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Nonlinear Waves and Dispersive Equations (hybrid meeting)

Organized by
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Abstract. Nonlinear dispersive equations are models for nonlinear waves in a wide range of physical contexts. Mathematically they display an interplay between linear dispersion and nonlinear interactions, which can result in a wide range of outcomes from finite time blow-up to solitons and scattering. They are linked to many areas of mathematics and physics, ranging from integrable systems and harmonic analysis to fluid dynamics, geometry and general relativity and probability.

Mathematics Subject Classification (2010): 35xx, 37K, 76xx,

Introduction by the Organizers

The workshop was organized by Herbert Koch (Bonn), Pierre Raphaël (Cambridge), Daniel Tataru (Berkeley) and Monica Vişan (Los Angeles). It had around 45 participants. It was one of the first workshops in-person for many participants. Twenty four talks were presented, with a focus on young mathematicians.

The workshop's goal was to highlight and discuss a variety of intertwined developments across the field of dispersive PDE. Important directions are

(1) Recent developments in the global solvability and asymptotic behavior of quasilinear systems, including presentations by Alazard, Ifrim, Oh, Stingo and Szeftel.
(2) Quasilinear local well-posedness results at low regularity, for instance Ai's talk on nonlinear wave equations

(3) Dispersive equations with noise, one example being Bringmann's presentation on the invariant Gibbs measure for the cubic nonlinear wave equation in 3d related to the Φ^4_3 model.
(4) Integrable PDEs via a mix of integrable and PDE techniques. Examples include an exposé of the Calegero-Moser system by Gérard; global well-posedness for the derivative cubic NLS presented by Ntekoume; recent developments in treating the entire KdV hierarchy uniformly presented by Klaus; the KdV equation with step-like and other exotic initial data by Laurens; conservation laws for the Gross–Pitaevskii hierarchy presented by Liao. Pausader also presented an interesting development in the use of integrable systems in the theory of the Vlasov-Poisson system.
(5) Stability analysis of particular solutions via detailed analysis of the associated linearized problem, including presentations by Chen, Lührmann and Schörkhuber.

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Workshop (hybrid meeting): Nonlinear Waves and Dispersive Equations

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Abstracts

Multi–soliton Dynamics for the Calogero–Moser DNLS equation

Patrick Gérard
(joint work with Enno Lenzmann)

We define Sobolev–Hardy spaces on the line as

$$H^s_+(\mathbb{R}) := \{f \in H^s(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subset [0, +\infty[\} \ , \ s \in \mathbb{R}$$

and we denote by $\Pi_+ : H^s(\mathbb{R}) \to H^s_+(\mathbb{R})$ the orthogonal projector.
Setting $D := -i\partial_x$, the equation

$$i\partial_t u + \partial_x^2 u + (D + |D|)(|u|^2)u = 0 \ , \ u(t, .) \in H^s_+(\mathbb{R}) \ , \tag{1}$$

was introduced by Abanov, Bettelheim and Wiegmann [1] as a continuous limit of Calogero–Moser systems of classical particles. A defocusing version of (1) was previously derived by Pelinovsky and Grimshaw [5] from the intermediate long wave equation.

Equation (1) is preserved by

- Gauge, translations and mass critical dilations
$$\mathrm{e}^{i\theta}\lambda^{1/2}u(\lambda^2(t - t_0), \lambda(x - x_0)) \ .$$
- Galilean transformations $\mathrm{e}^{i\eta x - it\eta^2}u(t, x - 2\eta t)$ with the constraint $\eta \geq 0$ in order to preserve $H^s_+(\mathbb{R})$. The corresponding semigroup $(S(\eta))_{\eta \geq 0}$ defined as
$$S(\eta)f(x) = \mathrm{e}^{i\eta x}f(x), \ f \in H^0_+(\mathbb{R})$$
is called the Beurling–Lax semigroup on the Hardy space $H^0_+(\mathbb{R})$.

Furthermore, local wellposedness in $H^s_+(\mathbb{R})$, $s > \frac{3}{2}$, can be obtained via Kato's classical iterative scheme.
Before stating our first result, let us recall the definition of Toeplitz operators on $H^0_+(\mathbb{R})$. Given $b \in L^\infty(\mathbb{R})$, we define

$$T_b(f) := \Pi_+(bf) \ , \ f \in H^0_+(\mathbb{R}) \ .$$

Theorem 1. *If $u \in C(I, H^s_+(\mathbb{R}))$ is a solution of (1), with s large enough, then*

$$\frac{d}{dt}L_u = [B_u, L_u] \ ,$$

where L_u, B_u are the following operators on $H^0_+(\mathbb{R})$,

$$L_u := D - T_u T_{\overline{u}} \ , \ B_u = T_u T_{\partial_x \overline{u}} - T_{\partial_x u} T_{\overline{u}} + i(T_u T_{\overline{u}})^2 \ .$$

Notice that the above Lax pair structure has some similarity with the one of the Benjamin–Ono equation, studied e.g. in [3] on the torus and [6] on the line. As a consequence, we infer

Corollary 2. *The spectrum of $L_u = D - T_u T_{\overline{u}}$ is conserved by the dynamics of* (1)*. Furthermore, the spectral measure of u for L_u is a conservation law. If $u_0 \in H^2_+(\mathbb{R})$ with $\|u_0\|^2_{L^2} < 2\pi$, then u_0 generates a global solution of* (1)*, globally bounded in H^2.*

The last statement follows from the sharp inequality

$$\|T_{\overline{u}} f\|^2_{L^2} \le (2\pi)^{-1} \|u\|^2_{L^2} \langle Df, f\rangle \ , \ u \in H^0_+(\mathbb{R}) \ , \ f \in H^{1/2}_+(\mathbb{R}) \tag{2}$$

which allows us to control the $H^{1/2}$ norm of u from the conservation laws $\|u\|^2_{L^2}$ and $\langle L_u u, u\rangle$. Then conservation laws $\langle L_u^2 u, u\rangle$ and $\langle L_u^4 u, u\rangle$ yield a bound in H^2.

Since the Calogero–Moser DNLS equation (1) is mass critical, it is natural to define the Lax operator at the critical regularity. This is performed in the next result, which is based on inequality (2) too.

Theorem 3. *For every $u \in H^0_+(\mathbb{R})$, the Lax operator $L_u = D - T_u T_{\overline{u}}$ is a semi-bounded selfadjoint operator on $H^0_+(\mathbb{R})$. Every eigenvalue of L_u is simple, and the number N of eigenvalues of L_u is finite, with*

$$N \le \frac{\|u\|^2_{L^2}}{2\pi} \ .$$

In particular, if $\|u\|^2_{L^2} < 2\pi$, L_u does not have point spectrum.

Next we introduce the class of multi–soliton potentials.

Proposition 4. *Given $u \in H^0_+(\mathbb{R})$ and $N \in \mathbb{N}_{\ge 1}$, the following properties are equivalent and are preserved by the (CMDNLS) dynamics.*

1. *The sum $\mathcal{E}_{pp}(u)$ of the L_u eigenspaces is preserved by the adjoint Beurling–Lax semigroup $(S(\eta)^*)_{\eta \ge 0}$, is N–dimensional and contains u.*
2. *There exists a polynomial Q of degree N, with all its zeroes in $\mathbb{C}_-$, and a polynomial P of degree at most $N-1$, such that*

$$u(x) = \frac{P(x)}{Q(x)} \ , \ P(x)\overline{P}(x) = i(Q'(x)\overline{Q}(x) - \overline{Q}'(x)Q(x)) \ .$$

It is easy to check that the case $N = 1$ in the above proposition corresponds to

$$u(x) = \frac{\sqrt{2\mathrm{Im}p}}{x+p} \ , \ \mathrm{Im}p > 0 \ ,$$

and that these functions are stationary solutions of (1).
Using the Lax pair, we can derive an inverse spectral formula for multi–soliton potentials, which allows us to study the long time dynamics of such solutions.

Theorem 5. *Define G such that $S(\eta)^* = \exp(-i\eta G)$ for every $\eta \ge 0$. Every N-soliton potential u such that L_u has eigenvalues $\lambda_0 = 0, \lambda_1, \dots, \lambda_{N-1}$ can be recovered through the inverse spectral formula*

$$u(x) = \frac{\hat{u}(0^+)}{2i\pi} \langle (M - x\mathrm{Id})^{-1} X, Y\rangle_{\mathbb{C}^N} \ , \ \mathrm{Im}x > 0 \ ,$$

where $X := (1, \dots, 1)^T$, $Y := (1, 0, \dots, 0)^T$ *and, for* $0 \le j, k \le N-1$,

$$M_{jk} = \frac{i}{\lambda_j - \lambda_k} \ , \ j \neq k \ , \ M_{jj} = \gamma_j - i\rho\delta_{j0} \ ,$$

$$\rho := \frac{|\hat{u}(0^+)|^2}{8\pi^2} \ , \ \gamma_j := \mathrm{Re}\langle G\psi_j, \psi_j\rangle \ , \ L_u\psi_j = \lambda_j\psi_j \ , \ \langle u, \psi_j\rangle = \sqrt{2\pi} \ .$$

Furthermore, the Calogero–Moser DNLS evolution reads as

$$\frac{d}{dt}\hat{u}(0^+) = 0 \ , \ \frac{d}{dt}\gamma_j = 2\lambda_j \ , \ j = 0, \dots, N-1 \ .$$

Moreover, multi–soliton solutions of (1) *are defined for all time, and display the following energy cascades if* $N \ge 2$,

$$\|u(t)\|_{H^s} \simeq |t|^{2s} \ , \ s > 0 \ .$$

The above results, as well as other features of the Calogero–Moser DNLS equation, will be discussed in detail in the forthcoming preprint [4]. Let us conclude with some perspectives.

- The first natural open question is the long time behaviour for general solutions. Can finite time blow up occur ? Can one expect weak soliton resolution? This is closely related to inverse scattering theory for the pair (L_u, D) and is the object of a work in progress.
- The similar equation on the torus is quite natural, and is called the Calogero–Sutherland DNLS equation. This equation is studied in the PhD thesis of Rana Badreddine [2], where a general inverse spectral formula is derived, and where the flow map is continuously extended to the open ball of $H^0_+(\mathbb{T})$ with subcritical mass.

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Well-posedness of the Zakharov-Kuznetsov equation

Sebastian Herr

(joint work with Shinya Kinoshita)

We have discussed recent results concerning the well-posedness theory for the Zakharov-Kuznetsov equation

$$(1) \qquad \begin{aligned} \partial_t u + \partial_x \Delta u &= \partial_x u^2 \text{ in } (-T,T)\times\mathbb{R}^d \\ u(0,\cdot) &= u_0 \in H^s(\mathbb{R}^d) \end{aligned}$$

where $u = u(t,x,\mathbf{y}) : (-T,T)\times\mathbb{R}\times\mathbb{R}^{d-1} \to \mathbb{R}$ and Δ denotes the Laplacian with respect to $(x,\mathbf{y}) \in \mathbb{R}\times\mathbb{R}^{d-1}$. It is a model for wave propagation in a plasma. In [5], it has been rigorously derived from the Euler-Poisson system as a long-wave and small-amplitude limit.

It generalizes the Korteweg-de Vries equation (KdV) to higher dimensions, e.g. the KdV solitons are solutions (constant in $\mathbf{y}$). Further, in $d = 3$, if Q is the ground state of $Q - \Delta Q = Q^2$, then $(t,x,y,z) \mapsto Q(x-t,y,z)$ is a solitary wave solution, and its asymptotic stability has been proven recently [1, 6], conditional on the well-posedness in $H^1(\mathbb{R}^3)$ (which is established in [4], see below).

For real-valued solutions of (1), the L^2-norm and the energy

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla_{x,\mathbf{y}} u(t,x,\mathbf{y})|^2 dx d\mathbf{y} + \frac{1}{3}\int_{\mathbb{R}^d} u(t,x,\mathbf{y})^3 dx d\mathbf{y}.$$

The equation is invariant with respect to the rescaling

$$u_\lambda(t,x,\mathbf{y}) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda \mathbf{y}),$$

which implies that $s_c := (d-4)/2$ is the critical Sobolev regularity for (1).

The following theorem has been obtained recently by Kinoshita [2].

Theorem 1. *Let $d = 2$. (1) is locally well-posed for $s > -1/4$.*

This subcritical threshold $s = -1/4$ is almost optimal in the sense that for $s < -1/4$ the flow map cannot be C^2. In particular, Theorem 1 implies global well-posedness in $L^2(\mathbb{R}^2)$.

In collaboration with Kinoshita, we have obtained the following extension to higher dimensions [4].

Theorem 2. *Let $d \geq 3$. (1) is locally well-posed for $s > s_c$.*

In particular, Theorem 2 implies global well-posedness in $H^1(\mathbb{R}^3)$, $L^2(\mathbb{R}^3)$, and under an additional smallness condition also in $H^1(\mathbb{R}^4)$.

In the talk, we have first reviewed a number of previous results concerning the local and global well-posedness of the Cauchy problem, we refer to [4] for references. Then, we have focussed on the key ideas in Kinoshita's proof of Theorem 1:

In dimension $d = 2$, it is known from earlier work that one can symmetrize the equation to

$$(2) \qquad \begin{aligned} \partial_t v + (\partial_x^3 + \partial_y^3) v &= c(\partial_x + \partial_y) v^2 \text{ in } (-T,T)\times\mathbb{R}^2 \\ v(0,\cdot) &= v_0 \in H^s(\mathbb{R}^2) \end{aligned}$$

for some irrelevant constant $c \neq 0$. By the standard approach via $X^{s,b}$-norms the proof of Theorem 1 reduces to proving that

$$\Big|\int\int \hat{w}(\tau_0,\xi_0,\eta_0)\hat{v}_1(\tau_1,\xi_1,\eta_1)\hat{v}_2(\tau_0-\tau_1,\xi_0-\xi_2,\eta_0-\eta_2)\mathrm{d}(\tau_1,\xi_1,\eta_1)\mathrm{d}(\tau_0,\xi_0,\eta_0)\Big|$$
$$\lesssim N_{max}^{-\frac{5}{4}}(L_0L_1L_2)^{\frac{1}{2}}\|w\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2},$$

for all L^2-functions with Fourier-support in the sets defined by

$$|\tau_j-(\xi_j^3+\eta_j^3)|\sim L_j,\ |(\xi_j,\eta_j)|\sim N_j \quad j=0,1,2.$$

A key quantity is the resonance function

$$\begin{aligned}\Phi =&\xi_0^3+\eta_0^3-(\xi_1^3+\eta_1^3+(\xi_0-\xi_1)^3+(\eta_0-\eta_1)^3)\\ =&3(\xi_0\xi_1(\xi_0-\xi_1)+\eta_0\eta_1(\eta_0-\eta_1)),\end{aligned}$$

whose zeros are the (time) resonances. Here, one can see the fundamental difference with KdV, where this method of proof has been successfully applied more than 25 years ago. In the case of KdV, the resonance function is simply $3\xi_0\xi_1(\xi_0-\xi_1)$, with only trivial zeros. Kinoshita's paper [2] is the first to exploit the structure of this significantly more complicated resonant set of (2). To deal with the most difficult cases, Kinoshita has devised an almost orthogonal decomposition (of Whitney-type) into certain tiles, where the interaction is either non-resonant (i.e. Φ bounded from below), or the interaction is transverse (so that a nonlinear Loomis-Whitney inequality can be applied). We refer to [2] for the details of this construction.

This has been the starting point for us to consider dimension $d=3$ and higher. The general strategy of proof of the results in [4] is similar and we construct, among other things, higher dimensional almost orthogonal decompositions to prove the subcritical results.

Finally, we have briefly discussed the following high-dimensional result in critical spaces obtained jointly with Kinoshita in [3].

Theorem 3. (1) *is globally well-posed for small initial data in* $\dot{B}^{s_c}_{2,1}(\mathbb{R}^5)$ *and in* $\dot{H}^{s_c}(\mathbb{R}^d)$ *for* $d\geq 6$*, respectively, and the solutions scatter to free solutions.*

There is an analogous result in the subspace of radial (with respect to $\mathbf{y}$) functions in $\dot{H}^{s_c}(\mathbb{R}^4)$.

One of the fundamental ingredients, besides transversal estimates, is the following Strichartz type estimate: Let $d\geq 3$ and (q,r) a Schrödinger admissible pair in dimension $(d-1)$, then

$$\big\|D_x^{\frac{1}{q}}e^{t\partial_x\Delta}f\big\|_{L^q_tL^r_{\mathbf{y}}L^2_x}\lesssim\|f\|_{L^2}.$$

For the proof, consider the rescaled Schrödinger propagator $V_\xi(t)f(\mathbf{y}) := e^{it\xi t\Delta_{\mathbf{y}}}f(\mathbf{y})$, where ξ is the (fixed) frequency in the x direction. We refer to [3] for details.

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Invariant Gibbs measures for the three-dimensional cubic nonlinear wave equation

Bjoern Bringmann

(joint work with Yu Deng, Andrea R. Nahmod, Haitian Yue)

In this talk, we discuss the invariance of the Gibbs measure for the three-dimensional cubic nonlinear wave equation. In the beginning of this talk, we discuss the connections of this problem with various problems in constructive quantum field theory, singular SPDEs, and random dispersive equations, which concern different Φ^4_d-models. The starting point of our discussion is the energy

$$E(\phi) = \int_{\mathbb{T}^d} \mathrm{d}x \bigg(\frac{|\phi|^2}{2} + \frac{|\nabla \phi|^2}{2} + \frac{|\phi|^4}{4} \bigg). \tag{1}$$

In (1), we omit the possible renormalization of the potential energy term, since its precise form depends on the dimension d. Equipped with (1), we formally define the Φ^4_d-measure as

$$\text{“}\mathrm{d}\Phi^4_d(\phi) = \mathcal{Z}^{-1} \exp\bigg(- \int_{\mathbb{T}^d} \mathrm{d}x \bigg(\frac{|\phi|^2}{2} + \frac{|\nabla \phi|^2}{2} + \frac{|\phi|^4}{4} \bigg)\bigg) \mathrm{d}\phi.\text{”} \tag{2}$$

In addition to the Φ^4_d-measure, the energy in (1) induces three different evolution equations, which are called dynamical Φ^4_d-models:

(i) A Langevin equation, which is given by the cubic heat equation with space-time white noise,
(ii) a real-valued Hamiltonian equation, which is given by the cubic wave equation,
(iii) and a complex-valued Hamiltonian equation, which is given by the cubic Schrödinger equation.

The evolution equations described in (i), (ii), and (iii) are also known as the parabolic, hyperbolic, and Schrödinger Φ^4_d-models, respectively. The main problems then concern the construction of the Φ^4_d-measure, the probabilistic well-posedness

Dimension	Measure	Heat	Wave	Schrödinger
$d=1$		[Iwa87]	[Zhi94]	[Bou94]
$d=2$	[Nel66]	[DPD03]	[Bou99]	[Bou96]
$d=3$	[GJ73]	[Hai14]	*This talk*	*Open*
$d=4$	[ADC21]			
$d\geq 5$	[Aiz81, Fro82]			

FIGURE 1. Existence and invariance of the Gibbs measure for the cubic stochastic heat, wave, and Schrödinger equations.

of the three evolution equations, as well as the invariance of (minor variants of) the Φ^4_d-measure under the three evolution equations. The extensive literature on Φ^4_d-models, which spans over six decades, is illustrated in Figure 1.

We now state the main result of this talk. To this end, let $N\geq 1$ be a frequency-truncation parameter. The frequency-truncated and renormalized three-dimensional cubic nonlinear wave equation is given by

$$\begin{cases} (\partial_t^2+1-\Delta)u_{\leq N} = -P_{\leq N}\Big[:(P_{\leq N}u_{\leq N})^3: + \gamma_{\leq N}\cdot P_{\leq N}u_{\leq N}\Big] \\ (u_{\leq N}, \langle\nabla\rangle^{-1}\partial_t u_{\leq N})\big|_{t=0} = (\phi^{\cos},\phi^{\sin}). \end{cases} \tag{3}$$

Here, $P_{\leq N}$ denotes a (sharp) frequency-truncation, the dots $:$ indicate the Wick-ordering, and $\gamma_{\leq N}$ denotes a (further) renormalization constant. Furthermore, let $\mu_{\leq N}$ be the corresponding frequency-truncated Gibbs measure, whose first marginal is given by a frequency-truncated Φ^4_3-measure. It is known that $(\mu_{\leq N})_N$ weakly converges to a unique limit, which is denoted by μ.

Theorem 1 (Global well-posedness and invariance, rigorous version)**.** *For any frequency-scale $N\geq 1$ and $(\phi^{\cos},\phi^{\sin})\in\mathscr{H}_x^{-1/2-\epsilon}(\mathbb{T}^3)$, let $u_{\leq N}$ be the solution of the frequency-truncated cubic wave equation* (3) *with initial data $u_{\leq N}[0] = (\phi^{\cos},\phi^{\sin})$. In addition, let μ be the Gibbs measure from above. Then, for μ-almost every $(\phi^{\cos},\phi^{\sin})$ and all $T\geq 1$, the limiting dynamics*

$$u[t] = \lim_{N\to\infty} u_{\leq N}[t] \tag{4}$$

exists in $C_t^0\mathscr{H}_x^{-1/2-\epsilon}([-T,T]\times\mathbb{T}^3)$. Furthermore, the Gibbs measure is invariant under the limiting dynamics, i.e.,

$$\mathrm{Law}_\mu\big(u[t]\big) = \mu \tag{5}$$

for all $t\in\mathbb{R}$.

In later parts of this talk, we discuss the following aspects of our proof: First, we discuss a caloric representation of the Gibbs measure. This caloric representation is inspired by Tao's caloric gauge [Tao04] and obtained using the parabolic Φ^4_3-model, i.e., the three-dimensional cubic stochastic heat equation.

Then, we discuss the para-controlled Ansatz for the solution $u_{\leq N}$, which takes the form

$$u_{\leq N} = \mathbf{I}_{\leq N} - \mathbf{Y}_{\leq N} - \mathbf{I}_{\leq N} + 3\,\mathbf{\Psi}_{\leq N} + X^{(1)}_{\leq N} + X^{(2)}_{\leq N} + Y_{\leq N}. \tag{6}$$

The first four summands in (6) are explicit stochastic objects, the fifth and sixth summands are para-controlled components, and the last summand is a nonlinear remainder. At the end of this talk, we discuss further aspects regarding explicit stochastic objects, such as a hidden cancellation between sextic stochastic objects.

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The 1d cubic NLS with a non-generic potential

Gong Chen
(joint work with Fabio Pusateri)

In this extended abstract, we report the work by the author and F. Pusateri on the long-time asymptotics of small solutions to the one-dimensional cubic nonlinear Schrödinger equation with a non-generic potential:

$$i\partial_t u - \partial_{xx} u \pm |u|^2 u + V(x)u = 0,\ u(0) = u_0, \tag{1}$$

where $u_0 \in H^{1,1}(\mathbb{R})$.

1. Main results

Denote the Schrödinger operator as

$$H := -\partial_{xx} + V.$$

The potential is said to be *non-generic*: there is a globally bounded solution $\varphi = \varphi(x)$ of $H\varphi = 0$ which is call the zero-energy resonance. Our assumptions on the potential are followings:

- We assume that V is non-generic and that the zero energy resonance is either even or odd;
- V has no discrete spectrum;
- Assume that for $\gamma > \frac{5}{2}$

  $$\langle x\rangle^{\gamma} V \in L^1.$$

  In particular, no regularity on V.

Theorem 1 (G.C.–F. Pusateri 2022). *Consider the cubic NLS* (1) *with a potential such that H satisfying our assumptions above. There exists $0 < \epsilon_0 \ll 1$ only depending on prescribed constants such that for all $\eta \le \epsilon_0$ and*

$$\|u_0\|_{H^{1,1}} = \|u_0\|_{H^1} + \|xu_0\|_{L^2} = \eta$$

the equation with initial data u_0 has a unique global solution satisfying the sharp decay rate

$$\|u(t)\|_{L^\infty_x} \lesssim \eta \frac{1}{(1+|t|)^{\frac{1}{2}}}.$$

Moreover, if we define the profile of the solution u as

$$f(t,x) := e^{-itH} u(t,x), \quad H := -\partial_{xx} + V, \tag{2}$$

then, for $\alpha \in (0, 1/4)$ we have the global bounds

$$\big\|(\mathcal{F}f)(t)\big\|_{L^\infty_k} + (1+|t|)^{-\alpha} \big\|\partial_k(\mathcal{F}f)(t)\big\|_{L^2_k} \lesssim \eta, \tag{3}$$

where, in the case of an even, respectively odd, zero energy resonance, $\mathcal{F}$ denotes the distorted Fourier transform $\widetilde{\mathcal{F}}$ associated to H, respectively the 'modified' distorted Fourier transform $\mathcal{F}^\sharp$.

Corollary 2. *With the notations above, we have the following asymptotic formulae for u.*

$$u\,(t,x)=\frac{e^{i\frac{x^2}{4t}}}{\sqrt{-2it}}(\mathcal{F}f)\left(t,-\frac{x}{2t}\right)+\mathcal{O}\big(\left\|\partial_k(\mathcal{F}f)(t)\right\|_{L^2_k}t^{-\frac{3}{4}}\big)$$

and

$$u\,(t,x)=\frac{e^{i\frac{x^2}{4t}}}{\sqrt{-2it}}\exp\left(-\frac{i}{2}\left|W_{+\infty}\left(-\frac{x}{2t}\right)\right|^2\log t\right)W_{+\infty}\left(-\frac{x}{2t}\right)+\mathcal{O}\left(\eta t^{-\frac{1}{2}-\delta}\right),\quad t\geq 1.$$

Here we record two important remarks.

- From the equation and the uniqueness of ODE, the parity of the resonance $\Longleftrightarrow$ the eveness of the potential (a.e.).
- Even potentials with zero energy resonances appear in the linearization around special solutions for many models. For example, Pöschl-Teller potentials

$$H_n:=-\partial_{xx}-n(n+1)\text{sech}^2(x),\qquad n\in\mathbb{N}_0,\tag{4}$$

  comes up in several classical asymptotic stability problems, such as the linearization around kink solutions of the sine-Gordon equation (H_1) and the ϕ^4 model (H_2), and solitons of quadratic (H_3) and cubic (H_2) Klein-Gordon equations. $V=0$ is not generic.

2. Main ideas

Following Germain-Pusateri-Rousset [3] and Chen-Pusateri [1], we use the "generalized plane waves" $\mathcal{K}(x,\lambda)$ such that one can define an L^2 unitary transformation $\widetilde{\mathcal{F}}$ by

$$\widetilde{\mathcal{F}}[f](\lambda):=\widetilde{f}(\lambda):=\int\overline{\mathcal{K}(x,\lambda)}f(x)\,dx,\quad\text{with}\quad\widetilde{\mathcal{F}}^{-1}\left[\phi\right](x)=\int\mathcal{K}(x,\lambda)\phi(\lambda)\,d\lambda.$$

The distorted transform $\widetilde{\mathcal{F}}$ diagonalizes the Schrödinger operator:

$$(-\partial_{xx}+V)=\widetilde{\mathcal{F}}^{-1}\lambda^2\widetilde{\mathcal{F}}.$$

Given a solution u, let $f=e^{-itH}u$ be its linear profile, so that $\widetilde{f}(t,k)=e^{itk^2}\widetilde{u}(t,k)$. A natural strategy, employed in several previous works, e.g. [1,3], is to prove bounds on $\widetilde{f}$ to infer bounds on u, using, for example, the basic linear estimate (stationary phase)

$$\|u(t,\cdot)\|_{L^\infty_x}\lesssim\frac{1}{|t|^{1/2}}\|\widetilde{f}(t)\|_{L^\infty_k}+\frac{1}{|t|^{3/4}}\|\partial_k\widetilde{f}(t)\|_{L^2_k}.\tag{5}$$

The immediate obstacle is the possible discontinuity of the distorted Fourier transform. The resonance ϕ is normalized s.t. $\phi(+\infty)=1$ and let $a:=\phi(-\infty)$. Then $\widetilde{f}(0-)=\frac{1}{a}\widetilde{f}(0+)$, whence one can not compute $\partial_k\widetilde{f}$ at $k=0$ when the zero energy resonance is odd. Note that

- In the generic setting: $\tilde{f}(t,0)=0$, $\forall t$.

Overall, in the non-generic setting, we have to face the following difficulties

- Potential discontinuity of the dFT at zero
- No local improved decay rate
- the structure of the nonlinear spectral measure is more complicated.

In this work, we introduce the following new ideas to resolve the difficulties above:

- a simple modification of the basis for the distorted Fourier transform to resolve the (possible) discontinuity at zero energy due to the presence of a resonance,
- a detailed analysis of the nonlinear spectral measure and, in particular, of its low-frequency structure, and
- general "smoothing estimates" for Schrödinger operators with non-generic potentials.

In the setting with an odd resonance, one has

$$\lim_{k\to 0^+} \tilde{f}(k) = -\lim_{k\to 0^-} \tilde{f}(k). \tag{6}$$

We introduce a *modified* version of the distorted Fourier base, $\mathcal{K}^{\#}(x,k) = \mathrm{sign}(k)\mathcal{K}(x,k)$ and its associated transform $\mathcal{F}^{\#}$. This transform has a nice Fourier theory.

We note that:

- this modified Fourier basis is Lipschitz at $k = 0$;
- the same linear estimate holds when we replace all $\widetilde{f}$ by $f^{\#} := \mathcal{F}^{\#}[f]$.

This gives us the basic setting to carry out our analysis. Then we perform careful and refined analysis to obtain

- $f^{\#}(t,k)$ uniformly in k and t.
- the L^2_k-norm of $\partial_k f^{\#}(t)$ with a small growth in t.

To overcome the slow decay, one of key tools is the smoothing estimate which can be stated as follows: assuming $\mathcal{Q}(x,k)$ is a bounded function, and letting $\phi = \phi(k)$ be a function such that $|\phi(k)| \lesssim \sqrt{|k|}$, we have

$$\left\| \int \left[\int e^{-ik^2 s} \phi(k) \overline{\mathcal{Q}}(y,k) F(s,y)\, dy \right] ds \right\|_{L^2_k} \lesssim \|F\|_{L^1_y L^2_s}. \tag{7}$$

This estimate can be proven using the Fourier transform in time, and may be regarded as the dual version of the classical Kato-$\frac{1}{2}$ smoothing estimate. The key point is the L^2_t norm on the right-hand side of (7), which is sufficient to make up for the lack of local decay.

These types of estimates and their refined versions have more applications. For example, in the proof of asymptotic stability of small solitons when the cubic NLS is perturbed by a trapping potential, see [2]

3. Application to the Klein-Gordon models

Our analysis naturally can be applied to Klein-Gordon models. Since the nonlinear spectral measures are the same and low-frequency structures of Klein-Gordon models are similar to the Schrödinger flow.

- Cubic Klein-Gordon equations:

$$\partial_t^2\phi - \partial_x^2\phi + m^2\phi + V(x)\phi = \phi^3, \qquad (\phi,\phi_t)(0) = (\phi_0,\phi_1), \tag{8}$$

with a potential $V = V(x)$ that is non-generic, and satisfies the assumptions. There exists $0 < \epsilon_0 \ll 1$ such that for all $\eta \leq \epsilon_0$ and initial data satisfying

$$\left\|(\sqrt{H+m^2}\phi_0, \phi_1)\right\|_{H^4} + \left\|\langle x\rangle(\sqrt{H+m^2}\phi_0, \phi_1)\right\|_{L^2} \leq \eta, \tag{9}$$

The solution satisfies the sharp decay rate

$$\|\phi(t)\|_{L^\infty_x} \lesssim \eta\,(1+|t|)^{-\frac{1}{2}} \tag{10}$$

and has a modified scattering behavior.

Extensions to quadratic KG models are more involved. It requires a normal form transformation.

- Consider the quadratic model

$$\partial_t^2\phi - \partial_x^2\phi + \phi + V(x)\phi = a(x)\phi^2 + b(x)\phi^3, \qquad (\phi, \phi_t)(0) = (\phi_0, \phi_1), \tag{11}$$

for ϕ, V and (ϕ_0, ϕ_1) as above, and a localized coefficient $a = a(x)$ (Possible to do nonlocalized coeff but more involved.).

Lindblad-Lührmann-Soffer [4] gives global bounds and asymptotics for solutions of the equation with $V = 0$ and under the 'non-resonance' assumption $\widehat{a\phi^2}(\pm\sqrt{3}) = 0$.

Lindblad-Lührmann-Schlag-Soffer [5] gives global bounds (for localized L^2-norms) and sharp linear decay for solutions with $b = 0$ and under the 'non-resonance' assumption $\big(\widetilde{\mathcal{F}}(a\phi^2)\big)(\pm\sqrt{3}) = 0$. (No symmetry assumptions on the resonance)

Our approach can extend the above results to include a non-trivial potential in [4] or, equivalently, cubic terms in the cited results from [5] provided the zero energy resonance is either even or odd (and the same non-resonance assumption $\big(\widetilde{\mathcal{F}}(a\phi^2)\big)(\pm\sqrt{3}) = 0$ holds).

Asymptotic stability of the kink solution to the sine-Gordon equation has special properties.

- In important recent work, Lührmann-Schlag [6] considered the sine-Gordon equation

$$\partial_t^2\phi - \partial_x^2\phi + \sin\phi = 0, \tag{12}$$

and proved asymptotic stability for small and localized odd perturbation of the kink $K(x) = 4\arctan(e^x)$. Essentially it is equivalent to analyze odd solutions of

$$\partial_t^2 u + H_1 u + u = a(x)u^2 + u^3, \tag{13}$$

The resonance of H_1 is odd. The key 'non-resonance' condition $(\widetilde{\mathcal{F}}a\phi^2)(\pm\sqrt{3}) = 0$ holds for this model.

In [6], Lührmann-Schlag made use of a factorization property, called the 'super-symmetric' factorization, of the linear operator that conjugates it to the flat one. (Still non-generic)

Our results show that it is actually possible to approach this type of problems without resorting to factorization properties.

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Well-posedness for the KdV hierarchy

Friedrich Klaus

(joint work with Herbert Koch, Baoping Liu)

The Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} - 6uu_x = 0 \tag{1}$$

is a fascinating, canonical dispersive equation, and one of the simplest completely integrable, nonlinear PDE. The complete integrability of KdV gives rise to an infinite sequence of formally conserved energies H_N^{KdV}. This makes KdV a member of the KdV hierarchy: Consider the Hamiltonian function

$$H_1^{\mathrm{KdV}}(u) = \frac{1}{2}\int u_x^2 + 2u^3 dx.$$

With this definition (1) can be written as the Hamiltonian equation

$$u_t = \partial_x \frac{\delta}{\delta u} H_N^{\mathrm{KdV}}(u), \tag{2}$$

where $N = 1$. Likewise we can define the Nth KdV equation as the Hamiltonian equation associated to the Hamiltonian function H_N^{KdV} by (2), and we call the family of these equations the KdV hierarchy.

It is instructive to write down the first few Hamiltonians in the hierarchy:

$$H_{-1}^{\mathrm{KdV}}(u) = \frac{1}{2}\int u\,dx, \qquad H_0^{\mathrm{KdV}}(u) = \frac{1}{2}\int u^2\,dx,$$

$$H_1^{\mathrm{KdV}}(u) = \frac{1}{2}\int u_x^2 + 2u^3\,dx, \qquad H_2^{\mathrm{KdV}}(u) = \frac{1}{2}\int u_{xx}^2 + 10uu_x^2 + 5u^4\,dx.$$

It becomes visible that the Nth KdV equation takes the form of a dispersive PDE of $2N+1$th order,

$$u_t = (-1)^N u^{(2N+1)} + \partial_x F_N(u),$$

for some differential polynomial $F(u)$.

In their seminal work [4], Killip-Visan showed that the KdV flow map $\Phi_1 : \mathbb{R}\times\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ extends continuously to a map $\mathbb{R}\times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$. This result is sharp: continuity breaks down in $H^s(\mathbb{R})$ when $s < -1$ [2,3].

They also proved that for the flow map of the second member of the KdV hierarchy, that is (2) with $N = 2$, there exists a continuous extension from $\mathcal{S}(\mathbb{R})$ to a Sobolev space at higher regularity, $\Phi_2 : \mathbb{R}\times L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Making use of local smoothing estimates, this result was improved to $H^{-1}(\mathbb{R})$ by Bringmann-Killip-Visan [1].

We extend these results to the whole hierarchy:

Theorem 1. *Let $N \geq 3$. Then there exists a well-defined flow map $\Phi_N : \mathbb{R}\times\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ associated to (2), which extends to a continuous map*

$$\mathbb{R}\times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R}).$$

The method developed in [4] and [1] is now called method of commuting flows. The central part in this method is played by the quantity

$$\alpha(\tau,u) = -\log T(i\tau,u) - (2\tau)^{-1}\int_{\mathbb{R}} u\,dx,$$

which is conserved by all KdV flows. Here, $T(z,u)$ is the KdV transmission coefficient. The quantity $\alpha(\tau,u)$ turns out to be analytic in $u\in H^{-1}(\mathbb{R})$, meaning that by the Cauchy-Lipschitz theorem an associated Hamiltonian flow can be defined. On the other hand, the KdV Hamiltonians are defined in terms of an asymptotic expansion,

$$-\log T(z,u) \sim 2i\sum_{k=-1}^{\infty}(2z)^{-2k-3}H_k^{\mathrm{KdV}}(u).$$

This motivates that the flow associated to the Hamiltonian

$$H_\tau(u) = 16\tau^5\big(\alpha(\tau,u) + (2\tau)^{-3}H_0^{\mathrm{KdV}}(u)\big)$$

is well-defined in $H^{-1}(\mathbb{R})$, and approximates the one of $H_1^{\mathrm{KdV}}(u)$ as $\tau\to\infty$. Similarly, if

$$\mathcal{T}_N^{\mathrm{KdV}}(i\tau,u) = \sum_{n=-1}^{N}(2i\tau)^{2N-2n}H_n^{\mathrm{KdV}}(u) + \frac{(2i\tau)^{2N+3}}{2i}\log T(i\tau,u),$$

then $-4\tau^2\mathcal{T}_N(i\tau,u) \to H_{N+1}(u)$ as $\tau\to\infty$. Thus if we already know that the first N KdV Hamiltonians give rise to a well-defined flow on $H^{-1}(\mathbb{R})$, then so will $\mathcal{T}_N(i\tau,u)$, and we may hope to show convergence of the flows as $\tau\to\infty$.

To this end the recipe developed in [1,4] involves proving that precompactness in $H^{-1}(\mathbb{R})$ is conserved along orbits of (2), and showing weak convergence to the identity of the flow associated to the difference Hamiltonian $H_{N+1}(u) - 4\tau^2\mathcal{T}_N(i\tau,u)$.

This means that we need local smoothing estimates both for the flow of (2) and the difference flow.

Our novel approach to this issue is to transform these questions formulated at low regularity for the KdV hierarchy into questions formulated at higher regularity for the Gardner hierarchy. The reason for doing so is because at $L^2(\mathbb{R})$ regularity, the issue of local smoothing can be resolved in a more elementary way using the equation.

Connecting the KdV and Gardner hierarchies is done by making use of the modified Miura map,

$$M(\tau_0, w) = w_x + 2\tau_0 w + w^2, \tag{3}$$

which formally maps solutions w of the Nth τ_0-Gardner equation to solutions $u = M(\tau_0, w)$ of the Nth KdV equation. It turns out that this map is a diffeomorphism from $L^2(\mathbb{R})$ to its range, which defines an open set in $H^{-1}(\mathbb{R})$, meaning that by scaling down the initial value in $H^{-1}(\mathbb{R})$, we can always find a preimage for (3). This shows that it is enough to prove a version of Theorem 1 on the Gardner side:

Theorem 2. *Let $N \geq 3$. Then there exists a well-defined flow map $\Phi_N^{\text{Gardner}} : \mathbb{R} \times \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ associated to the Nth τ_0-Gardner equation, which extends to a continuous map*

$$\mathbb{R} \times L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Solutions obtained by Theorem 1 and Theorem 2 can shown to be distributional solutions, because the local smoothing effect associated to the equations is strong enough to make sense of the nonlinearity. This raises the question whether such distributional solutions are unique. We show that they are:

Theorem 3. *Let $N \geq 1$. Given $w_0 \in L^2(\mathbb{R})$, there exists a unique weak solution $w \in C(\mathbb{R}, L^2(\mathbb{R}))$ to the Nth Gardner equation with $w(0) = w_0$.*

Here a weak solution is defined to be a distributional solution satisfying an additional uniform decay assumption with respect to the local smoothing effect. This is needed to gain continuity in $L^2(\mathbb{R})$ and to make sense of initial values.

Making use of the modified Miura map, we can obtain a similar statement for the KdV hierarchy. We need slightly more regularity for the uniqueness statement due to mapping properties of the modified Miura map on weak solutions.

Theorem 4. *Let $N \geq 1$. Given $u_0 \in L^2(\mathbb{R})$, there exists a unique weak solution $u \in C(\mathbb{R}, L^2(\mathbb{R}))$ to the Nth KdV equation (2) with $w(0) = w_0$.*

Finally, by connecting the good variables introduced in [4] to the Gardner variables w in terms of the transformation

$$w = W(v) = \tau_0 v - \frac{1}{2}\partial_x \log(1 + v),$$

which is an analytic diffeomorphism as well, we can transform Theorem 4 into a result for the good variable hierarchy.

Theorem 5. *Let $N \geq 1$. Given $v_0 \in H^1(\mathbb{R})$, there exists a unique weak solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ to the Nth good variable equation with $v(0) = v_0$.*

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Global well-posedness for the derivative nonlinear Schrödinger equation

Maria Ntekoume
(joint work with Benjamin Harrop-Griffiths, Rowan Killip, Monica Visan)

We consider the derivative nonlinear Schrödinger equation

$$i\partial_t q + q'' + i\big(|q|^2 q\big)' = 0 \tag{DNLS}$$

which describes the evolution of a complex-valued field q defined either on the line $\mathbb{R}$ or the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. This equation is L^2-critical with respect to the scaling

$$q(t,x) \mapsto q_\lambda(t,x) := \sqrt{\lambda}\, q(\lambda^2 t, \lambda x). \tag{1}$$

The L^2 norm is also preserved under the (DNLS) evolution. In fact, this equation has an infinite family of conservation laws, the first two being the L^2 norm and the Hamiltonian:

$$M(q) = \int |q(x)|^2\, dx,$$
$$H(q) = -\tfrac{1}{2} \int i(q\bar{q}' - \bar{q}q') + |q|^4\, dx.$$

We say that the equation is *completely integrable.*

The above suggest L^2 as a natural and interesting space to study the global well-posedness of this equation. Nevertheless, not much was known below $H^{\frac{1}{2}}$. Local well-posedness is known in H^s for $s \geq \frac{1}{2}$ ([8]), but uniformly continuous dependence on the initial data has been shown in [2] to fail when $s < \frac{1}{2}$. One would hope that the existence of infinitely many conserved quantities would make it easy to upgrade local well-posedness results to global. However, all conserved quantities except the L^2 norm are not coercive. As a result, until recently all global well-posedness results in H^s, $s \geq \frac{1}{2}$, required a smallness assumption on the L^2 norm of the initial data, namely that $\|q(0)\|_{L^2}^2 < 4\pi$ ([3,7]). We will see that 4π is precisely the threshold past which the conservation laws fail to control the Sobolev norms H^s for any $s > 0$. A big breakthrough in this direction came in the work of Bahouri and Perelman [1] who showed global well-posedness in $H^{\frac{1}{2}}$ on the line for arbitrarily large initial data.

In this talk we outline some results from a series of works exploring the well-posedness of (DNLS) below $H^{\frac{1}{2}}$.

First, in a joint work [6] with Rowan Killip and Monica Visan, we identified equicontinuity as an essential step towards understanding well-posedness in low regularity. We proved L^2-equicontinuity of (DNLS) orbits, under some assumption on the L^2 norm of the initial data.

Theorem 1 (Killip–N.–Visan)**.** *Let $Q \subseteq L^2$ be an equicontinuous set satisfying*

$$\sup\Big\{ \|q_0\|_{L^2}^2 : q_0 \in Q \Big\} < 4\pi.$$

Then the totality of states reached by (DNLS) *orbits originating from Q*

$$Q_* = \{ e^{tJ\nabla H} q_0 : q_0 \in Q \text{ and } t \in \mathbb{R} \}$$

is also L^2-equicontinuous.

This allowed us to obtain a priori H^s estimates for $0 < s < \frac{1}{2}$ and establish global well-posedness in $H^{\frac{1}{6}}$, both on the line and the circle.

Theorem 2 (Killip–N.–Visan)**.** *Fix $0 < s < \frac{1}{2}$ and let Q be a bounded subset of H^s satisfying*

$$\sup\Big\{ \|q_0\|_{L^2}^2 : q_0 \in Q \Big\} < 4\pi.$$

Then

$$\sup_{q \in Q_*} \|q\|_{H^s} < C\left(\sup_{q_0 \in Q} \|q_0\|_{L^2}^2, \sup_{q_0 \in Q} \|q_0\|_{H^s}^2 \right)$$

Moreover, if Q is H^s-equicontinuous, then so is Q_*.

Theorem 3 (Killip–N.–Visan)**.** *Fix $\frac{1}{6} \leq s < \frac{1}{2}$. The* (DNLS) *evolution is globally well-posed in $\mathbb{R}$ and $\mathbb{T}$ for all*

$$q_0 \in H^s \quad \text{with} \quad \|q_0\|_{L^2}^2 < 4\pi.$$

Note that the L^2 norm assumption in Theorems 2 and 3 is required to ensure the equicontinuity of the orbits. Although this remains the best known result on the circle, it was not long before an equicontinuity result without this assumption was obtained on the line in [5], also yielding large data global well-posedness in $H^{\frac{1}{6}}(\mathbb{R})$.

Theorem 4 (Harrop-Griffiths–Killip–Visan)**.** *Let $Q \subseteq L^2(\mathbb{R})$. If Q is equicontinuous, then Q_* is also equicontinuous.*

Corollary 5. *Fix $\frac{1}{6} \leq s < \frac{1}{2}$. The* (DNLS) *evolution is globally well-posed in $H^s(\mathbb{R})$.*

More recently, in [4] we were able to prove global well-posedness in the critical space $L^2(\mathbb{R})$.

Theorem 6 (Harrop-Griffiths–Killip–N.–Visan)**.** *The* (DNLS) *evolution is globally well-posed in $L^2(\mathbb{R})$.*

Our methods rely on the completely integrable structure of the equation. Based on the Lax pair for (DNLS), we consider the perturbation determinant

$$a(\kappa; q) = \det[1 - i\kappa\Lambda\Gamma]$$

where $\Lambda(\kappa; q) = (\kappa - \partial)^{-\frac{1}{2}} q (\kappa + \partial)^{-\frac{1}{2}}$, $\Gamma(\kappa; q) = (\kappa + \partial)^{-\frac{1}{2}} \bar{q} (\kappa - \partial)^{-\frac{1}{2}}$, as well as its logarithm

$$\alpha(\kappa; q) = -\log\det(1 - i\kappa\Lambda\Gamma) = \sum_{\ell \geq 1} \tfrac{1}{\ell} \operatorname{tr}\left\{(i\kappa\Lambda\Gamma)^{\ell}\right\}.$$

This is preserved under (DNLS) for all κ sufficiently large, giving us a new family of conserved quantities which are better adapted to the study of this equation at low regularity. For instance, defining $\beta(\kappa; q) = \|q\|_{L^2}^2 - 2\operatorname{Im}\alpha(\kappa; q)$ it becomes clear that these new conservation laws hold the key to equicontinuity: $\beta(\kappa)$ is conserved and its leading term is $\beta^{[2]}(\kappa; q) = \int_{\mathbb{R}} \frac{\xi^2 |\hat{q}(\xi)|^2}{4\kappa^2 + \xi^2} d\xi$ which captures the L^2 norm of the high frequency part of q.

The new conserved quantities $\alpha(\kappa)$ are also essential for proving well-posedness. We follow the commuting flows method. Inspired by the asymptotic expansion

$$\alpha(\kappa; q) = \tfrac{i}{2} M(q) + \tfrac{1}{4\kappa} H(q) + O(\kappa^{-2}),$$

we are led to consider the Hamiltonians

$$H_\kappa(q) := 4\kappa \operatorname{Re}\alpha(\kappa; q) = H(q) + O(\kappa^{-1}).$$

The flows induced by these Hamiltonians commute with each other and with (DNLS). Moreover, it is not hard to prove that they are globally well-posed in H^s for all $s > 0$. The main difficulty of the proof lies in showing that the flow induced by the difference of the Hamiltonians $H - H_\kappa$ converges to the identity as $\kappa \to \infty$. This is what dictates the threshold $s = \frac{1}{6}$ in Theorem 3.

In order to reach the critical space L^2 and prove Theorem 6, we discover and exploit a local smoothing property, both for (DNLS) and for the difference flows. One notable feature of our local smoothing estimates is that they hold only for equicontinuous families. Our conserved quantities $\alpha(\kappa)$ play an important role once again in proving local smoothing, but this time we rely on a microscopic version of these conservation laws. The proof follows the same general idea both for (DNLS) and the difference flows, but the latter is much more challenging: the difference flows become increasingly non-dispersive as $\kappa \to \infty$ so we have to exploit an enormous amount of cancellations to exhibit local smoothing.

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The nonlinear stability of Kerr for small angular momentum

Jérémie Szeftel

The celebrated Kerr metrics, discovered by R. Kerr in 1963, form a 2-parameter family $\mathbf{g}_{m,a}$, $|a| \leq m$, of solutions to the Einstein vacuum equations. This family contains the 1-parameter family of Schwarzschild metrics - discovered by K. Schwarzschild in 1915 - as a special case (i.e. $a = 0$, $m > 0$) and corresponds to rotating black holes (while the Schwarzschild family corresponds to the particular case of non rotating black holes). A central conjecture in the field of general relativity states that the exterior region of Kerr - i.e. the region outside of the Kerr black hole - is nonlinearly stable under the subextremal restriction $|a| < m$.

This conjecture has generated an intense activity first in physics in the sixties and seventies with black holes becoming a mainstream subject of research, and then in mathematical GR. The physics literature has focused on the problem of mode stability for the linearized Einstein equations culminating with the work of B. Whiting in 1989 showing the absence of exponentially growing modes. In the wake of the monumental proof of the stability of Minkowski by D. Christodoulou and S. Klainerman, mathematical GR has investigated the derivation of quantitative decay for model problems of increasing difficulty starting with the scalar wave equation on Kerr, then moving to the linearized Einstein equations around Kerr (the so-called linearized gravity system), and finally considering nonlinear model problems such as the nonlinear stability of Schwarzschild.

In joint works with S. Klainerman [3] [4] [5], a joint work with E. Giorgi and S. Klainerman [1], and a work of D. Shen [6], we have recently proved the Kerr stability conjecture in the particular case of slowly rotating black holes, i.e. for $|a| \ll m$. More precisely:

- The decay estimates are proved in [5].
- The modulation procedure to track the final parameters (a_f, m_f), as well as the final location of the black hole is treated in [3] [4] [6]. In particular, in order to deal with general covariance, i.e. the fact that Einstein vacuum equations possess a huge gauge group containing all diffeomorphisms of the spacetime, we rely on the concept of General Covariant Modulated (GCM) spheres, first introduced in restricted perturbations of Schwarzschild in [2], and then extended to general perturbations of Kerr in [3] [4].
- Finally, the hyperbolic estimates are proved in [1].

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The energy critical Zakharov equation below the ground state

Timothy Candy

We describe recent results obtained in [2] on the Zakharov system in the energy critical dimension $d = 4$ with energy below the ground state. These results build on earlier joint work of the author with Sebastian Herr and Kenji Nakanishi [3, 4]. The Zakharov system was introduced by Vladimir Zakharov [8] to model Langmuir turbulence in plasma. In the first order formulation, the Zakharov system is given by

$$\begin{aligned} &i\partial_t u + \Delta u = \Re(V)u \\ &\frac{1}{\alpha} i\partial_t V + |\nabla| V = -|\nabla||u|^2 \\ &(u, V)(0) = (f, g) \in H^s \times H^\ell \end{aligned} \tag{1}$$

where $u(t, x) : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C}$ models the magnetic field, and $V(t, x) : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C}$ is the ion density. The Zakharov system is a Hamiltonian system with (conserved) energy

$$\mathcal{E}_Z(u, V) = \int_{\mathbb{R}^4} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|V|^2 - \frac{1}{2}\Re(V)|u|^2 dx$$

and a conserved mass

$$\mathcal{M}(u) = \frac{1}{2} \int_{\mathbb{R}^4} |u|^2.$$

The system (1) is not scaling invariant, but should be thought of as energy critical in $4d$ as:

(1) The potential energy is just barely controlled by the kinetic energy in $4d$, namely via the sharp Sobolev embedding theorem

$$\Big| \int_{\mathbb{R}^4} \Re(V)|u|^2 dx \Big| \leqslant \|V\|_{L^2(\mathbb{R}^4)} \|u\|^2_{L^4(\mathbb{R}^4)} \lesssim \|V\|_{L^2(\mathbb{R}^4)} \|u\|^2_{\dot{H}^1(\mathbb{R}^4)}.$$

(2) If we let the wave speed $\alpha \to \infty$, the Zakharov system formally reduces to the focussing NLS

$$i\partial_t u + \Delta u = -|u|^2 u$$

which is energy critical in $4d$. This convergence can be made rigourous [7].

(3) The energy regularity $H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ is on the boundary of the admissible local well-posedness region.

The small data theory in 4d has been largely resolved in [4], where (in particular) it was shown that any small data $(f,g) \in H^s \times L^2$ with $\frac{1}{2} \leqslant s < 2$ leads to a globally well-posed solution which scatters to a free solution as $t \to \pm\infty$. The wave regularity L^2 is sharp, and cannot be reduced. In particular, the energy regularity $H^1 \times L^2$ is on the boundary of admissible regularities for local well-posedness.

In the large data setting, the Aubin-Talenti function $Q(x) = \frac{8}{8+|x|^2}$ plays an important role in the global dynamics for (1). In particular it leads to a (non-scattering) stationary solution $(u,V) = (Q, -Q^2)$, and is conjectured to provide a sharp threshold between (linear) scattering, and non-scattering. More precisely, it is known that *radial data* satisfying the energy constraint

$$\mathcal{E}_Z(f,g) < \mathcal{E}_Z(Q,-Q^2), \qquad \|g\|_{L^2} \leqslant \|Q\|_{L^2} \tag{2}$$

leads to global well-posedness and scattering [5]. Moreover, if the condition (2) fails, solutions can grow [5]. In the non-radial setting, it is known that under the constraint (2) we have global well-posedness [3], however the question of scattering below the ground state (i.e. for data satisfying (2)) is an open question.

The main result obtained recently in [2] is the following.

Theorem 1 ([2]). *Suppose scattering below the ground state fails. Then there exists a mass/energy threshold (M_c, E_c) with $E_c < \mathcal{E}_Z(Q,-Q^2)$ and a solution $(\psi,\phi) \in C(\mathbb{R}, H^1 \times L^2)$ such that*

$$\mathcal{E}_Z(\psi,\phi) = E_c, \qquad \mathcal{M}(\psi) = M_c, \qquad \|\phi\|_{L^\infty_t L^2_x} \leqslant \|Q\|_{L^2}$$

and

$$\|\psi\|_{L^2_t W^{\frac{1}{2},4}_x((0,\infty)\times\mathbb{R}^4)} = \|\psi\|_{L^2_t W^{\frac{1}{2},4}_x((-\infty,0)\times\mathbb{R}^4)} = \infty.$$

Moreover, there exists $x(t) : \mathbb{R} \to \mathbb{R}^4$ such that $\{(\psi,\phi)(t, x - x(t))\}$ is precompact in $H^1 \times L^2$.

Scattering for (1) is equivalent to a finite $L^2_t W^{\frac{1}{2},4}_x(\mathbb{R} \times \mathbb{R}^4)$ norm. Theorem 1 implies that there is no concentration below the ground state. This is unsurprising for the Schrödinger component (as it is in some sense subcritical), but the wave component is L^2 critical. Theorem 1 reduces the problem of scattering below the ground state to ruling out the existence of solutions of the form (ψ,ϕ). This still seems very challenging, as there does not currently seem to be a way to gain control over the speed of the translation parameter $x(t)$.

The proof of Theorem 1 proceeds by first obtaining a refined small data theory using the energy dispersed norm

$$\|u\|_Y = \sup_{\lambda \in 2^{\mathbb{N}}} \lambda^{-4} \|u_\lambda\|_{L^\infty_{t,x}}$$

where u_λ denotes the restriction of frequencies to the dyadic scale $|\xi| + 1 \approx \lambda$. The refined small data theory is needed due to the lack of a profile decomposition in the end point Strichartz space $L^2_t L^4_x$. The small data theory in $\|\cdot\|_Y$ is then combined with the standard profile decomposition of Bahouri-Gérard [1], and the concentration compactness argument of Kenig-Merle [6].

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Soliton resolution for the radial quadratic wave equation in six space dimensions

Charles Collot

(joint work with Thomas Duyckaerts, Carlos Kenig and Frank Merle)

1. Introduction to the soliton resolution conjecture

Prototypes of semilinear wave equations are that with quadratic nonlinearities:

$$\partial_t^2 u - \Delta u = |u|u, \tag{1}$$

$$\text{and} \qquad \partial_t^2 u - \Delta u = u^2 \tag{2}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, and $(u(0), \partial_t u(0)) \in \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. They admit a scaling invariance: if u is a solution then so is $u_\lambda(t,x) = \lambda^2 u(\lambda t, \lambda x)$ for any $\lambda > 0$. The energy $E(u) = \frac{1}{2}\int |\nabla_{t,x} u|^2 - \frac{1}{3}\int u^3$ is conserved $E(u)(t) = E(u)(0)$. We consider the case $N = 6$ where the equation is energy critical since the scaling invariance preserves the energy $E(u_\lambda) = E(u)$. We consider further spherically symmetric solutions $u(t,x) = u(t,|x|)$. Among them is the stationary state:

$$W(x) = \left(1 + \frac{|x|^2}{24}\right)^{-2}, \qquad -\Delta W = W^2,$$

which generates the full family $(W_\lambda)_{\lambda>0}$ of scaled stationary states.

Equation (1) belongs to a large collection of semilinear equations, including for example the general semilinear wave equation ((1) with nonlinearity $|u|^{p-1}u$ in any dimension), the wave maps equation, and the Yang-Mills equations to name a few. The *soliton resolution conjecture* predicts that any global in time solution evolves asymptotically as a sum of decoupled solitons (meaning here stationary states) plus a radiative term (here a linear solution) and a term going to zero in the energy space. For finite time blow-up solutions, a similar decomposition should hold, depending on the nature of the blow-up. We refer to the introduction of [5] for a historic perspective on this conjecture, and also to [9, 10] for additional reviews. For the radial energy critical generalisation of (1) in other dimensions $N \geq 3$, the resolution has first been proved in [3] for $N = 3$, then for all $N \geq 5$ odd in [4], for $N = 4$ in [5], and the present work [2] for $N = 6$. After this work was made public, another work [7] proved the resolution for $N \geq 8$. The related resolution problem for the wave maps equation was first solved in [2] for the co-rotational case, and then in [6] for the general k-equivariant case.

2. Main result

Theorem 1 (Soliton resolution for radial $6D$ critical waves [2]). *Let u be a radial solution of* (1) *(respectively, of* (2)*) and T_+ be its maximal time of existence. Assume*

$$\sup_{t\in[0,T_+)} \int_{\mathbb{R}^6} \left((\partial_t u(t,x))^2 + |\nabla_x u(t,x)|^2\right) dx < \infty. \tag{3}$$

Then if $T_+ < \infty$, there exist $(v_0, v_1) \in \dot{H}^1 \times L^2$, an integer $J \geq 1$, and for each $j \in \{1, \dots, J\}$, a positive function $\lambda_j(t)$ such that

$$0 < \lambda_J(t) \ll \dots \ll \lambda_1(t) \ll (T_+ - t), \quad \text{as } t \to T_+,$$

and signs $(\iota_j)_{1\leq j\leq J} \in \{-1, +1\}^J$ (respectively, the signs are $(\iota_j)_{1\leq j\leq J} \equiv (1, ..., 1)$ by convention for Equation (2)*), such that*

$$\left\| (u(t), \partial_t u(t)) - \left(v_0 + \sum_{j=1}^{J} \frac{\iota_j}{\lambda_j^2(t)} W\left(\frac{x}{\lambda_j(t)}\right), v_1 \right) \right\|_{\dot{H}^1\times L^2} \underset{t\to T_+}{\longrightarrow} 0. \tag{4}$$

If $T_+ = +\infty$, there exists a solution v_L of the linear wave equation $\partial_t^2 v - \Delta v = 0$ with $\vec{v}(0) \in \dot{H}^1 \times L^2$, an integer $J \in \mathbb{N}$, and for each $j \in \{1, \dots, J\}$, a positive function $\lambda_j(t)$ such that

$$0 < \lambda_J(t) \ll \dots \ll \lambda_1(t) \ll t, \quad \text{as } t \to \infty$$

and signs $(\iota_j)_{1\leq j\leq J} \in \{-1, +1\}^J$ (respectively, the signs are $(\iota_j)_{1\leq j\leq J} \equiv (1, ..., 1)$ by convention for Equation (2)*), such that*

$$\left\| (u(t), \partial_t u(t)) - \left(v_L(t) + \sum_{j=1}^{J} \frac{\iota_j}{\lambda_j^2(t)} W\left(\frac{x}{\lambda_j(t)}\right), \partial_t v_L(t) \right) \right\|_{\dot{H}^1\times L^2} \underset{t\to\infty}{\longrightarrow} 0. \tag{5}$$

In [2] we obtain in addition another closely related result: that the collision of several solitons is inelastic.

3. Ideas of the proof and novelties

The proof of Theorem 1 follows the strategy of [4], and the main difficulty is the lack of good channels of energy estimates in even dimensions. We reason by contradiction, and, already knowing that the soliton resolution holds true along a subsequence by [8] we obtain that on an infinite succession of time intervals, the solution must enter and then leave the neighbourhood of a multisoliton. The pieces of the profile decomposition [1] along subsequence of times in such intervals are then showed to be non-radiative for $|x| > |t|$ in the sense that they correspond to solutions u to (1) or (2) such that $E_{out}(0) = 0$ where

$$E_{out}(R) = \sum_{t\to\pm\infty} \lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\partial_t u(t,x)|^2 + |\nabla_x u(t)|^2 dx = 0. \tag{6}$$

A first novelty is the classification at infinity of such solutions:

Proposition 2. *There exists $\epsilon, \epsilon' > 0$ such that the following holds true.*

(i) Existence. *Assume that $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$ satisfies $|\mathbf{c}| \le \epsilon$. Then for Equations* (1)-(2)*, there exists a solution $a[\mathbf{c}]$ that is non-radiative for $|x| > 1 + |t|$, i.e. $E_{out}(1) = 0$, with*

$$a = \frac{c_1}{|x|^4} + \frac{c_2 t}{|x|^4} + \tilde a \quad \textit{with} \quad \sup_{t\in\mathbb{R}} \int_{|x|>1+|t|} |\partial_t \tilde a(t)|^2 + |\nabla_x \tilde a(t)|^2 dx \le C|\mathbf{c}|^4,$$

and $\int_{|x|>1} \nabla \tilde a(0).\nabla(|x|^{-4})dx = \int_{|x|>1} \partial_t \tilde a(0)|x|^{-4}dx = 0$.

(ii) Uniqueness. *Conversely, for any solution u to Equations* (1)-(2) *that is non-radiative for $|x| > 1 + |t|$, i.e. $E_{out}(1) = 0$, with $\int_{|x|>1} |\partial_t u(0)|^2 + |\nabla_x u(0)|^2 dx \le \epsilon'^2$, then there exists $a[\mathbf{c}]$ described in (i) above such that*

$$\forall |x| > 1 + |t|, \qquad u(t,x) = a[\mathbf{c}](t,x).$$

This proposition is used to obtain a lower bound on the largest scale of the multisoliton. Then, a contradiction is obtained by combining this lower bound with the main order system of ODEs governing the evolution of the scale parameters. To justify these modulation equations, another novelty is a new weakened channel of energy estimate for the linearisation of the wave equation around a soliton:

$$\partial_t^2 u - \Delta u - 2Wu = 0, \qquad (u(0), \partial u(0)) \in \dot H^1 \times L^2. \tag{7}$$

Proposition 3 (Channels of energy around the ground state)**.** *There exists $C > 0$ such that any radial solution u with $\int_{\mathbb{R}^6} \nabla u(0).\nabla(2W + x.\nabla W)dx = \int_{\mathbb{R}^6} \partial_t u(0)(2W + x.\nabla W)dx$ of* (7) *satisfies:*

$$\int_{\mathbb{R}^6} (\partial_t u(0))^2 dx + \sup_{R>0} \frac{1}{\langle \log R\rangle^2} \int_{R<|x|<2R} |\nabla_x u(0)|^2 dx \le C E_{out}(0). \tag{8}$$

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Global stability of Kaluza-Klein Theories: a toy model

Annalaura Stingo
(joint work with Cécile Huneau)

The classical mathematical approach to the unification of gravity and electromagnetism is represented by the so-called *Kaluza-Klein theory*, which takes the name from physicists Kaluza and Klein who first formulated it in the 1920s. In this theory, which is the first of several theories of higher-dimensional gravity, general relativity is formulated in $1 + 3 + 1$ dimensions and the additional dimension is compactified on a circle to obtain at low energies a $(3 + 1)$-dimensional Einstein-Maxwell-Scalar system. In a seminal work by Witten [5], it is proved that the ground state solution of the Kaluza-Klein unified theory, that is the 5-dimensional Minkowski solution $\bar{g}$ to the Einstein vacuum equations $Ric_{\mu\nu}[g] = 0$, is unstable against a process of semiclassical barrier penetration but heuristic arguments supporting classical stability are given.

The global nonlinear stability of the 4-dimensional Minkowski solution to the Einstein equations is a well known result, which goes back to the monumental work of Christoudoulou-Klainerman [1] and the simplified proof of Lindblad-Rodnianski

[4] for the vacuum case, and to the works of LeFloch-Ma [3] and Ionescu-Pausader [2] for Einstein-Klein-Gordon systems describing self-gravitating massive fields.

In the Kaluza-Klein setting, however, the nonlinear stability of Minkowski solution has only been proved for perturbations that do not depend on the compact direction, see the work of Wyatt [6]. In *harmonic coordinates*, the Einstein vacuum equations for the metric perturbation $h = g - \bar{g}$ take the form of a quasilinear system of wave equations on the product space $\mathbb{R}^{1+3} \times \mathbb{S}^1$

$$\tilde{\Box}_g h_{\mu\nu} := g^{\rho\sigma}\partial_\rho\partial_\sigma h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h), \quad \mu, \nu = 0, \dots, 4 \tag{1}$$

where $F_{\mu\nu}(h)(\partial h, \partial h)$ contains quadratic and cubic semilinear interactions. In the case where the metric perturbation does not depend on the periodic coordinate y, the above system simply reduces to a system of wave equations on $\mathbb{R}^{1+3}$. If instead general perturbations are considered, the Fourier expansion of the solution h along the periodic direction

$$h_{\mu\nu}(t,x,y) = \sum_{k\in\mathbb{Z}} \exp(iky) h^k_{\mu\nu}(t,x)$$

shows that the underlying structure of (1) is that of a system on $\mathbb{R}^{1+3}$ strongly coupling quasilinear wave equations (satisfied by the zero modes $h^0_{\mu\nu}$) with an infinite sequence of quasilinear Klein-Gordon equations (each non-zero mode $h^k_{\mu\nu}$ is a Klein-Gordon solution of mass $|k|$).

In this talk, we discuss a recent result obtained in collaboration with C. Huneau in which we study a toy model for (1). We consider a system of quasilinear wave equations on the product space $\mathbb{R}^{1+3} \times \mathbb{S}^1$ that has the following vector form

$$(-\partial_t^2 + \Delta_x + \partial_y^2)U + u\partial_y^2 U = Q(\partial U, \partial U) \tag{2}$$

where $U = (u, v)^T : \mathbb{R}^{1+3} \times \mathbb{S}^1 \to \mathbb{R}^2$ and $Q(\partial U, \partial U)$ is linear combination of the classical quadratic null forms

$$\begin{aligned} Q_0(\partial\phi, \partial\psi) &= \partial_t\phi\partial_t\psi - \nabla_x\phi \cdot \nabla_x\psi \\ Q_{ij}(\partial\phi, \partial\psi) &= \partial_i\phi\partial_j\psi - \partial_j\phi\partial_i\psi, \quad i, j = 1, 2, 3 \\ Q_{0i}(\partial\phi, \partial\psi) &= \partial_t\phi\partial_i\psi - \partial_i\phi\partial_t\psi, \quad i = 1, 2, 3 \end{aligned}$$

System (2) only keeps the quadratic semilinear null interactions and the quasilinear term $u\partial_y^2 U$, which is the analogue of $h^{44}\partial_y^2 h_{\mu\nu}$, from (1). Our primary goal in this work is in fact to fully understand the underlying wave-Klein-Gordon nature of (1) and to prove a small data global existence result without making use of the additional structure of the Einstein equation, inherited from the choice of gauge. The full problem, that is the global stability of the Kaluza-Klein ground state, is the object of an upcoming work in collaboration with C. Huneau and Z. Wyatt. A simplified version of the main theorem we will discuss states the following:

Theorem [Huneau-Stingo '20]. Let $(u[t], v[t]) = (u(t), \partial_t u(t), v(t), \partial_t v(t))$ denote the initial data for (2) and $\mathcal{H}^0$ be the energy space with norm

$$\|(u[t], v[t])\|^2_{\mathcal{H}^0} := \|u(t)\|^2_{H^1} + \|\partial_t u(t)\|^2_{L^2} + \|v(t)\|^2_{H^1} + \|\partial_t v(t)\|^2_{L^2}.$$

Let also fix $\alpha > 0$. There exists $\epsilon_0 \ll 1$ sufficiently small such that, if the initial data satisfy the following

$$\sum_{k=0}^{5} \|\langle x\rangle^{k+\alpha}\partial_{xy}^k(u[2], v[2])\|_{\mathcal{H}^0} \leq \epsilon \leq \epsilon_0,$$

then there is a unique global solution $U = (u, v)$ to (2) with $\partial_{xy}^k U \in \mathcal{H}^0$ for all $k = 0, \dots, 5$. Moreover, there exists some constant $C > 0$ such that $U^\flat = \fint_{\mathbb{S}^1} U dy$ and $U^\natural = U - U^\flat$ satisfy the following sharp pointwise bounds

$$|\partial U^\flat(t,x)| \leq \frac{C\epsilon}{\langle t+|x|\rangle\langle t-|x|\rangle^{1/2}},$$
$$\|U^\natural(t,x,\cdot)\|_{L^2(\mathbf{S}^1)} \leq \frac{C\epsilon}{\langle t+|x|\rangle^{3/2}}.$$

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Quasilinear and time-harmonic Maxwell equations

Robert Schippa

(joint work with Roland Schnaubelt, Rainer Mandel)

We consider Maxwell equations in media, which read in three dimensions:

$$(1) \qquad \begin{cases} \partial_t \mathcal{D} &= \nabla \times \mathcal{H}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ \partial_t \mathcal{B} &= -\nabla \times \mathcal{E}, \qquad \nabla \cdot \mathcal{D} = \rho_e, \ \nabla \cdot \mathcal{B} = 0. \end{cases}$$

The electric and displacement field $(\mathcal{E}, \mathcal{D}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ and magnetic and magnetizing field $(\mathcal{B}, \mathcal{H}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ are related via pointwise constitutive relations:

$$\mathcal{D}(t,x) = \varepsilon(t,x)\mathcal{E}(t,x), \quad \mathcal{B}(t,x) = \mu(t,x)\mathcal{H}(t,x).$$

$(\rho_e, \rho_m) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}$ denote electric and magnetic charges.

ε, $\mu : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}_{\text{sym}}$ are required to be uniformly elliptic:

$$(2) \qquad \exists \lambda, \Lambda > 0 : \forall \xi \in \mathbb{R}^3 \backslash \{0\} : \quad \lambda|\xi|^2 \leq \nu^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \nu \in \{\varepsilon, \mu\}.$$

In the two-dimensional case, $(\mathcal{E},\mathcal{D}):\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2\times\mathbb{R}^2$, $(\mathcal{H},\mathcal{B}):\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}\times\mathbb{R}$, the system is given by

$$\left\{\begin{array}{ll} \partial_t\mathcal{D} &= \nabla_\perp\mathcal{H}, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^2, \\ \partial_t\mathcal{B} &= -\partial_1\mathcal{E}_2+\partial_2\mathcal{E}_1, \qquad \nabla\cdot\mathcal{D}=\rho_e. \end{array}\right.$$

We denote $\nabla_\perp = (\partial_2,-\partial_1)^t$, $\varepsilon:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^{2\times 2}_{\text{sym}}$, and $\mu(t,x)=1$. We prove sharp Strichartz estimates in [1], which resemble Strichartz estimates for the scalar variable-coefficient wave equation:

Theorem 1. *Let $\varepsilon:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^{2\times 2}_{sym}$ be a matrix-valued function with Lipschitz coefficients, satisfying (2) and $\partial^2_{t,x}\varepsilon\in L^1_tL^\infty_x$. Let $u=(\mathcal{D}_1,\mathcal{D}_2,\mathcal{H}):\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^3$ with $\partial_1\mathcal{D}_1+\partial_2\mathcal{D}_2=\rho_e$, and*

$$\rho = 2\big(\frac{1}{2}-\frac{1}{q}\big)-\frac{1}{p}, \quad p,q\geq 2, \quad \frac{2}{p}+\frac{1}{q}\leq\frac{1}{2}.$$

Then, we find the following estimate to hold

$$\begin{aligned} \||\partial_x|^{-\rho}u\|_{L^p(0,T;L^q)} &\lesssim \kappa^{\frac{1}{p}}\|u\|_{L^\infty L^2}+\kappa^{-\frac{1}{p'}}\|P(x,D)u\|_{L^1L^2} \\ &\quad + T^{\frac{1}{2}}\||D'|^{-\frac{1}{2}}\rho_e(0)\|_{L^2_x}+T^{\frac{1}{2}}\||D'|^{-\frac{1}{2}}\partial_t\rho_e\|_{L^1_tL^2_x} \end{aligned} \tag{3}$$

whenever the right-hand side is finite, provided that $\kappa\geq 1$ and

$$T\|\partial^2_{x,t}\varepsilon\|_{L^1_tL^\infty_x}\leq\kappa^2.$$

The key ingredient is a microlocal diagonalization of the time-dependent Maxwell operator

$$P(x,D)=\begin{pmatrix} \partial_t & 0 & -\partial_2 \\ 0 & \partial_t & \partial_1 \\ \partial_1(\varepsilon_{21}\cdot)-\partial_2(\varepsilon_{11}\cdot) & \partial_1(\varepsilon_{22}\cdot)-\partial_2(\varepsilon_{12}\cdot) & \partial_t \end{pmatrix}.$$

This conjugates the evolutions in phase space to a degenerate component corresponding to the charges and to two non-degenerate half-wave equations. Phase space analysis was decisively used by Tataru [5–7] to prove the sharp Strichartz estimates for the scalar wave equation with rough coefficients in a series of papers. The precise dependence on $\|\partial^2_{x,t}\varepsilon\|_{L^1_tL^\infty_x}$ in (3) allows the proof of Strichartz estimates for even rougher coefficients via paradifferential decomposition.

Corollary 2. *Assume that $\|\partial_{x,t}\varepsilon\|_{L^2_tL^\infty_x}\lesssim 1$. Then the solution u to*

$$\left\{\begin{array}{ll} P(x,D)u &= f, \quad \partial_1u_1+\partial_2u_2=0, \\ u(0) &= u_0 \end{array}\right.$$

satisfies

$$\|\langle D'\rangle^{-\alpha}u\|_{L^p(0,T;L^q)}\lesssim_T\|u_0\|_{L^2}+\|f\|_{L^1(0,T;L^2)}$$

for $\alpha>\rho+\frac{1}{3p}$.

These Strichartz estimates allow to prove new local well-posedness results for the quasilinear Maxwell equation:

$$(4) \qquad \left\{ \begin{array}{ll} \partial_t u_1 & = \partial_2 u_3, \qquad u(0) = u_0 \in H^s(\mathbb{R}^2;\mathbb{R}^3)^3, \\ \partial_t u_2 & = -\partial_1 u_3, \qquad \partial_1 u_1 + \partial_2 u_2 = 0, \\ \partial_t u_3 & = \partial_2(\varepsilon^{-1}(u)u_1) - \partial_1(\varepsilon^{-1}(u)u_2), \end{array} \right.$$

where $\varepsilon^{-1}(u) = \psi(|u_1|^2 + |u_2|^2)$, $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ is smooth, monotone increasing, and $\psi(0) = 1$. This covers the Kerr nonlinearity

$$\mathcal{D} = (1 + |\mathcal{E}|^2)\mathcal{E}.$$

We show the following:

Theorem 3. (4) *is locally well-posed for* $s > \frac{11}{6}$.

This improves on the result by the energy method by 1/6 derivatives and is the limit of local well-posedness via Strichartz estimates.

The dispersive properties in higher dimensions, already in the constant-coefficient case, depend on the number of different eigenvalues of $\varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ when we let $\mu = 1_{3\times 3}$ for simplicity. The characteristic surface in the constant-coefficient case was analyzed in the context of time-harmonic equations in [3] and [2]. Under the monochromatic ansatz

$$\mathcal{D}(t,x) = e^{i\omega t} D(x), \quad \mathcal{B}(t,x) = e^{i\omega t} B(x), \dots$$

for the fields and currents, Maxwell equations become

$$\left\{ \begin{array}{ll} i\omega D & = \nabla \times H - J_e, \\ i\omega B & = -\nabla \times E + J_m \end{array} \right.$$

for stationary fields $(E, D) : \mathbb{R}^3 \to \mathbb{C}^3 \times \mathbb{C}^3$, $(B, H) : \mathbb{R}^3 \to \mathbb{C}^3 \times \mathbb{C}^3$, and currents $(J_e, J_m) : \mathbb{R}^3 \to \mathbb{C}^3 \times \mathbb{C}^3$. For the isotropic $\varepsilon = (\varepsilon_1, \varepsilon_1, \varepsilon_1)$ and partially anisotropic case $\varepsilon = (\varepsilon_1, \varepsilon_1, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 > 0$ we can diagonalize the time-independent Maxwell operator with L^p-bounded Fourier multipliers to two degenerate (corresponding to the contribution of electric and magnetic charges) and four non-degenerate Half-Laplacians. This shows equivalence of resolvent estimates with resolvent estimates for the Half-Laplacian (cf. [3]).

After discarding the degenerate components, in the isotropic case we find the characteristic set to be a sphere. In the partially anisotropic case the characteristic set is given as the union of two ellipsoids intersecting at one point.

In the fully anisotropic case $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3 \neq \varepsilon_1$, $\mu = 1_{3\times 3}$ the characteristic surface changes from two smooth ellipsoids to the Fresnel wave surface, which contains conical singularities. It is a classical result due to Darboux that the Fresnel wave surface has a regular component with two principal curvatures bounded from below, smooth one-dimensional submanifolds with one principal curvature bounded from below (the so-called Hamiltonian circles), and neighbourhoods of the conical singularities. A more quantitative analysis of the curvature was carried out in [2] (see also [4]). Via new Bochner-Riesz estimates with negative index, suitable to estimate the conical singularities, we showed the following existence result:

Theorem 4. *Let $1 \leq p_1, p_2, q \leq \infty$, $\varepsilon = diag(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\mu = diag(\mu_1, \mu_2, \mu_3)$ satisfy $\varepsilon_1/\mu_1 \neq \varepsilon_2/\mu_2 \neq \varepsilon_3/\mu_3 \neq \varepsilon_1/\mu_1$ and $(J_e, J_m) \in L^{p_1}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3)$ divergence-free. If*

$$\frac{1}{p_1} > \frac{3}{4}, \quad \frac{1}{q} < \frac{1}{4}, \quad \frac{1}{p_1} - \frac{1}{q} \geq \frac{2}{3},$$
$$\text{and } 0 \leq \frac{1}{p_2} - \frac{1}{q} \leq \frac{1}{3}, \quad (p_2, q) \notin \{(1,1), (1, \frac{3}{2}), (3, \infty), (\infty, \infty)\},$$

then, for any given $\omega \in \mathbb{R}\backslash\{0\}$ there exists a distributional time-harmonic solution to the fully anisotropic Maxwell equations that satisfies

$$\|(E, H)\|_{L^q(\mathbb{R}^3)} \lesssim_{p_1, p_2, q, \omega} \|(J_e, J_m)\|_{L^{p_1}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3)}.$$

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On the long-time behavior of scale-invariant solutions to the 2d Euler equation

Ayman R. Said
(joint work with Tarek M. Elgindi, Ryan Murray)

We give a complete description of the long-time behavior of scale-invariant solutions the 2d Euler equation. We show that all scale-invariant solutions relax in infinite time to rigidly rotating or steady states, which are fully classified and shown to be piece-wise constant profiles with countably many jumps. Consequently, all non-constant scale-invariant solutions that are smooth on $\mathbb{S}^1$ become singular in infinite time. On $\mathbb{R}^2$, this corresponds to generic infinite time spiral and cusp formation. In the process, we also show that for scale invariant solutions, the measure (on $\mathbb{S}^1$) of particles moving away from the origin and toward spatial infinity is a strictly increasing function of time.

The scale invariant solutions solve a relatively simple equation on $\mathbb{S}^1$:

$$\begin{cases} \partial_t g + 2G\partial_\theta g = 0 \\ -(4+\partial_{\theta\theta})G = g. \end{cases}$$

Now the main result states that for $g_0 \in L^\infty(\mathbb{S}^1)$, there exist two constants $a_{\pm\infty} \in \mathbb{R}$ and two piece-wise constant profiles $g_{\pm\infty} \in BV(\mathbb{S}^1)$ so that

$$|g(\cdot,t) - g_{\pm\infty}(\cdot - a_{\pm\infty}t)|_{L^p} \to 0$$

as $t \to \pm\infty$, for any $1 \leq p < \infty$. Moreover, the asymptotic profiles $g_{\pm\infty}(\cdot - a_{\pm\infty}t)$ are travelling wave solutions with speeds $a_{\pm\infty}$. When $g_0 \in C(\mathbb{S}^1)$, the asymptotic profiles $g_{\pm\infty}$ are identically constant and equal to $\frac{1}{2\pi}\int_{\mathbb{S}^1} g_0$.
We moreover give a complete classification of the profiles $g_{\pm\infty}$ and there basin of attraction. Finally to each solution of the equation we associate a strictly increasing function of time which quantifies the conjectured "ageing" phenomena for 2d Euler in the special case of scale invariant solutions and bounded m-fold (m$\geq$ 3) symmetric solutions not continuous at 0.

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Late time tails of waves on dynamic asymptotically flat spacetimes with odd space dimension

Sung-Jin Oh
(joint work with Jonathan Luk)

This extended abstract describes an upcoming work of the author and J. Luk, in which we put forth a method for understanding the late time tail (i.e., asymptotics of the solution along causal curves) for solutions to wave equations on asymptotically flat spacetimes with spatial dimension 3, or more generally, an odd number greater than 1.

In the context of the linear wave equation (Laplace–Beltrami) in the exterior of a $(3+1)$-dimensional stationary black hole, the investigation of this problem started in a seminal work of the physicist R. Price [12]. Numerous contributions to this problem from the physics and mathematics communities have ensued. Compared to the existing works, a distinguishing feature of our method is that it rigorously identifies the leading order late time tail for solutions to wave equations that *are nonlinear* and/or *set on a dynamic (i.e., non-stationary) spacetime.* Interestingly, our work shows that the introduction of even the mildest nonlinearity and/or non-stationarity gives rise to late time tails are in general different(!) from the linear and stationary case (see [2] for a formal/numerical work on this phenomenon).

A few selected examples. We will illustrate the last remark by exhibiting some examples. We begin by introducing some notation and conventions. Let $d \geq 3$ be an odd number, and consider $\mathbb{R}^{1+d}$ equipped with the rectangular coordinates $(t, x^1, \dots, x^d)$. Let $\Box$ denote the d'Alembertian $-\partial_t^2 + \Delta$. We introduce *Bondi coordinates* (u, r, θ), where $u = t - |x| \in \mathbb{R}$, $r = |x| \in (0, \infty)$, and $\theta = \frac{x}{|x|} \in \mathbb{S}^{d-1}$. In terms of the angular variables θ, we define $\mathring{\slashed{\Delta}}$ to be the Laplace-Beltrami operator corresponding to the round metric, and $\mathbb{S}_{(\ell)}$ the L^2-projection to the eigenspace associated with the ℓ-th smallest eigenvalue of $-\mathring{\slashed{\Delta}}$, (i.e., $\ell(\ell + d - 2)$). For the sake of simplicity, in the examples below, we shall only consider the asymptotics of $\phi(t, x)$ as $t \to \infty$ while x is fixed (but in fact, our main theorem may be used to identify the leading order asymptotics of ϕ on a wide class of causal curves). We use $\mathfrak{L}$ to denote a functional of the solution that is generically[1] nonzero that may vary from line to line.

We consider two examples, one linear and one nonlinear.

Example 1. Consider the following linear wave equation (written in Bondi coordinates)[2]:

$$P\phi := \left(\Box - \tfrac{2M}{r}\partial_r^2\right)\phi = 0 \quad \text{in } \{r \geq 1\},$$

where we extend the definition of P to $\{|x| \leq 1\}$ so that $P - \Box = MQ$ with Q a rotational invariant second order operator with smooth coefficients on $\mathbb{R}^d_x$ (the precise expression of Q does not affect the results below). We consider initial data $(\phi, \partial_t\phi)|_{\{t=0\}} = (g, h)$ that are smooth and compactly supported, since we are interested in the late time tail arising from dynamics and not from the spatial tail of the initial data. Then the following statements hold:

(1) When $M = 0$, we have $P = \Box$. Hence, by the strong Huygens Principle, $\phi(t, x) = 0$ for sufficiently large t. In short, *there is no late time tail.*
(2) (*Price's law*; [1,4,5,8,9,13]). However, when $M = M_0$ with $0 \neq |M_0| \ll 1$, we have $\phi(t, x) \sim \mathfrak{L}t^{-d}$ as $t \to \infty$. This is the celebrated Price's law.
(3) (*Luk-O.*). Interestingly, Price's law may break down if the stationarity assumption is even slightly broken! Consider $M = M_0 + \epsilon u^{-N}$, where $0 < |M_0| \ll 1$, N is an arbitrarily large positive number, and $0 \neq |\epsilon| \ll 1$. Then for $d \neq 3$ (i.e., $d \geq 5$ and odd), we have $\phi(t, x) \sim \mathfrak{L}t^{-(d-1)}$ as $t \to \infty$. Moreover, if $\phi = \mathbb{S}_{(\ell)}\phi$ (note that $[P, \mathbb{S}_{(\ell)}] = 0$), then for $(d, \ell) \neq (3, 0)$, we have $\phi(t, x) \sim \mathfrak{L}t^{-(d+2\ell-1)}$ as $t \to \infty$.

Example 2. Consider the wave maps equation on $\mathbb{R}^{1+3}$ taking values in a 2-surface $(\mathcal{N}, \mathbf{h})$. Fix $p \in \mathcal{N}$, and let Φ be the solution to the IVP with $(\Phi, \partial_t\Phi)|_{\{t=0\}}$ a small, smooth and compactly supported perturbation of $(p, 0)$. It is well-known that, by the vector field method, such a solution is global and decays to p as

[1] That is, for solutions corresponding to initial data in an open and dense subset of the set of admissible initial data in a suitable topology.

[2] The expression $\Box - \frac{2M}{r}\partial_r^2$ has the same principal part as the d'Alembertian on the Schwarzschild spacetime written in the Bondi coordinates. While we chose to work with P to keep this example simple, results analogous to (2) and (3) below hold on black hole spacetimes.

$t \to \pm\infty$ under an appropriate smallness condition. Concerning the late time tail of this solution, the following statements hold:

(1) First, when the Gauss curvature K vanishes in a neighborhood of p, then the wave maps equation reduces to the classical wave equation on $\mathbb{R}^{1+3}$. Hence, by the Strong Huygens Principle, there is no late time tail.
(2) (*Luk-O.*) On the other hand, when $K(p) \neq 0$, then a nontrivial tail appears: $\mathrm{dist}(\Phi(t,x),p) \sim \mathfrak{L}t^{-3}$ as $t \to \infty$.

In fact, when $K(p) = 0$, it can be shown that $\mathrm{dist}(\Phi(t,x),p) \le Ct^{-4}$ for some C that may depend on x. It is natural to conjecture that the leading order asymptotics of $\mathrm{dist}(\Phi(t,x),p)$ is dictated by the order of vanishing of K at p.

While space constraints force us to end the list of examples here, we remark that our method can be used to identify the leading order late time tail of global dispersive solutions to many other equations, such as *the time-like vanishing mean curvature hypersurface equation on* $\mathbb{R}^{1+3}$ and *semilinear wave equation with null structure in the exterior of a* $(3+1)$*-dimensional Kerr black hole.*

Leading order late time tails and higher radiation fields. As the above examples show, the late time tail of a wave in an asymptotically flat spacetime with odd spatial dimension depends delicately on the equation. Roughly, it is because the "base case" $\Box\phi = 0$ is highly degenerate due to the strong Huygens principle. Key to understanding the late time tail is connecting it to another problem where computation is easier. Indeed, our main theorem relates the leading order late time tail with another asymptotic expansion of the solution, namely, the *higher radiation fields* (see [7] for a prior instance of this idea).

A brief description of the result is as follows. Let $d \ge 3$ be odd and consider

$$P\phi := (\mathbf{g}^{-1})^{\alpha\beta}\nabla_\alpha\partial_\beta\phi + \mathbf{A}^\alpha\partial_\alpha\phi + V\phi = \mathcal{N}(t,x,\phi,\mathrm{d}\phi,\nabla\mathrm{d}\phi) \quad \text{on } \mathbb{R}^{1+d},$$

where ∇ is the Levi-Civita connection associated with a Lorentzian metric $\mathbf{g}$. Let (u,r,θ) be a Bondi-type coordinate system, i.e., $\mathbf{g}^{-1}$ asymptotes in a suitable fashion to the (inverse) Minkowski metric written in Bondi coordinates. Consider the following formal expansion:

$$\phi(u,r,\theta) = r^{-\frac{d-1}{2}}\left(\mathring{\Phi}_0(u,\theta) + r^{-1}\mathring{\Phi}_1(u,\theta) + \cdots\right). \tag{1}$$

The first term $\mathring{\Phi}_0(u,\theta) = \lim_{r\to\infty} r^{\frac{d-1}{2}}\phi(r,u,\theta)$ is the *Friedlander radiation field*; this limit is known to exist and also obey $\mathring{\Phi}_0 \to 0$ as $u \to \infty$ under fairly general assumptions [10]. The later terms $\mathring{\Phi}_1(u,\theta)$, ... are called *higher radiation fields.* An important observation regarding higher radiation fields is that given $\mathring{\Phi}_0$, and under appropriate assumptions for P and $\mathcal{N}$, the higher radiation fields may be *formally* determined by solving recurrence equations. From the formally determined higher radiation fields, we define J_f to be *the smallest order* j *for which* $\mathring{\Phi}_j(u,\theta) \not\to 0$ *as* $u \to \infty$. Then, in simple terms:

Main Theorem (Luk–O.)**.** *Under suitable assumptions on P, $\mathcal{N}$, and the solution ϕ, we have, for any $\eta > 0$,*

$$\phi(t,x) = c_d \mathfrak{L} t^{-\frac{d-1}{2}-J_f}\psi(x) + o_\eta(t^{-\frac{d-1}{2}-J_f}) \quad \text{in } \{r \le \eta u,\, t \ge 1\},$$

where $\mathfrak{L}$ is an (explicit) functional of $\mathring{\Phi}_0$ and ψ is the unique solution to a PDE derived from P with boundary condition $\psi \to 1$ as $r \to \infty$. Moreover, for each u, the expansion (1) *is justified up to order J_f.*

Among the assumptions of the theorem (which are too long to detail here) are *asymptotic flatness*, *ultimate stationarity*, (dynamic formulation of) *no resonance/eigenvalue at zero energy*, *null structure* when $d = 3$ and some *non-sharp upper bounds for ϕ* which follow from the vector field method [3,6,10,11]. We also remark that there is a version of the main theorem for $\phi = \mathbb{S}_{(\ell)}\phi$ on spherically symmetric spacetimes. All the examples above follow from the main theorem.

Heuristically, the key reason why an expansion as $r \to \infty$ such as (1) determines the leading order late time tail is the strong Huygens principle for $\Box$. To illustrate, consider the asymptotics of $r^{\frac{d-1}{2}}\phi(t, \ell t)$ as $t \to \infty$ with $\ell \in \mathbb{R}^d$, $0 \neq |\ell| < 1$, which is a part of the conclusion above. Writing $\Box\phi = (\Box - P)\phi + \mathcal{N}(\phi)$ with characteristic data on $\{u = U_0\}$ for some U_0 and ignoring $(\Box - P)\phi + \mathcal{N}(\phi)$ (which may eventually be justified thanks to asymptotic flatness and $\ell \neq 0$), the strong Huygens principle tells us that this asymptotics is determined by $\phi|_{u=U_0}$ for r large.

Finally, we also mention the interesting fact that, under the assumptions of the theorem, $\mathbb{S}_{(\ell)}\mathring{\Phi}_j(u) \to 0$ as $u \to \infty$ for $j \le \frac{d+2\ell-3}{2}$ ($\mathbb{S}_{(\ell)}$ is defined using (r, u, θ)). This statement, which we refer to as *strong Huygens correction*, follows from special cancellations for $\Box$ responsible for the strong Huygens principle. In particular, $J_f \ge \frac{d-1}{2}$, which is consistent with the examples above.

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Global solutions for 1D cubic defocusing dispersive equations

Mihaela Ifrim
(joint work with Daniel Tataru)

This work is devoted to a general class of one dimensional NLS problems with a cubic nonlinearity. The question of obtaining scattering, global in time solutions for such problems has attracted a lot of attention in recent years, and many global well-posedness results have been proved for a number of models under the assumption that the initial data is both *small* and *localized*. However, except for the completely integrable case, no such results have been known for small but non-localized initial data.

Here we consider instead the much more difficult case where the initial data which is just *small*, without any localization assumption. Then it is natural to restrict the analysis to defocusing problems, as focusing one-dimensional cubic NLS type problems typically admit small solitons and thus, generically, the solutions do not scatter at infinity. For this class of problems formulate the following broad conjecture:

Conjecture 1. One dimensional dispersive flows with cubic defocusing nonlinearities and small initial data have global in time, scattering solutions.

Our objective is to prove the first global in time well-posedness result of this type, assuming a Schrödinger type dispersion relation and a cubic nonlinearity with a smooth and bounded symbol. As part of our results, we also prove that our global solutions are scattering at infinity in a very precise, quantitative way, in the sense that they satisfy both L^6 Strichartz estimates and bilinear L^2 transversality bounds. This is despite the fact that the nonlinearity is non-perturbative on large time scales. Our method is based on a robust reinterpretation of the idea of interaction Morawetz estimates, developed almost 20 years ago by the I-team.

Our estimates are new even in the case of the classical cubic NLS problem; there they improve earlier estimates of Planchon and Vega.

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On asymptotic stability of classical solitons in 1D nonlinear scalar field theories

Jonas Lührmann
(joint work with Wilhelm Schlag, Yongming Li)

We consider the asymptotic stability properties of solitons in classical $(1+1)$-dimensional scalar field theories

$$(\partial_t^2 - \partial_x^2)\phi + W'(\phi) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

where $W\colon \mathbb{R} \to \mathbb{R}$ is a scalar interaction potential. Prime examples include the ϕ^4 model and the sine-Gordon model, as well as the quadratic and cubic 1D Klein-Gordon equations. The main goal of this talk is to survey recent advances in the study of the (conditional) asymptotic stability of the solitons in these classical models, and to explain the role that threshold resonances of the linearized operators can play for the long-time dynamics of perturbations of these solitons.

More specifically, we discuss two types of scalar potentials:

- single-well potentials $W(\phi) = \frac{1}{2}\phi^2 - \frac{1}{p+1}|\phi|^{p+1}$, $p > 1$, leading to the 1D focusing Klein-Gordon equations (KG_p),
- double-well potentials $W(\phi)$ featuring two consecutive global minima $\phi_- < \phi_+$ with $W(\phi_\pm) = 0$. The minima $\phi_\pm$ are sometimes referred to as vacuum states. The classical examples are the sine-Gordon model with $W(\phi) = 1 - \cos(\phi)$ and the ϕ^4 model with $W(\phi) = \frac{1}{4}(1-\phi^2)^2$.

Both types of models admit non-trivial static solutions $-\partial_x^2 Q + W'(Q) = 0$, $x \in \mathbb{R}$, satisfying $\lim_{x\to\pm\infty} Q(x) = 0$ for the single-well potentials and $\lim_{x\to\pm\infty} Q(x) = \phi_\pm$ for the double-well potentials. In the latter case these static solutions are called *kinks* since they connect the two distinct vacuum states $\phi_\pm$.

Due to the invariance of the models under translations and Lorentz transformations, arbitrary perturbations may cause the solitons $Q(x)$ to start moving. To reduce the complexity of the asymptotic stability problems for them, in a first step one imposes symmetry assumptions about the perturbations to prevent the translational mode from entering the dynamics. Then the evolution equation for a small perturbation $u(t,x) := \phi(t,x) - Q(x)$ is of the schematic form

$$(\partial_t^2 - \partial_x^2 + V(x) + m^2)u = \alpha(x)u^2 + \beta_0 u^3, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{2}$$

where $V(x)$ is a smooth localized potential, $m > 0$ is a mass parameter, $\alpha(x)$ is a smooth (bounded, possibly localized) variable coefficient, and $\beta_0 \in \mathbb{R}$ is a constant coefficient. Depending on the specific asymptotic stability problem, the linearized operator $-\partial_x^2 + V(x) + m^2$ has additional spectral features such as threshold resonances (sG, ϕ^4, KG_2, KG_3), positive gap eigenvalues (ϕ^4, KG_2), and negative eigenvalues (KG_2, KG_3). The zero eigenvalue (translational mode) is not relevant due to the symmetry assumptions on the perturbations.

The (conditional) asymptotic stability problems for the solitons of the ϕ^4 model, the sine-Gordon model, and for $\mathrm{KG}_{2/3}$ are deeply related by the fact that the potentials in the linearized operators all belong to the hierarchy of reflectionless

Pöschl-Teller potentials [20] given by $V(x) = -\ell(\ell+1)\mathrm{sech}^2(x)$, $\ell \in \mathbb{N}_0$ ($\ell = 1$: sG; $\ell = 2$: ϕ^4, KG_3; $\ell = 3$: KG_2).

Proving the (conditional) asymptotic stability of $Q(x)$ then consists in proving the decay to zero of small solutions to (2). Key difficulties in the analysis of the long-time behavior of small solutions to (2) are the slow dispersive decay of 1D Klein-Gordon waves, the low power nonlinearities (leading to modified scattering), and the consequences of threshold resonances or internal modes exhibited by the linearized operators.

A main goal of this talk is to explain the phenomena and difficulties caused by threshold resonances in the analysis of the long-time behavior of small solutions to (2). Formally, the linearized operator $L = -\partial_x^2 + V(x) + m^2$ exhibits a threshold resonance if there exists a bounded non-trivial function $\varphi \neq 0$ such that $L\varphi = m^2\varphi$. The significance of the presence of a threshold resonance for the dynamics of perturbations of a soliton is primarily that the corresponding Klein-Gordon waves only have slow local decay. In fact, the bulk of the Klein-Gordon waves still have improved local decay, only a projection to the threshold resonance exhibits the slow local decay, see the refined local decay estimate (2.62) in [14, Corollary 2.17].

One generally distinguishes the notion of *local asymptotic stability*, referring to the convergence to zero of small solutions to (2) locally in the energy space, and the notion of *full asymptotic stability*, referring to sharp $L_x^\infty(\mathbb{R})$ decay estimates and asymptotics for small solutions to (2).

An approach to proving (conditional) local asymptotic stability via virial-type (or positive commutator) methods has been pioneered in [8–12]. See also [4,17] for related recent contributions. We highlight that [9,10] establish the local asymptotic stability of the ϕ^4 kink under odd perturbations. An outstanding difficulty for this local asymptotic stability approach is that integrated local energy decay estimates for the (dispersive part of) perturbations of the soliton do not seem to be possible due to the failure of L_t^2-integrability in time of the contributions of the threshold resonances to the local decay of the corresponding Klein-Gordon waves.

Several approaches towards proving full asymptotic stability results have been put forth over the last few years: using the distorted Fourier transform, see e.g. [3, 6, 7], using the wave operator, see e.g. [5], or using super-symmetric factorization properties of the linearized operators [18]. The key difficulty that the presence of a threshold resonance poses for full asymptotic stability results is the formation of a singular quadratic source term due to the slow local decay of the Klein-Gordon waves. This type of source term is highly problematic for (distorted) vector field based methods to derive decay, and can potentially lead to a slow-down of the decay rate of the perturbations [14, 16].

In [14, 18] a remarkable non-resonance property of the corresponding quadratic nonlinearity in (2) for perturbations of the sine-Gordon kink was uncovered, which suppresses the worst effects of the source term. This led to a perturbative proof via super-symmetry by the author and W. Schlag [18] of the full asymptotic stability of the sine-Gordon kink under odd perturbations, for which the proof ideas are discussed in more detail in this talk. See [2] for a proof using inverse scattering techniques for the completely integrable sine-Gordon model.

If no favorable cancellation structures are present, further ideas are needed at this point to deal with the quadratic source term caused by the threshold resonances to obtain full (conditional) asymptotic stability results. This is for instance the main difficulty to establish the full conditional asymptotic stability of the soliton for KG_3 under even perturbations. Interestingly, an internal mode creates a related (but even worse) source term in the nonlinear Klein-Gordon equation for the dispersive part of perturbations of the ϕ^4 kink. It is the main challenge towards establishing a full asymptotic stability result for odd perturbations of the ϕ^4 kink, which remains a major open problem. See [5] for long-time decay estimates up to times ε^{-4+c}.

We conclude by referring to the sample of very recent works [1, 2, 4–6, 8, 10, 13, 14, 18, 19] for further references and perspectives on the study of the asymptotic stability of solitons, or solitary waves, for 1D wave-type or 1D Schrödinger models.

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Singularity formation for the three-dimensional Keller-Segel system

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(joint work with Irfan Glogić)

We consider the following system of equations

$$\begin{cases} \partial_t u = \Delta u + \nabla \cdot (u \nabla V), \\ \Delta V = u, \\ u(0, \cdot) = u_0, \end{cases} \tag{1}$$

where $V = V(t,x), u = u(t,x) \in \mathbb{R}$ and $(t,x) \in [0,T) \times \mathbb{R}^d$. Eq. (1) is frequently referred to as the parabolic-elliptic Keller-Segel system, named after E. Keller and L. Segel [6], who, in the 1970s, introduced a model to describe chemotactic aggregation phenomena in biology, see also [7]. In the three dimensional case, the above system also arises as a simplified model for self-gravitating matter in stellar dynamics, with u representing the gas density and V its gravitational self-potential, see [9] an the references therein. Note that on $\mathbb{R}^d$, the equation for V can be solved explicitly in terms of u by convolution with the fundamental solution of the Laplace operator.

Eq. (1) is invariant under the scaling transformation $(u, V) \mapsto (u_\lambda, V_\lambda)$, where

$$u_\lambda(t,x) = \lambda^{-2} u(t/\lambda^2, x/\lambda), \quad V_\lambda(t,x) = V(t/\lambda^2, x/\lambda), \quad \lambda > 0.$$

Assuming sufficient decay at infinity, the total mass

$$\mathcal{M}(u)(t) = \int_{\mathbb{R}^d} u(t,x) dx,$$

is conserved. Since $\mathcal{M}(u_\lambda) = \lambda^{d-2} \mathcal{M}(u)(\cdot/\lambda^2)$, the problem is mass critical for $d = 2$.

It is well known that Eq. (1) exhibits finite-time blowup in $L^\infty(\mathbb{R}^d)$ in all space dimensions $d \geq 2$. Thereby, if $\limsup_{t \to T^-} (T-t) \|u(t, \cdot)\|_{L^\infty} < C$, for some $C > 0$, the blowup is said to be of *type I*, and of *type II* otherwise.

In the past decades, the main focus has been on the investigation of the mass critical case. In comparison, the dynamics in higher space dimensions are less understood. In $d = 3$, the existence of type II blowup solutions was formally derived in [8]. Only recently, a solution of this type has been constructed rigorously in [1] in all space dimensions $d \geq 3$. It behaves like a collapsing shell and moreover, it is proven to be stable under small radial perturbations. Examples for type I blowup are typically provided by self-similar solutions, which are known to exist in space dimensions $d \geq 3$, see [2], [3]. From [4], it is known that any radial non-negative type I solution, which blows up solely at the origin, is asymptotically self-similar. An explicit self-similar blowup solution for $d \geq 3$ was given in [5],

$$u_T(t,x) := \frac{1}{T-t} U\left(\frac{x}{\sqrt{T-t}}\right) \quad \text{where} \quad U(x) = \frac{4(d-2)(2d+|x|^2)}{(2(d-2)+|x|^2)^2},$$

for $T > 0$. Note that $\lim_{t \to T^-} u_T(t,x) = C_d |x|^{-2}$ away from the origin.

According to numerical experiments presented in [5], u_T is conjectured to describe the generic blowup profile for large classes of radial blowup solutions. As the first natural step towards a rigorous verification of this conjecture, we investigate the nonlinear asymptotic stability of u_T under small radial perturbations in the lowest supercritical dimension and obtain the following result.

Theorem 1. *Let $d = 3$. There is an $\varepsilon > 0$ such that for any initial datum*

$$u(0,\cdot) = u_1(0,\cdot) + \varphi_0,$$

where φ_0 is a radial Schwartz function for which

$$\|\varphi_0\|_{L^2(\mathbb{R}^3) \cap \dot{H}^3(\mathbb{R}^3)} < \varepsilon,$$

there exists $T > 0$ and a classical radial solution $u \in C^\infty([0,T) \times \mathbb{R}^3)$ to (1), which blows up at the origin as $t \to T^-$. Furthermore, the following profile decomposition holds

$$u(t,x) = \frac{1}{T-t}\left[U\left(\frac{x}{\sqrt{T-t}}\right) + \varphi\left(t, \frac{x}{\sqrt{T-t}}\right)\right],$$

where $\|\varphi(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^3)} \to 0$ as $t \to T^-$ for all $s \in [0,3]$. In particular,

$$\|\varphi(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \to 0 \quad \text{as} \quad t \to T^-.$$

Our proof is based on the assumption of radial symmetry, $u(t,x) = \tilde{u}(t,|x|)$, and the reformulation of Eq. (1) in terms of the reduced mass

$$\tilde{w}(t,r) := \frac{1}{2r^2}\int_0^r \tilde{u}(t,s)s^2 ds,$$

which yields an evolution equation on $\mathbb{R}^5$ for $w(t,x) = \tilde{w}(t,|x|)$ given by

$$\begin{cases} \partial_t w = \Delta w + \Lambda w^2 + 6w^2, \\ w(0,\cdot) = w_0, \end{cases}$$

with $\Lambda f(x) := x \cdot \nabla f(x)$. Using self-similar variables $(\tau,\xi) \in [0,\infty) \times \mathbb{R}^5$,

$$\xi = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t) + \log T,$$

and the rescaled variable $\Psi(\tau,\xi) := (T-t)w(t,x)$, the pertubation ansatz $\Psi(\tau,\cdot) = U + \Phi(\tau)$ yields

$$\begin{cases} \partial_\tau \Phi(\tau) = L\Phi(\tau) + N\big(\Phi(\tau)\big), \\ \Phi(0) = Tw_0(\sqrt{T}\cdot) - U, \end{cases} \tag{2}$$

with

$$L := \Delta - \frac{1}{2}\Lambda - 1 + 2\Lambda(Uf) + 12Uf \quad \text{and} \quad N(f) = \Lambda f^2 + 6f^2.$$

We study Eq. (2) in intersection Sobolev spaces $X^k := \dot{H}^1(\mathbb{R}^5) \cap \dot{H}^k(\mathbb{R}^5)$ for $k \geq 3$. By combining the analysis of the linear evolution in X^k with a thorough spectral analysis for L in a suitably weighted L^2-space, we prove the existence of a rank-one projection P on X^k, such that the linear evolution decays exponentially on ker$P$ and grows exponentially on rg$P$. The instability arises naturally due to the time translation invariance of the problem. The nonlinear problem is treated via an integral formulation by using fixed point methods. In view of the derivative nonlinearity, this is subtle and crucially relies on smoothing estimates for the linear evolution. The unstable behavior is controlled by suitably adjusting the blowup time $T > 0$ of the solution.

We note that our methods can easily be generalized to higher space dimensions. In particular, for a fixed space dimension, the underlying spectral problem can be solved rigorously.

Our result sheds some more light on the highly complex dynamics for the parabolic-elliptic Keller-Segel system in three space dimensions. However, still many open questions remain. This concerns for example the in [5] observed convergence to u_T of solutions, which did not start off close. Other observations made in [5] raise the question of threshold dynamics between stable regimes of the system. In the radial case, the latter include blowup via u_T and via the collapsing ring solution constructed in [1], respectively, as well as convergence to zero for small initial conditions. Finally, the details of the dynamics outside radial symmetry are completely open.

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The time-like minimal surface equation in Minkowski space: low regularity solutions

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(joint work with Mihaela Ifrim, Daniel Tataru)

Let $n \geq 2$, and $\mathfrak{M}^{n+2}$ be the $n+2$ dimensional Minkowski space-time. A codimension one time-like submanifold $\Sigma \subset \mathfrak{M}^{n+2}$ is called a minimal surface if it is locally a critical point for the area functional

$$\mathcal{L} = \int_\Sigma dA,$$

where the area element is measured relative to the Minkowski metric. Representing Σ as a graph over $\mathfrak{M}^{n+1}$, the minimal surface equation can be thought of as the associated Euler-Lagrange equation, which takes the form

$$-\frac{\partial}{\partial t}\left(\frac{u_t}{\sqrt{1-u_t^2+|\nabla_x u|^2}}\right)+\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1-u_t^2+|\nabla_x u|^2}}\right)=0. \tag{1}$$

The main geometric object is the metric g which is the trace of the Minkowski metric in $\mathfrak{M}^{n+2}$ on Σ, and which, expressed in the $(t = x_0, x_1, \cdots, x_n)$ coordinates, has the form

$$g_{\alpha\beta} := m_{\alpha\beta} + \partial_\alpha u \partial_\beta u, \tag{2}$$

where $m_{\alpha\beta}$ denotes the Minkowski metric with signature $(-1, 1, ..., 1)$ in $\mathfrak{M}^{n+1}$. Since Σ is time-like, this is also a Lorentzian metric. Relative to this metric, the equation (1) can be expressed in the form

$$\Box_g u = 0, \tag{3}$$

where $\Box_g$ is the covariant d'Alembertian. Our objective is to study the local well-posedness of the associated Cauchy problem with initial data at $t = 0$ where the initial data (u_0, u_1) is taken in classical Sobolev spaces,

$$u[0] := (u_0, u_1) \in \mathcal{H}^s := H^s \times H^{s-1}, \tag{4}$$

and is subject to the constraint

$$u_1^2 - |\partial_x u_0|^2 < 1. \tag{5}$$

We aim to investigate the range of exponents s for which local well-posedness holds, and significantly improve the lower bound for this range.

The hyperbolic minimal surface equation (1) can be seen as a special case of more general scalar quasilinear wave equations, which have the form

$$g^{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u = N(u, \partial u), \tag{6}$$

where, again, $g^{\alpha\beta}$ is assumed to be Lorentzian, but without any further structural properties, and where u may be a vector valued function. This generic equation will serve as a reference.

As a starting point, we note that the equation (1) (and also (6) if $N = 0$) admits the scaling law

$$u(t, x) \to \lambda^{-1} u(\lambda t, \lambda x).$$

This allows us to identify the critical Sobolev exponent as

$$s_c = \frac{n+2}{2}.$$

Heuristically, s_c serves as a universal threshold for local well-posedness, i.e. we have to have $s > s_c$. Taking a naive view, one might think of trying to reach the scaling exponent s_c. However, this is a quasilinear wave equation, and getting to s_c has so far proved impossible in any problem of this type.

As a good threshold from above, one might start with the classical well-posedness result, due to Hughes, Kato, and Marsden [4], and which asserts that local well-posedness holds for $s > s_c + 1$. This applies to all equations of the form (6), and can be proved solely by using energy estimates. These have the form

$$\|u[t]\|_{\mathcal{H}^s} \lesssim e^{\int_0^t \|\partial^2 u(s)\|_{L^\infty} ds} \|u[0]\|_{\mathcal{H}^s}. \tag{7}$$

To close the energy estimates, it then suffices to use Sobolev embeddings, which allow one to bound the above L^∞ norm in terms of the $\mathcal{H}^s$ Sobolev norm provided that $s > \frac{n}{2} + 2$, which is one derivative above scaling.

The reason a derivative is lost in the above analysis is that one would only need to bound $\|\partial^2 u\|_{L^1 L^\infty}$, whereas the norm that is actually controlled is $\|\partial^2 u\|_{L^\infty L^\infty}$. This suggests that the natural way to improve the classical result is to control the $L^p L^\infty$ norm directly. This is indeed possible in the context of the Strichartz estimates. When true, Strichartz bounds yield well-posedness for $s > \frac{n+3}{2}$, which is 1/2 derivatives above scaling.

The difficulty in using Strichartz estimates is that, while these are well known in the constant coefficient case [2,6] and even for smooth variable coefficients [5,9], that is not as simple in the case of rough coefficients. Indeed, as it turned out, the full Strichartz estimates are true for C^2 metrics, see [12] ($n = 2, 3$), [17] (all n), but not, in general, for C^σ metrics when $\sigma < 2$, see the counterexamples of [13,14].

The result in [15] represents the starting point of the present work, and is concisely stated as follows:

Theorem 1 (Smith-Tataru [15]). (6) *is locally well-posed in* $\mathcal{H}^s$ *provided that*

$$s > s_c + \frac{3}{4}, \qquad n = 2, \tag{8}$$

respectively

$$s > s_c + \frac{1}{2}, \qquad n \geq 3. \tag{9}$$

The optimality of this result, at least in dimension three, follows from work of Lindblad [8], see also the more recent two dimensional result in [11]. However, this counterexample should only apply to "generic" models, and the local well-posedness threshold might possibly be improved in problems with additional structure, i.e. some form of null condition. We recall that in [18], a null condition was formulated for quasilinear equations of the form (6).

Definition 2 ([18]). *The nonlinear wave equation* (6) *satisfies the* nonlinear null condition *if*

$$\frac{\partial g^{\alpha\beta}(u,p)}{\partial p_\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0 \qquad in \qquad g^{\alpha\beta}(u,p)\xi_\alpha\xi_\beta = 0. \tag{10}$$

Here we use the terminology "nonlinear null condition" in order to distinguish it from the classical null condition, which is relative to the Minkowski metric, and was heavily used in the study of global well-posedness for problems with small localized data, see [10] as well as the books [3, 16]. In geometric terms, this null condition may be seen as a cancellation condition for the self interactions of wave packets traveling along null geodesics. It is straightforward to verify that the minimal surface equation indeed satisfies the nonlinear null condition.

Further, it was conjectured in [18] that, for problems satisfying (10), the local well-posedness threshold can be lowered below the one in [15]. This conjecture has remained fully open until now, though one should mention two results in [7] and [1] for the Einstein equation, respectively the minimal surface equation, where the endpoint in Theorem 1 is reached but not crossed.

Our recent work provides the first positive result in this direction, specifically for the minimal surface equation. Indeed, not only are we able to lower the local well-posedness threshold in Theorem 1, but in effect we obtain a substantial improvement, namely by 3/8 derivatives in two space dimensions and by 1/4 derivatives in higher dimension. Our main result, stated in a succinct form, is as follows:

Theorem 3. *The Cauchy problem for the minimal surface equation* (3) *is locally well-posed for initial data* $u[0]$ *in* $\mathcal{H}^s$ *which satisfies the constraint* (5), *where*

$$s > s_c + \frac{3}{8}, \qquad n = 2, \tag{11}$$

respectively

$$s > s_c + \frac{1}{4}, \qquad n \geq 3. \tag{12}$$

The result is valid regardless of the $\mathcal{H}^s$ size of the initial data. Here we interpret local well-posedness in a strong Hadamard sense, including existence and uniqueness of solutions, higher regularity, and continuous dependence on data.

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Rigidity of blowup solutions to the mass-critical nonlinear Schrödinger equation with mass equal to the mass of the ground state

Benjamin Dodson

In this talk we discuss the proof of rigidity for the focusing mass-critical NLS with mass equal to the mass of the ground state. recently, in [1] and [2], we proved that any solution at mass equal to the mass of the soliton that failed to scatter in both time directions must either be a soliton or the pseudoconformal transformation of the soliton. For technical reasons, the results only hold in dimensions $1 \leq d \leq 15$, although it is likely that they can be extended to all dimensions.

This result extend previous work on this problem that achieved partial results. Specifically, [4] proved this result in all dimensions for finite energy solutions with finite time blow-up. Later, [3] proved rigidity for radially symmetric solutions that blow up in both time directions in dimensions $d \geq 4$,

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Nontrivial global solutions to some quasilinear wave equations in three space dimensions

Dongxiao Yu

In this talk, we discuss the construction of nontrivial global solutions to some quasilinear wave equations in three space dimensions. For simplicity, we focus on a simple example

$$\Box u = (-\partial_t^2 + \Delta_x)u = u_t u_{tt}. \tag{1}$$

Here the unknown u is a real-valued function defined in $\mathbb{R}^{1+3}$. Unlike many other quasilinear wave equations, we do not have a global existence theory for (1), not even if the initial data are small, smooth and localized. In fact, Fritz John [4, 5] proved that any nonzero solution to (1) with C_c^∞ initial data must blow up in finite time. We also refer to Speck [6] for the blowup mechanism for (1).

Despite the previous discussions, there do exist global solutions to (1). For example, any polynomial of degree one is a global solution. In this talk, we present a new method to construct nontrivial global solutions to (1) which are not linear polynomials. Note that these global solutions do not have localized initial data, so their existence does not contradict John's blowup result.

The main tool in our construction is the *geometric reduced system.* In [7,8], I derived this type of asymptotic equations for general quasilinear wave equations by modifying Hörmander's derivation from [1–3]. To derive the geometric reduced system for general quasilinear wave equations

$$g^{\alpha\beta}(u,\partial u)\partial_\alpha\partial_\beta u^{(I)} = F^{(I)}(u,\partial u), \qquad I=1,\dots,M, \tag{2}$$

we make the ansatz

$$u(t,x) = \varepsilon r^{-1}U(s,q,\omega). \tag{3}$$

Here $0<\varepsilon\ll 1$, $r=|x|$, $s=\varepsilon\ln t$, $\omega = x/|x|$ and q is an optical function:

$$g^{\alpha\beta}(u,\partial u)q_\alpha q_\beta = 0. \tag{4}$$

By setting $\mu = q_t - q_r$ and viewing both U and μ as unknown functions of (s,q,ω), we obtain the following system (called the geometric reduced system) for (2):

$$\begin{cases} \partial_s(\mu U_q^{(I)}) = S^{(I)}(U,\mu U_q,\omega), \quad I=1,2,\dots,M \\ \partial_s\mu = Q(\mu,\mu U_q,\partial_q(\mu U_q),\omega) \end{cases} \tag{5}$$

Here $S^{(I)}$ is only determined by $F^{(I)}(u,\partial u)$, and Q is only determined by $g^{\alpha\beta}(u,\partial u)$. For example, the geometric reduced system for (1) is

$$\begin{cases} \partial_s(\mu U_q) = 0 \\ \partial_s\mu = \frac{1}{8}\mu^2\partial_q(\mu U_q) \end{cases} \tag{6}$$

Since a nonzero global solution to (6) for $s\geq 0$ does not have good decaying properties as $|q|\to\infty$, we will not construct global solutions to (1) directly. Instead, we will construct global solutions to the following equations for $v=(v_\alpha)=(\partial_\alpha u)$:

$$(\Box - v_{(0)}\partial_t^2)v_{(\alpha)} = \frac{1}{2}(\partial_\alpha v_{(0)} + \partial_t v_{(\alpha)})\partial_t v_{(0)}; \tag{7}$$

$$\partial_\alpha v_{(\beta)} = \partial_\beta v_{(\alpha)}. \tag{8}$$

We obtain these equations by differentiating (1). If v solves (7) and (8), then we can recover a solution to (1). By setting $\omega_0=-1$, we obtain the geometric reduced system for (7):

$$\begin{cases} \partial_s(\mu\partial_q V_{(\alpha)}) = \frac{1}{8}(\omega_\alpha\mu\partial_q V_{(0)} - \mu\partial_q V_{(\alpha)})\mu\partial_q V_{(0)}, \\ \partial_s\mu = -\frac{1}{4}\mu^2\partial_q V_{(0)}, \end{cases} \tag{9}$$

along with the constraint equations from (8):

$$\omega_\alpha\partial_q V_{(\beta)} = \omega_\beta\partial_q V_{(\alpha)}. \tag{10}$$

This system can be solved explicitly. In fact, if $(\mu,\partial_q V_{(0)})|_{s=0} = (-2,A)$ where $A\in C_c^\infty(\mathbb{R}\times\mathbb{S}^2)$, we obtain a solution:

$$V_{(\alpha)} = \int_{-\infty}^q -A(s,p,\omega)\omega_\alpha\, dp, \qquad \mu = \frac{4}{As-2}. \tag{11}$$

This is a global solution for $s\geq 0$ as long as $A\leq 0$ everywhere.

Let us now briefly discuss how the construction works. Given a global solution (11) for all $s \geq 0$, we construct an approximate optical function $q = q(t, x)$ by solving $q_t - q_r = \mu$. Both s and q are functions of (t, x), so we can view V as a function of (t, x). By localizing $\varepsilon r^{-1} V$ in a conic neighborhood of the light cone $r = t$, we obtain an approximate solution $v_{app} = v_{app}(t, x)$ to the wave equations in (7). With this approximate solution, fixing a large time $T \geq 0$, we now solve a certain backward Cauchy problem for v. This Cauchy problem for v is well chosen so that $v \equiv v_{app}$ for $t \geq 2T$ and that v solves the wave equations in (7) for $t \leq T$. By proving that $v = v^T$ converges as $T \to \infty$, we obtain a global solution to (7). Finally, we check that this solution v also satisfies (8) by deriving the equations $w = (\partial_\alpha v_{(\beta)} - \partial_\beta v_{(\alpha)})$. This ends our construction.

Our final remark is that this method to construct global solutions works not just for (1) but for a large class of quasilinear wave equations. For example, we can also use it to construct global solutions to the 3D compressible Euler equations without vorticity. For more details, we refer to [9].

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KdV with exotic spatial asymptotics

Thierry Laurens

A fundamental line of investigation for the Korteweg–de Vries (KdV) equation

$$\frac{d}{dt} u = -u''' + 6uu' \tag{1}$$

(where $u' = \partial_x u$) has been well-posedness in the L^2-based Sobolev spaces H^s on both the line $\mathbb{R}$ and the circle $\mathbb{R}/\mathbb{Z}$. Over the past 50 years, numerous cutting-edge methods have been introduced in the search for the lowest attainable regularity s. Recently, this effort culminated in the work [2] of Killip and Vişan who obtained global well-posedness in $H^{-1}(\mathbb{R})$ and $H^{-1}(\mathbb{R}/\mathbb{Z})$, a result that is sharp in both geometries.

Solutions in $H^s(\mathbb{R}/\mathbb{Z})$ spaces are spatially periodic and solutions in $H^s(\mathbb{R})$ spaces decay at infinity; however, there are other waveforms which are of physical interest. One such class of functions is initial data that is asymptotically periodic in x. This most commonly appears in the context of stability for the periodic traveling wave solutions, or *cnoidal waves*, of KdV. Such initial data also arises in the study of wave dislocation where the periods as $x \to \pm\infty$ may not align, and waves with altogether different periodic asymptotics as $x \to \pm\infty$.

Another important class of initial data are waveforms that are step-like, in the sense that $u(0, x)$ approaches different constant values as $x \to \pm\infty$. These arise in the study of bore propagation and rarefaction waves, and their long-time behavior has been heavily studied via the inverse scattering transform.

Although such waveforms are of great interest, their well-posedness theory is comparatively lacking. Our objective in this talk is to discuss the extension of low-regularity methods for well-posedness to the regime of exotic spatial asymptotics. Specifically, we will employ the method of commuting flows from [2]. This method relies on the existence of a generating function $\alpha(\kappa, u)$ for the polynomial conserved quantities of KdV, with the asymptotic expansion

$$\alpha(\kappa, u) = \tfrac{1}{4\kappa^3} P(u) - \tfrac{1}{16\kappa^5} H_{\mathrm{KdV}}(u) + O(\tfrac{1}{\kappa^7}) \tag{2}$$

for Schwartz u. Here, P and H_{KdV} denote the momentum and energy functionals

$$P(u) := \tfrac{1}{2} \int u(x)^2 \, dx, \qquad H_{\mathrm{KdV}}(u) := \int \left(\tfrac{1}{2} u'(x)^2 + u(x)^3\right) dx.$$

Rearranging the expansion (2), we might expect that the dynamics of the Hamiltonians

$$H_\kappa(u) := -16\kappa^5 \alpha(\kappa, u) + 4\kappa^2 P(u)$$

approximate that of KdV as $\kappa \to \infty$. In [2] the authors demonstrated that the flow induced by the Hamiltonian H_κ is well-posed in H^{-1} and converges to that of KdV in H^{-1} as $\kappa \to \infty$.

Our goal is to show that given a sufficiently regular solution $V(t, x)$ to KdV, KdV is also well-posed for $H^{-1}(\mathbb{R})$ perturbations of V. Specifically, we assume that the background wave V satisfies the following hypotheses:

Definition 1. *We call the background wave* $V(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ admissible *if for every* $T > 0$ *it satisfies the following:*

(i) V *solves KdV* (1) *and is bounded in* $W^{2,\infty}(\mathbb{R}_x)$ *uniformly for* $|t| \le T$*,*
(ii) The H_κ *flows* $V_\kappa(t, x)$ *with initial data* $V(0, x)$ *are bounded in* $W^{4,\infty}(\mathbb{R}_x)$ *uniformly for* $|t| \le T$ *and* $\kappa > 0$ *sufficiently large,*
(iii) $V_\kappa - V \to 0$ *in* $W^{2,\infty}(\mathbb{R}_x)$ *as* $\kappa \to \infty$ *uniformly for* $|t| \le T$ *and initial data in the set* $\{V_\varkappa(t) : |t| \le T, \ \varkappa \ge \kappa\}$*.*

The focus of this talk is the following general well-posedness result from [3]:

Theorem 2 (Global well-posedness)**.** *Given V admissible in the sense of Definition 1, the KdV equation* (1) *with initial data $u(0) \in V(0) + H^{-1}(\mathbb{R})$ is globally well-posed in the following sense: $u(t) = V(t) + q(t)$ and $q(t)$ is given by a jointly continuous data-to-solution map $\mathbb{R}_t \times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$ for the equation*

$$\frac{d}{dt} q = -q''' + 6qq' + 6(Vq)' \tag{3}$$

with initial data $q(0) = u(0) - V(0)$.

As we cannot make sense of the nonlinearity of KdV for $H^{-1}(\mathbb{R})$ solutions (even in the distributional sense), the solutions in Theorem 2 are constructed as limits of the H_κ flows as $\kappa \to \infty$. This is the right notion of solution because it coincides with the classical notion on a dense subset of initial data and Theorem 2 guarantees continuous dependence of the solution upon the initial data. Indeed, in [3] we show that for initial data $u(0) \in V(0) + H^3(\mathbb{R})$ our solution $u(t)$ solves KdV and is unique:

Theorem 3. *Fix V admissible and $T > 0$. Given initial data $u(0) \in V(0) + H^3(\mathbb{R})$, the solution $u(t)$ constructed in Theorem 2 lies in $V(t) + (C_t H^2 \cap C_t^1 H^{-1})([-T, T] \times \mathbb{R})$ for all $T > 0$, solves KdV* (1)*, and is unique in this class.*

There are rich classes of background waves that are admissible according to Definition 1. In [3], we cover the important case of smooth periodic initial data $V(0, x)$ (which includes cnoidal waves):

Corollary 4 (Periodic background)**.** *Given $V(0) \in H^5(\mathbb{R}/\mathbb{Z})$, the KdV equation* (1) *is globally well-posed for $u(0) \in V(0) + H^{-1}(\mathbb{R})$ in the sense of Theorem 2.*

In the companion paper [4], we show that Theorem 2 also applies to smooth step-like initial data. Let

$$W(x) = c_1 \tanh(x) + c_2 \quad \text{with } c_1, c_2 \in \mathbb{R} \text{ fixed}$$

denote a smooth step function which exponentially decays to its asymptotic values. As $-u$ is proportional to the water wave height by our convention, W models an incoming tide if $c_1 > 0$ and an outgoing tide if $c_1 < 0$. In both cases, we show that KdV is well-posed for $H^{-1}(\mathbb{R})$ perturbations of W:

Corollary 5 (Step-like background)**.** *Given initial data W, there is a unique global solution $V(t)$ to KdV* (1) *that lies in $W + H^s(\mathbb{R})$ for all $s \geq 3$. Moreover, KdV is globally well-posed for $u(0) \in W + H^{-1}(\mathbb{R})$ in the sense of Theorem 2.*

The main thrust of the talk will be to discuss the proof of Theorem 2. A major obstacle we encounter is that the presence of the background wave V breaks the macroscopic conservation laws of KdV. Indeed, if q is a regular solution to the equation (3) for the perturbation, then the momentum evolves according to

$$\frac{d}{dt} \int q(t, x)^2 \, dx = 6 \int V'(t, x) q(t, x)^2 \, dx.$$

In particular, for increasing step-like initial data $V(0, x)$ the term V' has a sign and there is no cancellation in the integral; the resulting growth in the momentum is manifested in a dispersive shock that develops in the long-time asymptotics [1, Fig. 1]. Despite this lack of conservation, we are able to adapt the method of commuting flows to (3) because these quantities do not blow up in finite time.

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The scattering map determines the nonlinearity

Jason Murphy

(joint work with Rowan Killip, Monica Visan)

We consider two-dimensional nonlinear Schrödinger equations of the form

$$i\partial_t u + \Delta u = F(u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \tag{1}$$

where we regard the nonlinearity $F : \mathbb{C} \to \mathbb{C}$ as an unknown parameter. We assume that $F(u) = h(|u|^2)u$ for some $h : [0,\infty) \to \mathbb{C}$ with

$$h(0) = 0 \quad \text{and} \quad |h'(\lambda)| \lesssim 1 + \lambda^{\frac{p}{2}-1}$$

for some $2 \le p < \infty$. We call such F *admissible*, with *growth parameter* p. A standard fixed-point argument using Strichartz estimates shows that this type of nonlinearity admits a small-data scattering theory in H^1:

Theorem 1 (Small data scattering). *Let $F : \mathbb{C} \to \mathbb{C}$ be an admissible nonlinearity with growth parameter $p \ge 2$. Define $s_p = 1 - \frac{2}{p}$ and*

$$B_\eta = \{f \in H^1 : \|f\|_{H^{s_p}} < \eta\}. \tag{2}$$

There exists $\eta > 0$ sufficiently small so that any initial data $u_0 \in B_\eta$ leads to a unique global solution u, which scatters in both time directions. That is, there exist $u_\pm \in H^1$ so that

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{H^1} = 0. \tag{3}$$

Additionally, for any $u_- \in B_\eta$ there exists a unique global solution u to (1) *and a unique $u_+ \in H^1$ such that* (3) *holds.*

The mapping $u_0 \mapsto u_+$ described in Theorem 1 is known as the (forward) *wave operator*, denoted $\Omega_F : B \to H^1$. The mapping $u_- \mapsto u_+$ is known as the *scattering map*, denoted $S_F : B \to H^1$. Our main result shows that knowledge of either the wave operator or the scattering map uniquely determines the nonlinearity in (1).

Theorem 2 (Scattering determines the nonlinearity, [3])**.** *Let* $F : \mathbb{C} \to \mathbb{C}$ *and* $\tilde{F} : \mathbb{C} \to \mathbb{C}$ *be admissible, with potentially distinct growth parameters. If* $\Omega_F = \Omega_{\tilde{F}}$ *or* $S_F = S_{\tilde{F}}$ *on* $B \cap \tilde{B}$*, then* $F = \tilde{F}$*.*

Previous works on the recovery of the nonlinearity from the scattering map have typically either considered analytic nonlinearities or made other strong structural assumptions on the nonlinearity (see e.g. [1, 2, 5–8]). Compared to the existing literature, our result requires only very mild assumptions about the nonlinearity.

The proof of Theorem 2 consists of two main steps. The first step is the reduction of the main problem to an inverse convolution problem. Specifically, we prove that if F and $\tilde{F}$ have the same scattering data, then

$$\int_{\mathbb{R}} [G'(e^{-k}) - \tilde{G}'(e^{-k})] e^{-k} w(k+\ell)\, dk = 0 \quad \text{for all} \quad \ell \in \mathbb{R}, \tag{4}$$

where $G(|u|^2) := F(u)\bar{u}$, $\tilde{G}(|u|^2) = \tilde{F}(u)\bar{u}$, and

$$w(k) := \big[(e^k - 1)^{\frac{3}{2}} + 6(e^k - 1)^{\frac{1}{2}} - 6\arctan\big((e^k-1)^{\frac{1}{2}}\big)\big]\chi_{(0,\infty)}(k)$$

The proof of (4) relies on the Born approximation to the scattering map evaluated on Gaussian data. The weight w is related to the distribution function for the linear Schrödinger flow with Gaussian data, which we can compute explicitly in two space dimensions.

The second step of the proof consists of showing that (4) implies $G' \equiv \tilde{G}'$. Under mild hypotheses on the nonlinearity, this can be derived from Wiener's Tauberian Theorem. To address the full range of admissible nonlinearities, we employ a theorem of Beurling and Lax characterizing shift-invariant subspaces of Hardy space (see [4]). This requires us to prove that the Laplace transform of w defines an outer function on some half-plane of $\mathbb{C}$. The proof of this latter fact also relies on an explicit computation, namely,

$$\int_0^\infty e^{-kz} w(k)\, dk = 9\sqrt{\pi}\, \frac{\Gamma(z+\frac{1}{2})}{\Gamma(z+1)}\, \frac{z-1}{z(2z-1)(2z-3)}$$

for $\operatorname{Re} z > \frac{3}{2}$, where $\Gamma(\cdot)$ is the Gamma function.

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Stability of a point charge for the repulsive Vlasov-Poisson system

Benoît Pausader
(joint work with Klaus Widmayer, Jiaqi Yang)

1. The Vlasov-Poisson system with a point charge

We consider solutions of the Vlasov-Poisson system of the form

$$M(t) := q_c\delta_{(\mathcal{X}(t),\mathcal{V}(t))}(\mathbf{x},\mathbf{v}) + q_g\mu^2(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v},$$

representing a smooth charge distribution $\mu^2 d\mathbf{x}d\mathbf{v}$ on $\mathbb{R}^3\times\mathbb{R}^3$ coupled with a point charge located at $(\mathcal{X},\mathcal{V}) : \mathbb{R}_t \to \mathbb{R}^3\times\mathbb{R}^3$. The system takes the form

(1)
$$\left(\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{x}} + \frac{q}{2}\frac{\mathbf{x}-\mathcal{X}}{|\mathbf{x}-\mathcal{X}|^3}\cdot\nabla_{\mathbf{v}}\right)\mu + Q\nabla_{\mathbf{x}}\phi\cdot\nabla_{\mathbf{v}}\mu = 0, \qquad \Delta_{\mathbf{x}}\phi = \int_{\mathbb{R}^3_{\mathbf{v}}}\mu^2 d\mathbf{v},$$
$$\frac{d\mathcal{X}}{dt} = \mathcal{V}, \qquad \frac{d\mathcal{V}}{dt} = \mathcal{Q}\nabla_{\mathbf{x}}\phi(\mathcal{X},t),$$

for positive constants $q, Q, \mathcal{Q} > 0$. The crucial qualitative feature of the forces in (1) is the *repulsive* nature of interactions between the gas and the point charge, i.e. the fact that $q > 0$. The attractive case $q < 0$ leads to a very different linearized system containing closed trajectories.

In the absence of a point charge (the *stability of vacuum*), the Vlasov-Poisson system has been extensively studied. Global existence is classical [10], decay of small perturbations has been established in various ways [1] and the asymptotic behavior was described in [8] and the scattering theory obtained in [5]. We also refer to [12] for recent progress on large perturbations.

Local and global existence in the case of a point charge was studied previously in [2–4, 9, 11]. Stability of a point charge for radial perturbations was established in [13], where several new components of our main strategy were intiated. We also mention the recent [6] which considers the transfer of energy between a fast point charge and a slow homogeneous background, as well as the recent [7] which establishes stability of an homogeneous equilibrium.

1.1. **Main result.** Our main result concerns (1) with sufficiently small and localized initial charge distributions μ_0. We establish the existence and uniqueness of global, strong solutions and we describe their asymptotic behavior as a modified scattering dynamic. While our full result can be most adequately stated in more adapted "action-angle" variables, for the sake of readability we give here a (weaker, slightly informal) version in standard Cartesian coordinates and refer to [14, Theorem 6.1] for the main result:

Theorem 1. *Given any* $(\mathcal{X}_0, \mathcal{V}_0) \in \mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}}$ *and any initial data* $\mu_0 \in C^1_c((\mathbb{R}^3_{\mathbf{x}} \setminus \{\mathcal{X}_0\}) \times \mathbb{R}^3_{\mathbf{v}})$*, there exists* $\varepsilon^* > 0$ *such that for any* $0 < \varepsilon < \varepsilon^*$*, there exists a unique global strong solution of* (1) *with initial data*

$$(\mathcal{X}(t=0), \mathcal{V}(t=0)) = (\mathcal{X}_0, \mathcal{V}_0), \qquad \mu(\mathbf{x}, \mathbf{v}, t=0) = \varepsilon\mu_0(\mathbf{x}, \mathbf{v}).$$

Moreover, the electric field decays pointwise at optimal rate, and there exists a point charge trajectory $(\mathcal{X}(t), \mathcal{V}(t))$*, an asymptotic profile* $\mu_\infty \in L^2((\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3)$ *and a Lagrangian map* $(\mathbf{Y}, \mathbf{W}) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^*_+ \to \mathbb{R}^3 \times \mathbb{R}^3$ *along which the particle distribution converges pointwise*

$$\mu(\mathbf{Y}(\mathbf{x}, \mathbf{v}, t), \mathbf{W}(\mathbf{x}, \mathbf{v}, t), t) \to \mu_\infty(\mathbf{x}, \mathbf{v}), \qquad t \to \infty.$$

The Lagrangian map can be expressed explicitly and involves logarithmic corrections to the linearized dynamics, revealing a modified scattering dynamic in (1).

2. The method of asymptotic action

For simplicity, we will focus here on the case when the charge does not move (this can be achieved using symmetry considerations). In this case, (1) becomes an equation for the gas distribution μ alone, which we can recast in Hamiltonian transport form:

$$\partial_t \mu - \{\mathbb{H}, \mu\} = 0, \qquad \mathbb{H} := \mathbb{H}_0 - \mathbb{H}_2, \qquad \mathbb{H}_0 := |\mathbf{v}|^2 + q/|\mathbf{x}| \tag{2}$$

where we have expanded the Hamiltonian $\mathbb{H}$ according to homogeneity in μ, so that $\mathbb{H}_0$ refers to terms independent of μ and will lead to the linearized equation, while the nonlinear term involves ϕ and is given in (6) below.

2.1. **General overview.** Our overall scheme follows the basic tenets of a small data/global existence problem. We first study the linearized problem, i.e. (2) without $\mathbb{H}_2$. The gas distribution is subject to the electrostatic field generated by the point charge and thus solves a singular transport equation by (3) which can be integrated explicitly through a canonical change of coordinates. Upon appropriate choice of unknown γ in these variables, we are reduced to the study of a purely nonlinear equation (6), given in terms of the electrostatic potential ϕ. This function and the derived electric field can be conveniently expressed (thanks to the canonical nature of the change of coordinates) as integrals of γ over phase space, and we study their boundedness properties. In particular, assuming moments and derivatives on γ, we establish that electrical functions decay pointwise. With this, we show how to propagate such moments and derivatives, relying heavily on the Poisson bracket structure in (2). Finally, this reveals the asymptotic behavior through an asymptotic shear equation that builds on a phase mixing property of asymptotic actions.

2.2. **Asymptotic action-angle coordinates.** One of the core principles introduced in this work is the importance of the *asymptotic action-angle coordinates* and an effective way to define them. Their main properties are the following:

Proposition 2. *There exists a smooth diffeomorphism* $\mathcal{T}$, $(\mathbf{x},\mathbf{v}) \mapsto (\Theta,\mathcal{A})$ *with inverse* $(\vartheta,\mathbf{a}) \mapsto (\mathbf{X},\mathbf{V})$, *which (i) is canonical, i.e.* $d\mathbf{x} \wedge d\mathbf{v} = d\Theta \wedge d\mathcal{A}$, *(ii) linearizes the flow of the Kepler ODE*

$$\dot{\mathbf{x}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = q\mathbf{x}/|\mathbf{x}|^3 \tag{3}$$

in the sense that $(\mathbf{x}(t),\mathbf{v}(t))$ *solves* (3) *if and only if*

$$\Theta(\mathbf{x}(t),\mathbf{v}(t)) = \Theta(\mathbf{x}(0),\mathbf{v}(0)) + t\mathcal{A}(\mathbf{x}(0),\mathbf{v}(0)), \quad \mathcal{A}(\mathbf{x}(t),\mathbf{v}(t)) = \mathcal{A}(\mathbf{x}(0),\mathbf{v}(0)), \tag{4}$$

(iii) satisfies the "asymptotic action property"

$$|\mathbf{X}(\vartheta + t\mathbf{a},\mathbf{a}) - t\mathbf{a}| = o(t), \qquad |\mathbf{V}(\vartheta + t\mathbf{a},\mathbf{a}) - \mathbf{a}| = o(1) \qquad \text{as } t \to +\infty. \tag{5}$$

It turns out that our change of variable also preserves the *angular momentum* and leads to a simple formula for the energy: $\mathbb{H}_0 = |\mathcal{A}|^2$. The *asymptotic action* property *(iii)* will be crucial to the asymptotic analysis. In short, it ensures that $\mathbf{a}$ parameterizes the trajectories that stay at a bounded distance from each other as $t \to \infty$ and connects in an effective way the trajectories of (3) to those of the free streaming $\ddot{\mathbf{x}} = 0$.

General dynamical systems are not, of course, completely integrable, and when they are, there are many different action-angle coordinates. Here the *asymptotic-action property* fixes the actions and helps restrict the set of choices. Besides, since the actions are defined in a natural way (as asymptotic velocities, see (5)), one can aim to find $\mathcal{T}$ through a *generating function* $\mathcal{S}(\mathbf{x},\mathbf{a})$ by solving a

Scattering problem: Given an asymptotic velocity $\mathbf{v}_\infty =: \mathbf{a} \in \mathbb{R}^3$ and a location $\mathbf{x}_0 \in \mathbb{R}^3$, find (if they exist) the trajectories $(\mathbf{x}(t),\mathbf{v}(t))$ through $\mathbf{x}_0$ with asymptotic velocity $\mathbf{a}$.

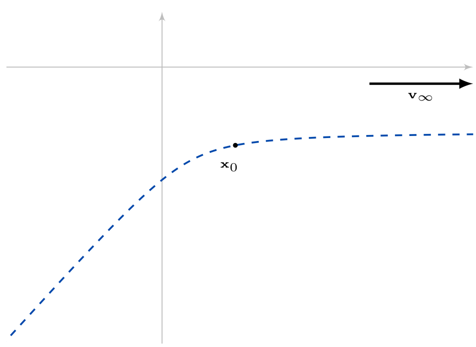

FIGURE 1. The scattering problem: Given a point $\mathbf{x}_0$ and asymptotic velocity $\mathbf{v}_\infty$, how to determine trajectories (one possibility dashed in blue) through $\mathbf{x}_0$ with asymptotic velocity $\mathbf{v}_\infty$?

Once such a trajectory has been found, one can define $\mathfrak{V}(\mathbf{x},\mathbf{a})$ as the velocity along the trajectory at $\mathbf{x}$, and look for a putative $\mathcal{S}$ such that $\mathfrak{V}(\mathbf{x},\mathbf{a}) = \nabla_{\mathbf{x}}\mathcal{S}$. By classical arguments, setting $\vartheta := \nabla_{\mathbf{v}}\mathcal{S}$ then yields a (local) canonical change of variables.

2.3. **Use of asymptotic action-angle coordinates.** Assuming that one can find appropriate action-angle coordinates, and control appropriate moments, the asymptotic behavior follows formally from the asymptotic action properties. Indeed, we can first conjugate by the linearized flow and define the new unknown

$$\gamma(\vartheta,\mathbf{a},t) := \mu(\mathbf{X}(\vartheta+t\mathbf{a},\mathbf{a}),\mathbf{V}(\vartheta+t\mathbf{a},\mathbf{a}),t)$$

which solves the purely nonlinear equation

$$\begin{aligned}&\partial_t\gamma+\{\mathbb{H}_2,\gamma\}=0,\qquad \mathbb{H}_2:=Q\psi(\mathbf{X}(\vartheta+t\mathbf{a},\mathbf{a}),t),\\ &\psi(\mathbf{y},t):=-\frac{1}{4\pi}\iint\frac{1}{|\mathbf{y}-\mathbf{X}(\vartheta+t\mathbf{a},\mathbf{a})|}\gamma^2(\vartheta,\mathbf{a},t)\,d\vartheta d\mathbf{a}.\end{aligned}\tag{6}$$

Now a consequence of the asymptotic action property is that

$$|\mathbf{X}(\vartheta+t\mathbf{a},\mathbf{a})-\mathbf{X}(\theta+t\alpha,\alpha)|=t|\mathbf{a}-\alpha|+O(\ln t)$$

and thus, looking at the formula, we see that $\psi(\mathbf{X}(\vartheta+t\mathbf{a},\mathbf{a}),t) = t^{-1}\Psi(\mathbf{a},t) + O(t^{-3/2})$ for some converging profile Ψ. Then the nonlinear equation reduces to

$$\partial_t\gamma+\frac{Q}{t}\{\Psi(\mathbf{a},t),\gamma\}=l.o.t.$$

which is transport by a shear flow which can easily be integrated leading to a logarithmic correction. This illustrates the power of the *asymptotic action-angle coordinates*: in these variables, the effective nonlinearity only depends on the asymptotic actions ("half" of the variables have been mixed out) and, through the properties of the Poisson bracket, can easily be integrated.

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Conserved energies for the one-dimensional Gross-Pitaevskii equation in general regularity setting

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(joint work with Herbert Koch)

We consider the one-dimensional Gross-Pitaevskii equation

$$i\partial_t q + \partial_{xx} q = 2(|q|^2 - 1)q, \tag{GP}$$

where $q = q(t,x) : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{C}$ denotes the unknown wave function and satisfies the following nonzero boundary condition at infinity

$$|q(t,x)| \to 1 \text{ as } |x| \to \infty.$$

We are interested in the global-in-time well-posedness and the (almost) conserved energies for the Cauchy problem of (GP) in the general regularity setting.

The general finite-energy spaces. Motivated by the conserved energy of the Gross-Pitaevskii equation (GP):

$$\mathcal{E}(q) = \int_{\mathbb{R}} \big((|q|^2-1)^2 + |\partial_x q|^2\big)dx =: \big\|\big(|q|^2-1, \partial_x q\big)\big\|^2_{L^2(\mathbb{R})},$$

we introduce more general finite-energy spaces

$$X^s = \big\{q \in H^s_{\mathrm{loc}}(\mathbb{R}) \,|\, E^s(q) := \big\|\big(|q|^2-1, \partial_x q\big)\big\|_{H^{s-1}(\mathbb{R})} < \infty\big\}/\mathbb{S}^1, \quad s \ge 0,$$

where $\mathbb{S}^1$ denotes the unit circle (that is, the functions which differ by a multiplicative constant of modulus 1 are identified in X^s).

There are some typical functions in the space X^s:

- The constant function $1 = -1 = i = e^{i\theta}$, $\forall\theta \in \mathbb{R}$ in X^s has zero energy norm: $E^s(1) = 0$;
- The function $\tanh(x) \in X^s$ has limits ±1 (up to $\mathbb{S}^1$) at $\pm\infty$;
- The function $e^{i\ln(1+|x|)} \in X^1$ has no limits at infinity. One can introduce a small parameter ε such that the energy $E^1(e^{i\ln(1+\varepsilon|x|)})$ is of size $\varepsilon^{\frac{1}{2}}$.

We have the following facts for the general finite-energy spaces:

(1) Equipped with the distance function

$$d^s(p,q) = \Big(\int_{\mathbb{R}} \inf_{|\lambda(y)|=1} \big\| \operatorname{sech}(x-y)(\lambda p(x) - q(x)) \big\|^2_{H^s_x(\mathbb{R})} dy \Big)^{\frac{1}{2}},$$

the space (X^s, d^s), $s \geq 0$ is a complete metric space, and the function set $1 + \mathcal{S}(\mathbb{R})$ is a dense subset in it.

(2) The subset consisting of all soliton solutions for (GP) and the constant function 1

$$Q = \{ q_c(x) = \sqrt{1-c^2}\tanh(\sqrt{1-c^2}x) + ic \,|\, -1 \leq c \leq 1 \}$$

is a strong deformation retract of (X^s, d^s), and hence (X^s, d^s) is homotopy equivalent to the circle.

(3) The asymptotic change of phase Θ is defined as

$$\Theta : \{ q \in X^0 \,|\, \partial_x q \in L^1 \} \to \mathbb{R}/(2\pi\mathbb{Z}).$$

The Hamiltonian

$$H_1(q) = -\operatorname{Im} \int_{\mathbb{R}} \big((|q|^2 - 1)\partial_x \log q \big)\, dx \in \mathbb{R}/(2\pi\mathbb{Z}),$$

which is defined on $\{ q \in X^{(\frac{1}{2})+} | q \neq 0 \}$, has a unique continuous and smooth extension to $X^{\frac{1}{2}}$ modulo $2\pi\mathbb{Z}$. It is conserved by the (GP) flow.

Conserved energies. Motivated by the reformulation of the rescaled energy norms

$$(E^s_\tau(q))^2 := \int_{\mathbb{R}} |(\widehat{|q|^2 - 1}(\xi), \widehat{\partial_x q}(\xi))|^2 (\tau^2 + \xi^2)^{s-1} d\xi, \quad s \in \mathbb{R}, \quad \tau \geq 2,$$

in terms of $E^0_\tau(q)$ as follows[1]

$$(E^s_{\tau_0}(q))^2 = -\frac{2}{\pi}\sin(\pi(s-1)) \int_{\tau_0}^{\infty} (\tau^2 - \tau_0^2)^{s-1} \tau (E^0_\tau(q))^2 d\tau, \quad \forall s \in (0,1),\ \tau_0 \geq 2,$$

we define our conserved energies

$$\mathcal{E}^0_\tau(q) = -\tau \operatorname{Re} \ln T_c^{-1}\Big(i\sqrt{\frac{\tau^2}{4} - 1} \Big),$$

and for $s \in (0,1)$,

$$\mathcal{E}^s_{\tau_0}(q) = \frac{2}{\pi}\sin(\pi(s-1)) \int_{\tau_0}^{\infty} (\tau^2 - {\tau_0}^2)^{s-1} \tau^2 \operatorname{Re} \ln T_c^{-1}\Big(i\sqrt{\frac{\tau^2}{4} - 1} \Big) d\tau,$$

such that the difference is of cubic error:

$$\big| -\mathcal{E}^s_\tau(q) + (E^s_\tau(q))^2 \big| \leq C \frac{E^0_\tau(q)}{\sqrt{\tau}} \big(E^s_\tau(q) \big)^2, \quad \forall s \in [0,1), \tag{1}$$

if the large parameter τ is chosen such that $\frac{E^0_\tau(q)}{\sqrt{\tau}} \leq \frac{1}{2C}$.

[1] One has to remove the part corresponding to $E^{[s]}_\tau(q)$ in the integrand for general $s \geq 1$. We restrict ourselves to the case $s \in [0,1)$ for simplicity.

Here the notation $T_c^{-1} = T_c^{-1}(\lambda; q)$ denotes the renormalized transmission coefficient associated to the Lax operator of the completely integrable (GP) system. It is defined on the upper sheet of the Riemannian surface

$$\mathcal{R} := \left\{(\lambda, z) \,|\, \lambda \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)), \quad z = z(\lambda) = \sqrt{\lambda^2 - 1} \text{ with } \mathrm{Im} z > 0\right\},$$

where λ is the coordinate on $\mathcal{R}$, and on the energy space $q \in X^0$. If $q \in X^0$ with $|q|^2 - 1, \partial_x q \in L^1$, the renormalized transmission coefficient T_c^{-1} is related to the original transmission coefficient T^{-1} as follows

$$T_c^{-1}(\lambda) = T^{-1}(\lambda) \exp\Big(-i\mathcal{M}(2z)^{-1} - i\Theta(2z(\lambda + z))^{-1}\Big),$$

where $\mathcal{M}$ denotes the mass $\mathcal{M}(q) = \int_{\mathbb{R}} (|q|^2 - 1)\, dx$, and $\Theta(q) \in \mathbb{R}/2\pi\mathbb{Z}$ is the asymptotic change of phase. By the density argument, one can show that $T_c^{-1}(\lambda, q) : \mathcal{R} \times X^0 \mapsto \mathbb{C}/e^{\frac{i}{z(\lambda+z)}\pi\mathbb{Z}}$ is conserved by (GP) flow. By the error estimate (1), one can show the uniform bound for the energy norm globally in time

$$E^s(q(t)) \le C_s(E^s(q_0)), \quad \forall s \ge 0, \quad \forall t \in \mathbb{R}.$$

This, together with the local-in-time well-posedness result, gives the global-in-time well-posedness result for the Cauchy problem of (GP) in X^s, $\forall s \ge 0$.

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The virial theorem for water-waves

Thomas Alazard

(joint work with Claude Zuily)

The water-waves equations dictate the dynamics of the interface separating air from a perfect fluid. This is a system of two coupled equations: the incompressible Euler equation inside the fluid domain, and a kinematic equation describing the evolution of the interface. Assuming that the flow is irrotational, the problem is described by two unknowns: the velocity potential ϕ, whose gradient gives the velocity, and the free surface elevation η, whose graph is the free surface. In this talk, for the sake of simplicity, we consider the gravity equations, without surface tension, and assume that the fluid domain has infinite depth. Then, at time t, the fluid domain is of the form

$$\Omega(t) = \{\, (x, y) \in \mathbb{T}^d \times \mathbb{R} \,:\, y < \eta(t, x) \,\},$$

and the conservation of energy reads $dE/dt = 0$ where the energy is the sum $E = E_k + E_p$ of the kinetic and potential energies:

$$E_k(t) = \frac{1}{2} \iint_{\Omega(t)} \left|\nabla_{x,y}\phi(t, x, y)\right|^2 dy\, dx, \quad E_p(t) = \frac{g}{2} \int_{\mathbb{T}^d} \eta(t, x)^2\, dx.$$

Zakharov ([7]) observed that the conservation of energy is in fact associated to an hamiltonian formulation:

$$\partial_t \eta = \frac{\delta E}{\delta \psi} \quad ; \quad \partial_t \psi = -\frac{\delta E}{\delta \eta},$$

where $\psi(t,x) = \phi(t,x,\eta(t,x))$.

This talk is motivated by the study of the principle of equipartition of energy. For linear and non linear wave-type equations, equipartition of kinetic and potential energy holds in various senses as shown by Brodsky [2], Lax and Phillips [4], Glassey and Strauss [3] and Strichartz [6] among many others. In particular, the principle of equipartition of energy is valid for many equations describing water-waves in certain asymptotic regimes or for the linearized water-waves equations.

For the full nonlinear problem, i.e. the incompressible Euler equation with free surface, the question of equipartition of energy was first studied by Rayleigh in 1911 [5]. He showed that equipartition does not hold in general. More precisely, he showed that it is not satisfied by the famous approximate solutions found by Stokes. Rayleigh result was later confirmed for standing water waves. It follows from these works that the difference between the mean kinetic energy and the mean potential energy is proportional to the fourth power of the wave amplitude.

Our main result states that, despite these negative results, there is an unexpected exact identity that states that, on average, the potential energy is equal to a modified kinetic energy. More precisely, we prove that for all regular solution (η, ψ) to the water-wave system, there holds

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} \eta\psi \, dx = \iint_{\Omega(t)} \left(\frac{3}{4}(\partial_y \phi)^2 + \frac{1}{4}|\nabla_x \phi|^2 \right) dy \, dx - \frac{g}{2}\int_{\mathbb{T}^d} \eta^2 \, dx.$$

The proof relies on a variant of the classical Rellich identity for harmonic function.

As an application, we discuss non-perturbative results justifying the formation of bubbles for the free-surface Rayleigh-Taylor instability, for any non-zero initial data.

We also discuss two related inequalities : a refined Rellich estimate proved with Siddhant Agrawal ([1]) and a trace estimate proved with Quoc-Hung Nguyen, controlling the $\dot{H}^{1/2}$-norm of ψ in terms of the $L^2(\Omega)$-norm of $\nabla_{x,y}\phi$, where the constant depends only on the BMO-norm of $\nabla\eta$.

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