal p-pair, giving rise to M and ${}^0M$ as above,
- σ is a discrete series representation of ${}^0M$,
- $\nu \in i\mathfrak{a}^*$, viewed as a unitary character of A.

Assume that $H^*(\mathfrak{g}, K; I_P(\sigma \otimes \nu)) \neq 0$. Then

1. $\nu = 0$,
2. P is fundamental,
3. the infinitesimal character of σ equals $-s\rho$ for a unique $s \in W^P$.

Moreover $\dim H^j(\mathfrak{g}, K; I_P(\sigma)) = \begin{cases} \binom{\ell_0}{j-q_0}, & j \in [q_0, q_0 + \ell_0], \\ 0, & \text{else.} \end{cases}$

This tells us about the cohomology of tempered representations because every tempered representation is a direct summand of $I_P(\sigma)$ as in the theorem. Even though $I_P(\sigma)$ may be reducible, the decomposition is well understood from the Knapp–Zuckerman theory of R-groups.

It is worth considering two "extreme" cases of the theorem. Firstly, Case (DS): if G is itself fundamental, then the nonvanishing of cohomology implies that $P = G$ (in particular $I_P(\sigma \otimes \nu) = \sigma$, $j = q = q_0$), and the cohomology is 1-dimensional (in the middle degree). Secondly, Case (SpT): if $G = A$ is a split torus, then again $P = G$, and it is directly computed from the definition that $H^j(\mathfrak{a}, \{1\}, \mathbb{C}) = \bigwedge^j \mathfrak{a}_{\mathbb{C}}^*$.

In the actual proof, (DS) and (SpT) are not corollaries but the initial input, to be proved independently. While (SpT) is elementary, the much deeper proof of (DS) ultimately relies on the identification of minimal K-types in discrete series representations. Once (DS) and (SpT) are in place, the proof for (P, A) is deduced from (DS) for ${}^0M$ and (SpT) for A, by applying standard tools in cohomology such as the Hochschild-Serre spectral sequence, Shapiro's lemma, Künneth formula, and Poincaré duality, together with Kostant's formula computing the cohomology of $\mathfrak{n} = \mathrm{Lie} N$.

We end with the following technical remark. For this proof to work, we need to actually state and prove the theorem in a more general form (at least in Case (DS)), to allow the twisted coefficient $I_P(\sigma) \otimes F_\lambda$ for a finite dimensional representation F_λ of G of highest weight λ.

B1. Mazur's universal deformation ring

Paulina Fust

The aim of this talk is to give an overview of deformation theory due to Mazur ([1], [2]), or more precisely, to define the universal deformation rings representing certain deformation functors of group representations and study some of their properties. For this, we fix a prime p, a finite field k of characteristic p and $\mathcal{O} = W(k)$ its ring of Witt vectors. Let $\widehat{\mathcal{C}_{\mathcal{O}}}$ be the category of complete local Noetherian rings R, with a fixed isomorphism $R/\mathfrak{m}_R \xrightarrow{\sim} k$ and let $\mathcal{C}_{\mathcal{O}}$ be the full subcategory of Artinian rings. Let G be a profinite group, with the property, that for any open subgroup $H \leq G$, one has

$$\dim_{\mathbb{F}_p} \mathrm{Hom}^{cts}(H, \mathbb{F}_p) < \infty$$

This will ensure that the universal deformation rings that we study are Noetherian.

We fix a residual representation

$$\bar{\rho} : G \to \mathrm{GL}_n(k)$$

and study the following deformation functors associated to $\bar{\rho}$.

The framed deformation functor:

$$\begin{aligned} D_{\bar{\rho}}^{\square} : \mathcal{C}_{\mathcal{O}} &\to \mathrm{Sets} \\ A &\mapsto \{\text{representations } \rho_A : G \to \mathrm{GL}_n(A) \text{ lifting } \overline{\rho}\} \end{aligned}$$

The universal deformation functor:

$$\begin{aligned} D_{\overline{\rho}} : \mathcal{C}_{\mathcal{O}} &\to \mathrm{Sets} \\ A &\mapsto D_{\bar{\rho}}^{\square}(A)/\sim \end{aligned}$$

where the equivalence relation on $D_{\bar{\rho}}^{\square}(A)$ is given by conjugation by an element in the kernel of $\mathrm{GL}_n(A) \to \mathrm{GL}_n(k)$.

Mazur proved (see [2]), using Schlessinger's criterion for pro-representability, that there exists a *universal framed deformation ring* $R_{\bar\rho}^{\square} \in \widehat{\mathcal{C}_{\mathcal{O}}}$, representing the functor $D_{\bar\rho}^{\square}$. And similarly, if one assumes further that $\mathrm{End}_G(\bar\rho) = k$, there is a *universal deformation ring* $R_{\bar\rho} \in \widehat{\mathcal{C}_{\mathcal{O}}}$ representing the functor $D_{\bar\rho}$.

In this case, the deformation ring $R_{\bar\rho}$ has relations to the continuous cohomology of G with coefficients in the adjoint representation $\mathrm{Ad}(\bar\rho) = \mathrm{End}_k(\bar\rho)$, for example:

Lemma 1 ([3, Lemma 1.4.3]). There is a natural bijection:

$$\mathrm{Hom}_k(\mathfrak{m}_{R_{\bar\rho}}/(\mathfrak{m}_{R_{\bar\rho}}^2, p), k) = D_{\bar\rho}(k[\epsilon]) \xrightarrow{\sim} H^1(G, \mathrm{Ad}(\bar\rho)),$$

where $k[\epsilon] = k[X]/(X^2)$ is the ring of dual numbers.

Furthermore, for a representation $\rho : G \to \mathrm{GL}_n(A)$ on some $A \in \mathcal{C}_{\mathcal{O}}$, and a morphism $B \to A$ in $\mathcal{C}_{\mathcal{O}}$, such that the kernel I is annihilated by the maximal ideal $\mathfrak{m}_B$ of B, one can define an obstruction class $O(\rho)$ of ρ in $H^2(G, \mathrm{Ad}(\bar\rho)) \otimes I$, which is zero precisely when there is a deformation of ρ to B.

Let $h^i := \dim_k H^i(G, \mathrm{Ad}(\bar\rho))$, then Lemma 1 implies that there is a surjection

$$S := \mathcal{O}[\![X_1, \dots, X_{h^1}]\!] \twoheadrightarrow R_{\bar\rho},$$

inducing an isomorphism on tangent spaces. Denote by J the kernel of this map.

Proposition 2 ([2, Proposition 2]). There is an injection of k-vector spaces

$$\begin{aligned}(J/\mathfrak{m}_S J)^* &\hookrightarrow H^2(G, \mathrm{Ad}(\bar\rho)),\\ f &\mapsto (1 \otimes f)(O(\rho^{\mathrm{univ}}))\end{aligned}$$

where $(-)^*$ denotes the k-linear dual, ρ^{univ} is the universal deformation of $\bar\rho$ to $R_{\bar\rho}$ and $O(\rho^{\mathrm{univ}})$ is its obstruction class in $H^2(G, \mathrm{Ad}(\bar\rho)) \otimes (J/\mathfrak{m}J)$ associated to the morphism $S/\mathfrak{m}J \to R_{\bar\rho}$.

Nakayama's lemma implies that the ideal J can be generated by h^2 elements and we obtain the following presentation of the universal deformation ring:

Corollary 3. There is an isomorphism

$$R_{\bar\rho} \cong \mathcal{O}[\![X_1, \dots, X_{h^1}]\!]/(f_1, \dots, f_{h^2}),$$

and in particular $\dim R_{\bar\rho} \geq 1 + h^1 - h^2$. Moreover, if $\dim R_{\bar\rho} = 1 + h^1 - h^2$, then $R_{\bar\rho}$ is a complete intersection.

Typical examples of groups to which one applies these results are the absolute Galois groups $G_K = \mathrm{Gal}(\overline{K}/K)$ of a finite extension $K/\mathbb{Q}_\ell$ for any prime ℓ or the Galois group $G_{F,S} = \mathrm{Gal}(F_S/F)$, where $F/\mathbb{Q}$ is a number field and F_S the maximal extension of F unramified outside a finite set of primes S.

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B3. Taylor–Wiles for GL_1 (minimal case)

DAVID SAVITT

The goal of this talk was to illustrate the classical ($\ell_0 = 0$) Taylor–Wiles method via the example of GL_1. Let F be a number field, with absolute Galois group G_F. The invariant ℓ_0 for the group $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_1$ is r_1+r_2-1, where r_1, r_2 are the number of real and complex places of F respectively; so the case $\ell_0 = 0$ corresponds to the situation where F is either $\mathbf{Q}$ or an imaginary quadratic field.

Fix a prime p and consider the deformation theory (valued in complete local Noetherian $\mathbf{Z}_p$-algebras) of the mod p trivial character of G_F. Let Q be a finite set of primes of F not dividing p. We saw in talk B1 that the universal deformation ring for deformations of the trivial character that are unramified away from Q is

$$R_Q := \mathbf{Z}_p[[\Gamma_Q]].$$

Here Γ_Q is the Galois group over F of the maximal abelian pro-p extension of F unramified away from Q, and the double-brackets denote the completed group ring.

On the automorphic side, let $U_Q \subset \mathbf{A}_F^\times$ be the tame level structure associated to our set of places Q. A feature of the $\ell_0 = 0$ setting is that the locally symmetric space $X_Q := F^\times\backslash\mathbf{A}_F^\times/U_Q$ splits as a product $\mathbf{R} \times \mathrm{RCl}(Q)$, where $\mathrm{RCl}(Q)$ is the ray class group of F of conductor Q. The cohomology of X_Q thus lives entirely in degree 0, and the Hecke algebra acting on $H^0(X_Q, \mathbf{Z}_p)$ that was associated to this setup in talk B2 is $\mathbf{Z}_p[\mathrm{RCl}(Q)]$. Taking the completion of this Hecke algebra with respect to the maximal ideal corresponding to the trivial character amounts to passing to the p-part of $\mathrm{RCl}(Q)$; that is, we obtain the algebra

$$\mathbf{T}_Q := \mathbf{Z}_p[\mathrm{RCl}(Q) \otimes \mathbf{Z}_p].$$

Thus a Taylor–Wiles-style isomorphism $R_Q \cong \mathbf{T}_Q$ will witness the familiar isomorphism $\Gamma_Q \cong \mathrm{RCl}(Q) \otimes \mathbf{Z}_p$ of class field theory.

To apply the Taylor–Wiles method to this setup, we take as a starting point that there is a Galois representation $G_F \to \mathrm{GL}_1(\mathbf{T}_Q)$ that is unramified outside Q and satisfies local-global compatibility at all places. This amounts to the existence of abelian extensions of F corresponding to the group $\mathrm{RCl}(Q)$, which can be seen when $\ell_0 = 0$ using either the theory of cyclotomic fields ($F = \mathbf{Q}$) or complex multiplication (F imaginary quadratic). From the existence of this Galois representation we obtain a surjection $R_Q \twoheadrightarrow \mathbf{T}_Q$.

Write $\mathcal{Q} \subset \mathcal{O}_F$ for the product of the primes in Q. There is an action of the group $\Delta_Q = \mathbf{Z}_p \otimes (\mathcal{O}_F/\mathcal{Q}\mathcal{O}_F)^\times$ on each of R_Q and $\mathbf{T}_Q$. For $R_Q = \mathbf{Z}_p[[\Gamma_Q]]$ the action comes from the natural maps $I_v \to \Gamma_Q$ for each $v \in Q$, while for $\mathbf{T}_Q = \mathbf{Z}_p[\mathrm{RCl}(Q) \otimes \mathbf{Z}_p]$ it comes from the exact sequence

$$\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathcal{Q}\mathcal{O}_F)^\times \to \mathrm{RCl}(Q) \to \mathrm{Cl}(Q) \to 0.$$

The two actions can be checked to be compatible with the map $R_Q \twoheadrightarrow \mathbf{T}_Q$. Moreover it is not difficult to see that $R_Q/\Delta_Q \cong R_\varnothing$ and $\mathbf{T}_Q/\Delta_Q \cong \mathbf{T}_\varnothing$. A crucial feature of the $\ell_0 = 0$ setting is that if we further assume $\zeta_p \notin F$ then $\mathcal{O}_F^\times$ is killed by the tensor product with $\mathbf{Z}_p$, so that $\mathbf{T}_Q$ is actually *free* over $\mathbf{Z}_p[\Delta_Q]$.

Let us then suppose that $\ell_0 = 0$ and $\zeta_p \notin F$, and try to "prove" a minimal modularity lifting theorem $R_\varnothing \cong \mathbf{T}_\varnothing$ (with "prove" in quotes because the argument will ultimately be circular).

The first step is to produce sets of *Taylor–Wiles primes.* Let $H^1_Q(G_F, \mathbf{F}_p)$ be the space of classes unramified away from Q, and write $q = \dim H^1_\varnothing(G_F, \mathbf{F}_p)$. Combining the Greenberg–Wiles formula, Kummer theory, and the Cebotarev density theorem, one shows for each $N > 0$ that there are infinitely many sets of primes Q of F with the properties:

- $\#Q = q$,
- $N(v) \equiv 1 \pmod{p^N}$ for each $v \in Q$, and
- $\dim H^1_Q(G_F, \mathbf{F}_p) = q$.

The first and third conditions together are equivalent to the vanishing of a suitable dual Selmer group. (When $\ell_0 > 0$, the first condition must be replaced with $\#Q = q + \ell_0$.) This application of Greenberg–Wiles is ultimately what makes this argument circular.

Fix a set of Taylor–Wiles primes Q_N for each N. The ring R_{Q_N} admits *two* maps from a power series ring over $\mathbf{Z}_p$ in q variables: one, a surjection, coming from the fact that the tangent space of R_{Q_N} has dimension q; and the other from writing $\mathbf{Z}_p[\Delta_{Q_N}]$ as the quotient of a power series ring S_∞ in q variables. For each N we thus obtain a diagram

$$\begin{tikzcd}
\mathbf{Z}_p[[x_1, \dots, x_q]] \arrow[r, two heads] & R_{Q_N} \arrow[r, two heads] \arrow[d, two heads] & \mathbf{T}_{Q_N} \arrow[d, two heads] \\
S_\infty \arrow[ur] & R_\varnothing \arrow[r, two heads] & \mathbf{T}_\varnothing .
\end{tikzcd}$$

The magic of the patching argument is that it allows one to "pass to the inverse limit" over (a subequence) of these diagrams, or more precisely of certain bounded finite quotients of these diagrams. In this limit one obtains surjections $\mathbf{Z}_p[[x_1, \dots, x_q]] \twoheadrightarrow R_\infty \twoheadrightarrow \mathbf{T}_\infty$ and a map $S_\infty \to R_\infty$ that factors through $\mathbf{Z}_p[[x_1, \dots, x_q]] \twoheadrightarrow R_\infty$ and over which $\mathbf{T}_\infty$ is *free.* Since the depth of $\mathbf{T}_\infty$ as an S_∞-module is $q + 1$, its depth as an $\mathbf{Z}_p[[x_1, \dots, x_q]]$-module is at least as large. The surjections $\mathbf{Z}_p[[x_1, \dots, x_q]] \twoheadrightarrow R_\infty \twoheadrightarrow \mathbf{T}_\infty$ are therefore isomorphisms. Taking the quotient of $R_\infty \cong \mathbf{T}_\infty$ by the augmentation ideal of S_∞ one finally obtains the desired isomorphism $R_\varnothing \cong \mathbf{T}_\varnothing$.

B4. Taylor–Wiles for $\mathbf{GL_1}$ (general case)

PRESTON WAKE

In this talk, we explain the Calegari-Geraghty modifications to the Taylor-Wiles method for the group GL_1. The main references are [C, Part II], [GV, Section 13], and [CG, Section 8].

REVIEW OF THE $\ell_0 = 0$ CASE

Set up. We recall the set up from David's talk:

- F is a number field,
- $\mathcal{O}$ is the valuation ring of a finite extension of $\mathbb{Q}_p$, and k its residue field,
- Q is a finite set of primes of F, none of which divide p,
- $\ell_0 = r_1 + r_2 - 1$ is the rank of the unit group of F,
- $\mathrm{RCl}(Q)$ is the ray class group of F with modulus Q,
- Γ_Q is the Galois group of the maximal abelian unramified-outside-Q extension of F,
- $R_Q = \mathcal{O}[\![\Gamma_Q]\!]$ is the unramified-outside-Q deformation ring of the trivial character of G_F,
- $\mathbb{T}_Q$ is the Hecke algebra acting on the space $X_Q = F^\times\backslash\mathbb{A}_F^\times/U_Q$, localized at the appropriate maximal ideal.

We assume that we are given a deformation of the trivial character with values in $\mathbb{T}_Q$ that satisfies a local-global compatibility condition. In particular, we have a homomorphism $R_Q \to \mathbb{T}_Q$.

As in Matt's talk, we can think of $\mathrm{Spec}(R_Q)$ as the space of deformations, and the subset

$$\mathrm{Spec}(\mathbb{T}_Q) \subset \mathrm{Spec}(R_Q)$$

as being the deformations that come from automorphic forms. Our goal is to show that $R_Q \cong \mathbb{T}_Q$, which is to say that all deformations come from automorphic forms.

As David explained, the Hecke algebra $\mathbb{T}_Q$ is the group ring $\mathcal{O}[\mathrm{RCl}(Q)]$ of the ray class group. So the isomorphism $R_Q \cong \mathbb{T}_Q$ amounts to the isomorphism of $\Gamma_Q \cong \mathrm{RCl}(Q)$ of global class field theory. Our proof of this isomorphism is ultimately circular, because we use global duality for Galois cohomology, which requires the full strength of class field theory. The purpose is to illustrate the Calegari-Geraghty method in the simplest case possible.

The Taylor-Wiles method. We review the Taylor-Wiles method from David's talk, paying special attention to the role of the assumption that ℓ_0 is zero. We specialize to the minimal case and take $R = R_\emptyset$ and $\mathbb{T} = \mathbb{T}_\emptyset$.

One key idea of the Taylor-Wiles method is that we know different kinds of properties about R and about $\mathbb{T}$; if they are to be equal, then both must have both kinds of properties. The properties we know are:

(1): $\mathbb{T}$ is a finite free $\mathcal{O}$-module.

(2): R has as many generators as relations.

Property **(1)** is self-explanatory. To understand property **(2)** we use a variant of the presentation proved in Paulina's talk:

$$R \cong \frac{\mathcal{O}[\![x_1, \ldots, x_{h^1_\emptyset}]\!]}{(f_1, \ldots, f_{h^2_\emptyset})}. \tag{2}$$

Here $h^1_\emptyset$ is the dimension of the Selmer group $H^1_\emptyset(G_F, k)$ from David's talk and $h^2_\emptyset$ is the dimension of a "higher Selmer group" that keeps track of obstructions both to lifting and to making the lift be unramified. Global duality implies that $h^2_\emptyset = \dim_k H^1_{\emptyset^*}(G_F, k(1)) =: h^1_{\emptyset^*}$, and, as in David's talk, the Greenberg-Wiles Euler characteristic formula gives

$$h^1_\emptyset - h^2_\emptyset = -\ell_0.$$

Using the assumption that $\ell_0 = 0$ and (2), we see **(2)**.

If $R = \mathbb{T}$, then **(1)** and **(2)** are true, so both R and $\mathbb{T}$ should be LCI rings. It is then natural to try to find a smooth ring in which they are cut out by a regular sequence. The Taylor-Wiles method produces this smooth ring by "patching" together R_Q's for auxiliary sets Q, carefully chosen so that $h^2_Q = 0$, and the regular sequence is provided by the diamond operators for these auxiliary levels. The key point to showing these sequences are regular is:

(1)': $\mathbb{T}_Q$ is a finite free module over the diamond operator algebra $S_Q = \mathcal{O}[\Delta_Q]$.

What does wrong when $\ell_0 > 0$? If $\ell_0 > 0$, the two key properties **(2)** and **(1)'** both fail! We have see why **(2)** fails: $h^1_\emptyset - h^2_\emptyset = -\ell_0$, so R as ℓ_0-many more generators than relations – it has "expected dimension $-\ell_0$".

To see why **(1)'** fails, note that, in order to make h^2_Q vanish, one has to have $h^2_\emptyset = h^1_\emptyset + \ell_0$ many primes in the set Q. The patched diamond operator algebra S_∞ has Krull dimension $\#Q + 1 = h^1_\emptyset + \ell_0 + 1$. But the presentation of R implies that the patched Hecke algebra $\mathbb{T}_\infty$ can have Krull dimension at most $h^1_\emptyset + 1$. Hence $\mathbb{T}_\infty$ cannot be finite free over S_∞.

To resolve this issue, we need to find a replacement for **(1)'** that reflects our expectation that R and $\mathbb{T}$ have expected dimension $-\ell_0$. Inspired by derived algebraic geometry, we study this overly intersected situation by replacing modules with complexes.

THE GENERAL CASE $\ell_0 > 0$

The freeness in **(1)'** comes the definition of $\mathbb{T}_Q$ as an algebra of endomorphisms of a module $M_Q = H^0(X_Q, \mathcal{O})$ and the fact that M_Q is free as a S_Q-module. The idea is to replace M_Q by a complex of free S_Q-modules.

Commutative algebra. To know what properties we want from the complex, we need the following lemma from commutative algebra.

Key Lemma 1. Let $\delta \geq 0$ and let $A \to B$ be a morphism of regular local rings with $\dim(B) = \dim(A) - \delta$. Let $C^\bullet$ be a complex of finitely generated free A-modules supported in degrees $[0, \delta]$, and suppose that the A-action on $H^*(C^\bullet)$

factors through B. Then $H^*(C^\bullet)$ is concentrated in degree δ and $H^\delta(C^\bullet)$ is a free B-module.

We apply this lemma to the case:

- $A = S_\infty$, the patched diamond-operator algebra,
- $B = \mathcal{O}[\![x_1, \ldots, x_{h^1_\emptyset}]\!]$, thought of as the numerator in the presentation (2) for the patched deformation ring R_∞,
- $\delta = \ell_0$,
- $C^\bullet = M_\infty^\bullet$, the new complex we need to construct.

In addition to $M_\infty^\bullet$ being a complex of finite free S_∞-modules in degrees $[0, \ell_0]$, we want the action of S_∞ on $H^*(M_\infty^\bullet)$ to factor through the maps

$$S_\infty \to \mathcal{O}[\![x_1, \ldots, x_{h^1_\emptyset}]\!] \twoheadrightarrow R_\infty \twoheadrightarrow \mathbb{T}_\infty. \tag{3}$$

The complex $M_\infty^\bullet$ should also have a base change property:

$$M_\infty^\bullet \otimes^{\mathbb{L}}_{S_\infty} \mathcal{O} \simeq M_\emptyset^\bullet. \tag{4}$$

for a complex $M_\emptyset^\bullet$ at minimal level (independent of the patching data).

Theorem 2. Assume there is a complex $M_\infty^\bullet$ with the above properties. Then the map $R \to \mathbb{T}$ is an isomorphism, and there is an isomorphism

$$H^*(M_\emptyset^\bullet) \cong \operatorname{Tor}_*^{S_\infty}(M_\infty, \mathcal{O})$$

for a free $\mathbb{T}_\infty$-module M_∞.

Proof. By the Key Lemma, the cohomology of the complex $M_\infty^\bullet$ is concentrated in degree ℓ_0 and $H^{\ell_0}(M_\infty^\bullet)$ is a free $\mathcal{O}[\![x_1, \ldots, x_{h^1_\emptyset}]\!]$-module. Since the $\mathcal{O}[\![x_1, \ldots, x_{h^1_\emptyset}]\!]$-action factors through the surjections in (3), we see that $H^{\ell_0}(M_\infty^\bullet)$ is a free $\mathbb{T}_\infty$-module, and that those surjections are isomorphisms. This implies that $R \to \mathbb{T}$ is an isomorphism by base-change.

Letting $M_\infty = H^{\ell_0}(M_\infty^\bullet)$, we see that there is a quasi-isomorphism $M_\infty^\bullet \simeq M_\infty[\ell_0]$. Now applying the base-change isomorphism (4) and taking cohomology, we get the desired description of $H^*(M_\emptyset^\bullet)$. □

The complex $M_\infty^\bullet$. The complex $M_\infty^\bullet$ is constructed by patching. For each choice of auxiliary prime set Q, we need

- a complex $M_Q^\bullet$ of finite free S_Q-modules supported in degrees $[0, \ell_0]$,
- a Galois action and Hecke action on $H^*(M_Q^\bullet)$ that are compatible with the given homomorphism $R_Q \to \mathbb{T}_Q$ (in other words, we want the $\mathbb{T}_Q$-valued deformation come from $H^*(M_Q^\bullet)$).

The complex $M_Q^\bullet$ is defined using the singular cohomology complex of the space X_Q. The component group of X_Q is $\mathrm{RCl}(Q)$, and, as explained in Matt's talk, the connect component of the identity is $\mathbb{R} \times (S^1)^{\ell_0}$. The Hecke action comes from the component group. The fact the singular cohomology complex can be represented by a prefect complex of S_Q-modules supported in degrees $[0, \ell_0]$ follows formally from the fact that the cohomological dimension of $(S^1)^{\ell_0}$ is ℓ_0 and that $X_Q \to X_\emptyset$ is a covering space with Galois group Δ_Q.

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C1. Simplicial sets and rings

Markus Szymik

This talk was an introduction to homotopical algebra in the style of Quillen. I started by introducing the standard cosimplicial category and simplicial objects in other categories.

Then I specialized the context to simplicial sets. Geometric realization allows us to define the homotopy groups and the weak equivalences via the well-known concepts for topological spaces. Cofibrations are degree-wise injections, and this determines the fibrations; the fibrant objects are the Kan complexes. I also introduced mapping spaces.

As for simplicial abelian groups, the normalized chain complex gives the Dold–Kan equivalence to the category of non-negative chain complexes. The homotopy groups become the usual homology groups. Therefore, the weak equivalences become the homology isomorphisms. We lift the fibrations from the underlying simplicial sets; all objects are now fibrant.

Simplicial rings are the monoids for the degree-wise tensor product of simplicial abelian groups. Without further mentioning, we understand everything possible to be commutative. The normalized chain complex of a simplicial ring is a differential graded algebra, but not every differential graded algebra has this form. For example, I mentioned the polynomial rings on a simplicial set of generators. We define the homotopy theory by lifting the weak equivalences and fibrations from simplicial sets or abelian groups. We can describe cofibrations by attaching cells.

Finally, I defined representable functors in the homotopical context. The forgetful functor is an example. I ended with two technical results: Representable functors are homotopy invariant. A functor is homotopy invariant if and only if it is equivalent to a Kan valued functor that admits a simplicial enrichment.

C2. Simplicial Artin rings and pro-representability

ARPON RASKIT

Just as classical Galois deformation rings arise as the representing objects of certain formal moduli problems, and are constructed by verifying representability criteria, derived Galois deformation rings arise as the representing objects of certain derived formal moduli problems (enhancing the classical formal moduli problems), and again are constructed by verifying representability criteria. In this lecture, our aim is to understand what a derived formal moduli problem is and the representability criteria for such. Our presentation largely follows Galatius–Venkatesh [1], to which one can refer for more information; see also the original work of Lurie [2].

ARTIN LOCAL SIMPLICIAL COMMUTATIVE RINGS

Classically, a formal moduli problem over a field k is a functor $\mathrm{Art}_k^0 \to \mathrm{Set}$, where Art_k^0 denotes the category of Artin local commutative rings with residue field identified with k. The main point of the derived enhancement is to enlarge our category of test objects.

Definition 1. Let A be a simplicial commutative ring. We say that A is *Artin local* if $\pi_0(A)$ is an Artin local ring and $\pi_*(A)$ is a finitely generated $\pi_0(A)$-module (in other words, $\pi_i(A)$ is a finitely generated $\pi_0(A)$-module for each i and vanishes for $i \gg 0$).

If A is Artin local, the *residue field* of A is defined to be the residue field of $\pi_0(A)$. For k a field, we let Art_k denote the full subcategory of $\mathrm{SCR}_{/k}$ consisting of those maps of simplicial commutative rings $\epsilon : A \to k$ where A is Artin local and ϵ exhibits k as the residue field of A.

Example. Any Artin local commutative ring can be regarded as an Artin local simplicial commutative ring. For any field k, this determines a fully faithful functor $\mathrm{Art}_k^0 \hookrightarrow \mathrm{Art}_k$.

Example. Let k be a field and let M be a simplicial k-module. Then we have the square-zero extension $k \oplus M \in \mathrm{SCR}_{/k}$, obtained by forming ordinary square-zero extensions at each simplicial level, and with zeroth homotopy group $\pi_0(k \oplus M)$ given by the ordinary square-zero extension $k \oplus \pi_0(M)$. If $\pi_*(M)$ is a finite-dimensional k-vector space, then $k \oplus M$ is Artin local, hence lies in Art_k.

There is an important special case of this construction. Let $n \geq 0$, and let S^n denote the simplicial n-sphere $\Delta^n/\partial\Delta^n$. We can form the free simplicial k-module $k[S^n]$, and then quotient out the basepoint to obtain a simplicial k-module that we will denote by $k[n]$ (note that $\pi_*(k[n])$ is k concentrated in degree n). This determines a square-zero Artin local simplicial commutative ring $k \oplus k[n] \in \mathrm{Art}_k$. When $n = 0$, this recovers the usual dual numbers $k[\epsilon]/(\epsilon^2)$.

In summary, just like an ordinary Artin local commutative ring, we think of an Artin local simplicial commutative ring as a finite infinitesimal/nilpotent thickening of its residue field, but now possibly with new kinds of nilpotents, namely elements in the higher homotopy groups.

Derived formal moduli problems

Definition 2. For k a field, a *derived formal moduli problem over k* is a homotopy invariant functor $\mathrm{Art}_k \to \mathrm{sSet}$.[1]

Example. Let k be a field and let R be a cofibrant simplicial commutative ring equipped with a map $\epsilon : R \to k$. Then $\mathcal{F}_R(A) := \mathrm{SCR}_{/k}(R, A)$ defines a derived formal moduli problem over k (encoding the formal completion of $\mathrm{Spec}(R)$ at the point ϵ).

More generally, suppose that $R = \{R_j\}_{j\in J}$ is a pro-object in $\mathrm{SCR}_{/k}$, i.e. a cofiltered diagram $J \to \mathrm{SCR}_{/k}$, with each R_j cofibrant. Then $\mathcal{F}_R(A) := \operatorname{colim}_{j\in J} \mathrm{SCR}_{/k}(R_j, A)$ defines a derived formal moduli problem over k. We say that a derived formal moduli problem is *pro-representable* if it is naturally weakly equivalent to one of this form where each R_j in fact lies in $\mathrm{Art}_k \subseteq \mathrm{SCR}_{/k}$.

In this derived setting, we have the following analogue of Schlessinger's criteria:

Theorem 3 (Lurie)**.** Suppose that $\Omega^1_{k/\mathbb{Z}} \simeq 0$. Then a derived formal moduli problem $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSet}$ is pro-representable if and only if the following three conditions hold:

(1) $\mathcal{F}(k)$ is weakly contractible;

(2) given a homotopy cartesian diagram in Art_k

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with $\pi_0(B) \to \pi_0(D)$ and $\pi_0(C) \to \pi_0(D)$ surjective, the induced diagram of simplicial sets

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r] \arrow[d] & \mathcal{F}(B) \arrow[d] \\ \mathcal{F}(C) \arrow[r] & \mathcal{F}(D) \end{tikzcd}$$

is also homotopy cartesian.

(3) $\mathcal{F}(k \oplus k[0])$ is homotopy discrete (i.e. its higher homotopy groups vanish).

Remark 4. Let us briefly discuss the notion of homotopy cartesian diagram invoked in 3. Up to weak equivalence, simplicial sets can be replaced by Kan complexes (even functorially). For $Y \to X \leftarrow Z$ a diagram of Kan complexes, the homotopy pullback $Y \times^h_X Z$ is defined to be the simplicial set $(Y \times Z) \times_{(X\times X)} X^{\Delta^1}$, so that a map of simplicial sets $S \to Y \times^h_X Z$ consists of maps $S \to Y$ and $S \to Z$

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[1] It may be more standard to require further conditions on a derived formal moduli problem (such as the first two appearing in 3 below). Our use of the terminology here is for purely expository purposes.

together with a homotopy of the two composites $S \to X$. In particular, a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

determines a map $S \to Y \times_X^h Z$, and we say that the diagram is homotopy cartesian if this map is a weak equivalence. We note also the following points:

(1) A weak equivalence of diagrams $Y \to X \leftarrow Z$ induces a weak equivalence of homotopy pullbacks (which is not true in general for usual pullbacks).

(2) If $Z = \Delta^0$, the homotopy pullback $Y \times_X^h Z$ is a homotopy fiber F of $Y \to X$ and we have a long exact sequence

$$\cdots \to \pi_1(X) \to \pi_0(F) \to \pi_0(Y) \to \pi_0(Z).$$

(3) Any simplicial abelian group, in particular any simplicial commutative ring, is a Kan complex, and the homotopy pullback of a diagram of such retains the same algebraic structure. For $Y \to X \leftarrow Z$ a diagram of simplicial abelian groups, there is a long exact sequence

$$\cdots \to \pi_1(X) \to \pi_0(Y \times_X^h Z) \to \pi_0(Y) \oplus \pi_0(Z) \to \pi_0(X).$$

In other words, the sequence $Y \times_X^h Z \to Y \oplus Z \to X$ is (up to a connectivity issue) a distinguished triangle on the other side of the Dold–Kan correspondence. This shows that for $B \to D \leftarrow C$ as in condition (2) of 3, the homotopy pullback $B \times_D^h C$ in is in fact an Artin local simplicial commutative ring. For example, for $n \geq 1$, there is an equivalence of simplicial k-modules $0 \times_{k[n]}^h 0 \simeq k[n-1]$, and therefore an equivalence of simplicial commutative rings $k \times_{k \oplus k[n]}^h k \simeq k \oplus k[n-1]$.

Structure of Artin local simplicial commutative rings

One ingredient in the proof of the classical Schlessinger theorem is the fact that Artin local commutative rings can be analyzed inductively: for A an Artin local commutative ring with residue field k, the quotient map $A \to k$ can be factored in Art_k^0 as a finite sequence of maps $A = A_m \to A_{m-1} \to \cdots \to A_0 = k$ where each $A_i \to A_{i-1}$ is a square-zero extension by k. This has the following generalization, which is used to prove 3:

Proposition 5. Let A be an Artin local simplicial commutative ring with residue field k. Then there is a sequence of maps $A \to A_m \to A_{m-1} \to \cdots \to A_0 \to k$, where the first and last are weak equivalences, together with weak equivalences from A_i to the homotopy pullback of a diagram $A_{i-1} \to k \oplus k[n_i] \leftarrow k$ for some $n_i \geq 1$.

Remark 6. If we just think additively, forming a homotopy pullback $A' \to k \oplus k[n] \leftarrow k$ is equivalent to forming an extension of A' by $k[n-1]$.

Remark 7. Let $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSet}$ be a derived formal moduli problem and let $A \to A'$ be a square-zero extension of ordinary Artin local commutative rings by k. Supposing that $\mathcal{F}$ satisfies the first two conditions of 3, $\mathcal{F}(A)$ is weakly equivalent to the homotopy fiber of a map $\mathcal{F}(A') \to \mathcal{F}(k \oplus k[1])$. This gives a long exact sequence

$$\cdots \to \pi_0(\mathcal{F}(A)) \to \pi_0(\mathcal{F}(A')) \to \pi_0(\mathcal{F}(k \oplus k[1])),$$

so that given a class $x \in \pi_0(\mathcal{F}(A'))$, there is an "obstruction class" in $\pi_0(\mathcal{F}(k \oplus k[1]))$ that vanishes if and only if x lifts to a class in $\pi_0(\mathcal{F}(A))$. Statements of this shape appear often in classical deformation theory, and indeed many can be explained from this perspective.

References

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C3. Cotangent complexes of rings and tangent complexes of functors.

Lukas Brantner

Overview. The goal of this talk is to introduce a derived generalisation of tangent spaces, using the cotangent complex formalism of André, Illusie, and Quillen. This is an important ingredient in Lurie's proof of the derived Schlessinger criterion.

Kähler differentials. Given a commutative ring A, an A-algebra C, and a C-module M, let $\mathrm{Der}_A(C, M)$ be the C-module of A-linear derivations $D : C \to M$. Writing $C \oplus M$ for the trivial square-zero extension of C by M, we can identify $\mathrm{Der}_A(C, M)$ with the space of A-algebra sections of the projection $C \oplus M \to C$.

The functor $M \mapsto \mathrm{Der}_A(C, M)$ is representable by the module $\Omega_{C/A}$ of Kähler differentials, and there is an adjunction

$$F : \{A\text{-algebras}/C\} \leftrightarrows \{C - \text{modules}\} : G$$

$$B \mapsto C \otimes_B \Omega_{B/A}$$

$$C \oplus M \leftarrow M$$

The cotangent complex. In homological algebra, we refine the tensor product $M \otimes_R N$ of two modules M, N over a ring R by picking a projective resolution $\ldots \to P_2 \to P_1 \to P_0$ of N and setting $M \otimes^{\mathbb{L}}_R N := (\ldots \to M \otimes_R P_1 \to M \otimes_R P_0)$, thereby computing the value of the left derived functor of $M \otimes_R (-)$ on N.

Quillen's *homotopical algebra* [4] lets us compute derived functors in the non-abelian setting of A-algebras, and refine the module of Kähler differentials $\Omega_{C/A}$. To pass from homological to homotopical algebra, we replace chain complexes by simplicial A-algebras, quasi-isomorphisms by weak equivalences, and levelwise projective chain complexes by cofibrant A-algebras.

If we want to compute the left derived functor of the functor F above on some B, we must first pick a weakly equivalent cofibrant simplicial A-algebra $P_\bullet$ over C, then apply the functor F levelwise to obtain the simplicial abelian group

$$C \otimes_{P_\bullet} \Omega_{P_\bullet/A},$$

and finally apply Dold-Kan to obtain a connective chain complex of C-modules.

The *(relative) cotangent complex* of $A \to C$ is then defined as $L_{C/A} := (\mathbb{L}F)(C)$; this construction natually extends to simplicial commutative rings.

We state three rules which often facilitate the computation of cotangent complexes:

(1) If $C = A[x_1, \dots, x_n]$ is a polynomial ring, then $L_{C/A}$ is equal to $\Omega_{C/A}$ concentrated in degree 0, which is a free C-module on generators $dx_1, \dots, dx_n$.

(2) Given homomorphisms $A \to B \to C$, there is a cofibre sequence

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}.$$

(3) If $A \to B$ is surjective with kernel I generated by a regular sequence, then $L_{B/A}$ is given by I/I^2 concentrated in homological degree 1.

Example. Given $C = A[x_1, ...x_n]/(f_1, ...f_m)$ with $f_1, ...f_m$ a regular sequence, we can compute $L_{C/A} \simeq (... \to 0 \to 0 \to I/I^2 \xrightarrow{f \mapsto 1 \otimes df} C \otimes_{A[x_1,...x_n]} \Omega_{A[x_1,...x_n]/A})$.

The tangent complex of pointed schemes. Fix a field k. Commutative rings R with a map to k correspond to affine schemes Spec(R) with a chosen k-point x.

Classically, the tangent space of Spec(R) at x is given by the k-vector space $(k \otimes_R \Omega_{R/\mathbb{Z}})^\vee \cong Map_{\mathrm{CRing}_{/k}}(R, k[\epsilon]/\epsilon^2)$, where $(-)^\vee$ denotes k-linear duality.

The *tangent complex* $\mathfrak{t}_R$ is a derived refinement of this construction, and given by the coconnective chain complex

$$\mathfrak{t}_R = (k \otimes^{\mathbb{L}}_R L_{R/\mathbb{Z}})^\vee.$$

For $n \geq 0$, the truncation $\tau_{\geq 0}(\mathfrak{t}_R[n])$ gives rise to a simplicial k-module via Dold-Kan; its underlying simplicial set models the *space* of morphisms $R \to k \oplus k[n]$. Here $C[n]$ denotes the n^{th} (homological) shift of a complex C, so that $C[n]_i = C_{i-n}$.

Remark. Let R, S be simplicial commutative rings with a map to k. If R is cofibrant, the space of maps $R \to S$ is modelled by a simplicial set with n-simplices corresponding to maps $R \to S^{\Delta^n}$, where Δ^n is the usual n-simplex.

The tangent complex of formally cohesive functors. We can recover the complex $\mathfrak{t}_R$ from the functor $\mathcal{F}_R = Map_{SCR_{/k}}(R, -) : \mathrm{Art}_k \to \mathcal{S}$ represented by R. Here $\mathrm{Art}_k \subset SCR_{/k}$ denotes full subcategory of simplicial Artin local rings $A \to k$ inducing a surjection on π_0, and $\mathcal{S}$ is the ∞-category of spaces.

To this end, we need two basic facts from higher category theory:

(1) The ∞-category $Mod_{k,\geq 0}$ of connective chain complexes over k is obtained from Vect_k^ω, the category of finite-dimensional k-vector spaces, by freely adjoining filtered colimits and geometric realisations, cf. [3, Section 1.3.3]. Hence $Mod_{k,\geq 0}$ is equivalent to $\mathcal{P}_\Sigma(\mathrm{Vect}_k^\omega)$, the ∞-category of

finite-product-preserving functors from $(\mathrm{Vect}_k^\omega)^{op}$ to spaces, cf. [2, Section 5.5.8].

(2) The ∞-category Mod_k of *all* chain complexes over k arises as the limit

$$Mod_k \simeq \lim\Big(\ldots \to Mod_{k,\geq 0} \xrightarrow{\Omega} Mod_{k,\geq 0} \xrightarrow{\Omega} Mod_{k,\geq 0}\Big),$$

where Ω is given by $M \mapsto \tau_{\geq 0}(M[-1])$, cf. [3, Proposition 1.4.2.24].

As $\mathcal{F}_R$ is formally cohesive, (1) and (2) imply that the sequence of functors

$$(\;\ldots,\;\; \mathcal{F}_R(k \oplus (-)^\vee[2]),\;\; \mathcal{F}_R(k \oplus (-)^\vee[1]),\;\; \mathcal{F}_R(k \oplus (-)^\vee))$$

defines a chain complex over k, which can be identified with the tangent complex $\mathfrak{t}_R$.

For a general formally cohesive functor $\mathcal{F} : \mathrm{Art}_k \to \mathcal{S}$, we define the tangent complex $\mathfrak{t}_\mathcal{F} \in Mod_k$ by simply applying the above construction to $\mathcal{F}$ instead of $\mathcal{F}_R$. Unravelling the definitions, we see that for any $V \in \mathrm{Vect}_k^\omega$ and any $n \geq 0$, we have

$$\mathrm{Map}_{Mod_k}(V, \tau_{\geq 0}(\mathfrak{t}_\mathcal{F}[n])) \simeq \mathcal{F}(k \oplus V^\vee[n]).$$

The tangent complex is a powerful invariant; for example, it detects equivalences:

Proposition. A natural transformation of formally cohesive functors $\mathcal{F} \to \mathcal{G}$ is an equivalence if and only if the induced map $\mathfrak{t}_\mathcal{F} \to \mathfrak{t}_\mathcal{G}$ is a quasi-isomorphism.

One can in fact equip the tangent complex $\mathfrak{t}_\mathcal{F}$ with additional Lie algebraic structure; we refer to cf. [1, Section 6] for a discussion of the resulting theory.

References

[1] Brantner, Lukas, and Akhil Mathew. *Deformation theory and partition Lie algebras.* arXiv preprint arXiv:1904.07352 (2019).
[2] Lurie, Jacob, *Higher topos theory.* Princeton University Press, 2009.
[3] Lurie, Jacob, *Higher Algebra*, Preprint. Available from the author's website.
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C4. Representation functors

Tony Feng

This talk defined the *derived representation functors* (without crystalline condition at p), which are (pro)-represented by the derived Galois deformation rings.

Overview

The classical deformation functor. We begin by recalling the form of the classical theory. The classical Galois deformation ring is constructed along the following lines. Let $k = \mathbf{F}_q$, Γ the relevant Galois group, and fix a "residual representation" $\overline{\rho}$, meaning a homomorphism $\Gamma \to G(k)$ up to conjugacy by $G(k)$.

Then one contemplates the functor $\mathrm{F}_{\Gamma}^{\overline{\rho}}$ on the category of Artinian algebras A augmented over k, sending A to the set of deformations of ρ into representations $G(A)$. This means equivalence classes of lifts

$$\begin{tikzcd} & G(A) \arrow[d] \\ \Gamma \arrow[ur, dashed, "\rho"] \arrow[r, "\overline{\rho}"'] & G(k) \end{tikzcd}$$

If G is adjoint and $\overline{\rho}$ is irreducible, then $\mathrm{F}_{\Gamma}^{\overline{\rho}}$ is pro-representable by Schlessinger's criterion, and the corresponding deformation ring is denoted $\mathrm{R}_{\Gamma}^{\overline{\rho}}$.

There is also a *framed* deformation functor $\mathrm{F}_{\Gamma}^{\overline{\rho},\square}$ where one does not quotient by conjugation, which is representable in greater generality.

The derived deformation functor. Now we want to make a derived version of the preceding construction. Very informally speaking, it should be a functor $\mathcal{F}_{\Gamma}^{\overline{\rho}}$ on Artinian simplicial commutative rings augmented over k, sending such an $A_{\bullet}$ to the "simplicial set of lifts"

$$\begin{tikzcd} & G(A_{\bullet}) \arrow[d] \\ \Gamma \arrow[ur, dashed, "\rho"] \arrow[r, "\overline{\rho}"'] & G(k) \end{tikzcd}$$

up to conjugacy.

Making this precise is a significant task. Let us point out some ways in which the interpretation needs to be careful. The basic principle is that all our constructions should be homotopy invariant; colloquially speaking, they need to be "derived". For example:

(1) To compute $G(A_{\bullet})$, we need to "derive" the formula $G(A) = \mathrm{Hom}(\mathcal{O}_G, A)$.
(2) The homomorphism space $\mathrm{Hom}(\Gamma, G(A_{\bullet}))$ need to be computed in a "derived" sense.
(3) To compute lifts, interpreted as the fiber of a map between spaces, we need to take a "homotopy fiber".

We will in fact take a slightly different approach, which combines steps (1) and (2). However, an alternative approach along the lines of the above is taken in [Zhu].

Desiderata. Before giving an "official" construction of $\mathcal{F}_{\Gamma}^{\overline{\rho}}$, let us enumerate its fundamental properties.

(1) (Homotopy invariance) Given a morphism of augmented Artinian local simplicial commutative rings

$$\begin{tikzcd} A_{\bullet} \arrow[r, "\sim"] \arrow[d, "\epsilon"] & A'_{\bullet} \arrow[d, "\epsilon"] \\ k \arrow[r, equal] & k \end{tikzcd}$$

which is a homotopy equivalence, then

$$\mathcal{F}_\Gamma^{\overline{\rho}}(f)\colon \mathcal{F}_\Gamma^{\overline{\rho}}(A_\bullet) \to \mathcal{F}_\Gamma^{\overline{\rho}}(A'_\bullet)$$

should also be a homotopy equivalence.

(2) (Compatibility with the non-derived theory) Recall that we have an inclusion of the category of Artinian local rings A into the category of Artinian local simplicial commutative rings. We want the following diagram to commute:

$$\begin{tikzcd}
\left\{\begin{matrix}\text{augmented Artinian local} \\ \text{commutative rings } A\end{matrix}\right\} \arrow[r, "\mathrm{F}_{\Gamma,\overline{\rho}}"] \arrow[d, hook] & \text{Set} \\
\left\{\begin{matrix}\text{augmented Artinian local simplicial} \\ \text{commutative rings } A_\bullet\end{matrix}\right\} \arrow[r, "\mathcal{F}_{\Gamma,\overline{\rho}}"] & \text{sSet} \arrow[u, "\pi_0"']
\end{tikzcd}$$

(3) (Tangent complex) $\mathcal{F}_\Gamma^{\overline{\rho}}$ should have the "right" tangent complex, which is $C^{\bullet+1}(\Gamma, \overline{\rho}^*\mathfrak{g})$, the (shifted) Galois cochains with coefficients of the pullback of the adjoint representation of G via $\overline{\rho}$.

Representability. We will take $\Gamma = \pi_1(\mathbb{Z}[1/N])$ or $\pi_1(\mathbb{Q}_v)$ or $\pi_1(\mathbb{Z}_v)$. Under the assumptions that $\overline{\rho}$ is irreducible and G is adjoint, the functor $\mathcal{F}_\Gamma^{\overline{\rho}}$ is representable by the *derived Galois deformation ring* (with no local conditions) $\mathcal{R}_\Gamma^{\overline{\rho}}$. We note that a conjecture of Mazur implies that this ring should in fact be homotopy discrete. (On the other hand, imposing local conditions should create higher homotopy groups whenever the invariant q, also known as ℓ_0, is positive.)

When G is not adjoint, one can still construct a deformation ring by modifying the deformation problem appropriately to account for the center; see [GV, §5.4].

Let $\mathrm{R}_\Gamma^{\overline{\rho}}$ be the commutative ring pro-representing $\mathrm{F}_\Gamma^{\overline{\rho}}$. We note that desideratum (2) implies that $\pi_0(\mathcal{R}_\Gamma^{\overline{\rho}}) \cong \mathrm{R}_\Gamma^{\overline{\rho}}$, i.e. the classical deformation ring can be recovered as the connected components of the derived one.

Further remarks on the tangent complex. Assume $p > 2$. Then the Galois cohomology of local or global fields with irreducible coefficients is supported in degrees $1, 2$. This plus desideratum (3) immediately implies that $\mathcal{R}_\Gamma^{\overline{\rho}}$ is *quasi-smooth* (a.k.a. "derived LCI"), meaning that the tangent complex is supported in degrees ≤ 1.

For the purposes of [GV], the desideratum (3) provides much of the "quantitative control" on the derived deformation ring. For example, the following argument is frequently used: a map of functors of Artinian simplicial commutative rings which induces a quasi-isomorphism of tangent complexes, is a weak equivalence [GV, Proposition 4.3].

To illustrate this, we will prove:

Lemma 1. The following are equivalent:

- $\mathrm{R}_\Gamma^{\overline{\rho}}$ is LCI of dimension $h^1(\Gamma, \overline{\rho}^*\mathfrak{g}) - h^2(\Gamma, \overline{\rho}^*\mathfrak{g})$ over $\mathbb{Z}_p$.
- The canonical map $\mathcal{R}_\Gamma^{\overline{\rho}} \to \pi_0(\mathcal{R}_\Gamma^{\overline{\rho}}) \cong \mathrm{R}_\Gamma^{\overline{\rho}}$ is a homotopy equivalence.

Proof. We will use two general facts about (simplicial) commutative algebra.

(1) For any $R_\bullet \in \mathrm{sArt}_k$, the map $R_\bullet \to \pi_0(R_\bullet)$ induces an isomorphism on $\mathfrak{t}^0$, and an inclusion on $\mathfrak{t}^1$.

(2) For any complete local noetherian ring R over $\mathbb{Z}_p$, $\dim_k \mathfrak{t}^0(R)$ is the minimal number of generators over $\mathbb{Z}_p$ and $\dim_k \mathfrak{t}^1(R)$ is the minimal number of relations.

If $\mathcal{R}_\Gamma^{\overline{\rho}} \to \mathrm{R}_\Gamma^{\overline{\rho}}$ is a homotopy equivalence, then we immediately obtain that the latter has tangent complex concentrated in degrees $0, 1$, so it is LCI with dimension over $\mathbb{Z}_p$ equal to the Euler characteristic of its tangent complex by (2).

Conversely, suppose it is LCI with dimension equal to $h^1(\Gamma, \overline{\rho}^*\mathfrak{g}) - h^2(\Gamma, \overline{\rho}^*\mathfrak{g})$. It remains to check that the map $\mathfrak{t}^1(\mathrm{R}_\Gamma^{\overline{\rho}}) \to \mathfrak{t}^1(\mathcal{R}_\Gamma^{\overline{\rho}})$ is an isomorphism. We know it is an injection by (1); if that injection were strick then the dimension of $\mathrm{R}_\Gamma^{\overline{\rho}}$ would be too large by (2).

□

CONSTRUCTION OF THE DERIVED REPRESENTATION FUNCTOR

Classifying spaces. Let H be a discrete group. The *classifying space* BH is the geometric realization of the simplicial set $N_\bullet(H)$

$$* \leftleftarrows H \mathrel{\substack{\leftarrow \\ \leftarrow \\ \leftarrow}} H \times H \dots \tag{5}$$

with maps as in [M99, p.128].

Remark 2. Another way to describe the simplicial set underlying BH is as the nerve of H viewed as a category (consisting of a single object, with endomorphisms given by H).

Let X be a "nice" space (e.g. a simplicial set). Then homotopy classes of maps from X to BH are in bijection with "H-local systems" on X. If X is connected, then this is equivalent to the set of homomorphisms $\pi_1(X, x) \to H$ up to conjugacy.[2]

If H is not discrete, then we can still form its classifying space H as the geometric realization of (5). More precisely, suppose H is a simplicial monoid. Then (5) is a bi-simplicial set, and by its geometric realization we mean the geometric realization of the diagonal simplicial set.

Mapping spaces. The preceding considerations motivate us to consider modeling the space of representations of $\pi_1(X, x)$ into H as the space of maps $X \to BH$.

Remark 3. Let X be a simplicial set. We write $\mathrm{Map}(X, BH)$ for the space of continuous maps from X to BH. The homotopy type of $\mathrm{Map}(X, BH)$ can be studied via its homotopy groups. We have

$$\pi_0 \mathrm{Map}(X, BH) = \mathrm{Hom}(\pi_1(X, x), H)/\mathrm{conj}.$$

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[2] Homomorphisms, i.e. "framed representations", are parametrized by maps of *pointed* spaces.

Furthermore, we have

- $\pi_1(\mathrm{Map}(X, BH), \rho\colon \pi_1 \to H) \approx Z_G(\mathrm{Im}\,\rho)$ (the centralizer of $\mathrm{Im}\,\rho$ in G).
- $\pi_i(\mathrm{Map}(X, BH), \rho) = 0$ for $i \geq 2$.

Thus we have

$$\mathrm{Map}(X, BH) = \coprod_{\rho\colon \pi_1(X) \to H/\mathrm{conj}} BZ_G(\rho).$$

This ceases to be the case for non-discrete groups, e.g. $H = \mathrm{GL}_1(\mathbf{C})$, and we regard the LHS as the better object.

Derived classifying spaces. Now suppose that $A_\bullet$ is a simplicial commutative ring and G is a reductive group. We want to make sense of "$BG(A_\bullet)$".

First we might try to make sense of $G(A_\bullet)$. If $A_\bullet = A$ were a discrete ring, this would be $\mathrm{Hom}(\mathcal{O}_G, A)$, so we might try to set "$G(A_\bullet) = \mathrm{Hom}(\mathcal{O}_G, A_\bullet)$". However, this is a bad idea because the result won't be homotopy invariant. To make it homotopy invariant, we should apply cofibrant replacement to $\mathcal{O}_G$. (In principle we should also apply fibrant replacement to $A_\bullet$, but in the usual model structure *all* simplicial commutative rings are fibrant, so this is unnecessary.)

Fact 4. There exists a functorial cofibrant replacement $c(A_\bullet) \xrightarrow{\sim} A_\bullet$.

However, we still have the problem that $\mathrm{Hom}(\mathcal{O}_G, A_\bullet)$ is not a priori a simplicial group, because the functor c is not guaranteed to be monoidal, i.e. preserve tensor products.

In fact, it is more convenient to define a space $BG(A_\bullet)$ directly, rather than as the classifying space of a simplicial group. To do so, we use the cosimplicial commutative ring

$$W(k) \rightrightarrows c(\mathcal{O}_G) \overset{\rightarrow}{\rightrightarrows} c(\mathcal{O}_G \otimes \mathcal{O}_G) \ldots, \tag{6}$$

where the maps are dual to (5).

The maps from (6) to a simplicial commutative ring $A_\bullet$ assemble into a bisimplicial set $N_\bullet(A_\bullet)$, and we define $BG(A_\bullet)$ to be the diagonal simplicial set.

Derived representation functor. Suppose Γ is a *discrete* group. Then we define $\mathcal{F}_\Gamma$ to be the functor sending an augmented simplicial Artinian ring $A_\bullet \xrightarrow{\epsilon} k$ to the simplicial set $\mathrm{Map}(B\Gamma, BG(A_\bullet))$.

Derived deformation functor. Continue to assume that Γ is a discrete group. Let $\overline{\rho}\colon \Gamma \to G(k)$ be a residual representation. We may regard it as a 0-simplex of $\mathcal{F}_\Gamma(k)$.

Now we define the derived deformation functor $\mathcal{F}_\Gamma^{\overline{\rho}}$ sending an augmented simplicial Artinian ring $A_\bullet \xrightarrow{\epsilon} k$ to the homotopy fiber of $\mathcal{F}_\Gamma(A_\bullet) \xrightarrow{\epsilon} \mathcal{F}_\Gamma(k)$ over the 0-simplex $\overline{\rho}$ of $\mathcal{F}_\Gamma(k)$.

Informally, $\mathcal{F}_X^{\overline{\rho}}(A_\bullet)$ is the space of lifts

$$\left\{ \begin{tikzcd} & BG(A_\bullet) \arrow[d, "\epsilon"] \\ B\Gamma \arrow[ur, dashed, "\rho"] \arrow[r, "\overline{\rho}"'] & BG(k) \end{tikzcd} \right\}.$$

and the content of saying "homotopy fiber" instead of "fiber" is intuitively that instead of asking $\overline{\rho}$ and $\epsilon \circ \rho$ to agree, we ask for a homotopy between their images.

Comparison to the classical deformation functor. Recall that π_0 is left adjoint to the inclusion of commutative rings as simplicial commutative rings. In other words,

$$\mathrm{Hom}_{\mathsf{SCR}}(\mathcal{R}, \mathrm{S}) = \mathrm{Hom}_{\mathsf{CR}}(\pi_0(\mathcal{R}), \mathrm{S}).$$

Hence if S is a classical ring, then $\pi_0(\mathcal{R})$ represents the classical deformation functor.

Derived Galois deformation functor. When studying representations of Galois groups, there is an additional wrinkle: the Galois group comes with a natural (profinite) topology, and we must account for that. So we write $\Gamma = \varprojlim \Gamma_\alpha$ where each Γ_α is finite.

When defining the deformation functor of a residual representation $\overline{\rho}$, we may (because $G(k)$ is finite) restrict ourselves to Γ_α over which $\overline{\rho}$ factors. The deformation functor is then defined as an ind-object,

$$\mathcal{F}_\Gamma^{\overline{\rho}} := \varinjlim \mathcal{F}_{\Gamma_\alpha}^{\overline{\rho}}.$$

If $\mathcal{F}_{\Gamma_\alpha}^{\overline{\rho}}$ is pro-represented by $\mathcal{R}_{\Gamma_\alpha}^{\overline{\rho}}$, then $\mathcal{F}_\Gamma^{\overline{\rho}}$ will be pro-represented by the pro-system $\{\mathcal{R}_{\Gamma_\alpha}^{\overline{\rho}}\}_\alpha$.

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D2. Fontaine–Laffaille theory

TOBY GEE

CRYSTALLINE REPRESENTATIONS

Let $K/\mathbf{Q}_p$ be finite, let $G/\mathbf{Q}_l$ be reductive, and consider a representation $\rho : \mathrm{Gal}_K \to G(\overline{\mathbf{Q}}_l)$. To say that ρ is "crystalline" is the analogue for $l = p$ of saying that it is "unramified" for $l \neq p$. We will take $l = p$ from now on; the actual definition of a crystalline representation is technical, and we will concentrate on examples, and the following properties:

- Crystalline representations form a Tannakian category.
- The p-adic étale cohomology groups of smooth projective varieties over $\mathcal{O}_K$ are crystalline.
- The p-adic Galois representations associated to automorphic representations unramified at p are (conjecturally) crystalline.
- e.g. for a newform f of level N with associated representation $\rho_f : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$, the restriction $\rho_f|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ is crystalline if and only if $p \nmid N$.

CRYSTALLINE CHARACTERS

Examples:

- unramified representations $\lambda_a : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{Gal}_{\mathbf{F}_p} \to \overline{\mathbf{Q}}_p^\times$, $\lambda_a(\mathrm{Frob}_p) = a$.
- $H^2_{\mathrm{et}}(\mathbf{P}^1)$ gives the cyclotomic character

$$\varepsilon : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p) \to \mathbf{Z}_p^\times,$$

  where $g(\zeta_{p^\infty}) = \zeta_{p^\infty}^{\varepsilon(g)}$.

The crystalline characters of $\mathrm{Gal}_{\mathbf{Q}_p}$ are exactly the characters $\lambda_a \varepsilon^i$, $i \in \mathbf{Z}$. More generally, for any $K/\mathbf{Q}_p$, the crystalline characters are determined by the data of $a \in \overline{\mathbf{Q}}_p^\times$, $(h_\sigma) \in (\mathbf{Z})^{\mathrm{Hom}_{\mathbf{Q}_p}(K,\overline{\mathbf{Q}}_p)}$, the Hodge–Tate weights.

FONTAINE–LAFFAILLE THEORY

If $E/\mathbf{Q}_p$ is an elliptic curve with good reduction, the corresponding p-adic representation is crystalline with Hodge–Tate weights $0, -1$. If E has ordinary reduction then this representation is reducible and can be studied "by hand". If it has supersingular reduction, the corresponding p-adic representation is irreducible, and If $a_p = 0$ then this representation is the induction of a crystalline character of $\mathrm{Gal}_{\mathbf{Q}_{p^2}}$, while if $a_p \neq 0$ I don't know how to describe it explicitly.

Instead we can use Fontaine–Laffaille theory in Hodge–Tate weights $0, -1$. Assume $p > 2$, and let $\mathrm{MF}^1_{\mathrm{tor}}$ be the abelian category consisting of:

- A finite, torsion $\mathbf{Z}_p$-module M.
- A submodule $M^1 \subseteq M$.
- $\mathbf{Z}_p$-linear maps $\varphi : M \to M$, $\varphi^1 : M^1 \to M$, such that:
  - $\varphi|_{M^1} = p\varphi^1$.
  - $\varphi(M) + \varphi^1(M^1) = M$.

Theorem 1. (Fontaine–Laffaille) There is an exact, fully faithful functor from $\mathrm{MF}^1_{\mathrm{tor}}$ to the category of finite torsion $\mathbf{Z}_p$-modules with a continuous $\mathrm{Gal}_{\mathbf{Q}_p}$-action.

The essential image is the quotients T/T' of lattices in crystalline representations of Hodge–Tate weights $0, -1$.

We can use this theory to define a deformation condition as follows: if A is Artin local with residue field $\mathbf{F}_p$, we ask that $\rho_A : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GL}_2(A)$ lifts $overline\rho$ and is in the essential image of the Fontaine–Laffaille functor. It is a good exercise to compute this explicitly in the case of rank 2 modules over $\mathbf{F}_p$.

More generally, Fontaine–Laffaille theory works for $K/\mathbf{Q}_p$ unramified, with Hodge–Tate weights in any range of length at most $p-2$. In the definition above, you replace $\mathbf{Z}_p$-modules with $W(k)$-modules, make φ-semilinear, and have a filtration of length (at most) $p-2$. It is not hard to show for $G = \mathrm{GL}_n$ that the corresponding deformation ring is formally smooth of the expected dimension $\mathrm{dim} G/B$.

The derived enhancement. Let $\mathcal{X}$ be the derived unrestricted deformation space for $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbf{F}_p)$, and write $\mathcal{X}^{\mathrm{cl}}$ for the underlying classical space. Write $\mathcal{X}^{\mathrm{cl}}_{\mathrm{FL}} \subseteq \mathcal{X}^{\mathrm{cl}}$ for the Fontaine–Laffaille subspace.

Then we define the derived Fontaine–Laffaille subspace $\mathcal{X}_{FL} \subseteq \mathcal{X}$ by $\mathcal{X}_{\mathrm{FL}} = \mathcal{X}^{\mathrm{cl}}_{\mathrm{FL}}$.

This is reasonable because we expect that $\mathcal{X} = \mathcal{X}^{\mathrm{cl}}$ (i.e. that $\mathcal{X}^{\mathrm{cl}}$ is lci of the expected dimension), and we know that $\mathcal{X}_{\mathrm{FL}}$ is formally smooth (of the expected dimension). This means that $\mathcal{X}^{\mathrm{glob}}_{\mathrm{FL}} = \mathcal{X}^{\mathrm{glob}} \times_{\mathcal{X}} \mathcal{X}_{\mathrm{FL}}$ has the expected tangent complex, i.e. in particular giving the Selmer group H^1_f (defined via Fontaine–Laffaille theory).

In general, we can define crystalline deformation rings via flat closure of the generic fibre. They have the expected dimensions $\mathrm{dim} G/B$, but in general little else is known about them (and they are usually not formally smooth). It is unclear how to define derived enhancements (similarly, it is unclear how to define an integral version without taking a flat closure).

THE NUMERICAL COINCIDENCE

Now let $F/\mathbf{Q}$ be a number field, $\overline{\rho} : \mathrm{Gal}_F \to \hat{G}(\mathbf{F})$, G split/$\mathbf{Z}$ with dual $\hat{G}$. Fix regular Hodge–Tate weights at each $v|p$. What is the "expected dimension" (from the tangent complex) over $\mathbf{Z}_p$ of the deformation ring for representations which are regular of these Hodge–Tate weights?

To compute this, note that the unrestricted local and global stacks have expected dimensions/$\mathbf{Z}_p$ given by the (negative) Euler characteristics χ_{F_v} and χ_F of the adjoint representation $\hat{\mathfrak{g}}$.

By the Euler characteristic formulas, for $v|p$, we have

$$\chi_{F_v} = [F_v : \mathbf{Q}_p] \mathrm{dim} G,$$

for $v \nmid p$, we have $\chi_{F_v} = 0$, and globally we have

$$\chi_F = [F : \mathbf{Q}]\mathrm{dim}G - \sum_{v|\infty} \mathrm{dim}\hat{\mathfrak{g}}^{\mathrm{Gal}(\overline{F_v}/F_v)}.$$

We saw above that the local (regular weight) crystalline stacks at $v|p$ have dimension $[F_v : \mathbf{Q}_p]\mathrm{dim}(\hat{G}/\hat{B})$.

Putting this together, the expected dimension of the global crystalline stack is

$$[F : \mathbf{Q}]\mathrm{dim}(\hat{G}/\hat{B}) - \sum_{v|\infty} \mathrm{dim}\hat{\mathfrak{g}}^{\mathrm{Gal}(\overline{F_v}/F_v)}.$$

For each $v|\infty$, this is at most $-(\mathrm{rank}(G_v) - \mathrm{rank}(K_v))$, with equality if and only if c_v is odd (e.g. for each complex place we get a contribution of $2\mathrm{dim}(\hat{G}/\hat{B}) - \mathrm{dim}\hat{G} = -\mathrm{dim}\hat{T}$.)

To get the expected dimension for the deformation ring, we have to add back on $\mathrm{dim}Z(\hat{G})$. In general we find that the expected dimension is $-l_0$, where $l_0 = \mathrm{rank}(G_\infty) - \mathrm{rank}(A_\infty K_\infty)$. This is sometimes called "the numerical coincidence".

D3. Setup for the Taylor–Wiles patching argument

STEFAN PATRIKIS

INTRODUCTION

The aim of this talk is to survey the input, from both the Galois and automorphic sides, to the Taylor-Wiles method, as extended by Calegari-Geraghty ([CG18]) to the $\ell_0 > 0$ setting. We primarily draw on [KT17], [Tho], [GV18], and [ACC$^+$18].

The precise setting in which we will work is the following: a positive integer d and a number field F, of signature (r_1, r_2), give rise to the symmetric space

$$\begin{aligned} Y_\infty &= \mathrm{GL}_d(F \otimes_{\mathbb{Q}} \mathbb{R})/K_\infty^0 A_\infty^0 \\ &\cong (\mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{C})^{r_2}) \,/\, \big(\mathrm{SO}(d)^{r_1} \times \mathrm{U}(d)^{r_2} \cdot \mathbb{R}_{>0}^\times\big) \end{aligned}$$

for $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_d)$. We find that

$$\ell_0 = \mathrm{rk}(G_\infty) - \mathrm{rk}(A_\infty K_\infty) = r_1 \frac{d'}{2} + r_2 d - 1,$$

where $d' = d$ if d is even and $d' = d + 1$ if d is odd. For a compact open subgroup $K = \prod_v K_v \subset \mathrm{GL}_d(\mathbb{A}_F^\infty)$, we write

$$Y(K) = \mathrm{GL}_d(F)\backslash(\mathrm{GL}_d(\mathbb{A}_F^\infty) \times Y_\infty)/K$$

for the associated locally symmetric space. We assume $K_v \subset \mathrm{GL}_d(\mathcal{O}_{F_v})$ for all v. Fix a prime p, which we assume unramified in F, and let S be a finite set of primes of F containing all those dividing p. Assume that for all $v \notin S$ and for all $v \mid p$, $K_v = \mathrm{GL}_d(\mathcal{O}_{F_v})$. Further assume $p > d$. We first state a variant of the conjecture from talk D.1. We write $C_\bullet(Y, R)$ for the singular chains with coefficients in a ring R on a topological space Y. We primarily take $R = \mathcal{O}$ the ring of integers, with residue field k, in a finite extension of $\mathbb{Q}_p$. Consider the unramified Hecke algebra at level K generated by $T_v^{(i)}$, $i = 1, \ldots, d$, $v \notin S$,

$$\mathbb{T}^S(K) \subseteq \mathrm{End}_{D(\mathcal{O})}(C_\bullet(Y(K), \mathcal{O})).$$

Conjecture 1. Let $\mathfrak{m} \subset \mathbb{T}^S(K)$ be a maximal ideal. Then there exists a continuous semisimple Galois representation

$$\bar{\rho}_{\mathfrak{m}} \colon G_{F,S} \to \mathrm{GL}_d(\mathbb{T}^S(K)_{\mathfrak{m}}/\mathfrak{m})$$

such that

1. $\bar{\rho}_{\mathfrak{m}}$ is characterized by the fact that for all $v \notin S$, the characteristic polynomial of $\bar{\rho}_{\mathfrak{m}}(\mathrm{Fr}_v)$ equals $\sum_{i=0}^{d}(-1)^i q_v^{\frac{i(i-1)}{2}} T_v^{(i)} X^{d-i} =: P_v(X)$, for Fr_v an arithmetic Frobenius element at v.
2. $\bar{\rho}_{\mathfrak{m}}$ is "as odd as possible:" for all real $v \mid \infty$, the numbers of $+1$ and -1 eigenvalues of the image $\bar{\rho}_{\mathfrak{m}}(c_v)$ of complex conjugation differ from each other by at most 1.
3. For $p \gg_d 0$ and $v \mid p$, $\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ is Fontaine-Laffaille type and has all d-tuples of "labeled Hodge-Tate weights" equal to $\{0, -1, \ldots, -d+1\}$.
4. When $\mathfrak{m}$ is non-Eisenstein (i.e., $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible), there exists a lift $\rho_{\mathfrak{m}} \colon G_{F,S} \to \mathrm{GL}_d(\mathbb{T}^S(K)_{\mathfrak{m}})$ of $\bar{\rho}_{\mathfrak{m}}$ satisfying properties analogous to (1)-(3).

From now on we assume this conjecture and let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}^S(K)$. We abbreviate $\bar{\rho} = \bar{\rho}_{\mathfrak{m}}$. Let R_S be the universal deformation ring of $\bar{\rho}$ parametrizing deformations that are unramified outside S and of Fontaine-Laffaille type (with the same labeled H-T weights as $\bar{\rho}$) at all places above p (see talk D.2). Then Conjecture 1 implies the existence of a surjective local $\mathcal{O}$-algebra homomorphism $R_S \twoheadrightarrow \mathbb{T}^S(K)_{\mathfrak{m}}$. The goal of the Taylor-Wiles method is to prove this map is an isomorphism by studying analogous maps "$R_{S\cup Q} \to \mathbb{T}_{\mathfrak{m}_Q}$" obtained by allowing auxiliary ramification at Taylor-Wiles sets Q of primes.

THE GALOIS SIDE

Under some simplifying assumptions ($H^2(G_{F_v}, \mathrm{ad}(\bar{\rho})) = 0$ for all $v \in S \setminus \{v|p\}$), the Greenberg-Wiles formula implies that $h^1_S - h^1_{S^*} = -\ell_0$ where

$$h^1_S = \dim(\ker\left(H^1(G_{F,S}, \mathrm{ad}(\bar{\rho})) \to \prod_{v\in S} H^1(G_{F_v}, \mathrm{ad}(\bar{\rho}))/L_v\right)),$$

where L_v is the tangent space of the Fontaine-Laffaille deformation functor for $v \mid p$ and is the full H^1 otherwise, and where $h^1_{S^*}$ denotes the dimension of the corresponding dual Selmer group. (The outcome of this Euler-characteristic calculation relies on oddness of $\bar{\rho}$ and on our assumption of "distinct H-T weights" at p.)

Definition 2. A Taylor-Wiles datum is

1. A finite set Q of primes away from S such that for all $v \in Q$, $N(v) \equiv 1 \pmod{p}$; and such that for all $v \in Q$, $\bar{\rho}(\mathrm{Fr}_v)$ has distinct eigenvalues in k.

(2) For all $v \in Q$, a choice of ordering $\alpha_{v,1}, \dots, \alpha_{v,d}$ of the eigenvalues of $\bar{\rho}(\mathrm{Fr}_v)$.

An allowable Taylor-Wiles datum is a Taylor-Wiles datum where additionally

(1) $\#Q = h^1_{S^*}$.
(2) $h^1_{(S\cup Q)^*} = 0$.

For any allowable Taylor-Wiles datum Q, the Greenberg-Wiles formula implies

$$h^1_{S\cup Q} - h^1_{(S\cup Q)^*} = -\ell_0 + \sum_{v\in Q} h^1(G_{F_v}, \mathrm{ad}(\bar{\rho})) - h^0(G_{F_v}, \mathrm{ad}(\bar{\rho})) = -\ell_0 + \#Q \cdot d,$$

by the local Euler-characteristic formula, local duality, and the assumption (1) on Taylor-Wiles primes. Thus $h^1_{S\cup Q} = -\ell_0 + \#Q \cdot d$, a quantity that is independent of the allowable T-W datum. The Čebotarev density theorem implies:

Proposition 3. Assume $\bar{\rho}(G_{F(\zeta_p)})$ is sufficiently large (see e.g. [KT17, 4.7]) and that $\zeta_p \notin F$. Then for any $n \geq 1$, there exist infinitely many allowable Taylor-Wiles data consisting of primes v with $N(v) \equiv 1 \pmod{p^n}$.

Talk B.3 explained this for $d = 1$. In general one also must contemplate the Galois extensions $F(\mathrm{ad}^0(\bar{\rho}), \zeta_{p^n}, \psi)/F$ for dual Selmer classes ψ for $\mathrm{ad}^0(\bar{\rho})$.

We move on to the structure of $R_{S\cup Q}$ for any T-W datum Q. The local deformation theory at T-W primes is particularly simple:

Lemma 4. Let $v \in Q$, and let $A \in \mathcal{C}_{\mathcal{O}}$. Any lift $\rho\colon G_{F_v} \to \mathrm{GL}_d(A)$ of $\bar{\rho}|_{G_{F_v}}$ is $\mathrm{GL}_d(A)$-conjugate to a diagonal representation $\oplus_{i=1}^d \chi_{v,i}$, where $\bar{\chi}_{v,i}(\mathrm{Fr}_v) = \alpha_{v,i}$. The characters $\chi_{v,i}$ thus obtained (by this ordering!) are unique.

One proves this by induction (on the length of an Artinian A) using the structure of tame inertia and the conditions $N(v) \equiv 1 \pmod{p}$, $\bar{\rho}(\mathrm{Fr}_v)$ regular semisimple. By the lemma, the restriction $\rho_{S\cup Q}|_{G_{F_v}}$ of the universal deformation (over $R_{S\cup Q}$) is $\mathrm{GL}_d(R_{S\cup q})$-conjugate to $\oplus_{i=1}^d \chi_{v,i}$ where $\bar{\chi}_{v,i}(\mathrm{Fr}_v) = \alpha_{v,i}$. Each $\chi_{v,i}$ factors

$$\begin{tikzcd}
G_{F_v} \arrow[rr, "\chi_{v,i}"] & & R^\times_{S\cup Q} \\
I_{F_v} \arrow[u] \arrow[r, two heads] & \mathcal{O}^\times_{F_v} \arrow[r, two heads] & k_v^\times(p), \arrow[u, dotted]
\end{tikzcd}$$

where $k_v^\times(p)$ is the maximal p-power quotient of the multiplicative group of the residue field k_v. Thus we obtain a homomorphism

$$\Delta_Q = \prod_{v\in Q} \prod_{i=1}^d k_v^\times(p) \xrightarrow{\prod_v \prod_i \chi_{v,i}} R^\times_{S\cup Q},$$

which induces an $\mathcal{O}$-algebra homomorphism $\mathcal{O}[\Delta_Q] \to R_{S\cup Q}$. Letting $\mathfrak{a}_Q$ be the augmentation ideal of $\mathcal{O}[\Delta_Q]$, the canonical map $R_{S\cup Q} \to R_S$ induces an isomorphism $R_{S\cup Q}/\mathfrak{a}_Q \xrightarrow{\sim} R_S$: the source is the quotient of $R_{S\cup Q}$ where Δ_Q acts trivially, while the target is the quotient of $R_{S\cup Q}$ where $\rho_{S\cup Q}$ becomes unramified at Q; but clearly from the construction of the Δ_Q-action these are identified.

The automorphic side

The constraint on the local deformation theory at T-W primes is mirrored on the automorphic side by a restriction (to "tamely ramified principal series") of the possible level-raising congruences at T-W primes. We write $\mathrm{Iw}_{v,1} \subset \mathrm{Iw}_v \subset \mathrm{GL}_d(\mathcal{O}_{F_v})$ for the maximal pro-v subgroup $\mathrm{Iw}_{v,1}$ inside the standard (for the upper-triangular Borel) Iwahori subgroup Iw_v. For a T-W datum Q, we consider a tower of locally symmetric spaces, abbreviating $K^Q = \prod_{v \notin Q} K_v$,

$$\begin{tikzcd}
Y(K^Q \times \prod_{v \in Q} \mathrm{Iw}_{v,1}) \arrow[r] \arrow[rr, bend left, "\prod_{v \in Q}(k_v^\times)^d"] & Y(K_1(Q)) \arrow[r, "\Delta_Q"] & Y(K_0(Q)) \arrow[r] & Y(K),
\end{tikzcd}$$

where: $K_0(Q) = K^Q \times \prod_{v \in Q} \mathrm{Iw}_v$, so $Y(K^Q \times \prod_{v \in Q} \mathrm{Iw}_{v,1}) \to Y(K_0(Q))$ is a Galois cover with group $\prod_{v \in Q}(k_v^\times)^d$, and $Y(K_1(Q))$ is by definition the intermediate Galois cover with group Δ_Q, the maximal p-power order quotient of $\prod_{v \in Q}(k_v^\times)^d$. We consider the following diagram of Hecke algebras, acting on singular chains, and maximal ideals:

$$\begin{tikzcd}
 & C_\bullet(Y(K), \mathcal{O}) \arrow[d, phantom, "\circlearrowleft"] \arrow[rr, dashed, bend left, "\frac{\mathrm{tr}}{[K:K_0(Q)]}"] & & C_\bullet(Y(K_0(Q)), \mathcal{O}) \arrow[ll] \arrow[d, phantom, "\circlearrowleft"] \\
\underset{\mathfrak{m}}{\mathbb{T}^S(K)} & \underset{\mathfrak{m}^Q}{\mathbb{T}^{S \cup Q}(K)} \arrow[l, hook'] & \underset{\mathfrak{m}_0^Q}{\mathbb{T}^{S \cup Q}(K_0(Q))} \arrow[l, two heads] \arrow[r, hook] & \underset{\mathfrak{n}_{0,\alpha}}{\mathbb{T}^{S \cup Q, Q\text{-aug}}(K_0(Q))} \\
 & & \underset{\mathfrak{m}_1^Q}{\mathbb{T}^{S \cup Q}(K_0(Q)/K_1(Q))} \arrow[u, two heads] \arrow[r, hook] & \underset{\mathfrak{n}_{1,\alpha}}{\mathbb{T}^{S \cup Q, Q\text{-aug}}(K_0(Q)/K_1(Q))} \arrow[u, two heads]
\end{tikzcd}$$

Explanation: The first two Hecke algebras in the middle row act on $C_\bullet(Y(K), \mathcal{O})$ and the next two act on $C_\bullet(Y(K_0(Q)), \mathcal{O})$. The "restriction" map $\mathbb{T}^{S \cup Q}(K_0(Q)) \to \mathbb{T}^{S \cup Q}(K)$ is induced by the splitting (the dashed arrow) of the canonical map $C_\bullet(Y(K_0(Q)), \mathcal{O}) \to C_\bullet(Y(K), \mathcal{O})$ (use that $p > d$ and $N(v) \equiv 1 \pmod{p}$ for $v \in Q$ implies $([K : K_0(Q)], p) = 1$). In the bottom row, the two Hecke algebras act on $C_\bullet(Y(K_1(Q)), \mathcal{O})$. The superscripts "$Q$-aug" indicate that we include the ramified Hecke operators $U_{\varpi_v}^{(i)}$ ($i = 1, \dots, d$, $v \in Q$) associated to a fixed uniformizer ϖ_v; and the "$K_0(Q)/K_1(Q)$" indicates that we take the $\mathcal{O}[\Delta_Q]$-algebra (Δ_Q acting via diamond operators/deck transformations) generated by the indicated Hecke operators. The vertical arrows are induced by the canonical *isomorphism* of complexes $C_\bullet(Y(K_1(Q)), \mathcal{O}) \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \xrightarrow{\sim} C_\bullet(Y(K_0(Q)), \mathcal{O})$ (use that $Y(K_1(Q)) \to Y(K_0(Q))$ is a Galois cover with group Δ_Q). Below each Hecke algebra is a maximal ideal: $\mathfrak{m}^Q$, $\mathfrak{m}_0^Q$, and $\mathfrak{m}_1^Q$ are induced from $\mathfrak{m}$ by pullback.

Less clear, but verified by an "oldform" calculation with the Iwahori-Hecke algebra over k, is that the ideal $\mathfrak{n}_{0,\alpha} \subset \mathbb{T}^{S\cup Q, Q\text{-aug}}(K_0(Q))$ generated by $\mathfrak{m}_0^Q$ and the elements $U_{\varpi_v}^{(i)} - \alpha_{v,1}\cdots\alpha_{v,i}$ $(i = 1,\dots,d,\ v \in Q)$ is maximal. The ideal $\mathfrak{n}_{1,\alpha} \subset \mathbb{T}^{S\cup Q, Q\text{-aug}}(K_0(Q)/K_1(Q))$ similarly constructed starting from $\mathfrak{m}_1^Q$ is also maximal.

The existence of the Borel-Serre compactification implies that each Hecke algebra is a finite $\mathcal{O}$-algebra, and more precisely that $C_\bullet(Y(K),\mathcal{O})$ (respectively, $C_\bullet(Y(K_1(Q)),\mathcal{O})$) is a perfect complex of $\mathcal{O}$-modules (respectively, of $\mathcal{O}[\Delta_Q]$-modules). We obtain an idempotent $e_\mathfrak{m} \in \mathbb{T}^S(K) \subset \mathrm{End}_{D(\mathcal{O})}(C_\bullet(Y(K),\mathcal{O}))$ associated to the maximal ideal $\mathfrak{m}$, and since $D(\mathcal{O})$ is idempotent complete, we obtain a canonical direct summand (in $D(\mathcal{O})$) $C_\bullet(Y(K))_\mathfrak{m}$ of $C_\bullet(Y(K),\mathcal{O})$. We similarly obtain direct summands $C_\bullet(Y(K_0(Q)))_{\mathfrak{n}_{0,\alpha}}$ and $C_\bullet(Y(K_1(Q)))_{\mathfrak{n}_{1,\alpha}}$. We now state the crucial analysis of level-raising congruences, which depends essentially on the use of Taylor-Wiles primes. This proposition makes precise the assertion that for our non-Eisenstein $\mathfrak{m}$ and T-W primes, only level-raising congruences, quantified here, to tamely ramified principal series can arise. (See [KT17, §6].)

Proposition 5. The canonical maps, equivariant for all $T_v^{(i)}$, $v \notin S \cup Q$,

$$C_\bullet(Y(K_1(Q)))_{\mathfrak{n}_{1,\alpha}} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \xrightarrow{\text{iso}} C_\bullet(Y(K_0(Q)))_{\mathfrak{n}_{0,\alpha}} \xrightarrow{\text{q-iso}} C_\bullet(Y(K))_\mathfrak{m}$$

are, as labeled, an isomorphism and quasi-isomorphism of complexes.

We now refine Conjecture 1 to include a local-global compatibility statement at primes in Q; see [ACC+18, Proposition 6.5.11] for an unconditional result.

Conjecture 6. There exists an $\mathcal{O}[\Delta_Q]$-algebra homomorphism

$$R_{S\cup Q} \to \mathbb{T}^{S\cup Q, Q\text{-aug}}(K_0(Q)/K_1(Q))_{\mathfrak{n}_{1,\alpha}}$$

mapping the characteristic polynomial of $\rho_{S\cup Q}(\mathrm{Fr}_v)$ to $P_v(X)$ for all $v \notin S \cup Q$.

From Proposition 5 and Conjecture 6, we deduce the commutative diagram

$$\begin{tikzcd}
R_{S\cup Q} \arrow[r] \arrow[d, "\mathrm{mod}\ \mathfrak{a}_Q"] & \mathrm{End}_{D(\mathcal{O}[\Delta_Q])}(C_\bullet(Y(K_1(Q)))_{\mathfrak{n}_{1,\alpha}}) \arrow[d, "\otimes^{\mathbb{L}}_{\mathcal{O}[\Delta_Q]}\mathcal{O}"] \\
R_S \arrow[r] & \mathrm{End}_{D(\mathcal{O})}(C_\bullet(Y(K))_\mathfrak{m}),
\end{tikzcd}$$

which will in the patching argument allow us to deduce that $R_S \to \mathbb{T}^S(K)_\mathfrak{m}$ is an isomorphism from the corresponding statement in the patching limit.

Minimal complexes for patching: So far we have worked with the (huge) singular chain complexes of our manifolds. Since they are perfect, we can replace them with the *minimal* complexes that are small enough to be used in the patching argument. Thus we fix minimal resolutions $C_Q \xrightarrow{\text{q-iso}} C_\bullet(Y(K_1(Q)))_{\mathfrak{n}_{1,\alpha}}$ (of complexes of $\mathcal{O}[\Delta_Q]$-modules) and $C_0 \xrightarrow{\text{q-iso}} C_\bullet(Y(K))_\mathfrak{m}$ (of complexes of $\mathcal{O}$-modules). Since C_Q and C_0 are minimal resolutions, they are supported in the same degrees as $H_\bullet(Y(K_1(Q)),k)_{\mathfrak{n}_{1,\alpha}}$ and $H_\bullet(Y(K),k)_\mathfrak{m}$. A fundamental **conjecture**, assumed in talk D.4, is that since $\mathfrak{m}$ is non-Eisenstein, these homology groups are non-zero only in degrees $[q_0, q_0 + \ell_0]$. The paper [ACC$^+$18] proves an automorphy lifting theorem without this conjecture, under the stronger assumption that $\bar{\rho}$ arises from an automorphic representation (not just a mod p homology class).

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D4. Main theorems and derived patching

CARL WANG-ERICKSON

In this concluding talk of the Arbeitsgemeinschaft, we state the main theorem of Galatius–Venkatesh [2, Thm. 14.1], discuss how its statement is more canonical than it might first appear according to an "independence theorem," and summarize its proof. This main theorem draws a graded isomorphism between the ring of homotopy groups $\pi_* \mathcal{R}$ of the derived Galois deformation ring $\mathcal{R}$ and the Tor-algebra $\mathrm{Tor}_*^{\mathrm{S}_\infty}(\mathrm{R}_\infty, \mathbb{Z}_p)$ arising from a patching technique in the obstructed Taylor–Wiles method of Calegari–Geraghty [1].

This "Taylor–Wiles patching" was previously discussed in Lectures B.3 and B.4, and involves adding sets of Taylor–Wiles primes Q_n to the level, resulting in classical deformation rings R_n parameterizing Galois representations of that level. The group algebra of diamond operators S_n, with augmentation map $\mathrm{S}_n \to \mathbb{Z}_p$, maps to R_n such that the intersection $\mathrm{R}_n \otimes_{\mathrm{S}_n} \mathbb{Z}_p$ recovers the base-level deformation ring R. By using the finiteness of these deformation rings modulo powers of p, patching produces a limit over n of these rings denoted $\mathrm{S}_\infty \to \mathrm{R}_\infty$, resulting in $\mathrm{R} \cong \mathrm{R}_\infty \otimes_{\mathrm{S}_\infty} \mathbb{Z}_p$. The ℓ_0-invariant appears in this situation: R_∞ and S_∞ are power series rings over $\mathbb{Z}_p$ with $\ell_0 = \dim \mathrm{S}_\infty - \dim \mathrm{R}_\infty$. This "resolution" of R as an intersection of smooth loci makes it possible to effectively compare R to Hecke algebras, which is one of the main upshots of [1].

On the other hand, the derived Galois deformation ring $\mathcal{R}$ enriches R in the sense that $\pi_0 \mathcal{R} \cong \mathrm{R}$, as discussed in lectures culminating in Lecture D.2. The additional information of $\pi_* \mathcal{R}$ reflects an overdetermined (or "non-transverse") intersection, in the moduli of local Galois representations, between the local restrictions of global Galois representations and the local motivically unramified sublocus. More specifically, the ℓ_0-invariant measures the degree of overdetermination, in that it equals the Euler characteristic $\dim \mathfrak{t}^0 - \dim \mathfrak{t}^1$ of the tangent complex of $\mathcal{R}$. The passage $\mathcal{R} \to \pi_0\mathcal{R} \cong \mathrm{R}$ loses information about the overdetermination when $\ell_0 > 0$; indeed, in this case, the obstructions represented by $\mathfrak{t}^1$ cannot be realized fully in a presentation for R (in the sense discussed in Lecture A.1).

The isomorphism of the main theorem $\pi_*\mathcal{R} \xrightarrow{\sim} \mathrm{Tor}_*^{\mathrm{S}_\infty}(\mathrm{R}_\infty, \mathbb{Z}_p)$ reflects that Taylor–Wiles patching realizes the overdetermined intersection in the context of classical deformation rings, and that the multitude of choices involved in patching reflects arithmetic meaning that is native to the base level. In particular, the action of $\mathrm{Tor}_*^{\mathrm{S}_\infty}(\mathrm{R}_\infty, \mathbb{Z}_p)$ on the homology H_* of the appropriate arithmetic manifold of base level, explained by Calegari–Geraghty [1], is given an interpretation native to base level. Namely, we have an action $\pi_*\mathcal{R} \curvearrowright H_*$.

Nonetheless, the multitude of choices involved in drawing the isomorphism of the main theorem implies that this action $\pi_*\mathcal{R} \curvearrowright H_*$ might depend on the choices involved in Taylor–Wiles patching. The "independence theorem" of [2, §15] demonstrates that this action is independent of these choices. The proof involves topics outside of the scope of this Arbeitsgemeinschaft, and works in a simplified setting where $\mathfrak{t}^0$ is assumed to vanish (which we call a "no congruences" condition, since it implies that the Hecke algebra is isomorphic to $\mathbb{Z}_p$); however, it is expected to work more generally. The argument relies on Venkatesh's construction of a derived Hecke algebra acting on H_*, along with a relationship between $\mathfrak{t}^1$ and the derived Hecke algebra [3].

The remainder of this talk dealt with the proof of the main theorem, following [2, §§11-12]. The greatest novelty in the proof lies in the use of a *derived patching method*, replacing the original Taylor–Wiles patching. The goal is to show that a natural map $\mathcal{R} \to \mathcal{C}_n := (\mathrm{R}_n/p^n\mathrm{R}_\mathrm{n})\underline{\otimes}_{(\mathrm{S}_n/p^n\mathrm{S}_n)}\mathbb{Z}_p/p^n\mathbb{Z}_p$ (where "$\underline{\otimes}$" denotes a derived tensor product) can be patched together into the desired map $\mathcal{R} \to \mathrm{R}_\infty \otimes_{\mathrm{S}_\infty} \mathbb{Z}_p$. Because there are no natural maps between levels (such as $\mathrm{R}_n \to \mathrm{R}_{n-1}$), we mimic the finiteness argument of (non-derived) Taylor–Wiles patching to show that a sensible limit over $n \to +\infty$ can be constructed. The new aspect is that we must now track π_0 of simplicial sets of maps $\mathcal{R} \to \mathcal{C}_n$, which we write as $[\mathcal{R}, \mathcal{C}_n]$. Crucially, we show that these $[\mathcal{R}, \mathcal{C}_n]$ have finite cardinality. Then we show that the following steps finish off the proof, each of which uses a substantial amount of the theory of simplicial commutative rings developed in the lectures of series C.

1. Prove that the natural map $\mathcal{R} \to \mathcal{C}_n$ induces an isomorphism on $\mathfrak{t}^0$ and a surjection on $\mathfrak{t}^1$.
2. Use the maps $\mathrm{R}_\infty \to \mathrm{R}_\mathrm{n}$, $\mathrm{S}_\infty \to \mathrm{S}_n$ (for all $n \in \mathbb{Z}_{\geq 1}$), obtained from classical Taylor–Wiles patching, to produce an arbitrary inverse system $[\mathcal{R}, \mathcal{C}_n]$ over n, preserving the property of (1).
3. Applying the finiteness of $[\mathcal{R}, \mathcal{C}_n]$ and (2), its subsystem satisfying the property of (1) has a non-empty limit. We use this to put together a map from $\mathcal{R}$ to a homotopy limit of the $\mathcal{C}_n$.
4. The comparison of Euler characteristics of tangent complexes – both are $-\ell_0$ – along with the property of (1) yields that $\mathcal{R} \to \mathrm{holim}_n\mathcal{C}_n$ induces

a weak equivalence on tangent complexes, and therefore (by Lecture C.3) is a weak equivalence, yielding the desired isomorphism of graded rings of homotopy groups.

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