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Embedding Spaces of Split Links

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EMBEDDING SPACES OF SPLIT LINKS

RACHAEL BOYD AND COREY BREGMAN

ABSTRACT. We study the homotopy type of the space $\mathcal{E}(L)$ of unparametrised embeddings of a split link $L = L_1 \sqcup \dots \sqcup L_n$ in \mathbb{R}^3 . Inspired by work of Brendle and Hatcher, we introduce a semi-simplicial space of separating systems and show that this is homotopy equivalent to $\mathcal{E}(L)$. This combinatorial object provides a gateway to studying the homotopy type of $\mathcal{E}(L)$ via the homotopy type of the spaces $\mathcal{E}(L_i)$. We apply this tool to find a simple description of the fundamental group, or motion group, of $\mathcal{E}(L)$, and extend this to a description of the motion group of embeddings in S^3 .

1. INTRODUCTION

In this paper we study the homotopy type of the unparametrised embedding space of a link L in \mathbb{R}^3 . We say L is *split* if it can be written as a disjoint union $L = L_1 \sqcup \dots \sqcup L_n$, where the L_i can be contained in disjoint balls. We call each sublink L_i a *piece* of L , and denote by $|L| = n$ the number of pieces. If the total number of circle components of L is m , let $\mathcal{E}(L)$ denote the connected component of

$$\text{Emb}(\sqcup_m S^1, \mathbb{R}^3) / \text{Diff}(\sqcup_m S^1)$$

containing the link L . We quotient by $\text{Diff}(\sqcup_m S^1)$ in order to consider unparametrised smooth embeddings.

Our main goal is to relate the homotopy groups of $\mathcal{E}(L)$ to those of the embedding spaces of individual pieces $\mathcal{E}(L_i)$. The main tool we develop is a semi-simplicial space of *separating systems* for $\mathcal{E}(L)$, which parametrises all of the ways of splitting a given link in $\mathcal{E}(L)$ into pieces. This reduces the codimension two problem of studying the homotopy groups of $\mathcal{E}(L)$ into

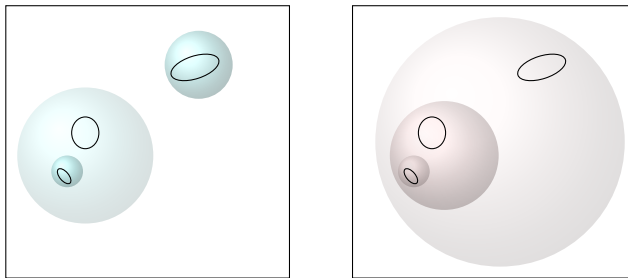
- (i) a codimension one problem - studying the space of separating 2-spheres;
- (ii) an easier codimension two problem - studying homotopy groups of $\mathcal{E}(L_i)$ for the pieces L_i .

We apply our method to express $\pi_1(\mathcal{E}(L))$ in terms of $\pi_1(\mathcal{E}(L_i))$ and $\pi_1(\mathbb{R}^3 \setminus L_i)$ for each piece L_i . In fact, our results give a framework to further study the homotopy groups of $\mathcal{E}(L)$ – for each homotopy class we are able to exhibit a representative with certain extra structure.

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1.1. Separating systems. Inspired by work of Brendle and Hatcher [BH13], we introduce a semi-simplicial space of *separating systems* associated to $\mathcal{E}(L)$. A separating system for a link $\rho \in \mathcal{E}(L)$ is a disjoint union of unparametrised embedded spheres in $\mathbb{R}^3 \setminus \rho$, which we denote by Σ , such that each connected component of $\mathbb{R}^3 \setminus \Sigma$ contains at most one piece of ρ . Some examples of separating systems for the 3 component unlink are shown below.



We build a semi-simplicial space Sep_\bullet of separating systems for $\mathcal{E}(L)$, where the 0-simplices Sep_0 is the space of pairs (Σ, ρ) such that $\rho \in \mathcal{E}(L)$ and Σ is a separating system for ρ . The p -simplices $\text{Sep}_p \subset (\text{Sep}_0)^{p+1}$ is the space of $(p+1)$ disjoint separating systems for the same link ρ .

There is a natural augmentation map $\varepsilon_\bullet: \text{Sep}_\bullet \rightarrow \mathcal{E}(L)$ which forgets the separating systems. Our first theorem concerns the induced map on the geometric realisation $|\text{Sep}_\bullet|$.

Theorem A. *The map $|\varepsilon_\bullet|: |\text{Sep}_\bullet| \rightarrow \mathcal{E}(L)$ is a homotopy equivalence.*

From this theorem we get a rigid structure on the homotopy classes of $\mathcal{E}(L)$. We show the theorem implies that each homotopy class in $\pi_k(\mathcal{E}(L))$ has a representative $f: S^k \rightarrow \mathcal{E}(L)$ such that the image of f exhibits what we call a *compatible separating triangulation*. This is a triangulation of the sphere S^k such that on the image of each simplex under f , there exists a separating system for the embedded link which changes by isotopy as the link does. Moreover on faces where two or more simplices meet, all separating systems exist simultaneously and disjointly.

In fact we go further, and prove that in each homotopy class there is a representative with compatible separating triangulation such that in addition all separating spheres are *round i.e.*, bound Euclidean balls in \mathbb{R}^3 .

1.2. Computation of the fundamental group. The fundamental group $\pi_1(\mathcal{E}(L))$ is known as the *motion group of L* . We use Theorem A to decompose the motion group $\pi_1(\mathcal{E}(L))$ (with basepoint L omitted from the notation) into an iterated semidirect product of three groups:

- (1) \mathbf{P}_L : The symmetric group S_n acts on the set of n link pieces by permutation, and P_L is the subgroup of permutations which only permute link pieces L_i and L_j when they are isotopic in \mathbb{R}^3 .
- (2) \mathbf{G}_L : is the direct product of the motion groups of the link pieces, *i.e.*,

$$G_L = \prod_{i=1}^n \pi_1(\mathcal{E}(L_i)).$$

- (3) $\mathcal{FR}_0(\mathbf{L})$: Let $H_i = \pi_1(\mathbb{R}^3 \setminus L_i)$ for each piece L_i of L and let $H_L = \pi_1(\mathbb{R}^3 \setminus L) \cong H_1 * \cdots * H_n$. We define $\mathcal{FR}(L)$ to be the subgroup of $\text{Aut}(H_L)$ generated by automorphisms that conjugate one factor H_j by an element $g \in H_i$, and call it the *Fouxe-Rabinovitch* group of L after [FR40, FR41]. Then $\mathcal{FR}_0(L) \leq \mathcal{FR}(L)$ is the subgroup generated by those conjugations where $i \neq j$.

Theorem B. *The motion group $\pi_1(\mathcal{E}(L))$ is isomorphic to $(\mathcal{FR}_0(L) \rtimes G_L) \rtimes P_L$.*

The action of P_L and G_L in this iterated semidirect product decomposition may be described as follows. P_L acts by permuting the link pieces, which has the effect of permuting the factors in $G_L = \prod_{i=1}^n \pi_1(\mathcal{E}(L_i))$, as well as the conjugating elements $g \in H_i$ and factors H_j in $\mathcal{FR}_0(L)$. For the inner semidirect product, each factor $\pi_1(\mathcal{E}(L_i))$ acts on $H_i = \pi_1(\mathbb{R}^3 \setminus L_i)$ (via the Dahm homomorphism [Dah62], see Section 6 for a definition). If an element $y \in \pi_1(\mathcal{E}(L_i))$ induces an automorphism φ of H_i , then y acts on $\mathcal{FR}_0(L)$ by sending the conjugation of H_j by $g \in H_i$ to the conjugation of H_j by $\varphi(g)$.

We prove this theorem by first exhibiting motions in $\pi_1(\mathcal{E}(L))$ that generate the various subgroups. We then construct two split exact sequences, the first of which splits off P_L . Using Theorem A, we show that $\mathcal{FR}_0(L)$ is the kernel of the second. We remark that the Fouxe-Rabinovitch group of a free product is finitely presented as long as each of the factors are. In particular, it follows from Theorem B that $\pi_1(\mathcal{E}(L))$ is finitely presented as long as $\pi_1(\mathcal{E}(L_i))$ is for each piece L_i .

Let us illustrate this semidirect product decomposition in the special case when $L = U_n$ is the n -component unlink. The motion group in this case was first computed by Goldsmith [Gol81], who identified $\pi_1(\mathcal{E}(U_n))$ with the *symmetric automorphism group* $\Sigma \text{Aut}(F_n)$ of the free group F_n . This is the subgroup of $\text{Aut}(F_n)$ whose elements send generators of F_n to conjugates of generators or their inverses. Goldsmith's identification can be understood as follows: F_n is the fundamental group of $\mathbb{R}^3 \setminus U_n$, and a motion in $\mathcal{E}(U_n)$ starting and ending at U_n acts as an automorphism of this fundamental group.

A presentation for $\Sigma \text{Aut}(F_n)$ was found by McCool [McC86] and later recovered by Brendle and Hatcher [BH13] in the context of motion groups. The decomposition given by Theorem B amounts to a description of $\Sigma \text{Aut}(F_n)$ as the semidirect product $\Sigma \text{Aut}(F_n) \cong (\mathcal{FR}_0(U_n) \rtimes (\mathbb{Z}/2\mathbb{Z})^n) \rtimes S_n$. Let us relate these 3 groups to motions of U_n :

- (1) View U_n is a split link with n unknot pieces. Then since any two pieces are isotopic, there is a subgroup of the motion group isomorphic to the symmetric group S_n given by motions which permute the pieces.
- (2) Since each unknot is unparametrised, $\pi_1(\mathcal{E}(L_i)) = \pi_1(\mathcal{E}(U)) = \mathbb{Z}/2\mathbb{Z}$. A generator for this is a 180° rotation of a round unknot. Since we can realise these motions disjointly, we get a copy of $(\mathbb{Z}/2\mathbb{Z})^n$ in $\pi_1(\mathcal{E}(U_n))$.
- (3) Since the complement of a single unknot has abelian fundamental group, there is an equality $\mathcal{FR}(U_n) = \mathcal{FR}_0(U_n)$. In this case, the Fouxe-Rabinovitch group is generated by motions where one unknot piece shrinks, moves through another, and returns to its original position. This corresponds

the pure symmetric automorphism group of F_n in which every generator is sent to a conjugate of itself.

1.3. Embeddings in S^3 . Thinking of S^3 as the 1-point compactification of \mathbb{R}^3 , there is a fibre sequence relating $\mathcal{E}(L)$ to $\underline{\text{Emb}}(L, S^3)$, which we define to be the path component of $\text{Emb}(\sqcup_m S^1, S^3)/\text{Diff}(\sqcup_m S^1)$ containing L . We explore this relationship in Section 6, and extend Theorem B to this setting. In fact, the fundamental group of $\underline{\text{Emb}}(L, S^3)$ is a quotient of $\pi_1(\mathcal{E}(L))$, by inner automorphisms of $\pi_1(S^3 \setminus L) \cong \pi_1(\mathbb{R}^3 \setminus L)$:

Corollary C. *Let $H_L = \pi_1(S^3 \setminus L)$. Then*

$$\pi_1(\underline{\text{Emb}}(L, S^3)) \cong (\mathcal{FR}_0(L) \rtimes G_L) / \text{Inn}(H_L) \rtimes P_L.$$

where $\text{Inn}(H_L)$ is the group of inner automorphisms of H_L .

1.4. Previous work on the homotopy type of knot and link embedding spaces. In low dimensions, the homotopy types of various embedding spaces have been studied and in some cases a formula for the homotopy type has been given. In this subsection we give a brief overview. The homotopy type of embedding spaces of knots in S^3 is understood, up to knowing the homotopy type of S^2 , following work of Hatcher and Budney. (In practice it is still non-trivial to compute specific homotopy groups for a given knot in S^3 .)

The starting point is Hatcher's proof of the Smale conjecture [Hat83], from which he later deduced the equivalent statement that the unknot component of the embedding space of *long knots* is contractible (see the appendix in [Hat83] and [Hat76]). Recall a long knot is a smooth embedding of \mathbb{R}^1 in \mathbb{R}^3 such that outside the unit ball it is standard, and isotopy classes of long knots correspond to connected components of the embedding space. Hatcher later showed that each connected component of the embedding space of long knots is an Eilenberg-MacLane space [Hat02], and in joint work with McCullough, that it has the homotopy type of a finite CW-complex [HM97].

Budney [Bud07] showed that the space of long knots is a free algebra over the *little 2-disks* operad, generated by the prime long knots. This gives a formula to compute the homotopy type of a long knot component, from the homotopy type of the pieces in its prime decomposition.

Hatcher [Hat02] computed the homotopy type of the connected component corresponding to a (long) torus knot – which has the homotopy type of S^1 – and a (long) hyperbolic knot – which has the homotopy type of $S^1 \times S^1$. This feeds into a second operadic description of the long knot space by Budney, who introduced a *splicing* operad that takes into account satellite moves on long knots. Using the JSJ decomposition of the knot complement, he was able to give a recursive formula for computing the homotopy type of a long knot component from hyperbolic and torus pieces and the splicing operad [Bud12]. This gives a closed formula for the homotopy type of any long knot component.

To pass from long knots in \mathbb{R}^3 to knots in S^3 , Budney and Cohen provide a formula giving the relationship between the homotopy type of the long knot component and the homotopy type of the connected component of the corresponding embedding

in S^3 , realised by taking one point compactification [BC09]. This is the stage at which the homotopy groups of $SO(4)$ are present.

For embedding spaces of links, much less is known. Early approaches focused on computing motion groups $\pi_1(\mathcal{E}(L))$ for specific links. Goldsmith [Gol81], building on work of Dahm [Dah62], computed the motion group of U_n (as noted above), and torus links [Gol82]. Damiani and Kamada computed the motion group for the round embedding space of the disjoint union of a Hopf link and an unknot [DK19].

In [BH13], Brendle and Hatcher showed that $\mathcal{E}(U_n)$ is homotopy equivalent to the embedding space of *round* unlinks, where round embeddings are such that each embedded S^1 bounds a Euclidean disk in some plane $\mathbb{R}^2 \subset \mathbb{R}^3$. The latter is a finite-dimensional manifold, hence in particular this shows $\mathcal{E}(U_n)$ has finite homotopy type. As mentioned above, on the level of π_1 this was used to give an explicit presentation for $\Sigma \text{Aut}(F_n)$ in terms of elements of $\pi_1(\mathcal{E}(U_n))$, recovering work of [McC86] and [Gol82]. Surya [Sur04] gave an iterated semidirect product description as Theorem B for the special case of $\pi_1(\mathcal{E}(U_n))$ (note that in [Sur04] the semidirect product symbol is written the wrong way around).

In recent work, Koytcheff [Koy22] studies the homotopy type of various spaces of links. This work focuses on long links defined in general dimension and codimension, and the results focus on long links of codimension at least 3.

1.5. Discussion and future directions. The proof of Theorem B in Section 6 suggests a strategy for computing the higher homotopy groups of $\mathcal{E}(L)$. In fact, we expect the higher homotopy groups $\pi_k(\mathcal{E}(L))$ to be built out of the classes in $\pi_k(\mathcal{E}(L_i))$ together with classes in π_k of the configuration space of p points in $\mathbb{R}^3 \setminus L_i$. If each piece is a knot, then the groups $\pi_k(\mathcal{E}(L_i))$ are understood, as discussed in the previous section, although concrete computations in the literature are sparse. Little is known in the case that the piece is a link, except in the case of $\pi_1(\mathcal{E}(T_{p,q}))$ by Goldsmith [Gol82]. In upcoming work, we will prove that the embedding space of an unparametrised Hopf link, $\mathcal{E}(HL)$, is homotopy equivalent to the round subspace $\text{REmb}(HL, \mathbb{R}^3)$, where each link component bounds a Euclidean disk in some $\mathbb{R}^2 \subset \mathbb{R}^3$. This has the homotopy type of a finite dimensional manifold which we describe.

Note that in the example of the unlink, the motion group is isomorphic to the symmetric automorphism group of $\pi_1(\mathbb{R}^3 \setminus L)$. One can build links for which $\pi_1(\mathbb{R}^3 \setminus L)$ is a right-angled Artin group with defining graph a disjoint union of trees. One can then define the analogue of the symmetric automorphism groups in this case, to be the motion groups of such links. The automorphism groups of RAAGs and their subgroups have been the topic of extensive study, and as in the case of the group $\Sigma \text{Aut}(F_n)$, $\mathcal{E}(L)$ provides a geometric model for a subgroup.

Finally, we point out an application of our work to diffeomorphisms of 4-manifolds. Regarding L as the locus of a collection of attaching spheres of 2-handles in S^3 , we can think of a point of $\text{Emb}(L, S^3)$, together with framing data, as describing a Kirby diagram of a 4-manifold M . If $\partial M = S^3$, we can cap off M by adding a 4-handle to obtain a closed 4-manifold \bar{M} .

A motion of L , representing an element of $\pi_1(\underline{\text{Emb}}(L, S^3))$, then gives rise to a diffeomorphism of M supported in a collar of the 0-handle. Since $\text{Diff}_0(S^3)$ is connected, this extends to a diffeomorphism of $M \cup (S^3 \times I)$ which is the identity on the boundary. Fixing an element of $\text{Diff}(D^4, \partial D^4)$ we can further extend to a diffeomorphism of \overline{M} . Using the Smale conjecture [Hat83], one can show that the isotopy class is independent of the choice made when extending to $M \cup (S^3 \times I)$. Therefore for each fixed class in $\pi_0(\text{Diff}(D^4, \partial D^4))$ we obtain a homomorphism

$$\pi_1(\underline{\text{Emb}}(L, S^3)) \rightarrow \pi_0(\text{Diff}(\overline{M})).$$

It would be interesting to compute when these classes are non-trivial, using Corollary C to understand the left hand side. Note that the image of this map consists of diffeomorphisms which act trivially on second homology, and thus are topologically isotopic to the identity [Qui86].

To give an example, when L is a disjoint union of n unknots with framing ± 1 and m Hopf links with zero-framing, \overline{M} is a connected sum of n $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$'s and m $S^2 \times S^2$'s. In this case, we can compute the fundamental group of $\underline{\text{Emb}}(L, S^3)$ explicitly using Corollary C (see Example 6.12 for a description of $\pi_1(\mathcal{E}(L))$).

One can similarly construct maps $\pi_k(\underline{\text{Emb}}(L, S^3)) \rightarrow \pi_{k-1}(\text{Diff}(\overline{M}))$, for each fixed class in $\pi_{k-1}(\text{Diff}(D^4, \partial D^4))$, but in this setting we do not have a version of Corollary C.

1.6. Outline. In Section 2 we define the spaces of interest, introduce separating systems, and prove some combinatorial properties which we need for the computation of the motion group. In Section 3 we introduce the semi-simplicial space and using semi-simplicial arguments we reduce the proof of Theorem A to a contractibility statement. Following this in Section 4 we prove the contractibility statement using geometric tools. Thus we have completed the proof of Theorem A, and in Section 5 we discuss the implications of this result for finding representatives of homotopy classes of $\mathcal{E}(L)$. Finally we use our tools to study $\pi_1(\mathcal{E}(L))$ in Section 6. We prove Theorem B and extend our result to the motion group of link configurations in S^3 as opposed to \mathbb{R}^3 , proving Corollary C.

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2. EMBEDDING SPACES AND SPHERE SYSTEMS

In this section, we set up some notation and define the basic objects of study that will be referred to throughout the paper.

2.1. Embedding spaces. Let M be a smooth manifold. The diffeomorphism group of M will be denoted $\text{Diff}(M)$, and the space of smooth embeddings of a smooth manifold N into M , endowed with the C^∞ Whitney topology, will be denoted $\text{Emb}(N, M)$. Recall that equipped with this topology $\text{Emb}(N, M)$ is a principal $\text{Diff}(N)$ -bundle where $\text{Diff}(N)$ acts by precomposition. The quotient is the space of unparametrised embeddings

$$\underline{\text{Emb}}(N, M) = \text{Emb}(N, M) / \text{Diff}(N).$$

We will typically work with $\underline{\text{Emb}}$ and thus often identify embeddings of spaces with their images.

A link L is an element of $\underline{\text{Emb}}(\sqcup_m S^1, \mathbb{R}^3)$ for some m . We denote the path component of $\underline{\text{Emb}}(\sqcup_m S^1, \mathbb{R}^3)$ containing L by $\mathcal{E}(L)$. L is called *split* if it can be written as a disjoint union $L = L_1 \sqcup \cdots \sqcup L_n$ where $n \geq 2$ and each L_i can be contained in a ball B_i such that $B_i \cap B_j = \emptyset$ for $i \neq j$. L is called *unsplit* otherwise. As in the introduction, the sublinks L_i will be called the pieces of L .

We now introduce the main spaces of study. For $i \in \{1, 2, \dots, n\}$, fix once and for all unsplit links $L_i \in \underline{\text{Emb}}(\sqcup_{m_i} S^1, \mathbb{R}^3)$ for some $m_i \in \mathbb{N}$. We choose L_i so that its image lies in the ball B_i of radius $\frac{1}{2}$ centered on $(2i, 0, 0) \in \mathbb{R}^3$. We also require that if L_i and L_j are isotopic, then the embeddings differ by a translation of \mathbb{R}^3 . Then for $m = \sum_{i=1}^n m_i$ we set L to be the image of the disjoint union $L_1 \sqcup \cdots \sqcup L_n$ in $\underline{\text{Emb}}(\sqcup_m S^1, \mathbb{R}^3)$. When we refer to $L \in \mathcal{E}(L)$ in the sequel, we mean this chosen unparametrised embedding, which will serve as the natural basepoint for $\mathcal{E}(L)$.

We also introduce the space $\mathcal{PE}(L)$ to be the component of the embedding space

$$\text{Emb}(\sqcup_m S^1, \mathbb{R}^3) / \prod_{i=1}^n \text{Diff}(\sqcup_{m_i} S^1)$$

containing $L_1 \sqcup \cdots \sqcup L_n$. This is the pure unparametrised embedding space, in analogy with the ‘‘pure’’ configuration space, where each piece L_i is unparametrised but labelled. Note that the basepoint $L \in \mathcal{PE}(L)$ is different from the basepoint $L \in \mathcal{E}(L)$ as in the latter isotopic pieces are not differentiated, whereas in the former the pieces are labelled with label set $\{1, \dots, n\}$. We use the notation L for both basepoints, as it is always clear as to which space we are working in.

As discussed at the end of Section 3, $\mathcal{PE}(L)$ is a finite cover of $\mathcal{E}(L)$.

2.2. Separating Systems.

Definition 2.1. A *sphere system* for an embedding $\rho \in \mathcal{E}(L)$ or $\mathcal{PE}(L)$ is an embedding of a disjoint union of spheres $\Sigma: \sqcup_{i=1}^k S^2 \hookrightarrow \mathbb{R}^3 \setminus \rho$ in $\underline{\text{Emb}}(\sqcup_{i=1}^k S^2, \mathbb{R}^3)$.

Remark 2.2. Note that although the pieces are labelled in $\mathcal{PE}(L)$, we do not require the spheres in a separating system for $\rho \in \mathcal{PE}(L)$ to be labelled.

Definition 2.3. A *separating system* Σ for $\rho \in \mathcal{E}(L)$ or $\mathcal{PE}(L)$ is a sphere system such that each component of $\mathbb{R}^3 \setminus \Sigma$ contains at most one piece of ρ . We call Σ *essential* if no connected component of $\mathbb{R}^3 \setminus (\rho \cup \Sigma)$ is homeomorphic to $\text{Int}(\mathbb{B}^3)$ nor $\text{Int}(S^2 \times I)$.

Note that the three requirements above correspond to the following attributes of an essential separating system:

- every sphere bounds a ball with at least one piece L_i inside it;
- no two spheres are isotopic;
- there is no sphere which bounds a ball containing the whole link.

For a link with n pieces, there must be at least n spheres in an essential separating system, but in general there may be more than n , subject to the rules above. Examples of separating systems for a link with 3 pieces are shown in Figure 1.

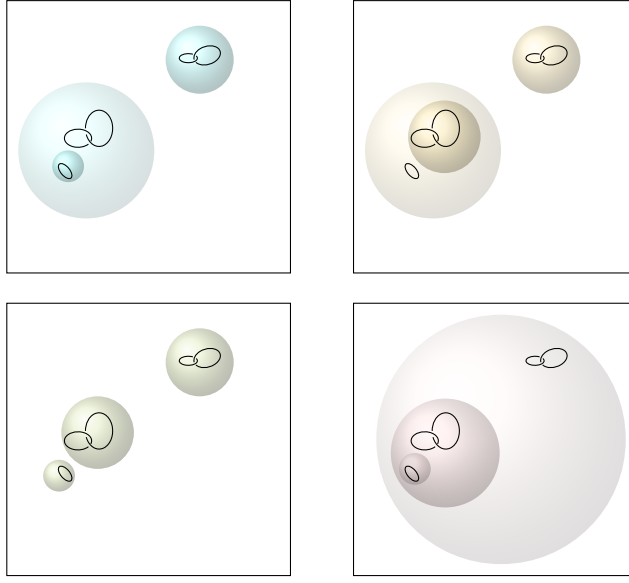


FIGURE 1. Separating systems for a link $\rho \in \mathcal{E}(L)$ with three pieces. Only the bottom right separating system is not essential, because the unbounded component is homeomorphic to $\text{Int}(S^2 \times I)$.

Example 2.4. For our chosen basepoint L in either $\mathcal{E}(L)$ or $\mathcal{PE}(L)$, recall that the pieces sit inside the balls B_i of radius $\frac{1}{2}$ centered on $(2i, 0, 0) \in \mathbb{R}^3$, for $1 \leq i \leq n$. Then taking the unparametrised embedding which has image the boundary spheres of the B_i gives a canonical separating system Σ_L for L .

We now consider embeddings in $\mathcal{E}(L)$ together with a separating system Σ . For k a positive integer, consider the subspace

$$\text{ESS}(L, k) \subseteq \text{Emb}(\sqcup_m S^1 \sqcup_k S^2, \mathbb{R}^3)$$

such that the image ρ of $\sqcup_m S^1$ is isotopic to L , and the image of $\sqcup_k S^2$ gives an essential separating system for ρ with k spheres *i.e.*,

$$\text{ESS}(L, k) = \{(\rho, \Sigma) \mid \rho \in \mathcal{E}(L), \Sigma \text{ essential, separating system for } \rho \text{ and } |\Sigma| = k\}.$$

Let $\text{ESS}(L)$ be the space

$$\text{ESS}(L) = \bigsqcup_{k=1}^{\infty} \text{ESS}(L, k)$$

of essential separating systems for links isotopic to L , *i.e.*,

$$\text{ESS}(L) = \{(\rho, \Sigma) \mid \rho \in \mathcal{E}(L), \Sigma \text{ essential separating system for } \rho\}.$$

We can similarly define a space of labelled links and separating systems:

$$\text{PESS}(L) = \{(\rho, \Sigma) \mid \rho \in \mathcal{PE}(L), \Sigma \text{ essential separating system for } \rho\}.$$

The point (L, Σ_L) is a natural basepoint for $\text{ESS}(L)$ and $\text{PESS}(L)$.

2.3. Combinatorics of separating systems. We associate a rooted tree to each essential separating system, which is labelled when the link is. Although there are infinitely many isotopy classes of essential separating systems, there are only finitely many possible such rooted trees up to isomorphism. We will make use of these rooted trees in Section 6 when we compute the fundamental group of $\mathcal{E}(L)$.

Definition 2.5. Recall that $\rho = \rho_1 \sqcup \dots \sqcup \rho_n \in \mathcal{PE}(L)$ has a labelling of its pieces by the set $\{1, \dots, n\}$. Without loss of generality, ρ_i is labelled by i and satisfies $\rho_i \simeq L_i$. The *combinatorial type* of a separating system Σ for ρ is the labelled rooted graph $\Gamma(\Sigma)$ whose vertices are connected components of $\mathbb{R}^3 \setminus \Sigma$, and where an edge connects two vertices if they are separated by a single sphere of Σ . Let the root of the graph be the unique unbounded component of the complement. The other vertices are labeled by i if the unique labelled link piece contained in that component is ρ_i (and hence labeled by i) and by \emptyset otherwise.

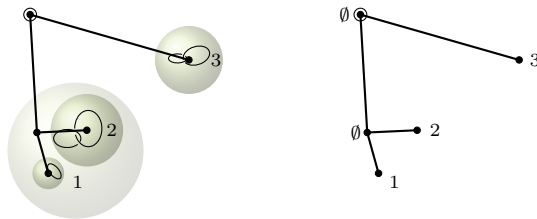


FIGURE 2. An essential separating system Σ for $\rho = \rho_1 \sqcup \rho_2 \sqcup \rho_3 \in \mathcal{PE}(L)$ and its dual rooted L -tree $\Gamma(\Sigma)$.

Remark 2.6. Since an embedded sphere separates \mathbb{R}^3 into exactly two components and the spheres of Σ are disjoint, $\Gamma(\Sigma)$ is always a tree. Observe that Σ is essential if and only if there does not exist a univalent vertex labeled by \emptyset , nor a bivalent vertex, unless that vertex is the root.

Remark 2.7. One could similarly define the combinatorial type of a separating system for $\rho \in \mathcal{E}(L)$, but some care must be taken with labels – for example one could label by isotopy classes of links, since we do not differentiate between isotopic

pieces. Since in Section 6 we only need the notion of combinatorial type for labelled links, we do not explore this further.

Definition 2.8. For $n \geq 1$, a *rooted L -tree* is a rooted tree whose vertices carry labels in the set $\{1, \dots, n, \emptyset\}$ such that

- (1) Labels $1, \dots, n$ appear exactly once.
- (2) There are no univalent or bivalent vertices labeled by \emptyset , unless it is a bivalent vertex which is also the root.

Any vertex can serve as the root of a rooted L -tree. The root is distinguished by a circled vertex as in Figure 2.

Lemma 2.9. *There are finitely many rooted L -trees.*

Proof. Observe that for any rooted L -tree T , the number of vertices and edges will each increase by 1 under the following operations

- Making an interior i -vertex into a leaf, as in Figure 3a,
- Splitting an interior \emptyset -vertex of valence at least 4 into two \emptyset -vertices of valence at least 3 with an edge between them, as in Figure 3b.
- Splitting an \emptyset -labeled root vertex of valence at least 3 into two \emptyset -labeled vertices, one of which is the new root and has valence at least 2, while the other has valence at least 3, as in Figure 3c.

As a consequence, the maximal number of edges occurs when each i -vertex is a leaf, each interior non-root \emptyset -vertex has valence 3, and the root is labeled by \emptyset and has valence 2. An Euler characteristic calculation shows that the maximal number of vertices in such a tree is $2|L| - 1$. Hence, there are only finitely many such T . \square

Lemma 2.10. *For any $\rho \in \mathcal{PE}(L)$, there is a bijection between combinatorial types of essential separating systems and rooted L -trees. Moreover, if Σ, Σ' are disjoint, essential, separating systems for ρ and $\Gamma(\Sigma) = \Gamma(\Sigma')$ then Σ is isotopic to Σ' rel ρ .*

Proof. Since any two embeddings $\rho, \rho' \in \mathcal{PE}(L)$ are ambient isotopic, there is a bijection between combinatorial types of essential separating systems for ρ and ρ' . Thus, it suffices to prove the lemma for the fixed embedding $L \in \mathcal{PE}(L)$. Recall that the image of L_i lies in a ball of radius $1/2$ centered on the point $(2i, 0, 0) \in \mathbb{R}^3$.

By Remark 2.6, for any essential separating system Σ , we know that $\Gamma(\Sigma)$ is a rooted L -tree. Now let T be any rooted L -tree. We claim that there exists an essential separating system Σ for L such that $\Gamma(\Sigma) = T$, and that if Σ, Σ' both have combinatorial type T and can be embedded disjointly, then Σ is isotopic to Σ' rel L . This will imply the lemma.

The proof is by induction on the number of pieces of L . If L has one piece, the only rooted L -tree is where T is a single rooted vertex. In this case the only essential separating system is empty. Hence this realises T and uniqueness up to isotopy is vacuous. Now suppose that there are $n \geq 2$ pieces. Then T has at least two vertices

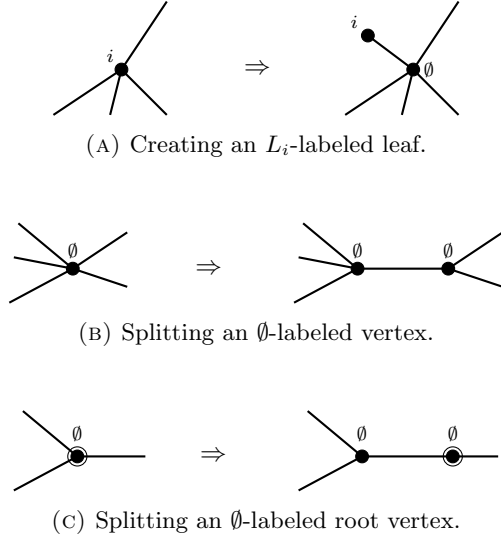


FIGURE 3. Three moves on L -trees.

and since it is a tree, it has at least leaf w which is not the root. Reordering the link pieces if necessary, we may assume that w is labeled by n . Let v be the vertex adjacent to w . Deleting w and its edge from T yields a tree T' labeled by $1, \dots, n-1$ and \emptyset . Condition (1) of Definition 2.8 is still satisfied, as well as condition (2), unless

- (i) v is not the root, is labeled by \emptyset , and has valence 3 in T , with adjacent vertices u_1, u_2 and w .
- (ii) v is the root, is labeled by \emptyset and has valence 2 in T .

If (i) occurs, we set T' to be the tree where the star of v has been replaced by a single edge between the vertices u_1 and u_2 . If (ii) occurs, we set T' to be the tree where v and w are deleted (along with the edges that have an end point at v) and the other vertex w' adjacent to v is now the root.

By induction, we can find Σ' realising T' for $L_1 \sqcup \dots \sqcup L_{n-1}$. First, assume that we do not fall into either of the cases (i) or (ii) above. Then v is still a vertex in T' . Let U be the component of $\mathbb{R}^3 \setminus \Sigma'$ corresponding to v . Now choose a point $p \in U \setminus (L_1 \sqcup \dots \sqcup L_{n-1})$ and a small closed ball $B \subset U$ centered on p . Choose $\tau \in \mathcal{E}(L_n)$ such that the image of τ lies in the interior of B . There is an isotopy simultaneously taking B to the ball of radius $1/2$ centered on $2n$ and the image of τ onto L_n . Now apply isotopy extension to obtain an ambient isotopy of \mathbb{R}^3 , from which we obtain a sphere system Σ realising T , where w corresponds to the interior of B , and we add the boundary of this ball to obtain Σ .

If we are in case (i), we take two parallel copies the sphere of Σ' corresponding to edge added between the vertices u_1 and u_2 and set U to be the component in between these two parallel copies. If we are in case (ii), then w' is the root, hence corresponds to the unbounded component. Add a large sphere surrounding all of Σ' and set U to be the unbounded complement of this sphere. In each case, we

then choose a ball $B \subset U$ and $\tau \in \mathcal{E}(L_n)$ whose image lies in B and proceed as above. This completes the proof of existence.

For uniqueness up to isotopy, if Σ, Σ' are two essential separating systems realising T which can be disjointly embedded, then consider the spheres S, S' respectively in each corresponding to the edge out of w . Because both S and S' separate L_n from the rest of L , they must be nested *i.e.*, $S \cup S'$ cobound an embedded $S^2 \times I$. Hence we can find an ambient isotopy from S to S' which is the identity outside of neighborhood of $S^2 \times I$. In particular, this isotopy is fixed on ρ . By induction, we can isotope each of the remaining spheres of Σ' to those of Σ as well. \square

3. SEMI-SIMPLICIAL SPACES OF SEPARATING SYSTEMS

This section introduces the main semi-simplicial space we work with, and contains the semi-simplicial arguments we require to prove Theorem A.

Recall that

$$\text{ESS}(L) = \{(\rho, \Sigma) \mid \rho \in \mathcal{E}(L), \Sigma \text{ essential separating system for } \rho\},$$

topologised as a subspace of the product of unparametrised embedding spaces, with the C^∞ Whitney topology, and $\text{PESS}(L)$ is the analogously defined space for links in $\mathcal{PE}(L)$.

Definition 3.1. The semi-simplicial space Sep_\bullet is defined as follows:

- $\text{Sep}_0 = \text{ESS}(L)$.
- $\text{Sep}_p \subseteq (\text{Sep}_0)^{p+1}$ consists of ordered $(p+1)$ tuples (ρ_i, Σ_i) such that $\rho_i = \rho_j$ for all $0 \leq i, j \leq p$ and $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$.
- Face maps $\partial_p^i: \text{Sep}_p \rightarrow \text{Sep}_{p-1}(L)$ for $0 \leq i \leq p$ are given by forgetting the i th entry in a tuple.

The semi-simplicial space PSep_\bullet is defined analogously, replacing $\text{ESS}(L)$ with $\text{PESS}(L)$.

There is a natural augmentation ε to the embedding space $\mathcal{E}(L)$ given by forgetting the sphere systems, and this is indeed an augmentation since it commutes with face maps. The fiber of this augmentation for a fixed $\rho \in \mathcal{E}(L)$ is itself a semi-simplicial space which levelwise is given by $\varepsilon_p^{-1}(\rho)$, and has face maps inherited from Sep_\bullet . We denote this semi-simplicial space by $\text{Sep}(\rho)_\bullet$.

Lemma 3.2. *For any two embeddings $\rho, \rho' \in \mathcal{E}(L)$, $\text{Sep}(\rho)_\bullet \cong \text{Sep}(\rho')_\bullet$.*

Proof. There exists an ambient isotopy Φ of \mathbb{R}^3 sending ρ to ρ' . Then Φ induces a bijective correspondence between separating systems for ρ and separating systems for ρ' . This correspondence clearly takes essential separating systems to essential separating systems, preserving disjointness and combinatorial type. Thus Φ induces an isomorphism $\Phi_\bullet: \text{Sep}(\rho)_\bullet \rightarrow \text{Sep}(\rho')_\bullet$. \square

Lemma 3.3. *The map $|\text{Sep}(\rho)_\bullet| \rightarrow \text{hofib}_\rho|\varepsilon_\bullet|$ is a weak equivalence.*

Proof. By isotopy extension, one can lift an open neighbourhood around any point ρ in $\mathcal{E}(L)$ to Sep_p , and it follows that the augmentation is a level-wise Serre fibration (compare with Section 4.4 in [Kup20]). Now Lemma 2.14 in [ERW19] shows that since the maps are levelwise quasifibrations then for each $\rho \in \mathcal{E}(L)$ the natural map

$$|\text{Sep}(\rho)_\bullet| = |\varepsilon_\bullet^{-1}(\rho)| \rightarrow \text{fib}_\rho|\varepsilon_\bullet|$$

is a weak equivalence. \square

Definition 3.4. We define some variations of $\text{Sep}(\rho)_\bullet$. Let $\text{Sep}(\rho)_\bullet^\delta$ be the semi-simplicial space where $\text{Sep}(\rho)_p^\delta$ is equal to $\text{Sep}(\rho)_p$ as a set, but is given the discrete topology. Fix a separating system $\hat{\Sigma}$ for ρ (not necessarily essential), and let $\text{Sep}(\rho, \hat{\Sigma})_p^\delta \subset \text{Sep}(\rho)_p^\delta$ be the subspace such that the tuples of separating systems $(\Sigma_0, \dots, \Sigma_p)$ for ρ are disjoint from $\hat{\Sigma}$. Let $\text{Sep}(\rho, \hat{\Sigma})_\bullet^\delta$ be the semi-simplicial space with these levels and face maps induced by those of $\text{Sep}(\rho)_\bullet^\delta$.

For the next proposition, we need the following lemma. We state it here, but reserve the proof until Section 4, when we have set up the appropriate notation.

Lemma 3.5. $|\text{Sep}(\rho, \hat{\Sigma})_\bullet^\delta|$ is contractible.

Proposition 3.6. If $|\text{Sep}(\rho)_\bullet^\delta|$ is k -connected, then so is $|\text{Sep}(\rho)_\bullet|$.

Proof. In this proof we use single bars for geometric realisation of a semi-simplicial set, and double bars for the iterated geometric realisation of a bi-semi-simplicial set.

Let ι be the map $|\text{Sep}(\rho)_\bullet^\delta| \rightarrow |\text{Sep}(\rho)_\bullet|$ induced by the identity map. We follow [GRW18]. Consider the bi-semi-simplicial space $D_{\bullet, \bullet}$ given by

$$D_{p,q} = \text{Sep}(\rho)_{(p+q+1)}$$

topologised as a subspace of $\text{Sep}(\rho)_p \times \text{Sep}(\rho)_q^\delta$, and with face maps inherited from $\text{Sep}(\rho)_\bullet$. That is, $D_{p,q}$ is the subspace of $\text{Sep}(\rho)_p \times \text{Sep}(\rho)_q^\delta$ such that all $(p+q+2)$ separating systems can be realised disjointly. Then there are two augmentation maps ϵ and δ :

$$\epsilon_p : |D_{p,\bullet}| \rightarrow \text{Sep}(\rho)_p \text{ and } \delta_q : |D_{\bullet,q}| \rightarrow \text{Sep}(\rho)_q^\delta$$

(i.e., set $D_{p,-1} = \text{Sep}(\rho)_p$ and $D_{-1,q} = \text{Sep}(\rho)_q^\delta$). Then by [GRW18, Lemma 5.8] the following diagram commutes up to homotopy.

$$\begin{array}{ccc} & \|\!|D_{\bullet,\bullet}\|\!| & \\ \swarrow |\epsilon_\bullet| & & \searrow |\delta_\bullet| \\ |\text{Sep}(\rho)_\bullet| & \longleftarrow & |\text{Sep}(\rho)_\bullet^\delta| \\ & \|\!|\iota_\bullet\|\!| & \end{array}$$

Now by [GRW18, Lemma 2.8], ϵ_p is a Serre microfibration, since $\text{Sep}(\rho)_p$ is Hausdorff, and for a fixed separating system $(\Sigma_0, \dots, \Sigma_p) \in \text{Sep}(\rho)_p$, the subspace $\{x \in \text{Sep}(\rho)_q | (\Sigma_0, \dots, \Sigma_p, x) \in \text{Sep}(\rho)_{p+q+1}\}$ is open in $\text{Sep}(\rho)_q$. Let $(\Sigma_0, \dots, \Sigma_p) \in \text{Sep}(\rho)_p$, and $\hat{\Sigma}$ be the image of the separating system $\Sigma_0 \sqcup \dots \sqcup \Sigma_p \in \mathbb{R}^3$. Then ϵ_p has fiber $\text{Sep}(\rho, \hat{\Sigma})_\bullet^\delta$, and the realisation of this space is contractible by Lemma 3.5.

It follows that $|\epsilon_\bullet|$ is a weak equivalence. Up to homotopy this map factors through $|\text{Sep}(\rho)_\bullet^\delta|$ (since $|\iota_\bullet| \circ |\delta_\bullet| \simeq |\epsilon_\bullet|$) and so we conclude that if $|\text{Sep}(\rho)_\bullet^\delta|$ is k -connected, then so is $|\text{Sep}(\rho)_\bullet|$. \square

Theorem 3.7. $|\text{Sep}(\rho)_\bullet^\delta|$ is contractible.

The proof of this theorem is the main focus of Section 4. Assuming this result, we can now prove Theorem A.

Remark 3.8. The spaces $\mathcal{E}(L)$, $\mathcal{PE}(L)$, Sep_\bullet , PSep_\bullet , and $\text{Sep}(\rho)_\bullet$ all have the homotopy type of CW-complexes by [Pal66, Mil59, Kur35]. Kuratowski [Kur35] showed if Y is an ANR and X is compact then the space of continuous maps from X to Y is an ANR. Milnor [Mil59] showed that ANR spaces are exactly those with the homotopy type of CW-complexes. Finally, Palais [Pal66] collected results on infinite dimensional manifolds, which included the space of smooth maps between manifolds. In particular they are ANRs, as are embedding spaces (since open subsets of ANRs are ANRs).

Theorem A. The map $|\epsilon_\bullet|: |\text{Sep}_\bullet| \rightarrow \mathcal{E}(L)$ is a homotopy equivalence.

Proof. We first show the map is a weak equivalence *i.e.*, for all k it induces isomorphisms

$$\pi_k(|\text{Sep}_\bullet|) \cong \pi_k(\mathcal{E}(L)).$$

By Lemma 3.3, $\text{hofib}_\rho|\epsilon_\bullet|$ is weakly equivalent to $|\text{Sep}(\rho)_\bullet|$, and combining Proposition 3.6 and Theorem 3.7 this is contractible. It follows that $\text{hofib}_\rho|\epsilon_\bullet|$ is weakly equivalent to a point and thus $|\epsilon_\bullet|$ is a weak equivalence. By Remark 3.8 the spaces both have the homotopy type of CW complexes and so $|\epsilon_\bullet|$ is a homotopy equivalence. \square

Recall from Definition 3.1 that PSep_\bullet is the semi-simplicial space constructed analogously to Sep_\bullet , but in which all link pieces are labeled. Then PSep_\bullet has an augmentation to $\mathcal{PE}(L)$ that forgets the separating systems, which we also denote by $\epsilon_\bullet: \text{PSep}_\bullet \rightarrow \mathcal{PE}(L)$. Consequently, there is a levelwise covering map $|\text{PSep}_\bullet| \rightarrow |\text{Sep}_\bullet|$ and a commutative pullback square

$$(1) \quad \begin{array}{ccc} |\text{PSep}_\bullet| & \longrightarrow & |\text{Sep}_\bullet| \\ \downarrow |\epsilon_\bullet| & & \downarrow |\epsilon_\bullet| \\ \mathcal{PE}(L) & \longrightarrow & \mathcal{E}(L) \end{array}$$

where the horizontal arrows are the covering maps induced by forgetting the labels on pieces. By Theorem A the right hand map is a homotopy equivalence.

Corollary 3.9. The map $|\epsilon_\bullet|: |\text{PSep}_\bullet| \rightarrow \mathcal{PE}(L)$ is a homotopy equivalence.

Proof. Since (1) is a pullback square, and the homotopy fibers of the right hand map are contractible it follows that the homotopy fibers of the left hand map are contractible and so it is a weak equivalence. By Remark 3.8 both spaces have the homotopy type of CW complexes, and so the statement of the corollary follows. \square

4. CONTRACTIBILITY OF $|\text{Sep}(\rho)_\bullet^\delta|$

The aim of this section is to prove Theorem 3.7. To do this, we introduce a larger contractible space, and show that the inclusion of $|\text{Sep}(\rho)_\bullet^\delta|$ into this space is a weak equivalence. We also prove Lemma 3.5. After writing the argument in this section we discovered that work of Mann and Nariman [MN20, Section 3] addresses a similar complex, by similar means. However, as our argument is considerably more detailed we felt it worth including in full.

Definition 4.1. Fix $\rho \in \mathcal{E}(L)$. The semi-simplicial space $\text{Sep}(\rho)_\bullet^\natural$ is defined as follows:

- The space $\text{Sep}(\rho)_0^\natural$ of zero-simplices is equal to $\text{Sep}(\rho)_0^\delta$, *i.e.*, it is given by the subspace

$$\{(\gamma, \Sigma) \in \text{ESS}(L) \text{ such that } \gamma = \rho\}$$

equipped with the discrete topology.

- The space Sep_p^\natural of p -simplices is the subspace of $(\text{Sep}(\rho)_0^\natural)^{p+1}$ consisting of ordered $(p+1)$ tuples (ρ, Σ_i) such that for all $0 \leq i < j \leq p$, either $\Sigma_i \cap \Sigma_j = \emptyset$ or the spheres of Σ_i and Σ_j intersect transversely.
- Face maps $\partial_p^i: \text{Sep}(\rho)_p \rightarrow \text{Sep}(\rho)_{p-1}$ for $0 \leq i \leq p$ are given by forgetting the i th entry in the tuple.

There is an inclusion $\iota_p: \text{Sep}(\rho)_p^\delta \hookrightarrow \text{Sep}(\rho)_p^\natural$, which induces an inclusion on geometric realisations

$$|\iota_\bullet|: |\text{Sep}(\rho)_\bullet^\delta| \hookrightarrow |\text{Sep}(\rho)_\bullet^\natural|.$$

In this section we will

- show $|\text{Sep}(\rho)_\bullet^\natural|$ is contractible, and
- show $|\iota_\bullet|$ is a homotopy equivalence.

Lemma 4.2. *For a finite set J let $\{\eta_j\}_{j \in J} \in \bigoplus_{j \in J} \underline{\text{Emb}}(S^2, \mathbb{R}^3 \setminus \rho)$ be a collection of embedded spheres in $\mathbb{R}^3 \setminus \rho$, with index set J . Then there exists a separating system $\Sigma \in \text{Sep}(\rho)_0^\natural$ with image transverse to every η_j .*

Proof. This follows from [Hir76, Theorem 2.1], which applied to our scenario shows that the subspace of separating systems transverse to a given embedded sphere η_j is residual in each connected component of $\text{Sep}(\rho)_0^\natural$. Since a countable intersection of residual subsets is residual, and J is finite and therefore countable, it follows that the subset of $\Sigma \in \text{Sep}(\rho)_0^\natural$ with image pairwise transverse to the collection $\{\eta_j\}_{j \in J}$ is not only non-empty but dense. \square

Remark 4.3. In fact, we can find a separating system for any isotopy class with a prescribed combinatorial type, since these correspond to different connected components of $\text{Sep}(\rho)_0^\natural$ for some fixed. This won't be relevant in our proof.

Recall from [ERW19] that points in $|X_\bullet|$ have the form $[x, \underline{t}]$, where for some $p \geq 0$, $x \in X_p$, $t \in \Delta^p$, and a quotient is taken which identifies faces in the appropriate way. Recall that

$$\Delta^p = \{\underline{t} = (t_0, \dots, t_p) \in \mathbb{R}^{p+1} \mid \sum_{i=0}^p t_i = 1\}$$

with the subspace topology. $|X_\bullet|$ is then topologised by taking the product topology on $\sqcup_{p \geq 0} X_p \times \Delta^p$ for each p , and then the quotient topology upon gluing faces.

Via the above, we can think of a point in $|\text{Sep}(\rho)_\bullet^\dagger|$ as a tuple $(\Sigma_0, \dots, \Sigma_p) \in \text{Sep}(\rho)_p^\dagger$, where each Σ_i has a *weight* t_i between 0 and 1 associated to it, and $\sum_{i=0}^p t_i = 1$. Passing to a face of the simplex corresponds to one of the t_i becoming zero, at which point the associated Σ_i is deleted from the tuple – that is, we pass to $\text{Sep}(\rho)_{p-1}^\dagger$. The weights can be thought of as ‘opacity’ of the separating systems. This discussion also holds for $|\text{Sep}(\rho)_\bullet^\delta|$, replacing $\text{Sep}(\rho)_p^\dagger$ with $\text{Sep}(\rho)_p^\delta$.

Let Y be a triangulated space. Then for any simplicial map $F : Y \rightarrow |\text{Sep}(\rho)_\bullet^\dagger|$ we define the set

$$\text{im}(F)_0 = \{[\Sigma, 1] \in |\text{Sep}(\rho)_\bullet^\dagger| \text{ s.t. } \exists p \in Y^{(0)} \text{ with } F(p) = [\Sigma, 1]\}$$

i.e., $\Sigma \in \text{im}(F)_0$ has weight 1, and is the image of a 0-simplex in Y . Since the image of a p simplex is determined by its vertices, $\text{im}(F)_0$ is exactly the set of separating systems in the image of the map F .

Proposition 4.4. $|\text{Sep}(\rho)_\bullet^\dagger|$ is contractible.

Proof. Consider a simplicial map $f : S^k \rightarrow |\text{Sep}(\rho)_\bullet^\dagger|$ from a triangulation of the sphere S^k , $k \geq 0$. Since S^k is finite dimensional and compact, it follows that $\text{im}(f)_0 = \{\Sigma_i\}_{i \in I}$ for I a finite set indexing $(S^k)^{(0)}$. This is a point in $\sqcup_{j \in J} \text{Emb}(S^2, \mathbb{R}^3 \setminus \rho)$ for some finite set J , *i.e.*, the image of a finite number of embedded spheres (individually embedded, not as a set) in $\mathbb{R}^3 \setminus \rho$. By Lemma 4.2 there exists a separating system $\Sigma \in \text{Sep}(\rho)_0^\dagger$ with image transverse to each of these spheres. Coning off the triangulation of S^k to a triangulation of D^{k+1} with cone point c , the map f extends to a map $F : D^{k+1} \rightarrow |\text{Sep}(\rho)_\bullet^\dagger|$ such that $F(c) = \Sigma$ with weight 1. Thus $|\text{Sep}(\rho)_\bullet^\dagger|$ is contractible as required. \square

We now digress slightly and prove Lemma 3.5. Recall $\text{Sep}(\rho, \hat{\Sigma})_p^\delta$ is the subspace of $\text{Sep}(\rho)_p^\delta$ where the $p+1$ mutually disjoint separating systems also do not intersect with a given separating system (not necessarily essential) $\hat{\Sigma}$.

Proof of Lemma 3.5. Consider a simplicial map f from a triangulation of S^k into $|\text{Sep}(\rho, \hat{\Sigma})_\bullet^\delta|$. Let $(-1, 1) \times \hat{\Sigma} \subset \mathbb{R}^3 \setminus \rho$ be a tubular neighbourhood of $\hat{\Sigma}$, such that $\hat{\Sigma} = (0 \times \hat{\Sigma}) \subset \mathbb{R}^3$. Then there exists an $\epsilon > 0$ such that $(\epsilon \times \hat{\Sigma})$ is disjoint from all separating systems in $\text{im}(f)_0$, since being disjoint from a finite number of separating systems is an open condition. Let $\Sigma' \subset (\epsilon \times \hat{\Sigma})$ be a sub-collection of the spheres of $(\epsilon \times \hat{\Sigma})$ such that Σ' is an essential separating system for ρ . Then, as in the proof of Proposition 4.4, Σ' acts as a cone point for $\text{im}(f)$, and thus f extends

to a map $F : c * S^k \cong D^{k+1} \rightarrow |\text{Sep}(\rho, \hat{\Sigma})_\bullet^\delta|$ such that $F(c) = \Sigma'$ with weight 1. Therefore $|\text{Sep}(\rho, \hat{\Sigma})_\bullet^\delta| \simeq *$. \square

It remains to show that $|\iota_\bullet|$ is a homotopy equivalence. The proof of this follows from the following lengthy discussion. Consider a representative f of a class in $\pi_k(|\text{Sep}(\rho)_\bullet^\delta|)$. Then, by Proposition 4.4, this extends to a map F from D^{k+1} to $|\text{Sep}(\rho)_\bullet^{\text{fl}}|$ as shown below.

$$(2) \quad \begin{array}{ccc} f : S^k & \longrightarrow & |\text{Sep}(\rho)_\bullet^\delta| \\ \partial \downarrow & & \downarrow |\iota_\bullet| \\ F : D^{k+1} & \longrightarrow & |\text{Sep}(\rho)_\bullet^{\text{fl}}| \end{array}$$

In the following, let a *pair of maps* be a pair (F, f) as in the above diagram. Our aim is to start with a generic pair of maps (F, f) and produce a pair (F', f') such that $f \simeq f'$ and $\text{im}(F') \subset |\text{Sep}(\rho)_\bullet^\delta|$. Moreover, because both right hand sides have the discrete topology, we can assume f and F are both simplicial maps, from triangulations of the sphere and disk respectively.

Let (F, f) be a pair of maps, p and q in $(D^{k+1})^{(0)}$, and Σ_p and Σ_q in $\text{im}(F)_0$ correspond to $F(p)$ and $F(q)$ respectively. If $\Sigma_p \cap \Sigma_q \neq \emptyset$ in \mathbb{R}^3 then the intersection is a union of embedded circles in \mathbb{R}^3 . We distinguish two types of intersections.

- (i) *Ghost intersections*: Intersections between spheres in Σ_p and Σ_q where $p \notin \text{lk}_{D^{k+1}}(q)$ ($\text{lk}_X(v)$ denotes the link of vertex v in simplicial complex X).
- (ii) *Real intersections*: Intersections between spheres in Σ_p and Σ_q where $p \in \text{lk}_{D^{k+1}}(q)$.

By the definition of $|\text{Sep}(\rho)_\bullet^{\text{fl}}|$, all real intersections in $\text{im}(F)_0$ are transverse. It is not the case that the ghost intersections are transverse, but the next lemma asserts we can always replace our pair with one for which this is true.

Definition 4.5. Let (F, f) be a pair of maps as in Equation (2). For $\Sigma \in \text{im}(F)_0$, let $tg_F(\Sigma)$ be 0 if Σ intersects all Σ' in $\text{im}(F)_0 \setminus \Sigma$ transversely, and 1 otherwise. Define the *tangential complexity* of F to be,

$$tc^F = \sum_{\Sigma \in \text{im}(F)_0} tg_F(\Sigma).$$

Let the tangential complexity of f be analogously described, *i.e.*,

$$tc^f = \sum_{\Sigma \in \text{im}(f)_0} tc_f(\Sigma).$$

In words, tc^F counts the number of separating systems in the image of F which have some non-transverse intersections with other separating systems in the image of F . These are necessarily ghost intersections. Note that since $\text{im}(f)_0 \subseteq \text{im}(F)_0$, it follows that $tc^F \geq tc^f$.

Proposition 4.6. *Given a pair of maps (F, f) as in Equation (2), there exists a pair of maps (G, g) with g homotopic to f , such that $tc^g = tc^G = 0$.*

Proof. We again use [Hir76, Theorem 2.1] to replace separating systems with tangential intersections with separating systems that intersect transversely.

Step 1: If $tc^f \geq 1$ we show that we can homotope f to a map f_1 and replace F with F_1 such that (F_1, f_1) are a pair, and $tc^{f_1} < tc^f$.

Let $\Sigma_p \in \text{im}(f)_0$ such that $f(p) = \Sigma_p$ and $tc_f(\Sigma_p) = 1$, i.e., Σ_p has a non-transverse intersection with some $\Sigma_q \in \text{im}(f)_0$ corresponding to $q \in (D^{k+1})^{(0)}$. Then since $f : S^k \rightarrow |\text{Sep}(\rho)_\bullet^\delta|$, Σ_p is disjoint from all $\Sigma_r \in \text{im}(f)_0$ such that r is a vertex in $\text{lk}_{S^k}(p)$. Being disjoint is an open condition in $\text{ESS}(L)$, so there is an open neighbourhood of Σ_p in $\text{ESS}(L)$ of separating systems which remain disjoint from the Σ_r and from ρ . By residuality of transverse embeddings, we can find a Σ'_p in this neighbourhood transverse to all separating systems in $\text{im}(F)_0$.

Now we create a new triangulation of D^{k+1} by adding a vertex p' and the simplices of $p' * \text{star}_{S^k}(p)$. Let F_1 be a simplicial map from this new triangulation of D^{k+1} to $|\text{Sep}(\rho)_\bullet^\delta|$ defined as follows. On 0-simplices let F_1 agree with F away from p' , and $F_1(p') = \Sigma'_p$. Since Σ'_p is transverse to $\text{im}(F)_0$ and disjoint from Σ_r when $r \in \text{lk}_{S^k}(p)$, we can extend this map over the simplices of the new triangulation to get a pair of maps (F_1, f_1) . (The map $f_1 : S^k \rightarrow |\text{Sep}(\rho)_\bullet^\delta|$ is given by F_1 restricted to ∂D^{k+1} .) It remains to show that f_1 is homotopic to f . We achieve this via the homotopy $h : ([0, 1] \times S^k) \rightarrow |\text{Sep}(\rho)_\bullet^\delta|$ with $h_0 = f$, $h_1 = f_1$ and such that as $t \in [0, 1]$ varies from 0 to 1 the weight from the separating system Σ_p shifts to Σ'_p . Formally $h_t = h_0$ away from $\text{star}_{S^k}(p)$, $h_t(p) = [(\Sigma_p, \Sigma'_p), (1-t, t)]$ and, for a k -simplex containing p in $\text{star}_{S^k}(p)$, h_t linearly shifts the weight t_i attributed to Σ_p when $t = 0$, to be attributed to Σ'_p when $t = 1$.

A schematic of this homotopy is depicted in the Figure 4, when $k = 1$.

The outcome is a pair of maps (F_1, f_1) such that $tc^{f_1} \leq tc^f - 1$, as required (since we have replaced Σ_p satisfying $tc_f(\Sigma_p) = 1$ with $\Sigma'_p = \Sigma_{p'}$ satisfying $tc_{f_1}(\Sigma_{p'}) = 0$). Furthermore, the point p lies in the interior of D^{k+1} , in the new triangulation.

Iterating Step 1 until no longer possible, we produce a pair (F_n, f_n) with f_n homotopic to f and $tc^{f_n} = 0$.

Step 2: Start with the pair (F_n, f_n) created in Step 1. We replace (F_n, f_n) with (F_{n+1}, f_n) such that $tc^{F_{n+1}} < tc^{F_n}$.

Consider $p \in (D^{k+1} \setminus \partial D^{k+1})^0$ such that $tc_{F_n}(\Sigma_p) = 1$. By residuality of transverse embeddings there exists a separating system Σ'_p transverse to all spheres in separating systems of $\text{im}(F_n)_0 \setminus \Sigma_p$. We modify the map F_n to a map F_{n+1} , by setting $F_{n+1}(p) = \Sigma'_p$ and for all simplices σ in $\text{star}_{D^{k+1}}(p)$ replacing the occurrence of Σ_p in $F_n(\sigma)$ with Σ'_p of the same weight. Then $F_{n+1} = f_n$ on $S^k = \partial D^{k+1}$, and $tc^{F_{n+1}} \leq tc^{F_n} - 1$ as required.

After a finite number of iterations of Step 2, we produce a pair $(F_m, f_n) = (G, g)$ such that $tc^g = tc^G = 0$, as required. \square

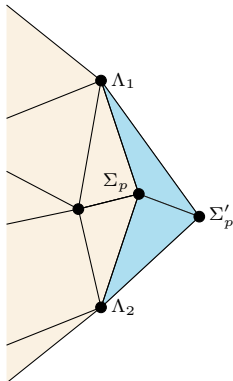


FIGURE 4. Replacing (F, f) with (F_1, f_1) when $k = 1$. The image of F is shown in tan, and the image of F_1 in tan and blue. The link of Σ_p in S^1 is Λ_1 and Λ_2 . The homotopy from f to f_1 replaces the star of Σ_p in the boundary of $\text{im}(F)$ with the star of Σ'_p in the boundary of $\text{im}(F_1)$. It linearly shifts the weight from Σ_p to Σ'_p , which pushes the arc $\Lambda_1 - \Sigma_p - \Lambda_2$ across the two rightmost blue triangles to $\Lambda_1 - \Sigma'_p - \Lambda_2$.

We now consider a pair of maps (G, g) with as produced by the previous lemma, *i.e.*, such that all separating systems in $\text{im}(G)$ intersect transversely, regardless of whether the intersections are ghost or real.

Let S_p and S_q be embedded spheres in \mathbb{R}^3 , which intersect in a disjoint union of circles. Let D be a disk in S_p with boundary one of these circles, such that the interior of D contains no other intersection circle with S_q . We recall what we mean by *surgery* on a S_q with respect to the disk $D \subset S_p$. Let $(-1, 1) \times S_q$ and $(-1, 1) \times S_p$ be tubular neighbourhoods of S_p and S_q , such that $-\epsilon \times S_p$ lies in the interior of S_p when $\epsilon > 0$ and likewise for S_q , and such that $\epsilon \times S_p$ intersects $\delta \times S_q$ transversely for all $\epsilon, \delta > 0$, with intersection pattern the same as that of S_p and S_q . These neighbourhoods can always be found with such properties, since transversality is an open condition in $\underline{\text{Emb}}(S^2, \mathbb{R}^3)$. Then there exists a tubular neighbourhood $(-1, 1) \times D$ of D satisfying that $(t \times D) \subset (t \times S_p)$ and $\partial(t \times D) \subset (-\delta \times S_q)$ for all $t \in (-1, 1)$, and some fixed $\delta \in (0, 1)$.

Since $\partial(t \times D) \subset (t \times S_p) \cap (-\delta \times S_q)$, for every t we can decompose $(-\delta \times S_q)$ into two disks $D_q^+(t)$ and $D_q^-(t)$, such that $(-\delta \times S_q) = D_q^+(t) \cup_{\partial(t \times D)} D_q^-(t)$ and a neighbourhood of $\partial D_q^+(t)$ (respectively $\partial D_q^-(t)$) lies in the interior (respectively exterior) of the ball bounded by $t \times S_p$. Then by *surgery on S_q with respect to D* we mean replacing S_q with the two spheres S_q^- and S_q^+ where

$$S_q^- = D_q^-(-\epsilon) \cup_{\partial(-\epsilon \times D)} (-\epsilon \times D) \text{ and } S_q^+ = D_q^+(\epsilon) \cup_{\partial(\epsilon \times D)} (\epsilon \times D).$$

These are manifolds with corners and therefore not elements of $\underline{\text{Emb}}(S^2, \mathbb{R}^3)$, but we can smooth the corners in a neighbourhood arbitrarily close to the corners, so we do this and by abuse of notation call the resulting smoothly embedded spheres S_q^+ and S_q^- . Note that by construction, S_q^+ and S_q^- are disjoint from S_q , and lie in the

interior of the ball bounded by S^q . A schematic of the surgery is shown in Figure 5.

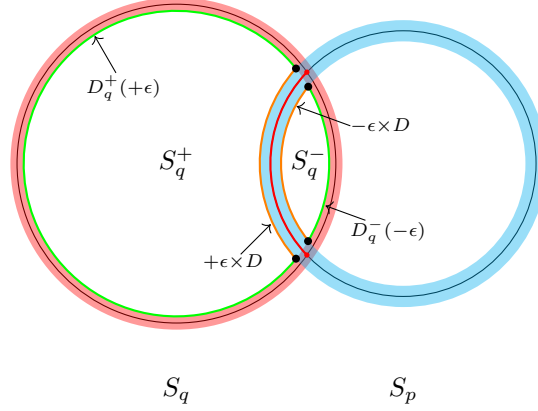


FIGURE 5. A schematic picture of surgery on S_q with respect to D . S_p , S_q are represented by the black circles on the left and right respectively, and D is the arc in red on S_p . The δ -neighbourhood of S_q is light red while the ϵ -neighbourhood of S_p is shown in light blue. The surgered sphere S_q^+ is the union of the green and orange arc in the interior of S_q , while S_q^- is the union of the orange and green arc in the interior of S_p .

Definition 4.7. Given a pair of maps (G, g) , with $tc^G = 0$, consider the image of all separating systems in $\text{im}(G)_0$. For each intersection circle in this image, the two intersecting spheres are divided by this circle into two disks. Let $\text{Disk}(G)$ be the set of all disks arising in this way in $\text{im}(G)_0$. We will be choosing a minimal disk in this set with respect to area, where area is calculated using the Euclidean metric on \mathbb{R}^3 .

Proof of Theorem 3.7. We show that $\iota_p : \text{Sep}(\rho)_p^\delta \hookrightarrow \text{Sep}(\rho)_p^{\text{th}}$ is a homotopy equivalence, by showing the relative homotopy groups vanish. We start with a pair of maps (F, f) , i.e.,

$$\begin{array}{ccc} f : S^k & \longrightarrow & |\text{Sep}(\rho)_\bullet^\delta| \\ \partial \downarrow & & \downarrow |\iota_\bullet| \\ F : D^{k+1} & \longrightarrow & |\text{Sep}(\rho)_\bullet^{\text{th}}| \end{array}$$

By Proposition 4.6 we can assume that the starting pair (F, f) satisfy the property that all separating systems in $\text{im}(F)$ intersect transversely. Let $\text{In}(F)$ be the total number of intersection circles between separating system of $\text{im}(F)_0$, and $\text{In}(f) \leq \text{In}(F)$ be the count of intersection circles where at least one of the intersecting spheres lies in some $\Sigma \in \text{im}(f)_0$. We will produce a pair of maps (F', f') such that $f \simeq f'$ and $\text{im}(F') \subset |\text{Sep}(\rho)_\bullet^\delta|$, i.e., $\text{In}(F')=0$.

Consider the sequence $\text{Disk}(F)$ from Definition 4.7, and let $D_{\min}(F) \in \text{Disk}(F)$ be minimal with respect to area. Then $D_{\min}(F)$ lies in some sphere $S_p \in \Sigma_p$ for $\Sigma_p \in \text{im}(F)_0$ the image of $p \in (D^{k+1})^{(0)}$. Moreover, the circle boundary of $D_{\min}(F)$ is an intersection circle of S_p with some $S_q \in \Sigma_q$ for $\Sigma_q \in \text{im}(F)_0$ the image of some $q \in (D^{k+1})^{(0)}$.

For $j = p$ or q , we choose the tubular neighbourhoods $(-1, 1) \times \Sigma_j$ such that for all t , $(t \times \Sigma_j)$ is an essential separating system for ρ , and has the same intersection pattern with $\text{im}(F)_0 \setminus \Sigma_j$ as S_j does, preserving transversality of intersections. This is possible since disjointness and transversality are open conditions in $\underline{\text{Emb}}(S^2, \mathbb{R}^3)$.

We surger S_q with respect to $D_{\min}(F)$, as detailed in the discussion preceding this proof, using the tubular neighbourhoods $(-1, 1) \times S_p$ and $(-1, 1) \times S_q$ given by restricting the tubular neighbourhoods of Σ_p and Σ_q to the spheres S_p and S_q . The output of the surgery is two spheres S_q^+ and S_q^- , which by construction are disjoint from ρ and lie in the interior of the ball bounded by S_q (recall they arise as disks in $(-\delta \times S_q)$).

Since Σ_q is an essential separating system, there is at most one link component in each connected component of $\mathbb{R}^3 \setminus \Sigma_q$. Suppose there is a link piece in the component with boundary S_q . Then one of S_q^+ and S_q^- bounds a ball containing this link piece – call this S'_q , and let Σ'_q be the separating system $(-\delta \times (\Sigma_q \setminus S_q)) \cup S'_q$. If there is no link piece in the component of $\mathbb{R}^3 \setminus \Sigma_q$ with boundary S_q , then let Σ'_q be the separating system $(-\delta \times (\Sigma_q \setminus S_q))$.

Similar to the proof of Proposition 4.6 we now have two cases, depending on whether q is in the boundary or interior of D^{k+1} . In the first case we homotope f and replace F , changing the triangulation of D^{k+1} in the process. In the second, we replace F with a map that sends q to a different vertex. Unlike the proof of Proposition 4.6, the order in which we do these relies on reevaluating D_{\min} after every step, so we cannot first fix the boundary and then the interior of D^{k+1} . (Note that this encompasses the case in which S_q^+ or S_q^- may be parallel to another sphere of Σ_q , so by construction Σ'_q does not contain parallel spheres).

Case 1: $q \in \partial D^{k+1}$. Proceed as in the proof of Proposition 4.6. Create a new triangulation of D^{k+1} by adding a vertex q' and the simplices of $q' * \text{star}_{S^k}(q)$. Let F_1 map from this new triangulation of D^{k+1} to $|\text{Sep}(\rho)_\bullet^\delta|$. On 0 simplices define F_1 to agree with F away from q' , and $F_1(q') = \Sigma'_q$. Since Σ'_q is disjoint from Σ_r when $r \in \text{star}_{S^k}(q)$, we can extend this map over the simplices of the new triangulation to get a pair of maps (F_1, f_1) . (The map $f_1 : S^k \rightarrow |\text{Sep}(\rho)_\bullet^\delta|$ is given by F_1 restricted to ∂D^{k+1} .) It remains to show that f_1 is homotopic to f . We achieve this via the homotopy $h : ([0, 1] \times S^k) \rightarrow |\text{Sep}(\rho)_\bullet|$ with $h_0 = f$, $h_1 = f_1$ and such that as t varies from 0 to 1 the weight from the separating system Σ_q shifts to Σ'_q in $\text{im}(h_t)$.

Case 2: $q \in D^{k+1} \setminus \partial D^{k+1}$. Proceed as in the proof of Proposition 4.6. Modify the map F to a map F_1 , by setting $F_1(q) = \Sigma'_q$ and for all simplices σ in $\text{star}_{D^{k+1}}(q)$ replacing the occurrence of Σ_q in $F(\sigma)$ with Σ'_q of the same weight. Then $F_1 = f$ on ∂D^{k+1} – rename f to f_1 .

In either case, we have replaced the pair (F, f) with a pair (F_1, f_1) such that either:

Case 1: $\text{In}(f_1) < \text{In}(f)$. Note that it is not necessarily the case that $\text{In}(F_1) \leq \text{In}(F)$, since we added an extra vertex.

Case 2: $\text{In}(F_1) < \text{In}(F)$. The reason we created no new intersections when doing our surgery was due to our choice of minimal area disk.

We now iterate the process with the pair (F_1, f_1) , considering a minimal area disk in the sequence $\text{Disk}(F_1)$. At each stage we either reduce the total number of intersections, or we possibly increase it, but reduce the number of total intersections involving separating systems in the image of S^k . Therefore, after a finite number of iterations we produce a pair of maps (F', f') satisfying $\text{In}(F') = 0$ and we are done.

Note that we only need to remove real intersections, but by considering both ghost and real intersections we ensure that no new intersections are created when doing the surgery (due to minimality of the area of the surgery disk). \square

Thus we have finished the proof of Theorem A.

5. HOMOTOPY GROUPS OF $\mathcal{E}(L)$

5.1. Implications of Theorem A. In this section we introduce a levelwise ‘round’ subspace of Sep , and use the previous sections to obtain nice representatives for classes in the homotopy groups of $\mathcal{E}(L)$.

Let $\underline{\text{REmb}}(\sqcup_{i=1}^k S^2, \mathbb{R}^3)$ be the space of unparametrised embeddings of $\sqcup_{i=1}^k S^2$ into \mathbb{R}^3 , for which the image of each sphere is *round*, i.e., it bounds a Euclidean ball in \mathbb{R}^3 . The following result is Lemma 4.2 of [BH13], but we include a proof here for the reader’s convenience.

Lemma 5.1. *The map $\iota: \underline{\text{REmb}}(\sqcup_{i=1}^k S^2, \mathbb{R}^3) \hookrightarrow \underline{\text{Emb}}(\sqcup_{i=1}^k S^2, \mathbb{R}^3)$ is a homotopy equivalence.*

Proof. We will prove the corresponding statement for parametrised embeddings, i.e., that the inclusion $\text{REmb}(\sqcup_{i=1}^k S^2, \mathbb{R}^3) \rightarrow \text{Emb}(\sqcup_{i=1}^k S^2, \mathbb{R}^3)$ is a homotopy equivalence. The statement will then follow from Smale’s theorem that $\text{Diff}(S^2) \simeq O(3)$. To simplify notation, we will abbreviate $\text{Emb}(\sqcup_{i=1}^k S^2, \mathbb{R}^3)$ to Emb and $\text{REmb}(\sqcup_{i=1}^k S^2, \mathbb{R}^3)$ to REmb . We will show that the relative homotopy groups $\pi_k(\text{Emb}, \text{REmb})$ vanish for all $k \geq 1$. Since both of these spaces have the homotopy type of CW complexes, the lemma will follow.

Consider a relative homotopy class $\xi: (D^k, \partial D^k) \rightarrow (\text{Emb}, \text{REmb})$. Since D^k is contractible, we may assume the spheres are labeled over all of D^k . We may also assume the spheres are round on a collar $\partial D^k \times [1, 1 - t_0]$ of ∂D^k . The configuration of spheres have the same combinatorial type over the whole disk D^k (see Definition 2.5). In particular there is a well-defined notion of nesting of spheres, as well as which spheres are outermost. We will homotope ξ into REmb first by making the outermost spheres round, then the second outermost spheres, etc. Let $S_1^2, \dots, S_{n_1}^2$ be the collection of outermost spheres. Use isotopy extension to get a family of embeddings $\hat{\xi}: D^k \rightarrow \text{Emb}(\sqcup_{i=1}^{n_1} B^3, \mathbb{R}^3)$. Restricting the map $\hat{\xi}$ to $\partial D^k \times [1, 1 - t_0]$ we get an $\partial D^k \times [1, 1 - t_0]$ -family of embedded balls, whose

image is round. By the proof of the Smale conjecture [Hat83] that $\text{Diff}(B^3 \text{rel} \partial B^3)$ is contractible, we can homotope $\hat{\xi}$ so that on $\partial D^k \times [1, 1 - t_0]$ it consists of round embeddings of $\sqcup_{i=1}^{n_1} B^3$.

Recall now that $\text{Diff}(\mathbb{R}^3)$ is homotopy equivalent to $O(3)$. Let $D_{1-t_0}^k$ denote the subdisk of radius $1 - t_0$. Since the interior of each ball B_i^3 is homeomorphic to \mathbb{R}^3 , we can further homotope $\hat{\xi}$ so that at each point $p \in D_{1-t_0}^k$, $\hat{\xi}(p)|_{B_i^3}$ is a rescaled isometry which agrees with $D\hat{\xi}(p)|_{B_i^3}(0)$ on a neighborhood of $0 \in B_i^3$. By compactness of $D_{1-t_0}^k$, there exists $r_0 > 0$ such that for each $p \in D^k$, $\hat{\xi}(p)$ is a rescaled isometry on the ball of radius r_0 about $0 \in B_i^3$.

We will define a homotopy $h_s: (D^k, \partial D^k) \times [0, 1] \rightarrow (\text{Emb}, \text{REmb})$ from $\hat{\xi}$ to a family of round embeddings, which is fixed on the boundary. Since we may rescale round embeddings and preserve roundness, the idea is to use the collar $D_{1-t_0}^k$ to partially shrink down each outermost ball and then restrict $\hat{\xi}$ to the rescaled balls. At the start of the homotopy, we do not shrink the balls at all; at the end of the homotopy, by the time we have reached $S^{k-1} \times \{1 - t_0\}$ the image of each outermost ball has been rescaled into the ball of radius r_0 , hence after restricting $\hat{\xi}$ it remains round over the interior of disk the whole disk. A schematic for this homotopy is shown in Figure 6.

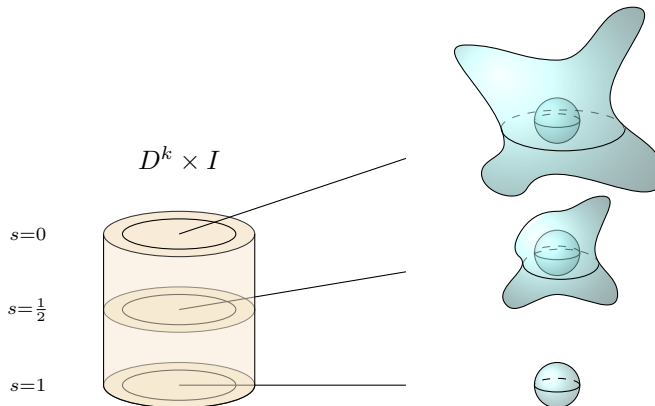


FIGURE 6. A schematic for the homotopy h_s . The cylinder $D^k \times I$ is shown at the left, with the vertical direction corresponding to the variable s . The collar on which the shrinking occurs is shown as the annulus between the two concentric circles at each level. On the right is shown the image of a ball as well as the inner round ball of radius r_0 . As s increases, $\hat{\xi}$ is restricted to a rescaled image of the outer ball. When $s = 1$ the rescaled radius is r_0 so the image is round.

In order to describe the homotopy we will need some auxillary functions. Let $\alpha(s) = (1 - s) + sr_0$. Then α linearly interpolates between 1 and r_0 as s goes from 0 to 1. Now define

$$\beta_s(t) = \begin{cases} \frac{1-\alpha(s)}{t_0}(t-1) + 1, & 1 - t_0 \leq t \leq 1 \\ \alpha(s), & 0 \leq t \leq 1 - t_0. \end{cases}$$

The function $\beta_s(t)$ describes the scaling factor as we move radially across the collar $S^{k-1} \times [1 - t_0, 1]$. For fixed s , the scaling factor is 1 at time $t = 1$, and $\alpha(s)$ by the time we reach $t = 1 - t_0$. In particular, $\beta_0(1 - t_0) = 1$ and $\beta_1(1 - t_0) = r_0$. To obtain the homotopy h_s we think of a point in D^k as a pair $(x, t) \in S^{k-1} \times [0, 1]$ where $S^{k-1} \times \{0\}$ is identified to a point. In these coordinates, we define a homotopy h_s by precomposing $\hat{\xi}(x, t)$ with the map that scales each B_i^3 by $\beta_s(t)$. Since β_s only depends on the radius t , this is well-defined on D^k . When $s = 0$, this is the identity composed with $\hat{\xi}$, while when $s = 1$, B_i^3 has been rescaled so that it lies in the ball of radius r_0 , hence is a round embedding. Moreover, h_s agrees with $\hat{\xi}$ on ∂D^k for all s . We now proceed to the next outermost spheres, doing the same procedure until all the spheres are round on all of D^k . \square

Remark 5.2. If $\sqcup_{i=1}^k B^3$ is a disjoint union of 3-balls we define in an analogous way $\underline{\text{REmb}}(\sqcup_{i=1}^k B^3, \mathbb{R}^3)$ to be the subspace of $\underline{\text{Emb}}(\sqcup_{i=1}^k B^3, \mathbb{R}^3)$ where the image of each ball is round. Then we also have an inclusion

$$\underline{\text{REmb}}(\sqcup_{i=1}^k B^3, \mathbb{R}^3) \rightarrow \underline{\text{Emb}}(\sqcup_{i=1}^k B^3, \mathbb{R}^3).$$

The same proof as in Lemma 5.1 shows that this inclusion is a homotopy equivalence. Moreover, as we argue in the proof, for each fixed relative class $(D^k, \partial D^k)$, we may assume the spheres/balls are labeled. Thus, the same argument also works for the ‘pure’ embedding space where all spheres/balls are labeled.

Definition 5.3. We define the semi-simplicial space RSep_\bullet by setting RSep_p to be the levelwise subspace of Sep_p consisting of embeddings such that the image of each embedded sphere in \mathbb{R}^3 is *round*, that is it bounds a Euclidean ball in \mathbb{R}^3 .

Proposition 5.4. *The inclusion map $\iota_\bullet : \text{RSep}_\bullet \hookrightarrow \text{Sep}_\bullet$ induced by the levelwise inclusion of embedding spaces, induces a homotopy equivalence on geometric realisations.*

Proof. From [ERW19, Lemma 2.4], it is enough to show that

$$\iota_p : \text{RSep}_p \hookrightarrow \text{Sep}_p$$

is a homotopy equivalence. A point in Sep_p may be regarded as an embedding $\rho \in \mathcal{E}(L)$ together with a $(p + 1)$ -tuple $(\Sigma_0, \dots, \Sigma_p)$, where each Σ_i is a separating system for ρ and all the Σ_i can be embedded disjointly. Consider the map

$$\text{Sep}_p \rightarrow \bigsqcup_{k=1}^{\infty} \underline{\text{Emb}}(\sqcup_{i=1}^k S^2, \mathbb{R}^3)$$

which sends each point in Sep_p to the union $\sqcup_{i=0}^p \Sigma_i$ of corresponding separating systems. This is a fibration over each connected component, hence it has the homotopy lifting property. We lift the deformation retraction from Lemma 5.1 to get a homotopy equivalence $\text{RSep}_p \simeq \text{Sep}_p$. \square

There is an augmentation map $\varepsilon_\bullet : \text{RSep}_\bullet \rightarrow \mathcal{E}(L)$ inherited from that of Sep_\bullet .

Theorem 5.5. *The realisation of the augmentation map*

$$|\varepsilon_\bullet| : |\text{RSep}_\bullet| \rightarrow |\mathcal{E}(L)|$$

is a homotopy equivalence.

Proof. From Proposition 5.4 we have that the map induced by inclusion

$$|\mathrm{RSep}_\bullet| \xrightarrow{|\iota_\bullet|} |\mathrm{Sep}_\bullet|$$

is a weak equivalence, and from Theorem A, the realisation of the augmentation map

$$|\mathrm{Sep}_\bullet| \xrightarrow{|\varepsilon_\bullet|} \mathcal{E}(L)$$

is also weak equivalence. Since all the spaces involved have the homotopy type of CW-complexes, we conclude that the composition of these two maps is a homotopy equivalence. \square

Definition 5.6. Suppose there exists a map $f : S^k \rightarrow \mathcal{E}(L)$ where S^k is triangulated such that:

- (i) on the image of each simplex there exists a separating system which varies by isotopy as the embedded link does, and,
- (ii) on shared faces of simplices all separating systems exist simultaneously and disjointly.

Then we will say the image of the triangulated sphere S^k under such a map f exhibits a *compatible separating triangulation* in $\mathcal{E}(L)$. If in addition every separating sphere is round, then we say the map f exhibits a *compatible round separating triangulation* in $\mathcal{E}(L)$.

Corollary 5.7. *Each homotopy class in $\pi_k(\mathcal{E}(L))$ has a representative $f : S^k \rightarrow \mathcal{E}(L)$ such that f exhibits a compatible round separating triangulation.*

Proof. From Theorem 5.5, $\pi_k(\mathcal{E}(L)) \simeq \pi_k(|\mathrm{RSep}_\bullet|)$, and so every map $f : S^k \rightarrow \mathcal{E}(L)$ lifts to a map $\hat{f} : S^k \rightarrow |\mathrm{RSep}_\bullet|$ and we consider the image $\hat{f}(S^k)$ of this map. Each point in the image is a point in $|\mathrm{RSep}_\bullet|$ and, as discussed in Section 4, the data associated to this point is a link, and a collection of disjoint separating systems $\Sigma_0, \dots, \Sigma_p$ for this link, such that each separating system has a weight (corresponding to the barycentric coordinates of the simplex direction of the geometric realisation). For each point $x \in S^k$, we choose a separating system $\Sigma(x)$ that has non zero weight at $\hat{f}(x)$. Then since having non-zero weight is an open condition, there exists an open set $U_x \subset S^k$ containing x for which Σ_x remains a separating system with non zero weight in $\hat{f}(U_x)$. The U_x give an open cover of S^k and we choose a finite subcover $\bigcup_{\alpha \in \Lambda} U_\alpha$. If $U_\alpha \cap U_\beta \neq \emptyset$ for $\alpha, \beta \in \Lambda$, it follows that the separating systems Σ_α and Σ_β both have non-zero weight on the intersection, which implies that the separating systems can be realised disjointly. We now triangulate S^k such that each simplex lies fully in some U_α , and this completes the proof. \square

5.2. Forgetting embedded balls. In this section we relate the homotopy type of $\mathcal{E}(L)$ to that of the space of embeddings of a link L together with a disjoint collection of points or 3-balls. We will use the results of this section to obtain our description of the fundamental group of $\mathcal{E}(L)$ in Section 6. However, we separate these results as they may be of independent interest.

Given a manifold M let $\text{Conf}^k(M)$ denote the configuration space of k *unordered* points in M . The configuration space of k *ordered* (or labeled) points will be denoted $\text{PConf}^k(M)$. Recall that $\text{PConf}^k(M)$ is a regular covering space of $\text{Conf}^k(M)$ with deck group equal to the symmetric group S_k , which acts by permuting the labels.

We will use the notation $\sqcup_k \{*\}$ for a disjoint union of k unordered points and reserve the notation $\{p_1, \dots, p_k\}$ for k ordered (labeled) points. We will also use the shorthand $\underline{\text{Emb}}(L \sqcup_k \{*\}, \mathbb{R}^3)$ for the space of unparametrised embeddings of $\sqcup_m S^1 \sqcup_k \{*\}$ where the image of $\sqcup_m S^1$ is isotopic to L . In contrast, we will use the notation $\underline{\text{Emb}}(L \sqcup \{p_1, \dots, p_k\}, \mathbb{R}^3)$ for the space of unparametrised embeddings of $\sqcup_m S^1 \sqcup \{p_1, \dots, p_k\}$ where the k points are ordered and the image of $\sqcup_m S^1$ is isotopic to L . In particular $\underline{\text{Emb}}(L \sqcup \{p_1, \dots, p_k\}, \mathbb{R}^3)$ is a regular covering of $\underline{\text{Emb}}(L \sqcup_k \{*\}, \mathbb{R}^3)$ with the analogous S_k -action.

By forgetting the points we obtain a commutative diagram where the rows are locally trivial fibre bundles and the middle and left vertical maps are the S_k -covering maps described above

$$\begin{array}{ccccc} \text{PConf}^k(\mathbb{R}^3 \setminus L) & \longrightarrow & \underline{\text{Emb}}(L \sqcup \{p_1, \dots, p_k\}, \mathbb{R}^3) & \longrightarrow & \mathcal{E}(L) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \text{Conf}^k(\mathbb{R}^3 \setminus L) & \longrightarrow & \underline{\text{Emb}}(L \sqcup_k \{*\}, \mathbb{R}^3) & \longrightarrow & \mathcal{E}(L). \end{array}$$

If instead of points we consider a disjoint union $\sqcup_k B^3$ of k 3-balls, embedded with a link isotopic to L we obtain a space $\underline{\text{Emb}}(L \sqcup_k B^3, \mathbb{R}^3)$ consisting of unparametrised embeddings of $\sqcup_m S^1 \sqcup_k B^3$ where the image of $\sqcup_m S^1$ is isotopic to L . If we want to consider the 3-balls as labeled we will write $B_1 \sqcup \dots \sqcup B_k$ and obtain a corresponding embedding space $\underline{\text{Emb}}(L \sqcup B_1 \sqcup \dots \sqcup B_k, \mathbb{R}^3)$. This is a regular S_k -cover of $\underline{\text{Emb}}(L \sqcup_k B^3, \mathbb{R}^3)$ and in terms of parametrised embeddings, we can describe $\underline{\text{Emb}}(L \sqcup B_1 \sqcup \dots \sqcup B_k, \mathbb{R}^3)$ as the component of

$$\text{Emb}(\sqcup_m S^1 \sqcup B_1 \sqcup \dots \sqcup B_k, \mathbb{R}^3) / (\text{Diff}(\sqcup_m S^1) \times \text{Diff}(B_1) \times \dots \times \text{Diff}(B_k))$$

where the image of $\sqcup_m S^1$ is isotopic to L . That is, we do not quotient by the elements of $\text{Diff}(\sqcup_k B^3)$ that permute distinct balls. We thus obtain another diagram of locally trivial fiber bundles

$$\begin{array}{ccccc} \underline{\text{Emb}}(B_1 \sqcup \dots \sqcup B_k, \mathbb{R}^3 \setminus L) & \longrightarrow & \underline{\text{Emb}}(L \sqcup B_1 \sqcup \dots \sqcup B_k, \mathbb{R}^3) & \longrightarrow & \mathcal{E}(L) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \underline{\text{Emb}}(\sqcup_k B^3, \mathbb{R}^3 \setminus L) & \longrightarrow & \underline{\text{Emb}}(L \sqcup_k B^3, \mathbb{R}^3) & \longrightarrow & \mathcal{E}(L). \end{array}$$

Lemma 5.8. *Let L be any (possibly empty) link. For any $k \geq 1$ there exist homotopy equivalences*

$$(3) \quad \underline{\text{Emb}}(L \sqcup_k B^3, \mathbb{R}^3) \xrightarrow{\simeq} \underline{\text{Emb}}(L \sqcup_k \{*\}, \mathbb{R}^3)$$

and

$$(4) \quad \underline{\text{Emb}}(L \sqcup B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3) \xrightarrow{\simeq} \underline{\text{Emb}}(L \sqcup \{p_1, \dots, p_k\}, \mathbb{R}^3)$$

Proof. The same proof works in each case; we will prove the homotopy in the case of Equation (4) since this is most relevant for Section 6. One could also obtain (3) from (4) by modding out by the S_k -action described above. Consider the fibration

$$\underline{\text{Emb}}(L \sqcup B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3) \rightarrow \underline{\text{Emb}}(B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3).$$

By Lemma 5.1 and Remark 5.2, we can round the B_i to obtain a deformation retraction from $\underline{\text{Emb}}(B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3)$ to $\underline{\text{REmb}}(B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3)$. We lift this deformation retraction via the fibration to obtain a homotopy equivalence

$$\phi: \underline{\text{Emb}}(L \sqcup B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3) \rightarrow \underline{\text{REmb}}(L \sqcup B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3)$$

where the target is the subspace of embeddings in which the image of each B_i is round (there is no restriction on the link embeddings). Once the B_i are round, we have a map

$$\psi: \underline{\text{REmb}}(L \sqcup B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3) \rightarrow \underline{\text{Emb}}(L \sqcup \{p_1, \dots, p_k\}, \mathbb{R}^3)$$

which sends the image of each B_i to the center point of the image. We claim that ψ is a homotopy equivalence. Indeed, ψ is a locally trivial fiber bundle and the fiber can be identified with the embedding space of round balls $B_1 \sqcup \cdots \sqcup B_k$ in $\mathbb{R}^3 \setminus L$ with a prescribed set $\{p_1, \dots, p_k\}$ of centers. Hence the only thing that may vary in the fiber is the radius of each ball. Fix a point in the fiber specified by radii (R_1, \dots, R_k) . For any other point (r_1, \dots, r_k) define a homotopy h_t by

$$h_t(r_1, \dots, r_k) = (tR_1 + (1-t)r_1, \dots, tR_k + (1-t)r_k)$$

This is a deformation retraction of the fiber to the embedding with radii (R_1, \dots, R_k) . Hence the fiber is contractible, and so ψ is a homotopy equivalence. Therefore

$$\psi \circ \phi: \underline{\text{Emb}}(L \sqcup B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3) \rightarrow \underline{\text{Emb}}(L \sqcup \{p_1, \dots, p_k\}, \mathbb{R}^3)$$

is the desired homotopy equivalence for Equation (4). \square

6. THE FUNDAMENTAL GROUP OF $\mathcal{E}(L)$

In this section we use the previous results to give a general presentation for the fundamental group of $\mathcal{E}(L)$, where L is the image in $\mathcal{E}(L)$ of the disjoint union $L_1 \sqcup \cdots \sqcup L_n$. Recall that the L_i satisfy the following criteria:

- (i) Each piece L_i is contained in the ball B_i of radius $\frac{1}{2}$ centered on $(2i, 0, 0) \in \mathbb{R}^3$.
- (ii) If $L_i \cong L_j$ for $i < j$ then $L_j = L_i + (2(j-i), 0, 0)$. That is, if two pieces are isotopic, then the unparametrised embeddings differ by a translation of \mathbb{R}^3 .

The embedding L will always serve as our chosen basepoint for π_1 , hence in what follows we will write $\pi_1(\mathcal{E}(L))$ in place of $\pi_1(\mathcal{E}(L), L)$, and similarly for $\mathcal{PE}(L)$. As a first step we pass to studying $\pi_1(\mathcal{PE}(L))$. The presentation of $\pi_1(\mathcal{PE}(L))$ will then depend on the following two families of groups:

- $G_i = \pi_1(\mathcal{E}(L_i))$, and
- $H_i = \pi_1(\mathbb{R}^3 \setminus L_i)$, for L_i a piece of L .

We will describe $\pi_1(\mathcal{PE}(L))$ in terms of the G_i and certain automorphisms of the fundamental group of the complement $\pi_1(\mathbb{R}^3 \setminus L) = H_L \cong H_1 * \cdots * H_n$.

The first subgroup we consider will be permutations of the link pieces. Let $S_{\{L_1, \dots, L_n\}}$ denote the symmetric group on the set of link pieces $\{L_1, \dots, L_n\}$ and define P_L to be the group of permutations satisfying $L_i \cong L_{\sigma(i)}$ for all $\sigma \in P_L$. That is, we only allow permutations in P_L when link pieces are isotopic.

Lemma 6.1. *There is a surjective homomorphism $\eta: \pi_1(\mathcal{E}(L)) \rightarrow P_L$.*

Proof. An element of $\pi_1(\mathcal{E}(L))$ is represented by a path of unparametrised embeddings $\gamma = \gamma_t: [0, 1] \rightarrow \mathcal{E}(L)$ such that $\text{im}(\gamma_0) = \text{im}(\gamma_1) = L$. Thus, γ induces a permutation of the pieces of L , and we define $\eta(\gamma) \in S_{\{L_1, \dots, L_n\}}$ to be this permutation. Since γ can only exchange L_i and L_j when $L_i \cong L_j$, the image of η lies in P_L . To see that η is surjective, we use criterion (ii) in the definition of L to produce a sequence of translations of the B_i which exchange L_i with L_j when $L_i \cong L_j$. Explicitly, for $L_i \cong L_j$, $i < j$: translate B_i up to $B_i + (0, 0, 2)$ then over to $B_i + (2(j-i), 0, 2)$ then down to $B_i + (2(j-i), 0, 0)$, while simultaneously translating B_j down to $B_j + (0, 0, -2)$ then to $B_j + (2(i-j), 0, -2)$ then up to $B_j + (2(i-j), 0, 0)$. \square

It follows from Lemma 6.1 that after passing to a finite cover of $\mathcal{E}(L)$ of degree $|P_L|$, we may assume that any loop of embeddings returns each piece of L back to itself. Otherwise stated, we can regard this cover as embeddings in $\mathcal{E}(L)$ where the pieces are labeled. This is precisely the space $\mathcal{PE}(L)$ with labelled basepoint L . In order to simplify exposition, we will now assume that the pieces of L are labeled, and revisit the unlabeled case at the end.

Let $\underline{\text{Emb}}(L_i, B_i)$ be the path component of the space of unparametrised embeddings of $\sqcup_{m_i} S^1$ into the ball B_i containing L_i . By identifying the interior of B_i with \mathbb{R}^3 , we obtain a homeomorphism $\mathcal{E}(L_i) \cong \underline{\text{Emb}}(L_i, B_i)$. Thus, the inclusion $B_i \rightarrow \mathbb{R}^3$ induces a weak equivalence $\underline{\text{Emb}}(L_i, B_i) \rightarrow \mathcal{E}(L_i)$, where the inverse map is given by shrinking \mathbb{R}^3 into B_i . Given any element of $\underline{\text{Emb}}(L_i, B_i)$ we obtain an element of $\mathcal{PE}(L)$ by mapping to the basepoint L_j on the pieces $L_j \neq L_i$. This defines an embedding $\beta_i: \mathcal{E}(L_i) \hookrightarrow \mathcal{PE}(L)$.

Lemma 6.2. *Let $(\beta_i)_*$ be the induced map on π_1 . Then $(\beta_i)_*$ is injective.*

Proof. As discussed above, since $B_i \cong \mathbb{R}^3$, the inclusion $B_i \hookrightarrow \mathbb{R}^3$ induces an isomorphism from $\pi_1(\underline{\text{Emb}}(L_i, B_i)) \cong \pi_1(\mathcal{E}(L_i))$. Now we have a composition

$$\underline{\text{Emb}}(L_i, B_i) \hookrightarrow \mathcal{PE}(L) \rightarrow \mathcal{E}(L_i)$$

where the first map is β_i and the second map forgets all pieces except L_i . Since the composition is the inclusion from above, $(\beta_i)_*$ must be injective. \square

In fact, putting the β_i maps together gives an embedding

$$\beta: \prod_{i=1}^n \underline{\text{Emb}}(L_i, B_i) \hookrightarrow \mathcal{PE}(L).$$

The image of this is the subspace of $\mathcal{PE}(L)$ where L_i lies in B_i . Let $G_L = \prod_{i=1}^n G_i$ be the product of the G_i . By Lemma 6.2, after identifying $\pi_1(\underline{\text{Emb}}(L_i, B_i))$ with G_i , β induces an injection

$$\beta_*: G_L \hookrightarrow \pi_1(\mathcal{PE}(L)).$$

Taking the product of the forgetful maps $\mathcal{PE}(L) \rightarrow \mathcal{E}(L_i)$, we obtain a homomorphism

$$(5) \quad r: \pi_1(\mathcal{PE}(L)) \rightarrow \prod_{i=1}^n \pi_1(\mathcal{E}(L_i)) = G_L,$$

which exhibits G_L as a retract.

Now consider the homomorphism

$$D: \pi_1(\mathcal{PE}(L)) \rightarrow \text{Aut}(\pi_1(\mathbb{R}^3 \setminus L)) \cong \text{Aut}(H_1 * \cdots * H_k) = \text{Aut}(H_L)$$

This homomorphism was first defined by Dahm [Dah62], and then later explored by Goldsmith [Gol81, Gol82], who computed its image when L is an (np, nq) -torus link.

The Dahm homomorphism can be defined as follows. Consider a family of embeddings $\gamma = \gamma_t$, $t \in [0, 1]$, such that $\gamma_0 = \gamma_1 = L$. Extend this to \mathbb{R}^3 via an ambient isotopy f_t with compact support. Choose now a basepoint $p \in \mathbb{R}^3$ that is fixed by f_t for all t . Thus $(f_1)_*$ defines an automorphism of $\pi_1(\mathbb{R}^3 \setminus L, p)$ and we set $D(\gamma) = (f_1)_*$. Although this definition depends on the choice of p , one can make it well-defined by choosing a sequence of basepoints along a ray out to infinity. Since the isotopies are compactly supported, points sufficiently far along the ray will be fixed. There is a canonical way to identify automorphisms based at points along this ray, and in fact along any ray out to infinity. See Section 4 of [Gol81] for details.

Lemma 6.3. *Any automorphism in the image of D sends H_i to a conjugate of itself.*

Proof. Recall $H_i = \pi_1(\mathbb{R}^3 \setminus L_i)$. Write

$$H_L = \pi_1(\mathbb{R}^3 \setminus L) = H_1 * \cdots * H_r * \overbrace{\mathbb{Z} * \cdots * \mathbb{Z}}^{n-r}$$

where H_1, \dots, H_r correspond to pieces that are not unknots, and the $(n-r)$ \mathbb{Z} factors correspond to unknot pieces. The uniqueness statement of the Grushko decomposition theorem (see e.g. [Sta77]) states that in any maximal free product decomposition of H_L , the factors $\{H_1, \dots, H_r\}$ are determined up to permutation and replacement by conjugates. Hence the fact that $\pi_1(\mathcal{PE}(L))$ preserves labels implies that any automorphism in the image of D preserves the conjugacy class of each H_i , $1 \leq i \leq r$. On the other hand, each of the \mathbb{Z} factors in H_L can be represented by a loop that goes through the corresponding unknot piece. The fact

that $\pi_1(\mathcal{PE}(L))$ preserves labels means that the free homotopy class of this loop is preserved. Hence each \mathbb{Z} factor is also taken to a conjugate of itself. \square

There are certain automorphisms, which we now describe, that lie naturally in the image of D .

Definition 6.4. Choose an element $g \in H_i$ and a factor H_j , where possibly $j = i$. Define the *partial conjugation of H_j by g* to be the automorphism $X(g, H_j) \in \text{Aut}(H)$ which sends $h \mapsto ghg^{-1}$ for every $h \in H_j$, and acts as the identity on all other factors H_k , $k \neq j$. We refer to g as the *acting element* of $X(g, H_j)$ and to H_j as the *support*. The *Fouxe-Rabinovitch group* $\text{FR}(H)$ is the subgroup of $\text{Aut}(H)$ generated by the partial conjugations $X(g, H_j)$.

We now describe loops in $\mathcal{PE}(L)$ that represent each $X(g, H_j)$, via the Dahm homomorphism D .

- (1) When $i \neq j$, $X(g, H_j)$ can be represented geometrically by a family of embeddings γ_t that shrink the piece L_j and move it along a path in the complement of L_i which represents $g \in H_i = \pi_1(\mathbb{R}^3 \setminus L_i)$. It is clear that we may choose γ_t in such a way that:
 - If $k \neq j$, γ_t fixes L_k for all t .
 - For every t , $\gamma_t(L_j)$ is a rescaling of L_j followed by a sequence of translations.
- (2) When $i = j$, we can represent $X(g, H_i)$ as follows. Choose a basepoint $p \in \mathbb{R}^3$ and represent $g \in H_i = \pi_1(\mathbb{R}^3 \setminus L_i)$ as a knot ℓ in \mathbb{R}^3 . We can thicken ℓ to a solid torus $\nu(\ell)$ that links L_i . Now consider a family of embeddings γ_t which send each point of L_i around the solid torus and back to itself, *i.e.*, such that every point of L_i moves along a loop in $\mathbb{R}^3 \setminus \nu(\ell)$ homotopic to the longitude $\ell \subset \nu(\ell)$. Since the basepoint p is fixed in this isotopy, this has the effect of conjugating each element of $H_i = \pi_1(\mathbb{R}^3 \setminus L_i)$ by $g = [\ell]$. We can choose $\nu(\ell)$ such that the whole motion takes place inside B_i , and thus lies in G_i .

Let $\chi(g, L_j)$ in $\pi_1(\mathcal{PE}(L))$ denote the homotopy class of γ_t as described in either (1) or (2). More generally, given $g \in H_i$ and any subset $A \subseteq \{L_1, \dots, L_n\} \setminus \{L_i\}$, let $\chi(g, A)$ denote the product $\prod_{L_j \in A} \chi(g, L_j)$. For $j_1 \neq j_2$ and not equal to i , $\chi(g, L_{j_1})$ and $\chi(g, L_{j_2})$ can be chosen to have disjoint support and thus commute. Thus $\chi(g, A)$ depends only on A .

Definition 6.5. Let $\mathcal{FR}(L)$ be the subgroup of $\pi_1(\mathcal{PE}(L))$ generated by the $\chi(g, L_j)$ as described above. Define $\mathcal{FR}_0(L) \leq \mathcal{FR}(L)$ to be the subgroup

$$\mathcal{FR}_0(L) = \langle \chi(g, L_j) \mid g \in H_i, i \neq j \rangle.$$

In other words, $\mathcal{FR}_0(L)$ is the subgroup generated by partial conjugations whose acting elements are disjoint from their support.

By definition, the image of $\mathcal{FR}(L)$ under D is $\text{FR}(H)$. Moreover, we have

Lemma 6.6. *The restriction of D to $\mathcal{FR}(L)$ is an isomorphism.*

Proof. By a result of [FR41] (see also [CG90], Proposition 3.1 and Remark (ii) on page 164) the relations in $\text{FR}(H_L)$ have the following three forms:

- (1) For all j and for $g, g' \in H_i$, $X(g, H_j)X(g', H_j) = X(gg', H_j)$. Here $i = j$ is allowed.
- (2) Suppose $g \in H_i$ and $g' \in H_{i'}$. Then $[X(g, H_j), X(g', H_{j'})] = 1$ if $j \neq j'$ and $\{i, i'\} \cap \{j, j'\} = \emptyset$. Here $i = i'$ is allowed.
- (3) $[X(g, H_j)X(g, H_k), X(g', H_j)] = 1$ where $g \in H_i$, $g' \in H_k$ and i, j, k are all distinct.

To prove the lemma, we verify that the chosen generators of $\mathcal{FR}(L)$ also satisfy these relations. For (1), realise g, g' as loops γ, γ' based at the center of B_j . Then the relation in (1) is simply the statement that $\gamma * \gamma'$ is homotopic to γ followed by γ' . For (2), since $j \neq j'$ and neither is equal to i nor i' , as we noted above the geometric representatives $\chi(g, L_j)$ and $\chi(g', L_{j'})$ can be chosen to have disjoint support, and thus can be realised simultaneously. Finally, for (3), since i, j , and k are distinct we shrink B_j and move it via a sequence of translations into B_k . Then moving B_k along the loop γ for $g \in H_i$ as in the definition of $\chi(g, L_k)$ will also move B_j . Now identify H_k with $\pi_1(B_k \setminus L_k)$. Since we have shrunk down B_j into B_k we are free to move B_j along the loop $\gamma' \in B_k \setminus L_k$ corresponding to $g' \in H_k$ at any point whilst completing the γ path with B_k . This means that $\chi(g, L_j)\chi(g, L_k)$ commutes with $\chi(g', L_j)$, so (3) is satisfied. Hence $\mathcal{FR}(L)$ satisfies all relations of $\text{FR}(L)$, which proves that D restricted to $\mathcal{FR}(L)$ is injective. Since it is surjective on generators, this proves the lemma. \square

Recall the map r from Equation (5). From the properties of our chosen representative for $\chi(g, L_j)$ with $g \in H_i$ and $j \neq i$, we deduce that $r(\chi(g, L_j)) = 1$. Therefore, we obtain a split short exact sequence

$$(6) \quad 1 \rightarrow \ker(r) \rightarrow \pi_1(\mathcal{PE}(L)) \xrightarrow{r} G_L \rightarrow 1$$

On the other hand, the restriction of $\chi(g, L_i)$ to L_i is in G_i when $g \in H_i$. It follows that $\ker(r) \cap \mathcal{FR}(L)$ is the subgroup $\mathcal{FR}_0(L)$. In particular $\mathcal{FR}_0(L) \cap G_L = \{1\}$. In what follows, we will show that $\ker(r) = \mathcal{FR}_0(L)$, and therefore obtain a complete description of $\pi_1(\mathcal{PE}(L))$ as a semidirect product.

Consider now an arbitrary embedding $\rho \in \mathcal{PE}(L)$ with separating system Σ . Let $\text{PSep}_0^{\rho \sqcup \Sigma}$ be the component of PSep_0 containing $\rho \sqcup \Sigma$. By Lemma 2.10, all combinatorial types of separating system are realised uniquely up to isotopy. Our goal will be to compute $\pi_1(\text{PSep}_0^{\rho \sqcup \Sigma})$. Recall from Definition 2.5 that the separating system Σ defines a rooted labeled tree $T = T_\Sigma$ where each vertex is a component of $\mathbb{R}^3 \setminus \Sigma$, labeled by $i \in \{1, \dots, n\}$ if it contains the piece ρ_i , and each edge is a sphere of Σ . The root is the unbounded component of $\mathbb{R}^3 \setminus \Sigma$.

The labels on the rooted tree T define a partial ordering on the set $\{1, \dots, n\}$ based on proximity to the root. Denote the set of descendants of vertex $v \in T$ by $\text{Desc}(v)$. If v is labeled by i then we also write $\text{Desc}(i) = \text{Desc}(v)$. We single out the direct descendants – or children – of vertex v as $\text{Desc}_1(v) = \text{Desc}_1(i)$. In terms of spheres,

children of an i -vertex are separated by a single sphere from the component of $\mathbb{R}^3 \setminus \Sigma$ containing ρ_i , and by at least one sphere from the unbounded component. Each vertex v defines a descending branch of T below v . Let $L(v)$ be the subset of basepoint pieces L_j such that $j \in \{1, \dots, n\}$ occurs as a label of this branch. Since Σ is an essential separating system, $L(v)$ is non-empty.

Definition 6.7. Let Σ be an essential separating system for ρ and T_Σ its dual tree. We define the *group of partial conjugations supported on T_Σ* to be

$$\chi(T_\Sigma) = \langle \chi(g, L(v)) \mid v \in \text{Desc}_1(L_i) \text{ and } g \in H_i \text{ for } 1 \leq i \leq n \rangle$$

Define the *motion group of T_Σ* to be the group $G(T_\Sigma)$ generated by G_L and $\chi(T_\Sigma)$.

Proposition 6.8. *For any separating system Σ , $\pi_1(\text{PSep}_0^{\rho \cup \Sigma}) \cong G(T_\Sigma)$. Moreover, $\chi(T_\Sigma) \leq G(T_\Sigma)$ is normalised by G_L .*

Proof. We prove this by induction on the number of pieces. For the base case $L = L_1$, $\mathcal{PE}(L_1) = \mathcal{E}(L_1)$, and by definition $\pi_1(\mathcal{E}(L_1)) = G_1$.

Now suppose $L = L_1 \sqcup \dots \sqcup L_n$ with $n > 1$. Let r be the root and $\text{Desc}_1(r) = \{v_1, \dots, v_k\}$. Each v_j together with its descendants induces a rooted subtree T_j of T , where v_j is the root. Since Σ is a separating system, each T_j has label set containing at least one i -labelled vertex for $i \in \{1, \dots, n\}$.

Furthermore, each $v_j \in \text{Desc}_1(r)$ defines an edge $\{r, v_j\}$ corresponding to a sphere S_j . Let B_j be the ball bounded by S_j . Let $K(j)$ be the union of all pieces in the interior of B_j and set $\rho(j)$ to be the restriction of ρ to $K(j)$. Since the interior of B_j is homeomorphic to \mathbb{R}^3 , we can regard the union of spheres in the interior of B_j as a separating system Σ_j for $\rho(j)$, with dual tree T_j . Denote by $\text{PSep}_0^{\rho(j) \sqcup \Sigma_j}$ the connected component of PSep_0 for $\rho(j)$ containing $\rho(j) \sqcup \Sigma_j$. Since the root r of T cannot be univalent, $K(j)$ has cardinality less than n . Therefore, by induction, $\pi_1(\text{PSep}_0^{\rho(j) \sqcup \Sigma_j}) \cong G(T_{\Sigma_j})$.

Now consider the map which forgets the link pieces and spheres in the interior of each B_j . The fiber is homeomorphic to the product $\prod_{j=1}^k \text{PSep}_0^{\rho(j) \sqcup \Sigma_j}$. There are two cases, depending on whether the root r is labeled by some $i \in \{1, \dots, n\}$ or not. First consider the case in which r is labeled by \emptyset . In this case, we obtain a fibration

$$\prod_{j=1}^k \text{PSep}_0^{\rho(j) \sqcup \Sigma_j} \hookrightarrow \text{PSep}_0^{\rho \sqcup \Sigma} \rightarrow \underline{\text{Emb}}(B_1 \sqcup \dots \sqcup B_k, \mathbb{R}^3).$$

By Lemma 5.8, the base space is homotopy equivalent to $\text{PConf}^k(\mathbb{R}^3)$, which is simply connected. Hence $\pi_1(\text{PSep}_0^{\rho \sqcup \Sigma}) \cong \prod_{j=1}^k G(T_{\Sigma_j})$. By considering generators, the latter is equal to $G(T_\Sigma)$, proving the proposition in this case.

Now suppose r is labeled by some $i \in \{1, \dots, n\}$. Relabeling if necessary, we may assume that r is labeled by n . In this case we have the following fibration

$$(7) \quad \prod_{j=1}^k \text{PSep}_0^{\rho(j) \sqcup \Sigma_j} \hookrightarrow \text{PSep}_0^{\rho \sqcup \Sigma} \rightarrow \underline{\text{Emb}}(L_n \sqcup B_1 \sqcup \dots \sqcup B_k, \mathbb{R}^3).$$

We claim that the fundamental group of the base space is isomorphic to the group $(\prod_{j=1}^k H_n) \rtimes G_n$. By Lemma 5.8, $\underline{\text{Emb}}(L_n \sqcup B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3)$ is homotopy equivalent to $\underline{\text{Emb}}(L_n \sqcup \{p_1, \dots, p_k\}, \mathbb{R}^3)$, where $\{p_1, \dots, p_k\}$ are k distinct labeled points. The latter fits into a fiber bundle

$$\text{PConf}^k(\mathbb{R}^3 \setminus L_n) \hookrightarrow \underline{\text{Emb}}(L_n \sqcup \{p_1, \dots, p_k\}, \mathbb{R}^3) \rightarrow \mathcal{E}(L_n).$$

The fundamental group of $\mathcal{E}(L_n)$ is G_n by definition. Since a point has codimension 3 in $\mathbb{R}^3 \setminus L_n$, it follows that $\pi_1(\text{PConf}^k(\mathbb{R}^3 \setminus L_n)) \cong \prod_{j=1}^k H_n$. Hence on the level of π_1 we obtain a short exact sequence

$$1 \rightarrow \prod_{j=1}^k H_n \rightarrow \pi_1(\underline{\text{Emb}}(L_n \sqcup \{p_1, \dots, p_k\}, \mathbb{R}^3)) \rightarrow G_n \rightarrow 1$$

which is split, since any 1-parameter family of embeddings of L can be chosen to avoid a finite collection of points. This proves the claim.

We return to Equation (7). By induction, the fundamental group of each $\text{PSep}_0^{\rho(j) \sqcup \Sigma_j}$ is generated by $G(T_{\Sigma_j})$, while $\pi_1(\underline{\text{Emb}}(L_n \sqcup B_1 \sqcup \cdots \sqcup B_k, \mathbb{R}^3)) \cong (\prod_{j=1}^k H_n) \rtimes G_n$. Moreover, the product $\prod_{j=1}^k H_n$ is generated by a family of embeddings that sends B_j along some loop in the complement of L_n representing an element $g_n \in H_n = \pi_1(\mathbb{R}^3 \setminus L_n)$. This induces the element $\chi(g_n, L(v_j))$ on the level of fundamental groups. (Recall $L(v)$ is the set of pieces L_j such that j is a label in the descending branch of T defined by v .) Hence $G(T_{\Sigma})$ is generated as claimed.

Finally, we prove that $\chi(T_{\Sigma})$ is normalised by G_L . By induction, the partial conjugations $\chi(T_{\Sigma_j})$ are normalised by each G_p if L_p is a piece in $\rho(j)$. Moreover, for each $j' \neq j$, elements of $G(T_{\Sigma_j})$ commute with $G(T_{\Sigma_{j'}})$ because they have disjoint support as automorphisms. In particular, if L_q is a piece in $\rho(j')$, then G_q commutes with $\chi(T_{\Sigma_j})$. Lastly, the existence of the semidirect product decomposition $(\prod_{j=1}^k H_n) \rtimes G_n$ now implies that the added partial conjugations are also normalised by G_L , since G_i commutes with $\prod_{j=1}^k H_n$ for each $i \neq n$. \square

Remark 6.9. Every element of $\mathcal{FR}_0(L)$ arises in some separating system, so it follows from Proposition 6.8 that G_L normalises $\mathcal{FR}_0(L)$.

Definition 6.10. An automorphism φ of H_L is said to *preserve the free splitting* $H_1 * \cdots * H_n$ if $\varphi(H_i) = H_i$ for each i .

Proposition 6.11. *Let $g \in \pi_1(\mathcal{PE}(L))$ and suppose $D(g)$ preserves the free splitting $H_1 * \cdots * H_n$. Then $g \in G_L$.*

Proof. By Corollary 3.9, $\mathcal{PE}(L) \simeq |\text{PSep}_{\bullet}|$. We can therefore lift any class in $\pi_1(\mathcal{PE}(L))$ to $|\text{PSep}_{\bullet}|$, and moreover we can find a representative $\gamma: S^1 \rightarrow \mathcal{PE}(L)$ such that the image of γ lifts to the 1-skeleton of $|\text{PSep}_{\bullet}|$. In particular we obtain a subdivision of $[0, 1]$ into intervals $I_0 J_1 \cdots I_{p-1} J_p I_p$ such that $\gamma(I_k)$ lifts to PSep_0 and $\gamma(J_l)$ lifts to PSep_1 for all k, l (by an abuse of notation, we will omit γ and think of the I_k, J_l as paths themselves). Since our basepoint (from Example 2.4) lies in $\text{PSep}_0^{L \sqcup \Sigma_L}$, both I_0 and I_p lift to $\text{PSep}_0^{L \sqcup \Sigma_L}$. We will show that g can be written

as a product of partial conjugations and elements of G_L . Since G_L normalises the partial conjugations by Remark 6.9, our assumption that $D(g)$ preserves the free splitting $H_1 * \cdots * H_n$ will imply the proposition.

For each $1 \leq l \leq p$, J_l connects two components of PSep_0 . Let $\rho_l \in \mathcal{PE}(L)$ be the link embedding determined by $J_l(0)$. At $J_l(0)$, there are two separating systems Σ_{l-1} and Σ_l for ρ_l . On the interior of J_l , both separating systems are disjointly embedded with the link and vary only by isotopy. Denote by $\text{PSep}_1^{\rho_l \sqcup \Sigma_{l-1} \sqcup \Sigma_l}$ the connected component of PSep_1 containing $\rho_l \sqcup \Sigma_{l-1} \sqcup \Sigma_l$.

Claim: On each J_l we may assume that the embeddings of L , and the associated separating systems Σ_{l-1} and Σ_l are fixed.

Proof of claim. There's a forgetful map $f : \text{PSep}_1^{\rho_l \sqcup \Sigma_{l-1} \sqcup \Sigma_l} \rightarrow \text{PSep}_0^{\rho_l \sqcup \Sigma_l}$, given by forgetting the embedding of Σ_{l-1} . We replace the segments J_l and I_{l+1} with segments J'_l and I'_{l+1} as follows. Let J'_l be the path in $\text{PSep}_1^{\rho_l \sqcup \Sigma_{l-1} \sqcup \Sigma_l}$ where L , remains fixed at $J_l(0)$ and Σ_{l-1} and Σ_l are fixed on the interior of J_l , with the weight shifting linearly from Σ_{l-1} to Σ_l . Let $J_l|_{L \cup \Sigma_l}$ be the isotopy of $L \cup \Sigma_l$ which occurs in the interior of J_l . We concatenate $f(J_l) = J_l|_{L \cup \Sigma_l}$ with the path I_{l+1} to get I'_{l+1} . By homotoping γ , we may replace the segment $J_l I_{l+1}$ with $J'_l I'_{l+1}$. Relabelling J' to J_l and I'_{l+1} to I_{l+1} proves the claim. \square

We next consider the contribution of each interval I_k . Denote the link embedding at $I_k(0)$ (respectively $I_k(1)$) by ρ_k (respectively ρ'_k), with separating system Σ_k (resp. Σ'_k). Choose a path α_k from ρ_k to L . For $I_0(0)$ and $I_p(1)$ we choose these paths to be constant at L . By the claim, $\rho'_{k-1} = \rho_k$ for $1 \leq k \leq p$, and the separating systems Σ'_{k-1} and Σ_k are disjointly embedded. Therefore we extend α_k to both separating systems by isotopy extension. The path $\overline{\alpha}_k * I_k * \alpha_{k+1}$ for $0 \leq k \leq p-1$ lifts to an element of $\pi_1(|\text{PSep}_\bullet|)$, and the concatenation

$$(I_0 \alpha_1)(\overline{\alpha}_1 J_1 \alpha_1)(\overline{\alpha}_1 I_1 \alpha_2) \overline{\alpha}_2 \cdots \alpha_p (\overline{\alpha}_p J_p \alpha_p)(\overline{\alpha}_p I_p)$$

is homotopic to γ . By the claim, each term $\overline{\alpha}_l J_l \alpha_l$ is trivial in $\pi_1(\mathcal{E}(L))$, since J_l is stationary on the link.

On the other hand, each term $\overline{\alpha}_k I_k \alpha_{k+1}$ can be regarded as an element of the motion group $\pi_1(\text{PSep}_0^{\rho_k \sqcup \Sigma_k})$. By Proposition 6.8, this is an element g_p of $G(T_{\Sigma_k})$ which is a product of elements of $\mathcal{FR}_0(L)$ and elements of G_L . Since G_L normalises $\mathcal{FR}_0(L)$ (Remark 6.9), we can write this as a product $g_k = h_k t_k$, where $h_k \in \mathcal{FR}_0(L)$ and $t_k \in G_L$. Doing this $\forall 0 \leq k \leq p$ gives

$$\begin{aligned} g &= g_0 \cdots g_p = (h_0 t_0)(h_1 t_1) \cdots (h_p t_p) \\ &= (h_0 h'_1 \cdots h'_p)(t_0 \cdots t_p) \end{aligned}$$

where on the last line we used the fact that G_L normalises $\mathcal{FR}_0(L)$ again. Since $D(G_L)$ preserves the free splitting $H_1 * \cdots * H_n$, the product $D(t_0 \cdots t_p)$ also does. Now since $D(g)$ preserves the free splitting by assumption we conclude that $D(h_0 h'_1 \cdots h'_p)$ preserves the free splitting as well. Under D , each element in $\mathcal{FR}_0(L)$ conjugates some H_j by $g \in H_i$ where $i \neq j$. Hence the only way that $D(h_0 h'_1 \cdots h'_p)$ preserves the free splitting is if $D(h_0 h'_1 \cdots h'_p) = 1$. By Lemma 6.6, this implies $h_0 h'_1 \cdots h'_p = 1 \in \mathcal{FR}_0(L)$, hence $g = t_0 \cdots t_p \in G_L$, as desired. \square

We are now ready to revisit the unlabeled case and prove Theorem B:

Theorem B. $\pi_1(\mathcal{E}(L))$ is isomorphic to $(\mathcal{FR}_0(L) \rtimes G_L) \rtimes P_L$.

Proof. Recall the short exact sequence from Equation (6) above:

$$1 \rightarrow \ker(r) \rightarrow \pi_1(\mathcal{PE}(L)) \xrightarrow{r} G_L \rightarrow 1.$$

Suppose $g \in \ker(r)$. By Lemma 6.3, $D(g)$ sends each factor H_i of H_L to a conjugate of itself. Since $\mathcal{FR}_0(L) \leq \ker(r)$, there exists an element $h \in \mathcal{FR}_0(L)$ such that $D(gh^{-1})$ preserves the free splitting $H_1 * \cdots * H_n$. But then by Proposition 6.11, this implies $gh^{-1} \in G_L \cap \ker(r) = \{1\}$. This means that $g = h \in \mathcal{FR}_0(L)$, proving that $\mathcal{FR}_0(L) = \ker(r)$.

This further implies that our choice of lifts of elements of P_L splits the surjection $\pi_1(\mathcal{E}(L)) \rightarrow P_L$. Let $\sigma_1 \cdots \sigma_q$ be a product of these lifts which induces the trivial permutation. Since each σ_i acts as a sequence of translations, we have that $r(\sigma_i) = 1$ for all i . On the other hand, the product of σ_i preserves the free splitting $H_1 * \cdots * H_n$. Hence $\sigma_1 \cdots \sigma_p = 1 \in \pi_1(\mathcal{E}(L))$ by Proposition 6.11. The description of $\pi_1(\mathcal{E}(L))$ as a semidirect product now follows from Equation (6) and Lemma 6.1. \square

We conclude this section with an example. Let H be the Hopf link. By work of Goldsmith, $\pi_1(\mathcal{E}(H))$ is isomorphic to the quaternion group Q_8 , since H is isomorphic to the torus link $T(2,2)$ [Gol82]. Using this and Theorem B, we can compute the fundamental group of $\mathcal{E}(L)$ when L is a disjoint union of an n -component unlink and m Hopf links. Following [DK19], we call such a link H -trivial.

Example 6.12 (H-trivial links). Let $L = H_{n,m}$ be a disjoint union of an n -component unlink and m Hopf links. Then

$$\pi_1(\mathbb{R}^3 \setminus L) = \overbrace{\mathbb{Z} * \cdots * \mathbb{Z}}^n * \overbrace{\mathbb{Z}^2 * \cdots * \mathbb{Z}^2}^m$$

and $\mathcal{FR}(L) = \mathcal{FR}_0(L)$ since each $\pi_1(\mathbb{R}^3 \setminus L_i)$ is abelian for each piece L_i . We can permute each of the unlink components and each of the Hopf links, so $P_L \cong S_n \times S_m$. Each unknot contributes a $\mathbb{Z}/2$ factor to G_L , while each Hopf link contributes a Q_8 factor. Thus $G_L \cong (\mathbb{Z}/2)^n \times (Q_8)^m$. Therefore we obtain

$$\pi_1(\mathcal{E}(H_{n,m})) \cong \left(\text{FR}(\overbrace{\mathbb{Z} * \cdots * \mathbb{Z}}^n * \overbrace{\mathbb{Z}^2 * \cdots * \mathbb{Z}^2}^m) \rtimes ((\mathbb{Z}/2)^n \times (Q_8)^m) \right) \rtimes (S_n \times S_m).$$

From personal correspondence, we believe that Damiani, Kamada and Piergallini have independently computed a presentation for this motion group.

6.1. Passing to S^3 . In this section we discuss the effect on homotopy groups, and specifically π_1 , when passing to embeddings in S^3 instead of \mathbb{R}^3 . Consider the space of embeddings $\text{Emb}(L \sqcup \{*\}, S^3)$, where $\{*\}$ is a disjoint point. Then, regarding \mathbb{R}^3

as $S^3 \setminus \{*\}$, we identify $\underline{\text{Emb}}(L, S^3 \setminus \{*\}) \cong \underline{\text{Emb}}(L, \mathbb{R}^3) = \mathcal{E}(L)$. We then obtain the following two fibrations.

$$\begin{array}{ccccc}
 & & \underline{\text{Emb}}(\{*\}, S^3 \setminus L) = S^3 \setminus L & & \\
 & & \downarrow & & \\
 \underline{\text{Emb}}(L, S^3 \setminus \{*\}) \cong \mathcal{E}(L) & \longrightarrow & \underline{\text{Emb}}(L \sqcup \{*\}, S^3) & \longrightarrow & \underline{\text{Emb}}(\{*\}, S^3) = S^3 \\
 & & \downarrow & & \\
 & & \underline{\text{Emb}}(L, S^3) & &
 \end{array}$$

Since S^3 is 2-connected, the long exact sequence for the horizontal fibration yields an isomorphism $\pi_1(\mathcal{E}(L)) \cong \pi_1(\underline{\text{Emb}}(L \sqcup \{*\}, S^3))$. Combined with the vertical fibration we then have a short exact sequence

$$(8) \quad \pi_2(\underline{\text{Emb}}(L, S^3)) \rightarrow \pi_1(S^3 \setminus L) \rightarrow \pi_1(\mathcal{E}(L)) \rightarrow \pi_1(\underline{\text{Emb}}(L, S^3)) \rightarrow 1.$$

Recall that for any group G the conjugation action of G on itself induces a homomorphism $G \rightarrow \text{Aut}(G)$ whose image is the subgroup of *inner automorphisms* $\text{Inn}(G)$. The kernel of this homomorphism is the center of G .

Proposition 6.13. *The image of $\pi_1(\underline{\text{Emb}}(\{*\}, S^3 \setminus L))$ in $\pi_1(\mathcal{E}(L))$ is isomorphic to the group of inner automorphisms of $\pi_1(S^3 \setminus L)$. In particular, the image of $\pi_2(\underline{\text{Emb}}(L, S^3))$ is the center of $\pi_1(S^3 \setminus L)$.*

Proof. Regard $\{*\} \in S^3 \setminus L$ as a basepoint for the fundamental group. An element of $\pi_1(\underline{\text{Emb}}(\{*\}, S^3 \setminus L))$ drags the basepoint around a loop γ representing a class in $\pi_1(S^3 \setminus L)$, which changes the identification of the fundamental group by conjugation by $[\gamma]$. The image of $\pi_1(\underline{\text{Emb}}(\{*\}, S^3 \setminus L))$ is therefore the subgroup of inner automorphisms which lies in $\mathcal{FR}(L) \leq \pi_1(\mathcal{E}(L))$. The statement about the image of $\pi_2(\underline{\text{Emb}}(L, S^3))$ follows from exactness of Equation (8). \square

Inputting Proposition 6.13 with Equation (8), Theorem B, and the fact that the image of $\pi_1(\underline{\text{Emb}}(\{*\}, S^3 \setminus L))$ is normal now implies the following corollary.

Corollary C. *Let $H_L = \pi_1(S^3 \setminus L)$. Then*

$$\pi_1(\underline{\text{Emb}}(L, S^3)) \cong (\mathcal{FR}_0(L) \rtimes G_L) / \text{Inn}(H_L) \rtimes P_L.$$

where $\text{Inn}(H_L)$ is the group of inner automorphisms of H_L .

We end this section with some remarks on Theorem B and Corollary C. Recall that if L has a single piece, then $S^3 \setminus L$ is aspherical. In particular, $H_L = \pi_1(S^3 \setminus L)$ is finitely presented, torsion-free and has cohomological dimension at most 2. If H_L is abelian, then it is isomorphic to either \mathbb{Z} or \mathbb{Z}^2 . The former occurs exactly when L is the unknot U , and the latter exactly when L is the Hopf link H .

If L has at least two pieces, H_L is a nontrivial free product, hence its center is trivial. In this case $\mathcal{FR}(L) \leq \mathcal{FR}_0(L) \rtimes G_L$ is always infinite. $\mathcal{FR}(L) / \text{Inn}(H_L)$ is also infinite, unless L is the two component unlink $U \sqcup U$ or a disjoint union of two Hopf links $H \sqcup H$. If L has a single piece, then P_L is trivial by definition and

$\mathcal{FR}(L) = \text{Inn}(L)$, so $\pi_1(\underline{\text{Emb}}(L, S^3))$ is equal to $G_L/\mathcal{FR}(L)$. On the other hand, in this case $\mathcal{FR}(L)$ is infinite if and only if H_L is not abelian. By the preceding discussion, $\pi_1(\mathcal{E}(L))$ is thus finite if and only if $L = U$ or $L = H$, and in this case $\pi_1(\mathcal{E}(U))$ is $\mathbb{Z}/2$ while $\pi_1(\mathcal{E}(H))$ is Q_8 .

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