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Groups and Dynamics: Topology, Measure, and Borel Structure

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Abstract. While the subjects of topological dynamics, ergodic theory, and descriptive set theory have long interacted in a variety of profitable ways, recent developments have ushered in a vigorous new phase of interplay between them, from the abstract transfer and coordinated development of ideas and methods (as in the theory of dynamical tilings) to the direct leveraging of technical points of contact (as in boundary theory). The workshop served as a platform for promoting and advancing these connections by bringing together researchers working on various facets of topological, measured, and Borel dynamics.

Mathematics Subject Classification (2020): 03E15, 20C, 20E, 20F, 28A05, 28D, 37A, 37B, 54H05, 60B.

Introduction by the Organizers

Over the last several years a number of remarkable developments in the study of groups and their actions have opened up exciting new directions and prospects in the interaction between topological dynamics, ergodic theory, and descriptive set theory. The workshop aimed to promote this interaction among researchers working in the three broad subfields of topological, measured, and Borel dynamics, and each talk at the workshop could roughly be categorized as instantiating work in one or more of these subfields.

One theme that was ubiquitous throughout the workshop was the question of the extent to which intrinsic algebraic and geometric properties of an acting group are reflected dynamically within each of the above paradigms. This theme appeared in the talk of Caprace through hyperbolicity, in the talk of Duchesne through property (T), in the talk of Barbieri through the symbolic-dynamical notion of self-simulability, in the talks of Geffen and Naryshkin through the topological-dynamical property of comparison, in the talk of Frisch on Poisson boundaries, and in the talks of Kida, Wrobel, and Zomback relating to dynamical properties of actions of free groups. It was also philosophically central to the talk of Panagiotopoulos on dynamical obstructions to classification by TSI-group invariants.

Another theme concerned the tension and interplay between dynamics and combinatorics. This could be seen in the guise of paradoxicality in Barbieri's talk on self-simulable groups, via the Ramsey property in Bartosova's talk on ultraproducts of finite structures, via the continuous Lovász Local Lemma in Bernshteyn's talk on constructing equivariant maps, and via the measurable combinatorial constructions described in Bowen's talk on combinatorics in hyperfinite graphs.

Several talks addressed asymptotic phenomena in topological dynamics. Li spoke on the relation between entropy, asymptotic pairs, and the topological Markov property with applications to the structure of algebraic actions of amenable groups, Zucker on minimal flows lacking a characteristic measure, and Glasner on the structure theory for tame minimal actions. In a complementary direction in which the resilience of asymptotic behavior is tested under weakenings of dynamical conjugacy, especially with a view towards structure theories of a different categorial nature, Le Maître discussed recent advances in the nascent area of quantitative orbit equivalence for measure-preserving actions and Melleray offered a new perspective on orbit equivalence for minimal Cantor systems and its characterization in terms of invariant measures.

An additional theme was the interaction between operator algebras and dynamics. On the topological side, this connection was exemplified in the talk of Geffen on purely infinite crossed products by non-amenable groups, the talk of Kennedy on proximality and higher order syndeticity, and the talk of Naryshkin on almost finite actions. The connection to operator algebras was also reified on the measured side in the talk of Boutonnet on infinite characters on groups and in the talk of Houdayer on noncommutative dynamics of lattices in higher rank simple algebraic groups.

The workshop was held in hybrid format, with more than half of the participants attending remotely. There were 31 talks in total, 13 of which were delivered in person. Despite the circumstances and reduced in-person crowd size, the on-site participants took full advantage of the opportunities that the superb Oberwolfach facilities provided for informal interaction, with many lively discussions running late into the evenings and an energizing Wednesday afternoon hike.

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Workshop: Groups and Dynamics: Topology, Measure, and Borel Structure

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Abstracts

Simplicity of automorphism groups of countable structures

Aleksandra Kwiatkowska
(joint work with Filippo Calderoni, Katrin Tent)

The program of understanding the normal subgroup structure of groups that arise as automorphism groups of countable structures dates back at least to the '50s, when Higman [3] described all proper normal subgroups of the automorphism group of rationals $(\mathbb{Q}, <)$.

In recent years, Macpherson and Tent [4] proved simplicity for a large collection of such groups. Their methods encompass a number of examples that had been considered before by various authors: the random graph (Truss [6]), the random K_n-free graphs and the random tournament (Rubin, unpublished). However, their framework does not apply to ordered or even partially ordered structures, in particular, it does not apply to the random poset whose automorphism group was proved to be simple in [2]. Tent and Ziegler [5] introduced the notion of a stationary independence relation and investigated automorphism groups of structures allowing for such a relation. Their approach is very general: apart from recovering the cases from [4], it applies to the bounded Urysohn metric space and its variations. However, ordered ultrahomogeneous structures like the ordered random graph and the random tournament do not carry such a stationary independence relation.

In a joint work with Filippo Calderoni and Katrin Tent [1], we weaken the notion of a stationary independence relation from [5]. We prove simplicity for the automorphism groups of order and tournament expansions of ultrahomogeneous structures like the bounded Urysohn metric space and the random graph. In particular, we show that the automorphism group of the linearly ordered random graph is a simple group.

Our main result is the following theorem.

Theorem. *Assume that* $\mathbf{M}$ *is one of the following:*

(1) *the Fraïssé limit of a free, transitive and nontrivial amalgamation class;*
(2) *the bounded rational Urysohn space; or*
(3) *the random poset.*

If $\mathbf{M}^*$ *is an order expansion of* $\mathbf{M}$*, then* $G := Aut(\mathbf{M}^*)$ *is simple. The same holds if* $\mathbf{M}^*$ *is a tournament expansion of* (1) *or* (2).

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Infinite characters on $\mathbf{SL}_n(\mathbb{Z})$

Rémi Boutonnet

1. Finite characters

In view of a subsequent talk of Cyril Houdayer on character rigidity and related topics, I first discussed classical characters on groups. Recall that a character on a group G is a positive definite function $\phi : G \to \mathbb{C}$ which takes the value 1 at the neutral element and is conjugation invariant.

We will be interested in the specific case where G is discrete. In this case, characters occur classically from three different contexts: finite dimensional unitary representations (giving the so called *almost periodic characters*), normal subgroups or more generally, stabilizers of probability measure preserving actions (IRS's).

It follows from the well known GNS construction that there are two other ways of viewing a character on G: either as a trace on the universal C*-algebra $C^*(G)$, or as a generating morphism $\pi : G \to \mathcal{U}(M)$ into the unitary group of a tracial von Neumann algebra (M, τ). Here generating means that $\pi(G)$ generates M as a von Neumann algebra.

The set of all characters on G is a weak-* closed convex subspace of the dual of $C^*(G)$. So understanding all characters boils down to understanding extremal ones. In the von Neumann algebraic picture this extremality condition is equivalent to factoriality of M; so an extremal ϕ corresponds to a generating representation into a II_1-factor or into a finite dimensional factor (if ϕ is almost periodic).

In the case of semi-simple groups and their lattices, striking rigidity results have been obtained over the past two decades. Let us mention explicitly a result of Bekka.

Theorem (Bekka, [1])**.** *Take $n \geq 3$. Every extremal character on $\mathrm{PSL}_n(\mathbb{Z})$, is either the regular character δ_e (corresponding to the regular representation) or an almost periodic character.*

2. Infinite characters

In view of the description of finite characters in terms of generating morphisms into finite von Neumann algebras, we will call *infinite character* a generating morphism $\pi : G \to \mathcal{U}(M)$ where (M, Tr) is an infinite semi-finite von Neumann algebra, with the condition that the C*-algebra generated by π contains a non-zero positive element with finite trace. We call this last condition *traceability*. Note that by composing with the semi-finite trace on M, such a representation gives a tracial weight Φ on $C^*(G)$ which is non-zero and not purely infinite (i.e. there exists $x \in C^*(G)$ such that $0 < \Phi(x) < \infty$). Conversely any such tracial weight arises this way, thanks to the GNS representation. However, there is no natural way to restrict such a weight to G itself, so an infinite character cannot be seen as a function on G in any reasonable sense.

Remark. Compared to the finite setting, this notion is not very well understood. For example, such infinite characters are not clearly connected to infinite measure preserving actions. For example, already for type I actions $G \curvearrowright G/H$, some pathologies may happen, especially if H is commensurated by G.

In contrast, if H is malnormal in G, things tend to be better behaved. This observation is due to Bekka [2], and allowed him to give examples of infinite characters on $\mathrm{GL}_n(K)$ for a global field K, as well as on $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, answering a question of Rosenberg [2, 3]. His construction gave examples of characters of type I, i.e. generating representations into type I factors, which are traceable. Motivated by his rigidity result on finite characters, he then asked the question whether such groups may admit characters of type II: do they admit traceable generating representations into II_∞-factors?

Adapting his argument, we prove that this is the case for $\mathrm{SL}_n(\mathbb{Z})$. The construction for $G = \mathrm{SL}_3(\mathbb{Z})$ is particularly simple: take a generating representation of $H := \mathrm{GL}_2(\mathbb{Z})$ into the hyperfinite II_1-factor and induce it to G (via the top-left embedding $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{SL}_3(\mathbb{Z})$). Then the induced representation generates a type II_∞-factor and it is traceable. The main reason behind this is that H is *a-normal* inside G: for every $g \in G \setminus H$, $H \cap gHg^{-1}$ is amenable. Then since H is a virtually free group, it has plenty of representations into the hyperfinite factor, which will all give rise to different characters.

We don't know if for $n \geq 4$, $\mathrm{SL}_n(\mathbb{Z})$ admits an a-normal subgroup at all, but we are able to adapt our argument to prove the following theorem. Note that the case $n = 2$ is easy: free groups admit plenty of representations.

Theorem. *For all $n \geq 2$, $\mathrm{SL}_n(\mathbb{Z})$ admits uncountably many traceable generating representations into II_∞-factors, none of which is weakly contained in any other.*

The construction is ad hoc, and doesn't say anything about other lattices in simple Lie/algebraic groups.

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Symbolic substitutions in dilation groups

Felix Pogorzelski
(joint work with Siegfried Beckus, Tobias Hartnick)

Tilings of spaces are nice to look at - on top of that, they play a central role in the mathematical theory of aperiodic order. One way to produce such tilings and to understand their properties is via iterated substitution systems, where local structures are systematically replaced according to a pre-fixed rule. In view of a new development of extending the theory of aperiodic order to non-abelian groups and symmetric spaces, intiated in [3, 4, 5, 6], it is natural to explore in this geometric framework the possibility of constructing "mathematical quasicrystals" via substitutions. This talk is devoted to a new formalism concerning symbolic substitutions producing an abundance of such examples in certain nilpotent Lie groups. There is a vast body of literature on substitution systems in abelian spaces. We refer to the monographs [15, 1] and the references therein. Moreover, we highlight the papers [16, 17, 8] as they contain some crucial ideas that can be adapted to the non-abelian setting.

In the following, G denotes a 1-connected Lie group endowed with a left-invariant, proper metric d that generates the topology of G. Moreover, we suppose that G admits a 1-parameter family $(D_\lambda)_{\lambda>0}$ of metric *dilations*, i.e. each D_λ is a continuous G-automorphism satisfying

$$d\big(D_\lambda(g),\, D_\lambda(h)\big) = \lambda \cdot d(g,h) \quad \text{for all} \quad g,h \in G.$$

Moreover, let $\Gamma \leq G$ be a uniform lattice along with a relatively compact fundamental domain V having non-empty interior. We fix some value $\lambda_0 > 0$, called *stretch parameter*, such that $D_{\lambda_0}(\Gamma) \subseteq \Gamma$ and $\lambda_0 \geq 1 + r_+/r_-$, where $r_+ > 0$ and $r_- > 0$ are an outer, respectively inner radius for V. The collection of aforementioned elements is called a *dilation datum (over G)* and we write $\mathcal{D} = \mathcal{D}(G, d, (D_\lambda), \Gamma, V, \lambda_0)$ for it. A typical example is given by the 3-dimensional Heisenberg group $G = H_3(\mathbb{R})$, along with the Cygan-Koranyi metric d_{CK} arising from the Cygan-Koranyi group norm $\|(x,y,z)\|_{CK} = \sqrt[4]{(x^2+y^2)^2+z^2}$, a (uniform) lattice of $\mathbb{Z}$-points and a family of dilations defined as $D_\lambda(x,y,z) = (\lambda x, \lambda y, \lambda^2 z)$. Our general focus is on *homogeneous dilation data* which can be realized in certain nilpotent Lie groups G that we call RAHOGRASPs (rationally homogeneous group with rational spectrum). These are groups admitting a rational form $\mathfrak{g}_{\mathbb{Q}}$ of its Lie algebra $\mathfrak{g}$ with a particular $\mathbb{Q}$-grading. With this at hand, there is a canonical way to define a homogeneous family of dilations. Via standard Lie group theory one obtains a homogeneous metric d compatible with the dilation structure. A suitable lattice is generated via an integral basis of $\mathfrak{g}_{\mathbb{Q}}$. The class of groups admitting homogeneous dilation data seems to be quite rich. For instance, it can be checked by a straightforward linear algebra algorithm that this is the case for at least 133 out of the 149 families of 7-dimensional nilpotent real Lie algebras as classified in [12]. We remark at this point that very similar dilation structures had been considered before in the construction of periodic self-similar tilings of nilpotent Lie groups, see e.g. [18, 11].

Having explained the geometric framework of our investigation, let us turn to the combinatorial side, leading to the concept of a *substitution datum* (based on a given dilation datum). Fix some homogeneous substitution datum $\mathcal{D}$ and set $D := D_{\lambda_0}$. Moreover consider a finite set $\mathcal{A}$, called *colors.* A *substitution rule* is a map $S_0 : \mathcal{A} \to \mathcal{A}^{\Gamma \cap D(V)}$. In analogy to the abelian situation, there is a canonical extension of S_0 to the set of all colored patches supported on Γ, called the *substitution map* S. Moreover, the restriction of S to $\mathcal{A}^\Gamma$ (in the following also denoted by S) is continuous. The collection of the aforementioned objects is called a *subsitution datum (over $\mathcal{D}$)* and is denoted by $\mathcal{S} = \mathcal{S}(\mathcal{D}, \mathcal{A}, S_0)$. We will need two additional assumptions on substitution data.

- $\mathcal{S}$ is said to be *primitive* if there is some $L \in \mathbb{N}$ such that every $a \in \mathcal{A}$ occurs in the patch $S^L(b)$ for each $b \in \mathcal{A}$;
- $\mathcal{S}$ is called *non-periodic* if S_0 is injective and

$$\gamma^{-1} S(a)|_{\gamma^{-1}D(V) \cap D(V)} \neq S(b)|_{\gamma^{-1}D(V) \cap D(V)}$$

for all $\gamma \in \big(\Gamma \cap D(V)\big) \setminus \{e\}$ and all $a, b \in \mathcal{A}$.

In order to obtain interesting colorings $\omega \in \mathcal{A}^\Gamma$ one now seeks for fixed points under some iteration of S, i.e. one checks whether there are $k \in \mathbb{N}$ and $\omega \in \mathcal{A}^\Gamma$ such that $S^k(\omega) = \omega$. This question can be answered in the affirmative if $\mathcal{S}$ is assumed to be primitive. Moreover, ω is linearly repetitive with respect to the restriction of the metric d to Γ, i.e. all colored patterns in ω with "support size" r occur in every patch of the form $\omega_{|T_r}$, where T_r is any "test window of size $\kappa \cdot r$", with $\kappa \geq 1$ being independent of r. If $\mathcal{S}$ is additionally assumed to be non-periodic, then ω has trivial Γ-stabilizer. Precisely, we have the following theorem.

Theorem. *Consider a RAHOGRASP G of dimension at least* 2*, along with a homogeneous dilation datum $\mathcal{D}$. Let $\mathcal{A}$ be finite with at least two elements. Then there exists a primitive, non-periodic substitution datum $\mathcal{S}$ over $\mathcal{D}$ with color set $\mathcal{A}$. In particular there exists $\omega \in \mathcal{A}^\Gamma$ such that*

- *ω is linearly repetitive with respect to $d_{|\Gamma \times \Gamma}$,*
- *ω has trivial Γ-stabilizer,*
- *the action $\Gamma \curvearrowright \Omega_\omega := \overline{\{\gamma.\omega : \gamma \in \Gamma\}}$ is uniquely ergodic.*

It is known from abelian settings that there are strong interrelations between uniform subadditive convergence theorems, linear repetitivity of aperiodic structures, and unique ergodicity of the associated dynamical systems, cf. [9, 13, 7, 14, 10]. This is also true in the nilpotent case as was shown recently in our companion paper [2], and so unique ergodicity in the above theorem is a consequence of linear repetitivity.

In the two 2-step case we can even guarantee ω to be strongly aperiodic which means that every $\eta \in \Omega_\omega$ has trivial Γ-stabilizer.

Theorem. *Consider a 2-step RAHOGRASP G. Let $\mathcal{A}$ be finite and containing at least two elements. Then there exists a uniform lattice $\Gamma \leq G$ and a coloring $\omega \in \mathcal{A}^{\Gamma}$ such that*

- *ω is linearly repetitive with respect to $d_{|\Gamma\times\Gamma}$,*
- *the action $\Gamma \curvearrowright \Omega_\omega$ is uniquely ergodic and free.*

It is a rather immediate consequence from the theorem that many 2-step groups, such as Heisenberg groups, contain strongly aperiodic Delone sets that are linearly repetitive with respect to the group metric. To the best of our knowledge these are the first explicit examples of this kind in non-abelian Lie groups.

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Purely infinite crossed products by non-amenable groups

Shirly Geffen

(joint work with Eusebio Gardella, Julian Kranz, Petr Naryshkin)

Let G be a countable discrete group acting by homeomorphisms on a compact metric space X. If $U \subseteq X$ is a non-empty set, we write $X \prec U$ if X admits a finite open cover whose elements can be transported via the group action to pairwise disjoint subsets of U.

In the absence of invariant measures for the system (which is always the case when the group G is non-amenable and the given action is topologically amenable), it is said that the system $G \to \mathrm{Homeo}(X)$ has *dynamical comparison* if $X \prec U$ for every non-empty open subset $U \subseteq X$ (see [2, Definition 3.2]).

We show in [1] that all amenable minimal actions of a large class of non-amenable countable groups on compact metric spaces have dynamical comparison. This class includes all non-amenable hyperbolic groups, many HNN-extensions, non-amenable Baumslag-Solitar groups, a large class of amalgamated free groups, lattices in many Lie groups, as well as direct products of the above with arbitrary countable groups. As a consequence, crossed product C^*-algebras by amenable minimal and topologically free actions of such groups on compact metric spaces are Kirchberg algebras in the UCT class, and are therefore classified by K-theory.

We conjecture that all minimal (amenable) actions of non-amenable groups on compact metric spaces have dynamical comparison.

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Ramsey properties of ultraproducts of finite structures

Dana Bartošová

(joint work with Mirna Džamonja, Rehana Patel, Lynn Scow)

The seminal work of Kechris, Pestov, and Todorcevic [3] revealed deep connections between abstract topological dynamics of infinite-dimensional groups and Ramsey theory of finitely-generated structures.

Definition 1 (Ramsey property)**.** Let $\mathcal{K}$ be a class of finitely-generated first order structures. We say that $A \in \mathcal{K}$ has the Ramsey property in $\mathcal{K}$ if for every $B \in \mathcal{K}$ which embeds A, there is $C \in \mathcal{K}$ such that for every colouring c of copies of A in C by finitely many colours, there is a copy of B in C whose all copies of A have the same colour, i.e., it is monochromatic.

For simplicity, we introduce the following notation. For two structures A and B, we write $A \leq B$ when A is a substructure of B. We denote by $\binom{B}{A}$ all substructure of B isomorphic to A (copies of A).

The classes of finite sets, finite linear orders, or finite Boolean algebras are examples of classes whose members all have the Ramsey property. However, most classes have objects that do not posses the Ramsey property. They might however have only finitely many "recognizable types" of embeddings within the class, leading to the notion of finite Ramsey degrees.

Definition 2 (Ramsey degrees)**.** We say that A in a class of finitely-generated structures $\mathcal{K}$ has a finite Ramsey degree if there exists $t \in \mathbb{N}$, such that for every B with $A \leq B$ and any $r \geq 2$, there is $C \in \mathcal{A}$ such that for any $c : \binom{C}{A} \to \{0, 1, \ldots, r-1\}$ there is $B' \in \binom{C}{B}$ such that c restricted to $\binom{B'}{A}$ takes on at most t colours. The smallest such t (if it exists) is called the Ramsey degree of A in $\mathcal{K}$.

Classes of finite (K_n-free) graphs and different types of finite hypergraphs are examples of classes whose all members have finite Ramsey degrees. In [3], the authors showed that if $\mathcal{K}$ has finite Ramsey degrees, it is the amalgamation property and up to isomorphism it is countable, then the automorphism group of its generic limit (Fraïsslímit) has only metrizable minimal flows. This striking connection fueled much new development in the study of Ramsey properties of classes of finitely-generated structures and their more difficult version when a monochromatic copy, or a copy with a bounded number colours, of the whole countable structure is required. For a countable structure $\mathcal{A}$, let Age($\mathcal{A}$) be the class of all finitely-generated structures isomorphic to a substructure of $\mathcal{A}$.

Definition 3 (Big Ramsey degrees)**.** Let $\mathcal{A}$ be a countable structure and let $A \in \text{Age}(\mathcal{A})$. We say that A has big finite Ramsey degree in $\mathcal{A}$ if there is $t \in \mathbb{N}$ such that for every $B \in \text{Age}(\mathcal{A})$ and every $c : \binom{\mathcal{A}}{A} \to \{0, 1, \ldots, r-1\}$ for r finite, there is $\mathcal{A}' \in \binom{\mathcal{A}}{A}$ such that c restricted to $\binom{A'}{A}$ takes on at most t colours. The smallest such t (if it exists) is the big Ramsey degree of A.

A countably infinite sets admit finite big Ramsey degrees for all finite subsets by Ramsey theorem. Exact big Ramsey degrees for finite linear orders in the countable linear order without endpoints $(\mathbb{Q}, <)$ were computed by Devlin in [2] and Sauer showed that finite graphs have finite big Ramsey degrees in the random graph in [4].

In this talk we present our current research on big Ramsey degrees of ultraproducts of finite structures. In order to ensure that our ultraproduct contains all structures from a class of interest in its Age we introduce the following notion.

Definition 4 ($\mathcal{D}$-trending)**.** Let $\mathcal{K}$ be a class of finite structures and let $\mathcal{D}$ be a non-principal ultrafilter on ω. We say that a sequence $(K_i)_{i\in\omega}$ of structures from $\mathcal{K}$ is $\mathcal{D}$-trending if for every $A \in \mathcal{K}$, the set $\{i \in \omega : A \text{ embeds into } K_i\}$ is in $\mathcal{D}$.

The natural starting point are colourings on the ultraproduct that are determined by a sequence of colouring on the coordinates.

Definition 5 (Internal colouring)**.** Let $\mathfrak{K} = \Pi_{i\in\omega} K_i/\mathcal{D}$ be an ultraproduct of finite structures from a class $\mathcal{K}$ over an ultrafilter $\mathcal{D}$. Let $A \in \mathcal{K}$ and suppose that A embeds into $\mathfrak{K}$. We say that a colouring $c : \binom{\mathfrak{K}}{A} \to \{0, 1, \ldots, r-1\}$ is internal if there are colourings $c_i : \binom{K_i}{A} \to \{0, 1, \ldots, r-1\}$ for $i \in \omega$ such that $c(A') = n$ if and only if $\{i \in \omega : c_i(A'(i)) = n\} \in \mathcal{D}$.

Our main theorem is the following.

Theorem 1 (BDPS)**.** *Let $\mathcal{K}$ be a class of finite structures and let $\mathcal{D}$ be a non-principal ultrafilter on ω. Let $\mathfrak{K}$ denote $\Pi_{i_\omega} K_i/\mathcal{D}$, where $(K_i)_{i\in\omega}$ is $\mathcal{D}$-trending. Suppose that $A \in \mathcal{K}$ has a Ramsey degree d in $\mathcal{K}$ and let $\mathcal{A}$ be any structure of cardinality at most $\aleph_1$ with Age($\mathcal{A}$) $\subset \mathcal{K}$. Then for any internal colouring $c : \binom{\mathfrak{K}}{A} \to \{0, 1, \ldots, r-1\}$ there is a copy $\mathcal{A}'$ of $\mathcal{A}$ in $\mathfrak{K}$ such that c restricted to $\binom{\mathcal{A}'}{A}$ assumes at most d colors.*

Since every countable ultraproduct of finite structures is either finite or of cardinality $2^{\aleph_0}$, we have the following corollary.

Corollary (BDPS)**.** *Under CH, if the conditions in Theorem 1 hold, then A has big Ramsey degree d in $\mathfrak{K}$.*

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Minimal flows without a characteristic measure

Andy Zucker

(joint work with Josh Frisch, Brandon Seward)

Given a countable group G and a faithful G-flow X, we write $\mathrm{Aut}(X, G)$ for the group of homeomorphisms of X which commute with the G-action. When G is abelian, $\mathrm{Aut}(X, G)$ contains a natural copy of G resulting from the G-action, but in general this need not be the case. Much is unknown about how the properties of X restrict the complexity of $\mathrm{Aut}(X, G)$; for instance, Cyr and Kra [1] conjecture that when $G = \mathbb{Z}$ and $X \subseteq 2^{\mathbb{Z}}$ is a minimal, 0-entropy subshift, then $\mathrm{Aut}(X, \mathbb{Z})$ must be amenable. In fact, no counterexample is known even when restricting to any two of the three properties "minimal," "0-entropy," or "subshift." In an effort to shed light on this question, Frisch and Tamuz [2] define a probability measure $\mu \in P(X)$ to be *characteristic* if it is $\mathrm{Aut}(X, G)$-invariant. They show that 0-entropy subshifts always admit characteristic measures, but along the way raise several questions about these objects. In particular, they asked whether there exists, for any countable group G, some minimal G-flow without a characteristic measure. We give a strong affirmative answer.

Theorem 1. *For any countably infinite group G, there is a free minimal G-flow X so that X does not admit a characteristic measure. More precisely, there is a free $G \times F_2$-flow X which is minimal as a G-flow and with no F_2-invariant measure.*

Over the course of proving Theorem 1, there are two main difficulties to overcome. The first difficulty is a collection of dynamical problems we refer to as *minimal subdynamics*. The general template of these questions is as follows. Consider a countably infinite group Γ and a collection $\{\Delta_i : i \in I\}$ of infinite subgroups. When is there a free Γ-flow which is minimal as a Δ_i-flow for every $i \in I$ simultaneously? In [4], the author showed that this was possible in the case $\Gamma = G \times H$ and $\Delta = G$ for any countably infinite groups G and H. We manage to strengthen this result considerably.

Theorem 2. *For any countably infinite group Γ and any collection $\{\Delta_n : n \in \mathbb{N}\}$ of infinite normal subgroups of Γ, there is a free Γ-flow which is minimal as a Δ_n-flow for every $n \in \mathbb{N}$.*

In fact, what we show when proving Theorem 2 is considerably stronger. Recall that given a countably infinite group Γ, a subshift $X \subseteq 2^\Gamma$ is *strongly irreducible* if there is some finite symmetric $D \subseteq \Gamma$ so that whenever $S_0, S_1 \subseteq \Gamma$ satisfy $DS_0 \cap S_1 = \emptyset$ (i.e. S_0 and S_1 are *D-apart*), then for any $x_0, x_1 \in X$, there is $y \in X$ with $y|_{S_i} = x_i|_{S_i}$ for each $i < 2$. Write $\mathcal{S}$ for the set of strongly irreducible subshifts, and write $\overline{\mathcal{S}}$ for its Vietoris closure. Frisch, Tamuz, and Vahidi-Ferdowsi [3] show that in $\overline{\mathcal{S}}$, the minimal subshifts form a dense G_δ subset. In our proof of Theorem 2, we show that the shifts in $\overline{\mathcal{S}}$ which are Δ_n-minimal for each $n \in \mathbb{N}$ also form a dense G_δ subset.

This brings us to the second main difficulty in the proof of Theorem 1. Using this stronger form of Theorem 2, one could easily prove Theorem 1 by finding a strongly irreducible F_2-subshift which does not admit an invariant measure. This would imply the existence of a strongly irreducible $(G \times F_2)$-subshift without an F_2-invariant measure. As not admitting an F_2-invariant measure is a Vietoris-open condition, the genericity of G-minimal subshifts would then be enough to obtain the desired result. Unfortunately whether such a strongly irreducible subshift can exist (for any non-amenable group) is a wide-open question. To overcome this, we introduce a flexible weakening of the notion of a strongly irreducible shift.

Definition. Let $\mathcal{P}_f(\Gamma)$ denote the collection of finite subsets of Γ, and fix a right-invariant subset $\mathcal{B} \subseteq \mathcal{P}_f(\Gamma)$. We say that a subshift $X \subseteq 2^\Gamma$ is *$\mathcal{B}$-irreducible* if there is a finite symmetric $D \subseteq \Gamma$ so that for any $m < \omega$, any $B_0, ..., B_{m-1} \in \mathcal{B}$, and any $x_0, ..., x_{m-1} \in X$, if the sets $\{B_0, ..., B_{m-1}\}$ are pairwise D-apart, then there is $y \in X$ with $y|_{B_i} = x_i|_{B_i}$ for each $i < m$. We call D the *witness* to $\mathcal{B}$-irreducibility. If we have D in mind, we can say that X is *$\mathcal{B}$-D-irreducible.*

When $\mathcal{B} = \mathcal{P}_f(\Gamma)$, we recover the ordinary notion of strongly irreducible shift. By then considering F_2-subshifts which encode paradoxical decompositions and considering the family $\mathcal{B} \subseteq \mathcal{P}_f(F_2)$ of finite sets which are connected in the standard left Cayley graph of F_2, we can obtain $\mathcal{B}$-irreducible shifts with no F_2-invariant measure. With this idea as our starting point, we prove the following about $G \times F_2$; with a bit of extra work to get the freeness result, the following implies Theorem 1.

Theorem 3. *For each $n \in \mathbb{N}$, set*

$$\mathcal{B}_n := \{B \in \mathcal{P}_f(G \times F_2) : \text{for every } F_2\text{-coset } C, \text{ different connected components of } B \cap C \text{ are at distance at least } n\}$$

Let $\mathcal{S}_n$ denote the collection of $\mathcal{B}_n$-irreducible $(G \times F_2)$-subshifts. Then there is some $X \in \bigcup_n \mathcal{S}_n$ with no F_2-invariant measure, and the collection of G-minimal members of $\overline{\bigcup_n \mathcal{S}_n}$ is dense G_δ.

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Tame dynamical systems and their relation to the nature of the acting group

Eli Glasner

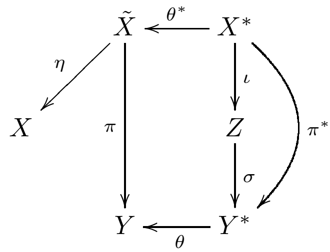
When the topology of the enveloping semigroup of a flow (X, G), for say a general countable group G, is determined by the convergence of sequences (rather than nets) we have at our disposal many powerful tools which are not available in the general case. Perhaps the most important one is the Lebesgue convergence theorem. We started our talk by considering the following statement.

Let $(X, G) \overset{\pi}{\to} (Y, G)$ be a proximal extension of minimal flows with (Y, G) an equicontinuous flow. Then if (X, G) admits an invariant measure it is unique.

This statement fails in general, but assuming that the enveloping semigroup $E(X, G)$ is Fréchet and using Lebesgue's convergence theorem, this becomes a theorem. A system whose enveloping semigroup is Fréchet is called *tame*.

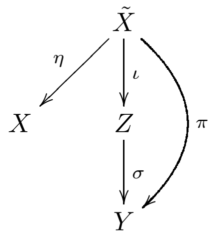
In my talk I defined the notion of tameness in topological dynamics, described several useful characterisations and then discussed several results which show how the nature of the acting group G is related to the structure of tame dynamical G-systems. The most prominent of these is the following structure theorem which demonstrates the role of the acting group being amenable:

Theorem. *For a general group G, a tame, metric, minimal dynamical system (X, G) has the following structure:*



Here (i) $\tilde{X}$ is a metric minimal and tame system (ii) η is a strongly proximal extension, (iii) Y is a strongly proximal system, (iv) π is a point distal and RIM extension with unique section, (v) θ, θ^* and ι are almost one-to-one extensions, and (vi) σ is an isometric extension.

When the map π is also open this diagram reduces to



In general the presence of the strongly proximal extension η is unavoidable. If the system (X, G) admits an invariant measure μ then Y is trivial and $X = \tilde{X}$ is an almost automorphic system; i.e. $X \xrightarrow{\iota} Z$, where ι is an almost one-to-one extension and Z is equicontinuous. Moreover, μ is unique and ι is a measure theoretical isomorphism $\iota : (X, \mu, G) \to (Z, \lambda, G)$, with λ the Haar measure on Z. Thus, this is always the case when G is amenable.

Belinskaya's theorem is optimal

François Le Maître

(joint work with Alessandro Carderi, Matthieu Joseph and Romain Tessera)

This talk revolved around two cornerstone theorems on the orbits of measure-preserving transformations. The first is Dye's theorem [3], which states that up to conjugacy, any two ergodic measure-preserving transformations of standard probability spaces share the same orbits. In particular, one has to ask for more stringent conditions than sharing the same orbits in order to get an interesting conjugacy invariant.

The second theorem, due to Belinskaya [1], provides such a condition. Given two ergodic measure-preserving transformations T_1 and T_2 of a standard probability space (X, μ) with the same orbits, one naturally gets two cocycles $c_{T_1}, c_{T_2} : X \to \mathbb{Z}$, uniquely defined by the equations

$$T_1(x) = T_2^{c_{T_1}(x)} \text{ and } T_2(x) = T_1^{c_{T_2(x)}}(x).$$

Belinskaya's theorem states that if c_{T_2} is moreover an L^1 map (meaning that $\int_X |c_{T_2}(x)| d\mu(x)$ is finite), then T_1 and T_2 are flip-conjugate, which is the strongest possible conclusion: up to conjugating T_1 by a measure-preserving transformation, we have $T_1 = T_2$ or $T_1 = T_2^{-1}$.

We investigated what happens when one replaces the L^1 condition by L^p, where $p \in (0, 1)$. Our main result is that Belinskaya's theorem becomes false [2]: given any ergodic transformation T_1, there is another transformation T_2 with the same orbits as T_1 whose cocycle satisfies $\int_X |c_{T_2}(x)|^p d\mu(x) < +\infty$ but which is not flip-conjugate to T_1. This result uses crucially the so-called L^p full groups of measure-preserving transformations, which are Polish groups for their natural L^p metric. We also discussed an application of our results which answers a question of Kerr and Li [4]: for general ergodic measure-preserving transformations, Shannon orbit equivalence does not imply flip conjugacy.

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Hyperbolic groups of type I

Pierre-Emmanuel Caprace

(joint work with Mehrdad Kalantar and Nicolas Monod)

1. A type I conjecture

Let G be a second countable locally compact group. We are interested in relating the algebraic structure of the group G with the properties of its unitary representations. Classical theory on unitary representations reveals a basic dichotomy between the so-called **type I groups** and the locally compact groups that are not type I. By definition, the group G is type I if for every continuous unitary representation π, the von Neumann algebra $\pi(G)''$ is of type I. The type I groups are precisely those whose unitary representations are well behaved: the direct integral decomposition of an arbitrary unitary representation into irreducible representations is essentially unique, and the classification problem of the irreducible unitary representations up to equivalence has moderate complexity (more precisely, the unitary dual of the group is a standard Borel space). For more information and background on this topic, we refer to [1]. The general theme of our investigations consists in elucidating the structural properties of type I groups.

A sufficient condition for a locally compact group G to be type I is that G be **liminal** (or **CCR**), which means that for every irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of G, the image $\pi(C^*(G))$ of the maximal C^*-algebra of G entirely consists of compact operators on the Hilbert space $\mathcal{H}_\pi$. A necessary condition for G to be type I is that the group von Neumann algebra $L(G) = \lambda_G(G)''$ be type I, where λ_G denotes the left regular representation.

The class of liminal groups includes compact groups, abelian groups, connected nilpotent Lie groups, semisimple Lie groups and semisimple algebraic groups over local fields, as well as some non-linear groups like the full automorphism group of a regular locally finite tree.

The larger class of type I groups includes all algebraic groups over local fields of characteristic 0. An example of a type I group that is not liminal is the affine group $\mathbf{Q}_p \rtimes \mathbf{Q}_p^*$.

The free group F_2 is not type I: indeed, its group von Neumann algebra is a factor of type II_1. Certain connected solvable Lie groups are not type I either; among them, the best known example is the Mautner group, which is a connected, simply connected 5-dimensional metabalian Lie group.

In the case of discrete groups, the relationship between the type I condition and the group structure is fully elucidated by the following result, due to E. Thoma.

Theorem 1 (Thoma [9])**.** *For a discrete group G, the following conditions are equivalent.*

(i) *G is type I.*
(ii) *G is virtually abelian.*
(iii) *$L(G)$ is type I.*
(iv) *G is liminal.*
(v) *Every irreducible unitary representation of G is finite-dimensional.*
(vi) *The supremum of the dimensions of irreducible unitary representations of G is finite.*

The following conjectural description of non-discrete type I groups is proposed in [4].

Conjecture 1. *Let G be a second countable locally compact group. If G is type I, then G has a cocompact amenable subgroup.*

This statement is of course meaningless if G is amenable. The existence of amenable groups that are not type I shows that the converse to Conjecture 1 need not hold. However, it might potentially hold under the additional assumption that G has a trivial amenable radical, i.e. that the only amenable closed normal subgroup of G is the trivial subgroup $\{e\}$.

2. Groups acting on trees

Beyond Lie, algebraic and discrete groups, the class of locally compact groups whose unitary representations are the most studied is the class of groups acting on trees, which we now briefly discuss.

Let T be a locally finite tree, each of whose vertices has degree at least 3. The set of ends ∂T carries a natural topology, which makes it a compact Hausdorff space. Throughout this section, we let G be a non-compact closed subgroup of $\mathrm{Aut}(T)$ whose action on ∂T is minimal. In this context, we have the following.

Conjecture 2 (Nebbia [8], Houdayer–Raum [7]). *The following conditions are equivalent.*

(i) *G is liminal.*
(ii) *G is type I.*
(iii) *The G-action on ∂T is 2-transitive.*

It is worth noting that, in this context, the condition (iii) is known to be equivalent to the existence of a cocompact amenable subgroup in G, see [3]. The implication from (i) to (ii) is valid in full generality. Nebbia's conjecture from [8] concerns the implication from (iii) to (i). The conjectural implication from (ii) to (iii) is proposed by Houdayer–Raum in [7], where they notably establish the following.

Theorem 2 (Houdayer–Raum [7]). *Let $G \leq \mathrm{Aut}(T)$ be as above. If $L(G)$ is amenable (e.g. if $L(G)$ is type I), then the G-action on T is locally 2-transitive, i.e. the stabiliser of every vertex acts 2-transitively on the set of neighbouring vertices.*

If the G-action on ∂T is 2-transitive, then the G-action on T is locally 2-transitive. The converse need not hold.

3. Hyperbolic groups

A locally compact group is called **hyperbolic** if it is word hyperbolic with respect to some compact generating set. Hyperbolic locally compact groups can be characterized as those groups acting continuously, properly, cocompactly by isometries on a proper hyperbolic geodesic metric space. All simple Lie groups of rank ≤ 1 are hyperbolic. If G contains a uniform lattice Γ, then G is hyperbolic if and only if Γ is so. If T is a locally finite tree, a closed subgroup $G \leq \mathrm{Aut}(T)$ acting minimally on ∂T is hyperbolic if and only if it is compactly generated. We refer to [3] for more information.

The main result presented in this talk is that Conjecture 1 holds for hyperbolic groups containing a uniform lattice.

Theorem 3 (Caprace–Kalantar–Monod [4]). *Let G be a hyperbolic group containing a uniform lattice. If G is type I, then G has a cocompact amenable subgroup.*

The structure of hyperbolic groups with a cocompact amenable subgroup has been studied in detail in [3]. In particular, Theorem 3 can be combined with the following.

Theorem 4 (Caprace–Cornulier–Monod–Tessera [3]). *Let G be a unimodular hyperbolic group. If G has a cocompact amenable subgroup, then G as a compact normal subgroup W such that the quotient G/W satisfies exactly one of the following descriptions.*

(i) *G/W is a simple Lie group of rank one.*
(ii) *G/W is a closed subgroup of the automorphism group of a locally finite non-elementary tree T, acting without inversions and with exactly two orbits of vertices, and acting 2-transitively on ∂T.*
(iii) *G/W is trivial or virtually isomorphic to $\mathbf{Z}$ or $\mathbf{R}$.*

Specializing to groups acting on trees, we obtain the following.

Corollary 1. *The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in Conjecture 2 hold.*

To establish this, we first invoke Theorem 2 to reduce to the case where G is unimodular and cocompactly generated. The main result of [2] then ensures that G contains a uniform lattice, and the required conclusion follows by invoking Theorems 3 and 4.

4. Boundary representations

The strategy of proof of Theorem 3 relies on two ingredients. The first one is Glimm's characterization of type I groups [6], according to which a group G is type I if and only if any two weakly equivalent irreducible unitary representations of G are unitarily equivalent. The second ingredient is provided by the work of Garncarek [5] on boundary representations of discrete hyperbolic groups. Given a discrete hyperbolic group Γ and a hyperbolic word metric d on Γ, one may construct a canonical Γ-invariant measure class ν_d on the Gromov boundary $\partial\Gamma$, called a Patterson–Sullivan measure. The associated Koopman representation κ_{ν_d} of Γ on $L^2(\partial\Gamma, \nu_d)$ is called a **boundary representation**. Garncarek proves that each boundary representation is irreducible. Moreover, if d, d' are different hyperbolic word metrics, then the representations κ_{ν_d} and $\kappa_{\nu_{d'}}$ are unitarily equivalent if and only if the metrics d, d' are **homothetic up to an additive constant**, i.e. there exist constants $L, C > 0$ such that

$$Ld(x,y) - C \leq d'(x,y) \leq Ld(x,y) + C$$

for all $x, y \in \Gamma$. While any two word metrics are quasi-isometric, the condition of being homothetic up to an additive constant is a very demanding one. In particular, varying the word metric, one obtains a wealth of inequivalent irreducible representations of the discrete group Γ.

In the context of Theorem 3, we have a non-discrete hyperbolic group G containing a uniform lattice Γ. The first step of the proof consists in showing that any two boundary representations of G are weakly equivalent. Restricting those representations of Γ and invoking Garncarek's results, we deduce that the boundary representations of G are irreducible. If G is type I, then by Glimm's theorem those boundary representations are all pairwise unitarily equivalent. Using again Garncarek's results for Γ, together with the fact that G/Γ is compact, we infer that any two hyperbolic word metrics on G are homothetic up to an additive constant. The final step of the proof is purely geometric: from that restriction on the word metrics on G, we deduce the existence of a cocompact amenable subgroup.

The hypothesis of existence of a uniform lattice in Theorem 3 is an ad hoc condition, which is necessary for us to invoke the work of Garncarek [5]. It is likely that Garncarek's work can be generalized to all non-amenable hyperbolic locally compact groups. Such a generalization would ensure that Theorem 3 holds for all hyperbolic groups.

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Settled elements of the group of automorphisms of trees.

María Isabel Cortez

(joint work with Olga Lukina)

For every $n \geq 1$, let $d > 1$ be an integer. The space $\Sigma = \{0, \cdots, d-1\}^{\mathbb{N}}$ is a Cantor set if we endow $\{0, \cdots, d-1\}$ with the discrete topology and Σ with the product topology. This is a metric space with the usual metric defined on a space of infinite sequences. The associated d-ary tree is the infinite graph T whose set of vertices V is equal to the disjoint union $\bigcup_{n\geq 0} V_n$, where V_0 contains only one element and V_n is the set of words of length n in $\{0, \cdots, d-1\}$, for every $n \geq 1$. The set of edges E of T is equal to the disjoint union $\bigcup_{n\geq 1} E_n$, where E_n contains exactly one edge e from $v \in V_{n-1}$ to $va \in V_n$, for every $a \in \{0, \cdots, d-1\}$ and $n \geq 1$. The set of infinite paths of T is given by

$$\partial T = \{(e_n)_{n\geq 1} \in \prod_{n\geq 1} E_n : t(e_n) = s(e_{n+1}), \forall n \geq 1\}.$$

The element $(x_n)_{n\geq 1} \in \Sigma$ is identified with $(e_n)_{n\geq 1} \in \partial T$ given by $t(e_1) = x_1$, $s(e_n) = x_1 \cdots x_{n-1}$ and $t(e_n) = x_1 \cdots x_n$, for every $n > 1$. This identification induces a bijection between Σ and ∂T, which is an isometry with the induced metric on ∂T.

An *automorphism* of T is a bijection $\tau : V \to V$ such that the restriction $\tau|_{V_n}$ is a permutation on V_n and such that preserves the structure of the tree. The collection of all the automorphisms of T is denoted by $\mathrm{Aut}(T)$. This is a group with the composition of functions which can be identified with the collection of isometries on ∂T. The group $\mathrm{Aut}(T)$ is a profinite group, and its topology coincide with the uniform and the pointwise convergence topologies.

We say that an element $\sigma \in \mathrm{Aut}(T)$ is *minimal* if the dynamical system given by the action of σ on ∂T is minimal. This is equivalent to say that for each $n \geq 1$, the permutation induced by σ on V_n has only one cycle. Given $\sigma \in \mathrm{Aut}(T)$, we say that $v \in V_n$ is in a stable cycle of σ if for every $m > n$, the vertices in V_m which are connected to the vertices in the cycle of σ to which v belongs, are in the same cycle in V_m. The automorphism σ is *settled* if the proportion of vertices in V_n which are in a stable cycle goes to 1 whenever n goes to infinity. If there exists $n \geq 1$ such that every vertex $v \in V_n$ is in a stable cycle, we say that σ is *strongly settled*. It is not difficult to see that strongly settled implies settled, and settled implies minimal.

Motivated by questions coming from Number Theory, Boston and Jones ([1]) asked under which conditions a subgroup Γ of $\mathrm{Aut}(T)$ is densely settled, i.e, under which conditions the set of settled elements of Γ is dense in Γ. Examples of densely settled groups are $\mathrm{Aut}(T)$ itself and the closure of the group generated by a settled automorphism.

In this talk we present the following result:

Theorem. *Let $d > 1$ be a prime number and let T be the d-ary tree. For every minimal $\sigma \in \mathrm{Aut}(T)$ the normalizer $N(\sigma)$ of $\overline{\langle\sigma\rangle}$ is densely settled.*

An important tool to show this results is the rational spanning of $N(\sigma)$, defined as the set $\{\sigma_{m,k} : m \geq 1, k \geq 1 \text{ not divisible by } d\}$, where $\sigma_{m,k}$ is the unique element σ in $N(a)$ verifying $\sigma a \sigma^{-1} = a^k$ and $\sigma(0^\infty) = a^m(0^\infty)$.

This result is part of the work [2].

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Quasi-invariant measures for self-similar groups

Volodymyr Nekrashevych

Hyperbolic groupoids (see [1]) are generalizations of various hyperbolicity conditions in dynamics and geometric group theory. One of them is the notion of a locally expanding self-covering of a compact metric space. Another is the notion of a *contracting self-similar group.* A *self-similar action* is defined as a faithful action of a group G on the space X^ω of infinite sequences over a finite alphabet, such that for every $g \in G$ and every $x \in X$ there is $g|_x \in G$ and $y \in X$ such that

$$g(xw) = yg|_x(w)$$

for all $w \in X^\omega$. We have then $g(vw) = ug|_v(w)$ for all $v \in X^*$ and $w \in X^\omega$.

We say that the action is *contracting* if there exists a finite set $N \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in N$ for all $v \in X^*$ such that $|v| \geq n$. Equivalently, $|g|_v| \leq \lambda^{|v|}|g| + C$ for some $\lambda \in (0,1)$.

Both give examples of *hyperbolic groupoids*: they are generated by a compact set S of germs of contractions (the germs of f^{-1} and the germs of $w \mapsto xg(w)$ for $x \in X$, $g \in N$), the *Cayley graphs* Γ_ξ of germs at a point ξ are Gromov-hyperbolic, and the negative paths are quasi-geodesics converging to one point of the boundary.

Let $\mathfrak{G}$ be a topological groupoid. A *quasi-cocycle* on $\mathfrak{G}$ is a map $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ such that there exists a constant $\eta > 0$ such that

- For every $g \in \mathfrak{G}$ there exists a neighborhood U of g such that $|\nu(g)-\nu(h)| < \eta$ for every $h \in U$.
- For any two $g_1, g_2 \in \mathfrak{G}$ for which g_1g_2 is defined we have

$$|\nu(g_1g_2) - (\nu(g_1) + \nu(g_2))| < \eta.$$

We consider quasi-cocycle up to the equivalence relation

$$\beta_1 \sim \beta_2 \Longrightarrow \sup_{g\in\mathfrak{G}} |\beta_1(g) - \beta_2(g)| < \infty.$$

Let $\mathfrak{G}$ be a hyperbolic groupoid, and let S be the corresponding generating set of germs of contractions. Then a *Busemann quasi-cocycle* on $\mathfrak{G}$ is a quasi-cocycle β such that there exist $C > 0, L > 1$ such that

$$L^{-1}n - C \leq \beta(s_1s_2\cdots s_n) \leq Ln + C$$

for all composable sequences of elements $s_i \in S$.

If β is a Busemann quasi-cocycle, then for all positive small enough α there exists a metric d on the unit space of $\mathfrak{G}$ such that every $g \in \mathfrak{G}$ there exists a neighborhood $U \subset \mathfrak{G}$ (seen as a partial homeomorphism of the unit space) such that

$$C^{-1}e^{-\alpha\cdot\beta(g)} \leq \frac{d(U(x), U(y))}{d(x,y)} \leq Ce^{-\alpha\cdot\beta(g)}.$$

There also exists a unique $\eta > 0$ such that there exists a measure on the unit space which is $\mathfrak{G}$-quasi-invariant and the Radon-Nikodim cocycle $\frac{dg^*(\mu)}{d\mu}$ satisfies

$$C^{-1}e^{-\eta\cdot\beta(g)} \leq \frac{dg^*(\mu)}{d\mu} \leq Ce^{-\eta\cdot\beta(g)}.$$

Hyperbolic groupoids come in pairs: a *dual hyperbolic groupoid* naturally acts on the Gromov boundary of the Cayley graph of the original groupoid, and vice versa. Given a Busemann quasi-cocycle on a hyperbolic groupoid, there is also a well defined (up to the mentioned above equivalence) dual Busemann quasi-cocycle on the dual groupoid.

Groupoids associated with contracting self-similar groups are dual to groupoids associated with expanding covering maps.

Our project to study Busemann cocycles on the hyperbolic groupoids associated with self-similar contracting groups. Every such a quasi-cocycle is uniquely determined by its values on germs of maps

$$S_v : w \mapsto vw,$$

i.e., essentially by a metric on the tree X^*.

For example, it is natural to try to describe the measures on X^ω associated with Busemann cocycles on the groupoids associated with a contracting self-similar group G. Another interesting project is finding the Hausdorff dimension of the natural metrics associated with the *dual* Busemann quasi-cocycles.

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Dynamical obstructions to classification by (co)homology and other TSI-group invariants.

Aristotelis Panagiotopoulos
(joint work with Shaun Allison)

One of the leading questions in many mathematical research programs is whether a certain classification problem admits a "satisfactory" solution. Hjorth's theory of turbulence provides conditions under which such a classification problem cannot be solved using only isomorphism types of countable structures as invariants. In the same spirit, we will introduce "unbalancedness": a new dynamical obstruction to classification by orbits of a Polish group which admits a two-side invariant metric (TSI). We will illustrate how "unbalancedness" can be used for showing that a classification problem cannot be solved by classical homology and cohomology invariants. We will finally apply these ideas to attain new anti-classification results for problem of classifying Hermitian line bundles up to isomorphism and the problem of classifying continuous trace C^*-algbras up to Morita equivalence. This is joint work with Shaun Allison.

Growth of actions of finitely generated solvable groups

Nicolás Matte Bon
(joint work with Adrien Le Boudec)

Let G be a finitely generated group endowed with a finite symmetric generating set S. Assume that X is a G-set, that is a set endowed with a G-action. We define the growth function of the action as

$$\mathrm{vol}_{G,X}(n) = \max_{x \in X} |B_{G,S}(n)x|.$$

Here $B_{G,S}(n)$ denotes the ball of radius n in the Cayley graph of (G, S). We note that this definition does note require the action to be transitive and, as a consequence, it remains well defined and decreases when passing to subgroups, in the sense that $\mathrm{vol}_{H,X}(n) \preceq \mathrm{vol}_{G,X}(n)$ for every finitely generated subgroup of G. In particular the study of this invariant can be used to rule out the existence of embeddings between groups.

It is clear that the function $\mathrm{vol}_{G,X}(n)$ is bounded above by the word of the group G, but many groups admit faithful actions with much smaller growth. For example non-abelian free groups admit faithful actions with $\mathrm{vol}_{G,X}(n) \simeq n$ (the slowest possible behaviour for a faithful action of an infinite group), and the same holds true for many other examples of groups. This raises the question whether there are interesting obstructions on certain (classes of) groups G to admit faithful actions of small growth. A basic example of this phenomenon are groups with property (T), for which it is well known that every faithful G-set X has exponential growth.

In this work we study the (non)-existence of faithful actions of small growth for finitely generated solvable groups. A first result concerns solvable groups of finite Prüfer rank, a class of groups which includes all polycyclic groups and all solvable groups which are linear over the field $\mathbb{Q}$ of rational numbers. For such groups we have the following.

Theorem. *If G is a finitely generated solvable group of finite Prüfer rank, then every faithful G-set satisfies* $\mathrm{vol}_{G,X}(n) \simeq \exp(n)$.

This phenomenon fails for more general solvable groups: there are example of solvable groups of exponential growth which admit faithful actions of polynomial growth. For example the lamplighter groups $G = (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}^d$ admit faithful actions with $\mathrm{vol}_{G,X}(n) \simeq n^d$, while $G = \mathbb{Z} \wr \mathbb{Z}^d$ admit faithful actions with $\mathrm{vol}_{G,X}(n) \simeq n^{d+1}$. For such wreath products, we show that these growth functions are optimal, in the sense that every faithhful G-set X satisifies $\mathrm{vol}_{G,X}(n) \succeq n^d$ for $G = (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}^d$ and $\mathrm{vol}_{G,X}(n) \succeq n^{d+1}$ for $G = \mathbb{Z} \wr \mathbb{Z}^d$. This is in fact a special case of a phenomoenon that holds for arbitrary metabelian groups. In fact for a metabelian group G, we establish a general lower bound on the growth of faithful G-sets in terms of the Krull dimension of G. The latter is a classical invariant of commutative rings which was introduced in the setting of metabelian groups by Cornulier, and further exploited by Jacoboni. More precisely we prove the following.

Theorem. *Let G be a finitely generated metabelian group which is not virtually abelian, and suppose that G has Krull dimension d. Then for every faithful G-set X we have* $\mathrm{vol}_{G,X}(n) \succeq n^d$.

We note that there are many finitely generated solvable group which admit faithful G-sets X whose growth satisfies the extreme behaviour $\mathrm{vol}_{G,X}(n) \simeq n$. Trivial examples are virtually abelian groups, but there are also examples of exponential growth, such as the lamplighter group $G = (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$. Nevertheless the following results show that this behaviour is forbidden in many situations.

Theorem. *Let G be finitely presented metabelian group. Then either G is virtually abelian or* $\mathrm{vol}_{G,X}(n) \succeq n^2$ *for every faithful G-set.*

Theorem. *Let G be a finitely generated torsion free solvable group. Assume that G is metabelian, or alternatively that it is linear over a field. Then either G is virtually abelian or* $\mathrm{vol}_{G,X}(n) \succeq n^2$ *for every faithful G-set.*

We conjecture that the previous result should hold for arbitrary torsion-free solvable groups.

Conjecture. *Let G be a finitely generated solvable torsion-free group. Then either G is virtually abelian or* $\mathrm{vol}_{G,X}(n) \succeq n^2$ *for every faithful G-set.*

We show that the conjecture is true if we restrict to transitive actions, and more generally to actions with finitely many orbits.

A crucial notion in the proof of the previous result is the one of *expanding subset* of a group. We say that a subset Σ of G is expanding if for every faithful G-set, and every finite subset of $\Delta \subset \Sigma$, there exists a point $x \in X$ such that the map $g \mapsto gx$ is injective on Δ. If Σ is expanding, then the Schreier graph of the action of G on X contains isometrically embedded copies of every finite subset of Σ. This of course can be used to bound from below the growth of faithful G-set, and also other geometric invariants such as the asymptotic dimension of the Schreier graph. Expanding subsets are tightly related to confined subgroups, which are the subgroups of a group G whose conjugacy class does not accumulate to the identity subgroup in the Chabauty space $\mathrm{Sub}(G)$. The proofs or the previous results are all based on the identification of good expanding subsets in the groups under consideration.

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Noncommutative dynamics of lattices in higher rank simple algebraic groups

Cyril Houdayer
(joint work with Uri Bader, Rémi Boutonnet)

In this talk, I present a noncommutative Nevo-Zimmer theorem for actions of (lattices in) higher rank simple algebraic groups on von Neumann algebras [1]. This extends to the realm of algebraic groups defined over arbitrary local fields the noncommutative Nevo-Zimmer theorem we obtained with Rémi Boutonnet in 2019 for real Lie groups [5].

I then discuss two applications of the above theorem for lattices in higher rank simple algebraic groups. The first application deals with the dynamics of positive definite functions and unitary representation theory. The second application is a noncommutative analogue of Margulis' factor theorem.

We use the following notation throughout. Let k be any local field, that is, k is a nondiscrete locally compact field. Let $\mathbf{G}$ be any almost k-simple connected algebraic k-group such that $\mathrm{rank}_k(\mathbf{G}) \geq 2$. We denote by $\mathbf{P} < \mathbf{G}$ a minimal parabolic k-subgroup.

Examples. For every $n \geq 3$, $\mathbf{G} = \mathrm{SL}_n$ is an absolutely almost simple connected algebraic k-group such that $\mathrm{rank}_k(\mathbf{G}) = n - 1 \geq 2$. Moreover, we can choose the minimal parabolic k-subgroup $\mathbf{P} < \mathbf{G}$ to be the subgroup of upper triangular matrices.

Denote by $G = \mathbf{G}(k)$ and $P = \mathbf{P}(k)$ the groups of k-points. We have $G/P = (\mathbf{G}/\mathbf{P})(k)$. We endow G/P with its unique G-invariant measure class. Let $\Gamma < G$ be any lattice, that is, $\Gamma < G$ is a discrete subgroup with finite covolume. Then we simply say that $\Gamma < G$ is a *higher rank lattice*.

Examples. Let $n \geq 3$. Here are examples of higher rank lattices:

- $\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$;
- $\mathrm{SL}_n(\mathbb{Z}[i]) < \mathrm{SL}_n(\mathbb{C})$;
- $\mathrm{SL}_n(\mathbb{F}_q[t^{-1}]) < \mathrm{SL}_n(\mathbb{F}_q((t)))$ where $q = p^r$, $r \geq 1$ and p is a prime.

Before stating our main result, we need to introduce some operator algebraic terminology. Let M be any von Neumann algebra and $\Gamma \curvearrowright M$ any action by automorphisms. We say that $\Gamma \curvearrowright M$ is ergodic if the fixed point von Neumann subagebra $M^\Gamma \subset M$ is trivial. Then we say that M is an ergodic Γ-von Neumann algebra. Let $\Theta : M \to L^\infty(G/P)$ be any Γ-equivariant faithful normal unital completely positive (ucp) map. Then we say that $\Theta : M \to L^\infty(G/P)$ is a Γ-*boundary structure*.

Note that the action $\Gamma \curvearrowright G/P$ is amenable and ergodic. Fix a probability measure $\nu_P \in \mathrm{Prob}(G/P)$ in the unique G-invariant measure class.

Examples. We provide two examples of Γ-boundary structures.

(1) Let $\Gamma \curvearrowright X$ be any minimal action on a compact metrizable space. Since $\Gamma \curvearrowright G/P$ is amenable, there exists a Γ-equivariant measurable map $\beta : G/P \to \mathrm{Prob}(X)$. By duality, we obtain a Γ-equivariant ucp map $\Theta_\beta : C(X) \to L^\infty(G/P)$. Letting $\nu = \nu_P \circ \Theta_\beta$, we may extend Θ_β to obtain a Γ-boundary structure $\Theta : L^\infty(X, \nu) \to L^\infty(G/P, \nu_P)$.

(2) Let $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ be any unitary representation and set $A := C^*_\pi(\Gamma) = C^*(\pi(\Gamma))$. Since $\Gamma \curvearrowright G/P$ is amenable, there exists a Γ-equivariant measurable map $\beta : G/P \to \mathfrak{S}(A)$. By duality, we obtain a Γ-equivariant ucp map $\Theta_\beta : A \to L^\infty(G/P)$. Letting $\varphi = \nu_P \circ \Theta_\beta$, we may extend Θ_β to obtain a Γ-boundary structure $\Theta : \pi_\varphi(A)'' \to L^\infty(G/P, \nu_P)$.

Our main result is the following noncommutative analogue of Nevo-Zimmer's structure theorem for stationary actions of higher rank simple Lie groups on standard measure spaces [10].

Theorem ([1, 5])**.** *Let $\Gamma < G$ be any higher rank lattice. Let M be any ergodic Γ-von Neumann algebra and $\Theta : M \to L^\infty(G/P)$ any Γ-boundary structure. The following dichotomy holds:*

- *Either* $\Theta(M) = \mathbb{C}1$.
- *Or there exist a proper parabolic* k*-subgroup* $\mathbf{P} < \mathbf{Q} < \mathbf{G}$ *and a* Γ*-equivariant unital normal embedding* $\iota : L^\infty(G/Q) \hookrightarrow M$ *such that* $\Theta \circ \iota : L^\infty(G/Q) \hookrightarrow L^\infty(G/P)$ *is the canonical normal embedding, where* $Q = \mathbf{Q}(k)$.

A few comments are in order. Our main theorem extends Nevo-Zimmer's structure theorem in two ways. Firstly, we deal with Γ-actions rather than G-actions. Secondly, we deal with arbitrary (noncommutative) von Neumann algebras. In the case when $k = \mathbb{R}$, the above theorem was proved in [5]. It was generalized in [1] to arbitrary local fields k.

Our first application deals with the dynamics of the space of positive definite functions. Denote by $\mathscr{P}(\Gamma) \subset \ell^\infty(\Gamma)$ the weak-$*$ compact convex set of all normalized positive definite functions on Γ endowed with the affine conjugation action $\Gamma \curvearrowright \mathscr{P}(\Gamma)$. A fixed point for this action is called a *character*. We denote by $\mathrm{Char}(\Gamma) \subset \mathscr{P}(\Gamma)$ the weak-$*$ closed convex subset of all characters.

Theorem ([1, 5])**.** *Let* $\Gamma < G$ *be any higher rank lattice.*

(1) *Let* $\mathscr{C} \subset \mathscr{P}(\Gamma)$ *be any nonempty* Γ*-invariant weak-*$*$ *closed convex subset. Then* $\mathscr{C}$ *contains a character.*
(2) *The group* Γ *is* character rigid *in the sense that for any extremal character* $\varphi \in \mathrm{Char}(\Gamma)$*, either* φ *is supported on the center* $\mathscr{Z}(\Gamma)$ *or its GNS representation* π_φ *is finite dimensional.*

Assume moreover that $\mathbf{G}$ *has trivial center.*

(3) *For any weakly mixing unitary representation* $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$*, the left regular representation* $\lambda : \Gamma \to \mathscr{U}(\ell^2(\Gamma))$ *is* weakly contained *in* π*, that is, the map* $\pi(\Gamma) \to \lambda(\Gamma) : \pi(\gamma) \mapsto \lambda(\gamma)$ *extends to a unital* $*$*-homomorphism* $\mathrm{C}^*_\pi(\Gamma) \to \mathrm{C}^*_\lambda(\Gamma)$*. Moreover,* $\mathrm{C}^*_\pi(\Gamma)$ *has a unique trace and a unique maximal proper ideal.*
(4) *For any minimal action by homeomorphisms* $\Gamma \curvearrowright X$ *on a compact metrizable space, either* X *is finite or* $\Gamma \curvearrowright X$ *is topologically free.*

Items (1) and (2) strengthen Margulis' celebrated normal subgroup theorem [9, Theorem IV.4.9] and character rigidity results by Bekka [3], Creutz-Peterson [6] and Peterson [11]. Item (3) gives a far reaching generalization of the simplicity and the unique trace property of the reduced C*-algebra $C^*_\lambda(\Gamma)$ by Bekka-Cowling-de la Harpe [4]. Item (4) provides a topological analogue of a result by Stuck-Zimmer [12] and gives a positive answer to a recent problem due to Glasner-Weiss [7].

Our second main application deals with the structure of intermediate von Neumann subagebras $L(\Gamma) \subset M \subset L(\Gamma \curvearrowright G/P)$. Here $L(\Gamma \curvearrowright G/P)$ denotes the *group measure space* von Neumann algebra associated with the ergodic action $\Gamma \curvearrowright G/P$. Our next result can be regarded as a noncommutative analogue of Margulis' factor theorem [9, Theorem IV.2.11].

Theorem ([1]). *Let $\Gamma < G$ be any higher rank lattice. Assume that $\mathbf{G}$ has trivial center. Let $L(\Gamma) \subset M \subset L(\Gamma \curvearrowright G/P)$ be any intermediate von Neumann subalgebra. Then there exists a unique parabolic k-subgroup $\mathbf{P} < \mathbf{Q} < \mathbf{G}$ such that $M = L^\infty(\Gamma \curvearrowright G/Q)$, where $Q = \mathbf{Q}(k)$.*

It is well known that there are exactly $2^{\mathrm{rank}_k(\mathbf{G})}$ intermediate parabolic k-subgroups $\mathbf{P} < \mathbf{Q} < \mathbf{G}$. The above theorem implies that the inclusion of von Neumann algebras $L(\Gamma) \subset L(\Gamma \curvearrowright G/P)$ retains $\mathrm{rank}_k(\mathbf{G})$. Thus, our result sheds some new light on Connes' rigidity conjecture which asks whether the group von Neumann algebra $L(\Gamma)$ retains $\mathrm{rank}_k(\mathbf{G})$.

We refer the reader to [2] and to the ICM survey [8] for further results regarding noncommutative ergodic theory of higher rank lattices.

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Almost finite actions and classifiable crossed products

Petr Naryshkin

After several decades of work by many people, the classification program for C^*-algebras recently culminated in the following theorem (see [2]).

Theorem. *Unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebras satisfying the UCT are classified by their Elliott invariant (which consists of K-theoric and tracial data).*

Moreover, the result is optimal: every possible Elliott invariant can be realized within this class (and thus "classifiable" C*-algebras are ones that satisfy the conditions of the Theorem). One of the main sources of examples of C*-algebras are crossed products of topological dynamical systems, and it has been a long standing problem in the field to find sufficient conditions under which they are classifiable ([17, 6, 14, 13, 18, 19]). For free minimal topological systems $G \curvearrowright X$ with G an amenable discrete group and X is a compact metrizable space, the crossed product $C(X) \rtimes G$ automatically satisfies all the classifiability assumptions except for $\mathcal{Z}$-stability (and, in fact, these dynamical conditions are very close to being necessary). Thus, from now on we only consider systems satisfying the above conditions and it remains to determine which of them give rise to $\mathcal{Z}$-stable crossed products.

To address this problem in full generality, Kerr introduced the purely dynamical notion of almost finiteness [10], and proved that it's sufficient to imply $\mathcal{Z}$-stability of the crossed product. This notion is a topological version of the Ornstein-Weiss tiling theorem (for actions) and asks for an existence of disjoint open "towers" in the space such that: (i) they are parameterized by sufficiently invariant Følner sets, (ii) the remainder is small in a strong sense, and (iii) the diameters of all the tower levels are small. In a joint work with Szabó [12] he proved that an action is almost finite if and only if it has the small boundary property and comparison (the latter roughly states that if one set is smaller than the other with respect to all invariant measures, it is also smaller combinatorially). We're now ready to state the expected answer to the question of $\mathcal{Z}$-stability of a crossed product.

Conjecture. *For a system $G \curvearrowright X$ the following are equivalent:*

(1) *$G \curvearrowright X$ has the small boundary property,*
(2) *$G \curvearrowright X$ is almost finite,*
(3) *the crossed product $C(X) \rtimes G$ is classifiable.*

As explained above, the implications (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are known in all cases. While the implication (3) $\Rightarrow$ (1) is interesting and there has been some progress ([8], [9]), in this talk we concentrate on the results showing almost finiteness for certain classes of actions. So far, most of the work in this direction has been done under the assumption that the space X is zero-dimension. We mention here the three results which are the best available to our knowledge (assuming that $\dim(X) = 0$): Conley–Jackson–Kerr–Marks–Seward–Tucker-Drob showed [3] that a generic action of any countable amenable group is almost finite, Downarowicz and Zhang proved [5] that all actions of groups with locally subexponential growth have comparison (and are therefore almost finite), and Kerr and the author [11] have shown that actions of elementary amenable groups are almost finite. In fact, the latter two results can be generalized to the case $\dim(X) < \infty$ (which is still a significantly stronger assumption than having the small boundary property) since Kerr and Szabó [12] also showed that if all actions of a group G on zero-dimensional spaces are almost finite, then all actions of G on spaces with finite covering dimension are almost finite.

Much less is known about the case when X is infinite-dimensional (and the action $G \curvearrowright X$ still has the small boundary property). The earliest result in that situation is one of Elliott and Niu [7], where they show implication (1) $\Rightarrow$ (3) for $G = \mathbb{Z}$ using the theory of subhomogeneous C*-algebras (and completely bypassing (2)). However, in a surprising turn of events, it has recently been observed that the comparison property mentioned above is in many cases automatic: for $G = \mathbb{Z}^d$ first in a weaker form by Niu [16] and then in a stronger form by Bosa, Perera, Wu and Zacharias [1], and for G a finitely generated group of polynomial growth by the author. Thus, combined with the result of Kerr and Szabó, when G is a group of polynomial growth, (1) implies (2).

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Ergodic theorems along trees

Jenna Zomback
(joint work with Anush Tserunyan)

The classical pointwise ergodic theorem, whose first instance dates back to Birkhoff [1], states that for any ergodic measure-preserving transformation $T : X \to X$ on a standard probability space (X, μ) and $f \in L^1(X, \mu)$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int f \, d\mu$ for a.e. $x \in X$.

In general, an ergodic measure-preserving action of a countable (discrete) semigroup G on a standard probability space (X, μ) is said to have the *pointwise ergodic property* along a sequence (F_n) of finite subsets of G, if for every $f \in L^1(X, \mu)$, for μ-a.e. $x \in X$,

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(g \cdot x) = \int f \, d\mu.$$

It is a celebrated theorem of Lindenstrauss [4] that the pointwise ergodic property is true for the pmp actions of all countable amenable groups along tempered Følner sequences and this was extended by Butkevich in [3] to all countable left-cancellative amenable semigroups.

Amenability, or rather the fact that the Følner sets F_n are almost invariant, is essential for the pointwise ergodic property as it ensures that the limit of averages is an invariant function. This is why, to obtain a version of the pointwise ergodic property for nonamenable (semi)groups, e.g. for the nonabelian free groups $\mathbb{F}_r$, one has to imitate the almost invariance of finite sets by taking weighted averages instead, so that the weight of the boundary is small. One of the most general results in this vein is due to Bufetov [2] which applies to pmp actions of a free semigroup on a finite set I. It states, for any L^1 function, the convergence (to the conditional expectation of f) of weighted ergodic averages taken over the balls $I^{\leq n} := \sum_{k\leq n} I^k$. The weights are assigned to the finite words in I by a Markov chain on I.

We prove the following new pointwise ergodic theorem for pmp actions of free groups, vastly strengthening the conclusion of [2]: we replace balls by arbitrary finite subtrees of the standard Cayley graph. However, unlike [2], our theorem is only for groups (not semigroups) and for a more restrictive class of Markov chains.

Theorem 1 (Pointwise ergodic along trees [5])**.** *Let $\mathbb{F}_r$ be the free group on $r < \infty$ generators and let $\mathbb{F}_r \curvearrowright (X, \mu)$ be a (not necessarily free) ergodic pmp action of $\mathbb{F}_r$. Let I be the standard symmetric set of generators of $\mathbb{F}_r$ and let $\mathfrak{m}$ be a stationary Markov measure on the set of finite words in I whose support is $\mathbb{F}_r$ (the set of reduced words). For every $f \in L^1(X, \mu)$,*

$$\frac{1}{\mathfrak{m}(S)} \sum_{w\in S} f(w \cdot x)\mathfrak{m}(w) \to \int f \, d\mu \text{ as } \mathfrak{m}(S) \to \infty, \quad \text{for a.e. } x \in X,$$

where $S \subseteq \mathbb{F}_r$ ranges over all finite subtrees of the (left) Cayley graph of $\mathbb{F}_r$ containing the identity.

The limit in this theorem is not taken over a single sequence, but rather over the set of all trees, which means that for any sequence (S_n) of (not necessarily increasing/coherent) trees with $\mathfrak{m}(S_n) \to \infty$, the limit is equal to $\int f \, d\mu$. Taking S_n to be the ball of radius n in $\mathbb{F}_r$ gives the conclusion of [2].

The main result underlying Theorem 1 is a backward pointwise ergodic theorem for a pmp Borel transformation T (Theorem 2), where the averages are taken along trees of possible pasts (in the direction of T^{-1}). Although T is pmp, the induced orbit equivalence relation E_T is not pmp, unless T is one-to-one, so the averages are weighted according to the corresponding Radon–Nikodym cocycle $\mathfrak{w} : E_T \to \mathbb{R}^+$.

Theorem 2 (Backward pointwise ergodic along trees [5])**.** *Let (X, μ) be a standard probability space and let $T : X \to X$ an ergodic aperiodic countable-to-one pmp Borel transformation. Let E_T denote the induced orbit equivalence relation and let $\mathfrak{w} : E_T \to \mathbb{R}^+$, $(x, y) \to \mathfrak{w}_y(x)$, the Radon–Nikodym cocycle of E_T with respect to μ. For every $f \in L^1(X, \mu)$,*

$$\frac{1}{\mathfrak{w}_x(S_x)} \sum_{y \in S_x} f(y)\mathfrak{w}_x(y) \to \int f \, d\mu \text{ as } \mathfrak{w}_x(S_x) \to \infty \text{ for a.e. } x \in X,$$

where S_x ranges over all (possibly infinite) subtrees of the graph of T of finite height rooted at x and directed towards x, and $\mathfrak{w}_x(S_x) \coloneqq \sum_{y \in S_x} \mathfrak{w}_x(y)$.

Thus, while the classical pointwise ergodic theorem for T says that to approximate $\int f \, d\mu$, we can start almost anywhere in the space and walk forward in time (in the direction of T), Theorem 2 allows us to walk back in time (in the direction of T^{-1}) scanning sufficiently heavy trees of possible pasts. Note that Theorem 2 implies the classical (forward) pointwise ergodic theorem for one-to-one transformations T when applied to T^{-1}. We obtain Theorem 1 by applying Theorem 2 to a specific choice of T.

Theorem 2 implies, in particular, convergence of the $\mathfrak{w}$-weighted averages along any sequence (S_n) of subtrees with $\mathfrak{w}(S_n) \to \infty$. When these trees S_n have bounded height to $\mathfrak{w}$-weight ratio (i.e. are "bushy"), we in addition prove L^p convergence of these averages for all $p \geq 1$. An obvious example of such a sequence of "bushy" trees is that of complete trees of height n, i.e. $S_n \coloneqq \bigcup_{i<n} T^{-i}(x)$, and we now state this important special case.

Corollary (Backward pointwise ergodic along complete trees [5])**.** *Let (X, μ) be a standard probability space and let $T : X \to X$ an ergodic aperiodic countable-to-one pmp Borel transformation. Let E_T denote the induced orbit equivalence relation and let $\mathfrak{w} : E_T \to \mathbb{R}^+$, $(x, y) \to \mathfrak{w}_y(x)$, the Radon–Nikodym cocycle of E_T with respect to μ. For every $f \in L^1(X, \mu)$,*

$$\frac{1}{n+1} \sum_{i=0}^{n} \sum_{y \in T^{-i}(x)} f(y)\mathfrak{w}_x(y) \to \int f \, d\mu \quad \text{as } n \to \infty \text{ for a.e. } x \in X,$$

Furthermore, for any $1 \leq p < \infty$, if $f \in L^p(X, \mu)$, we also have convergence to $\int f \, d\mu$ in the L^p norm.

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Entropy, asymptotic pairs, and topological Markov properties

Hanfeng Li

(joint work with Sebastian Barbieri, Felipe Garcia-Ramos)

The talk is based on results in [1]. Let $\Gamma \curvearrowright X$ be a continuous action of a countably infinite discrete group Γ on a compact metrizable space X, and let ρ be a compatible metric of X.

Let $k \geq 2$ be an integer. A tuple $(x_1, \dots, x_k) \in X^k$ is called *asymptotic* or *homoclinic* if $\max_{1\leq i<j\leq k} \rho(sx_i, sx_j) \to 0$ as $s \to \infty$. Denote by $\mathrm{A}_k(\Gamma \curvearrowright X)$ the set of all asymptotic k-tuples.

The group Γ is called *amenable* if there is a sequence $\{F_n\}$ of nonempty finite subsets of Γ such that $\frac{|sF_n \Delta F_n|}{|F_n|} \to 0$ as $n \to \infty$ for every $s \in \Gamma$. Such a sequence is called a left Følner sequence. When Γ is amenable, one has the *topological entropy* $\mathrm{h}_{\mathrm{top}}(\Gamma \curvearrowright X) \in [0, \infty]$ defined as follows. For any finite open cover $\mathcal{U}$ of X, denote by $N(\mathcal{U})$ the minimal cardinality of subcovers of $\mathcal{U}$. For finite open covers $\mathcal{U}$ and $\mathcal{V}$ of X, the joint $\mathcal{U} \vee \mathcal{V}$ is the cover $\{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Given any finite open cover $\mathcal{U}$ of X, the limit $\lim_{n\to\infty} \frac{1}{|F_n|} \log N(\bigvee_{s\in F_n} s^{-1}\mathcal{U}) \in [0, |\mathcal{U}|]$ exists and does not depend on the choice of the left Følner sequence $\{F_n\}$. Then $\mathrm{h}_{\mathrm{top}}(\Gamma \curvearrowright X)$ is defined as the supremum of such limits for $\mathcal{U}$ ranging over finite open covers of X.

The *integral group ring* $\mathbb{Z}\Gamma$ is the set of all finitely supported functions $f : \Gamma \to \mathbb{Z}$. We write f formally as $\sum_{s\in\Gamma} f_s s$, where $f_s \in \mathbb{Z}$ for each $s \in \Gamma$ and $f_s = 0$ except for finitely many s. The addition and multiplication of $\mathbb{Z}\Gamma$ are give by $\sum_{s\in\Gamma} f_s s + \sum_{s\in\Gamma} g_s s = \sum_{s\in\Gamma}(f_s + g_s)s$ and $\big(\sum_{s\in\Gamma} f_s s\big)\big(\sum_{t\in\Gamma} g_t t\big) = \sum_{s,t\in\Gamma} f_s g_t (st)$ respectively.

The action $\Gamma \curvearrowright X$ is said to be *algebraic* if X is a compact metrizable abelian group and Γ acts on X via continuous automorphisms [13, 8]. Given an algebraic action $\Gamma \curvearrowright X$, there is a unique left $\mathbb{Z}\Gamma$-module structure on $\widehat{X}$ satisfying $(s\varphi)(sx) = \varphi(x)$ for $\varphi \in \widehat{X}, x \in X$ and $s \in \Gamma$. Here the Pontryagin dual $\widehat{X}$ of X is the countable abelian group consisting of all continuous group homomorphisms $X \to \mathbb{R}/\mathbb{Z}$. In fact, this yields a natural one-to-one correspondence between isomorphism classes of algebraic actions of Γ and isomorphisms classes of countable left $\mathbb{Z}\Gamma$-modules.

When $\Gamma \curvearrowright X$ is algebraic and Γ is amenable, the topological entropy $\mathrm{h}_{\mathrm{top}}(\Gamma \curvearrowright X)$ coincides with the measure-entropy $\mathrm{h}_{\mu_X}(\Gamma \curvearrowright X)$ for the normalized Haar measure μ_X of X [5], thus we shall denote this comment value by $\mathrm{h}(\Gamma \curvearrowright X)$.

The action $\Gamma \curvearrowright X$ is said to be *expansive* if there is some $c > 0$ such that $\sup_{s\in\Gamma} \rho(sx, sy) > c$ for all distinct $x, y \in X$. For expansive algebraic actions $\Gamma \curvearrowright X$, the left $\mathbb{Z}\Gamma$-module $\widehat{X}$ is finitely generated [13].

The question we investigate here is the relation between positive topological entropy and the existence of non-diagonal asymptotic pairs. Blanchard-Host-Ruette [3] showed that for $\Gamma = \mathbb{Z}$, if $\mathrm{h}_{\mathrm{top}}(\Gamma \curvearrowright X) > 0$, then there are non-diagonal pairs which are positively asymptotic, i.e. distinct $x, y \in X$ satisfying $\rho(sx, sy) \to 0$ as $s \to +\infty$. On the other hand, Lind and Schmidt constructed an example of an algebraic action $\mathbb{Z} \curvearrowright X$ with positive entropy but no non-diagonal asymptotic pairs [9]. These motivate the following question:

Question. Assume that Γ is amenable and that $\Gamma \curvearrowright X$ is an expansive algebraic action. Is is true that $\mathrm{h}(\Gamma \curvearrowright X) > 0$ if and only if there are non-diagonal asymptotic pairs?

The "if" part is relatively easy. It was proven by Lind-Schmidt for $\Gamma = \mathbb{Z}^d$ [9] and by Chung-Li in general [4].

The "only if" part is much more subtle. It was proven by Lind-Schmidt for $\Gamma = \mathbb{Z}^d$ [9] and by Chung-Li for the case $\mathbb{Z}\Gamma$ is left Noetherian [4]. A unital ring R is called left Noetherian if every left ideal of R is finitely generated. The group Γ is called *polycyclic-by-finite* if there is a finite sequence of subgroups $\Gamma = \Gamma_1 \rhd \Gamma_2 \rhd \cdots \rhd \Gamma_n = \{e\}$ such that Γ_j/Γ_{j+1} is either finite or cyclic for every $j = 1, \dots, n-1$. It is known that when Γ is polycyclic-by-finite, $\mathbb{Z}\Gamma$ is left Noetherian [7], and it is a long-standing conjecture that the converse holds. On the other hand, Meyerovitch showed that the "only if" part fails for some abelian group Γ [12].

The new tool we use to study Question is the following notion. For any $K \subseteq \Gamma$ and $x, y \in X$, put $\rho_K(x, y) = \sup_{s\in K} \rho(sx, sy)$.

Definition. We say $\Gamma \curvearrowright X$ has the *strong topological Markov property (strong TMP)* if for any $\varepsilon > 0$ there are $\delta > 0$ and a finite set $F \subseteq \Gamma$ containing the identity element e such that for any finite set $A \subseteq \Gamma$ and any $x, y \in X$ with $\rho_{FA\setminus A}(x, y) \le \delta$, there is some $z \in X$ satisfying $\rho_{FA}(x, z) \le \varepsilon$ and $\rho_{\Gamma\setminus A}(y, z) \le \varepsilon$.

For any nonempty finite set Λ, one has the full shift action $\Gamma \curvearrowright \Lambda^\Gamma$ given by $(sx)_t = x_{s^{-1}t}$ for $x \in \Lambda^\Gamma$ and $s, t \in \Gamma$. A closed Γ-invariant subset of Λ^Γ is called a *subshift.* In the case of subshifts, the strong TMP can be described as follows.

Proposition. *Let $X \subseteq \Lambda^\Gamma$ be a subshift. Then $\Gamma \curvearrowright X$ has the strong TMP if and only if there is some finite set $F \subseteq \Gamma$ containing e such that for any finite set $A \subseteq \Gamma$ and any $x, y \in X$ with $x = y$ on $AF \setminus A$ the element $z \in \Lambda^\Gamma$ satisfying $z|_A = x|_A$ and $z|_{\Gamma\setminus A} = y|_{\Gamma\setminus A}$ lies in X.*

The strong TMP for subshifts described as above was introduced by Gromov [6, Section 8.C] and called *splicable*, and independently by Barbieri-Gómez-Marcus-Taati [2].

Let $k \geq 2$ be an integer. The *independence density* of a k-tuple $(U_1, \ldots, U_k)$ of subsets of X is the largest $q \geq 0$ such that for every nonempty finite set $F \subseteq \Gamma$ there is some $K \subseteq F$ with $|K| \geq q|F|$ so that for every map $\sigma : K \to \{1, \ldots, k\}$ one has $\bigcap_{s \in K} s^{-1}U_{\sigma(s)} \neq \emptyset$. A tuple $(x_1, \ldots, x_k) \in X^k$ is called an *IE-tuple*, or *IE-pair* when $k = 2$, if for any neighborhood U_i of x_i for $i = 1, \ldots, k$, the tuple $(U_1, \ldots, U_k)$ has positive independence density. Denote by $\mathrm{IE}_k(\Gamma \curvearrowright X)$ the set of all IE-k-tuples. When Γ is amenable, $\mathrm{h}_{\mathrm{top}}(\Gamma \curvearrowright X) > 0$ if and only if there are non-diagonal IE-pairs [8, Theorem 12.19].

Theorem. *Suppose that* Γ *is amenable and that* $\Gamma \curvearrowright X$ *is expansive and has the strong TMP. Then* $\mathrm{IE}_k(\Gamma \curvearrowright X) \subseteq \overline{\mathrm{A}_k(\Gamma \curvearrowright X)}$ *for every* $k \geq 2$*. In particular, if* $\mathrm{h}_{\mathrm{top}}(\Gamma \curvearrowright X) > 0$*, then there are non-diagonal asymptotic pairs.*

For any positive integers m, n and $a \in M_{m,n}(\mathbb{C}\Gamma)$, the *von Neumann dimension* $\dim_{\mathrm{vN}} \ker a$ of the kernel $\ker a$ of the bounded linear operator $(\ell^2(\Gamma))^n \to (\ell^2(\Gamma))^m$ sending z to az is defined as $\sum_{j=1}^{n} \langle P\delta_{e,j}, \delta_{e,j} \rangle \in [0, n]$, where P is the orthogonal projection $(\ell^2(\Gamma))^n \to \ker a$ and $\delta_{s,j}$ for $s \in \Gamma$ and $j = 1, \ldots, n$ are the canonical orthonormal basis of $(\ell^2(\Gamma))^n$. The group Γ is said to satisfy the *strong Atiyah conjecture* if $\dim_{\mathrm{vN}} \ker a$ lies in the subgroup of $\mathbb{Q}$ generated by $1/|H|$ for H ranging over finite subgroups of Γ [11]. The strong Atiyah conjecture fails for the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. So far there is no counterexample for the strong Atiyah conjecture in the case there is an upper bound on the orders of finite subgroups. The strong Atiyah conjecture holds for elementary amenable groups with an upper bound on the orders of finite subgroups [10] and left-orderable amenable groups.

Theorem. *Suppose that* Γ *is amenable and that* $\Gamma \curvearrowright X$ *is an expansive algebraic action. Assume further that at least one of the following conditions holds:*

(1) $\mathbb{Z}\Gamma$ *is left Noetherian;*
(2) Γ *satisfies the strong Atiyah conjecture, there is an upper bound on the orders of finite subgroups of* Γ*, and* $\widehat{X}$ *is a finitely presented left* $\mathbb{Z}\Gamma$*-module.*

Then $\Gamma \curvearrowright X$ *has the strong TMP. Consequently, the "only if" part of Question holds for* $\Gamma \curvearrowright X$*.*

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On treeings arising from Baumslag-Solitar groups

Yoshikata Kida

For integers $2 \leq p \leq q$, the Baumslag-Solitar (BS) group $\mathrm{BS}(p,q)$ is defined by the presentation

$$\mathrm{BS}(p,q) = \langle\, a, t \mid ta^p t^{-1} = a^q \,\rangle.$$

This group was introduced by Baumslag-Solitar and is today known to have interesting features in various contexts. We study these groups in orbit equivalence.

Let G be a countable group and $G \curvearrowright (X,\mu)$ a probability-measure-preserving (p.m.p.) action on a standard probability space. Given such two actions $G \curvearrowright (X,\mu)$, $H \curvearrowright (Y,\nu)$, we say that they are *orbit equivalent (OE)* if there exists a measure space isomorphism $f\colon (X,\mu) \to (Y,\nu)$ such that $f(Gx) = Hf(x)$ for a.e. $x \in X$. We say that two countable groups G, H are *OE* if they have free p.m.p. actions which are OE.

We are interested in whether $\mathrm{BS}(p,q)$ and $\mathrm{BS}(p',q')$ are OE or not for distinct (p,q) and (p',q'). In [4], we introduced the flow associated to a free p.m.p. action $\mathrm{BS}(p,q) \curvearrowright (X,\mu)$ and proved that the isomorphism class of the flow is an OE invariant among actions of BS groups. This implies that $\mathrm{BS}(p,q)$ and $\mathrm{BS}(r,r)$ are not OE if $2 \leq p < q$. However the flow is not enough to distinguish other distinct BS groups up to OE (see [3] for some results toward this direction under assuming ergodicity for the action of some subgroups).

Let $G = \mathrm{BS}(p,q)$ and let $m\colon G \to \mathbb{R}^+$ be the modular homomorphism defined by $m(a) = 1$ and $m(t) = q/p$. Given a free p.m.p. action $G \curvearrowright (X,\mu)$, we have the cocycle $\log \circ m\colon G \times X \to \mathbb{R}$, $(g,x) \mapsto \log(m(g))$. The flow associated to the action $G \curvearrowright (X,\mu)$, mentioned above, is defined as the Mackey range of this cocycle. In [4], we showed that not only the range of the cocycle but also the kernel is an OE-invariant. Namely, we proved the following: Let $G = \mathrm{BS}(p,q)$ and $H = \mathrm{BS}(p',q')$, denote by m_G and m_H the modular homomorphisms for G and H, respectively, and suppose that G and H are OE. Then $\ker m_G$ and $\ker m_H$ are OE.

My talk is focused on the OE class of $\ker m$. My main result is stated as follows: If $2 \leq p < q$, then $\ker m$ is OE to $\mathbb{Z} \times F_\infty$, where F_∞ is the free group of countably infinite rank. Thus looking at only $\ker m$ is not enough to distinguish BS groups up to OE, unfortunately.

The group $\ker m$ has the following presentation:

$$\langle\, a_n \ (n \in \mathbb{Z}) \mid a_n^q = a_{n+1}^p \,\rangle,$$

where a_n corresponds to $t^n a t^{-n} \in G$. In other words, $\ker m$ is the amalgamated free product of a bi-infinite sequence of copies of $\mathbb{Z}$, where two successive $\mathbb{Z}$'s are amalgamated over the subgroups $q\mathbb{Z}$, $p\mathbb{Z}$.

There are two ingredients in the proof of the main result. One is normal subequivalence relations and quotient groupoids, introduced by Feldman-Sutherland-Zimmer [1]. Another is Gaboriau's induction argument for treeings [2].

For some action $G \curvearrowright (X, \mu)$, letting $\mathcal{G}$ and $\mathcal{E}$ be the orbit equivalence relations of the actions of G and $E = \langle a \rangle$, respectively, we can show that $\mathcal{E}$ is normal in $\mathcal{G}$ (while E is not normal in G). We then construct a treeing of the quotient groupoid $\mathcal{G}/\mathcal{E}$. Here we mean by a treeing a measurable bundle of trees such that the vertex set of each tree is a fiber of the range map of $\mathcal{G}/\mathcal{E}$. This construction is the first step toward the proof of the main result. Details will appear in [5].

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Characterizing orbit equivalence via invariant measures: a new proof of a theorem of Giordano, Putnam and Skau

Julien Melleray

(joint work with Simon Robert)

Given two homeomorphisms φ, ψ of the Cantor space $X = \{0,1\}^{\mathbb{N}}$, one says that φ and ψ are *orbit equivalent* if the equivalence relations induced by the associated $\mathbb{Z}$-actions are isomorphic. More precisely: φ and ψ are orbit equivalent iff there exists a homeomorphism g of X such that

$$\forall x, y \in X\ (\exists n \in \mathbb{Z}\ \varphi^n(x) = y) \Leftrightarrow (\exists m \in \mathbb{Z}\ \psi^m(g(x)) = g(y))$$

Denote by $M(\varphi)$ the set of all Borel probability measures on X which are φ-invariant. It is straightforward to check that if g witnesses that φ and ψ are orbit equivalent then $\{g_*\mu \colon \mu \in M(\varphi)\} = M(\psi)$. A remarkable result of Giordano, Putnam and Skau asserts that the converse implication holds.

Theorem (Giordano–Putnam–Skau [1])**.** *Assume that φ, ψ are minimal homeomorphisms of the Cantor space such that there exists $g \in \mathrm{Homeo}(X)$ satisfying $\{g_*\mu \colon \mu \in M(\varphi)\} = M(\psi)$. Then φ and ψ are orbit equivalent.*

This result is somewhat mysterious, in particular because it is quite possible to have $M(\varphi) = M(\psi)$ even though the relations induced by φ and ψ are distinct (one may even be strictly coarser than the other). The original proof of this result used some homological algebra; since then other arguments were proposed, but (to the author of this abstract at least) the underlying phenomenon was still quite hard to grasp. We propose a new approach, based on an improvement of a theorem of Krieger [4].

Given a minimal homeomorphism φ of X, its *topological full group* is

$$[[\varphi]] = \{g \in \mathrm{Homeo}(X) \colon \exists n_1, \dots, n_k \in \mathbb{Z}\ \forall x \in X\ \exists i\ g(x) = \varphi^{n_i}(x)\}$$

We will also make use of the *full group* $[\varphi]$, that is, the group of all homeomorphisms of X which map each φ-orbit onto itself.

Fix $x \in X$, denote $\mathcal{O}^+(x) = \{\varphi^n(x) \colon n \geq 0\}$ and let

$$\Gamma_x(\varphi) = \left\{g \in [[\varphi]] \colon g(\mathcal{O}^+(x)) = \mathcal{O}^+(x)\right\}$$

This is a countable, locally finite group, and the orbits under the action of $\Gamma_x(\varphi)$ are almost the same as those of φ: the orbit of x splits in two pieces (the positive and negative semi-orbits of x) and all the other orbits are the same.

Using a lemma due to Glasner and Weiss [3], the aforementioned theorem of Giordano, Putnam and Skau can be reformulated as the statement that, whenever φ and ψ are minimal homeomorphisms of X such that $\overline{[\varphi]}$ and $\overline{[\psi]}$ are conjugated subgroups of $\mathrm{Homeo}(X)$, the full groups $[\varphi]$ and $[\psi]$ are themselves conjugated (note that an orbit equivalence between φ and ψ is the same thing as a homeomorphism g of X such that $g[\varphi]g^{-1} = [\psi]$).

It is then quite interesting to observe that a theorem of Krieger [4] implies that for any two minimal homeomorphisms φ, ψ of X and any $x, y \in X$, assuming that $\overline{\Gamma_x(\varphi)}$ and $\overline{\Gamma_y(\psi)}$ are conjugated implies that $\Gamma_x(\varphi)$ and $\Gamma_y(\psi)$ are conjugated.

Say that a minimal homeomorphism φ is *saturated* if for some (eq., for any) $x \in X$ one has $\overline{\Gamma_x(\varphi)} = \overline{[\varphi]}$. Given the close relationship between the $\Gamma_x(\varphi)$ orbits and the φ-orbits, it is possible (using an improvement on Krieger's theorem which we describe below in a bit more detail) to prove that, if φ and ψ are saturated minimal homeomorphisms of X such that $M(\varphi) = M(\psi)$ then φ and ψ are orbit equivalent. In other words, one can prove the Giordano-Putnam-Skau theorem in the particular case of saturated minimal homeomorphisms using a variation on Krieger's theorem, whose proof is based on a relatively simple back-and-forth argument.

To obtain the full result, it is then enough to prove that any minimal homeomorphism is orbit equivalent to a saturated minimal homeomorphism; we achieve this by a careful construction of *Kakutani-Rokhlin* partitions. Since this part is quite technical (hopefully it can be simplified, though we do not know how), I will say no more on this here, though it is interesting to note that our variation on Krieger's theorem also plays a key role in this part of our argument.

To close this abstract, we explain this variation, which seems to us to be our main new contribution. Following Krieger [4], say that a subgroup of Homeo(X) is *ample* if it satisfies the following conditions:

- Γ is countable and locally finite.
- Γ is a full group (i.e. whenever $U_1, \dots, U_n$ are a clopen partition of X and $g \in \mathrm{Homeo}(X)$ is such that for all i g coincides with some $\gamma_i \in \Gamma$ on U_i, then $g \in \Gamma$)
- For all $\gamma \in \Gamma$ the set $\{x \in X \colon \gamma(x) = x\}$ is clopen.

Using our notations from above, any $\Gamma_x(\varphi)$ is ample; actually, any ample subgroup of Homeo(X) acting minimally can be realized as a $\Gamma_x(\varphi)$ for some $x \in X$ and some minimal $\varphi \in$ Homeo(X).

Given a minimal ample group Γ, and a closed subset K of X, say that K is Γ-*sparse* if K intersects each Γ-orbit in at most one point. Here is a statement of our improvement on Krieger's theorem.

Theorem (Melleray-Robert)**.** *Assume that* Γ *is a minimal ample subgroup of* Homeo(X)*, and that* K, L *are two* Γ*-sparse closed subsets of* X*. Assume also that* $h \colon K \to L$ *is a homeomorphism. Then there exists* $g \in \mathrm{Homeo}(X)$ *such that* $g\Gamma g^{-1} = \Gamma$ *and* $g_{|K} = h$*.*

This is already interesting, and useful, in the case where K and L both consist of two points belonging to distinct Γ-orbits. It can be seen as a strong homogeneity statement on the quotient space X/Γ. Using this result, we are also able to recover some *absorption theorems* in a elementary way, which are particular cases of results from [2], [5].

The interested reader may consult the preprint [6] for a detailed discussion.

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The geography of the space of subgroups

Alessandro Carderi
(joint work with Damien Gaboriau, François Le Maître, Yves Stalder)

Let Γ be a countable group. The space of subgroups $\mathrm{Sub}(\Gamma)$ of Γ is a compact subset of the Cantor space $\{0,1\}^{\Gamma}$ on which Γ acts continuously by conjugation. The Cantor-Bendixson theorem gives us a partition of the compact set $\mathrm{Sub}(\Gamma) = \mathcal{C} \sqcup \mathcal{K}$ such that $\mathcal{C}$ is countable and $\mathcal{K}$ is either empty or perfect. Moreover this decomposition is Γ-invariant. It is hard to find what this decomposition is for a general groups, however there are some explicit examples:

- for every group which have countably many subgroups $\mathrm{Sub}(\Gamma) = \mathcal{C}$;
- for $\Gamma = \mathbf{F}_2$, we have that $\mathcal{K}$ consists in the subset of infinite index subgroups;
- $\mathrm{Sub}(\mathbf{F}_\infty) = \mathcal{K}$.

The topology on $\mathcal{C}$ can be somewhat understood with the *Cantor-Bendixson rank* of the space. Whenever the space $\mathcal{K}$ is not empty, it is a Cantor space. We are interested in the question of whether the action of Γ on it is topologically transitive or if there are some open invariant subsets. In the case of the Baumslag-Solitar group we can prove the following theorem.

Theorem (Carderi-Gaboriau-Le Maître-Stalder)**.** *Denote by* Γ *the Baumslag-Solitar group* $\mathrm{BS}(2,3) = \langle b, t \colon tb^2t^{-1} = b^3 \rangle$. *Then*

- *$\mathcal{K}$ consists in the subset of infinite index subgroups;*
- *$\mathcal{K} = \cup_n A_n \cup A_\infty$ where A_n is Γ-invariant open not closed and A_∞ is closed not open;*
- *for each $n \neq \infty$, A_∞ admits a unique closed Γ-invariant subset consisting of a point;*
- *for each $n \leq \infty$, the action of Γ on A_n is topologically transitive.*

Bounded Harmonic Functions on Linear Groups

Joshua Frisch
(joint work with Anna Erschler)

Given a countable group G and a probability measure μ we define a function from $G \to R$ be μ-harmonic if $f(k) = \Sigma_{g \in G} f(kg)\mu(g)$. The main question we are interested in answering is for which pairs G, μ there exist non-constant bounded harmonic functions. We will assume for the rest of this abstract that the μ discussed are non-degenerate (meaning that their support generates G as a semi-group). The first main result in this direction is that for any non-amenable group there are always non-trival bounded harmonic functions [1]. The converse of this statement was eventually proved by Kaimanovich and Vershik [2] and Rosenblatt [3] who showed that for any amenable group there exist a measure with only trivial harmonic functions.

For finitely supported measures, however, the situation is much more subtle. We still do not have a clear characterization of when the boundary is trivial is when it is non-trivial. In work in progress with Anna Erschler we give a complete characterization (which is somewhat too technical to be properly included in this report) for the case of linear groups over fields of positive characteristic. We also give partial results over fields of characteristic 0. The following theorem is representative of the results we obtain.

Theorem. *Let G be an amenable finitely generated linear group over K a field of transcendence degree at most 1 or of positive characteristic and degree 2. Let μ be a symmetric finitely supported measure on G then all μ harmonic functions are constant.*

We remark that the above results are sharp. For fields F of transcendence degree at least 3 or degree at least 2 in characteristic 0 there exist a finitely generated amenable linear group G_F over F and a finitely supported symmetric μ on G such that the bounded μ-harmonic functions are not all constants.

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Amenability, proximality and higher order syndeticity

Matthew Kennedy
(joint work with Sven Raum, Guy Salomon)

We show that the universal minimimal proximal flow and the universal minimal strongly proximal flow of a discrete group can be realized as the Stone spaces of translation invariant Boolean algebras of subsets of the group satisfying a higher order notion of syndeticity. We establish algebraic, combinatorial and topological dynamical characterizations of these subsets that we use to obtain new necessary and sufficient conditions for strong amenability and amenability. We also characterize dense orbit sets, answering a question of Glasner, Tsankov, Weiss and Zucker.

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Constructing equivariant maps to free (and almost free) subshifts

Anton Bernshteyn

We investigate certain combinatorial problems arising from the study of the dynamics of countable group actions. To set the stage, let Γ be a countably infinite (discrete) group and let q be a positive integer. We follow the standard convention and identify q with the q-element set $\{0, 1, \ldots, q-1\}$ equipped with the discrete topology. By a *q-coloring* of a set S we simply mean a mapping $S \to q$. The *shift action* $\Gamma \curvearrowright q^\Gamma$ of Γ on the product space q^Γ of all q-colorings of Γ is given by

$$(\gamma \cdot x)(\delta) \coloneqq x(\delta\gamma), \quad \text{for all } x \in q^\Gamma \text{ and } \gamma, \delta \in \Gamma.$$

We are particularly interested in *free* actions of Γ, i.e., actions $\Gamma \curvearrowright X$ such that $\mathrm{Stab}(x) = \{\mathbf{1}\}$ for all $x \in X$, where $\mathrm{Stab}(x)$ denotes the stabilizer of x and $\mathbf{1}$ is the identity element of Γ. Note that the shift action $\Gamma \curvearrowright q^\Gamma$ is not free; however, when $q \geq 2$, its *free part* $\mathrm{Free}(q^\Gamma) \coloneqq \{x \in q^\Gamma \,:\, \mathrm{Stab}(x) = \{\mathbf{1}\}\}$ is a dense G_δ set.

Given an action $\Gamma \curvearrowright X$, there is a natural one-to-one correspondence between equivariant maps $X \to q^\Gamma$ and q-colorings of X. Specifically, given a q-coloring $f \colon X \to q$, we define a map $\pi_f \colon X \to q^\Gamma$ via

$$(\pi_f(x))\,(\gamma) \coloneqq f(\gamma \cdot x), \quad \text{for all } x \in X \text{ and } \gamma \in \Gamma.$$

We call π_f the *coding map* of f. Conversely, for every equivariant map $\pi \colon X \to q^\Gamma$, there is a unique q-coloring $f \colon X \to q$ with $\pi = \pi_f$, namely $f = (x \mapsto (\pi(x))(\mathbf{1}))$. We can now analyze q-colorings of X according to the dynamical properties of their coding maps. For example, we can introduce the following definition:

Definition 1 (Aperiodic and hyper-aperiodic colorings)**.** Let $\Gamma \curvearrowright X$ be an action of Γ. We say that a q-coloring $f \colon X \to q$ is:

- *aperiodic* if $\pi_f(X) \subseteq \mathrm{Free}(q^\Gamma)$;
- *hyper-aperiodic* if the topological closure of $\pi_f(X)$ is included in $\mathrm{Free}(q^\Gamma)$.

Theorem 1 (Seward–Tucker-Drob [6])**.** *Every free Borel action $\Gamma \curvearrowright X$ on a Polish space X admits a Borel hyper-aperiodic 2-coloring $f \colon X \to 2$.*

Our goal is to strengthen Theorem 1 by adding extra combinatorial constraints on f. The general class of combinatorial constraints we will be considering is introduced in the next pair of definitions:

Definition 2 (Subshifts of finite type)**.** A *subshift* is a closed subset $\mathcal{S} \subseteq q^\Gamma$ invariant under the shift action. A subshift $\mathcal{S}$ is *of finite type* if there exist a finite set $W \subset \Gamma$, called a *window*, and a family $\Phi \subseteq q^W$ of q-colorings of W such that

$$\mathcal{S} \,=\, \{x \in q^\Gamma \,:\, (\gamma \cdot x)|_W \in \Phi \text{ for all } \gamma \in \Gamma\}.$$

For brevity, we use the acronym *SFT* to mean “subshift of finite type.”

Definition 3 ($\mathcal{S}$-colorings)**.** Let $\Gamma \curvearrowright X$ be an action of Γ and let $\mathcal{S} \subseteq q^\Gamma$ be an SFT. An *$\mathcal{S}$-coloring* of X is a q-coloring $f \colon X \to q$ such that $\pi_f(X) \subseteq \mathcal{S}$.

As a motivating example, let $F \subset \Gamma$ be a finite subset such that $\mathbf{1} \notin F$ and $F = F^{-1}$. The *Cayley graph* of Γ corresponding to F is the graph $G(\Gamma, F)$ with vertex set Γ and edge set $\{\{\gamma, \sigma\gamma\} : \gamma \in \Gamma,\ \sigma \in F\}$. More generally, for an action $\Gamma \curvearrowright X$, the *Schreier graph* $G(X, F)$ has vertex set X and edge set $\{\{x, \sigma \cdot x\} : x \in X,\ \sigma \in F\}$. Recall that a *proper q-coloring* of a graph G is a function $f\colon V(G) \to q$ such that $f(x) \neq f(y)$ whenever the vertices x and y are adjacent in G. The following observation exemplifies the relationship between SFTs and combinatorial problems:

Observation 1. Let $\mathsf{Col}(F, q)$ denote the set of all proper q-colorings of the Cayley graph $G(\Gamma, F)$. Then $\mathsf{Col}(F, q)$ is an SFT, and a $\mathsf{Col}(F, q)$-coloring of an action $\Gamma \curvearrowright X$ is the same thing as a proper q-coloring of the Schreier graph $G(X, F)$.

The question of the existence of proper q-colorings of Schreier graphs is a subject of intense study (see [3, 5] for surveys of some relevant results). In particular, we have the following:

Theorem 2 (Kechris–Solecki–Todorcevic [4, Proposition 4.6]). *Let $F \subset \Gamma$ be a finite set such that $\mathbf{1} \notin F$ and $F = F^{-1}$. Set $d := |F|$. Let $\Gamma \curvearrowright X$ be a free Borel action of Γ on a Polish space X. Then the Schreier graph $G(X, F)$ admits a Borel proper $(d + 1)$-coloring.*

In view of Theorems 1 and 2, it is natural to ask whether, in the setting of Theorem 2, there exists a Borel *aperiodic* (or even hyper-aperiodic) proper $(d+1)$-coloring of $G(X, F)$. Our main result yields an affirmative answer and in fact provides a general sufficient condition that guarantees the existence of Borel hyper-aperiodic $\mathcal{S}$-colorings, for a given SFT $\mathcal{S}$:

Theorem 3 (Bernshteyn [2]). *Let $\mathcal{S} \subseteq q^\Gamma$ be an SFT and let $f\colon \mathrm{Free}(k^\Gamma) \to q$ be a continuous $\mathcal{S}$-coloring, for some $k \geq 2$. Set $\pi := \pi_f$ and suppose that no non-identity element of Γ fixes every point in $\pi(\mathrm{Free}(k^\Gamma))$. Then every free Borel action $\Gamma \curvearrowright X$ of Γ on a Polish space X admits a Borel hyper-aperiodic $\mathcal{S}$-coloring.*

Corollary 1 (Bernshteyn [2]). *Let $F \subset \Gamma$ be a finite set such that $\mathbf{1} \notin F$ and $F = F^{-1}$. Set $d := |F|$. Let $\Gamma \curvearrowright X$ be a free Borel action of Γ on a Polish space X. Then the graph $G(X, F)$ admits a Borel hyper-aperiodic proper $(d+1)$-coloring.*

Proof. By Theorem 3, it suffices to construct, for some $k \geq 2$, a continuous proper $(d + 1)$-coloring f of the graph $G(\mathrm{Free}(k^\Gamma), F)$ such that no non-identity group element fixes every point in the image of π_f. For $k = 3$, this can be achieved by a modification of the proof of Theorem 2; for details, see [2, §2.4]. □

It is particularly easy to apply Theorem 3 in the case when Γ has no non-trivial finite normal subgroups. Indeed, for an equivariant map $\pi\colon \mathrm{Free}(k^\Gamma) \to q^\Gamma$, let $\mathrm{Stab}(\pi)$ be the set of all group elements that fix every point in $\pi(\mathrm{Free}(k^\Gamma))$. We then have the following:

Proposition 1 ([2, Proposition 1.3]). *Let $\pi\colon \mathrm{Free}(k^\Gamma) \to q^\Gamma$ be a non-constant continuous equivariant map. Then $\mathrm{Stab}(\pi)$ is a finite normal subgroup of Γ.*

Corollary 2. *Let $\mathcal{S} \subseteq q^\Gamma$ be an SFT. Assume that:*

- Γ *has no non-trivial finite normal subgroups, and*
- *for some $k \geq 2$, $\mathrm{Free}(k^\Gamma)$ admits a non-constant continuous $\mathcal{S}$-coloring.*

Then every free Borel action $\Gamma \curvearrowright X$ of Γ on a Polish space X admits a Borel hyper-aperiodic $\mathcal{S}$-coloring.

It is possible to further generalize Theorem 3 to construct Borel $\mathcal{S}$-colorings that are, in some sense, as close to being hyper-aperiodic as one can hope:

Theorem 4 (Bernshteyn [2])**.** *Let $\mathcal{S} \subseteq q^\Gamma$ be an SFT and let $f \colon \mathrm{Free}(k^\Gamma) \to q$ be a continuous $\mathcal{S}$-coloring, for some $k \geq 2$. Set $\pi := \pi_f$. Then there exists a subshift $\mathcal{S}' \subseteq \mathcal{S}$ with the following properties:*

- *the stabilizer of every point in $\mathcal{S}'$ is precisely $\mathrm{Stab}(\pi)$;*
- *every free Borel action $\Gamma \curvearrowright X$ of Γ on a Polish space X admits a Borel equivariant map $X \to \mathcal{S}'$.*

The proofs of our results rely on combinatorial methods, in particular on the continuous version of the Lovász Local Lemma developed in [1].

We conclude with a couple of open problems.

Problem 1. Is there a version of Theorem 3 that only needs f to be Borel (rather than continuous)?

The difficulty with Problem 1 is that it is not even clear what the right statement of the desired result should be. It is certainly not enough to simply replace the word "continuous" by "Borel" in the statement of Theorem 3—some further assumptions are necessary. Perhaps the following concrete question can serve as a test for our understanding of this problem:

Problem 2. Let $\mathcal{S} \subseteq q^\Gamma$ be an SFT and let $f \colon \mathrm{Free}(k^\Gamma) \to q$ be a Borel aperiodic $\mathcal{S}$-coloring. Must exist a Borel hyper-aperiodic $\mathcal{S}$-coloring of $\mathrm{Free}(k^\Gamma)$ as well?

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Measurable combinatorics in hyperfinite graphs

Matt Bowen
(joint work with Gabor Kun and Marcin Sabok; Felix Weilacher; Antoine Poulin and Jenna Zomback)

Given a Polish space (X, τ) with Borel probability μ, a **Borel graph** on X is a graph G with vertex set $V(G) = X$ and edge set $E(G) \subseteq X^2$ a Borel set. Such graphs arise naturally in many contexts, such as Schreier graphs of group actions, limits of finite graphs, dynamics, etc. [10].

A natural question is which results from finite combinatorics have analogues 'definable' (i.e., Borel, μ-measurable, τ-Baire measurable) counterparts for Borel graphs. In this talk, the most relevant notion is that of the definable edge chromatic numbers. More specifically, the Borel edge chromatic number of a Borel graph, $\chi'_B(G)$, is the smallest cardinal κ so that G admits Borel proper edge coloring using κ colors. The Baire measurable edge chromatic number, $\chi'_{BM}(G)$, is the minimum of $\chi'_B(G \upharpoonright C)$, where C varies over τ-comeagre G-invariant Borel sets, and similarly the μ-measurable chromatic number, $\chi'_{BM}(G)$, which is just $\chi'_B(G \upharpoonright C)$, where C varies over μ-conull G-invariant Borel sets.

The study of these parameters was begun in [7], where it was shown that the greedy upper bound $\chi'_B(G) \leq 2\Delta(G) - 1$ holds, while the irrational rotation graph, G_α, gives a simple example of a 2-regular acylic graph where $2 = \chi'(G_\alpha) < \chi'_B(G_\alpha) = \chi_\mu(G_\alpha) = \chi_{BM}(G_\alpha) = 3$. Constructions of Marks in [9] show that the greedy bound cannot always be improved, even in the acyclic case, in stark contrast with Kőnig's theorem that $\chi'(G) = \Delta(G)$ for finite bipartite graphs. In the special case of μ-preserving graphs better positive results are known: Csóka, Lippner, and Pikhurko showed that bipartite μ-invariant graphs satisfy $\chi'_\mu(G) \leq \Delta(G)+1$ in [5], and this was recently improved by Pikhurko and Grebík to work for not necessarily bipartite graphs in [6], matching the optimal general bound in the finite case due to Vizing. However, outside of the pmp measurable setting few other general results seem to have been known.

In this talk we discuss several new results of a similar flavor for hyperfinite graphs, which in many cases gave the first improvement on the bound of $2\Delta(G)-1$ originally proved in [7]. The first of these is the following version of Kőnig's theorem:

Theorem (B., Weilacher [4])**.** *Let G be a bipartite Borel graph with maximum degree $\Delta(G)$. Then $\chi'_{BM}(G) \leq \Delta(G)+1$. If G is μ-hyperfinite then $\chi'_\mu(G)$ satisfies the same bound.*

For non-bipartite graphs, Mark's question [9] regarding the Baire measurable version of Vizing's theorem remains open. While the bounds we can show are still far from resolving this, we can prove a bound on the Baire measurable edge chromatic number of multigraphs that comes close to the optimal bound in the finite setting due to Shannon:

Theorem (B. [1])**.** *Every Borel multigraph with maximum degree $\Delta(G)$ satisfies*

$$\chi'_{BM}(G) \leq \lceil \frac{3\Delta(G)}{2} \rceil + 6.$$

The same bound holds for χ'_μ if G is μ-null preserving and hyperfinite.

We are also able to use similar ideas to obtain results on definable matchings for **one-ended** Borel graphs. (A connected graph G is one-ended if $G - F$ has at most one infinite connected component for any finite subgraph F, and a Borel graph is one-ended if each of its connected components is. Natural examples include the Cayley graph of any finitely generated amenable group of superlinear growth). In particular, we show the following:

Theorem (B., Kun, Sabok [2] for measure, and B., Poulin, and Zomback [3] for Baire measure)**.** *Let G be a regular degree one-ended bipartite Borel graph. Then G admits a Borel perfect matching on an invariant comeagre Borel set. If G is μ-hyperfinite and measure preserving then the same holds for an invariant conull set.*

The above, together with earlier work of Lyons and Nazarov [8], allows us to characterize which bipartite Cayley graphs admit factor of iid perfect matchings:

Theorem (B., Kun, Sabok [2])**.** *Let Γ be a finitely generated group.*

- *If Γ is isomorphic to $\mathbb{Z} \ltimes \Delta$ for a finite normal subgroup Δ of odd order, then no bipartite Cayley graph of Γ admits a factor of iid perfect matching a.s.*
- *Else, if Γ is not isomorphic to $\mathbb{Z} \ltimes \Delta$ for a finite normal subgroup Δ of odd order, then every bipartite Cayley graph of Γ admits a factor of iid perfect matching a.s.*

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Maximal pronilfactors and a topological Wiener-Wintner theorem

Yonatan Gutman
(joint work with Zhengxing Lian)

In recent years there has been an increase in interest in pronilfactors both for measure-preserving systems (m.p.s.) and topological dynamical systems (t.d.s.). Pronilfactors of a given system are either measurable or topological (depending on the category) factors given by an inverse limit of nilsystems. A t.d.s. (m.p.s.) is called a topological (measurable) *d-step pronilsystem* if it is a topological (measurable) inverse limit of nilsystems of degree at most d. In the theory of measure preserving systems $(X, \mathcal{X}, \mu, T)$ maximal measurable pronilfactors appear in connection with the L^2-convergence of the nonconventional ergodic averages

$$\frac{1}{N}\sum f_1(T^n x)\dots f_k(T^{kn}x) \tag{1}$$

for $f_1, \dots, f_k \in L^\infty(X, \mu)$ ([1, 2]). In the theory of topological dynamical systems maximal topological pronilfactors appear in connection with the higher order regionally proximal relations ([3, 4, 5]).

When a system possesses both measurable and topological structure, it seems worthwhile to investigate pronilfactors both from a measurable and topological point of view. A natural meeting ground are strictly ergodic systems - minimal topological dynamical systems (X, T) possessing a unique invariant measure μ. For $k \in \mathbb{Z}$ let us denote by $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ respectively $(W_k(X), T)$ the maximal k-step measurable respectively topological pronilfactor of (X, T). Clearly $(W_k(X), T)$ has a unique invariant measure ω_k. We thus pose the question when is $(W_k(X), \mathcal{W}_k(X), \omega_k, T)$ isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ as m.p.s.? We call a t.d.s. which is strictly ergodic and for which $(W_k(X), \mathcal{W}_k(X), \omega_k, T)$ is isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ as m.p.s., a *CF-Nil*(k) system. Note that $(W_k(X), \mathcal{W}_k(X), \omega_k, T)$ is always a measurable factor of $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$. At first glance it may seem that CF-Nil(k) systems are rare however a theorem by Benjamin Weiss regarding topological models for measurable extensions implies that every ergodic m.p.s. is measurably isomorphic to a CF-Nil(k) system.

We give two characterizations of CF-Nil(k) systems. The first characterization is related to the Wiener-Wintner theorem while the second characterization is related to *k-cube uniquely ergodic* systems - a class of topological dynamical systems introduced in [6]. The Wiener-Wintner theorem ([7]) states that for an ergodic system $(X, \mathcal{X}, \mu, T)$, for μ-a.e. $x \in X$, *any* $\lambda \in \mathbb{S}^1$ and any $f \in L^\infty(\mu)$, the following limit exists:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda^n f(T^n x) \tag{2}$$

Denote by $M_T \subset \mathbb{S}^1$ the set of measurable eigenvalues of $(X, \mathcal{X}, \mu, T)$. Let $P_\lambda f$ be the projection of f to the eigenspace corresponding to λ (in particular for $\lambda \notin M_T$, $P_\lambda f \equiv 0$). For fixed $\lambda \in \mathbb{S}^1$, one can show (2) converges a.s. to $P_\lambda f$.

For topological dynamical systems one may investigate the question of *everywhere* convergence in the Wiener-Wintner theorem. In [8], Robinson proved that for an uniquely ergodic system (X, μ, T), for any $f \in C(X)$, if every measurable eigenfunction of $(X, \mathcal{X}, \mu, T)$ has a continuous version then the limit (2) converges everywhere. He noted however that if $P_\lambda f \neq 0$ for some $\lambda \in M_T$, then the convergence of (2) is not uniform in (x, λ), since the limit function $P_\lambda f(x)$ is not continuous on $X \times \mathbb{S}^1$ as M_T is countable. Moreover Robinson constructed a strictly ergodic system (X, T) such that (2) does not converge for some continuous function $f \in C(X)$, some $\lambda \in \mathbb{C}$ and some $x \in X$.

The first main result of this article is the following theorem:

Theorem 1. *Let (X, T) be a minimal system. Then for $k \geq 0$ the following are equivalent:*

(I). *(X, T) is a CF-Nil(k) system.*

(II). *For any k-step nilsequence $\{a(n)\}_{n\in\mathbb{Z}}$, any continuous function $f \in C(X)$ and any $x \in X$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a(n) f(T^n x)$$

exists.

We remark that the direction (I)⇒(II) of Theorem 1 follows from [9] whereas the case $k = 1$ of Theorem 1 follows from [8, Theorem 1.1]. As part of the proof of Theorem 1 we established a fundamental property for pronilsystems:

Theorem 2. *Let (Y, ν, T) be a minimal (uniquely ergodic) k-step pronilsystem. Then*

(I). *(Y, ν, T) is measurably coalescent, that is, if $\pi : (Y, \nu, T) \to (Y, \nu, T)$ is a measurable factor map, then π is a measurable isomorphism, and*

(II). *(Y, T) is topologically coalescent, that is, if $\Phi : (Y, T) \to (Y, T)$ is a topological factor map, then Φ is a topological isomorphism.*

As part of the the theory of higher order regionally proximal relations, Host, Kra and Maass introduced in [3] the *dynamical cubespaces* $\mathrm{C}_T^n(X) \subset X^{2^n}$, $n \in \mathbb{N} := \{1, 2, \ldots\}$. These compact sets enjoy a natural action by the *Host-Kra cube groups* $\mathcal{HK}^n(T)$. According to the terminology introduced in [6], a t.d.s. (X, T) is called *k-cube uniquely ergodic* if $(\mathrm{C}_T^k(X), \mathcal{HK}^k(T))$ is uniquely ergodic. The third main result of this article is the following theorem:

Theorem 3. *Let (X, T) be a minimal t.d.s. Then the following are equivalent for any $k \geq 0$:*

(I). *(X, T) is a CF-Nil(k) system.*

(II). *(X, T) is $(k+1)$-cube uniquely ergodic.*

We remark that the implication (I) ⇒ (II) follows from [10].

In the context of various classes of strictly ergodic systems, several authors have investigated the question of whether every measurable eigenfunction has a continuous version. Famously in [11], Host established this is the case for *admissible substitution dynamical systems*. In [12, Theorem 27] an affirmative answer was given for strictly ergodic *Toeplitz type systems of finite rank*. In [13], the continuous and measurable eigenvalues of minimal Cantor systems were studied. It is easy to see that for strictly ergodic systems (X, T) the condition that every measurable eigenfunction has a continuous version is equivalent to the fact that (X, T) is CF-Nil(1). Thus Theorem 3 provides for strictly ergodic systems a new condition equivalent to the property that every measurable eigenfunction has a continuous version. Namely this holds iff $(\mathrm{C}^2_T(X), \mathcal{HK}^2(T))$ is uniquely ergodic. As the last condition seems quite manageable one wonders if this new equivalence may turn out to be useful in future applications.

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A closed subgroup of the homeomorphism group of the circle with property (T)

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The Zimmer program aims to understand actions of lattices of semi-simple Lie groups on compact manifolds. The simplest compact manifold is without any doubt the circle since it is the unique connected compact manifold of dimension 1.

So, actions of lattices of semi simple Lie groups on the circle attracted some attention in the last decades. It was recently proved by Deroin and Hurtado that higher rank lattices can act by homeomorphisms on the circle only via a finite group. This puts an end to different previous work like the ones of Witte-Morrins and Ghys.

Since higher rank lattices have Property (T) and this rigidity property is often thought as an obstruction to act on one dimensional objects. Ghys ask if there are groups with Property (T) acting non-elementarily on the circle. Here a non elementary action is an action without invariant probability measure. Margulis proved some "Tits alternative" in this case: an action of a group G by homeomorphisms on the circle is elementary or G contains a non-abelian free group.

The question about groups with Property (T) was implicitly asked for discrete groups but Property (T) is a topological property and $\mathrm{Homeo}(\mathbf{S}^1)$ is topological group with the compact-open topology. So, one can ask if there are interesting topological groups with Property (T) acting on the circle.

Let us recall that a topological group G has Property (T) if there is a pair (K, ε) where $K \subset G$ is compact and $\varepsilon > 0$ such that for any continuous unitary representation π on some some Hilbert $\mathcal{H}$, if there is a unit vector $u \in \mathcal{H}$ such that $||\pi(g)u - u|| < \varepsilon$ for any $g \in K$ then there is an invariant unit vector.

This is a topological group property and some non-locally compact group like the group of all permutation of an infinite countable set (endowed with the pointwise convergence topology) have this topology. In fact, the group $\mathrm{Homeo}(\mathbf{S}^1)$ itself has this property but it is a void property since there no non-trivial irreducible unitary representation for this group.

The idea of this work is to exhibit a closed topological group of $\mathrm{Homeo}(\mathbf{S}^1)$ that has this property and has a very large dual. For example, there is a faithful unitary representation and infinitely many non isomorphic unitary representations.

This group is constructed using homeomorphisms of dendrites. Let us recall that a dendrite is a compact connected locally connected metrizable space with the property that any two points are joined by a unique arc (i.e. the image of an injective continuous maps from the unit interval.).

The link with the circle is given by the *Carathéodory loop*. Any dendrite can be embedded in the plane and there is a Riemann mapping between the complement of the unit disk in the plane and the complement of the dendrite since both are simply connected. In this particular case, this map extends to the boundaries and gives the so-called "Carathéodory loop" $\mathrm{S}^1 \to D$ where D in a dendrite.

Choosing well D and a subgroup $G < \mathrm{Homeo}(D)$, one can lift the action $G \curvearrowright D$ and find the group with the desired properties.

Orbit equivalence and wreath products

Konrad Wróbel
(joint work with Robin Tucker-Drob)

Two groups are said to be *orbit equivalent* if they admit essentially free probability measure preserving (p.m.p.) actions on standard probability spaces that generate isomorphic orbit equivalence relations.

Our first main theorem is an antirigidity result in the nonamenable setting. Fix a wreath product group $\Lambda \wr \Gamma = (\bigoplus_\Gamma \Lambda) \rtimes \Gamma$ and fix an essentially free ergodic p.m.p. action $\Lambda \curvearrowright Z$. Define the wreath product action $\Lambda \wr \Gamma \curvearrowright Z^\Gamma$ by $((f,\gamma)\cdot x)(\delta) = f(\gamma^{-1}\delta)\cdot x(\gamma^{-1}\delta)$. It is easy to see that when Λ is amenable, this action does not depend on the initial action $\Lambda \curvearrowright Z$ up to orbit equivalence.

Theorem 1. *Let Γ be a countable group that contains an infinite amenable group as a free factor. The wreath product actions of $A \wr \Gamma$ and $B \wr \Gamma$ are orbit equivalent for all nontrivial (possibly finite) amenable groups A, B.*

This is most interesting in the case when at least one of the groups A or B is finite, since if A and B are both infinite amenable groups, then they are orbit equivalent by Ornstein-Weiss, so a simple direct argument shows that $A \wr \Gamma$ and $B \wr \Gamma$ are also orbit equivalent.

Theorem 1 In particular implies $C_n \wr \mathbb{F}_2$ is orbit equivalent to $C_m \wr \mathbb{F}_2$, where $\mathbb{F}_2$ is the free group on 2 generators and C_k is the cyclic group of order k. This was previously unknown, although a consequence of work of Bowen implies that the group von Neumann algebras $L(C_n \wr \mathbb{F}_2)$ and $L(C_m \wr \mathbb{F}_2)$ are isomorphic[1].

We can strengthen this result via several nice tricks to get the following corollary.

Corollary. *Let Γ be a countable group that contains an infinite amenable group as a free factor, and let A and B be amenable groups.*

(1) *The groups $\Lambda \wr \Gamma$ and $(\Lambda \times A) \wr \Gamma$ are orbit equivalent for every nontrivial group Λ.*
(2) *If Λ_0 and Λ_1 are nontrivial groups such that $\Lambda_0 \times A$ and $\Lambda_1 \times B$ are measure equivalent, then $\Lambda_0 \wr \Gamma$ and $\Lambda_1 \wr \Gamma$ are orbit equivalent.*

In particular, if Λ_0 and Λ_1 are nontrivial groups which are measure equivalent, then $\Lambda_0 \wr \Gamma$ and $\Lambda_1 \wr \Gamma$ are orbit equivalent.

We prove Theorem 1 by showing the canonical wreath product actions are orbit equivalent. By contrast, we also show the following rigidity.

Theorem 2. *If Γ is a sofic Bernoulli superrigid group with no nontrivial finite normal subgroups and A and B are amenable groups of different cardinalities, then the wreath product actions of $A \wr \Gamma$ and $B \wr \Gamma$ are not stably orbit equivalent.*

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Cardinal algebras and invariant measures

Ruiyuan Chen

Tarski's 1949 theory of cardinal algebras [8] is an axiomatic approach to "constructive" aspects of cardinal arithmetic. A **cardinal algebra** $A = (A, +, 0, \sum)$ consists of a commutative monoid $(A, +, 0)$ together with an infinite addition operation $\sum : A^{\mathbb{N}} \to A$, required to obey four simple axioms. Examples are $[0, \infty]$ with the usual addition; cardinal numbers (under a weak form of the axiom of choice, e.g., countable dependent choice); and the algebra of countable equidecomposition types in a measurable space under a group action or equivalence relation.

As a concrete instance of this last example, fix a standard Borel space X and countable Borel equivalence relation $E \subseteq X^2$ (i.e., with countable equivalence classes). Two Borel sets $A, B \subseteq X$ are called ***E*-equidecomposable**, denoted

$$A \sim_E B,$$

if there is a Borel bijection $f : A \cong B$ with graph contained in E. If E is induced by the Borel action of a countable group $\Gamma \curvearrowright X$, then this is equivalent to the existence of countable Borel partitions $A = \bigsqcup_n A_n$ and $B = \bigsqcup_n B_n$ and group elements $\gamma_n \in \Gamma$ such that $\gamma_n \cdot A_n = B_n$. The quotient set

$$\mathcal{K}(E) := \{\text{Borel sets} \subseteq X\}/{\sim_E}$$

forms a cardinal algebra under union of pairwise disjoint representative Borel sets, modulo the fact that it may not be possible to pick such pairwise disjoint representatives in general; this can be worked around by replacing (X, E) with its "amplification" $(X \times \mathbb{N}, E \times \mathbb{N}^2)$, or equivalently by considering Borel maps $X \to \mathbb{N} \sqcup \{\infty\}$ instead of Borel sets. The algebraic structure of $\mathcal{K}(E)$ conveniently captures many combinatorial properties of countable Borel equivalence relations [2], and has recently seen several applications to Borel and measurable combinatorics [5, 6].

A main reason behind the fruitfulness of the theory of cardinal algebras has been the large collection of general results Tarski [8] and later authors proved from the four basic axioms. Indeed, these results suggest that all "natural" properties of the particular algebra $[0, \infty]$ seem to hold in all cardinal algebras. We prove the following precise form of this idea:

Theorem. *Every cardinal algebra A obeys all countable universal axioms that hold in all powers $[0, \infty]^X$ of $[0, \infty]$, i.e., all axioms of the form*

$$\forall \vec{x}\, \phi(\vec{x})$$

where $\vec{x}$ is a countable list of variables, and $\phi(\vec{x})$ is a quantifier-free infinitary first-order statement over those free variables, built using countable $\bigwedge, \bigvee, \neg$.

The proof is "model-theoretic", by showing that A embeds into such a power $[0, \infty]^X$. In fact, we prove the result for a broader class of algebras, called **regular positive σ-DCPO-monoids**, which includes all cardinal algebras but is itself axiomatized by certain universal axioms (in an extended language including a partial order $\leq$ and countable directed suprema) known to follow from the non-universal axioms of cardinal algebras. Similar embedding theorems have been shown by Wehrung [11] in a context with only finite addition, as well as Tix [9] under the additional domain-theoretic assumption of *continuity*. Our result replaces these assumptions with the algebraic condition of **countable presentability** (via countably many generators and relations):

Theorem. *Every countably presented regular positive σ-DCPO-monoid embeds into a power $[0,\infty]^X$, in fact into the subalgebra of Borel maps $X \to [0,\infty]$, for some standard Borel space X.*

The proof passes through point-free topology/descriptive set theory (otherwise known as *locale theory*) [3], specifically Vickers' theory of *valuation locales* [10]. In fact, the most general form of the result is formulated entirely in the point-free context and is independent of countability restrictions, which only enter via the known correspondence between countably presented topologies and quasi-Polish spaces [4]. The previously stated result follows due to the countability restriction on the universal axioms.

By applying our main result to the algebra $\mathcal{K}(E)$ defined above, we obtain a quick proof of Nadkarni's classical theorem [1, 4.5] which states (in one equivalent formulation) that for any countable Borel equivalence relation E and two Borel sets A, B, A is E-equidecomposable with a Borel subset of B iff every E-invariant measure μ has $\mu(A) \leq \mu(B)$. Moreover, by suitably modifying the algebra $\mathcal{K}(E)$, we also recover B. Miller's [7] quasi-invariant version of Nadkarni's theorem, where equidecomposability is defined with scaling by a Borel cocycle $E \to (0,\infty)$.

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On invariant random orders

Frank Lin
(joint work with Yair Glasner, Tom Meyerovitch)

All groups we consider are countable. An invariant random order (IRO) on a countable group Γ is an invariant probability measure on the space of all total orders on that group, viewed as a subspace of the product space $2^{\Gamma\times\Gamma}$ with Γ acting by diagonal left-shift. The notion of an IRO generalizes the classical notion of a left-invariant order on a group (see [2] for background). To our knowledge early authors who considered IROs include Stepin, Tagi-Zade [6], and Kieffer [5]. Other authors have also applied IROs in the study of entropy theory.

In contrast to the deterministic case, every group has an invariant random order by independently giving every group element a number uniformly randomly from the unit interval. In particular torsion, which is an obstruction to left-orderability, is not an obstruction in the case of IROs.

On the question of extendability, it is known in the deterministic case that every left-invariant partial order on any torsion-free locally nilpotent group has an extension to a total order [3]. Examples of non-extendable partial orders can be found in [4].

We say that a group has the IRO-extension property if every invariant random partial order on the group can be extended to an invariant random (total) order. It is known that all amenable groups have the IRO-extension property. In our talk we present an example, inspired by work from Witte [7], showing that $SL_3(\mathbf{Z})$ does not have the IRO-extension property, answering a question of Alpeev, Meyerovitch, and Ryu [1]. The example is in fact the deterministic partial order $\prec_0$ associated with the semigroup generated by the following unipotent matrices

$$(1)\qquad \begin{aligned} a_1 &= \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix} & a_2 &= \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix} & a_3 &= \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix} \\ a_4 &= \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} & a_5 &= \begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix} & a_6 &= \begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}\end{aligned}.$$

As with Witte, our proof uses the commutator relations of unipotent elements and ideas related to a group element being infinitely larger than another element. We show that the set of total orders extending $\prec_0$, though not necessarily empty, is *weakly wandering* in a dynamical sense and thus cannot support any invariant probability measure. The following open question remains: does there exist a non-amenable group with the IRO-extension property? If the answer is negative then the IRO-extension property would be another characterization of amenability.

Our upcoming paper includes other results and questions not presented in the talk.

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Allosteric actions for surface groups

Matthieu Joseph

The aim of this talk is to compare two notions of freeness for actions of countable groups. A minimal ergodic action of a countable group Γ is an action by homeomorphisms on a compact Hausdorff space C, which is minimal (every orbit is dense), with an ergodic Γ-invariant probability measure on C. A minimal ergodic action is

- topologically free if the set of points with trivial stabilizer is comeager,
- essentially free if the set of points with trivial stabilizer is conull.

Essential freeness implies topological freeness, but the converse is false in general. A minimal ergodic action which is topologically free but not essentially free is called *allosteric*. A group which admits allosteric actions is called *allosteric*. There are several reasons for a group Γ not to be allosteric. They are related to the dynamics of Γ on its Chabauty space, which is the space $\mathrm{Sub}(\Gamma)$ of subgroups of Γ, on which Γ acts by conjugation. For instance, if $\mathrm{Sub}(\Gamma)$ is countable, then one can show that Γ is not allosteric. Similarly, if the only ergodic Invariant Random Subgroups (IRS) of Γ are atomic, then Γ is not allosteric. Here, an IRS is a Γ-invariant probability measure on $\mathrm{Sub}(\Gamma)$.

In the survey [3, Prob. 7.3.3], Grigorchuk, Nekrashevych and Suschanskii asked whether allosteric group exists. The first examples of allosteric groups are due to Bergeron and Gaboriau: if Γ and Λ are two residually finite, nontrivial groups, then the free product $\Gamma * \Lambda$ is allosteric (except if Γ and Λ are both isomorphic to the cyclic group of order two), see [2]. A similar result was proved independently for free groups by Abért and Elek [1]. In the talk, we explain the main result of [4]: the fundamental group of any hyperbolic closed surface is allosteric.

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