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Oberwolfach Preprints

OWP 2022 - 16

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

DOI 10.14760/OWP-2022-16

Root Cycles in Coxeter Groups

Sarah Hart, Veronica Kelsey and Peter Rowley

Abstract

For an element w of a Coxeter group W there are two important attributes, namely its length, and its expression as a product of disjoint cycles in its action on Φ, the root system of W. This paper investigates the interaction between these two features of w, introducing the notion of the crossing number of w, $\kappa(w)$. Writing $w = c_1 \cdots c_r$ as a product of disjoint cycles we associate to each cycle c_i a 'crossing number' $\kappa(c_i)$, which is the number of positive roots α in c_i for which $w \cdot \alpha$ is negative. Let $\mathrm{Seq}_\kappa(w)$ be the sequence of $\kappa(c_i)$ written in increasing order, and let $\kappa(w) = \max \mathrm{Seq}_\kappa(w)$. The length of w can be retrieved from this sequence, but $\mathrm{Seq}_\kappa(w)$ provides much more information.

For a conjugacy class X of W let $\kappa_{\min}(X) = \min\{\kappa(w) \mid w \in X\}$ and let $\kappa(W)$ be the maximum value of $\kappa_{\min}$ across all conjugacy classes of W. We call $\kappa(w)$ and $\kappa(W)$, respectively, the crossing numbers of w and W. Here we determine the crossing numbers of all finite Coxeter groups and of all universal Coxeter groups. We also show, among other things, that for finite irreducible Coxeter groups if u and v are two elements of minimal length in the same conjugacy class X, then $\mathrm{Seq}_\kappa(u) = \mathrm{Seq}_\kappa(v)$ and $\kappa_{\min}(X) = \kappa(u) = \kappa(v)$.

Keywords: Coxeter group, root system, root cycles, length function

1 Introduction

The symbiotic relationship between a Coxeter group W and its root system Φ is most evident in the interplay between the length of an element w of W and the action of w on Φ. The set $N(w) = \{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^-\}$, where Φ^+ is the set of positive roots and Φ^- is the set of negative roots, is central here. Recall that the length $\ell(w)$ of w is equal to $|N(w)|$. When w is viewed as a permutation of Φ, the orbit structure of the cyclic group $\langle w \rangle$ on Φ is an important attribute: Φ is partitioned into a number of $\langle w \rangle$-orbits, some of which will be contained entirely in Φ^+ or Φ^-, and some of which will contain both positive and negative roots. The length $\ell(w)$ cannot be recovered purely by counting the number of orbits that cross the boundary from positive to negative roots, because within a given orbit, the boundary may be crossed several times. We need, rather, to consider the number of crossings from positive to negative roots that occur. This means that we need to consider not orbits, but cycles, of roots. The sum of these 'crossing numbers' across all the root cycles of w will be $\ell(w)$. But the sequence of crossing numbers gives finer detail. This paper focuses on this interaction between the cycles of w on Φ, referred to as root cycles, and the set $N(w)$.

For a recap of basic terminology concerning Coxeter groups and root systems see Section 2. Here, we define crossing numbers and state our main results.

Definition 1.1. Let W be a Coxeter group with root system Φ and let $w \in W$. Then, viewing W as a permutation group of Φ, the element w may be expressed as a product of disjoint cycles of roots called *root cycles.* For a given root cycle c of w, the *crossing number* of c is

$$\kappa(c) = |\{\alpha \in \operatorname{supp}(c) \mid \alpha \in \Phi^+, w \cdot \alpha \in \Phi^-\}| = |\operatorname{supp}(c) \cap N(w)|.$$

The sequence of crossing numbers of the distinct cycles of w, arranged in increasing order, is called the *crossing number sequence* of w and denoted by $\operatorname{Seq}_\kappa(w)$. Finally, the *crossing number* $\kappa(w)$ of w is defined to be $\max(\operatorname{Seq}_\kappa(w))$; that is, the largest crossing number of a root cycle of w.

Note that $\operatorname{supp}(c)$ denotes the support of c and $\ell(w) = \sum_{k \in \operatorname{Seq}_\kappa(w)} k$.

Example 1.2. Let W be the Coxeter group A_6, which we view as $\operatorname{Sym}(7)$. Setting $V = \mathbb{R}^7$, with orthonormal basis $\{e_1, \ldots, e_7\}$, we can define the root system for W as $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq 7\}$, $\Phi^- = -\Phi^+$, and $\Phi = \Phi^+ \cup \Phi^-$. Then the action of W on Φ can be seen as $\operatorname{Sym}(7)$ permuting the subscripts of the basis vectors. Let $u = (1, 2, 3, 4, 5, 6)$, $v = (1, 2, 3, 4, 5, 7)$, $w = (1, 3, 2, 4, 5, 6) \in W$. Then, for example, the following cycle c is a root cycle of u.

$$c = (e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 - e_6, -e_1 + e_6).$$

Just looking at the signs of the roots (positive or negative), this is $(+, +, +, +, +, -)$, and so $\kappa(c) = 1$. It can be shown that all root cycles of u have crossing number 0 or 1, and in fact $\operatorname{Seq}_\kappa(u) = [0, 0, 1, 1, 1, 1, 1]$, with $\ell(u) = 5$. On the other hand, the cycle

$$c' = (e_1 - e_3, -e_2 + e_3, e_2 - e_4, e_4 - e_5, e_5 - e_6, -e_1 + e_6)$$

of w has sign pattern $(+, -, +, +, +, -)$, so that $\kappa(c') = 2$. It is not hard to calculate that v and w, which both have length 7 and are conjugate to u, have different crossing number sequences. Specifically, $\operatorname{Seq}_\kappa(v) = [1, 1, 1, 1, 1, 1, 1]$ and $\operatorname{Seq}_\kappa(w) = [0, 0, 1, 1, 1, 2, 2]$. Hence, $\kappa(u) = \kappa(v) = 1$, whereas $\kappa(w) = 2$.

Many different crossing numbers of elements are possible. However, as we will show for $\operatorname{Sym}(n)$ (the Coxeter group A_{n-1}), every non-identity element w is conjugate to some w' such that $\kappa(w') = 1$. This does not hold in all Coxeter groups. With this in mind, we make a further definition.

Definition 1.3. Let W be a Coxeter group and X a conjugacy class of W. The *minimum crossing number* of X, denoted $\kappa_{\min}(X)$, is

$$\kappa_{\min}(X) := \min\{\kappa(w) \mid w \in X\}.$$

The *crossing number* of W, denoted $\kappa(W)$, is defined to be

$$\kappa(W) := \max\{\kappa_{\min}(X) \mid X \text{ a conjugacy class of } W\},$$

with $\kappa(W) = \infty$ if this maximum value does not exist.

We may now state our main results. As Example 1.2 shows, crossing number sequences for elements of the same length in a given conjugacy class can be different. Our first result for finite irreducible Coxeter groups is that the crossing number sequences of all minimal length elements in a given conjugacy class are the same, and that their crossing numbers equal the minimal crossing number of the class. For a conjugacy class X of W, we denote the set of elements of minimal length in X by $X_{\min}$.

Theorem 1.4. *Let W be a finite irreducible Coxeter group, X a conjugacy class of W, and u be any element of $X_{\min}$. Then $\kappa_{\min}(X) = \kappa(u)$. Moreover, for all $v \in X_{\min}$ we have $Seq_{\kappa}(v) = Seq_{\kappa}(u)$.*

Theorem 1.4 enables us to find $\kappa(W)$ for all finite irreducible Coxeter groups W.

Theorem 1.5. *Let W be a finite irreducible Coxeter group.*

(i) $\kappa(W) = 1$ *if and only if W is one of* $\mathrm{A}_n (n \geq 1)$, $\mathrm{B}_2, \mathrm{B}_3, \mathrm{B}_4, \mathrm{D}_4, \mathrm{F}_4$, *or* G_2.

(ii) *If W is B_n with $n \geq 3$, or D_n with $n \geq 4$, then* $\kappa(W) = \lfloor \frac{n-1}{2} \rfloor$.

(iii) $\kappa(\mathrm{E}_6) = 2$; $\kappa(\mathrm{E}_7) = 3$; $\kappa(\mathrm{E}_8) = 3$; $\kappa(\mathrm{H}_3) = 3$; $\kappa(\mathrm{H}_4) = 7$.

(iv) *If W is $\mathrm{I}_2(n)$ (so $W \cong \mathrm{Dih}(2n)$) and $n \geq 3$, then*

$$\kappa(W) = \begin{cases} \frac{n}{2} - 2 & \text{if } n \equiv 2 \bmod 4; \\ \lfloor \frac{n-1}{2} \rfloor & \text{otherwise.} \end{cases}$$

Our next theorem is at the opposite end of the spectrum from finite Coxeter groups.

Theorem 1.6. *Let W be a finite rank universal Coxeter group. Then $\kappa(W) = 1$.*

The proof of Theorem 1.4 relies on results that hold only for finite irreducible Coxeter groups. However, for arbitrary Coxeter groups W we can obtain lower bounds for $\kappa(W)$ by looking at its standard parabolic subgroups, using the following result.

Theorem 1.7. *Let W be an arbitrary Coxeter group, and W_I a standard parabolic subgroup of W. Then $\kappa(W) \geq \kappa(W_I)$.*

Corollary 1.8. *Let n be a positive integer.*

(i) *There exists a finite irreducible Coxeter group W with $\kappa(W) = n$. If n is even and greater than two, there are exactly five such groups. If n is odd and greater than seven, there are exactly seven such groups. However there are infinitely many finite reducible Coxeter groups W with $\kappa(W) = n$.*

(ii) *There exists an infinite irreducible Coxeter group W such that $\kappa(W) \geq n$.*

In Section 2 we begin with some standard notation and background material, and then prove some preliminary lemmas. In Section 3 we prove Theorem 1.4, and in Section 4 we prove Theorem 1.5. Then, in Section 5, we prove Theorem 1.6, and in Section 6 we establish Theorem 1.7 and Corollary 1.8.

2 Notation and Background

Let W be a finite rank Coxeter group with its set of fundamental (or simple) reflections being R. Then the *length function* ℓ on W is defined by $\ell(1) = 0$ and for $w \in W \setminus \{1\}$

$$\ell(w) = \min\{\ell \in \mathbb{N} \mid w = r_1 r_2 \cdots r_\ell \text{ where } r_i \in R\}.$$

Let V be a vector space over $\mathbb{R}$ with basis $\Pi = \{\alpha_r \mid r \in R\}$. Let $r, s \in R$ with corresponding $\alpha_r, \alpha_s \in \Pi$ and denote the order of rs by m_{rs}. Then we define the following inner product on V

$$\langle \alpha_r, \alpha_s \rangle = \begin{cases} -\cos(\pi/m_{rs}) & \text{if } m_{rs} < \infty, \\ -1 & \text{if } m_{rs} = \infty. \end{cases}$$

We can now define a faithful action of W on V which preserves the inner product. For $r \in R$ and $v \in V$ let

$$r \cdot v = v - 2\langle v, \alpha_r \rangle \alpha_r.$$

The *root system* of W is $\Phi = \{w \cdot \alpha_r \mid w \in W, r \in R\}$, The set of *positive roots* is $\Phi^+ = \{\sum_{r\in R} \lambda_r \alpha_r \mid \lambda_r \geq 0 \text{ for all } r \in R\}$, and the *negative roots* are $\Phi^- = -\Phi^+$. It can be shown that $\Phi = \Phi^+ \dot{\cup} \Phi^-$. We define one further function: for $w \in W$ let

$$N(w) = \{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^-\},$$

and so $N(w)$ is the number of positive roots taken negative by w. It is well known that $\ell(w) = |N(w)|$, see for example [8, §5.6 Proposition(b)]. In the proofs of a number of our results we will need to consider how a given element g of W act on the cycles of w. For a root cycle $c = (\alpha_1, \alpha_2 \ldots, \alpha_m)$ of w and an element g of W, we define $g \cdot c$ in the natural way: $g \cdot c = (g \cdot \alpha_1, g \cdot \alpha_2, \ldots, g \cdot \alpha_m)$. For $J \subseteq R$ we define the corresponding *standard parabolic* subgroup to be $W_J = \langle J \rangle$. Then W_J is a Coxeter group with root system

$$\Phi_J = \{w \cdot \alpha_r \mid r \in J, w \in W_J\}.$$

Let W be a Coxeter group and $g, h \in W$. Then

$$N(gh) = \big[N(h) \setminus \big(-h^{-1} \cdot N(g)\big)\big] \mathbin{\dot\cup} h^{-1} \cdot \big(N(g) \setminus N(h^{-1})\big). \tag{1}$$

(This is well-known, but for a proof see, for example [7, Lemma 2.2].) Hence,

$$\ell(gh) = \ell(g) + \ell(h) - 2|N(g) \cap N(h^{-1})|. \tag{2}$$

Lemma 2.1. *Let $w \in W$, let $r \in R$, and suppose that $rwr \neq w$. Then $\ell(rwr) = \ell(w)$ if and only if α_r is contained in exactly one of $N(w)$ and $N(w^{-1})$, while $\ell(rwr) < \ell(w)$ if and only if $\alpha_r \in N(w) \cap N(w^{-1})$.*

Proof. Setting $g = rw$ and $h = r$ in Equation (2) gives

$$\ell(rwr) = \ell(rw) + \ell(r) - 2|N(rw) \cap N(r)|. \tag{3}$$

Since $N(r) = \{\alpha_r\}$, now setting $g = r$ and $h = w$ in Equation (1) gives

$$N(rw) = \begin{cases} N(w) \cup \{w^{-1} \cdot \alpha_r\} & \text{if } \alpha_r \notin N(w^{-1}), \\ N(w) \setminus \{-w^{-1} \cdot \alpha_r\} & \text{if } \alpha_r \in N(w^{-1}). \end{cases}$$

Hence, $\ell(rw) = \ell(w) + 1 - 2|N(w^{-1}) \cap \{\alpha_r\}|$. Since we are assuming that $rwr \neq w$, we have that $w^{-1} \cdot \alpha_r \neq \pm\alpha_r$. Thus, $\alpha_r \in N(rw)$ if and only if $\alpha_r \in N(w)$. Equation (3) now becomes

$$\ell(rwr) = \ell(w) + 2 - 2|N(w^{-1}) \cap \{\alpha_r\}| - 2|N(w) \cap \{\alpha_r\}|.$$

The result follows immediately. ∎

Finally, we note the following well-known observations.

Lemma 2.2. *Let (W, R) be a Coxeter system, with $w \in W$, and suppose $w = r_1 r_2 \cdots r_\ell$, with $r_i \in R$ and $\ell = \ell(w)$, is a reduced expression for w. Write α_{r_i} for the fundamental root corresponding to r_i. Then*

$$N(w) = \{\alpha_{r_\ell}, r_\ell \cdot \alpha_{r_{\ell-1}}, \ldots, (r_\ell \cdots r_{\ell-1}) \cdot \alpha_{r_1}\}.$$

For $I \subseteq R$, let $W^I = \{w \in W \mid \ell(rw) > \ell(w) \text{ for all } r \in I\}$. Then the following holds, see [8, §1.10-Proposition(c)] for example.

Theorem 2.3. *Let W be a Coxeter group with W_I a standard parabolic subgroup of W. Then $W = W_I W^I$. Moreover, every element w of W has a unique expression $w = w_I w^I$ such that $w_I \in W_I$, $w^I \in W^I$, and $\ell(w) = \ell(w_I) + \ell(w^I)$.*

3 Finite Irreducible Coxeter Groups

In this section we prove Theorem 1.4. Throughout, W will denote a finite irreducible Coxeter group. We begin with notation to be found in Definitions 3.2.3 and 3.2.4 of [4].

Definition 3.1. Let $u, v \in W$.

(a) We write $u \to v$ if there exist $r_1, \ldots r_m \in R$ and distinct $w_0, w_1, \ldots, w_m \in W$, such that $w_0 = u$, $w_m = v$, with $w_i = r_i w_{i-1} r_i$ and $\ell(w_i) \leq \ell(w_{i-1})$ for $1 \leq i \leq m$.

(b) We say that u and v are *elementarily strongly conjugate* if $\ell(u) = \ell(v)$ and there exists w in W such that either $uw = wv$ and $\ell(uw) = \ell(u) + \ell(w)$, or $wu = vw$ and $\ell(wu) = \ell(u) + \ell(w)$.

The following result of Geck and Pfeiffer is crucial here, as it allows us to choose conjugating elements that give us considerable control over the behaviour of the crossing number.

Theorem 3.2 ([4, Theorem 3.2.9])**.** *Let X be a conjugacy class of W.*

(a) For each u in X there exists an element v of $X_{\min}$ such that $u \to v$.

(b) Let $u, v \in X_{\min}$. Then there exists w in $X_{\min}$ with $u \to w$ such that w and v are elementarily strongly conjugate.

Theorem 3.2(a) tells us that we may conjugate any element of a conjugacy class to an element of minimal length in the class, by a series of conjugations with fundamental reflections where at each stage the length either stays the same or decreases.

In the following lemma we consider how conjugation by a fundamental reflection affects the crossing number based on the change in length.

Lemma 3.3. *Let $w \in W$ and $r \in R$. If $\ell(rwr) = \ell(w)$, then $Seq_\kappa(rwr) = Seq_\kappa(w)$, and hence $\kappa(rwr) = \kappa(w)$. If $\ell(rwr) < \ell(w)$, then $\kappa(rwr) \leq \kappa(w)$.*

Proof. If $rwr = w$, the result is obvious. So we may assume that $rwr \neq w$. Then $w \cdot \alpha_r \notin \{\pm\alpha_r\}$, and $w^{-1} \cdot \alpha_r \notin \{\pm\alpha_r\}$. The root cycles of rwr are of the form $r \cdot c$, where c is a root cycle of w. Since $N(r) = \{\alpha_r\}$, the only roots whose sign changes under the action of r are α_r and $-\alpha_r$. Thus, whenever a root cycle c does not contain $\pm\alpha_r$, the pattern of positive and negative signs of roots in $r \cdot c$ is equal to that in c, meaning that $\kappa(c) = \kappa(r \cdot c)$.

Now consider the root cycle c containing α_r, and suppose first that $\ell(rwr) = \ell(w)$. By Lemma 2.1, exactly one of $w \cdot \alpha \in \Phi^-$ and $w^{-1} \cdot \alpha \in \Phi^-$ holds. That is, we have $(\ldots, w^{-1} \cdot \alpha_r, \alpha_r, w \cdot \alpha_r, \ldots)$, with the corresponding sign pattern $(\ldots, -, +, +, \ldots)$ or $(\ldots, +, +, -, \ldots)$. Thus, the corresponding part of $r \cdot c$ has sign pattern $(\ldots, -, -, +, \ldots)$ or $(\ldots, +, -, -, \ldots)$. Either way, $\kappa(r \cdot c) = \kappa(c)$. A similar observation holds for $-c$, the cycle containing $-\alpha_r$. We have shown that the crossing number of each root cycle c is unchanged by the action of r. Hence $\mathrm{Seq}_\kappa(rwr) = \mathrm{Seq}_\kappa(w)$.

The remaining case to consider is where $\ell(rwr) < \ell(w)$. Again, we only need to look at the root cycle containing $\pm\alpha_r$. If c is the root cycle containing α_r, then by Lemma 2.1 again c contains $(\ldots, w^{-1} \cdot \alpha_r, \alpha_r, w \cdot \alpha_r, \ldots)$, with sign pattern $(\ldots, -, +, -, \ldots)$. The corresponding part of $r \cdot c$ has sign pattern $(\ldots, -, -, -, \ldots)$. The part of the cycle $-c$ containing $-\alpha_r$ has sign pattern $(\ldots, +, -, +, \ldots)$, which becomes $(\ldots, +, +, +, \ldots)$ in $r \cdot (-c)$. The net effect is that $\kappa(r \cdot c) < \kappa(c)$ and $\kappa(r \cdot (-c)) < \kappa(-c)$. Hence, $\kappa(rwr) \leq \kappa(w)$. □

Remark 3.4. Note that even if $\ell(rwr) < \ell(w)$, we cannot conclude that $\kappa(rwr) < \kappa(w)$. The crossing number of the cycle c containing α_r decreases, as does the crossing number of $-c$, but this will not affect $\kappa(rwr)$ unless $\kappa(c) = \kappa(w)$, and $\pm c$ are the only cycles with this crossing number. Should this happen, then $\kappa(rwr) \leq \kappa(w) - 1$. However, if $c = -c$, then $\kappa(r \cdot c) = \kappa(c) - 2$. All we can say in the general case is that $\kappa(w) - 2 \leq \kappa(rwr) \leq \kappa(w)$.

Lemma 3.5. *Let X be a conjugacy class of W, and suppose $u, v \in X_{\min}$. Then $Seq_\kappa(u) = Seq_\kappa(v)$.*

Proof. By Theorem 3.2(b), there exists u' in $X_{\min}$ with $u \to u'$ such that u' and v are elementarily strongly conjugate. By definition, there exist $r_1, \ldots r_m \in R$ and distinct $w_0, w_1, \ldots, w_m \in W$, such that $w_0 = u$, $w_m = u'$, and, for all i, $w_i = r_i w_{i-1} r_i$ and $\ell(w_i) \leq \ell(w_{i-1})$. The fact that u has minimal length in its conjugacy class forces $\ell(w_i) = \ell(w_{i-1})$ for all i. Therefore, by Lemma 3.3, we have $\mathrm{Seq}_\kappa(w_i) = \mathrm{Seq}_\kappa(w_{i-1})$. Hence, $\mathrm{Seq}_\kappa(u') = \mathrm{Seq}_\kappa(u)$. Thus, it suffices to prove the lemma in the case where u and v are elementarily strongly conjugate. So, suppose u and v are elementarily strongly conjugate. Then there exists x in W such that either $ux = xv$ and $\ell(ux) = \ell(u) + \ell(x)$, or $xu = vx$ and $\ell(xu) = \ell(u) + \ell(x)$. Replacing u, v, x with their inverses, if necessary, we may assume without loss of generality that $ux = xv$ and $\ell(ux) = \ell(u) + \ell(x)$. That is, $v = x^{-1}ux$. By Equation (2),

$$\ell(ux) = \ell(u) + \ell(x) - 2|N(u) \cap N(x^{-1})|.$$

Thus,

$$N(u) \cap N(x^{-1}) = \varnothing. \tag{4}$$

Also, again by Equation (2),

$$\ell(x^{-1}ux) = \ell(x) + \ell(ux) - 2|N(x^{-1}) \cap N((ux)^{-1})|.$$

Now $\ell(x^{-1}ux) = \ell(v) = \ell(u)$, and so $|N(x^{-1}) \cap N((ux)^{-1})| = \ell(x)$. This forces

$$N(x^{-1}) \subseteq N((ux)^{-1}). \tag{5}$$

Since $\ell((ux)^{-1}) = \ell(ux) = \ell(u) + \ell(x)$, we observe from Equation (1), with $g = u^{-1}$ and $h = x^{-1}$, that there is only one way for $|N((ux)^{-1})|$ to attain the required cardinality, and that is for

$$N((ux)^{-1}) = N(x^{-1}u^{-1}) = N(u^{-1}) \mathbin{\dot\cup} uN(x^{-1}).$$

Note here that $u \cdot N(x^{-1})$ really is a set of positive roots, because $N(x^{-1}) \cap N(u) = \varnothing$. Thus,

$$\alpha \in N((ux)^{-1}) \text{ if and only if either } u^{-1} \cdot \alpha \in \Phi^- \text{ or } u^{-1} \cdot \alpha \in N(x^{-1}). \tag{6}$$

Note that c is a root cycle of u if and only if $x^{-1} \cdot c$ is a root cycle of v. Therefore, to show that $\mathrm{Seq}_\kappa(v) = \mathrm{Seq}_\kappa(u)$, it is sufficient to prove that $\kappa(x^{-1} \cdot c) = \kappa(c)$ for all root cycles c of u. Let c be a root cycle of u. If $\mathrm{supp}(c)$ does not intersect $\pm N(x^{-1})$, then $x^{-1} \cdot \alpha$ has the same sign as α, for each α in $\mathrm{supp}(c)$. Hence c and $x^{-1} \cdot c$ have the same sign pattern, and $\kappa(c) = \kappa(x^{-1} \cdot c)$. At the other extreme, if $\mathrm{supp}(c) \subseteq (N(x^{-1}) \cup -N(x^{-1}))$, then the action of x^{-1} will change the sign of every root in c, and so (because $\mathrm{supp}(c)$ is finite), $\kappa(x^{-1} \cdot c) = \kappa(-c) = \kappa(c)$. The remaining case to consider is root cycles c of u whose support intersects, but is not contained entirely within, $N(x^{-1}) \cup -N(x^{-1})$. Let c be such a cycle. Replacing c with $-c$ if necessary, we can assume without loss of generality that c contains some $\alpha \in N(x^{-1})$, and some other root $\beta \notin N(x^{-1}) \cup -N(x^{-1})$. Let i be the least positive integer such that $u^i \cdot \alpha \notin N(x^{-1})$. Then α, $u \cdot \alpha$, …, $u^{i-1} \cdot \alpha$ are elements of $N(x^{-1})$, so in particular they are positive roots and, by (4), they are not contained in $N(u)$. Thus, $u^i \cdot \alpha = u \cdot (u^{i-1}\alpha) \in \Phi^+$. That is, the root cycle c contains

$$(\dots, \alpha, u \cdot \alpha, \dots, u^{i-1} \cdot \alpha, u^i \cdot \alpha, \dots)$$

with sign pattern

$$(\dots, \underbrace{+, +, \dots, +}_{\text{in } N(x^{-1})}, \underbrace{+}_{\notin N(x^{-1})}, \dots).$$

Now we consider the roots preceding α in c. Let j be the least positive integer such that $u^{-j} \cdot \alpha \notin N(x^{-1})$. Then α, $u^{-1} \cdot \alpha$, …, $u^{-(j-1)} \cdot \alpha$ are all positive roots contained in $N(x^{-1})$. By (5) we have that $u^{-(j-1)} \cdot \alpha \in N((ux)^{-1})$. Since $u^{-1}(u^{-j-1} \cdot \alpha) \notin N(x^{-1})$, we see from (6) that $u^{-j} \cdot \alpha \in \Phi^-$. Moreover, $u^{-j} \cdot \alpha$ is not contained in $-N(x^{-1})$, because $u^{-j} \cdot \alpha \in -N(u)$, and $N(u) \cap N(x^{-1}) = \varnothing$. That is, the root cycle c contains

$$(\dots, u^{-j} \cdot \alpha, u^{-(j-1)} \cdot \alpha, \dots, u^{-1} \cdot \alpha, \alpha, \dots)$$

with sign pattern

$$(\dots, \underbrace{-}_{\notin -N(x^{-1})}, \underbrace{+, +, \dots, +}_{\text{in } N(x^{-1})}, \dots).$$

Combining these observations we see that for every place where $\mathrm{supp}(c)$ intersects $N(x^{-1})$, the corresponding sign pattern is of the form

$$(\dots, \underbrace{-}_{\notin -N(x^{-1})}, \underbrace{+, +, \dots, +}_{\text{in } N(x^{-1})}, \underbrace{+}_{\notin N(x^{-1})}, \dots).$$

This means that the corresponding part of $x^{-1} \cdot c$ will contain $(\dots, -, -, \dots, -, +, \dots)$. This clearly does not affect the crossing number. Similarly, each place where $\mathrm{supp}(c)$ intersects $-N(x^{-1})$ will consist of a sequence

$$(\dots, \underbrace{+}_{\notin N(x^{-1})}, \underbrace{-, -, \dots, -}_{\text{in } -N(x^{-1})}, \underbrace{-}_{\notin -N(x^{-1})}, \dots).$$

In $x^{-1} \cdot c$, this will become $(\ldots, +, +, \ldots, +, -, \ldots)$. Again, the crossing number is unaffected. Therefore, $\kappa(x^{-1} \cdot c) = \kappa(c)$.

We have shown that each root cycle c of u has the same crossing number as the corresponding root cycle $x^{-1} \cdot c$ of v. Therefore, $\mathrm{Seq}_\kappa(v) = \mathrm{Seq}_\kappa(u)$. □

Proof of Theorem 1.4. Let W be a finite irreducible Coxeter group and X a conjugacy class of W. Let $w \in X$. Then by Theorem 3.2(a) there exists $w' \in X_{\min}$ such that $w \to w'$. It follows by repeated application of Lemma 3.3 that $\kappa(w') \leq \kappa(w)$. Hence,

$$\kappa_{\min}(X) = \min\{\kappa(v) \mid v \in X_{\min}\}.$$

Let $u \in X_{\min}$. By Lemma 3.5, for all $v \in X_{\min}$, $\mathrm{Seq}_\kappa(v) = \mathrm{Seq}_\kappa(u)$. In particular, $\kappa(v) = \kappa(u)$. Hence, $\kappa_{\min}(X) = \kappa(u)$, thus completing the proof of Theorem 1.4. □

4 Proof of Theorem 1.5

Let W be a finite irreducible Coxeter group with root system Φ. For $w \in W$ and $\alpha \in \Phi$, let $\Phi_w(\alpha)$ be the root cycle of w which contains α. That is, $\Phi_w(\alpha) = (\ldots, w^{-1}\cdot\alpha, \alpha, w\cdot\alpha, \ldots)$.

4.1 W of type A

We begin by describing representative elements of minimal length in the conjugacy classes of $\mathrm{A}_{n-1} \cong \mathrm{Sym}(n)$, for $n \geq 2$.

Let $W = \mathrm{A}_{n-1} \cong \mathrm{Sym}(n)$ with positive roots $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}$, and negative roots $\Phi^- = -\Phi^+$. The conjugacy classes of $\mathrm{Sym}(n)$ are parameterised by compositions of n; that is, integer tuples $\lambda = (b_1, b_2, \ldots, b_p)$ where $1 \leq b_p \leq \cdots \leq b_1 \leq n$, $1 \leq p \leq n$, and $\sum_{i=1}^p b_i = n$. Let Λ be the set of all such λ. For a given $\lambda \in \Lambda$, let $\{\beta_i \mid 0 \leq i \leq p\}$ be the set of partial sums of the b_i. That is, $\beta_0 = 0$ and $\beta_j = \sum_{i=1}^j b_i$ for $1 \leq j \leq p$. Then we can associate to each b_i an element u_i of $\mathrm{Sym}(n)$, given by $u_i = (\beta_{i-1} + 1, \ldots \beta_i)$. Define

$$w_\lambda = u_1 u_2 \cdots u_p = (1, \ldots, \beta_1)(\beta_1 + 1, \ldots, \beta_2) \cdots (\beta_{p-1} + 1, \ldots, \beta_p). \tag{7}$$

The following is a well known result for symmetric groups.

Theorem 4.1. *The set $\{w_\lambda\}_{\lambda \in \Lambda}$ is a complete set of representatives of minimal length of the conjugacy classes of W.*

We now prove Theorem 1.5(i) in the case of A_n.

Lemma 4.2. *Let $n \geq 1$ and $W = \mathrm{A}_n$. Then $\kappa(W) = 1$.*

Proof. It is more convenient to work with $\mathrm{Sym}(n)$, so we will actually prove the equivalent result that if $n \geq 2$ and $W = A_{n-1} \cong \mathrm{Sym}(n)$, then $\kappa(W) = 1$. Let $w = u_1 \cdots u_p \in \{w_\lambda\}_{\lambda \in \Lambda}$ and $\alpha = e_i - e_j \in \Phi^+$ with $i \in \mathrm{supp}(u_m)$ and $j \in \mathrm{supp}(u_k)$ for some $m, k \in \{1, \ldots, p\}$. Then it is immediate that $i < j$ and so $m \leq k$.

First, suppose that $m < k$. For all $r \in \mathrm{supp}(u_m)$ and $s \in \mathrm{supp}(u_k)$ it follows from (7) that $r < s$ and so $e_r - e_s \in \Phi^+$. Thus $\mathrm{supp}(\Phi_w(\alpha)) \subseteq \Phi^+$ and so $\kappa(\Phi_w(\alpha)) = 0$.

Now suppose that $m = k$. Then

$$\Phi_w(\alpha) = (\underbrace{e_1 - e_{j-i+1}, \ldots, e_i - e_j, \ldots, e_{i+(\beta_m - j)} - e_{\beta_m}}_{+} \underbrace{e_{i+1+(\beta_m-j)} - e_1, \ldots, e_{\beta_m} - e_{j-i}}_{-})$$

and so $\kappa(\Phi_w(\alpha)) = 1$.

Recall that $\kappa(W) = \max\{\kappa(w_\lambda) \mid \lambda \in \Lambda\}$ by Theorem 1.4, and so we have shown that $\kappa(\mathrm{A}_n) = 1$ for all $n \geq 1$. $\square$

4.2 W of type B or D

We begin by introducing some notation and preliminary results. We use the notation of signed cycles as in [5], and so

$$\mathrm{B}_n = \langle (\overset{-}{1}), (\overset{+}{1}, \overset{+}{2}), \ldots, (\overset{+}{n-1}, \overset{+}{n}) \rangle \quad \text{and} \quad \mathrm{D}_n = \langle (\overset{-}{1}, \overset{-}{2}), (\overset{+}{1}, \overset{+}{2}), \ldots, (\overset{+}{n-1}, \overset{+}{n}) \rangle.$$

Recall that with this scenario D_n has the following sets of positive and negative roots:

$$\Phi^+ = \{\pm e_i + e_j \mid 1 \leq i < j \leq n\} \text{ and } \Phi^- = \{\pm e_i - e_j \mid 1 \leq i < j \leq n\}.$$

The roots of B_n are given by the union of those of D_n with positive roots $\{e_i \mid 1 \leq i \leq n\}$, and negative roots $\{-e_i \mid 1 \leq i \leq n\}$. Then, for example, $w = (\overset{-}{1}, \overset{+}{2}, \overset{-}{3}, \overset{+}{4})(\overset{-}{6}) \in \mathrm{B}_6$ represents the element $(1, -2, -3, 4)(-1, 2, 3, -4)(6, -6)$ and

$$\Phi_w(e_1 - e_2) = (e_1 - e_2, -e_2 - e_3, -e_3 + e_4, e_4 + e_1)$$

We now describe representative elements of minimal length of each conjugacy class using [4] (in fact we give the inverse of the elements in [4] as they write their elements on the right of the roots, and use the following change of notation $(\alpha, \beta) \mapsto (b, c)$). For integers m, p with $0 \leq m \leq n$ and $0 \leq p \leq n$, let $\lambda = (c_1, \ldots c_m; b_1, \ldots, b_p) \in \mathbb{Z}^{m+p}$ be such that $1 \leq c_1 \leq c_2 \leq \cdots \leq c_m$ and $1 \leq b_p \leq \cdots \leq b_2 \leq b_1$ with $\sum_{i=1}^m c_i + \sum_{i=1}^p b_i = n$. Let Λ be the set of all such sequences. We now define elements of B_n associated to these sequences.

Set $\gamma_0 = 0$, for $1 \leq j \leq m$ let $\gamma_j = \sum_{i=1}^j c_i$ and $u_j^- = (\overset{-}{\gamma_{j-1}+1}, \overset{+}{\gamma_{j-1}+2}, \ldots, \overset{+}{\gamma_j})$, and for $1 \leq j \leq p$ let $\beta_j = \gamma_m + \sum_{i=1}^j b_i$ and $u_j^+ = (\overset{+}{\beta_{j-1}+1}, \ldots, \overset{+}{\beta_j})$. We then associate to $\lambda \in \Lambda$ the following element of B_n.

$$w_\lambda = u_1^- \cdots u_m^- u_1^+ \cdots u_p^+ \tag{8}$$

We may view D_n as the subgroup of index 2 in B_n consisting of elements whose signed cycles contain an even number of minus signs. Conjugacy classes of these elements in D_n are the same as in B_n, except where every signed cycle has even length and an even number of minus signs. In that case, there are two conjugacy classes. In order to describe minimal length conjugacy class representatives in D_n, we thus need to define two subsets of Λ. Let $\Delta \subseteq \Lambda$ be the set of sequences $(c_1, \ldots c_m; b_1, \ldots, b_p)$ with m even. Let $\Gamma \subseteq \Delta$ be the set of sequences $(\varnothing; b_1, \ldots, b_p)$ such that b_i is even for $1 \leq i \leq p$.

Lemma 4.3. *[4, Proposition 3.4.7 and 3.4.12]*

(i) If $W = \mathrm{B}_n$, then $\mathcal{M}(\mathrm{B}_n) := \{w_\lambda\}_{\lambda \in \Lambda}$ is a complete set of representatives of minimal length of the conjugacy classes of W.

(ii) If $W = \mathrm{D}_n$, then the following is a complete set of representatives of minimal length of the conjugacy classes of W

$$\mathcal{M}(\mathrm{D}_n) := \{w_\delta\}_{\delta \in \Delta} \cup \{(\overset{-}{b_1}) w_\gamma (\overset{-}{b_1})\}_{\gamma \in \Gamma}.$$

Note that for $\gamma \in \Gamma$, we have $(\overset{-}{b_1})w_\gamma(\overset{-}{b_1}) = (\overset{-}{1}, \overset{+}{2}, \ldots, \overset{+}{b_1 - 1}, \overset{-}{b_1})u_2^+ \cdots u_p^+$. We are now able to prove Theorem 1.5(ii).

Lemma 4.4. *Let $W = \mathrm{B}_n$ $(n \geq 3)$ or D_n $(n \geq 4)$. Then $\kappa(W) = \lfloor \frac{n-1}{2} \rfloor$.*

Proof. We begin by considering $W = \mathrm{B}_n$. Let $w = u_1^- \cdots u_m^- u_1^+ \cdots u_p^+ \in \{w_\lambda\}_{\lambda \in \Lambda}$. For each $\alpha \in \Phi^+$ we consider the possibilities for $\Phi_w(\alpha)$.

First let $\alpha = e_i$. If $i \in \mathrm{supp}(u_k^+)$, then $\mathrm{supp}(\Phi_w(e_i)) \subseteq \Phi^+$, and if $i \in \mathrm{supp}(u_k^-)$, then $\Phi_w(e_i) = (\underbrace{e_i, e_{i+1}, \ldots, e_{\gamma_k}, e_{\gamma_{k-1}+1}}_{+}, \underbrace{-e_{\gamma_{k-1}+2}, \ldots, -e_{\gamma_k}, -e_{\gamma_{k-1}+1}}_{-}, \underbrace{e_{\gamma_{k-1}+2}, \ldots, e_{i-1}}_{+})$. Hence $\kappa(\Phi_w(\alpha))$ is 0 or 1.

Now let $i < j$ and consider $\alpha = \pm e_i + e_j$. If $i, j \in \mathrm{supp}(u_k^+)$, then $\Phi_w(e_i + e_j) \subseteq \Phi^+$, and $\Phi_w(-e_i + e_j)$ is

$$(\underbrace{-e_i + e_j, \ldots, -e_{i-j+\beta_k} + e_{\beta_k}}_{+}, \underbrace{-e_{i-j+\beta_k+1} + e_1, \ldots, -e_{\beta_k} + e_{j-i}}_{-}, \underbrace{-e_1 + e_{j-i+1}, \ldots, -e_{i-1} + e_{j-1}}_{+}).$$

Hence $\kappa(\Phi_w(\alpha)) = 0$ or 1.

On the other hand, if $i, j \in \mathrm{supp}(u_k^-)$, then $\Phi_w(e_i + e_j)$ is

$$(\underbrace{e_i + e_j, \ldots, e_{i+\gamma_k-j} + e_{\gamma_k}, e_{i+\gamma_k-j+1} + e_{\gamma_{k-1}+1}, e_{i+\gamma_k-j+2} - e_{\gamma_{k-1}+2}, \ldots, e_{\gamma_k} - e_{\gamma_{k-1}+j-i}}_{+},$$

$$\underbrace{e_{\gamma_{k-1}+1} - e_{\gamma_{k-1}+j-i+1}, -e_{\gamma_{k-1}+2} - e_{\gamma_{k-1}+j-i+2}, \ldots, -e_{i+\gamma_k-j} - e_{\gamma_k}}_{-},$$

$$\underbrace{-e_{i+\gamma_k-j+1} - e_{\gamma_{k-1}+1}, -e_{i+\gamma_k-j+2} + e_{\gamma_{k-1}+2}, \ldots, -e_{\gamma_k} + e_{\gamma_{k-1}+j-i}}_{-},$$

$$\underbrace{-e_{\gamma_{k-1}+1} + e_{\gamma_{k-1}+j-i+1}, e_{\gamma_{k-1}+2} + e_{\gamma_{k-1}+j-i+2} \ldots, e_{i-1} + e_{j-1}}_{+}),$$

and $\Phi_w(-e_i + e_j)$ is

$$(\underbrace{-e_i + e_j, \ldots, -e_{i+\gamma_k-j} + e_{\gamma_k}}_{+}$$

$$\underbrace{-e_{i+\gamma_k-j+1} + e_{\gamma_{k-1}+1}, -e_{i+\gamma_k-j+2} - e_{\gamma_{k-1}+2}, \ldots, -e_{\gamma_k} - e_{\gamma_{k-1}+j-i}}_{-},$$

$$\underbrace{-e_{\gamma_{k-1}+1} - e_{\gamma_{k-1}+j-i+1}, e_{\gamma_{k-1}+2} - e_{\gamma_{k-1}+j-i+2}, \ldots, e_{i+\gamma_k-j} - e_{\gamma_k}}_{-},$$

$$\underbrace{e_{i+\gamma_k-j+1} - e_{\gamma_{k-1}+1}, e_{i+\gamma_k-j+2} + e_{\gamma_{k-1}+2}, \ldots, e_{\gamma_k} + e_{\gamma_{k-1}+j-i}}_{+},$$

$$\underbrace{e_{\gamma_{k-1}+1} + e_{\gamma_{k-1}+j-i+1}, -e_{\gamma_{k-1}+2} + e_{\gamma_{k-1}+j-i+2} \ldots, -e_{i-1} + e_{j-1}}_{+}),$$

Hence $\kappa(\Phi_w(\alpha)) = 1$.

If $i \in \mathrm{supp}(u_k^\pm)$ and $j \in \mathrm{supp}(u_m^+)$, then all elements in $\mathrm{supp}(\Phi_w(\alpha))$ look like $\pm e_r + e_s$ for some $r \in \mathrm{supp}(u_k^\pm)$ and $s \in \mathrm{supp}(u_m^+)$. Since $r < s$, it follows that $\mathrm{supp}(\Phi_w(\alpha)) \subseteq \Phi^+$, and so $\kappa(\Phi_w(\alpha)) = 0$.

Finally suppose that $i \in \text{supp}(u_k^{\pm})$ and $j \in \text{supp}(u_m^-)$. Then in fact, since $k < m$ we must have $i \in \text{supp}(u_k^-)$. Similarly to the previous case, all elements in $\text{supp}(\Phi_w(\alpha))$ look like $\pm e_r \pm e_s$ for some $r \in \text{supp}(u_k^-)$ and $s \in \text{supp}(u_m^-)$, with sign equal to the sign of $\pm e_s$. The order of $\Phi_w(\alpha)$ is $\text{lcm}(|u_k^-|, |u_m^-|)$ and the sign will change after each string of $\frac{|u_m^-|}{2}$ elements (note that $|u_m^-| = 2(\gamma_m - \gamma_{m-1})$). Exactly half of these changes will be from positive to negative. Hence

$$\kappa(\Phi_w(\pm e_i + e_j)) = \frac{\text{lcm}(|u_k^-|, |u_m^-|)}{|u_m^-|} = \frac{|u_k^-|}{\text{gcd}(|u_k^-|, |u_m^-|)}.$$

By Lemma 1.4, $\kappa(\mathrm{B}_n) = \max\{\kappa(w) \mid w \in \mathcal{M}(\mathrm{B}_n)\}$. Therefore the conjugacy class with the largest $\kappa_{\min}$ value will be the one which maximises $\frac{\text{lcm}(|u_k^-|,|u_m^-|)}{|u_m^-|}$ (and note that because $k < m$ we have $|u_k^-| \leq |u_m^-|$). The conjugacy class parametrised by $\lambda = (c_1, c_2; b_1) = (\lfloor \frac{n-1}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor; 1)$ if n is even, and by $\lambda' = (c_1', c_2'; \varnothing) = (\lfloor \frac{n-1}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor; \varnothing)$ if n is odd will achieve this maximum. Hence the result follows for B_n.

Now we consider D_n, for $n \geq 4$. Observe that for $w \in \mathrm{D}_n$, every root cycle of w is also a root cycle of w when viewed as an element of B_n. Therefore, $\kappa(\mathrm{D}_n) \leq \kappa(\mathrm{B}_n)$. Moreover, the elements λ and λ' which achieve the maximum values for $\kappa(\mathrm{B}_n)$ satisfy (in the notation of Equation (8)) $m = 2$, so are contained in Δ, and hence w_λ and $w_{\lambda'}$ are elements of D_n. Thus $\kappa(\mathrm{D}_n) = \kappa(\mathrm{B}_n)$. □

4.3 W of type I_2

We now move onto $\mathrm{I}_2(n)$, the dihedral group $\text{Dih}(2n)$.

Lemma 4.5. *Let $W = \mathrm{I}_2(n)$ for some $n \geq 3$. Then*

$$\kappa(W) = \begin{cases} \frac{n}{2} - 2 & \text{if } n \equiv 2 \bmod 4, \\ \lfloor \frac{n-1}{2} \rfloor & \text{otherwise.} \end{cases}.$$

Proof. Let $R = \{r, s\}$, so that $m_{rs} = n$. Every element of W is either a reflection (conjugate to r or s or both) or an element of $\langle rs \rangle$. Note that if c is a root cycle, then $\kappa(c) \leq \lfloor \frac{|c|}{2} \rfloor$. If w is a reflection, then all of its root cycles have length 1 or 2, and hence $\kappa(w) = 1$. Therefore, in order to determine $\kappa(W)$ it suffices to consider $w \in \langle rs \rangle$. Now, rs does not fix any root α, for, if it did, then rs would commute with the corresponding reflection r_α of W, and this does not happen as $n \geq 3$. Hence, for all $\alpha \in \Phi$, the root cycle $\Phi_{rs}(\alpha)$ has length n, the order of rs. Thus, for $w \in \langle rs \rangle$, the root cycles of w have length dividing n, with equality if and only if $w = (rs)^i$ for some i coprime to n. Hence, $\kappa(c) \leq \lfloor \frac{n}{2} \rfloor$ for each root cycle c of w, and so $\kappa(w) \leq \lfloor \frac{n}{2} \rfloor$. Therefore $\kappa(W) \leq \lfloor \frac{n}{2} \rfloor$.

If n is odd, then we have $\kappa(W) \leq \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Consider $w = (rs)^{(n-1)/2}$. Since $\frac{n-1}{2}$ is coprime to n, the root cycles of w all have length n. As there are $2n$ roots in total, w therefore has exactly two root cycles, c_1 and c_2 say, each of length n, and for each c_i we have $\kappa(c_i) \leq \frac{n-1}{2}$. Now, the longest element w_0 of W is $(rs)^{(n-1)/2}r$, and it has Coxeter length n. Thus w, which equals $w_0 r$, has Coxeter ength $n-1$. Since $\ell(w) = \kappa(c_1) + \kappa(c_2)$, we must have $\kappa(c_1) = \kappa(c_2) = \frac{n-1}{2}$ and so $\kappa(w) = \frac{n-1}{2}$. Hence, $\kappa(W) = \frac{n-1}{2} = \lfloor \frac{n-1}{2} \rfloor$, as required.

Now suppose n is even and consider $w = (rs)^i$. Since $(rs)^{n/2} = (sr)^{n/2}$, and $(rs)^i = ((rs)^{n-i})^{-1}$, we can replace w with w^{-1} if necessary, in order to assume that $i < \frac{n}{2}$. We have

$$N(w) = \{(sr)^j\alpha_s, (sr)^j s \cdot \alpha_r \mid 0 \leq j \leq i\}$$

and $\ell(w) = 2i$. Since n is even, the root system has two orbits under the action of W, meaning that elements of $W \cdot \alpha_r$ and $W \cdot \alpha_s$ can never lie in the same root cycle of any w in W. Moreover, exactly half the elements of $N(w)$ lie in $W \cdot \alpha_r$, and half lie in $W \cdot \alpha_s$. If i is not coprime to n, then the root cycles of w have length at most $\frac{n}{2}$, and hence crossing number at most $\frac{n}{4}$. Thus $\kappa(w) \leq \frac{n}{4}$. which, since $n \geq 4$, certainly does not exceed either $\frac{n}{2} - 2$ or $\lfloor \frac{n-1}{2} \rfloor$. But if i is coprime to n and $w = (rs)^i$, then w has exactly two root cycles, both of length n, and they are $c_1 = \Phi_w(\alpha_r)$ and $c_2 = \Phi_w(\alpha_s)$. Moreover, $|\mathrm{supp}(c_i) \cap N(w)| = \frac{1}{2}\ell(w) = i$. Thus $\kappa(w) = i$. Hence $\kappa(W) = \max\{i \in \{1, \ldots, \frac{n}{2}\} \mid \gcd(i, n) = 1\}$. If $n \equiv 2 \mod 4$, then $\kappa(W) = \frac{n}{2} - 2$, and if $n \equiv 0 \mod 4$, then $\kappa(W) = \frac{n}{2} - 1 = \lfloor \frac{n-1}{2} \rfloor$. This completes the proof of Lemma 4.5. □

4.4 The exceptional groups

The remaining finite irreducible Coxeter groups are the exceptional groups E_6, E_7, E_8, F_4, H_3, and H_4, which we consider here using MAGMA [1]. In the following, for brevity, we use exponent notation to indicate repetitions in the crossing sequences. For example we understand $[1^2, 3^3]$ to denote a sequence $[1, 1, 3, 3, 3]$. We begin with a remark.

Remark 4.6. If W is reducible, then $\kappa(W)$ is simply the maximum crossing number amongst the irreducible components of W. Therefore, if $\kappa(W) = n$, then $\kappa(W \times \mathrm{A}_m) = n$, for any $m \geq 1$.

Lemma 4.7. *Let W be a finite irreducible Coxeter group. Then*

$$\kappa(W) = \begin{cases} 1 & \text{if } W = \mathrm{F}_4, \\ 2 & \text{if } W = \mathrm{E}_6, \\ 3 & \text{if } W \in \{\mathrm{E}_7, \mathrm{E}_8, \mathrm{H}_3\}, \\ 7 & \text{if } W = \mathrm{H}_4. \end{cases}$$

Proof. Recall from Theorem 1.4 that if X is a conjugacy class of W with $u, v \in X_{\min}$, then $\mathrm{Seq}_\kappa(u) = \mathrm{Seq}_\kappa(v)$ and $\kappa_{\min}(X) = \kappa(u)$. Every conjugacy class of W either intersects with a maximal parabolic subgroup of W or is a cuspidal class. For each W, it is therefore sufficient to calculate the crossing number (or in some cases, for more detail, the crossing sequence) of each maximal parabolic subgroup and of each cuspidal classes. We use Remark 4.6 and Theorem 1.5 to calculate crossing numbers of the maximal parabolic subgroups. For the labelling of the cuspidal classes we use [6, Tables 2-7] (see also [2], [3] and [4]).

If W is F_4 then all maximal parabolic subgroups have crossing number 1, as do all cuspidal classes. Thus $\kappa(\mathrm{F}_4) = 1$.

Let W be E_6. If w is an element of minimal length in the maximal parabolic subgroup D_5, then $\mathrm{Seq}_\kappa(w) = [0^4, 1^3, 2^2]$. For w an element of minimal length in a cuspidal class, $\mathrm{Seq}_\kappa(w) = [1^{\ell(w)}]$. Hence, $\kappa(\mathrm{E}_6) = 2$.

Let W be E_7. Then the largest crossing number among the maximal parabolic subgroups is achieved by E_6. Let w be a minimal element in a cuspidal class. If $w \in \mathrm{E}_7(a_3)$, then $\mathrm{Seq}_\kappa(w) = [1^4, 3^3]$, and otherwise $[1^{\ell(w)}]$. Therefore, $\kappa(\mathrm{E}_7) = 3$.

If W is E_8, then E_7 gives the largest crossing number of the maximal parabolic subgroups. There are two cuspidal classes whose crossing sequences are not $[1^{\ell(w)}]$. These are $\mathrm{D}_8(a_2)$ with sequence $[1^{11}, 3^5]$, and $\mathrm{D}_5(a_1) + \mathrm{A}_3$ with sequence $[1^{22}, 2^{12}]$. Thus, $\kappa(\mathrm{E}_8) = 3$.

If W is H_3, then the largest crossing number arising from the maximal parabolic subgroups is 2, corresponding to $\mathrm{I}_2(5)$. If w is an element of minimal length in cuspidal class 9, then $\mathrm{Seq}_\kappa(w) = [3^3]$, and otherwise $\mathrm{Seq}_\kappa(w) = [1^{\ell(w)}]$. Therefore, $\kappa(\mathrm{H}_3) = 3$.

Finally, let W be H_4. Then H_3 is the maximal parabolic subgroup with the highest crossing number. The elements of minimal length in cuspidal classes 19, 22, 23, 27, 28, 30, 31, or 33 have respective crossing sequences $[1^{11}, 3^1]$, $[2^8]$, $[3^6]$, $[1^{22}, 2^2]$, $[7^4]$, $[3^{12}]$, $[1^5, 3^{11}]$ and $[2^{24}]$. All others have $[1^{\ell(w)}]$. Hence, $\kappa(\mathrm{H}_4) = 7$. □

We now prove Theorem 1.5.

Proof of Theorem 1.5. Parts (ii), (iii) and (iv) follow by Lemmas 4.4, 4.7 and 4.5 respectively. Part (i) follows from Lemma 4.2, along with parts (ii) – (iv) and the observation that $\mathrm{G}_2 = \mathrm{I}_2(6)$ and $\mathrm{B}_2 = \mathrm{I}_2(4)$. □

5 Universal Coxeter groups

Let R be a set of size n, and let $W = W_n = \langle R \mid r^2 = 1 \text{ for all } r \in R\rangle$ be the universal Coxeter group on n generators (for example W_2 is the infinite dihedral group). It is well known that any two reduced expressions for an element w of an arbitrary Coxeter group can be transformed into each other by use only of braid relations $(rs)^{m_{rs}} = 1$, where $r, s \in R$, $r \neq s$ and $m_{rs} < \infty$. Since no such relations exist in universal Coxeter groups, every element has a unique reduced expression. Therefore, if $r_1 \cdots r_k$ is a reduced expression for w in W, and if r is a fundamental reflection with $r \neq r_k$, then $\ell(wr) > \ell(w)$, meaning that $w \cdot \alpha_r \in \Phi^+$.

Lemma 5.1. *Let W be the universal Coxeter group on n generators. Then $\kappa(W) = 1$.*

Proof. Let $w \in W$ be an element of minimal length within its conjugacy class and let $r_1 r_2 \cdots r_m$ be the unique reduced expression for w, so that $\ell(w) = m$. If $r_1 = r_m$, then we must have $m = 1$, otherwise $\ell(w^{r_1}) < \ell(w)$, contradicting the minimality of w. If $m = 1$, then w is a fundamental reflection, and clearly $\kappa(w) = 1$ in this case. So we can assume that $r_1 \neq r_m$. By Lemma 2.2, we have

$$N(w) = \{\alpha_{r_m}, r_m \cdot \alpha_{r_{m-1}}, \ldots, (r_m \cdots r_2) \cdot \alpha_{r_1}\}.$$

Let $Q = w \cdot N(w) = \{-(r_1 \cdots r_{m-1}) \cdot \alpha_m, -(r_1 \cdots r_{m-2}) \cdot \alpha_{m-1}, \ldots, -\alpha_1\} \subseteq \Phi^-$. Since $r_1 \neq r_m$, it follows that for all positive integers k and all j with $1 \leq j \leq m$, the unique reduced expression for $w^k r_1 \cdots r_{j-1}$ does not end in r_j. Consequently, $(w^k r_1 \cdots r_{j-1}) \cdot r_j \in \Phi^+$ for all positive integers k and all $j \in \{1, \ldots, m\}$. Therefore, no element of Q is sent positive by any power of w. Hence each root cycle contains at most one element of $N(w)$. Thus $\kappa(w) = 1$, and the result follows. □

6 Parabolic Subgroups

In this section we look at the relationship between the crossing number of a Coxeter group and the crossing numbers of its standard parabolic subgroups. Also in this section we exhibit infinitely many examples of infinite Coxeter groups W for which $\kappa(W) = 1$. We begin by proving Theorem 1.7, which stated that if W is an arbitrary Coxeter group, and W_I is any standard parabolic subgroup of W, then $\kappa(W) \geq \kappa(W_I)$.

Proof of Theorem 1.7. Let W be a Coxeter group, $W_I \leq W$ a standard parabolic subgroup, Y a conjugacy class of W_I and $y \in Y$ such that $\kappa(y) = \kappa_{\min}(Y) = \kappa(W_I)$.

Set $X = Y^W$ and let $x \in X$. There exists $g \in W$ such that $x = y^g$, and by Theorem 2.3 we may write $g = g_I g^I$ for some $g_I \in W_I$ and $g^I \in W^I$. Thus $w := y^{g_I} \in Y$ and $x = w^{g^I}$. Note that c is a root cycle of w if and only if $(g^I)^{-1} \cdot c$ is a root cycle of x. Let c be a root cycle of w for which $\kappa(c) = \kappa(w)$. Observe that $N(w) \subseteq \Phi_I^+$ and $N((g^I)^{-1}) \cap \Phi_I = \varnothing$. Hence for $\alpha \in \mathrm{supp}(c)$ it follows that $(g^I)^{-1} \cdot \alpha \in \Phi^+$ if and only if $\alpha \in \Phi^+$. Therefore, c and $(g^I)^{-1} \cdot c$ have the same sign pattern and so

$$\kappa(x) \geq \kappa((g^I)^{-1} \cdot c) = \kappa(c) = \kappa(w).$$

Since $x \in X$ was chosen arbitrarily, and the crossing number of a group is the maximum of the $\kappa_{\min}$ over conjugacy classes, it follows that

$$\kappa(W) \geq \kappa_{\min}(X) \geq \kappa(w). \tag{9}$$

By choice of Y and y we have

$$\kappa(w) \geq \kappa(y) = \kappa_{\min}(Y) = \kappa(W_I). \tag{10}$$

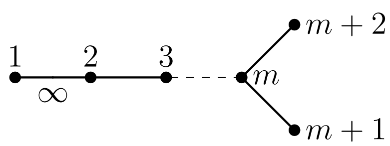
Combining (9) and (10) gives the result. □

Proof of Corollary 1.8. We first prove part (i). By Theorem 1.5 the only finite irreducible Coxeter groups with crossing number even and greater than two, or odd and greater than seven, are $\mathrm{B}_n, \mathrm{D}_n$, and $\mathrm{I}_2(n)$ (for appropriate choices of n).

If n is odd and greater than seven and $W \in \{\mathrm{B}_{2n+1}, \mathrm{D}_{2n+1}, \mathrm{B}_{2n+2}, \mathrm{D}_{2n+2}, \mathrm{I}_2(2n+1), \mathrm{I}_2(2n+2), \mathrm{I}_2(2n+4)\}$, or if n is even and greater than two and $W \in \{\mathrm{B}_{2n+1}, \mathrm{D}_{2n+1}, \mathrm{B}_{2n+2}, \mathrm{D}_{2n+2}, \mathrm{I}_2(2n+1)\}$, then $\kappa(W) = n$, and these constitute the only finite irreducible Coxeter groups with that crossing number.

Combining the above with Remark 4.6 implies the existence of infinitely many finite Coxeter groups with crossing number n.

We now prove part (ii). Let W be the Coxeter group corresponding to the following diagram.



(11)

Then W has $\mathrm{A}_1 \times \mathrm{D}_m$ as a parabolic subgroup. Hence

$$\begin{aligned}
\kappa(W) &\geq \kappa(\mathrm{A}_1 \times \mathrm{D}_m) && \text{by Theorem 1.7,}\\
&= \max\left\{\kappa(\mathrm{A}_1),\ \kappa(\mathrm{D}_m)\right\} && \text{by Remark 4.6,}\\
&= \max\left\{1,\ \left\lfloor \frac{m-1}{2} \right\rfloor\right\} && \text{by Theorem 1.5,}\\
&= \left\lfloor \frac{m-1}{2} \right\rfloor.
\end{aligned}$$

Taking $m = 2n+1$ gives the result. □

7 Acknowledgements

We thank the London Mathematical Society for funding a research visit in summer 2021 during which this work was started (LMS Research Reboot grant, Ref No: 42036). We were exceptionally grateful for a chance to collaborate when so many other opportunities had to be cancelled due to Covid-19. This work was then finished during a visit to the Mathematisches Forschungsinstitut Oberwolfach, as part of their Oberwolfach Research Fellows program.

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