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Automorphic Forms, Geometry and Arithmetic (hybrid meeting)

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Abstract. The workshop on automorphic forms, geometry and arithmetic focused on important recent developments within the research area, in particular, on the different recent approaches towards the Langlands functoriality principle and the Langlands correspondence, on their *relative* analogues, and on the relations between those advances and more arithmetic questions.

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Introduction by the Organizers

This workshop will concentrate on several aspects of the theory of automorphic forms, with an emphasis on the different recent approaches towards the Langlands functoriality principle and the Langlands correspondence, on their *relative* analogues, and on the relations between those advances and more arithmetic questions.

The program initiated by Langlands in the 1960's and 70's envisions a remarkable correspondence between the infinite-dimensional representation theory of reductive groups and the arithmetic of local and global number fields. It provides a promising approach to difficult number theoretic questions by bringing to bear the technical tools of harmonic and functional analysis as well as those of algebraic geometry that are inherent in representation theory of reductive groups.

While the development of this program has been a massive undertaking over the intervening half century, we are at present entering a particularly significant phase in its history. On the one hand, important parts of the theory are now being completed, such as the stabilized Arthur-Selberg trace formula, the classification of automorphic representations of classical groups, description of local Arthur packets, etc. So it is an ideal moment to survey what has been achieved. On the other hand, several exciting new directions are rapidly opening up, for example, new methods for attacking the problem of general functoriality ("beyond endoscopy"), a relative version of Langlands correspondence for spherical varieties, dramatic new results in the function field case, and tantalizing possibilities arising from Scholze's revolutionary ideas in p-adic geometry.

This workshop will focus on the following topics where important recent developments suggest that new progress is now possible.

Workshop (hybrid meeting): Automorphic Forms, Geometry and Arithmetic

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Abstracts

Construction of local A-packets

Hiraku Atobe

Fix a p-adic field F with the Weil group W_F. Let $G = \mathrm{Sp}_{2n}(F)$ or $G = \mathrm{SO}_{2n+1}(F)$ be split. We denote by $\widehat{G} = \mathrm{SO}_{2n+1}(\mathbb{C})$ or $\widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$ the Langlands dual group of G. A *local A-parameter* for G is a homomorphism

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$$

such that $\psi(W_F)$ is bounded and consists of semisimple elements.

Let $\mathrm{Irr}_{(\mathrm{unit})}(G)$ be the set of equivalence classes of irreducible smooth (unitary) representations of G. For any A-parameter ψ for G, Arthur [1, Theorem 2.2.1] defined a finite multi-set Π_ψ over $\mathrm{Irr}_{\mathrm{unit}}(G)$, which called the *(local) A-packet* associated with ψ. Arthur's multiplicity formula [1, Theorem 1.5.2] (together with the generalized Ramanujan conjecture) says that the local A-packets classify the local components of discrete spectrum of square-integrable automorphic forms.

In [2], as a refinement of Mœglin's construction (see [5]), we construct Π_ψ explicitly. To do this, we consider the following filtration of A-parameters:

$$\begin{aligned}(\text{discrete } L\text{-parameters}) &\subset (\text{non-negative DDR})\\ &\subset (\text{of good parity})\\ &\subset (\text{general}).\end{aligned}$$

Here, we say that ψ is *of good parity* if it is a sum of irreducible self-dual representations of the same type as ψ.

To state our main results, we recall some basic notations. A *segment* is a set $[x, y]_\rho = \{\rho|\cdot|^x, \rho|\cdot|^{x-1}, \dots, \rho|\cdot|^y\}$ with an irreducible unitary cuspidal representation ρ of some $\mathrm{GL}_d(F)$, and $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}_{\geq 0}$. It gives an essentially discrete series representation

$$\Delta_\rho[x, y] = \mathrm{soc}(\rho|\cdot|^x \times \cdots \times \rho|\cdot|^y)$$

of $\mathrm{GL}_{d(x-y+1)}(F)$, which is called the *Steinberg representation*.

The Langlands classification says that any irreducible representation π of G is a unique irreducible subrepresentation of $\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi_0$, where

- ρ_i is an irreducible unitary cuspidal representation of some $\mathrm{GL}_{d_i}(F)$;
- $x_1 + y_1 \leq \cdots \leq x_r + y_r < 0$;
- π_0 is an irreducible tempered representation of a classical group G_0 of the same type as G.

In this case, we write

$$\pi = L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]; \pi_0),$$

and call $(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]; \pi_0)$ the *Langlands data* for π.

We define our parameters.

Definition 1.1.

(1) *An* extended segment *is a triple* $([A,B]_\rho, l, \eta)$, *where*
- $[A,B]_\rho = \{\rho|\cdot|^A, \dots, \rho|\cdot|^B\}$ *is a segment;*
- $l \in \mathbb{Z}$ *with* $0 \le l \le \frac{b}{2}$, *where* $b \coloneqq \#[A,B]_\rho = A - B + 1$;
- $\eta \in \{\pm 1\}$.

(2) *An* extended multi-segment *for G is an equivalence class of multi-sets of extended segments*

$$\mathcal{E} = \bigcup_\rho \{([A_i,B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$$

such that
- *ρ runs over the set of equivalence classes of irreducible self-dual cuspidal representations of several* $\mathrm{GL}_d(F)$;
- *I_ρ is a totally ordered finite set with a fixed order $>$ which is called admissible;*
- $A_i + B_i \ge 0$ *for all ρ and $i \in I_\rho$;*
- *as a representation of* $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$,

$$\psi_\mathcal{E} \coloneqq \bigoplus_\rho \bigoplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}$$

is an A-parameter for G of good parity, where $a_i \coloneqq A_i + B_i + 1$ *and* $b_i \coloneqq A_i - B_i + 1$;
- *a sign condition*

$$\prod_\rho \prod_{i \in I_\rho} (-1)^{[\frac{b_i}{2}] + l_i} \eta_i^{b_i} = 1$$

holds.

(3) *Two extended segments* $([A,B]_\rho, l, \eta)$ *and* $([A',B']_{\rho'}, l', \eta')$ *are* equivalent *if*
- $[A,B]_\rho = [A',B']_{\rho'}$;
- $l = l'$; *and*
- $\eta = \eta'$ *whenever* $l = l' < \frac{b}{2}$.

Similarly,
$\mathcal{E} = \cup_\rho \{([A_i,B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ *and* $\mathcal{E}' = \cup_\rho \{([A'_i,B'_i]_\rho, l'_i, \eta'_i)\}_{i \in (I_\rho, >)}$ *are* equivalent *if* $([A_i,B_i]_\rho, l_i, \eta_i)$ *and* $([A'_i,B'_i]_\rho, l'_i, \eta'_i)$ *are equivalent for all ρ and $i \in I_\rho$.*

For an extended multi-segment $\mathcal{E} = \cup_\rho \{([A_i,B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ for G, we can define a representation $\pi(\mathcal{E})$ of G. It is irreducible or zero. Moreover, if $\pi(\mathcal{E}) \ne 0$, one can compute the Langlands data for $\pi(\mathcal{E})$.

The first main theorem, which is a refinement of Mœglin's construction of A-packets of good parity, is as follows:

Theorem 1.2 ([2, Theorem 1.2]). *For an A-parameter* $\psi = \oplus_\rho(\oplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i})$ *for G of good parity, fixing a "very admissible" order $>$ on I_ρ, we have*

$$\Pi_\psi = \left\{ \pi(\mathcal{E}) \;\middle|\; \mathrm{supp}(\mathcal{E}) = \cup_\rho \{[A_i,B_i]_\rho\}_{i \in (I_\rho, >)} \right\} \setminus \{0\},$$

where $A_i = (a_i + b_i)/2 - 1$, $B_i = (a_i - b_i)/2$, *and* $\mathrm{supp}(\mathcal{E}) = \cup_\rho \{[A_i, B_i]_\rho\}_{i \in (I_\rho, >)}$ *if* $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$.

By refining Xu's algorithm [6], we obtain the following.

Theorem 1.3 ([2, Theorem 1.4])**.** *We have a combinatorial criterion for* $\pi(\mathcal{E}) \neq 0$. *In particular, we can compute the cardinality* $|\Pi_\psi|$ *for any A-parameter* ψ *of good parity.*

Aubert [4] defined an involution $\pi \mapsto \hat{\pi}$ on $\mathrm{Irr}(G)$. We call $\hat{\pi}$ the *Aubert dual* of π. It is known by Xu [5] that for $\psi = \oplus_\rho(\oplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i})$ of good parity, we have

$$\{\hat{\pi} \mid \pi \in \Pi_\psi\} = \Pi_{\hat{\psi}},$$

where $\hat{\psi} = \oplus_\rho(\oplus_{i \in I_\rho} \rho \boxtimes S_{b_i} \boxtimes S_{a_i})$. In [3], the author and Mínguez established an algorithm to compute $\hat{\pi}$ for any $\pi \in \mathrm{Irr}(G)$, but it is hard to carry out. We give a more efficient formula for $\hat{\pi}$ when $\pi = \pi(\mathcal{E})$.

Theorem 1.4. *Let* $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ *be an extended multi-segment for* G. *Suppose that* $>$ *on* I_ρ *is very admissible, i.e.,* $B_i < B_j \implies i < j$. *Define* δ, α_i, β_i *and* $\hat{\mathcal{E}} = \cup_\rho \{([A_i, -B_i]_\rho, \hat{l}_i, \hat{\eta}_i)\}_{i \in (I_\rho, \hat{>})}$ *as follows.*

- $\delta \in \{0, 1/2\}$ *with* $B_i \equiv \delta \bmod \mathbb{Z}$;
- $\alpha_i = \sum_{j \in I_\rho, j < i}(A_j + B_j + 1)$ *and* $\beta_i = \sum_{j \in I_\rho, j > i}(A_j - B_j + 1)$;
- $i \hat{>} j \iff i < j$;
- $\hat{l}_i = l_i + B_i + \delta(-1)^{\alpha_i}\eta_i$ *and* $\hat{\eta}_i = (-1)^{\alpha_i + \beta_i + 2\delta}\eta_i$. *Here, if* $\delta = 1/2$ *and* $2l_i = A_i - B_i + 1$, *we regard* $\eta_i = -(-1)^{\alpha_i}$.

Then $\hat{\pi}(\mathcal{E}) \cong \pi(\hat{\mathcal{E}})$.

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On the formal degree conjecture for classical groups

Raphaël Beuzart-Plessis

Let F be a local field of characteristic zero and $\psi : F \to \mathbb{C}^\times$ be a continuous non-trivial unitary character. Let $\underline{H}$ be a connected semisimple group defined over F and set $H = \underline{H}(F)$. To each discrete series σ of H, we can associate its formal degree $d_H(\sigma) \in \mathbb{R}_{>0}$ characterized by the relation

$$\int_H \langle \sigma(h)u, u^\vee \rangle \langle v, \sigma^\vee(h)v^\vee \rangle d_\psi h = d_H(\sigma)^{-1} \langle u, v^\vee \rangle \langle v, u^\vee \rangle$$

for every $u, v \in \sigma$ and $u^\vee, v^\vee \in \sigma^\vee$ (the smooth contragredient of σ). Here the Haar measure $d_\psi h = |\omega_H|_\psi$ is given, through the choice of the ψ-delf-dual Haar measure in local coordinates and following a construction of Weil [19, §2.2], by an invariant volume form ω_H on $\underline{H}_{\overline{F}}$ (the base-change of $\underline{H}$ to an algebraic closure of F) that we choose to be the pull-back of an invariant everywhere nonzero form $\omega_\mathbb{Z}$ on the split form $\underline{H}_\mathbb{Z}$ of $\underline{H}$ over $\mathbb{Z}$ by an isomorphism

$$\iota : \underline{H}_{\overline{F}} \simeq \underline{H}_\mathbb{Z} \times \overline{F}.$$

(This actually only characterize ω_H up to some root of unity but the measure $d_\psi h$ doesn't see this ambiguity).

A conjecture due to Hiraga-Ichino-Ikeda (the so-called formal degree conjecture) [7] predicts a formula for $d_H(\sigma)$ in terms of data associated to σ by the local Langlands correspondence (LLC). For simplicity, we assume that $\underline{H}$ admits a pure inner which is quasi-split (this is always the case for the so-called classical groups: symplectic, unitary and special orthogonal groups). Then the LLC, in its most recent formulation due to Vogan [18], is supposed to attach to every admissible irreducible representation σ of H two kinds of invariants: first a L-parameter $\phi_\sigma : W_F' \to {}^L H$ (where W_F' stands for the Weil group or the Weil-Deligne group of the field F depending on whether it is Archimedean or not) and secondly an irreducible representation ρ_σ of the finite component group $S_\sigma := \pi_0(Cent_{\hat{H}}(\phi_\sigma))$. Actually, the representation ρ_σ is supposed to depend on the auxilliary choice of a Whittaker datum (of a quasi-split pure inner form of $\underline{H}$) but only up to a twist (see [4, §9]) and in particular its dimension $\dim(\rho_\sigma)$ ought to be completely canonical. Let Ad_H denote the adjoint representation of ${}^L H$ on $Lie(\hat{H})$. Then, we can associate to representation $Ad_H \circ \phi_\sigma$ of W_F' its γ-factor:

$$\gamma(s, \sigma, Ad_H) := \epsilon(s, Ad_H \circ \phi_\sigma, \psi) \frac{L(1-s, Ad_H \circ \phi_\sigma)}{L(s, Ad_H \circ \phi_\sigma)}.$$

One basic expectation of the local Langlands correspondence is that σ is a discrete series if and only if ϕ_σ is a *discrete* L-parameter that is to say the centralizer $Cent_{\hat{H}}(\phi_\sigma)$ is finite. This last condition implies that the γ-factor $\gamma(s, \sigma, Ad_H)$ is regular and nonzero at $s = 0$. We can now state the formal degree conjecture of Hiraga-Ichino-Ikeda [7]:

Conjecture 1.1 (Hiraga-Ichino-Ikeda)**.** *For every discrete series σ of H, we have*

$$d_H(\sigma) = \frac{\dim(\rho_\sigma)}{|S_\sigma|}|\gamma(0, \sigma, Ad_H)|.$$

As already explained above, it is expected that $\dim(\rho_\sigma)$ does not depend on the choice of a Whittaker datum needed to normalize the LLC. Moreover, if $\underline{H}$ is a classical group (in the sense given above), S_σ is abelian and therefore we always have $\dim(\rho_\sigma) = 1$.

The above conjecture is actually pleasantly aligned with another conjecture of Langlands on the normalization of standard intertwining operators [11, Appendix II] and the two can be combined to give a, conjectural, formula for the *Plancherel density* $\mu_H(\sigma)$ of H. Recall that this density is the unique (up to modification on a measure-zero subset) measurable function on the tempered dual $Temp(H)$ of H such that for every test function $f \in C_c^\infty(H)$ we have

$$f(1) = \int_{Temp(H)} \Theta_\sigma(f)\mu_H(\sigma)d\sigma. \tag{1}$$

Here Θ_σ stands for the distribution-character of σ and $d\sigma$ is some "elementary" measure on $Temp(H)$ coming, by twisting, from normalized Haar measures on the tori of unitary unramified characters of the Levi subgroups of H (this measure actually implicitely depends on the choice of ψ, see [2, §2.7] for details). When σ is a discrete series, we simply have $\mu_H(\sigma) = d_H(\sigma)$ whereas for a tempered representation σ in generic position, which is parabolically induced $\sigma = I_L^H(\tau)$ from a discrete series τ of a Levi L of H, $\mu_H(\sigma)$ can be written as the product of the formal degree of τ with a certain (scalar) composition of standard intertwining operators. Then, combining Langlands expected normalization of those [11, Appendix II] with Conjecture 1.1 (for all Levi subgroups of H), we arrive at the following prediction:

Conjecture 1.2 (Hiraga-Ichino-Ikeda, Langlands)**.** *For almost all $\sigma \in Temp(H)$, we have*

$$d_H(\sigma) = \frac{\dim(\rho_\sigma)}{|S_\sigma|}|\gamma^*(0, \sigma, Ad_H)|$$

where

$$\gamma^*(0, \sigma, Ad_H) = \lim_{s\to 0} \gamma(s, \mathbf{1}_F, \psi)^{-n_\sigma}\gamma(s, \sigma, Ad_H), \quad n_\sigma = ord_{s=0}\gamma(s, \sigma, Ad_H)$$

is the first non-zero term in the Laurent expansion of $\gamma(s, \sigma, Ad_H)$ at $s = 0$ (suitably normalized).

In the Archimedean case, both Conjectures 1.1 and 1.2 are known thanks to the work of Harish-Chandra [6] on discrete series and of Knapp and Stein [10] on intertwining operators. Although the LLC has not yet been established for all p-adic groups, there has been a lot of partial results in the direction of Conjecture 1.1 among which we can cite:

- The case of $H = PGL_n$ (or more generally GL_n) can be settled using works of Silberger and Zink [17]. Other proofs have been provided by

Hiraga-Ichino-Ikeda [7, §4] (using Shahidi's γ-factors) and Ichino-Lapid-Mao [8, §2] (using Rankin-Selberg γ-factors). Once combined with the work of Shahidi [15], this also gives a proof of Conjecture 1.2 in this case.

- For the Steinberg representation, Conjecture 1.1 has been verified by Hiraga, Ichino and Ikeda [7, §3.3].
- The case of odd special orthogonal groups is proven in [8] by Ichino, Lapid and Mao.
- In [2], I've established Conjecture 1.1 for unitary groups. Subsequently, Morimoto [12] gave another proof for unitary groups of even rank following the method of Ichino-Lapid-Mao.
- Using's Kaletha parameterization [9], David Schwein [14] was recently able to verify Conjecture 1.1 for all regular supercuspidal representations. This result was subsequently extended by Kazuma Ohara [13] to all the so-called non-singular supercuspidal representations.
- In [3], Feng, Opdam and Solleveld have established the formal degree conjecture for unipotent supercuspidal representations.

In a work in progress, I have obtained the following new cases of Conjectures 1.1 and 1.2.

Theorem 1.3. *Conjectures 1.1 and 1.2 hold whenever H is a symplectic or an even special orthogonal group.*

This theorem is based on the version of LLC for those groups due to Arthur [1]. We can remark that for even special orthogonal groups, the work of Arthur only gives a weak form of the correspondence where everything is determined up to the outer automorphism coming from conjugation by the corresponding full orthogonal group. However, it is easy to see that Conjecture 1.1 is insensible to this undeterminacy. Moreover, the work of Arthur [1] contains a proof of Langlands conjecture on the normalization of standard intertwining operators. Thus, for the groups considered we know a priori that Conjecture 1.1 implies Conjecture 1.2 but nevertheless both results are actually established at the same time. Finally, the basic idea of the proof goes back to an argument already developed in the original paper of Hiraga-Ichino-Ikeda [7, §7] in the context of stable discrete series of even unitary groups and uses in a new way the Goldberg-Shahidi's method [16], [5] of computing residues of certain intertwining operators.

Actually, the proof of Theorem 1.3 is based on the spectral expansion of certain (singular) orbital integrals on twisted general linear groups. More precisely, assume for simplicity that H is split from now on and let V be an N-dimensional vector space over F where

$$N = \begin{cases} 2n+1 & \text{if } H = Sp(2n), \\ 2n & \text{if } H = SO(2n). \end{cases}$$

Set $M = GL(V)$, $A = Z(M)$ (the center of M) and $\widetilde{M} = Isom(V, V^*)$ the space of linear isomorphisms of V with its dual. Then, $\widetilde{M}$ can be naturaly identified with the set of nondegenerate bilinear forms on V and moreover it is a twisted space under M (in the sense of Labesse; i.e. it is both a left and a right M-torsor). Let $\gamma \in \widetilde{M}$ be a nondegenerate bilinear form that can be written as a sum of a symplectic form of maximal rank and a quadratic form of rank one. Let $\widetilde{f} \in C_c^\infty(A\backslash\widetilde{M})$ and let

$$O(\gamma, \widetilde{f}) = \int_{AM_\gamma\backslash M} \widetilde{f}(m^{-1}\gamma m)dm$$

be the integral orbital over the conjugacy class of γ (where M_γ stands for the centralizer of γ).

Theorem 1.4. *For every $\widetilde{f} \in C_c^\infty(A\backslash\widetilde{M})$ we have the identity*

$$O(\gamma, \widetilde{f}) = \int_{Temp(H)/stab} \Theta_{\widetilde{\pi}}(\widetilde{f}) \times \frac{2}{|S_\sigma^+|}\gamma^*(0, \sigma, Ad_H)d\sigma$$

where:

- *$Temp(H)/stab$ denotes the quotient of $Temp(H)$ by the relation "being in the same L-packet" (the "elementary" measure $d\sigma$ descends to this quotient);*
- *π is the functorial lift of σ to M and $\widetilde{\pi}$ is the* Whittaker-normalized *extension of π to a representation of the twisted space $\widetilde{M}$;*
- *$\Theta_{\widetilde{\pi}}$ is the distribution-character of the twisted representation $\widetilde{\pi}$;*
- *$S_\sigma^+ = \pi_0(Cent_{O_N(\mathbb{C})}(\phi_\sigma))$ if the centralizer of the L-parameter of σ after composition with the natural embedding $\widehat{H} \hookrightarrow O_N(\mathbb{C})$.*

Finally, let me comment briefly on the proofs. First, we can deduce Theorem 1.4 from Theorem 1.3 using the twisted endoscopic character relations of [1, §2.2] and that characterize the L-packets for H. More precisely, these relations are based on the Langlands-Kottwitz-Shelstad transfer of functions $\widetilde{f} \in C_c^\infty(A\backslash\widetilde{M}) \to f \in C_c^\infty(H)$ defined through certain relations between (stable) orbital integrals. It can be shown that for such a pair of matching functions, we have

$$O(\gamma, \widetilde{f}) = f(\epsilon) \tag{2}$$

where $\epsilon = (-1)^N \in Z(H)$. On the other hand, the endoscopic relations of Arthur state that

$$\Theta_{\widetilde{\pi}}(\widetilde{f}) = \frac{|S_\sigma^+|}{2|S_\sigma|} \sum_{\sigma \to \widetilde{\pi}} \Theta_\sigma(f). \tag{3}$$

Combining (2) and (3) with the identity of Theorem 1.4, we obtain a spectral expansion for the Dirac distribution $f \in C_c^\infty(H) \mapsto f(\epsilon)$ which, when compared with the Plancherel formula (1), gives the desired formula for the Plancherel density (that of Conjecture 1.2).

As already mentioned, the proof of Theorem 1.4 is based on a method developed by Shahidi in [16] that relate the residues of certain intertwining operators to the twisted orbital integral $O(\gamma, \widetilde{f})$. To be more specific, set $W = V \oplus V^* \oplus F$ that we equip with the quadratic form $q(v, v^*, \lambda) = \langle v, v^*\rangle + \lambda^2$. Let $G = SO(W, q)$ be the corresponding (odd) special orthogonal group. Then, the stabilizers $P = Stab_G(V)$

and $\overline{P} = Stab_G(V^*)$ of V and V^* respectively in G are opposite maximal parabolic subgroups with common Levi factor $P \cap \overline{P} \simeq M$. Moreover, we also have an identification $\widetilde{M} \simeq Norm_G(M) \setminus M$ of the twisted space $\widetilde{M}$ with the non-neutral component of the normalizer of M in G. Let π be a supercuspidal representation of M which admits an extension $\widetilde{\pi}$ to $\widetilde{M}$. Then, in [16] Shahidi relates the residue at $s = 0$ of the standard intertwining operator

$$M(\pi, s) : I_P^G(\pi \otimes |\det|^{s/2}) \to I_{\overline{P}}^G(\pi \otimes |\det|^{s/2})$$

to the orbital integral $O(\gamma, .)$ applied to matrix coefficient of $\widetilde{\pi}$. To get Theorem 1.4, we imitate Shahidi's computation for the (highly non admissible) regular representation $\pi = C_c^\infty(A\backslash M)$. More precisely, this entails computing in two different ways the residue

$$Res_{s=0} \int_{\overline{U}} f_s(\overline{u}) d\overline{u} \tag{4}$$

where $\overline{U}$ is the unipotent radical of $\overline{P}$ and $s \mapsto f_s \in C_c^\infty(AU\backslash G, \delta_P^{1/2}|\det|^{s/2})$ is a "nice" holomorphic family of functions (e.g. one given by the Mellin transform $f_s(x) = \int_A f(ax)\delta_P(a)^{-1/2}|\det a|^{-s/2} da$ of a test function $f \in C_c^\infty(U\backslash G)$). On the one hand, mimicking Shahidi calculation in [16] we can write this residue as $O(\gamma, f^{\widetilde{M}})$ where $f^{\widetilde{M}} := f_0 \,|_{\widetilde{M}} \in C_c^\infty(A\backslash \widetilde{M})$. On the other hand, using Shahidi's normalization of intertwining operators for generic representations [15] as well as a long but explicit computation of certain residual distributions very close to that of [2, §3], we can also show that this residue is equal to the spectral side of Theorem 1.4 applied to the same function $f^{\widetilde{M}}$.

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Character sheaves on loop Lie algebras

Alexis Bouthier

1. Introduction

Let k be an algebraically closed field, G a connected reductive group over k, (B,T) a Borel pair and W the associated Weyl group. Usual Springer theory constructs $\mathrm{Ad}(G)$-équivariant perverse sheaves on $\mathrm{Lie}(G)$ as well as representations of W. Let

$$\pi : \tilde{\mathfrak{g}} = \{(g,\gamma) \in G/B \times \mathfrak{g}, \mathrm{ad}(g)^{-1}\gamma \in \mathrm{Lie}(B)\} \to \mathfrak{g},$$

be the Grothendieck-Springer fibration, which is small, projective and finite etale of group W above the regular semisimple locus $j : \mathfrak{g}^{rs} \hookrightarrow \mathfrak{g}$. We consider $\mathcal{S}_{fin} = \pi_*\overline{\mathbb{Q}}_\ell[\dim(\tilde{\mathfrak{g}})]$ and obtain a perverse sheaf which is the extension of its restriction to $\mathfrak{g}^{rs}$ and such that $\mathrm{End}(\mathcal{S}_{fin}) = \overline{\mathbb{Q}}_\ell[W]$. In particular, $\mathcal{S}_{fin}$ is equipped with an action of W. For each irreducible representation V of W, we obtain a V-isotypical component $\mathcal{S}_{fin,V}$ which is also perverse and $\mathrm{Ad}(G)$-equivariant. These sheaves are analogs for the Lie algebra of Lusztig's character sheaves. More generally, to obtain enough character sheaves, it is important to consider the generalizations $\mathcal{S}_{fin,\mathcal{L}} = \pi_*\mathcal{L}[\dim(\tilde{\mathfrak{g}})]$ where $\mathcal{L}$ is a local system.

In [2], Bouthier, Kazhdan and Varshasky extend this theory in the affine case, i.e. G is replaced by the loop group $G((t))$, $\mathfrak{g}$ by $\mathfrak{g}((t))$ and W by the extended affine Weyl group $\widetilde{W} = X_*(T) \rtimes W$, where $X_*(T)$ are the cocharacters of the maximal torus T. The Grothendieck-Springer fibration is replaced by its affine analog:

$$f : \widetilde{\mathfrak{C}} = \{(g,\gamma) \in G((t))/I \times \mathfrak{g}((t)), \mathrm{ad}(g)^{-1}\gamma \in \mathrm{Lie}(I)\} \to \mathfrak{g}((t))$$

where I is the Iwahori associated to B. As a matter of fact, the fibration factors through the sub-ind-scheme $\mathfrak{C}$ of compact elements.

In this context, all the object considered are ind-schemes, so a priori, smallness or perversity do not make sense. In addition to that, the reduced geometric fibers are k-schemes locally of finite type whose homology is infinite dimensional. Finally, as $\widetilde{W}$ is now infinite, the coinvariants functor is not exact and one has to consider derived coinvariants for which it is not clear whether we keep a perverse sheaf. In [2, Thm. 7.1.4], part of these difficulties are overcome. We have the following theorem, let denote :

$$[f] : [\widetilde{\mathfrak{C}}_\bullet/G((t))] \to [\mathfrak{C}_\bullet/G((t))],$$

the induced fibration on $G((t))$-equivariant objects.

Theorem 1.1. *We have the following assertions:*

(1) *In a reasonable sense, f is a small morphism.*
(2) *We have a well-defined theory of $G((t))$-equivariant sheaves on both stacks of the fibration, equipped with t-structures and cohomological functors, and the complex $\mathcal{S} = [f]_!\omega_{\widetilde{\mathfrak{C}}_\bullet}$, where $\omega_{\widetilde{\mathfrak{C}}_\bullet}$ is the dualizing sheaf, is perverse and obtained as the intermediate extension of its restriction to a regular semisimple locus $\mathfrak{C}_{rs}$, above which, up to nilpotents, the fibration is Galois of group $\widetilde{W}$.*
(3) *We have that* $\mathrm{End}(\mathcal{S}) = \overline{\mathbb{Q}}_\ell[\widetilde{W}]$.

Nevertheless, the second step which consists in considering the isotypical components is not studied as well as the necessary finiteness statements. Moreover, as in the classical case, it is important to consider more general sheaves than $\omega_{\widetilde{\mathfrak{C}}_\bullet}$ for which we have perversity statements.

1.1. **Local symmetries and perversity.** To be able to compute the derived coinvariants under $\widetilde{W}$, we need to know how $\widetilde{W}$ acts. By analogy with the global Springer theory, initiated by Yun, which concerns the parabolic Hitchin fibration $f^{par} : \mathcal{M}^{par} \to \mathcal{A}^{par}$, the spherical part, i.e. the action of the subalgebra $\overline{\mathbb{Q}}_\ell[X_*(T)]^W$ of $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$ factors through the action of the sheaf of connected components $\pi_0(\mathcal{P}^{glob}/\mathcal{A}^{par})$ of the Picard stack $\mathcal{P}^{glob}$ acting on $\mathcal{M}^{par}$ over $\mathcal{A}^{par}$. Moreover, by Lusztig [3], we have an a priori different action of $\widetilde{W}$ on the homology of the affine Springer fibers than the one constructed in [2] that we need to compare. We first construct a local Picard $\mathcal{P}$ that acts on $\widetilde{\mathfrak{C}}$ over $\mathfrak{C}$ and we have the following theorem [1, 3.3.3, 3.4.2]:

Theorem 1.2. *The affine Grothendieck-Springer sheaf $\mathcal{S}$ is naturally $\mathcal{P}$-equivariant and this action commutes with the one of $\widetilde{W}$. Moreover, the action of $\widetilde{W}$, constructed by intermediate extension coincides with the one of Lusztig.*

At the same time, we take the opportunity to generalize the perversity statement of [2] for local systems. More specifically, we have an equivalence of étale sheaves $[\widetilde{\mathfrak{C}}/G((t))] \cong [\mathrm{Lie}(I)/I]$. The stack $\mathcal{X} = [\mathrm{Lie}(I)/I]$ can be written as a projective limit of smooth Artin stacks of finite type $\mathcal{X} \simeq \varprojlim \mathcal{X}_i$ with smooth transition morphisms and we consider a category of renormalized local systems :

$$\mathrm{Loc}^{ren}(\mathcal{X}) \simeq \mathrm{colim}_{f^!} \mathrm{Loc}^{ren}(\mathcal{X}_i),$$

with $\mathrm{Loc}^{ren}(\mathcal{X}_i) = \{\mathcal{L} \otimes_{\overline{\mathbb{Q}}_\ell} \omega_{\mathcal{X}_i}, \mathcal{L} \in \mathrm{Loc}(Y)\}$ (we show that this definition is independent of choices). We have the following theorem [1, Thm. 3.2.9]:

Theorem 1.3. *The functor $[f]_!$ is left t-exact and for every $\mathcal{L} \in \mathrm{Loc}^{ren}([\widetilde{\mathfrak{C}}_\bullet/G((t))])$, $[f]_! \mathcal{L}$ is perverse and can be obtained as the intermediate extension of its restriction to $\mathfrak{C}_{rs}$.*

1.2. **Constructibility.** In general the homology of affine Springer fibers is infinite dimensional. Nevertheless, once we take the derived coinvariants under $\widetilde{W}$, one expects to obtain constructible perverse sheaves. We thus prove a stronger finiteness statement before taking the coinvariants [1, Thm. 4.3.15]:

Theorem 1.4. *The affine Grothendieck-Springer sheaf $\mathcal{S}$ is constructible as a sheaf of $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$-modules. In particular, for every finite dimensional representation τ of $\widetilde{W}$, the sheaf $\mathcal{S}_\tau$ of τ-coinvariants is a constructible sheaf over $[\mathfrak{C}_\bullet/G((t))]$.*

Now we have a finiteness statement for the τ-coinvariants, we are interested in their perversity.

1.3. **Homotopy lemma and perversity of coinvariants.** In the case of the global Hitchin fibration, we have by the homotopy lemma that the action of $\mathcal{P}^{glob}$ on the cohomology sheaves $R^i f_*^{par} \overline{\mathbb{Q}}_\ell$ factors through $\pi_0(\mathcal{P}^{glob})$. Moreoveor, Yun defines a morphism of sheaves:

$$\sigma : \overline{\mathbb{Q}}_\ell[X_*(T)]^W \to \pi_0(\mathcal{P}^{glob})$$

and shows ([4, Thms. 1, 2 ,3]) that the spherical part of the action of $\widetilde{W}$ on the cohomology with compact support $R^i f_*^{par} \overline{\mathbb{Q}}_\ell$of the Hitchin fibres factors through σ. He also shows a local statement, but that holds stalks by stalks, that is to say, for $\gamma \in \mathfrak{C}_\bullet(k)$, we have a local morphism:

$$\sigma_\gamma : \overline{\mathbb{Q}}_\ell[X_*(T)]^W \to \pi_0(\mathcal{P}_\gamma) \tag{1}$$

and again the spherical part acts on $H_c^i(f^{-1}(\gamma), \overline{\mathbb{Q}}_\ell)$ through σ_γ. Concerning homology, one has a weaker statement. In our case, one needs a generalization of Yun's result in various directions. First, to compute derived coinvariants, we need to work with ∞-categories and second, we need an homotopy lemma that holds on the complex rather than on its cohomology sheaves. We thus show an homotopy statement for the action of a commutative smooth group scheme on a complex, from which we deduce the following corollary that applies both in the global and local case [1, Thms. 5.1.5, 5.2.1]:

Theorem 1.5.

(1) *The action of $\mathcal{P}^{glob}$ on the complex $f_*^{par} \overline{\mathbb{Q}}_\ell$ factors through an action of $\pi_0(\mathcal{P}^{glob}/\mathcal{A})$.*

(2) *For every geometric point* $\gamma \in \mathfrak{C}_\bullet(k)$*, the action of* $\mathcal{P}_\gamma$ *on* $R\Gamma(f^{-1}(\gamma), \omega_{X_{f^{-1}(\gamma)}})$ *factors through* $\pi_0(\mathcal{P}_\gamma)$*.*

This allows to formulate a conjecture that generalizes Yun's theorem:

Conjecture 1.6. *For every algebraically closed field* K *and* $\gamma \in \mathfrak{C}_\bullet(K)$*, the action fo the spherical part* $\overline{\mathbb{Q}}_\ell[X_*(T)]^W$ *of* $\widetilde{W}$ *on* $R\Gamma(f^{-1}(\gamma), \omega_{f^{-1}(\gamma)})$ *factors through* (1)*.*

Assuming this conjecture, we prove a perversity statement for the coinvariants. If τ is a finite-dimensional representation that is torsion, i.e. its restriction $\tau_{|X_*(T)}$ is a sum of torsion characters. We prove [1, Thm. 5.3.1]:

Theorem 1.7. *Let* G *be a simply connected group,* τ *a finite-dimensonal torsion representation of* $\widetilde{W}$*, we suppose that 1.6 holds, then the sheaf of* τ*-coinvariants* $\mathcal{S}_\tau$ *is perverse.*

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From representations of p-adic groups to congruences of automorphic forms

Jessica Fintzen

(joint work with Sug Woo Shin)

In the talk I presented new results about the representation theory of p-adic groups and demonstrated how these can be used to obtain congruences between arbitrary automorphic forms and automorphic forms which are supercuspidal at p based on joint work with Sug Woo Shin ([FS21]). I outlined how this simplified earlier constructions of attaching Galois representations to automorphic representations, i.e. the global Langlands correspondence, for general linear groups. Since our results apply to general p-adic groups, they have the potential to become widely applicable beyond the case of the general linear group.

Let $\mathcal{G}$ be a (connected) reductive group over $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R})$ is compact mod center and let $\mathbb{A}$ denote the ring of adèles of $\mathbb{Q}$. (Our results work also over arbitrary (totally real) number fields, but for ease of notation we restricted our attention to $\mathbb{Q}$ for the talk.) Let p be a prime number, U^p a compact open subgroup of $\prod'_{\ell \neq p} \mathcal{G}(\mathbb{Q}_\ell)$, and U_p a compact open subgroup of $\mathcal{G}(\mathbb{Q}_p)$.

If Λ is a finite free $\mathbb{Z}_p$-module with a smooth action of U_p, then we write $M(U_pU^p, \Lambda)$ for the space of Λ-valued functions on $\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U^p\mathcal{G}(\mathbb{R})^\circ$ satisfying $f(gu_p) = u_p^{-1}.f(g)$ for $g \in \mathcal{G}(\mathbb{A}), u_p \in U_p$. We write $\mathbb{Z}_p$ for the free, one-dimensional $\mathbb{Z}_p$-module with trivial U_p-action. Then

$$M(U_pU^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U_pU^p\mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$$

is the space of *algebraic automorphic forms*. For an integer m, we write $A_m = \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m-1})$.

Theorem 1.1 ([FS21])**.** *Let p be larger than the Coxeter number of $\mathcal{G}$. Then there exist explicitly constructed compact open subgroups $U_{p,m} \subset \mathcal{G}(\mathbb{Q}_p)$ with a smooth action of $U_{p,m}$ on A_m such that*

(1) $U_{p,1} \rhd U_{p,2} \rhd \cdots$, *and* $\{U_{p,m}\}_{m\geq 1}$ *forms a basis of open neighborhoods of* $1 \in \mathcal{G}(\mathbb{Q}_p)$

(2) $M(U_{p,m}U^p, \mathbb{Z}_p)/(p^m) \simeq M(U_{p,m}U^p, A_m)/(1-T)$ *(compatible with the action of isomorphic Hecke algebras)*

(3) *If* $\Pi = \prod'_\ell \Pi_\ell \otimes \Pi_\infty$ *is an automorphic representation of* $\mathcal{G}(\mathbb{A})$ *contributing to* $M(U_{p,m}U^p, A_m) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}_p}$, *then* Π_p *is a supercuspidal representation of* $\mathcal{G}(\mathbb{Q}_p)$.

A special case in which $\mathcal{G}(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)$ is due to Scholze ([Sch18]) and was generalized by Kegang Liu to the case where $\mathcal{G}(\mathbb{Q}_p) = \mathrm{GL}_n(\mathbb{Q}_p)$ ([Liu]) independently of our work. Moreover, Raphaël Beuzart-Plessis ([FS21, Appendix D]) discovered an alternative approach that allows to remove the assumption on p for a bare existence statement without an explicit construction of the compact open subgroups $U_{p,m}$ with their action on A_m. The advantage of our approach is a precise understanding of the supercuspidal representations occurring as Π_p.

Theorem 1.1 is proven using new results about representations of p-adic groups. More precisely, the following notion forms a key ingredient. Let k be a finite extension of $\mathbb{Q}_p$ and let G be a reductive group over k. For an integer n, we call a pair (U, λ) consisting of a compact, open subgroup $U \subset G(k)$ and a smooth surjectiv group morphisms $\lambda : U \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ an *omni-supercuspidal type* (of level p^n) if the following holds: For every nontrivial character $\chi : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{C}^*$, every smooth, irreducible representation π of $G(k)$ that contains $\chi \circ \lambda$ when restricted to U is supercuspidal. This definition is designed with the application to Theorem 1.1 in mind. The key ingredient for the proof of Theorem 1.1 is the following result.

Theorem 1.2 ([FS21])**.** *Let p be larger than the Coxeter number of G. Then there exists an explicitly constructed sequence $\{(U_m, \lambda_m)\}_{m\geq 1}$ such that*

- (U_m, λ_m) *is an omni-supercuspidal type of level* p^m,
- $G \supset U_1 \rhd U_2 \rhd \cdots$, *and* $\{U_m\}_{m\geq 1}$ *forms a basis of open neighborhoods of* 1.

The proof of Theorem 1.2 combines a variety of different techniques and relies on the construction of supercuspidal representations by J-K Yu ([Yu01], [Fin]) and results from [Fin21].

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Twisted GGP problems and conjectures

Wee Teck Gan

(joint work with Benedict Gross, Dipendra Prasad)

In this talk, we explain a twisted version of the basic Fourier-Jacobi case of the GGP conjecture. When E and K are two étale quadratic algebras over a local or global field F, one considers a skew-Hermnitian space V relative to E/F, and let V_K be its base change to K, so that V_K is a skew-Hermitian space relative to $E \otimes K/K$. For an irreducible representation Π of $\mathrm{U}(V_K)$ and a Weil representation $\omega_{V,\psi,\mu}$ of $\mathrm{U}(V)$, the twisted GGP problem considers the the branching problem $\mathrm{Hom}_{\mathrm{U}(V)}(\Pi, \omega_{V,\psi,\mu})$ locally and the corresponding period integral globally. We formulate precise conjectural answers to this branching problem, both locally and globally, discuss some evidences for it in low rank and for unitary principal series representations, and highlight some ongoing work.

The generalized Fourier transforms on a basic affine space and the Whittaker model

Nadya Gurevich

(joint work with David Kazhdan)

Let G be a group of F-points of simply-connected quasi-split group defined over a local non-archimedean field F. We fix a Borel subgroup B of G and the decomposition $B = T \cdot U$. The basic affine space $X = U \backslash G$ admits unique (up to a scalar) G-invariant measure ω_X. We define a unitary representation θ of the group $G \times T$ on $L^2(X, \omega_X)$ by:

$$\theta(g,t)f(Uh) = \delta_B^{1/2}(t) f(Ut^{-1}hg)$$

for the modular character δ_B. Let $W := N_G(T)/T$ be the Weyl group.

For split groups Gelfand and Graev in [1], see also [3],[2], extended the action of $G \times T$ to the representation of $G \times (T \rtimes W)$, so that every element w of W acts on $L^2(X, \omega_X)$ by an operator Φ_w, called a generalized Fourier transform. We extend their construction to quasi-split groups and show that the action of W is compatible with the Whittaker map, as we describe below.

- We denote by $\mathcal{S}_c(X)$ (resp. $\mathcal{S}_c(T)$) the space of smooth functions of compact support.

  Fix a non-degenerate character Ψ of U. The Whittaker map

$$\mathcal{W}_\Psi : \mathcal{S}_c(X) \to \mathcal{S}_c(T), \quad \mathcal{W}_\Psi(f)(t) = \int_U f(Ut^{-1}n_0u)\Psi^{-1}(u)du \tag{1}$$

  defines an isomorphism $\mathcal{S}_c(X)_{U,\Psi} \simeq \mathcal{S}_c(T)$.
- We define an action of W on $\mathcal{S}_c(T)$. For split groups set

$$w \cdot \varphi(t) = \delta_B^{1/2}(w^{-1}tw \cdot t^{-1})\varphi(w^{-1}tw).$$

  For quasi-split groups there is a minor twist in this action.
- We define a $G \times T$ submodule $\mathcal{S}_0(X)$ that is dense in $L^2(X)$ and put

$$\mathcal{S}_0(T) = \mathcal{W}_\Psi(\mathcal{S}_0(X)) \simeq \mathcal{S}_0(X)_{U,\Psi}.$$

- There is a natural map $\kappa_\Psi : \mathrm{End}_G(\mathcal{S}_0(X)) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}_0(X)_{U,\Psi}) = \mathrm{End}_{\mathbb{C}}(\mathcal{S}_0(T))$ such that for every $B \in \mathrm{End}_G(\mathcal{S}_0(X))$ the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{S}_0(X) & \xrightarrow{B} & \mathcal{S}_0(X) \\ \downarrow{\scriptstyle \mathcal{W}_\Psi} & & \downarrow{\scriptstyle \mathcal{W}_\Psi} \\ \mathcal{S}_0(T) & \xrightarrow{\kappa_\Psi(B)} & \mathcal{S}_0(T) \end{array}$$

  We prove that the map κ_Ψ is injective.

With these notations we formulate the main result.

Theorem 1.1. *There exists* **unique** *family of unitary operators $\Phi_w, w \in W$, on $L^2(X, \omega_X)$, preserving the space $\mathcal{S}_0(X)$ and satisfying:*

$$\begin{cases} \Phi_w \circ \theta(g,t) = \theta(g,t^w) \circ \Phi_w & \forall w \in W, t \in T, g \in G \\ \Phi_{w_1}\Phi_{w_2} = \Phi_{w_1w_2} & \forall w_1, w_2 \in W \\ \kappa_\Psi(\Phi_w)(\varphi) = w \cdot \varphi & \forall w \in W, t \in T, \varphi \in \mathcal{S}_0(T) \end{cases} \tag{2}$$

The injectivity of the map κ_Ψ is crucial. First, it implies the uniqueness of the family $\{\Phi_w\}$. Second, it reduces the construction of Φ_w to the construction of the operators Φ_s where $s \in W$ is a simple reflection. This in turn reduces the construction to the quasi-split groups of rank one, i.e. $G = SL_2$ and $G = SU_3$. In both cases we write the operator Φ_s explicitly, using the minimal of a group H containing $G \times W$ as a commuting pair.

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Adams' conjecture on theta correspondence-the discrete diagonal restriction case

Marcela Hanzer

(joint work with Petar Bakić)

In recent years, there was a significant breakthrough in Langlands program, facilitated by the work of Arthur ([1]) and others ([2],[3]), where the classification of the automorphic discrete spectrum of (certain classes of) classical groups is obtained and this has given rise to the proofs of local Langlands conjecture in the appropriate situations. Now one can ask how global Arthur parametrization of automorphic discrete spectrum of the groups involved (and their their local components) behave under theta correspondence. More precisely, if an automorphic representation is given through its Arthur parameter can its global theta lift (assume that it is defined and automorphic) have a global Arthur parameter of similar form (prescribed in a certain way prescribed below) to an original one? And we can pose an analogous question for the local components of those automorphic representations, (i.e. for their local Arthur parameters) and local theta correspondence. We will be interested here in the local situation. Let us recall of the main notions we mentioned.

Globally, theta correspondence, gives us, by the integration of automorphic forms on one group against the theta kernel, one of the few direct ways to construct new automorphic forms (on the other group). We will now explain which are the two groups appearing in this situation. We will dedicate ourselves here to the local situation, i.e. to reductive (mostly classical or their covers) groups defined over p-adic fields, and all the representation we are considering are over the complex numbers. Theta correspondence has its origins in the work of Howe on dual reductive pairs and of Weil who constructed a unique non-trivial two-fold cover of a symplectic group (the metaplectic group) and a distinguished representation of it, the Weil representation. The Weil representation of a metaplectic group is infinite-dimensional, but small (actually, more precisely, minimal representation) in terms of Gelfand-Kirillov dimension; we just note that its "minimality" dictates that the branching on the specific subgroups does not behave wildly. The dual reductive pairs in a symplectic group (i.e. the subgroups which are commutants of each other) can split in the metaplectic cover or they might not. We are now concerned with the dual reductive pairs (these are the pairs of groups mentioned above) consisting of a symplectic group $Sp(W)$ and $O(V)$, an orthogonal group, and they form a dual reductive pair in the symplectic group $Sp(W \otimes V)$. If the dimension of V is even, both $O(V)$ and $Sp(W)$ split in the metaplectic cover $Mp(W \otimes V)$ of $Sp(W \otimes V)$ so that their lifts commute in $Mp(W \otimes V)$ and then we can view the metaplectic representation ω_ψ of $Mp(W \otimes V)$ (it depends on the choice of an additive character ψ) as a representation of $Sp(W) \times O(V)$. We denote this kind of a dual reductive pair (G, H) (symmetrically). Having this in mind, we denote this (restricted) representation of the dual reductive pair as $\omega_{m,n,\psi}$. Now, for an irreducible representation of π of G, we denote the maximal π-isotypic quotient of $\omega_{m,n,\psi}$ by $\Theta(\pi, m)$ and call it the full theta lift of π to H. This representation of H, when non-zero, has a unique irreducible quotient, denoted $\theta(\pi, m)$—the small theta lift of π. This basic fact, called the Howe duality conjecture, was first formulated by Howe, proven by Waldspurger (for an odd residue characteristic) and by Gan and Takeda([4]) in general. The Howe duality establishes a map $\pi \mapsto \theta(\pi)$ which is called the theta correspondence. The study of theta correspondence started by Roger Howe, and further developed by Kudla, Rallis, Moeglin, Vigneras, Waldspurger and many others. In recent years, there were many significants developments: e.g., the conservation conjecture is proved by Sun and Zhu ([5]). The research then moved to the determination of the theta lifts explicitly (i.e. in terms of their Langlands parameters, using the Langlands classification obtained by Arthur ([1])), firstly for the cases where the lifted representation π is of a certain sort (e.g. discrete series and tempered representations in the works of Muić ([6]), and then for general tempered representations of the classical dual reductive pairs ([7]). The Langlands parameters of theta lifts of general representations of those classical dual pairs are obtained by Bakć and Hanzer ([8]).

In the series of papers (e.g. [9]), Mœglin explicitly constructed members of local Arthur packets (i.e. irreducible representations which are attached to local Arthur parameters; so they are local components of the (discrete) automorphic representations at the p–adic places). On the other hand, in his paper *L-functoriality for Dual Pairs*, Asterisque 171-172, 1989, 85-129, J. Adams loosely formulates a conjecture regarding representations in local Arthur packets. He predicts that their theta lifts on the groups of bigger rank are going to be, in certain situations, also members of Arthur packets of a similar form. There, he is concerned with the archimedean cases. In her paper *Conjecture d'Adams pour la correspondance de Howe et filtration de Kudla,* Arithmetic geometry and automorphic forms, 445–503, Adv. Lect. Math. (ALM), 19, Int. Press, Somerville, MA, 2011, Mœglin is concerned with the p–adic case of this conjecture where she made it more precise. She resolved there the case of discrete series representations. To be able to roughly explain the Adams conjecture in the p–adic case, let us describe the form of the local Arthur parameter for a group G, where, for simplicity, we take G to be p–adic symplectic or even orthogonal. It is an admissible homomorphism (we will not explain all the relevant properties)

$$\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \to O(l, \mathbb{C}),$$

which is semisimple (W_F is the Weil group of a p–adic field F of characteristic zero); here $O(l,\mathbb{C})$ is identified with the L–group of G. Assume that π belongs to the A-packet Π_ψ; then $\pi = \pi(\psi, t, \eta)$ where are t, η are additional parameters which are related to the commutator of the image of ψ in $O(l,\mathbb{C})$ and which determine π uniquely inside Π_ψ. For an odd positive integer l and appropriate character χ the parameter $\tilde{\psi} = \psi \oplus \chi \otimes S_1 \otimes S_l$ is an A-parameter of H_l, the other group in a dual reductive pair. Here S_k denotes the unique algebraic representation of $SL(2,\mathbb{C})$ of dimension k. Adams conjecture predicts that, for certain $\tilde{t}, \tilde{\eta}$, $\theta(\pi) = \pi(\tilde{\psi}, \tilde{t}, \tilde{\eta})$; i.e. theta lift of G to H_l is again attached to Arthur parameter of the prescribed form. We want to know for which l this holds, and if it holds the specific l, how to find the correct $\tilde{t}$ and $\tilde{\eta}$.

The answers to these questions bear not only local significance in understanding of how the local A-packets behave under local theta correspondence, but, because of the obvious relation with the global situation, we can predict application in improving our understanding of global theta correspondence. Also, the complete understanding of this phenomenon will also advance our understanding of the behavior of general unitary representations under theta correspondence, not only those representations which are in the Arthur class.

We were able to completely answer the above questions for a wide class of parameters ψ, the so-called parameters with the discrete diagonal restriction; the general representations in the Arthur class are the Jacquet modules of the representations with the discrete diagonal restriction.

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Around local Langlands correspondences

Michael Harris
(joint work with Böckle, Feng, Khare, and Thorne)

My talk in August was a report on three recent projects related to the construction of local Langlands correspondences: work in progress with Gan and Sawin on the one hand, with Böckle, Feng, Khare, and Thorne, on the other hand, and a paper with Khare and Thorne on the group G_2. The results primarily apply to groups over non-archimedean local fields of positive characteristic and to the parametrization of representations of these groups by Genestier and Lafforgue.

Let G be a connected reductive group over the non-archimedean local field F, with residue field k of characteristic p. Denote by $\Phi(G/F)$ the set of equivalence classes of Weil-Deligne parameters for G/F and let $\Pi(G/F)$ denote the set of (equivalence classes of) irreducible admissible representations of $G(F)$, both taken over a fixed algebraically closed field C of characteristic 0. In its simplest form, the local Langlands conjecture asserts the existence of a canonical parametrization of one set by the other set:

Conjecture 1.1.

(a) *There is a canonical parametrization*

$$\mathcal{L} = \mathcal{L}_{G/F} : \Pi(G/F) \to \Phi(G/F).$$

(b) *For any $\varphi \in \Phi(G/F)$, the* L**-packet** $\Pi_\varphi := \mathcal{L}^\varphi$ *is finite.*

(c) *For any $\varphi \in \Phi(G/F)$ the L-packet Π_φ is non-empty.*

The first part of the talk was a review of the properties expected of any such correspondence. In general it is not known that this list of desirable properties suffices to characterize the correspondence. The conjecture is known to hold when $G = GL(n)$, and in this case the word "canonical" in the above formulation has a precise meaning. Versions of the conjecture have also been established for classical groups over p-adic fields. Semisimple parametrizations have been constructed by means of arithmetic geometry by Genestier and Lafforgue [6], when $F = k((t))$ is of positive characteristic, and by Fargues and Scholze [2] for p-adic fields; these parametrizations satisfy various desirable properties but they are not known to have properties (b) and (c) above. Kaletha has defined parametrizations for large classes of supercuspidal representations [8], using the work of J.K. Yu and Fintzen; these parametrizations also satisfy desirable properties, but it is not known in most cases that they coincide with those defined by arithmetic geometry.

Say the representation $\pi \in \Pi(G/F)$ is *pure* if all the Frobenius eigenvalues of its parameter in $\Phi(G/F)$ are Weil q-numbers of the same weight. Assume G is split semisimple. The main body of the talk was devoted to explaining the following result with Gan and Sawin:

Theorem 1.2. *Let $F = k((t))$ and let π be a pure supercuspidal representation of $G(F)$. Suppose π is compactly induced from a compact open subgroup $U \subset G(F)$. (For example, if p does not divide the order of the Weyl group $W(G)$, this follows* *from* **Fintzen's theorem***.) Suppose moreover that $q > 3$. Then the Genestier-Lafforgue semisimple parameter $\mathcal{L}^{ss}(\pi)$ is not unramified.*

The proof is based on the construction of appropriate Poincaré series, following a method of Gan and Lomelí [5], for the group G over the curve $\mathbb{P}^1$. A similar argument shows that $\mathcal{L}^{ss}(\pi)$ is even wildly ramified if the group U is sufficiently small.

Using the compatibility of $\mathcal{L}^{ss}$ with parabolic induction, we conclude

Corollary 1.3. *Assume p does not divide the order of the Weyl group $W(G)$. Let π be a pure representation of $G(F)$. Suppose $\mathcal{L}^{ss}(\pi)$ is unramified. Then π is an irreducible constituent of an unramified principal series.*

For general connected reductive groups, we can show

Theorem 1.4. *For any supercuspidal π, $\mathcal{L}^{ss}(\pi)$ can be completed (necessarily uniquely) to a Weil-Deligne parameter $(\mathcal{L}^{ss}(\pi), N)$ that satisfies purity of the monodromy weight filtration.*

The talk also reported briefly on applications of automorphy lifting theorems, in the spirit of the Taylor-Wiles method, to the local Langlands correspondence. The following result is specific to the split group G_2 over a p-adic field:

Theorem 1.5. [7] *For F p-adic and $G = G_2$, there is a natural bijection*

$$\mathcal{L}_0^{generic} : \Pi_0^{generic}(G_2/F) \xrightarrow{\sim} \Phi_0(G_2/F)$$

where $\Pi_0^{generic}$ is the set of generic supercuspidals and Φ_0 is the set of irreducible parameters.

Finally, in joint work in progress with Böckle, Feng, Khare, and Thorne, we prove

Theorem 1.6. *Let K be a global function field over a finite field. Let Π be a cuspidal automorphic representation of $G(\mathrm{ad}_K)$. Suppose the Mumford-Tate group of Π is $\hat{G}$. Then there is an integer $b(\Pi)$, depending only on Π, such that, for any good odd prime ℓ for G such that $\ell > b(\Pi)$, and any cyclic extension K'/K of degree ℓ, there is an automorphic representation Π' of $G(\mathrm{ad}_{K'})$ such that, at all unramified places v of K, the representation Π'_v of $G(K'_v)$ is the base change of Π_v.*

When K is a number field, one can prove such a result using the stable trace formula, but this is not yet available over function fields. Instead the proof is based on Feng's construction of cyclic base change for automorphic representations with coefficients in characteristic ℓ, using Smith theory [3], together with the automorphy lifting theorems of [1].

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Certain Poisson Summation Formulae on $\mathbf{GL_1}$ and Langlands Automorphic L-functions

Dihua Jiang

(joint work with Zhilin Luo)

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. For simplicity, we may take G to be a k-split reductive algebraic group, and denote $G^\vee(\mathbb{C})$ its complex dual group. Let $\rho\colon G^\vee(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ be any finite dimensional representation of $G^\vee(\mathbb{C})$. For any $\sigma = \otimes_\nu \sigma_\nu \in \mathcal{A}_{\mathrm{cusp}}(G)$, the set of equivalence classes of irreducible cuspidal automorphic representations of $G(\mathbb{A})$, R. Langlands ([11]) defines automorphic L-functions by the following Euler product of the local L-functions:

$$L(s,\sigma,\rho) = \prod_\nu L(s,\sigma_\nu,\rho).$$

When ν is archimedean, the local L-functions $L(s,\sigma_\nu,\rho)$ can be defined via the Langlands classification ([12]). When ν is finite, they can be defined via the Satake isomorphism ([14]) when the local components σ_ν are unramified. However, if ν is finite and σ_ν is ramified, the definition of the local L-function $L(s,\sigma_\nu,\rho)$ can be regarded as part of the local Langlands conjecture for G over k_ν, which is known for many important cases, but is still widely open in general. The recent work of L. Fargues and P. Scholze ([4]) presents impressive progress towards the local Langlands conjecture in general.

Langlands proved that the Euler product that defines $L(s,\sigma,\rho)$ converges absolutely for $\Re(s)$ sufficiently positive, and made the following conjecture ([11]).

Conjecture 1.1 (L-functions)**.** *The automorphic L-function $L(s,\sigma,\rho)$ has meromorphic continuation and satisfies functional equation*

$$L(s,\sigma,\rho)=\epsilon(s,\sigma,\rho)L(1-s,\widetilde{\sigma},\rho)$$

where $\widetilde{\sigma}$ *is the contragredient of* σ.

When $G=\mathrm{GL}_1$ and $\sigma=\chi$ is a character of the idele class group $k^\times\backslash\mathbb{A}^\times$, Conjecture 1.1 is a classical theory of E. Hecke. In his 1950 Princeton Thesis ([16]), J. Tate proved the meromorphic continuation and functional equation of the Hecke L-functions $L(s,\chi)$ by using the classical Fourier transform and the associated Poisson summation formula on the quotient $k\backslash\mathbb{A}$. This idea was taken up by R. Godement and H. Jacquet in their work ([7]), where they established the meromorphic continuation and functional equation for the standard L-functions $L(s,\pi)$ attached to any $\pi\in\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_n)$, by using the Fourier transform and the associated Poisson summation formula for $\mathrm{M}_n(k)\backslash\mathrm{M}_n(\mathbb{A})$. Here M_n denotes the space of all $n\times n$ matrices. Conjecture 1.1 has been studied by the Langlands-Shahidi method and the Rankin-Selberg method, for many important families of cases. However, the general situation remains widely open.

In 2000, A. Braverman and D. Kazhdan ([1]) proposed that there should exist a generalized Fourier transform $\mathcal{F}_{\rho,\psi}$ on $G(\mathbb{A})$ for any reductive group G defined over k and any finite dimensional complex representation ρ of the L-group ${}^LG$; and if the associated Poisson summation formula could be established, then there is a hope to prove Conjecture 1.1 for automorphic L-function $L(s,\pi,\rho)$ attached to the pair (π,ρ), where $\pi\in\mathcal{A}_{\mathrm{cusp}}(G)$. In his 2020 paper ([13]), B. C. Ngô suggests that such generalized Fourier transforms could be put in a framework that generalizes the classical Hankel transform for harmonic analysis on GL_1 and might be more useful in the trace formula approach to establish the Langlands conjecture of functoriality beyond the case of endoscopy.

In this talk, I explain my recent work joint with Zhilin Luo ([8] and [9]) on the possibility that utilizes harmonic analysis on GL_1 to understand Conjecture 1.1, as a vast generalization of the classical work of Tate ([16]).

For any pair (G,ρ) as before, our papers introduce, for any $\sigma\in\mathcal{A}_{\mathrm{cusp}}(G)$, the space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ of (σ,ρ)-Schwartz functions on $\mathbb{A}^\times$ and the (σ,ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ that takes $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$, where ψ is a nontrivial character of $k\backslash\mathbb{A}$.

Theorem 1.2. *The* (σ,ρ)*-theta functions*

$$\Theta_{\sigma,\rho}(x,\phi):=\sum_{\alpha\in k^\times}\phi(\alpha x)$$

converge absolutely for all $\phi\in\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$.

The key in the GL_1-theory developed in [8] and [9] is the following conjecture.

Conjecture 1.3 ((σ,ρ)-Poisson Summation Formula)**.** *Given a pair* (G,ρ) *as before, for any* $\sigma\in\mathcal{A}_{cusp}(G)$*, there exist nontrivial* $k^\times$*-invariant linear functionals* $\mathcal{E}_{\sigma,\rho}$ *and* $\mathcal{E}_{\widetilde{\sigma},\rho}$ *on* $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ *and* $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$*, respectively, such that the* (σ,ρ)*-Poisson Summation Formula:*

$$\mathcal{E}_{\sigma,\rho}(\phi)=\mathcal{E}_{\widetilde{\sigma},\rho}(\mathcal{F}_{\sigma,\rho,\psi}(\phi))$$

holds for $\phi\in\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$.

It is expected that the (σ,ρ)-Poisson summation formula on GL_1 should be responsible for Conjecture 1.1 associated the pairs (σ,ρ). It is important to point out that Conjecture 1.3 holds for (G,ρ) and an irreducible cuspidal automorphic representation σ of $G(\mathbb{A})$ if the global Langlands functoriality is known for the pair (G,ρ) and the functorial transfer to GL_n of σ is cuspidal.

Theorem 1.4. *For $G=\mathrm{GL}_n$ and any $\pi\in\mathcal{A}_{cusp}(\mathrm{GL}_n)$, the π-theta function*

$$\Theta_\pi(x,\phi):=\sum_{\alpha\in k^\times}\phi(\alpha x)$$

with $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$ satisfies the following identity

$$\Theta_\pi(x,\phi)=\Theta_{\widetilde{\pi}}(x^{-1},\mathcal{F}_{\pi,\psi}(\phi)).$$

It is expected that Conjecture 1.3 can be proved directly for a split classical group G and the standard representation ρ of the complex dual group $G^\vee(\mathbb{C})$, by using the doubling method of I. Piatetski-Shapiro and S. Rallis in [5]) and the recent work of L. Zhang and the authors in [10] and of J. Getz and B. Liu in [6].

As one of the applications of the GL_1-theory as developed as above, [9] provides a spectral interpretation of the critical zeros of the automorphic L-functions $L(s,\pi)$ for any $\pi\in\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_n)$, which is a reformulation of C. Soulé ([15, Theorem 2]) and also that of A. Deitmar ([3]) in the exact adelic framework of A. Connes in [2]), and can be viewed as an extension of [2, Theorem III-1] from the Hecke L-functions $L(s,\chi)$ to the automorphic L-function $L(s,\pi)$.

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Lusztig correspondence and exotic Fourier transforms

Emmanuel Letellier
(joint work with Gérard Laumon)

Assume given a connected reductive group G over a finite field $\mathbb{F}_q$ with its dual group $G^\vee$ which is also a connected reductive algebraic group over $\mathbb{F}_q$. We will use the same letter F to denote the Frobenius on G and $G^\vee$. Let ℓ be a prime which does not divide q. Lusztig considered a partition of the set $\mathrm{I}rr(G^F)$ of irreducible $\overline{\mathbb{Q}}_\ell$-characters of the finite group G^F

$$\mathrm{I}rr(G^F) = \coprod_{(s)} \mathcal{E}_G(s)$$

where (s) runs over the set of F-stable semisimple conjugacy classes of $G^\vee$. We call $\mathcal{E}_G(s)$ the Lusztig series of G^F corresponding to (s) and we denote by $\mathrm{L}S(G^F)$ the set of Lusztig series of G^F.

If $\rho^\vee : G^\vee \to H^\vee$ is a morphism of algebraic groups defined over $\mathbb{F}_q$, then we have a map (lifting)

$$t_\rho : \mathrm{L}S(G^F) \to \mathrm{L}S(H^F)$$

given by $\mathcal{E}_G(s) \mapsto \mathcal{E}_H(\rho^\vee(s))$.

We now consider the case $H = \mathrm{G}L_n$ with its standard $\mathbb{F}_q$-structure. We let $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ be a non-trivial additive character of $\mathbb{F}_q$ and we consider the Fourier transform

$$\mathcal{F}^{\mathrm{GL}_n}(f)(x) = \sum_{y \in \mathrm{GL}_n^F} \psi\big(\mathrm{T}race(xy)\big) f(y),$$

for all function $f : \mathrm{GL}_n^F \to \overline{\mathbb{Q}}_\ell^\times$ and $x \in \mathrm{GL}_n^F$.

The gamma function $\gamma^{\mathrm{GL}_n} : \mathrm{I}rr(\mathrm{GL}_n^F) \to \overline{\mathbb{Q}}_\ell$ is defined by

$$\mathcal{F}^{\mathrm{GL}_n}(\chi) = \gamma^{\mathrm{GL}_n}(\chi)\chi^\vee$$

where χ is an irreducible character of GL_n^F and $\chi^\vee$ is the dual character.

The function γ^{GL_n} turns out to be constant on Lusztig series, and so one can transfer it into a gamma function $\gamma_\rho^G : \mathrm{I}rr(G^F) \to \overline{\mathbb{Q}}_\ell$ defined by

$$\gamma_\rho^G = c_{G,\mathrm{GL}_n}\gamma^{\mathrm{GL}_n} \circ t_\rho$$

for some explicit constant c_{G,GL_n}. This define an operator $\mathcal{F}_\rho^G$ (considered by Braverman-Kazhdan) on the space of functions on G^F that we call the *exotic Fourier transform.* More precisely, if we define $\phi_\rho^G : G^F \to \overline{\mathbb{Q}}_\ell$ by

$$\phi_\rho^G(g) = \sum_{\chi \in \mathrm{I}rr(G^F)} \gamma_\rho^G(\chi)\overline{\chi(g)}\chi(1)$$

then

$$\mathcal{F}_\rho^G(f)(g) = \sum_{h \in G^F} \phi_\rho^G(gh)f(h).$$

Problem: Compute ϕ_ρ^G explicitly and extends $\mathcal{F}_\rho^G$ to an involutive operator.

Let T be an F-stable maximal torus of G and let W be the Weyl group of G with respect to T. Denote by $[T/W]$ and $[G/G]$ the quotient stacks for the conjugation actions.

Theorem 1.1 (Laumon-Letellier)**.** *(1) There exists a pair of adjoint functors (R, I) between categories of perverse sheaves*

$$\mathcal{M}([T/W]) \underset{R}{\overset{I}{\rightleftarrows}} \mathcal{M}([G/G])$$

such that $R \circ I \simeq 1$.

(2) If moreover G is of type A with connected center then $I \circ R \simeq 1$.

A similar result for D-modules was obtained by S. Gunningham.

Assume for convenience that $\rho^\vee$ restricts to a morphism $T^\vee \to T_n$ where T_n is the maximal torus of diagonal matrices. Then we have a morphism $\rho_T : T_n \to T$ which is W-equivariant. We then obtain a W-equivariant perverse sheaves Φ_ρ^T on T (assuming that ρ_T is surjective) by twisting by a sign character the obvious W-equivariant structure on the proper pushforward along ρ_T of the Artin-Schreier sheaf on T_n. This W-equivariant perverse sheaf descends thus to a perverse sheaf $\Phi_\rho^{[T/W]}$ on $[T/W]$.

Theorem 1.2 (Laumon-Letellier)**.** *Assume that ρ_T is surjective. We have*

$$\phi_\rho^G = X_{I(\Phi_\rho^{[T/W]})}$$

where X_K means the characteristic function of a complex K equipped with $F^*K \simeq K$.

This theorem is a reformulation of a conjecture of Braverman-Kazhdan (2003).

Extending $\mathcal{F}_\rho^G$ to an involutive operator ?

Assume that G is of type A and has a connected center and that ρ_T is surjective.

From $\Phi_\rho^{[T/W]}$ we define a functor $F_\rho^{[T/W]} : \mathcal{M}([T/W]) \to \mathcal{M}([T/W])$ and a functor $F_\rho^{[G/G]}$ (which give back $\mathcal{F}_\rho^G$ on functions) that makes the right square of the diagram below commutative

$$\begin{array}{ccccc}
\mathcal{M}([\mathbb{A}^n/\mathrm{K}er(\rho_T) \rtimes W]) & \xleftarrow{j_!} & \mathcal{M}([T/W]) & \xleftarrow{R} & \mathcal{M}([G/G]) \\
\big\downarrow \text{(involutive)} & & \big\downarrow F_\rho^{[T/W]} & & \big\downarrow F_\rho^{[G/G]} \\
\mathcal{M}([\mathbb{A}^n/\mathrm{K}er(\rho_T) \rtimes W]) & \xrightarrow{j^*} & \mathcal{M}([T/W]) & \xrightarrow{I} & \mathcal{M}([G/G])
\end{array}$$

where $j : [T/W] \to [\mathbb{A}^n/\mathrm{K}er(\rho_T) \rtimes W]$ and where the left vertical arrow arises from the classical Fourier transform on $\mathbb{A}^n$

Homological branching: recent results and beyond

Dipendra Prasad

If H is a subgroup of a group G, π_1 an irreducible representation of G, one is often interested in decomposing the representation π_1 when restricted to H, called the branching laws.

However, since π_1 restricted to H is usually not semi-simple for non-compact H, there is often no meaning to "decomposing the representation" restricted to H, or a meaning has to be assigned in some precise way, such as the Plancherel decomposition for unitary representations of G restricted to H. For smooth representations of p-adic groups, the problem most studied is of understanding $\mathrm{Hom}_H(\pi_1, \pi_2)$, for π_2 an irreducible representation of H. Much less is studied of an apparently similar looking question about $\mathrm{Hom}_H(\pi_2, \pi_1)$.

The Bernstein decomposition of the smooth category of representations of a group G, denoted here by $\mathcal{M}(G)$, gives a framework to study such questions more completely, and it seems possible to fully understand the question "how does a representation of G restrict to H" from this point of view, such as for the GGP branching problems. We will restrict ourselves in this lecture only to the pair of groups $(G, H) = (\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$, where F is a non-archimedean local field.

1. First result on the restriction problem

Proposition 1.1. *Let π_1 be an irreducible generic representation of $\mathrm{GL}_{n+1}(F)$, and let (M,ρ) be a cuspidal datum for $\mathrm{GL}_n(F)$ for which we assume that no nontrivial element of $N_G(M)/M$ preserves ρ up to an unramified twist. It is easy to see that ρ restricted to M^0 is a finite direct sum of irreducible representations, each appearing with multiplicity 1. Let ρ^0 be an irreducible representation of ρ restricted to M^0. Then, the (M,ρ) Bernstein component of π_1 restricted to $\mathrm{GL}_n(F)$ is the universal principal series representation:*

$$Ind_{P^0}^{G}(\rho^0),$$

where $P^0 = M^0N$.

Proof. By a theorem of Roche, parabolic induction gives an equivalence of categories. Next, we appeal to the theorem of Aizenbud-Sayag for finite generation, and the theorem of multiplicity = 1 for generic representations, to complete the proof of this proposition using the following lemma in Commutative algebra. □

Lemma 1.2. *Let R be a reduced Noetherian ring, M a finitely generated module over R such that for each maximal ideal $\mathfrak{m}$ of R, $M/\mathfrak{m}M$ is free of rank 1 over $R/\mathfrak{m}$, then M is free of rank 1 over R.*

Conjecture: Suppose π is a tempered representation of $\mathrm{GL}_{n+1}(F)$. Then if the cuspidal support of π does not contain an unramified character of $\mathrm{GL}_1(F)$, or if π is an essentially discrete series representation, then the Iwahori component of the restriction of π to $\mathrm{GL}_n(F)$ is projective and is $\mathrm{ind}_K^{\mathrm{GL}_n(F)}(\mathrm{St})$.

Remark 1.3. *By a theorem of Chan-Savin, cf.* [2], *for generic representations π of $\mathrm{GL}_{n+1}(F)$ which are projective restricted to $\mathrm{GL}_n(F)$, classified by Chan in* [1], *their restriction to $\mathrm{GL}_n(F)$ is independent of π, in particular, it is the same as that of a cuspidal representation of $\mathrm{GL}_{n+1}(F)$ restricted to $\mathrm{GL}_n(F)$, which is answered by a theorem of Chan-Savin that we discuss next, and which therefore answers the above conjecture in the affirmative for all such representations of $\mathrm{GL}_{n+1}(F)$.*

2. Epilogue

This lecture deals with situations when the restriction problem brings one to a projective representation inside a particular Bernstein component of the subgroup H, and one hopes that projective modules of a given rank are all isomorphic, therefore the restriction problem has as complete an understanding as one might desire.

At the other extreme is the question of understanding the restriction problem for an unramified generic representation of $\mathrm{GL}_{n+1}(F)$, and to consider its component in the Iwahori block of $\mathrm{GL}_n(F)$. In this case, one will not be dealing with a projective module. How should one 'model' the restriction? For example, is the isomorphism class of this restriction in the Iwahori block independent of the unramified generic representation chosen? What happens to products of the Steinbergs?

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***p*-adic shtukas and Shimura varieties**

Michael Rapoport

(joint work with George Pappas)

It is a long-standing problem to construct good p-integral models of Shimura varieties. After the modular curve, the emphasis has been on constructing such models in the case when the p-component of the level structure is of *parahoric type*. The bulk of the work on this problem has been in the *classical case*, i.e., for Shimura varieties of PEL-type. For Shimura varieties of abelian type such models have been constructed in many cases by Kisin-Pappas [2] and, more generally, by Kisin-Zhou [3].

It is also a long-standing problem to uniquely characterize these models. In the *tame case*, such a characterization is due to G. Pappas [4] who used $\mathcal{G}$-displays. In the talk I gave a different characterization which works in general. It is based on Scholze's theory of diamonds and v-sheaves [6]. It is remarkable that to obtain a characterization, even for the classical case, one has to use such sophisticated tools.

1. p-adic shtukas

We recall Scholze's p-adic analogue of Drinfeld's notion of a shtuka in the function field setting.

Definition 1.1 (Scholze). *Let $S = Spa(R, R^+) \in Perf_{\mathbb{F}_p}$, and let $S^\sharp$ be an untilt of S. A shtuka over S with leg along $S^\sharp$ is a vector bundle $\mathcal{V}$ on the analytic adic space $S \times \mathbb{Z}_p = S \times Spa(\mathbb{Z}_p) = Spa(W(R^+)) \setminus \{[\varpi] = 0\}$ equipped with an isomorphism*

$$\phi_{\mathcal{V}} : Frob_S^*(\mathcal{V})|_{S\times\mathbb{Z}_p\setminus S^\sharp} \xrightarrow{\sim} \mathcal{V}|_{S\times\mathbb{Z}_p\setminus S^\sharp}$$

which is meromorphic along the closed Cartier divisor $S^\sharp$ of $S \times \mathbb{Z}_p$. Here $[\varpi]$ denotes the Teichmuller lift of a pseudo-uniformizer of R^+.

Note that here S lives in characteristic p but $S^\sharp$ is mostly in characteristic zero.

Variants 1.2.
(i) Let $\mathcal{G}/\mathbb{Z}_p$ be a connected smooth group scheme such that its generic fiber G is a reductive group over $\mathbb{Q}_p$. Then there is a notion of a $\mathcal{G}$-shtuka over S with leg along $S^\sharp$.
(ii) In the setting of (i), let μ be a conjugacy class of cocharacters of G. Then there is the notion of a $\mathcal{G}$-shtuka over S with leg along $S^\sharp$ bounded by μ.
(iii) Recall the diamond functor $X \mapsto X^\diamondsuit$ of Scholze from the category of schemes over Spec($\mathbb{Z}_p$) or of formal schemes over Spf($\mathbb{Z}_p$) to v-sheaves on Perf$_{\mathbb{F}_p}$. There is the notion of a shtuka over X*: roughly speaking, a rule that associates to any S-valued point $(S^\sharp, x)$ of $X^\diamondsuit$, where $x : S^\sharp \to X$ is a morphism, a shtuka over S with leg along $S^\sharp$.*

Examples 1.3.
a) (characteristic 0). Let $(\mathcal{G}, G, \mu)$ as above such that μ is minuscule. Let $E = E(G, \mu)$ be the corresponding reflex field. Let X/E be a locally noetherian adic space. Then there is an equivalence between the following two kinds of objects:

(1) *A $\mathcal{G}$-shtuka on X bounded by μ*
(2) *A pair $(\mathbb{P}, H)$, where $\mathbb{P}$ is a pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$-torsor on $X^\diamondsuit$ and $H : \mathbb{P} \to \mathcal{F}_{\mu^{-1}}$ is a $\mathcal{G}(\mathbb{Z}_p)$-equivariant map into the flag variety attached to (G, μ^{-1}).*

b) (characteristic p). Let A be a perfect $\mathbb{F}_p$-algebra essentially of finite type (no topology on A). Then there is an equivalence between the following two kinds of objects:

(1) *A $\mathcal{G}$-shtuka on Spec(A)*
(2) *A pair $(\mathcal{V}, \phi_{\mathcal{V}})$, where $\mathcal{V}$ is a vector bundle on Spec($W(A)$) and $\phi_{\mathcal{V}}$: Frob$^*(\mathcal{V})[\frac{1}{p}] \xrightarrow{\sim} \mathcal{V}[\frac{1}{p}]$ is an isomorphism of vector bundles on Spec($W(A)[\frac{1}{p}]$).*

c) Any abelian variety of dimension n over a $\mathbb{Z}_p$-scheme X defines a shtuka over X of rank $2n$ bounded by the minuscule coweight $(1^{(n)}, 0^{(n)})$ of GL_{2n}.

2. Global Shimura varieties and their p-integral models

Let (G, X) be a Shimura datum, let E be its reflex field and let v be a p-adic place of E. We denote by $\mathrm{Sh}(\mathsf{G}, \mathsf{X})$ the corresponding Shimura variety and by $\mathrm{Sh}(\mathsf{G}, \mathsf{X})_{\mathsf{E}}$ its canonical model over $E = \mathsf{E}_{\mathsf{v}}$. We assume that $\mathsf{K} \subset \mathsf{G}(\mathbb{A}_{\mathsf{f}})$ is of the form $\mathsf{K} = \mathsf{K}_{\mathsf{p}}\mathsf{K}^{\mathsf{p}}$, with $\mathsf{K}_{\mathsf{p}} = \mathcal{G}(\mathbb{Z}_{\mathsf{p}})$, where $\mathcal{G}$ is a smooth model of $G = \mathsf{G} \otimes_{\mathbb{Q}} \mathbb{Q}_{\mathsf{p}}$. We assume that the $\mathbb{Q}$-rank of Z^o coincides with the $\mathbb{R}$-rank of Z^o. Then there is a pro-étale $\mathcal{G}(\mathbb{Z}_p)$-local system $\mathbb{P}_{\mathsf{K}}$ over $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X})_{\mathsf{E}}$ obtained by the system of covers

$$\mathrm{Sh}_{\mathsf{K}'}(\mathsf{G}, \mathsf{X})_{\mathsf{E}} \to \mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X})_{\mathsf{E}},$$

where $\mathsf{K}' = \mathsf{K}'_{\mathsf{p}}\mathsf{K}^{\mathsf{p}} \subset \mathsf{K} = \mathsf{K}_{\mathsf{p}}\mathsf{K}^{\mathsf{p}}$, with K'_{p} running over all compact open subgroups of $\mathsf{K}_{\mathsf{p}} = \mathcal{G}(\mathbb{Z}_{\mathsf{p}})$. By Liu-Zhu, the pro-étale $\mathcal{G}(\mathbb{Z}_p)$-local system $\mathbb{P}_{\mathsf{K}}$ over $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X})_{\mathsf{E}}$ is deRham (and bounded by μ_X). Using the Example a) above, we obtain a $\mathcal{G}$-shtuka $\mathcal{P}_{\mathsf{K},\mathsf{E}}$ over $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X})_{\mathsf{E}}$ with leg bounded by μ_X. Furthermore, $\mathcal{P}_{\mathsf{K},\mathsf{E}}$ are supporting prime-to-p Hecke correspondences, i.e., for $g \in \mathsf{G}(\mathbb{A}_{\mathsf{f}}^{\mathsf{p}})$ and K'^{p} with $g\mathsf{K}'^{\mathsf{p}}\mathsf{g}^{-1} \subset \mathsf{K}^{\mathsf{p}}$, there are compatible isomorphisms $[g]^*(\mathcal{P}_{\mathsf{K},\mathsf{E}}) \simeq \mathcal{P}_{\mathsf{K}',\mathsf{E}}$ which cover the natural morphisms $[g] : \mathrm{Sh}_{\mathsf{K}_{\mathsf{p}}\mathsf{K}'^{\mathsf{p}}}(\mathsf{G}, \mathsf{X})_{\mathsf{E}} \to \mathrm{Sh}_{\mathsf{K}_{\mathsf{p}}\mathsf{K}^{\mathsf{p}}}(\mathsf{G}, \mathsf{X})_{\mathsf{E}}$.

In the next conjecture, there appears the *integral model* $\mathcal{M}^{\rm int}_{\mathcal{G},b,\mu}$ of the *local Shimura variety* $\mathrm{Sh}_{G,b,\mu}$ attached to *local Shimura data* (G,b,μ), in the sense of Scholze [6]: the first object is a v-sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$ and the second object is its generic fiber, a diamond represented by a rigid-analytic variety. The v-sheaf $\mathcal{M}^{\rm int}_{\mathcal{G},b,\mu}$ represents the functor of $\mathcal{G}$-shtukas bounded by μ equipped with a *framing* compatible with b. Scholze conjectures that $\mathcal{M}^{\rm int}_{\mathcal{G},b,\mu}$ is represented by a formal scheme which is normal and flat locally formally of finite type over $\mathrm{Spf}(\mathbb{Z}_p)$. If (G,b,μ) is of RZ-type, then $\mathcal{M}^{\rm int}_{\mathcal{G},b,\mu}$ is representable by the corresponding RZ formal scheme (Scholze [6]).

Conjecture 2.1. *Assume that $\mathcal{G}$ is a parahoric group scheme. There exists a pro-system of normal and flat integral models $\mathcal{S}_{\mathsf{K}}$ over O_E with generic fiber $Sh_{\mathsf{K}}(\mathsf{G},\mathsf{X})_{\mathsf{E}}$, with finite étale transition maps for varying K^{p}, with the following properties.*

a) *For every dvr R of characteristic $(0,p)$ over O_E,*
$$(\varprojlim_{\mathsf{K}^{\mathsf{p}}} Sh_{\mathsf{K}}(\mathsf{G},\mathsf{X})_{\mathsf{E}})(\mathsf{R}[1/\mathsf{p}]) = (\varprojlim_{\mathsf{K}^{\mathsf{p}}} \mathcal{S}_{\mathsf{K}})(\mathsf{R}).$$
b) *The $\mathcal{G}$-shtuka $\mathcal{P}_{\mathsf{K},\mathsf{E}}$ extends to a $\mathcal{G}$-shtuka $\mathcal{P}_{\mathsf{K}}$ on $\mathcal{S}_{\mathsf{K}}$.*

c) *Let $k=\bar{\kappa}_E$. For $x\in\mathcal{S}_{\mathsf{K}}(k)$, with associated $b_x\in G(\breve{\mathbb{Q}}_p)$, there exists an isomorphism of formal completions*
$$\Theta_x\colon \widehat{\mathcal{M}^{int}_{\mathcal{G},b_x,\mu}}_{/x_0} \xrightarrow{\sim} \widehat{\mathcal{S}_{\mathsf{K}_{/x}}},$$
such that the pullback shtuka $\Theta_x^*(\mathcal{P}_{\mathsf{K}})$ coincides with the tautological shtuka on $\mathcal{M}^{int}_{\mathcal{G},b_x,\mu}$ that arises from Scholze's definition of $\mathcal{M}^{int}_{\mathcal{G},b_x,\mu}$ as a v-sheaf moduli space of shtukas. Here x_0 denotes the base point of $\mathcal{M}^{int}_{\mathcal{G},b_x,\mu}$. In particular, $\widehat{\mathcal{M}^{int}_{\mathcal{G},b_x,\mu}}_{/x_0}$ is representable.

Here the element $b_x\in G(\breve{\mathbb{Q}}_p)$ is well-defined up to σ-conjugacy by $\mathcal{G}(\breve{\mathbb{Z}}_p)$, and is determined by the fiber of $\mathcal{P}_{\mathsf{K}}$ at x, as follows. The pull-back $x^*(\mathcal{P}_{\mathsf{K}})$ is a $\mathcal{G}$-shtuka over $\mathrm{Spec}(k)$, and yields, by Example b) above, a $\mathcal{G}$-torsor $\mathcal{P}_x$ over $\mathrm{Spec}(W(k))$ with an isomorphism
$$\phi_{\mathcal{P}_x}\colon \mathrm{Frob}^*(\mathcal{P}_x)[1/p] \xrightarrow{\sim} \mathcal{P}_x[1/p].$$
The choice of a trivialization of the $\mathcal{G}$-torsor $\mathcal{P}_x$ then defines $b_x\in G(\breve{\mathbb{Q}}_p)$.

Theorem 2.2. *Any two such pro-systems of integral models are uniquely isomorphic.*

The main point in the proof of this theorem is a rigidity property of the isomorphisms Θ_x in (c). One shows that the diamond automorphism group of the $\mathcal{G}$-shtuka given by the trivial $\mathcal{G}$-torsor and the Frobenius $\phi_b = b\times\mathrm{Frob}$ has as its global sections over the completed local ring $\widehat{\mathcal{O}}_{\mathcal{S}_{\mathsf{K}},x}$ only the obvious ones, i.e. the σ-centralizer group $J_b(\mathbb{Q}_p)$.

Theorem 2.3. *Let (G,X) be of Hodge type. Also exclude the cases $p=2$ when G is an wildly ramified odd unitary group and $p=3$ when G is a wildly ramified triality group. Then the conjecture above holds true.*

The construction of $\mathcal{S}_{\mathsf{K}}$ is quite straightforward. The Hodge embedding defines a closed embedding of Shimura varieties into the Siegel type Shimura variety,

$$\mathrm{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X})_{\mathsf{E}} \hookrightarrow \mathrm{Sh}_{\mathsf{K}^\flat}(\mathrm{GSp}_{\mathsf{2g}},\mathsf{S}^{\pm}_{\mathsf{2g}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathsf{E}.$$

Here, we need to choose the Siegel moduli level structure $\mathsf{K}^\flat$ so that $\mathsf{K}=\mathsf{K}^\flat \cap \mathsf{G}(\mathbb{A}_{\mathsf{f}})$. After identifying $\mathrm{Sh}_{\mathsf{K}^\flat}(\mathrm{GSp}_{2g},S^{\pm}_{2g})_{\mathbb{Q}}$ with the generic fiber of the Siegel moduli space $\mathcal{A}_{\mathsf{K}^\flat}$ over $\mathbb{Z}_{(p)}$, one defines $\mathcal{S}_{\mathsf{K}}$ as the normalization of the Zariski closure of $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X})_{\mathsf{E}}$ inside $\mathcal{A}_{\mathsf{K}^\flat}\otimes_{\mathbb{Z}_{(p)}} O_E$. Then property (a) for the model $\mathcal{S}_{\mathsf{K}}$ follows from the Néron-Ogg-Shafarevich criterion of good reduction for abelian varieties. For property (b), one realizes the $\mathcal{G}$-shtuka $\mathcal{P}_{\mathsf{K},\mathsf{E}}$ in the generic fiber through some tensors in the pull-back to $\mathcal{S}_{\mathsf{K},\mathsf{E}}$ under the Hodge embedding of the shtuka defined by the Tate module of the universal abelian scheme. Using an analogue for shtukas of Tate's theorem on the extension of homomorphisms of p-divisible groups, these tensors extend over $\mathcal{S}_{\mathsf{K}}$. The crux is now to show that these tensors indeed define a $\mathcal{G}$-shtuka over $\mathcal{S}_{\mathsf{K}}$. Here the main tool is the Anschütz theorem [1] that for an algebraically closed non-archimedean field C of characteristic p, any $\mathcal{G}$-torsor on the punctured spectrum $\mathrm{Spec}(W(O_C))\setminus\{s\}$ is trivial. Finally, property (c) comes down to showing that the pull-back of the $\mathcal{G}$-shtuka $\mathcal{P}_{\mathsf{K}}$ to the completed local ring $\widehat{\mathcal{O}}_{\mathcal{S}_{\mathsf{K}},x}$ admits a framing.

Remark 2.4. *If the conjecture above holds, we obtain a map*

$$\Upsilon_{\mathsf{K}}:\mathcal{S}_{\mathsf{K}}(k)\to G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)_\sigma.$$

The fibers of this map would give the definition of *central leaves* for general Shimura varieties, generalizing the case of Shimura varieties of PEL-type. One would similarly define *Newton strata*, resp. *EKOR strata*, resp. *KR strata* of the special fiber.

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Geometric Langlands and Automorphic Forms

Sam Raskin
(joint work with D. Arinkin, D. Gaitsgory, D. Kazhdan, N. Rozenblyum, and Y. Varshavsky)

1. Introduction

1.1. The aim of this talk is to survey our works [2], [3], and [4], joint with D. Arinkin, D. Gaitsgory, D. Kazhdan, N. Rozenblyum, and Y. Varshavsky.

Related ideas have been pursued by Zhu [12] and Fargues-Scholze [7].

1.2. **Starting point.**

1.2.1. *Arithmetic Langlands.* Let G be a (split, connected) reductive group. Recall that the Langlands conjectures for $G \neq GL_n$ take a subtle, and somewhat unsatisfying form. They do not describe any space of automorphic forms, but *packets* of automorphic representations. Moreover, these packets should correspond to maps from the Langlands group $\mathcal{L}_F$ (F our global field) to the Langlands dual group $\check{G}$. Here the Langlands group is an object whose bare existence amounts to the Langlands functoriality conjectures, which is roughly the best conjecture for general G.

1.2.2. *Geometric Langlands.* By contrast, the geometric Langlands conjecture of Beilinson-Drinfeld is expected to work systematically for general reductive G, and takes a more satisfying form. It asserts:

$$\mathsf{D}(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp}(\mathrm{LocSys}_{\check{G}}).$$

Here we have:

- X is a smooth projective curve over a field k of characteristic zero, all implicit in the above.
- Bun_G is the moduli stack of G-bundles on X.
- $\mathrm{LocSys}_{\check{G}}$ is the moduli stack of *de Rham* (see below) $\check{G}$-local systems on X (see below).
- IndCoh is a variant on QCoh, and the same is true with the subscript $\mathcal{N}ilp$ (though one may refer to [1] for details).

Suffice it to say that there are many compatibilities such an equivalence should satisfy; see [8] for an overdetermined list.

The analogy is that a category of sheaves on a space should be an analogue of functions on the $\mathbb{F}_q$-points of the "corresponding" variety. Therefore, $\mathsf{D}(\mathrm{Bun}_G)$ is an analogue of the space of functions on $G(\mathbb{F}_q(X)\backslash G(\mathbb{A})/G(\mathbb{O}))$, i.e., the space of unramified automorphic forms for some function field.

Therefore, the brief motto is that the geometric Langlands conjecture *completely describes some object analogous to automorphic forms*, even for general reductive G.

1.2.3. *Our work.* In the above works, we formulate a geometric Langlands conjecture that is reasonable over general fields k and for general *sheaf theories*. Working over a field field and with ℓ-adic sheaves, we obtain a conjecture that straddles the arithmetic and geometric settings. Taking traces of Frobenius, we extract a meaningful, novel conjecture for unramified automorphic forms over function fields, refining the usual Langlands conjectures.

We detail this work below.

2. Local systems with restricted variation

One of the main technical accomplishments in our work is the introduction of moduli spaces of local systems for ℓ-adic sheaves. We begin our survey of this work here; see also §4.3.2.

2.1. **Notation.** There are two fields in the story.

First, there is the *geometric field* k, assumed algebraically closed. The (smooth, projective, geometrically connected) curve X lives over k, as do G (which lives over $\mathbb{Z}$, but k is enough here) and Bun_G. For this talk, one should imagine $k = \overline{\mathbb{F}}_q$, with X defined over $\mathbb{F}_q$.

Second, there is the *spectral field* e of characteristic zero, where coefficients live. We assume we have a theory $\mathsf{Shv}(-)$, which is a *sheaf theory* for varieties over k, taking values in e-linear DG categories. Our sheaf categories allow infinite direct sums, so are more like ind-constructible than constructible. The reader should imagine $\mathsf{e} = \overline{\mathbb{Q}}_\ell$ for $\ell \in k^\times$ and Shv = ind-constructible ℓ-adic sheaves. The dual group $\check{G}$ lives over e, as will our moduli space of local systems. All the derived categories we consider are enriched over e.

(In the de Rham setting, this distinction does not occur: $\mathsf{e} = k$.)

We work in the setting of derived algebraic geometry (over e) without mention; if the reader is uncomfortable with this, it can largely be ignored with the only cost that our conjectures lose their validity.

For X of genus 0, we elide a technical distinction about the non-left completeness of $\mathsf{Lisse}(X)$; we refer to [2] for details.

Finally, we assume that the characteristic of k is greater than the Coxeter number of G.

2.2. The moduli space.

2.2.1. For us, moduli spaces are *prestacks*, meaning functors:

$$\mathsf{AffSch}^{op} \to \mathsf{Gpd}.$$

We define $\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}$ to be the prestack over e with S-points the groupoid of right t-exact symmetric monoidal DG functors:

$$\mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(S) \otimes \mathsf{Lisse}(X).$$

Here if $S = \mathrm{Spec}(A)$, then the right hand side is the DG category of A-modules in the category of ind-lisse sheaves on X.

2.2.2. The first basic geometric result describing $\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}$ is:

Proposition 2.1. $\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}$ *is the quotient of an ind-affine indscheme by an action of* $\check{G}$.

In fact, one can say more: each connected component of $\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}$ is the quotient by $\check{G}$ of the formal completion of a scheme almost of finite type along a closed subscheme. This is related to the finite-dimensionality of tangent spaces. This finer description is key in some technical arguments related to later results.

2.2.3. *Examples.* We now briefly, and informally, describe some examples of $\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}$. We pretend there is an actual Artin stack $\mathrm{LocSys}_{\check{G}}^{\mathrm{true}}$ of *all* local systems; this is the case in the de Rham and Betti settings, where the below become precise descriptions.

(1) For $\check{G} = \mathbb{G}_m$, $\mathrm{LocSys}_{\mathbb{G}_m}^{\mathrm{restr}}$ is the disjoint union of the formal completions of $\mathrm{LocSys}_{\check{G}}^{\mathrm{true}}$ at all its closed points.
(2) For $\mathbb{G}_a$ (which of course, is not reductive, but the definition still makes sense), we have:

$$\mathrm{LocSys}_{\mathbb{G}_a}^{\mathrm{restr}} = \mathrm{LocSys}_{\mathbb{G}_a}^{\mathrm{true}} = C^\bullet(X, \mathsf{e})[1].$$

Here the right hand side is thought of as a stack over e encoding the cohomology of X; explicitly, it looks like $\mathbb{B}\mathbb{G}_a \times H^1(X) \times \Omega H^2(X)$. In particular, for genus > 0, there are non-trivial physical directions coming from $H^1(X)$.

For general reductive $\check{G}$, the result mixes the above two examples. Specifically, we show that $\pi_0(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}})$ is indexed by *possible semi-simplification types* of $\check{G}$-local systems, and, roughly speaking, looks like the disjoint union of $\mathrm{LocSys}_{\check{G}}^{\mathrm{true}}$ along the locally closed substacks given by fixed semi-simplification type.

3. Restricted geometric Langlands

3.1. **Formulation of the conjecture.** We can now briefly state:

Conjecture 3.1 (Restricted geometric Langlands conjecture) *In the above setting, there is a canonical equivalence of DG categories*:

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}).$$

Again, there are many compatibilities, generally extracted from following ones nose in reading [8]: all well-posed statements should still hold.

In the above, $\mathsf{Shv}_{\mathcal{N}ilp}$ indicates *sheaves with singular support in the global nilpotent cone.* For ℓ-adic sheaves, the relevant singular support theory was developed by Beilinson in [5].

The formulation of this conjecture follows that of Ben-Zvi–Nadler [6] in the Betti setting.

3.1.1. *Evidence.* The restricted geometric conjecture as formulated above is true for $G = \mathbb{G}_m$ (or any torus). It is also true for arbitrary G and $X = \mathbb{P}^1$. Finally, we show that its truth for $\text{char}(k) = 0$ is equivalent to that of the Beilinson-Drinfeld conjecture.

Finally, we prove that there is a canonical action of $\mathsf{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}})$ on $\mathsf{Shv}_{\mathcal{N}ilp}(\text{Bun}_G)$ that (informally speaking) refines the action of Hecke functors on $\mathsf{Shv}(\text{Bun}_G)$. The proof uses the same method as the similar Betti result, which is due to Nadler-Yun [9]. Below, we refer to this action as the *spectral decomposition* of $\mathsf{Shv}_{\mathcal{N}ilp}(\text{Bun}_G)$.

4. Automorphic functions

4.1. **Setting.** In this section, we discuss consequences for unramified automorphic forms over function fields.

It is now important that $k = \overline{\mathbb{F}}_q$, $\mathsf{e} = \overline{\mathbb{Q}}_\ell$, sheaves are ℓ-adic, and X is defined over $\mathbb{F}_q$. All Frobenii are geometric, which we consider as a map $\text{Frob}_X : X \to X$ over $\text{Spec}(k)$.

4.2. **Categorical traces.** DG categories form a symmetric monoidal category, so it makes sense to speak of dualizability, traces, and other familiar concepts from linear algebra.

For DG categories of modules over rings, trace constructions have easy explicit meaning. In particular, this applies for D-modules. However, for DG categories of ℓ-adic sheaves, the situation is much more difficult; the results we detail below are quite specific to Bun_G.

4.3. Arithmetic local systems.

4.3.1. In the above setting, we obtain a "Frobenius" automorphism $\text{Frob} : \text{LocSys}^{\text{restr}}_{\check{G}} \to \text{LocSys}^{\text{restr}}_{\check{G}}$ defined by pullback of local systems along Frob_X. We emphasize that this is a map of prestacks over $\mathsf{e} = \overline{\mathbb{Q}}_\ell$. For $\mathbb{G}_a$, $\text{LocSys}^{\text{restr}}_{\mathbb{G}_a}$ encodes the étale cohomology of X, and the resulting automorphism corresponds to the Frobenius automorphism acting on étale cohomology.

4.3.2. We define:

$$\text{LocSys}^{\text{arthm}}_{\check{G}} := (\text{LocSys}^{\text{restr}}_{\check{G}})^{\text{Frob}=\text{id}}.$$

That is, the left hand side parametrizes $\sigma \in \text{LocSys}^{\text{restr}}_{\check{G}}$ plus an isomorphism $\text{Frob}^*_X(\sigma) \simeq \sigma$. One should think of $\text{LocSys}^{\text{arthm}}_{\check{G}}$ as the moduli of *Weil representations.*

We show that, unlike $\text{LocSys}^{\text{restr}}_{\check{G}}$, $\text{LocSys}^{\text{arthm}}_{\check{G}}$ has quite favorable geometric properties: it is a quasi-compact algebraic stack. From this perspective, elliptic Langlands parameters correspond to isolated smooth points in $\text{LocSys}^{\text{restr}}_{\check{G}}$.

Remark 4.3.2.1. Even though $\text{LocSys}^{\text{arthm}}_{\check{G}}$ is *nicer* in many respects than $\text{LocSys}^{\text{restr}}_{\check{G}}$, its geometry is more opaque. For instance, what are its connected components?

4.3.3. The formalism of traces easily implies:

(1) $$\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}) \simeq \mathrm{tr}_{\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}})-\mathsf{mod}}(\mathrm{Frob}^*).$$

Informally, the point is that categorical traces relate the the circle $S^1 = \mathbb{BZ}$, and $\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}$ is by definition the fixed points of a $\mathbb{Z}$-action on $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}$.

4.4. **The trace conjecture.**

4.4.1. Pull-back along $\mathrm{Frob}_{\mathrm{Bun}_G}$ defines an endofunctor (in fact, autoequivalence):

$$\mathrm{Frob}_{\mathrm{Bun}_G,*} : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G).$$

We prove:

Theorem 4.4.1.1 ([4]) *There is a canonical isomorphism*:

$$\mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q)) \simeq \mathrm{tr}_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)}(\mathrm{Frob}_{\mathrm{Bun}_G,*}) \in \mathsf{Vect}_{\mathsf{e}}.$$

We show that the isomorphism comes from the Grothendieck-Deligne sheaves-to-functions construction. Our theorem can be imagined as a higher categorical analogue of their idea, except that their idea is general, and ours is particular to the automorphic setting.

4.4.2. Using the spectral action from §3.1.1, one can form an enriched trace:

$$\mathrm{tr}^{\mathrm{enh}}_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)}(\mathrm{Frob}_{\mathrm{Bun}_G,*}) \in \mathrm{tr}_{\mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}})-\mathsf{mod}}(\mathrm{Frob}^*) \simeq \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}).$$

We refer to the result object as the *Drinfeld sheaf* $\mathrm{Drinf} \in \mathsf{QCoh}(\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}})$. Its basic property is:

$$\Gamma(\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}, \mathrm{Drinf}) \simeq \mathrm{tr}_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)}(\mathrm{Frob}_{\mathrm{Bun}_G,*}) \simeq \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

The resulting action of $\mathrm{Fun}(\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}})$ on $\mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ is given by V. Lafforgue's *excursion operators* in a certain sense.

4.4.3. In some sense, the logic of the previous subsection is misleading.

In point of fact, we prove Theorem 4.4.1.1 by a result of Xue [11], valid for arbitrary non-proper curves. Roughly speaking, we deduce a *reciprocity law for shtuka cohomologies* from her theorem; this result yields non-obvious isomorphisms between some shtuka cohomology groups. Our proof of Theorem 4.4.1.1 proceeds by first calculating $\mathrm{tr}_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)}(\mathrm{Frob}_{\mathrm{Bun}_G,*})$ to obtain a certain shtuka cohomology group (related to Beilinson's construction of spectral projectors), and then applying reciprocity to yield the desired answer.

We say this in fact to highlight: Xue's theorem and the reciprocity law yield a direct construction of the Drinfeld sheaf also for non-proper curves; in particular, it may be constructed directly, without mention of the category $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$. It is, of course, desirable to better understand the meaning of this sheaf in the ramified setting in terms of some automorphic sheaves.

4.5. **Conjectures and speculations.**

4.5.1. We make a (quite mild) conjecture:

$$\mathrm{tr}_{\mathsf{IndCoh}_{\mathcal{N}ilp}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}})}(\mathrm{Frob}^*) \xrightarrow{\simeq} \mathrm{tr}_{\mathsf{IndCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}})}(\mathrm{Frob}^*) \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}, \omega).$$

The second isomorphism is a standard calculation. The first should be a general result related to the contracting property of Frobenius on the spectral nilpotent cone.

4.5.2. Assuming the above, we obtain:

Conjecture 4.5.1: *There is an isomorphism*:

$$\mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q)) \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}, \omega).$$

For instance, the isomorphism should be compatible with Hecke operators.

Example 4.5.2.1. For $X = \mathbb{P}^1$, one can explicitly verify the above conjecture. Here $\mathrm{Bun}_G(\mathbb{F}_q) = \check{\Lambda}^+$ (up to negligable stackyness), while $\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}} \simeq \check{G}/\check{G}$; the latter is well-known to have global functions:

$$\Gamma(\check{T}, \mathcal{O})^W = \mathsf{e}[\check{\Lambda}]^W = \mathsf{e}[\check{\Lambda}] = \mathrm{Fun}_c(\check{\Lambda}^+).$$

Finally, we remark that $\mathcal{O} = \omega$ on $\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}$.

One may similarly treat $G = \mathbb{G}_m$ explicitly; we omit the details here.

Thea above conjecture describes the space of unramified automorphic functions in spectral terms. There are no L-packets, or cuspidal/discrete series/temperedness restrictions, or subtleties about multiplicities.

For now, one can only dream what a generalization of this conjecture might say in the ramified setting, or over number fields; for some speculations, see [10].

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Categorical Künneth formula for Weil sheaves

Timo Richarz
(joint work with Tamir Hemo, Jakob Scholbach)

Künneth-type formulas relate the cohomology of product spaces to the tensor product of the cohomology of the single factors. In [3, Section A.2], this is upgraded to an equivalence of categories in the setting of topological sheaves and D-modules on varieties in characteristic 0. In characteristic $p > 0$, such categorical Künneth formulas fail for (pro-)étale sheaves by the example below, in fact already for étale fundamental groups. This failure can be rectified by introducing equivariance data under partial Frobenius morphisms [2, Theorem 2.1]. This is known as Drinfeld's lemma and a crucial stepping stone in recent approaches to Langlands parametrizations. When adding such equivariance data in a sheaf-theoretic context, one arrives at the notion of Weil sheaves. In my talk, I reported on recent joint work with Tamir Hemo and Jakob Scholbach [4] where we upgrade Drinfeld's lemma to a categorical Künneth formula for constructible Weil sheaves.

Main result. Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$, and fix an algebraic closure $\mathbb{F}$. Let $\ell \neq p$ be a prime. For a scheme X over $\mathbb{F}_q$, we denote by $\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \overline{\mathbb{Q}}_\ell)$ the category of Weil sheaves, that is, constructible complexes of $\overline{\mathbb{Q}}_\ell$-sheaves M on $X_{\mathbb{F}}$ together with an isomorphism $M \cong \phi_X^* M$ where $\phi_X = \mathrm{Frob}_X \times \mathrm{id}_{\mathbb{F}}$ is the partial q-Frobenius. The following theorem is referred to as the categorical Künneth formula (or derived Drinfeld's lemma):

Theorem. *Let X_1, X_2 be finite type schemes over $\mathbb{F}_q$. Then the external product functor $(M_1, M_2) \mapsto M_1 \boxtimes M_2$ induces an equivalence*

$$\mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell} \mathrm{D}_{\mathrm{cons}}(X_2^{\mathrm{Weil}}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}} \times X_2^{\mathrm{Weil}}, \overline{\mathbb{Q}}_\ell), \tag{1}$$

where the target is the category of sheaves M on $X_{1,\mathbb{F}} \times X_{2,\mathbb{F}}$ equipped with commuting isomorphisms $M \cong \phi_{X_i}^* M$ for $i = 1, 2$.

So, colloquially speaking, a constructible sheaf on a product of varieties in positive characteristic equipped with partial Frobenii is generated by external products of sheaves on the single factors. In [4], we prove variants of the theorem for torsion coefficients, for integral coefficients, for lisse sheaves and their ind-variants. The latter variant applies to the cohomology of shtuka spaces when combined with recent results of Xue [7]. The presence of the partial Frobenii is necessary as the following example shows:

Example. Let $X_1 = X_2 = \mathbb{A}^1_{\mathbb{F}_q}$ be the affine line with coordinates denoted by x_1 and x_2. Then the Artin-Schreier equation

$$U := \{t^p - t = x_1 \cdot x_2\} \longrightarrow \mathbb{A}^2_{\mathbb{F}}$$

defines a finite étale $\mathbb{Z}/p$-cover. Let $M \in \mathrm{D}_{\mathrm{lis}}(\mathbb{A}^2_{\mathbb{F}}, \overline{\mathbb{Q}}_\ell)$ be the sheaf associated with some non-trivial character $\mathbb{Z}/p \to \overline{\mathbb{Q}}_\ell^\times$. For $\lambda, \mu \in \mathbb{F}$ not differing by a scalar in $\mathbb{F}_p^\times$, the fibers $U|_{\{x_1=\lambda\}}$, $U|_{\{x_1=\mu\}}$ are not isomorphic over $\mathbb{A}^1_{\mathbb{F}}$ by Artin-Schreier theory. Hence, $M \not\cong \phi^*_{X_i} M$ and one can show that $M \not\cong M_1 \boxtimes M_2$ for any $M_i \in \mathrm{D}(\mathbb{A}^1_{\mathbb{F}}, \mathbb{Q}_\ell)$.

Roughly, the fully faithfulness of (1) is implied by the Künneth formula for constructible $\overline{\mathbb{Q}}_\ell$-sheaves whereas the essential surjectivity relies on Drinfeld's lemma as explicated in [5, Theorem 8.1.4] and [6, Lemma 3.3.2] for $\overline{\mathbb{Q}}_\ell$-coefficients. The tensor product in (1) is induced from Lurie's tensor product which requires the use of suitable ∞-categorical enrichments. We use the proétale topology introduced by Bhatt–Scholze [1] to define the above categories of Weil sheaves.

Categories of constructible sheaves. Let X be a qcqs scheme and Λ a condensed ring. Let $\Lambda_X = p_X^{-1}\Lambda$ be the pullback along the map of proétale sites $p_X\colon X_{\text{proét}} \to *_{\text{proét}}$. We denote by $\mathrm{D}(X, \Lambda)$ the derived category of sheaves of Λ_X-modules on $X_{\text{proét}}$.

Definition. A sheaf $M \in \mathrm{D}(X, \Lambda)$ is *lisse* if it is dualizable for the derived tensor product $\otimes_{\Lambda_X}$, and *constructible* if it is lisse along a constructible stratification.

The resulting full subcategories are denoted by $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$. The definition allows extremely general rings of coefficients including all T1-topological rings. The following lemma allows for the comparison with the categories of ℓ-adic sheaves defined by Deligne, Ekedahl and Bhatt–Scholze:

Key lemma. *For any w-contractible affine scheme X, there is an equivalence*

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \xrightarrow{\cong} \mathrm{Perf}_{\Gamma(X,\Lambda)}, \;\; M \mapsto \mathrm{R}\Gamma(X, M),$$

where $\mathrm{Perf}_{\Gamma(X,\Lambda)}$ *is the category of perfect complexes of* $\Gamma(X, \Lambda)$*-modules.*

Proof. As X is w-contractible affine, the functor $\mathrm{R}\Gamma(X, \text{-}) = \Gamma(X, \text{-})$ is monoidal. So, if M is lisse, the complex $\mathrm{R}\Gamma(X, M)$ is dualizable, hence perfect. For fully faithfulness, we observe that $\underline{\mathrm{Hom}}(M, N) \cong M^\vee \otimes N$ by dualizability of M. We conclude by applying the monoidal functor $\mathrm{R}\Gamma(X, \text{-})$. For essential surjectivity, pick $K \in \mathrm{Perf}_{\Gamma(X,\Lambda)}$ and define $K_X := \underline{K} \otimes_{\underline{\Gamma(X,\Lambda)}} \Lambda_X$. Then $\mathrm{R}\Gamma(X, K_X) \cong K$ as desired. □

Lisse and constructible sheaves start out life in a derived setting. The standard t-structure on $\mathrm{D}(X, \Lambda)$ restricts to a t-structure on the subcategories whenever the underlying topological space $|X|$ is Noetherian and the underlying ring $\Lambda_* = \Gamma(*, \Lambda)$ is semi-hereditary. This applies, for example, to algebraic field extensions $E \supset \mathbb{Q}_\ell$ or their rings of integers $\mathcal{O}_E$. For geometrically unibranch schemes X, Section 7 of [1] induces an equivalence of categories

$$\mathrm{D}_{\mathrm{lis}}(X, \overline{\mathbb{Q}}_\ell)^\heartsuit \cong \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(X)), \tag{2}$$

where $\pi_1(X)$ denotes the étale fundamental groupoid. For a derived statement in the spirit of Barwick–Glasman–Haine's exodromy theorem, the reader is referred to [4, Appendix A]. The equivalence (2) allows to use Drinfeld's lemma for étale covers when proving the essential surjectivity of (1).

We pursue the following site-theoretic approach to Weil sheaves which slightly differs from earlier work of Lichtenbaum and Geisser on the Weil-étale topology:

Definition. For a scheme X over $\mathbb{F}_q$, the *Weil-proétale site* $X^{\mathrm{Weil}}_{\mathrm{proét}}$ is the site with objects (U, φ) where $U \in (X_{\mathbb{F}})_{\mathrm{proét}}$ and $\varphi\colon U \to U$ is an endomorphism of $\mathbb{F}$-schemes covering the partial Frobenius ϕ_X, with morphisms given by equivariant maps and with covers induced from $(X_{\mathbb{F}})_{\mathrm{proét}}$.

We let $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ be the resulting derived category. The subcategories of lisse, respectively constructible sheaves $\mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda)$ are defined as the dualizable objects, respectively those being dualizable along a constructible stratification of X.

Proposition. *For* $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$*, pullback of sheaves along the map of sites* $(X_{\mathbb{F}})_{\mathrm{proét}} \to X^{\mathrm{Weil}}_{\mathrm{proét}}$ *induces an equivalence*

$$\mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)^{\phi_X = \mathrm{id}} \tag{3}$$

where the target denotes the homotopy fixed points.

Amongst other things, the proof uses that the projection $X_{\mathbb{F}} \to X$ induces a homeomorphism $|X_{\mathbb{F}}|_{\mathrm{cons}}/\phi_X \cong |X|_{\mathrm{cons}}$ in the constructible topology. We conclude from (3) that for geometrically unibranch schemes X, the equivalence (2) applied to $X_{\mathbb{F}}$ descends to an equivalence

$$\mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \overline{\mathbb{Q}}_\ell)^{\heartsuit} \cong \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Weil}(X)), \tag{4}$$

where $\mathrm{Weil}(X) = \pi_1(X_{\mathbb{F}})/\phi_X^{\mathbb{Z}}$ is the Weil groupoid defined by Deligne.

Similarly, for two schemes X_1, X_2 over $\mathbb{F}_q$, we introduce the site $(X_1^{\mathrm{Weil}} \times X_2^{\mathrm{Weil}})_{\mathrm{proét}}$ consisting of triples $(U, \varphi_1, \varphi_2)$ where $U \in (X_{1,\mathbb{F}} \times X_{2,\mathbb{F}})_{\mathrm{proét}}$ and $\varphi_i\colon U \to U$ covers the partial Frobenius ϕ_{X_i}. Following the above recipe, this allows to define the category of constructible Λ-sheaves on $X_1^{\mathrm{Weil}} \times X_2^{\mathrm{Weil}}$ appearing in the target of (1). The variant of (3) for two factors holds true and relies on decompositions of partial Frobenius invariant cycles on $X_{1,\mathbb{F}} \times X_{2,\mathbb{F}}$ in terms of cycles on the single factors. From here, dévissage arguments reduce the fully faithfulness of (1) to the Künneth formula for ℓ-adic cohomology and the essential surjectivity to the classical Drinfeld's lemma.

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A Geometric Approach to the Ramanujan Conjecture for Automorphic Forms over Function Fields

Will Sawin
(joint work with Nicolas Templier)

In joint work with Nicolas Templier [7], we proved a special case of the Ramanujan conjecture over function fields. This was done for automorphic representations that, at one place, satisfy a strong local condition we call monomial geometric supercuspidal.

Let $\mathbb{F}_q$ be a finite field and G a reductive group. Let π be an admissible irreducible representation of $G(\mathbb{F}_q((t)))$. We say that π is *monomial geometrically supercuspidal* if there exists an algebraic subgroup H of $G[[t]]$ and character sheaf $\mathcal{L}$ on H such that

(1) π is a quotient of $\text{c-Ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q((t)))}\left(\chi^{\mathcal{L},\mathbb{F}_q}\right)$.
(2) For all $e \in \mathbb{N}$ and for all parabolic subgroups of $G_{\mathbb{F}_{q^e}}$, the Jacquet module of $\text{c-Ind}_{H(\mathbb{F}_{q^e})}^{G(\mathbb{F}_{q^e}((t)))}\left(\chi^{\mathcal{L},\mathbb{F}_{q^e}}\right)$ vanishes.

Our main result is that, for G a split semisimple group over a function field $\mathbb{F}_q(X)$ of odd characteristic and π an automorphic representation of $G(\mathbb{A}_F)$ which satisfies the following:

(1) At one place, π_v is a monomial geometric supercuspidal representation.
(2) For each e, the cyclic base change Π^e of π to $\mathbb{F}_{q^e}(X)$ exists. At each ramified place away from v, the depth of Π^e_v is bounded independently of e. At v, Π^e_v is a quotient of $\text{c-Ind}_{H(\mathbb{F}_{q^e})}^{G(\mathbb{F}_{q^e}((t)))}\left(\chi^{\mathcal{L},\mathbb{F}_{q^e}}\right)$.

is tempered at every unramified place.

This result is mainly novel in the case of G an exceptional group, because L. Lafforgue [4] established the Ramanujan conjecture for GL_r over function fields (building on earlier work of Drinfeld [3]) and Lomelí handled the case of split classical groups [5].

The methods of proof are novel in every case. The proof in the GL_r case [4] relies on the construction of Langlands parameters from the cohomology of moduli spaces of shtukas, along the lines of the proof of the classical Ramanujan conjecture by Deligne [2], and the proof in the classical group case [5] reduces to the GL_r case by endoscopy.

Our proof does not rely on Langlands parameters, shtukas, or any case of functoriality aside from cyclic base change. Instead, we study the traces of Hecke operators on a family of automorphic forms.

We first set up a family of automorphic forms satisfying local conditions. We can express this family more classically in terms of a space of automorphic forms, the space of functions on an adelic double coset space that transform according to a certain character along a certain group action. We prove a bound for the trace of a Hecke operator acting on this space by geometry.

Specifically, we fix geometric data defining our family and, using this, express the adelic double coset space as the $\mathbb{F}_q$-points of a stack $\mathrm{Bun}_G(D)$, and the kernel of a Hecke operator T_λ as the trace function of a complex of sheaves on $\mathrm{Bun}_G(D) \times \mathrm{Bun}_G(D)$. We prove that this complex is in fact a perverse sheaf by compactifying the Hecke correspondence and proving a cleanness property. The perversity then implies a bound for the sum of the square of the trace function of the sheaf, equivalenty, the integral of the square of the Hecke kernel of T_λ, which equals the trace of the Hecke operator $T_\lambda T_\lambda^*$.

Our bound for the trace of $T_\lambda T_\lambda^*$ then implies a bound for the eigenvalue of the Hecke operator T_λ acting on an individual form generating the representation π. This bound, however, is not strong enough for our purposes. To remedy this, we fix all the geometric data but then pass from the base finite field $\mathbb{F}_q$ to a larger field $\mathbb{F}_{q^e}$. Since our geometric argument is insensitive to the choice of field, we obtain the same bound for a larger family of automorphic forms in this setting. By assumption on cyclic base change, the original form π has a base change Π_e in this new family, so we obtain a bound for the eigenvalue of T_λ on the cyclic base change. Combining this information for all e, we get a better bound for the eigenvalue of T_λ on the original π, and this implies Ramanujan.

I close with some questions, mostly on generalizations or strengthenings of this work.

Question 1: In [7], we showed that the simple supercuspidal representations constructed by Reeder and Yu [6] as well as many of the toral supercuspidal representations constructed by Adler [1] are monomial geometric supercuspidal. How many of the representations constructed by Yu [8] can be shown to be monomial geometric supercuspidal?

Question 2: To what extent can one generalize the proof from monomial representations to other representations?

Question 3: Can the assumption on cyclic base change be removed or replaced with other local assumptions by a function field proof of cyclic base change?

Question 4: When, using Hecke operators at ramified places, can one prove temperedness at ramified placed by this method?

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Relative perversity

Peter Scholze
(joint work with David Hansen)

We observe that for a family of schemes $f : X \to S$ of finite presentation, there is a relative perverse t-structure on ℓ-adic sheaves on X, with relative perversity equivalent to perversity of the restrictions to all geometric fibres of f.

More precisely, fix a prime ℓ. We assume that all schemes are qcqs, and live over $\mathbb{Z}[\frac{1}{\ell}]$. Let $f : X \to S$ be a morphism of finite presentation between such schemes. In this note, we will state the results with $\overline{\mathbb{Q}}_\ell$-coefficients, although everything works similarly for example with $\mathbb{Z}_\ell$ or $\mathbb{Z}/\ell^n$-coefficients. We denote $D(X) = D^b_c(X, \overline{\mathbb{Q}}_\ell)$. For technical reasons having to do with this choice of coefficients, we have to assume that all constructible subsets of S have only finitely many connected components (without this condition, the theorem is demonstrably false already for $X = S$).

Theorem 1.1. *There is a (necessarily unique) t-structure* $({}^{p/S}D^{\leq 0}, {}^{p/S}D^{\geq 0})$ *on* $D(X)$*, called the relative perverse t-structure, with the following property:*

An object $A \in D(X)$ *lies in* ${}^{p/S}D^{\leq 0}$ *(resp.* ${}^{p/S}D^{\geq 0}$*) if and only if for all geometric points* $\overline{s} \to S$ *with fibre* $X_{\overline{s}} = X \times_S \overline{s}$*, the restriction* $A|_{X_{\overline{s}}} \in D(X_{\overline{s}})$ *lies in* ${}^pD^{\leq 0}$ *(resp.* ${}^pD^{\geq 0}$*), for the usual (absolute) perverse t-structure.*

The proof has two ingredients: v-descent, and the theory of nearby cycles. Roughly speaking, v-descent allows us to reduce to the case that S is the spectrum of a valuation ring V, and one can even assume that its fraction field K is algebraically closed. In that case the theorem is closely related to the t-exactness properties of nearby cycles, with respect to the perverse t-structure.

Let us first state the result regarding v-descent.

Theorem 1.2 (Bhatt–Mathew [1], Gabber [2])**.** *The association* $X \mapsto \mathcal{D}(X)$ *defines an v-sheaf (even arc-sheaf) of* ∞*-categories.*

A related result of Gabber's is worth stating separately, as it is about general étale sheaves (without abelian group structure).

Theorem 1.3 ([2])**.** *Sending any scheme X to the category of étale sheaves on X defines a stack with respect to universal submersions.*[1] *In particular, sending any*

[1]In [2], Gabber also sketches an extension of this result to the case where one sends X to the $(2,1)$-category of ind-finite étale stacks.

scheme X to the category of separated étale maps of schemes $Y \to X$ defines a stack with respect to universal submersions, and in particular a v-stack.

This strengthens some previous descent results, notably by Rydh [4], [1, Theorem 5.6].

Using these descent results and some approximation arguments, we can reduce Theorem 1.1 to the case that $S = \mathrm{Spec} V$ where V is a valuation ring with algebraically closed fraction field K; one can even assume that V is of rank 1.

In that case, we use the theory of nearby cycles. The following two theorems are essentially due to Lu–Zheng [3].

Theorem 1.4. *Let $f : X \to S$ be a map of finite presentation between qcqs schemes and let $A \in D(X)$. The following conditions are equivalent.*

(i) *The pair (X, A) defines a dualizable object in the symmetric monoidal 2-category of cohomological correspondences over S.*

(ii) *The following condition holds after any base change in S. For any geometric point $\overline{x} \to X$ mapping to a geometric point $\overline{s} \to S$, and a generization $\overline{t} \to S$ of $\overline{s}$, the map*
$$A|_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}}, A)$$
is an isomorphism.

(iii) *The following condition holds after any base change in S. For any geometric point $\overline{x} \to X$ mapping to a geometric point $\overline{s} \to S$, and a generization $\overline{t} \to S$ of $\overline{s}$, the map*
$$A|_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} \overline{t}, A)$$
is an isomorphism.

(iv) *After base change along $\mathrm{Spec} V \to S$ for any rank 1 valuation ring V with algebraically closed fraction field K and any geometric point $\overline{x} \to X$ mapping to the special point of $\mathrm{Spec} V$, the map*
$$A|_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \to R\Gamma(X_{\overline{x}} \times_{\mathrm{Spec} V} \mathrm{Spec} K, A)$$
is an isomorphism.

Moreover, these conditions are stable under any base change, and can be checked arc-locally on S.

In particular, this shows that the key to understanding universal local acyclicity is the case where the base is the spectrum of a (rank 1) valuation ring with algebraically closed fraction field. The key result is the following, which rederives all the basic properties of the nearby cycles functor.

Theorem 1.5. *Let X be a scheme of finite presentation over $S = \mathrm{Spec} V$, where V is a valuation ring with algebraically closed fraction field K. Let $j : X_K \subset X$ be the inclusion of the generic fibre. The restriction functor*
$$j^* : D^{\mathrm{ULA}}(X/S) \to D(X_K)$$
is an equivalence, whose inverse is given by Rj_*.

In particular, the formation of Rj_* preserves constructibility, and commutes with any flat base change $V \to V'$ of valuation rings with algebraically closed fraction fields, with relative Verdier duality, and satisfies a Künneth formula.

Moreover, relative perversity interacts well with universal local acyclicity. More precisely:

Theorem 1.6. *Relative perverse truncation preserves universal local acyclicity.*

By Theorem 1.6, we get in particular a well-behaved abelian category $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ of relatively perverse universally locally acyclic sheaves over S. Our final result concerns properties of this abelian category.

Theorem 1.7. *Assume that S is connected.*

(i) *Let $\overline{s} \to S$ be any geometric point, with $i = i_{\overline{s}} : X_{\overline{s}} \to X$ the inclusion of the fibre. The restriction functor*

$$i^* : \mathrm{Perv}^{\mathrm{ULA}}(X/S) \to \mathrm{Perv}(X_{\overline{s}})$$

is an exact and faithful functor of abelian categories, and the category $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is noetherian and artinian.

(ii) *Assume that S is geometrically unibranch; let $\eta = \mathrm{Spec} K \subset S$ be the (necessarily unique) generic point, with $j : X_\eta \subset X$ the inclusion. The restriction functor*

$$j^* : \mathrm{Perv}^{\mathrm{ULA}}(X/S) \to \mathrm{Perv}(X_\eta)$$

is exact and fully faithful, and its image is stable under subquotients.

This allows one to recover some results of Gaitsgory, on preservation of universal local acyclicity under passage to perverse cohomologies, and perverse subquotients.

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Orthogonal root numbers and the refined formal degree conjecture

David Schwein

Let k be a local field. If k is nonarchimedean, let p be its residue characteristic and let q be the cardinality of the residue field of k. Let $\underline{G}$ be a reductive k-group, let $G = \underline{G}(k)$, and let $\underline{A}_G$ the split component of the center of $\underline{G}$. Let Γ_k be the Galois group, let W_k be the Weil group, let $\mathrm{WD}_k \stackrel{\mathrm{def}}{=} W_k \times \mathrm{SL}_2(\mathbb{C})$ be the Weil-Deligne group, and let

$$L_k \stackrel{\mathrm{def}}{=} \begin{cases} \mathrm{WD}_k & \text{if } k \text{ is nonarchimedean} \\ W_k & \text{if } k \text{ is archimedean} \end{cases}$$

be the Langlands group. For all local (that is, L-, ε-, and γ-) factors, we use the additive character of k for which $\mathcal{O}_k$ is the largest fractional ideal of k in the kernel and use the Haar measure on k that assigns volume one to $\mathcal{O}_k$.

Raphaël's talk introduced us to the following conjecture of Hiraga, Ichino, and Ikeda.

Conjecture 0.1 ([5, Conjecture 1.4]). *Let π be a(n essentially) discrete series representation of G with extended parameter (φ, ρ). Then*

$$d(\pi, \nu_G) = \frac{\dim \rho}{|S_\varphi|} \cdot |\gamma(0, \varphi, \mathrm{Ad}_G)|.$$

Here $d(\pi)$ is the formal degree of π,

$$\gamma(s, \varphi, \mathrm{Ad}_G) \overset{\mathrm{def}}{=} \varepsilon(s, \mathrm{Ad}_G \circ \varphi) \cdot \frac{L(1-s, \mathrm{Ad}_G \circ \varphi)}{L(s, \mathrm{Ad}_G \circ \varphi)},$$

ν_G is the measure on G defined in Raphaël's talk using a volume form on the Chevalley model of $\underline{G}$, and Ad_G is the adjoint representation of ${}^L G$ on $\widehat{\mathfrak{g}}/\hat{\mathfrak{z}}^\Gamma$. The precise definitions of ρ and S_φ are not so important to us; let me say only that S_φ is built from the centralizer of φ and ρ is a finite-dimensional irreducible representation of a group similar to S_φ.

In this talk I'll explain some of my work on understanding the conjecture more fully.

1. Refined formal degree

In this section, we assume k is nonarchimedean. Gross and Reeder [3, Section 7] refined Conjecture 0.1 to remove the absolute value bars and interpret the sign of the γ-factor. This sign is a quotient of root numbers. Recall that the root number of a representation $\varphi\colon W_k \to \mathrm{GL}(V)$ is the number $\omega(\varphi)$ defined by the formula

$$\varepsilon(s, \varphi) = \omega(\varphi) q^{\mathrm{cond}(\varphi)(1/2-s)}$$

where $\mathrm{cond}(\varphi)$ is the Artin conductor. The root number has modulus one, and even better, when φ is self-dual, it is a fourth root of unity because

$$\omega(\varphi)^2 = (\det \varphi)(-1).$$

(Use Artin reciprocity to make sense of the righthand side.) Deligne has given an interpretation of the sign in terms of Stiefel-Whitney classes. His theorem has several variations, the simplest of which says that when $\det \varphi = 1$, the root number $\omega(\varphi)$ is $+1$ if and only if φ lifts to the spin double-cover $\mathrm{Spin}(V)$ of $\mathrm{SO}(V)$ (and otherwise the root number is -1).

Since there are good formulas to compute the Artin conductor [19, Chapter VI], most of the mystery of the ε-factor lies in the root numbers, and these numbers typically carry deep information. For instance, we saw in Wee Teck's talk that symplectic root numbers are expected to carry information about branching problems for classical groups. In this talk, we will see that orthogonal root numbers carry information about central characters of irreducible admissible representations of G.

The starting point for Gross and Reeder's refinement is to normalize the Haar measure so that the Steinberg representation St_G has formal degree one. Such a measure, called the Poincaré measure, had already been studied by Serre [18]. It satisfies the following properties.

First, for every discrete, cocompact, torsion-free subgroup $\Lambda \subseteq G$, the Euler characteristic of (the rational group cohomology of) Λ is $\chi\big(\mathrm{H}^\bullet(\Lambda, \mathbb{Q})\big) = \mathrm{vol}(\Lambda\backslash G, \mu_G)$. This property was Serre's motivation.

Second, $\mu_G \neq 0$ if and only if $A_G = 1$. So we must assume that $A_G = 1$ for μ_G to be of interest.

Third, the Poincaré measure is a Haar measure up to sign: specifically, $(-1)^{r(G)}\mu_G$ is a Haar measure, where $r(G)$ is the split rank of G.

Fourth, $d(\mathrm{St}_G, \mu_G) = (-1)^{r(G)}$, where St_G is the Steinberg representation [6].

Conjecture 1.1 ([3, Conjecture 7.1(5)]). *In the setting of Conjecture 0.1, if $A_G = 1$ then*

$$(-1)^{r(G)} \deg(\pi, \mu_G) = \pm\frac{\dim\rho}{|S'_\varphi|} \cdot \frac{\gamma(0, \mathrm{Ad}_G \circ \varphi)}{\gamma(0, \mathrm{Ad}_G \circ \varphi_{\mathrm{prin}})}$$

with sign $\omega(\mathrm{Ad}_G \circ \varphi)/\omega(\mathrm{Ad}_G \circ \varphi_{\mathrm{prin}}) = \pm 1$.

Here φ_{prin} is the principal parameter, the parameter whose L-packet contains the Steinberg representation. This parameter is trivial on W_k and its restriction to the Deligne SL_2 corresponds under the Jacobson-Morozov theorem to the sum of the Lie algebra elements (for a basis of roots) in any pinning of $\widehat{G}$. Let $\underline{Z}$ be the center of $\underline{G}$.

Conjecture 1.2 ([3, Conjecture 8.3]). *In the setting of Conjecture 0.1,*

$$\frac{\omega(\mathrm{Ad}_G \circ \varphi)}{\omega(\mathrm{Ad}_G \circ \varphi_{\mathrm{prin}})} = \chi_\varphi(z_{\mathrm{Ad}_G}),$$

where $z_{\mathrm{Ad}_G} \in Z$ is a certain involution to be defined momentarily.

Here χ_φ is the character of Z, where $\underline{Z}$ is the center of $\underline{G}$, corresponding to the parameter φ, as originally constructed by Langlands. It is one of Borel's desiderata for the local Langlands correspondence [1, III.10] that χ_φ, which has a simple definition via the correspondence for tori, be the central character of the representations in the L-packet of φ.

2. Orthogonal root numbers

Gross and Reeder proved Conjecture 1.2 when G is split (and $A_G = 1$ and k is nonarchimedean) using an argument in Galois cohomology. I was able to generalize their theorem by generalizing their proof. This proves Conjecture 1.2 modulo Borel's desiderata.

Theorem 2.1 ([17, Theorem A]). *Let k be a local field, let $r\colon {}^L G \to \mathrm{O}(V)$ be an orthogonal representation, and let $\varphi\colon L_k \to {}^L G$ be a tempered L-parameter. Then*

$$\frac{\omega(r \circ \varphi)}{\omega(r \circ \varphi_{\mathrm{prin}})} = \chi_\varphi(z_r).$$

Since this theorem lives on the Galois side of the local Langlands correspondence, nothing is lost in assuming that G is quasi-split, and we add this as a standing hypothesis. Here T is a minimal Levi of G and the element $z_r \in Z$ is the involution defined by

$$z_r \stackrel{\text{def}}{=} \prod_{0<\varpi\in X_*(Z)} \varpi(-1)^{\dim V_\varpi},$$

where V_φ is the ϖ weight space for the action of $\widehat{T}$ on V by r.

Remark 2.2. *I have not defined the root number of a Weil-Deligne representation. But such a definition exists, and for orthogonal representations $\varphi\colon \mathrm{WD}_k \to \mathrm{O}(V)$,*

$$\omega(\varphi) = \omega(\varphi|_{W_k}).$$

In particular, $\omega(r \circ \varphi_{\mathrm{prin}}) = \omega(r|_{W_k})$. (Here we use the Weil form of ${}^L G$.)

The key lemma in the proof of the theorem is the following statement in group cohomology.

Lemma 2.3. *Consider the following commutative diagram, where $c_G \in \mathrm{H}^2_{\mathrm{Borel}}({}^L G, \pi_1(\widehat{G}))$ and $c_{\mathrm{pin}} \in \mathrm{H}^2_{\mathrm{Borel}}(\mathrm{O}(V), \{\pm 1\})$ classify the top and bottom group extensions.*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\widehat{G}) & \longrightarrow & \widehat{G}_{\mathrm{univ}} \rtimes W_k & \longrightarrow & {}^L G & \longrightarrow & 1 \qquad (c_G) \\
 & & \downarrow{\scriptstyle e_r} & & & & \downarrow{\scriptstyle r} & & \\
1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Pin}(V) & \longrightarrow & \mathrm{O}(V) & \longrightarrow & 1 \qquad (c_{\mathrm{pin}}).
\end{array}$$

Then $r^*(c_{\mathrm{pin}}) = e_{r,*}(c_G) \cdot (p^* r|_{W_k}^*)(c_{\mathrm{pin}})$ in $\mathrm{H}^2_{\mathrm{Borel}}({}^L G, \{\pm 1\})$ where $p\colon {}^L G \to W_k$ is projection.

To prove Theorem 2.1, we pull back the conclusion of Lemma 2.3 along φ, using in particular that $\mathrm{H}^2(W_k, \{\pm 1\}) \simeq \mathbb{Z}/2\mathbb{Z}$. By Deligne's theorem, the two factors with c_{pin} become the root numbers (normalized by the determinant) of $r \circ \varphi$ and $r \circ \varphi_{\mathrm{prin}}$. To identify $\varphi^* e_{r,*}(c_G)$ with $\chi_\varphi(z_r)$ requires the following computation.

Lemma 2.4. *Let $\underline{T}$ be a k-torus. Then*

$$\mathrm{H}^2(W_k, X^*(\underline{T})) = \mathrm{Hom}_{\mathrm{cts}}(T^1, \mathbb{C}^\times)$$

where $T^1 \subseteq T$ is the maximal bounded subgroup of T.

Remark 2.5. *In the statement of Lemma 2.3, $\mathrm{H}^\bullet_{\mathrm{Borel}}$ denotes the Borel cohomology groups (sometimes called Moore cohomology), a variant of continuous group cohomology defined using Borel-measurable cochains. This generalization is needed because the spin-cover of the orthogonal group does not admit a continuous set-theoretic section. The theory of Borel cohomology was largely worked out in the 60's and 70's by Calvin C. Moore* [11, 12, 13, 14]*, and it would be very interesting to revisit the subject with modern techniques.*

3. Yu supercuspidals

There are several approaches to proving the formal degree conjecture. In Raphaël's talk, for instance, we used twisted endoscopy to deduce the conjecture for classical groups from the conjecture for GL_n. Another approach is to compute everything directly. This approach, while less sophisticated, has the advantage that it produces explicit formulas for the formal degree, which may be of interest in applications. In this final section, I want to explain what happens for Yu supercuspidals [21]. Jessica's talk reviewed the history of these supercuspidals; in particular, every supercuspidal is of the type constructed by Yu if p does not divide the order of the Weyl group of G [10, 2].

Theorem 3.1 ([16, Theorem A]). *Let $\underline{G}$ be semisimple and let Ψ be a Yu datum with associated supercuspidal representation π. Then*

$$\deg(\pi,\mu) = \frac{\dim\rho}{[G^0_{[y]} : G^0_{y,0+}]}\exp_q \tfrac{1}{2}\Big(\dim\underline{G} + \dim\underline{G}^0_{y,0:0+} + \sum_{i=0}^{d-1} r_i(|R_{i+1}| - |R_i|)\Big).$$

The formula of the theorem is rather complicated. I have not defined several of its constituents because that would require me to explain what a "Yu datum" is. The main takeaway from the formula is that the formal degree is the product of two factors, one coming from depth-zero objects and one coming from positive-depth objects, the latter of which is a power of q. The more "ramified" the representation, the higher its power of q.

To compute the formal degree, we use a general formula for the formal degree of a compactly-induced representation:

$$d(\text{c-Ind}_K^G\,\sigma,\mu) = \frac{\dim\sigma}{\mathrm{vol}(K,\mu)}.$$

(This is the version of the formula when K is compact-open; in general, one needs a modification in which K is open and compact-mod-center.) Both σ and K derive from the Yu datum. The main difficulty is to compute $\mathrm{vol}(\mu, K)$.

After Yu's construction of supercuspidals, a natural next step was to match the supercuspidals with L-parameters. In a series of recent papers [7, 8, 9], Kaletha has accomplished this matching in increasing generality for most of Yu's supercuspidals. His most general construction, for the non-singular supercuspidals, matches those whose L-packet does not contain a discrete series representation, at least if p does not divide the order of the Weyl group of G.

Corollary 3.2 ([16, Theorem B], [15]). *The formal degree conjecture holds for Kaletha's non-singular L-packets.*

I was able to prove the formal degree conjecture for regular supercuspidals, a slightly less general class of supercuspidals than the non-singular supercuspidals, and Ohara generalized this work to the non-singular supercuspidals.

4. Coda: future work

The formal degree conjecture describes the discrete part of the tempered dual, but the tempered dual is not entirely discrete. Its other components, coming from parabolic inductions of discrete series of Levi subgroups, are finite-index orbifold quotients of real compact tori [20]. Inspired by Langlands's conjecture on Plancherel measures, Hiraga, Ichino, and Ikeda proposed a description of the Plancherel measure on these nondiscrete components of the tempered dual [5, Conjecture 1.4]. I hope to verify their conjecture for the components that come from the parabolic induction of a non-singular Yu supercuspidal.

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Symmetric Power Functoriality for $\mathbf{GL_2}$

Jack A. Thorne

If K is a number field and π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ then there is a restricted tensor product decomposition $\pi = \otimes'_v \pi_v$, where for each finite place v of K, π_v is an irreducible admissible representation of $\mathrm{GL}_n(K_v)$ [8].

Langlands's functoriality conjecture [9] predicts the existence, for any algebraic representation $R : \mathrm{GL}_n \to \mathrm{GL}_m$, of a 'functorial lift along R': this should be another automorphic representation $R_*(\pi)$ of $\mathrm{GL}_m(\mathbb{A}_K)$, characterized by the requirement that the local components $R_*(\pi)_v$ are related, under the local Langlands correspondence, to the images under R of the Langlands parameters of the local components π_v.

The first non-trivial cases of this conjecture arise when $n = 2$ where the irreducible representations are given, up to twist, by the symmetric powers $\mathrm{Sym}^m : \mathrm{GL}_2 \to \mathrm{GL}_{m+1}$ of the standard representation of GL_2. The existence of the symmetric power lifting has been established when $m = 2, 3$, or 4 [7, 10, 11].

One can go further for those automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_K)$ which admit associated Galois representations. Indeed, if the Galois representation $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is associated to π then the existence of $\mathrm{Sym}^m(\pi)$ is equivalent to the automorphy of the symmetric power Galois representation $\mathrm{Sym}^m \rho$. This point of view has been used by Clozel–Thorne and Dieulefait to prove the existence of $\mathrm{Sym}^m_*(\pi)$ for most π associated to Hilbert modular forms for $5 \leq m \leq 8$ [3, 4, 5, 6].

In 2020, James Newton and I used this point of view to prove the existence of all symmetric power liftings for those cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ which are regular algebraic (i.e. associated to holomorphic newforms of weight $k \geq 2$) [12, 13]. We first proved the vanishing of the adjoint Bloch–Kato Selmer group $H^1_f(K^+, \mathrm{ad}\, \mathrm{Sym}^m \rho)$ for any conjugate self-dual Galois representation ρ associated to a cuspidal, regular algebraic automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_K)$ (and where K is a CM number field) [14].

We then applied this to prove two general theorems that are applied repeatedly. The first is a principle of 'analytic continuation': for a fixed prime p, integer $m \geq 1$, and unitary group G_2 in 2 variables, the existence of the degree m symmetric power lifting is a property which is 'constant on irreducible components of the eigenvariety of G_2'. Taking inspiration from [1], we use the vanishing of the adjoint Bloch–Kato Selmer group to study the immersion of the eigenvariety for an $(m+$

1)-variable unitary group in a corresponding moduli space of trianguline Galois representations.

The second is a kind of 'functoriality lifting theorem', which shows (under some rather general conditions) that the existence of the symmetric power lifting $\mathrm{Sym}^m_*(\pi)$ is a property that can be propagated along congruences modulo p. This result is proved using a variation on the Taylor–Wiles–Kisin patching method, in which the vanishing of the adjoint Bloch–Kato Selmer group intervenes to get control on the generic fibre of a patched pseudodeformation ring.

Our results rely on the explicit description of the geometry of the 2-adic, tame level 1 eigencurve for $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ given by [2], which seems rather special. It remains an interesting problem to explore the full reach of our methods in the context, for example, of the symmetric power problem for Hilbert modular forms.

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Irreducible components of affine Deligne–Lusztig varieties

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1. Results for affine Deligne–Lusztig varieties

Affine Deligne–Lusztig varieties (ADLV's) were introduced by M. Rapoport in [16], motivated by the study of reductions of Shimura varieties. Fix a local field F, with completed maximal unramified extension $\breve{F}$. Let k be the residue field of $\breve{F}$, which is algebraically closed. In this report we only discuss ADLV's at hyperspecial level. Thus an ADLV is a geometric object $X_\mu(b)$ attached to a triple (G, b, μ), where G is a (connected) reductive group scheme over $\mathcal{O}_F$, b is an element of $G(\breve{F})$, and μ is a cocharacter of $G_{\overline{F}}$ up to conjugacy. At the level of k-points, $X_\mu(b)(k)$ consists of $g \in G(\breve{F})/G(\mathcal{O}_{\breve{F}})$ such that the $G(\mathcal{O}_{\breve{F}})$-double coset of $g^{-1}b\sigma(g)$ corresponds to the conjugacy class of μ under the Cartan decomposition of $G(\breve{F})$. Here σ is the absolute arithmetic Frobenius in $\mathrm{Aut}(\breve{F}/F)$.

An important fact is that $X_\mu(b)$ has the geometric structure of a k-scheme locally of finite type (resp. a perfect k-scheme locally of perfectly finite type) if F has equal (resp. mixed) characteristic. The geometric structure comes from the geometric structure of the affine Grassmannian and the Witt vector affine Grassmannian (constructed by X. Zhu [21] and Bhatt–Scholze [2]) in the two cases. In the mixed characteristic setting, the geometry of ADLV's is closed related to Shimura varieties via the Rapoport–Zink uniformization (cf. [17, 5, 8]). For instance, information about connected components of ADLV's has played an important role in the version of Langlands–Rapoport Conjecture proved in [10] and the subsequent strengthening in [12].[1]

In this report we present results on the set $\Sigma^{\mathrm{top}}(X_\mu(b))$ of top-dimensional irreducible components of $X_\mu(b)$. (It is conjectured [16, Conj. 5.10] that $X_\mu(b)$ is equidimensional, cf. [6, §3].) The motivation for studying this set comes from the relation with certain cycles in the basic locus of the special fiber of Shimura varieties, see for instance [20]. The scheme $X_\mu(b)$ is equipped with an action of $J_b(F)$, the F-rational points of the Frobenius-centralizer J_b of b. This induces an action of $J_b(F)$ on $\Sigma^{\mathrm{top}}(X_\mu(b))$. We are interested in the following two questions.

(1) Classify the $J_b(F)$-orbits in $\Sigma^{\mathrm{top}}(X_\mu(b))$.
(2) For each $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, determine the stabilizer of Z in $J_b(F)$.

For question (i), M. Chen and X. Zhu conjectured a natural bijection between $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$ and the Mirkovic–Vilonen basis $\mathbb{MV}_\mu(\lambda_b)$ for the λ_b-weight space of the representation of the dual group $\widehat{G}$ corresponding to μ. Here λ_b is a character on a subtorus of $\widehat{G}$ (which is a maximal torus if G_F is split over F) defined in terms of the two discrete invariants (a la Kottwitz) of the σ-conjugacy class of b in $G(\breve{F})$ by a simple recipe. In particular, this conjecture gives an elementary way of computing the cardinality of the finite set $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$.

[1] Strictly speaking, the application to Langlands–Rapoport Conjecture is independent of the Rapoport–Zink uniformization per se, but it follows the same spirit.

Special cases of this conjecture were proved in [20, 6, 15] (based on a common idea of reduction to the superbasic case, which goes back to [3]). The general case was finally proved in the joint work of R. Zhou and the author, and in the work of S. Nie using different methods.

Theorem A ([22], [14])**.** *There is a natural bijection* $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b)) \cong \mathbb{MV}_\mu(\lambda_b)$.

For question (ii), it was conjectured by X. Zhu that every stabilizer should be a parahoric subgroup of $J_b(F)$ of maximal volume. In joint work with X. He and R. Zhou we confirm this conjecture. Generalizing the results in [1, A.4], we have a distinguished class of parahoric subgroups of $J_b(F)$, called *very special*, which can be read off from the relative local Dynkin diagram of J_b with the decorating integers $d(v)$ as in [18, §§1.8, 1.11]. These subgroups are equivalently characterized as the parahoric subgroups having the maximal volume.

Theorem B ([7])**.** *The stabilizers for the* $J_b(F)$*-action on* $\Sigma^{\mathrm{top}}(X_\mu(b))$ *are all very special parahoric subgroups of* $J_b(F)$.

For a reductive group H over F with no factors of type $C\text{-}BC_n$, very special parahoric subgroups of $H(F)$ are unique up to conjugation by $H^{\mathrm{ad}}(F)$. In our current setting of hyperspecial level, J_b never has factors of type $C\text{-}BC_n$, so Theorem B determines the stabilizers up to conjugation by $J_b^{\mathrm{ad}}(F)$. It would be an interesting problem to determine the stabilizers up to $J_b(F)$-conjugacy.

2. Application to Shimura varieties

Let $(\mathbf{G}, X)$ be a Shimura datum of Hodge type, and let $\mathrm{K} \subset \mathbf{G}(\mathbb{A}_f)$ be a compact open subgroup. Let $p > 2$ be a prime and assume that $\mathrm{K} = \mathrm{K}_p\mathrm{K}^p$ where K^p is a sufficiently small compact open subgroup of $\mathbf{G}(\mathbb{A}_f^p)$ and $\mathrm{K}_p = G(\mathbb{Z}_p)$ is a hyperspecial subgroup of $\mathbf{G}(\mathbb{Q}_p)$, where G is a reductive model over $\mathbb{Z}_p$ of $\mathbf{G}_{\mathbb{Q}_p}$. By the work of Kisin [9], for any prime v of the reflex field E of (G, X) above p, there is a smooth canonical integral model $\mathcal{S}_{\mathrm{K}}(\mathbf{G}, X)$ of $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ over $\mathcal{O}_{E,(v)}$. The geometric special fiber of $\mathcal{S}_{\mathrm{K}}(\mathbf{G}, X)$ has a *Newton stratification* indexed by the Kottwitz set $B(G_{\mathbb{Q}_p}, \mu) \subset G(\breve{\mathbb{Q}}_p)/\sigma\text{-conj}$, where μ is a Hodge cocharacter of X. The unique basic element $[b]$ of $B(G_{\mathbb{Q}_p}, \mu)$ corresponds to a stratum denoted by $\mathrm{Sh}_{\mathrm{K,bas}}$. This is a generalization of the supersingular locus in the case of a modular curve. Define $X_\mu(b)$ with respect to G. By a "reduced version" of the Rapoport–Zink uniformization (see e.g. [20, Cor. 7.2.16]), the perfection of $\mathrm{Sh}_{\mathrm{K,bas}}$ is isomorphic to

$$I(\mathbb{Q})\backslash X_\mu(b) \times \mathbf{G}(\mathbb{A}_f^p)/\mathrm{K}^p.$$

Here I is a certain reductive group over $\mathbb{Q}$ equipped with isomorphisms $\iota_p : I \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong J_b$ and $\iota^p : I \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{A}_f^p$, and the left action of $I(\mathbb{Q})$ on $X_\mu(b) \times \mathbf{G}(\mathbb{A}_f^p)$ is defined using ι_p and ι^p. The following result follows from Theorems A and B, and the equidimensionality proved in [6, Thm. 3.4].

Theorem C ([7]). *The set* $\Sigma(\mathrm{Sh}_{\mathrm{K,bas}})$ *of irreducible components of* $\mathrm{Sh}_{\mathrm{K,bas}}$ *is in prime-to-p-Hecke-equivariant bijection with*

$$\coprod_{\mathbf{a}\in\mathbb{MV}_\mu(\lambda_b)} I(\mathbb{Q})\backslash I(\mathbb{A}_f)/I_p^{\mathbf{a}} I^p,$$

where $I^p = (\iota^p)^{-1}(\mathrm{K}^p) \subset I(\mathbb{A}_f^p)$ *and* $I_p^{\mathbf{a}}$ *is a very special parahoric subgroup of* $I_p(\mathbb{Q}_p)$ *for each* $\mathbf{a}$.

We refer the reader to [19, 5, 13] for results similar to Theorem C for special cases of Shimura varieties. For certain arithmetic applications, such as the arithmetic level raising in [13], Theorem C allows one to interpret functions on $\Sigma(\mathrm{Sh}_{\mathrm{K,bas}})$ as automorphic forms on the group I. The knowledge about $I_p^{\mathbf{a}}$ is thus crucial in controlling the level of these automorphic forms.

In [22] and [7] we have also proved generalizations of Theorems A and B beyond the setting of hyperspecial level, to cases where G is a special parahoric group scheme over $\mathcal{O}_F$ and G_F is quasi-split but not necessarily unramified over F. In order to obtain the corresponding generalization of Theorem C, one needs to generalize the above description of the perfection of $\mathrm{Sh}_{\mathrm{K,bas}}$ in terms of $X_\mu(b)$, where $\mathrm{Sh}_{\mathrm{K,bas}}$ is defined using the Kisin–Pappas integral model [11]. This is done in [7].

2.1. **Some features of the proofs.** The key idea in our proof of Theorem A is to use the Lang–Weil bound to relate point counting on $X_\mu(b)$ to the cardinality of $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$. Based on this idea, we show that this cardinality is related to the asymptotic behavior, as n grows, of a certain twisted orbital integral over the degree n unramified extension of F. We study the latter using explicit methods from local harmonic analysis and representation theory, including the base change fundamental lemma of Clozel and Labesse, and the Kato–Lusztig formula.

In our proof, certain linear combinations of the q-analogue of Kostant partition functions appear, and it is key to estimate their sizes. It seems reasonable to expect that a more thorough study of the combinatorial and geometric properties of these objects could lead to interesting results about ADLV's.

As a byproduct of our proof of Theorem A, we have the following intermediate result: In the "essential cases" that one reaches after some reductions steps, the *average* of the inverses of the volumes of the stabilizers depends only on b, not on μ. This result allows one to reduce Theorem B to the statement that there is at least *one* top-dimensional irreducible component whose stabilizer is very special parahoric. We prove the last statement in [7] by an explicit construction, employing a refinement of the Deligne–Lusztig reduction method in [4],

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Arthur multiplicity formula for unitary and even orthogonal groups via theta lifts

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(joint work with Rui Chen)

Let G be an even orthogonal or unitary group over a number field. Based on the same method used in [5], we proved the Arthur's multiplicity formula for the generic part of the automorphic discrete spectrum of G. Enhancing this method, we also obtained a description of the full automorphic discrete spectrum of even orthogonal or unitary groups with Witt index less than or equal to one. The main idea is to transfer the Arthur's multiplicity formula for symplectic or quasi-split unitary groups, which is proved by Arthur [3] and Mok [8], to even orthogonal or unitary groups by using J.-S. Li's results [6] on global theta lifts in the stable range.

Let F be a number field and $\mathbb{A}_F$ be the Adele of F. Here we shall focus on the even orthogonal group case to illustrate our results. The unitary group case is parallel. So from now on $G = \mathrm{O}(V)$ is an even orthogonal group over F. Our ultimate goal is to study the decomposition of

$$L^2_{\mathrm{disc}}(G) \coloneqq L^2_{\mathrm{disc}}\left(G(F)\backslash G(\mathbb{A}_F)\right).$$

The very first step is to decompose $L^2_{\mathrm{disc}}(G)$ into a direct sum of different near equivalent classes. When G is quasi-split, Arthur's results asserted that each near equivalent class can be "represented" by an elliptic A-parameter (cf. [2, §6.5]). Our first theorem generalizes this result of Arthur to the case that G is non-quasi-split:

Theorem 1.1 (Chen-Zou)**.** *There exists a decomposition*

$$L^2_{\mathrm{disc}}(G) = \bigoplus_{\psi\in\Psi_{ell}(G)} L^2_{\psi}(G),$$

where $\Psi_{ell}(G)$ is the set of elliptic A-parameters for G and $L^2_{\psi}(G)$ is a full near equivalence class of irreducible representations π in $L^2_{\mathrm{disc}}(G)$ represented by ψ.

The main ingredient for proving Theorem 1.1 is J.-S. Li's results [6] on global theta lifts in the stable range. Let H be a large symplectic group such that (G, H) is a reductive dual pair in the stable range. Given an irreducible unitary representation $\pi = \otimes'\pi_v$ of $G(\mathbb{A}_F)$, we define

$$\theta^{\mathrm{abs}}(\pi) = \otimes'\theta(\pi_v),$$

which is an irreducible unitary representation of $H(\mathbb{A}_F)$. Here $\theta(\pi_v)$ is the local theta lift of π_v to $H(F_v)$. Let

$$m_{\mathrm{aut}}(\pi) = \dim \mathrm{Hom}_{G(\mathbb{A})}\left(\pi, \mathcal{A}(G)\right);$$
$$m_{\mathrm{disc}}(\pi) = \dim \mathrm{Hom}_{G(\mathbb{A})}\left(\pi, \mathcal{A}^2(G)\right),$$

where $\mathcal{A}(G)$ (resp. $\mathcal{A}^2(G)$) is the space of automorphic forms (resp. square-integrable automorphic forms) on G. Likewise, one can define the multiplicities $m_{\text{aut}}(\theta^{\text{abs}}(\pi))$ and $m_{\text{disc}}(\theta^{\text{abs}}(\pi))$. Then J.-S. Li's results [6] asserted that

$$m_{\text{disc}}(\pi) \leq m_{\text{disc}}(\theta^{\text{abs}}(\pi)) \leq m_{\text{aut}}(\theta^{\text{abs}}(\pi)) \leq m_{\text{aut}}(\pi). \tag{1}$$

These inequalities together with some easy computations at unramified places imply our Theorem 1.1.

To investigate the structure of each near equivalent class $L^2_\psi(G)$, we need to enhance J.-S. Li's results. We made the following conjecture.

Conjecture 1.2. *Let ψ be an elliptic A-parameter of G and $\pi = \otimes'_v \pi_v$ is a representation of $G(\mathbb{A}_F)$ in the near equivalent class represented by ψ. Then*

$$m_{\text{disc}}(\pi) = m_{\text{aut}}(\pi).$$

If this conjecture holds, then by (1) we have $m_{\text{disc}}(\pi) = m_{\text{disc}}(\theta^{\text{abs}}(\pi))$. With this equality at hand, we can transfer the Arthur's multiplicity formula for H to G. We show that:

Theorem 1.3 (Chen-Zou)**.** *Assume that one of the following two conditions holds:*

1. *G is any even orthogonal groups and ψ is a generic elliptic A-parameter for G;*
2. *G has Witt index less than or equal to 1 and ψ is any elliptic A-parameter for G.*

Then Conjecture 1.2 holds.

The proof of the first case is similar to [5, Proposition 4.1]. The proof of the second case involves Mœglin-Waldspurger's square-integrability criterion and some knowledges on the Jacquet module of certain unramified representations.

In these two cases, we can write down an elliptic A-parameter $\theta(\psi)$ for H in terms of ψ explicitly. After checking certain compatibility conditions, we can transfer the Arthur's multiplicity formula for $L^2_{\theta(\psi)}(H)$ to $L^2_\psi(G)$ as follows.

- For each place v of F, Arthur has defined the local A-packet

$$\Pi^A_{\theta(\psi)_v}(H) = \{\sigma_{\eta_v} | \eta_v \in \widehat{\overline{\mathcal{S}_{\theta(\psi)_v}}}\},$$

  where $\overline{\mathcal{S}_{\theta(\psi)_v}}$ is the component group of $\theta(\psi)_v$ (cf.[3, §1.4]) and σ_{η_v} is an unitary representation of H_v of finite length. We set

$$\Pi^\theta_{\psi_v}(G_v) = \left\{ \pi_{\eta_v} = \bigoplus_{\ell^*_v(\eta'_v)=\eta_v} \theta\left(\sigma_{\eta'_v}\right) | \eta_v \in \widehat{\mathcal{S}_{\psi_v}} \right\},$$

  where $\ell_v : \mathcal{S}_{\psi_v} \to \overline{\mathcal{S}_{\theta(\psi)_v}}$ is the natural map between these local component groups.

- There is a canonical map between the global and local components groups:
$$\Delta : \mathcal{S}_\psi \longrightarrow \mathcal{S}_{\psi,\mathbb{A}_F} = \prod_v \mathcal{S}_{\psi_v}.$$
Let ϵ_ψ be the canonical sign character of $\mathcal{S}_\psi$ defined by Arthur (cf.[3, §1.5]). For each $\eta = \otimes' \eta_v \in \widehat{\mathcal{S}_{\psi,\mathbb{A}_F}}$, we set $\pi_\eta = \otimes'_v \pi_{\eta_v}$.

Theorem 1.4 (Chen-Zou)**.** *Assume that one of the two conditions in Theorem 1.3 holds. Then we have:*
$$L^2_\psi(G) = \bigoplus_{\eta \in \widehat{\mathcal{S}_{\psi,\mathbb{A}_F}}} m_\eta \cdot \pi_\eta,$$
with
$$m_\eta = \begin{cases} 1, & \text{if } \Delta^*(\eta) = \epsilon_\psi; \\ 0, & \text{otherwise.} \end{cases}$$

However these local packets $\Pi^\theta_{\psi_v}(G_v)$ are very artificial. So finally to make our results more convincing, we would like to investigate these local packets.

Theorem 1.5 (Chen-Zou)**.** *Let v be a local place of F.*

(1) *If ψ_v is generic, then the local packet $\Pi^\theta_{\psi_v}(G_v)$ coincides with the local L-packet $\Pi^L_{\psi_v}(G_v)$ defined in Atobe-Gan* [2] *and Chen-Zou* [4]*.*
(2) *If v is non-Archimedean, then the the local packet $\Pi^\theta_{\psi_v}(G_v)$ is independent of the choice of H. Moreover, if we further assume that G_v is quasi-split, then $\Pi^\theta_{\psi_v}(G_v)$ coincides with the local A-packet $\Pi^A_{\psi_v}(G_v)$ defined by Arthur* [3] *and Atobe-Gan* [2]*.*

Remark 1.6.

(1) *In* [1]*, Adams has proposed a conjecture on describing the local theta correspondence in terms of local A-parameters. Theorem 1.5 can be regarded as an refined version of Adams conjecture in the case of stable range: we not only prove the bijection between packets as (multi) sets, we also show the consistency of "labellings".*
(2) *Mœglin's work* [7] *on Adams conjecture already implies the assertion* (2) *in the even orthogonal group case (except for the labeling, one also needs B. Xu's work* [9]*). We provide an independent proof to avoid using endoscopic theory for non-quasi-split groups. Our proof use global methods and the local intertwining relations.*
(3) *Recently, Hanzer and Bakić has made some progress on Adams conjecture in non-Archimedean cases. We refer readers to Hanzer's talk in this conference on their work.*

We end up this extended abstract by the following two questions.

Question 1.7.

(1) *Can one prove that $\Pi^\theta_{\psi_v}(G_v)$ satisfies endoscopy character identities?*
(2) *Can one prove Theorem 1.5* (2) *when v is Archimedean?*

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