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Character Theory and Categorification

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Abstract. Over a hundred years after the work of Frobenius and Schur, the sheer enormity of what is *not known* about the character theory of symmetric and alternating groups continues to surprise and awe the uninitiated. How does one decompose the tensor product of a pair of complex characters? Or the restriction of a complex character to a Sylow or wreath product subgroup? Can we understand the vanishing sets of complex characters? What about the asymptotic behaviour of complex characters? What are the dimensions of the modular characters? These questions have been hailed as some of the definitive open problems in representation theory and algebraic combinatorics, they have deep connections with Lie theory, group theory, local-global conjectures in representation theory of finite groups, symplectic geometry, complexity theory, statistical mechanics and quantum information theory. The overarching theme of this proposal is the use of hidden, richer representation theoretic structures arising in modular, local-global, and categorical representation theory in order to prove and disprove conjectures concerning characters of symmetric and alternating groups.

Mathematics Subject Classification (2020): 05E10, 20C30, 20C08.

Introduction by the Organizers

The workshop *Character Theory and Categorification* was organised by Christine Bessenrodt (Leibniz University Hannover), Chris Bowman (University of York), Eugenio Giannelli (University of Florence) and Alexander Kleshchev (University of Oregon), and was attended by 48 researchers from across Europe, North America, Australia, Chile, and Japan. This workshop was entirely in person (no zoom participation) and it surprised us just how enthusiastic everyone was about this decision. Dozens of participants repeatedly told us how much they loved the atmosphere and the buzz. One professor (who has been chair of department for several COVID years) told the organisers that this workshop had gotten him excited about research again, for the first time in a long time. A young researcher was offered a postdoctoral position through the connections made at the workshop.

Over the course of 24 lectures, much recent progress was presented and many new conjectures proposed across categorical and combinatorial representation theory. Remarkable new advances in our understanding have come from complexity theorists, number theorists, combinatoricists and even from Google's artificial intelligence! Two big existential questions arose as recurring themes of the conference: *"what representation theoretic questions can be completely solved?"* and *"at what structural level should we attempt to answer these questions?"*

This year one of the organisers, Christine Bessenrodt, sadly passed away. Chris Bowman gave a talk about Christine's work and legacy; the abstract of this talk is below.

"What questions can be completely solved?". This first question was discussed from both from a formal algorithmic complexity theory perspective and a more intuitive hierarchy of age-old combinatorial and number theoretic questions.

A stand alone highlight was Ikenmeyer's talk on the first day of the conference: informally, he gave the first ever example of a provably impossible question in combinatorial representation theory. In more detail, Ikenmeyer discussed a 2022 preprint of Ikenmeyer–Pak–Panova in which they prove that if the problem of computing the squares of $\mathfrak{S}_n$-characters belongs to class $\sharp\mathsf{P}$, then the polynomial time hierarchy collapses to the second level. This exciting work was initiated at our prequel Oberwolfach mini-workshop *"Kronecker, plethysm, and Sylow branching coefficients and their applications to complexity theory"* in 2018.

De Visscher, Libedinsky, and Hazi each emphasised the impossibility of understanding (p)-Kazhdan–Lusztig polynomials in general: this subsumes the problem of determining prime divisors of Fibonacci numbers; this is a notoriously difficult problem in number theory, for which an algorithmic solution is almost certainly impossible. De Visscher's talk took this as motivation to ask which p-Kazhdan–Lusztig polynomials can and cannot be understood combinatorially and to focus on those of Hermitian symmetric pairs, which she proved to be characteristic-free. Libedinsky proposed a new iterative construction of "pre-canonical bases" in order to factorise this impossible problem into an iterative series of (combinatorially manageable?!) steps.

Number theory did not arise *solely* as a harbinger of impossible problems. Motivated by the study of some of the central local-global conjectures in group representation theory (namely the McKay Conjecture and its Galois refinement), Navarro presented several results on fields of values of irreducible characters and discussed new conjectures connecting character values with Sylow branching coefficients. In particular, he concluded his talk by proposing a new refinement of the Galois-McKay Conjecture involving restriction of irreducible characters to Sylow subgroups. In the same direction, Alex Miller explained how number theory came to the rescue in the recent resolution of Miller's own conjecture concerning the vanishing of symmetric group characters. This conjecture (now a theorem) stated that *for any prime p, all character values of symmetric groups are zero modulo p in the limit as the rank tends to ∞*. In a light-hearted aside, Miller measured the the importance of this result in terms of its 10 Tweets from Tim Gowers. On similar lines, in his talk Pacifici focused on arithmetical properties of irreducible character degrees. More precisely, he announced significant advances towards a complete proof of the long standing Huppert's $\rho - \sigma$ Conjecture, and explained recent work on the character degree graph, an extremely useful combinatorial tool used to extract information on irreducible character degrees of a finite group.

Fayers and Brundan emphasised another lens through which to view the (p)-Kazhdan–Lusztig polynomial problem. Lusztig and James conjectured that affine p-Kazhdan–Lusztig polynomials should be independent of the prime $p \geq 0$ under certain conditions. While these conjectures are now known to be false, Chuang–Tan and Fayers–Kleshchev–Morotti have verified that they are true in the case of RoCK blocks of symmetric groups and their double covers. These RoCK blocks are, in some sense, generic blocks and provide "almost all blocks". Brundan developed this idea further: every block is derived equivalent to one of these generic RoCK blocks thanks to very recent work of Brundan–Kleshchev and Ebert–Lauda–Vera in 2022, building on the ideas of Chuang–Rouquier.

Ariki, Bowman, and Geranios took as a-given the impossibility of answering certain structural questions and instead proposed new milestones for our understanding. Ariki and Geranios both sought answers to modular theoretic questions: classifications of endo-trivial Specht modules, tame and wild behaviour, and appropriate ways of understanding wild representation categories. Bowman's talk was dedicated to the work of Christine Bessenrodt and emphasised her ability to carve out the separation between the knowable and unknowable problems of algebraic combinatorics, in part through her classification of irreducible, homogeneous, and multiplicity-free instances of these combinatorial questions.

"At what structural level should we answer these questions?". In a surprising and eye-opening talk, Malle postulated that the structure obscuring local-global group conjectures was the groups themselves! Researchers in local-global group theory have long searched for a richer, structural interpretation for these conjectures. Following this approach, Kessar–Malle–Semeraro have used topological structures in order to generalise Alperin's Weight conjecture beyond the realms of finite groups of Lie type to the more exotic setting of (perhaps non-existent!) objects known as Spetses. In so doing, Kessar–Malle–Semeraro actually resolve this conjecture for previously *unknown cases in finite group theory*, thus providing ample evidence for Malle's seemingly bizarre postulate.

Brundan and Webster both expounded the virtue of Heisenberg categorical actions as a dramatic simplification and neat tying together of many of the most important results in the past 15 years. Such categories admit beautiful diagrammatic visualisations, which were emphasised as the easiest and most natural way of grasping the underlying ideas. Ostrik discussed natural questions surrounding limiting behaviour of tensor products of cyclic groups. This allowed him to discuss the construction of “Verlinde categories” which should be thought of as the replacement of vector spaces within the theory of modular categorification.

On the other hand, Law, Gurevich, Gerber, Orellana, Muth, Wildon, and Zabrocki emphasised the importance of concrete combinatorics such as graph- and partition-theoretic constructions. Gurevich explained new combinatorial conjectures in Kazhdan–Lusztig theory which were recently formulated with the help of Google’s Artificial Intelligence. Law, Orellana, and Zabrocki discussed combinatorial algorithms for calculating plethysm and branching coefficients for symmetric and general linear groups and predicting the existence of richer structures behind stability phenomena. Muth discussed how cuspidal combinatorics allows for the computation of new decomposition numbers for symmetric groups and its pivotal role in the spectacular resolution of Broué’s abelian defect conjecture for spin covers of symmetric groups.

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Workshop: Character Theory and Categorification

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Abstracts

Some aspects of Kazhdan–Lusztig polynomials

Nicolas Libedinsky
(joint work with Leonardo Patimo, David Plaza and Geordie Williamson)

The problem of understanding Kazhdan–Lusztig polynomials is of fundamental importance to representation theory. An impressive amount of results in the representation theory of "Lie-type" objects (such as character formulas or multiplicities of relevant objects in other relevant objects) are expressed in terms of Kazhdan–Lusztig polynomials, often evaluated in 1. We present three important problems related to these polynomials and new approaches to their solution.

(1) The first problem is that of calculating Kazhdan–Lusztig polynomials appearing in representation theory. Towards this end we present a new concept introduced by D. Plaza, L. Patimo and the author in [2] called the "pre-canonical bases". These are a new family of bases N^i_λ (with λ a dominant weight and i a varying positive integer) of the spherical Hecke algebra that can also be seen as elements of the affine Hecke algebra. The key point of why these objects are appealing is that they divide the extremely difficult problem of calculating Kazhdan–Lusztig polynomials into a big number of much easier steps. This is because when i is big, N^i_λ is the Kazhdan–Lusztig basis element associated to λ and for $i = 1$ this is just the standard basis associated to λ. We give explicit formulas in low ranks, thus explaining how this simplifies considerably the combinatorics. Once one evaluates in $q = 1$, these bases give an interpolation between the Weyl characters and Hall-Littlewood polynomials. In affine type A we conjecture that the decomposition of N^{i+1}_λ in terms of N^i_μ is positive.

(2) The second problem is that of proving positivity properties for Kazhdan–Lusztig polynomials. Wolfgang Soergel proved that in Soergel category of bimodules $\mathcal{B}$ associated to a Coxeter group W, the indecomposable objects B_x are indexed by elements $x \in W$. Furthermore, he proved that if H is the Hecke algebra of W then the map

$$\langle B\rangle \mapsto \sum_{y\in W} \underline{\mathrm{rk}}\ \mathrm{Hom}_{\not< x}(B_y, B_x)h_y$$

from the split Grothendieck ring $\langle\mathcal{B}\rangle$ to H is an isomorphism (here h_y is the element of the standard basis associated to y and $\mathrm{Hom}_{\not< x}$ means that we consider the full Hom space in $\mathcal{B}$ and we take the quotient by any map that can be factored through $B_z(i)$ with $z < x$ and i an integer). In a huge breakthrough by Ben Elias and Geordie Williamson they proved that under this map, the class of the indecomposable object $\langle B_x\rangle$ has image the Kazhdan–Lusztig basis element b_x. This immediately implies the positivity of Kazhdan–Lusztig polynomials (its coefficients are graded ranks of these $\mathrm{Hom}_{\not<x}$ spaces). With Geordie Williamson we proved a generalization of this result in [3] to the parabolic setting. More precisely, we prove it for the anti-spherical Kazhdan–Lusztig polynomials. The problem is still open in the case of general spherical Kazhdan–Lusztig polynomials (some particular groups are known in this case).

(3) The last problem is that of producing a counting formula for Kazhdan–Lusztig polynomials. For $\underline{y}$ an expression of $y \in W$ and $y > x \in W$, let us denote by $\mathrm{SE}_{x,\underline{y}}$ the set of subexpressions of $\underline{y}$ for x. Vinay Deodhar had the following idea. Find a subset $\mathrm{SE}^L_{x,\underline{y}} \subset \mathrm{SE}_{x,\underline{y}}$ such that the Kazhdan–Lusztig polynomial

$$h_{x,y} = \sum_{e \in \mathrm{SE}^L_{x,\underline{y}}} v^{\mathrm{df}(e)},$$

where $\mathrm{df}(e)$ is Deodhar's defect. The main problem with this proposal is that there are many possible choices of $\mathrm{SE}^L_{x,\underline{y}}$, so the choice is far from being canonical.

In [4] we construct the "canonical light leaves" which give a *canonical* isomorphism of graded vector spaces

$$\mathrm{CLL} : \bigoplus_{e \in \mathrm{SE}_{x,\underline{y}}} \mathbb{R}e \xrightarrow{\sim} \mathrm{Hom}_{\not<x}(B_{\underline{y}}, B_x).$$

where the left hand side is graded by Deodhar's defect, i.e. the generator $e \in \mathrm{SE}_{x,\underline{y}}$ has degree $\mathrm{df}(e)$. This theorem leads to a natural refinement of Deodhar's proposal:

Find a subset $\mathrm{SE}^L_{x,\underline{y}} \subset \mathrm{SE}_{x,\underline{y}}$ such that the composition of the inclusion, canonical light leaves and the canonical surjection

$$\bigoplus_{e \in \mathrm{SE}^L_{x,\underline{y}}} \mathbb{R}e \hookrightarrow \bigoplus_{e \in \mathrm{SE}_{x,\underline{y}}} \mathbb{R}e \xrightarrow{\mathrm{CLL}} \mathrm{Hom}_{\not<x}(B_{\underline{y}}, B_x) \twoheadrightarrow \mathrm{Hom}_{\not<x}(B_y, B_x)$$

is an isomorphism of graded vector spaces. If the choice of the subset $\mathrm{SE}^L_{x,\underline{y}}$ could be made canonically we would regard it as a solution to the counting problem above. In [1] we give a solution following these lines of the $\widetilde{A}_2$ case. The result is surprisingly simple and gives hope that this method might work in general.

References

[1] N. Libedinsky, L. Patimo *On the affine Hecke category for SL_3*, preprint arXiv:2005.02647.

[2] N. Libedinsky, L. Patimo, D. Plaza *Pre-canonical bases on affine Hecke algebras*, Advances in Mathematics **399** (2022).

[3] N. Libedinsky, G. Williamson *The anti-spherical category*, Advances in Mathematics **405** (2022).

[4] N. Libedinsky, G. Williamson *Kazhdan–Lusztig polynomials and subexpressions*, Journal of Algebra **568** (2021), 181–192.

Characters and Sylow p-subgroups

Gabriel Navarro

This is the continuation of the talk I gave in the mini workshop in February 2020. It is dedicated to Christine Bessenrodt.

1. Introduction. Our main goal has always been to better understand the McKay conjecture: If p is a prime, G is a finite group, then this conjecture asserts that

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|\,,$$

where P is a Sylow p-subgroup of G and $\mathrm{Irr}_{p'}(G)$ is the subset of the irreducible complex characters χ of G, $\chi \in \mathrm{Irr}(G)$, such that p does not the degree $\chi(1)$.

As happens in the case where P is normal in G, could it be that the restriction χ_P contains some special orbit of $\mathbf{N}_G(P)$-irreducible characters that can help us to construct some kind of McKay bijection? To fully understand the restriction χ_P is equivalent to know the so called *Sylow Branching Coefficients*, which are the multiplicities $[\chi_P, \gamma]$, for $\gamma \in \mathrm{Irr}(P)$. Work in symmetric groups [1] show that to know these numbers is extremely difficult.

If p is odd and $\mathbf{N}_G(P) = P$, we did prove in [8] that if $\chi \in \mathrm{Irr}_{p'}(G)$, then $\chi_P = \lambda + \Delta$ for a unique linear $\lambda \in \mathrm{Irr}(P)$, and where all the irreducible constituents of Δ have degree divisible by p. Hence a canonical McKay bijection exists in this case. This already does not happen in S_5 for $p = 2$. (Although in the case of S_n and $p = 2$ more sophisticated canonical bijections exist by work in [4] and [2].) But in general, I do know that there is no McKay canonical bijection in S_n for p odd.

Here is where Galois action become fundamental. A fact that I take for granted is that any *canonical* (choice free, natural) character bijection, should commute with the action of the absolute group $\mathcal{G} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Hence, a canonical McKay bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ should preserve the number of rational-valued characters, say. Now take $G = S_5$ and $p = 5$, and the possibility of a natural McKay bijection disappears. This led to my version of McKay in [6].

Galois-McKay Conjecture. *There should be a bijection*

$$^* : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$$

conmuting with $\mathcal{G}_p$-action.

(Here $\mathcal{G}_p$ is the Galois automorphisms that send p'-roots of unity to some fixed but arbitrary p-power.)

This conjecture implies that there is a connection between values of characters of G and of $\mathbf{N}_G(P)$ (over the p-adic field).

2. New Results on Fields of Values. My new results and conjectures today are about field of values.

In what follows, if ψ is a character of a group X, then $\mathbb{Q}(\psi)$ is the smallest field containing the values of ψ. In particular, $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_{|X|}$, the cyclotomic field. We let $c(\psi)$, the conductor of ψ, to be the smallest number n such that $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_n$. Is there anything special about the field of values $\mathbb{Q}(\chi)$ whenever $\chi \in \mathrm{Irr}(G)$?

We know that $\mathbb{Q}(\chi)/\mathbb{Q}$ is an abelian extension, but the converse is also true: If $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}_n$, then the semidirect product of $C_n : \mathrm{Gal}(\mathbb{Q}_n/F)$ has irreducible characters with field of values F. As you have seen, my interest (surely by the McKay conjecture) is about characters of p'-degree, where p is a fixed prime.

Let us fix $p = 2$ and start looking for the fields $\{\mathbb{Q}(\chi) \mid \chi \in \mathrm{Irr}_{p'}(G)\}$. A big surprise is that $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{-5})$, say, are not showing up. In fact, we proved the following using the classification ([5]):

Theorem. *Suppose that $\chi \in \mathrm{Irr}(G)$ has odd-degree. Then $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_n$, where n is odd, or $i \in \mathbb{Q}(\chi)$. In particular, if $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{d})$, where $d > 1$ is squarefree integer, then $d \equiv 1$ mod 4.*

This was the *baby* case of the theorem that I later proved with Tiep in [7].

Theorem. *Suppose that χ has odd degree and conductor $c(\chi) = 2^a m$, where m is odd. Then $\mathbb{Q}_{2^a} \subseteq \mathbb{Q}(\chi)$.*

Moreover, the converse is true: if $F/\mathbb{Q}$ is an abelian extension with conductor $2^a m$, then $F = \mathbb{Q}(\chi)$ for some odd-degree χ if and only if $\mathbb{Q}_{2^a} \subseteq F$. This converse is also complicated. We were struggling to produce a group G with an irreducible character of odd degree whose field of values was, for instance, $\mathbb{Q}(\sqrt{57})$. We proved that you cannot do this with solvable groups. We had to use adhoc large general linear groups.

We have analogs of the previous results for p odd, but conjecturally only: if $\chi \in \mathrm{Irr}_{p'}(G)$ and $c(\chi) = p^a m$, where p does not divide m, then $|\mathbb{Q}_{p^a} : \mathbb{Q}_{p^a} \cap \mathbb{Q}(\chi)|$ is not divisible by p.

What all this has to do with restrictions of characters? The only way to reduce all these theorems to quasi-simple groups (and prove them for $p = 2$) was to show that the p-irrationalities were taken by p-elements, that is in χ_P. We concluded that a possible conjecture could explain what we were seeing (and more).

Suppose that $c(\chi) = p^a m$, where p does not divide m. Since $\mathbb{Q}(\chi_P) \subseteq \mathbb{Q}(\chi) \subseteq \mathbb{Q}_{|G|}$, we have that $\mathbb{Q}(\chi_P) \subseteq \mathbb{Q}_{|P|} \cap \mathbb{Q}_{p^a m}$, and therefore $\mathbb{Q}(\chi_P) \subseteq \mathbb{Q}_{p^a}$. In [7], we made the following:

Conjecture A. *If $\chi \in \mathrm{Irr}(G)$ has p'-degree with conductor $p^a m$, then $\mathbb{Q}_{p^a}/\mathbb{Q}(\chi_P)$ has degree not divisible by p.*

For instance, if $p = 2$, then this is telling us that $\mathbb{Q}(\chi_P) = \mathbb{Q}_{2^a}$ is always cyclotomic! Or that χ is 2-rational if and only if χ_P is rational.

We proved this conjecture for many quasi-simple groups. But what about p-solvable groups? This has been a complete nightmare. To have a conjecture open for p-solvable groups is to be crying for a big problem... However, recently, M. Isaacs and myself have proved the p-solvable case. We needed to consider a new invariant.

If ψ is a character of a p-group P with degree not divisible by p, it is clear that ψ contains linear constituents. Write $L(\psi)$ for the linear part, and $NL(\psi)$ for the non-linear part. Hence $L(\psi)$ has degree not divisible by p. Let's order them according to their order:

$$L(\psi) = p^0(\psi) + p^1(\psi) + \ldots + p^t(\psi)\,.$$

That is,

$$p^i(\psi) = \sum_{\substack{\lambda\in \mathrm{Irr}(P),\lambda(1)=1\\ o(\lambda)=p^i}} [\psi,\lambda]\lambda\,.$$

Notice that some $p^i(\psi)$ has degree not divisible by p, and let p^e be the largest such.

Theorem. (Isaacs-N). *Let $\chi \in \mathrm{Irr}(G)$, $c(\chi)_p = p^a$. Assume that G is p-solvable. If $e, a \geq 1$, then $e = a$ and $[\mathbb{Q}_{p^a} : \mathbb{Q}(\chi_P)]$ is not divisible by p.*

(In the degenerate cases $e = 0$ or $a = 1$, the second conclusion is also true.) This proved the above mentioned conjecture for p-solvable groups.

This new work has led to the following natural questions.

Question A. *Is the theorem above true for non p-solvable groups?*

Notice that we can put the conjecture above and Galois-McKay in a single conjecture:

Conjecture. *There should be a bijection*

$$^* : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$$

conmuting with $\mathcal{G}_p$-action such that $\mathbb{Q}_p(\chi_P) = \mathbb{Q}_p(\chi^*_P)$.

In my Oberwolfach 2020, I stated this conjecture with $\mathbb{Q}(\chi_P) = \mathbb{Q}(\chi^*_P)$, but in the degenerate cases this can be wrong: we have to consider $\mathbb{Q}_p(\chi_P)$ instead of $\mathbb{Q}(\chi_P)$.

Question B. *Suppose that $\chi \in \mathrm{Irr}(G)$ with conductor $p^a m$, $a \geq 1$, p does not divide m. Is it true that $|\mathbb{Q}_{p^a} : \mathbb{Q}_p(\chi_P)| \leq \chi(1)_p$?*

References

[1] E. Giannelli, S. Law, Sylow branching coefficients for symmetric groups, J. Lond. Math. Soc. (2) **103** (2021), no. 2, 697–728.

[2] E. Giannelli, A. Kleshchev, G. Navarro, P. H. Tiep, Restriction of odd degree characters and natural correspondences, International Math. Research Notices, Vol. 2017, No. 20, 6089–6118.

[3] I. M. Isaacs, G. Navarro, Conductors of characters, preprint.

[4] I. M. Isaacs, G. Navarro, J. Olsson, P. H. Tiep, Character Restrictions and Multiplicities in Symmetric Groups, J. Algebra **478** (2017) 271–282.

[5] I. M. Isaacs, M. W. Liebeck, G. Navarro, P. H. Tiep, Fields of values of odd-degree irreducible characters, Adv. Math. **354** (2019) 106757, 26pp.

[6] G. Navarro, The McKay conjecture and Galois automorphisms. Ann. of Math. (2) **160** (2004), no. 3, 1129–1140.

[7] G. Navarro, P. H. Tiep, The fields of values of characters of degree not divisible by p, Forum Math. Pi **9** (2021), vol 9, 1–28.

[8] G. Navarro, P. H. Tiep, C. Vallejo, McKay natural correspondences of characters, Algebra Number Theory **8** (2014), 1839–1856.

Decomposition numbers for spin RoCK blocks of symmetric groups

Matt Fayers

(joint work with Sasha Kleshchev & Lucia Morotti)

In the representation theory of the symmetric group $\mathfrak{S}_n$ in characteristic p, a typical approach is to study blocks of fixed p-weight (or equivalently, fixed defect) as n varies. The verification by Scopes [8] of Donovan's conjecture for the symmetric groups shows that there are only finitely many Morita equivalence classes of blocks of p-weight w, and one of these classes (whose members are now called *RoCK blocks*) was singled out by Rouquier as potentially being easier to understand. This potential was realised by Chuang and Kessar, who proved a Morita equivalence result in the case of abelian defect (that is, the case $w < p$) which in particular showed that Broué's abelian defect group conjecture holds for RoCK blocks. Derived equivalence results of Chuang and Rouquier [2] then extended this to show that Broué's conjecture hold for all blocks of symmetric groups. Chuang and Tan [3] then gave a formula for the decomposition numbers for RoCK blocks of abelian defect, which was extended to the non-abelian case by Turner [9].

In recent years there has been a surge of interest in the representation theory of double covers of symmetric groups, and it is natural to ask for analogues of known results for symmetric groups. A recent paper of Kleshchev and Livesey [6] provides a natural analogue of RoCK blocks for double covers, and gives a Morita equivalence result, showing that a RoCK blocks of abelian defect is Morita equivalent to a certain wreath product algebra $A_{(p-1)/2} \wr \mathfrak{S}_w$, which in turn shows that Broué's conjecture holds for RoCK blocks of double covers. This has since been extended by Brundan–Kleshchev [1] and by Ebert–Lauda–Vera [4] to prove Broué's conjecture for all blocks of double covers. However, the results of Kleshchev–Livesey do not show how to find decomposition numbers for spin RoCK blocks, and this is the subject of the talk.

From calculation in the quantised Fock space of type $A^{(2)}_{p-1}$ and results of Leclerc–Thibon [7] relating the Fock space to decomposition numbers for symmetric groups, the speaker was able to formulate a conjecture for the decomposition numbers [5]. This was proved in joint work with Kleshchev and Morotti. This involved two substantial ingredients:

- constructing projective characters and comparing them with the conjectural indecomposable projective characters to show that the conjecture is true up to a non-negative unitriangular adjustment;
- constructing simple modules and indecomposable projectives for the wreath product algebra $A_{(p-1)/2} \wr \mathfrak{S}_w$ to determine its Cartan invariants (and hence the Cartan invariants of a RoCK block), and showing that these agree with the conjecture.

The first of these tasks is more difficult than in the symmetric group case, since the standard set of labels for the simple modules in characteristic p (the restricted p-strict partitions) is no longer a subset of the the standard set of labels for the simple module in characteristic 0 (the strict partitions). But a delicate argument involving the dominance order and using the Brundan–Kleshchev regularisation theorem allows this to work.

References

[1] J. Brundan & A. Kleshchev, *Odd Grassmannian bimodules and derived equivalences for spin symmetric groups*, arXiv:2203.14149.

[2] J. Chuang & R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. of Math. **167** (2008), 245–298.

[3] J. Chuang & K. M. Tan, *Filtrations in Rouquier blocks of symmetric groups and Schur algebras*, Proc. London Math. Soc. (3) **86** (2003), no. 3, 685–706.

[4] M. Ebert, A. Lauda & L. Vera, *Derived superequivalences for spin symmetric groups and odd $\mathfrak{sl}(2)$-categorifications*, arXiv:2203.14153.

[5] M. Fayers, *Comparing Fock spaces in types $A^{(1)}$ and $A^{(2)}$*, arXiv:2207.01879.

[6] A. Kleshchev & M. Livesey, *RoCK blocks for double covers of symmetric groups and quiver Hecke superalgebras*, arXiv:2201.06870

[7] B. Leclerc & J.–Y. Thibon, *q-deformed Fock spaces and modular representations of spin symmetric groups*, J. Physics A **30** (1997), 6163–6176.

[8] J. Scopes, *Cartan matrices and Morita equivalence for blocks of the symmetric groups*, J. Algebra **142** (1991), 441–455.

[9] W. Turner, *Rock blocks*, Mem. Amer. Math. Soc. **202**, no. 947 (2009), viii+102.

An introduction to modular plethysms

Mark Wildon

(joint work with Eoghan McDowell, Rowena Paget)

As motivation I began my talk with the observation that the vector spaces $\mathrm{Sym}^2\,\mathbb{C}^{d-1}$ and $\bigwedge^2 \mathbb{C}^d$ both have the same dimension, namely $\binom{d}{2}$. An appealing explanation is that if E is the natural 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ then $\mathrm{Sym}^2\,\mathrm{Sym}^{d-1}E \cong \bigwedge^2 \mathrm{Sym}^d\, E$ as representations of $\mathrm{SL}_2(\mathbb{C})$. This is generalized by the Wronskian isomorphism

$$\mathrm{Sym}^r\,\mathrm{Sym}^\ell E \cong \bigwedge^r \mathrm{Sym}^{l+r-1} E,$$

categorifying the counting identity that the number of r-multisubsets of $\{1,\ldots,\ell+1\}$ is the number of r-subsets of $\{1,\ldots,\ell+r\}$. It is natural to ask if the Wronskian isomorphism holds over fields other than $\mathbb{C}$. The answer is 'yes', provided that a suitable duality is introduced, replacing a Sym^r with its dual functor Sym_r; this corresponds to taking invariants rather than coinvariants in a tensor power.

Theorem (McDowell–W, Theorem 1.4 in [4])**.** *Let F be a field and let E be the natural representation of $\mathrm{SL}_2(F)$. For $r,\ \ell \in \mathbb{N}$ there is an isomorphism of $\mathrm{SL}_2(F)$-representations*

$$\mathrm{Sym}_r\,\mathrm{Sym}^\ell E \cong \bigwedge^r \mathrm{Sym}^{r+\ell-1} E.$$

Isomorphisms of $\mathrm{SL}_2(\mathbb{C})$-modules such as $\bigwedge^r \mathrm{Sym}^{r+\ell-1} E$ were studied systematically in my joint paper [5] with Rowena Paget. An essential result was the following equivalent characterisations:

(i) $\nabla^\lambda \mathrm{Sym}^\ell E \cong_{\mathrm{SL}_2(\mathbb{C})} \nabla^\mu \mathrm{Sym}^m E$;
(ii) $(s_\lambda \circ s_{(\ell)})(q, q^{-1}) = (s_\mu \circ s_{(m)})(q, q^{-1})$;
(iii) $s_\lambda(q^\ell, q^{\ell-2}, \ldots, q^{-\ell}) = s_\mu(q^m, q^{m-2}, \ldots, q^{-m})$;
(iv) $s_\lambda(1, q, \ldots, q^\ell) = s_\mu(1, q, \ldots, q^m)$ up to a power of q

Here ∇^λ is the Schur functor for the partition λ and $s_\lambda \circ s_\ell$ is the *plethysm product* of the two Schur functions, defined by substituting the monomials in s_ℓ for the variables in s_λ. In this case, the chosen variables are q and q^{-1}, and since $s_\ell(x, y) = x^\ell + x^{\ell-1}y + \cdots + y^\ell$, the monomials are $q^\ell, q^{\ell-2}, \ldots, q^{-\ell}$. Because of this connection with symmetric functions, we refer to isomorphisms, such as those in the theorem stated above, as *modular plethysms*. For further background and some motivation for why composition of Schur functors corresponds to the plethysm product on Schur functions, see [3].

Example. *Hermite reciprocity is the isomorphism*

$$\mathrm{Sym}^r \mathrm{Sym}^\ell E \cong \mathrm{Sym}^\ell \mathrm{Sym}^r E.$$

By the equivalence of (i) *and* (iv)*, taking $\lambda = (r)$ and $m = r$, it is equivalent to prove that $s_{(r)}(1, q, \ldots, q^\ell) = s_{(\ell)}(1, q, \ldots, q^r)$. Remembering that $s_{(n)}$ is the complete symmetric function, this follows by interpreting the left-hand side as the generating function enumerating partitions whose Young diagram is contained in a box with r rows and ℓ columns, and the right-hand side similarly, using the box with ℓ rows and r columns.*

In this example we saw a combinatorial proof of an algebraic isomorphism. To continue in this theme, a very useful result is Stanley's Hook Content Formula [6, Theorem 7.21.2], which states that there is a power $q^{b(\lambda)}$ such that

$$s_\lambda(1, q, \ldots, q^m) = q^{b(\lambda)} \frac{\prod_{(i,j)\in[\lambda]} [j - i + m + 1]_q}{\prod_{(i,j)\in[\lambda]} [h_{(i,j)}]_q}$$

where $[r]_q$ is the quantum integer $(q^r - 1)/(q - 1)$ and $h_{(i,j)}$ is the hook length of the box (i, j) of the Young diagram $[\lambda]$. Note that $j - i$ is the content of the box $(i, j) \in [\lambda]$, so the quantum integers in the numerator are the contents of $[\lambda]$, shifted by $\ell + 1$. Using this result Paget and I proved the following simultaneous generalization of Hermite reciprocity and the Wronksian isomorphism.

Theorem (Paget–W, Theorem 1.6 in [5])**.** *Let λ be a partition with at most ℓ parts. There is an isomorphism $\nabla^\lambda \mathrm{Sym}^\ell E \cong \mathrm{Sym}^a \mathrm{Sym}^b E$ of representations of $\mathrm{SL}_2(\mathbb{C})$ if and only if λ is obtained by adding columns of length $\ell + 1$ to one of the partitions $(a), (1^a), (b), (1^b), (a^b), (b^a)$, and ℓ is respectively b, $a + b - 1$, a, $a + b - 1$, b, a.*

A related result is the converse of a theorem of King. For a fixed s, let $\lambda^{\bullet d}$ denote the complement of the partition λ having largest part at most s in a $d \times s$ box. For example if $s = 5$ then $(4, 3, 3, 1)^{\bullet 4} = (4, 2, 2, 1)$.

Theorem (King 1985 [if], Paget–W 2019 [only if])**.** *Let λ have at most d parts. Then*

$$\nabla^\lambda \operatorname{Sym}^\ell E \cong \nabla^{\lambda^{\bullet d}} \operatorname{Sym}^\ell E$$

if and only if $\lambda = \lambda^{\bullet d}$ or $\ell = d - 1$.

Using Stanley's Hook Content Formula one can obtain an attractive combinatorial interpretation of this theorem. To illustrate it by example, again take $s = 5$ and $\lambda = (4, 3, 3, 1)$. By the theorem, $\nabla^\lambda \operatorname{Sym}^3 E \cong \nabla^{\lambda^{\bullet 4}} \operatorname{Sym}^3 E$. The two tableaux below show the hook lengths of $[\lambda]$ and $[\lambda^{\bullet 4}]$ in ordinary type numbers, and the shifted contents in bold. Please ignore the subscripts for the moment.

$\mathbf{4}_0$	$\mathbf{5}_1$	$\mathbf{6}_2$	$\mathbf{7}_3$	1_0
$\mathbf{3}_0$	$\mathbf{4}_1$	$\mathbf{5}_2$	1_0	3_1
$\mathbf{2}_0$	$\mathbf{3}_1$	$\mathbf{4}_2$	2_0	4_1
$\mathbf{1}_0$	1_0	2_1	5_2	7_3

7_3	5_2	4_1	1_0	$\mathbf{1}_0$
5_2	3_1	2_0	$\mathbf{3}_1$	$\mathbf{2}_0$
4_2	2_1	1_0	$\mathbf{4}_1$	$\mathbf{3}_0$
1_0	$\mathbf{7}_3$	$\mathbf{6}_2$	$\mathbf{5}_1$	$\mathbf{4}_0$

Thus the left-hand tableau has the quantum integers appearing in

$$\prod_{(i,j)\in[\lambda]} [i - j + 4]_q \prod_{(i,j)\in[\lambda^{\bullet 4}]} [h_{(i,j)}(\lambda)]_q$$

and the right-hand tableaux has the quantum integers appearing in the analogous product swapping λ and $\lambda^{\bullet 4}$. By Stanley's formula, the two products are equal. Hence, by a unique factorization result for quantum integers, proved as Lemma 3.2 in [5], the multisets of entries in the two tableaux are equal. I gave a combinatorial proof of this fact in [7]. After seeing this paper, Prof. Christine Bessenrodt [2] observed that by [1] a stronger combinatorial result holds, in which the hooks and shifted contents are paired with their corresponding arm lengths, as shown in the tableaux above as subscripts.

Problem. *Give an algebraic proof of Bessenrodt's observation using Jack symmetric functions.*

Returning to the original algebraic theorem, it is natural to ask when its isomorphism holds over other fields. The following result gives, we believe, the more precise possible answer.

Theorem (McDowell–W, Theorem 1.2 in [4])**.** *Let G be a group. Let V be a d-dimensional representation of G over an arbitrary field. Let $s \in \mathbb{N}$, and let λ be a partition with $\ell(\lambda) \le d$ and first part at most s. There is an explicit isomorphism*

$$\nabla^\lambda V \cong \nabla^{\lambda^{\bullet d}} V^\star \otimes (\det V)^{\otimes s}.$$

In the final part of my talk I emphasised that the existence of such modular plethysms is far from obvious, and there are many cases where an isomorphism known to hold for $\mathrm{SL}_2(\mathbb{C})$ does not generalize to arbitrary fields.

Theorem (McDowell–W 2020, Theorem 1.6 in [4]). *Let F be an infinite field of prime characteristic p. There exist infinitely many pairs (a, b) such that, provided e is sufficiently large, the eight representations of* $\mathrm{SL}_2(F)$ *obtained from* $\nabla^{(a+1,1^b)} \operatorname{Sym}^{p^e+b} E$ *by*

- *Replacing* ∇ *with* Δ *(duality)*
- *Replacing* $(a+1, 1^b)$ *with* $(b+1, 1^a)$ *and* $p^e + b$ *with* $p^e + a$ *(King conjugation);*
- *Replacing* $\operatorname{Sym}^\ell E$ *with* $\operatorname{Sym}_\ell E$ *(another duality);*

are all non-isomorphic.

Even establishing the non-existence of an isomorphism is not easy, because the existence of an isomorphism over $\mathrm{SL}_2(\mathbb{C})$ means that many of the standard techniques, for example, considering the image of representations in the Grothendieck ring, are inapplicable. I recommend the further study of these modular plethysms.

References

[1] Christine Bessenrodt, *On hooks of Young diagrams*, Ann. Comb. **2** (1998), no. 2, 103–110.

[2] Christine Bessenrodt, personal communication, April 2019.

[3] Melanie de Boeck, Rowena Paget, and Mark Wildon, *Plethysms of symmetric functions and highest weight representations*, Submitted. ArXiv:1810.03448 (September 2018), 35 pages.

[4] Eoghan McDowell and Mark Wildon, *Modular plethystic isomorphisms for two-dimensional linear groups*, arXiv:2105.00538 (May 2021), 40 pages.

[5] Rowena Paget and Mark Wildon, *Plethysms of symmetric functions and representations of* $\mathrm{SL}_2(\mathbf{C})$, arXiv:1907.07616 (July 2019), 51 pages.

[6] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[7] Mark Wildon, *A corollary of Stanley's Hook Content Formula*, arXiv:1904.08904, April 2019, 8 pages.

Squares of characters of the symmetric group

Christian Ikenmeyer

(joint work with Igor Pak, Greta Panova)

This is based on the paper [2].

Consider the following two classical identities from the representation theory of the symmetric group:

$$n! \;=\; \sum_{\lambda \vdash n} \big(\chi^\lambda(1)\big)^2\,, \quad \text{and} \tag{1}$$

$$n! \;=\; \sum_{\pi \in \mathfrak{S}_n} \big(\chi^\lambda(\pi)\big)^2 \quad \text{for all} \;\; \lambda \vdash n. \tag{2}$$

Here χ^λ is the irreducible character of the symmetric group $\mathfrak{S}_n$ of the representation indexed by λ, and $\chi^\lambda(\pi) \in \mathbb{Z}$ is its evaluation. Both identities arise in a similar manner, as squared norms of row and column vectors in the character table of $\mathfrak{S}_n$.

Equalities such as these, are an invitation for combinatorialists to search for natural bijections between the sets of combinatorial objects counting both sides. In both cases, the LHS is the set $\mathfrak{S}_n$ of permutations of n symbols. For (1), the RHS is the set of pairs of *standard Young tableaux* of the same shape with n boxes. The bijection between the set of permutations and the set of pairs of Young tableaux is the celebrated *Robinson–Schensted correspondence*, which is fundamental in Algebraic Combinatorics.

Similarly, for (2), one would want to give a bijection between $\mathfrak{S}_n$ and a set of $n!$ many combinatorial objects that are partitioned naturally into subsets of sizes $\big(\chi^\lambda(\pi)\big)^2$. In this paper we prove that this approach would fail for the fundamental reason that the RHS of (2) does not admit such an interpretation. As the following theorem implies, it is unlikely that there exist "sets of $\big(\chi^\lambda(\pi)\big)^2$ many combinatorial objects".

Theorem 1. Let $\chi^2 : (\lambda, \pi) \mapsto \big(\chi^\lambda(\pi)\big)^2$, where $\lambda \vdash n$ and $\pi \in \mathfrak{S}_n$. If the function χ^2 is contained in the complexity class $\#\mathsf{P}$, then $\mathsf{coNP} = \mathsf{C_=P}$. Consequently, if $\chi^2 \in \#\mathsf{P}$, then the polynomial hierarchy collapses to the second level: $\mathsf{PH} = \Sigma_2$.

The assumption $\mathsf{PH} \neq \Sigma_2$ in the theorem is a widely believed standard complexity theoretic assumption, which formally implies $\mathsf{P} \neq \mathsf{NP}$. From a combinatorial perspective, this theorem is much stronger than just saying that the character squares are hard to compute. The theorem rules out that there exists any positive combinatorial interpretation for the character squares, even if "positive combinatorial interpretation" is interpreted in the wide sense of $\#\mathsf{P}$. Large parts of Enumerative and Algebraic Combinatorics deal with finding explicit (positive) combinatorial interpretations of quantities, while impossibility results are extremely rare.

Note also how close the upper and lower bounds are. Recall that the character square is in $\mathsf{GapP} = \#\mathsf{P} - \#\mathsf{P}$, is always nonnegative, and yet is not in $\#\mathsf{P}$ by the theorem unless the polynomial hierarchy collapses. Our proof goes via showing that deciding the vanishing of $\chi^\lambda(\pi)$ is $\mathsf{C_=P}$-complete:

Theorem 2. The language $\{(\lambda, \pi) \mid \chi^\lambda(\pi) = 0\}$ is $\mathsf{C_=P}$-complete under many-one reductions.

Theorem 1 then follows from Theorem 2 using similar ideas as in [1]. The proof of Theorem 2 goes via an almost gap-parsimonious (up to a well-controlled constant factor) polynomial-time many-one reduction from GapCircuitSAT.

References

[1] C. Ikenmeyer, I. Pak, *What is in #P and what is not?*, arXiv:2204.13149, accepted for the IEEE Symposium on Foundations of Computer Science (FOCS) 2022.

[2] C. Ikenmeyer, I. Pak, G. Panova, *Positivity of the symmetric group characters is as hard as the polynomial time hierarchy*, arXiv:2207.05423

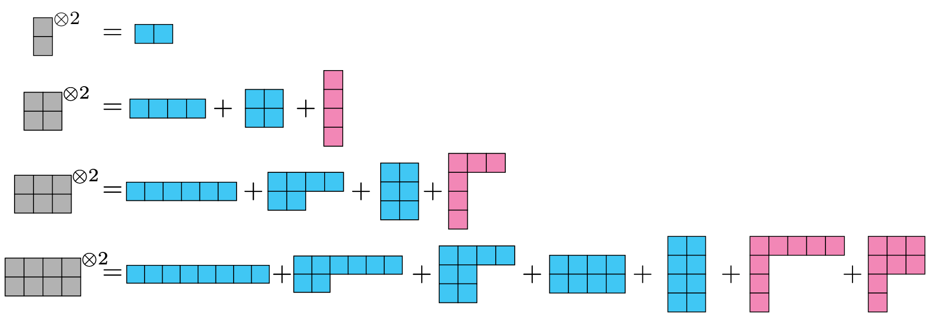
Tensor products, modular representations, and character vanishing

Chris Bowman

This talk was dedicated to the work of Christine Bessenrodt, a dear friend as well as an organiser of this workshop, who sadly passed away in January 2022. At our prequel Oberwolfach mini-workshop *"Kronecker, plethysm, and Sylow branching coefficients and their applications to complexity theory"*, Igor Pak had joked that Christine was the only person in the audience who could possibly have understood every talk. Christine's mathematics wove through almost all of the talks of the current workshop: the connections between tensor products, modular representations and character vanishing which she forged, as well as the shared stories of Christine's warmth and generosity and her willingness to freely share spectacular mathematical insights.

One of the recurring themes, of both my own talk and the wider conference, was the question *"what representation theoretic questions can be completely solved?"* and the related question *"what milestones should we set for our understanding?"*. Christine tackled many "impossible" problems in representation theory and algebraic combinatorics in this manner: carving out the boundaries of our understanding through her classifications of irreducibility, homogeneity, and multiplicity-free phenomena within these difficult-to-impossible combinatorial problems. In this abstract, we will focus our attention on Kronecker products and modular representation theory.

Multiplicity-free products. The first of these problems on which I was privileged to work with Christine was her multiplicity-free classification conjecture for products of $\mathbb{C}\mathfrak{S}_n$-characters, from 1999. Christine had the big ideas of the proof, as well as the roadmap for the case-by-case analysis, already concretely laid out before I entered the scene. As a young postdoctoral researcher at the time, I was incredibly grateful to Christine for inviting me onto the project and to help work through the incredibly intricate details of the proof. Before stating the theorem, here are some examples of multiplicity-free Kronecker products:



From which one can perhaps guess the following, rather charming, theorem:

Theorem 1 ([GWXZ, Ma])**.** *We have that*

$$\chi_{(k,k)}^{\otimes 2} = \sum_{\substack{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\vdash 2k \\ \lambda_i \in 2\mathbb{N}}} \chi_\lambda + \sum_{\substack{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\vdash 2k \\ \lambda_i \in 2\mathbb{N}+1}} \chi_\lambda.$$

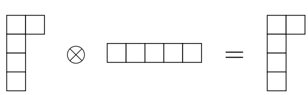
All the coefficients in the expansion of the above product are either 0 or 1 and so we refer to such a product as multiplicity-free. Multiplicity-free products can be classified and are detailed below. The important point is perhaps not exactly the list of pairs (λ, μ) which appear on the list (which we have blurred slightly for readability) but rather (i) that all of these products admit closed combinatorial descriptions (ii) classifying these products took 15 years and the use of the full spectrum of tools we have for studying Kronecker products. Thus the following theorem serves as a very concrete milestone of what can (and cannot?) be known about Kronecker products.

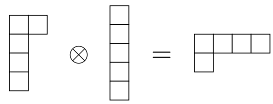
Theorem 2. *The product $\chi_\lambda \otimes \chi_\mu$ is multiplicity-free if and only if one of the following holds*

- $\lambda = (n-1, 1)$ *and μ has 2 removable nodes;*
- $\lambda = (k, k) = \mu$ *and some other pairs of 2-line partitions;*
- $\lambda = (k, k)$ *and μ is a hook partition;*
- $\lambda = (a^b)$ *and $\mu = (n-2, 2)$ or $(n-2, 1^2)$;*
- *some exceptional small rank cases for $n \leq 12$.*

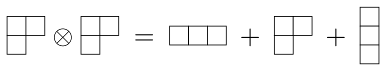
The proof of this theorem proceeds by complementary use of Dvir recursion and the semigroup property. The combinatorics of Dvir allows one to delicately peal-off small partitions from arbitrary points within Young diagrams, whereas the semigroup property allows one to add partitions together and hence propagate (generic) behaviour. The proof begins with the partitions "most likely to appear on the list" and then works its way towards partitions which are less and less likely to appear in a multiplicity-free product: hooks and 2-line partitions (for which the proof is essentially trivial), then rectangles, then fat hooks, and then arbitrary partitions. One has to prove the corresponding result for skew partitions in tandem, so that one has access to these skew shapes when "breaking partitions into smaller parts" using the Dvir combinatorics. This all then fits into an inductive proof on the rank. Christine had this entire proof structure in place and raring to go before I even entered the picture. However, it took a year for us to go through the careful case-by-case analysis... and though Christine was the one with the big picture viewpoint, she also loved getting her hands dirty with the details, the hard work and the grind. After knowing Christine for longer, I realised this was how she approached everything she did: working exceptionally hard and giving a great deal of herself into her work for the European Women in Mathematics, the DFG, running the department at Hannover, and her dedication to her students.

Homogeneous Products. Christine classified the homogeneous and irreducible products of $\mathbb{C}\mathfrak{S}_n$-characters in joint work with Sasha Kleshchev, another organiser of this conference. Coming up with examples of homogenous tensor products is not difficult, for example





and finding examples of inhomogeneous products is also not difficult. Indeed the smallest non-linear example is inhomogeneous



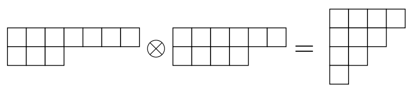
and, in fact, Sasha and Christine proved the following:

Theorem 3. *The only irreducible or homogeneous* $\mathbb{C}S_n$*-products are these 1-dimensional ones*

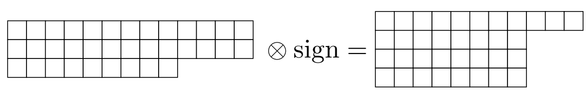
$$\chi_\lambda \otimes \chi_{(n)} = \chi_\lambda \qquad \chi_\lambda \otimes \chi_{(1^n)} = \chi_{\lambda^T}$$

where λ^T *denotes the transpose partition of* λ*.*

The corresponding classification for modular $\mathfrak{S}_n$-characters was conjectured by Kleshchev–Gow [GK]. An example of one of these moduar homogenous tensor products is as follows



for the prime $p = 2$. This is certainly a step-up in difficulty, as even tensoring with the sign representation becomes combinatorially significant in positive characteristic. Indeed this is the subject of Mullineux's famous conjecture, which was one of Christine's favourite combinatorial results. An example of such a modular product is as follows



for characteristic $p = 7$. Christine and Sasha made a great deal of headway on Kleshchev–Gow's conjecture in [BK], but the conjecture was only finally proven in its entirety by Lucia Morotti in [M]. It was Christine who directed Lucia toward this problem – which has since flourished into a hugely successful collaborative project, together with Sasha Kleshchev and Pham Tiep, with important applications to the Aschbacher–Scott programme.

Applications of modular representation theory to combinatorics. A recent example of Christine's enduring influence comes from a famous conjecture of Saxl, which asks if there exist tensor squares containing all $\mathbb{C}\mathfrak{S}_n$-characters. In more detail Saxl concretely conjectured that $\chi^{\otimes 2}_{(k,k-1,k-2,\dots,2,1)}$ contains every irreducible $\mathbb{C}\mathfrak{S}_n$-character with non-zero multiplicity.

At the AIM conference on Kronecker coefficients in 2015, Christine proposed attacking this conjecture using ideas from modular representation theory. In 2022, Harman–Ryba used this idea of Christine's together with a result of Serre in order to prove the following:

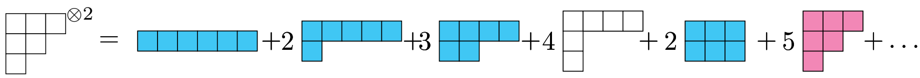
Theorem 4 ([HR]). *Let $n = \frac{1}{2}k(k+1)$. The tensor cube $\chi^{\otimes 3}_{(k,k-1,k-2,\dots,2,1)}$ contains every irreducible $\mathbb{C}\mathfrak{S}_n$-character with non-zero multiplicity.*

In light of a theorem of Burnside (which states that the mth tensor power of a G-faithful irreducible character contains every character for $m \gg 0$) we can view the above theorem as being an "almost" proof of Saxl's conjecture (which says $m = 2$, but the above is for $m = 3$).

One of the simplest examples of modular representation theory being used to understand the $m = 2$ case is the following theorem:

Theorem 5. *Every $\mathbb{C}S_n$-character, $\chi^{\mathbb{C}}_\lambda$, of odd degree appears in Saxl's tensor square $\chi^{\otimes 2}_{(k,k-1,k-2,\dots,2,1)}$.*

For example we have that



(along with their transposes occurring with the same multiplicities). All the blue characters can be detected with the above theorem. The pink character is "as odd as possible within its 2-block", in other words it is of height zero. This can be detected with the more general form of Theorem 5, for which we refer to [BBS, Theorem B].

Character vanishing. The talk ended by looking forward, with an unpublished conjecture of Christine's concerning character vanishing. In order to state this conjecture, we first define the character vanishing sets to be the $V^G_\lambda = \{g \in G \mid \chi_\lambda(g) = 0\}$ associated to χ_λ any irreducible G-character. We first recall a theorem of Belonogov, which asks (and answers!) the following question: *"is a character uniquely determined by its zero values?"* (modulo silliness: i.e. tensoring with a linear character).

Theorem 6 ([B]). *The vanishing sets V_λ of simple $\mathbb{C}S_n$-characters determine χ_λ up to linear twist. That is, $V^{S_n}_\lambda = V^{S_n}_\mu$ implies $\lambda = \mu$ or $\lambda = \mu^T$.*

For alternating groups, things get trickier. Christine and I tried to prove the following conjecture of hers (with little success!) which she formulated during one of my visits to Hannover. For characters of alternating groups, there is a fundamental splitting (upon restriction from the symmetric group) into symmetric and non-symmetric partitions. Here a symmetric partition is λ fixed by taking transposes, that is $\lambda = \lambda^T$. Given λ a symmetric partition, we have a decomposition $\chi_\lambda{\downarrow_{A_n}} = \chi^+_\lambda + \chi^-_\lambda$ where χ^+_λ and χ^-_λ are irreducible and have the same vanishing sets as $\chi_\lambda{\downarrow}$. However, we can throw such pairs into our loosely defined notion of "silliness" (noting for good measure that A_n has no non-trivial linear characters) and repose Belonogov's question in the case of alternating groups as follows:

Conjecture 7 (Bessenrodt)**.** *The pairs of χ_λ, χ_μ such that $V_\lambda^{A_n} = V_\mu^{A_n}$ and $\lambda \neq \mu$ are as follows*

- *Pairs λ, μ such that $3-\mathrm{core}(\lambda) = 3-\mathrm{core}(\mu)$ is symmetric and $3-\mathrm{quo}(\lambda) = ((1), \varnothing, \varnothing)$ and $3-\mathrm{quo}(\mu) = (\varnothing, (1), \varnothing)$.*
- *Pairs λ, μ such that $3-\mathrm{core}(\lambda) = 3-\mathrm{core}(\mu)$ is symmetric and $3-\mathrm{quo}(\lambda) = ((1), (1), \varnothing)$ and $3-\mathrm{quo}(\mu) = (\varnothing, (2), \varnothing)$.*
- *Some small rank exceptions for $n \leq 23$.*

References

[B] V. A. Belonogov, On irreducible characters of the groups Sn and An. Sibirsk. Mat. Zh. 45 (2004), no. 5, 977–994;

[BBS] C. Bessenrodt, C.Bowman, L. Sutton, Kronecker positivity and 2-modular representation theory, Trans. Amer. Math. Soc. Ser. B 8 (2021), 1024–1055.

[BK] C. Bessenrodt, A. Kleshchev, On tensor products of modular representations of symmetric groups. Bull. London Math. Soc. 32 (2000), no. 3, 292–296.

[GWXZ] A. Garsia, N. Wallach, G. Xin, M. Zabrocki, Kronecker coefficients via symmetric functions and constant term identities, Internat. J. Algebra Comput. 22 (2012), no. 3, 1250022, 44 pp. 05E05 (20C30)

[GK] R. Gow, A. Kleshchev, Connections between the representations of the symmetric group and the symplectic group in characteristic 2. J. Algebra 221 (1999), no. 1, 60–89.

[HR] N. Harman, C. Ryba, A Tensor-Cube Version of the Saxl Conjecture, arXiv:2206.13769

[Ma] L. Manivel, A note on certain Kronecker coefficients., Proc. Amer. Math. Soc. 138 (2010), no. 1, 1–7.

[M] L. Morotti, Irreducible tensor products for symmetric groups in characteristic 2, Proc. Lond. Math. Soc. (3) 116 (2018), no. 6, 1553–1598.

Vanishing results for character tables

Alexander R. Miller

Interest in zeros of characters goes back to the beginning of character theory with Burnside's result that each nonlinear irreducible character χ of a finite group G has zeros. But how many zeros? In particular, we ask the following [3].

Question 1. What is the chance that a character value $\chi(g)$ equals 0?

The two most natural ways of choosing a character value $\chi(g)$ are as follows.

(1) Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$ uniformly at random, and then evaluate $\chi(g)$. The chance that $\chi(g)$ equals zero will be denoted by $\mathrm{Prob}(\chi(g) = 0)$, so

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|}.$$

(2) Choose $\chi \in \mathrm{Irr}(G)$ and a class $K = g^G \in \mathrm{Cl}(G)$ uniformly at random, and then evaluate $\chi(K) := \chi(g)$. In other words, choose an entry $\chi(K)$ uniformly at random from the character table of G. The chance that $\chi(K)$ equals zero will be denoted by $\mathrm{Prob}(\chi(K) = 0)$, so

$$\mathrm{Prob}(\chi(K) = 0) = \frac{|\{(\chi, K) \in \mathrm{Irr}(G) \times \mathrm{Cl}(G) : \chi(K) = 0\}|}{|\mathrm{Irr}(G) \times \mathrm{Cl}(G)|}.$$

Example 2. If $G = S_4$, then

$$\mathrm{Prob}(\chi(g) = 0) = \frac{28}{120} \approx 0.194 \quad \text{and} \quad \mathrm{Prob}(\chi(K) = 0) = \frac{4}{25} = 0.16.$$

§1. It turns out that many characters have many zeros. The first result in this direction is for symmetric groups [3, Theorem 1].

Theorem 3. *If $\chi \in \mathrm{Irr}(S_n)$ and $g \in S_n$ are chosen uniformly at random, then $\chi(g) = 0$ with probability $\to 1$ as $n \to \infty$.*

One of the two proofs given in [3] proceeds by showing that a vanishingly small proportion of classes covers almost all of S_n. The key ingredient here is the following inequality [3, Proposition 3].

Lemma 4. *For any finite group G and any collection $\mathcal{K} \subseteq \mathrm{Cl}(G)$,*

$$\mathrm{Prob}(\chi(g) = 0) \geq \frac{|\{g \in G : g^G \in \mathcal{K}\}|}{|G|} - \frac{|\mathcal{K}|}{|\mathrm{Cl}(G)|}.$$

Different groups require different tools. The following bound in terms of character degrees and class sizes was established in joint work of P. X. Gallagher, M. J. Larsen, and the speaker [1].

Lemma 5. *For each finite group G and each $\epsilon > 0$,*

$$\mathrm{Prob}(\chi(g) \neq 0) \leq \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \gcd(\chi(1), |g^G|) \geq \epsilon\chi(1)\}|}{|\mathrm{Irr}(G) \times G|} + \epsilon^2.$$

Lemma 5 was used in [1] to establish the $\mathrm{GL}(n, q)$ analogue of Theorem 3.

Theorem 6. *For $G = \mathrm{GL}(n, q)$, the proportion $P_{n,q}$ of pairs $(\chi, g) \in \mathrm{Irr}(G) \times G$ with $\chi(g) = 0$ satisfies*

$$\inf_q P_{n,q} \to 1 \text{ as } n \to \infty.$$

So for any sequence of prime powers $q_1, q_2, \ldots$, we have $P_{n,q_n} \to 1$ as $n \to \infty$.

We also have the following result about the sparsity of the character tables of finite simple groups of Lie type due to M. Larsen and the speaker [2, Theorem 1.1].

Theorem 7. *If G_n is any sequence of finite simple groups of Lie type with rank tending to ∞, then almost every entry in the character table of G_n is zero as $n \to \infty$. In other words, the fraction of the character table of G_n that is covered by zeros tends to 1 as $n \to \infty$.*

For S_n, however, we do not know the limiting behavior of $\mathrm{Prob}(\chi(K) = 0)$, which we shall denote by $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$ with the understanding that λ and μ are chosen uniformly at random from the partitions of n and $\chi_\lambda(\mu)$ is shorthand for the value of χ_λ at any permutation of cycle type μ. See [3] and [4, Table 3]. The following question [3, Question 2] is wide open.

Question 8. What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

§2. Instead of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$, what happens if we work mod 2? One might guess that $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \mod 2) \to 1/2$ as $n \to \infty$. And for $n = 4, 5, \ldots, 10$, the proportions are approximately 0.24, 0.33, 0.36, 0.40, 0.55, 0.56, 0.55. But the values keep growing. At $n = 76$, for example, roughly 87% of the more than 86 trillion entries in the character table of S_n are even. See [4] for more data. The speaker also carried out computations for other primes and prime powers and conjectured that, for any positive integer m, $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \mod m) \to 1$. The case where m is prime was recently confirmed by Peluse and Soundararajan.

§3. Let $P(G)$ denote a fixed choice of either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$. So far, we have discussed starting with a sequence of groups $G_1, G_2, \ldots$ and then studying the corresponding fractions $P(G_1), P(G_2), \ldots \in [0, 1]$.

3.1. $P(G)$ can also be studied as a random variable itself, with G chosen from some distribution. For example, we can choose a random Young subgroup S_λ of S_n. The natural question is then: What is the expected value of $P(S_\lambda)$ when λ is chosen uniformly at random from the partitions of n? The answer is the following [5, Theorem 2].

Theorem 9. *The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.*

3.2. We can also ask what the sequences $P(G_1), P(G_2), \ldots$ can possibly look like. The following answer is Theorem 1 in [5].

Theorem 10. *If $a_1, a_2, \ldots \in [0, 1]$ and $\epsilon_1, \epsilon_2, \ldots \in (0, \infty)$, then for each prime p there exists an ascending chain of p-groups $G_1 < G_2 < \ldots$ such that, for each i,*

$$|P(G_i) - a_i| < \epsilon_i.$$

In particular, the set $\{P(G) : |G| < \infty\}$ is dense in $[0, 1]$.

§4. There are also the classical results of J. G. Thompson and P. X. Gallagher involving both zeros and roots of unity. Thompson proved that if $\chi \in \mathrm{Irr}(G)$, then $\chi(g)$ is either zero or a root of unity for more than a third of the elements $g \in G$. In terms of the function

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{|\{g \in G : \chi(g) \text{ is zero or a root of unity}\}|}{|G|},$$

Thompson's result says

$$\theta(G) > 1/3.$$

Gallagher proved similarly that if K is a larger than average conjugacy class of a finite group G, then $\chi(K)$ is either zero or a root of unity for more than a third of the characters $\chi \in \mathrm{Irr}(G)$. In terms of the function

$$\theta'(G) = \min_{K} \frac{|\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is zero or a root of unity}\}|}{|\mathrm{Irr}(G)|},$$

with the minimum taken over all larger than average conjugacy classes K of G, Gallagher's result says

$$\theta'(G) > 1/3.$$

We ask in [6] if the Thompson and Gallagher lower bounds are the best possible. More specifically, we ask the following [6, Questions 1 and 2].

Question 11. What are the greatest lower bounds $\inf_G \theta(G)$ and $\inf_G \theta'(G)$?

The main conjecture of [6] is the following answer.

Conjecture 12. $\inf \theta(G) = 1/2$ and $\inf \theta'(G) = 1/2$.

The greatest lower bounds can not be greater than 1/2 [6, Eqs. (20) and (21)].

Proposition 13. *For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n) \to \frac{1}{2}^+$ and $\theta'(G_n) \to \frac{1}{2}^+$.*

So Conjecture 12 is equivalent to the following [6, Conjecture 1].

Conjecture 14. $\theta(G) \geq 1/2$ and $\theta'(G) \geq 1/2$ for every finite group G.

As evidence for the conjecture, we have the following [6, Cor. 3 and Thm. 10].

Theorem 15. *Conjecture 14 holds for the following groups.*

- *All finite groups of order $< 2^9$.*
- *All simple groups of order $\leq 10^9$.*
- *All sporadic groups.*
- *A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$.*
- *All finite nilpotent groups.*

In fact for finite nilpotent groups we have the following much stronger results [6, Theorems 1 and 2].

Theorem 16. *Each nonlinear irreducible character of a finite nilpotent group is zero on more than half of the group elements.*

Theorem 17. *More than half of the nonlinear irreducible characters of a finite nilpotent group are zero on any given larger than average class.*

The main new ingredient for these vanishing results is [6, Theorem 8], which is a very strong improvement of a classical result of Siegel [7] about totally positive algebraic integers, but in the context of the totally positive integers $|\chi(g)|^2$ coming from finite nilpotent groups.

References

[1] P. X. Gallagher, M. J. Larsen, and A. R. Miller, *Many zeros of many characters of $GL(n, q)$*, Int. Math. Res. Not. IMRN **2022** (2022), 4376–4386.
[2] M. J. Larsen and A. R. Miller, *The sparsity of character tables of high rank groups of Lie type*, Represent. Theory **25** (2021), 173–192.
[3] A. R. Miller, *The probability that a character value is zero for the symmetric group*, Math. Z. **277** (2014), 1011–1015.
[4] A. R. Miller, *On parity and characters of symmetric groups*, J. Combin. Theory Ser. A **162** (2019), 231–240.
[5] A. R. Miller, *Dense proportions of zeros in character values*, C. R. Math. Acad. Sci. Paris **357** (2019), 771–772.
[6] A. R. Miller, *Zeros and roots of unity in character tables*, Enseign. Math., to appear.
[7] C. L. Siegel, *The trace of totally positive and real algebraic integers*, Ann. Math. **46** (1945), 302–312.

The anti-spherical Hecke categories for Hermitian symmetric pairs

Maud De Visscher
(joint work with C. Bowman, A. Hazi and E. Norton)

Kazhdan–Lusztig polynomials are remarkable polynomials associated to pairs of elements in a Coxeter group W. They describe the base change between the standard and Kazhdan–Lusztig bases for the corresponding Hecke algebra. They were discovered by Kazhdan and Lusztig [6] in 1979 and have found applications throughout representation theory and geometry. In 1987, Deodhar introduced in [3] parabolic Kazhdan–Lusztig polynomials associated to a Coxeter group W and a standard parabolic subgroup P. These describe the base change between the standard and Kazhdan–Lusztig bases for the anti-spherical module for the Hecke algebra. (We recover the original definition of Kazhdan and Lusztig by taking the trivial parabolic subgroup). (Anti-spherical) Hecke categories first rose to mathematical celebrity as the centrepiece of the proof of the (parabolic) Kazhdan–Lusztig positivity conjecture. The Hecke category categorifies the Hecke algebra and the anti-spherical Hecke category categorifies the anti-spherical module. More precisely, it was shown in [5] and [7] that the (parabolic) Kazhdan–Lusztig polynomials are precisely the graded decomposition numbers for the (anti-spherical) Hecke categories over fields of characteristic zero, hence proving positivity of their coefficients.

The (anti-spherical) Hecke categories can be defined over any field. Their graded decomposition numbers over fields of positive characteristic p, the so-called (parabolic) p-Kazhdan–Lusztig polynomials, have been shown to have deep connections with the modular representation theory of reductive groups and symmetric groups (see [8] and [1]). These polynomials are notoriously difficult to compute. Unlike in the case of the ordinary (parabolic) Kazhdan–Lusztig polynomials, there is not even a recursive algorithm to compute them in general.

Over fields of characteristic zero, the families of Kazhdan–Lusztig polynomials which are best understood combinatorially are those for Hermitian symmetric pairs (W, P) (see for example [4] and references therein). So it seems to be the natural starting point to study their generalisation to positive characteristic. Our first main result is the following.

Theorem A. *Let k be a field of characteristic $p \geq 0$ and let (W, P) be a Hermitian symmetric pair. The anti-spherical Hecke category $\mathcal{H}_{(W,P)}$ over k is standard Koszul and the associated p-Kazhdan–Lusztig polynomials are independent of p.*

This provides the first family of p-Kazhdan–Lusztig polynomials to be calculated since Williamson's famous torsion explosion examples to the Lusztig's conjecture. Our proof of Koszulity explicitly constructs linear projective resolutions of standard modules by induction on the rank of the Coxeter system. This is possible from our construction of singular Hecke categories for simply laced Hermitian symmetric pairs - a very difficult problem in general.

Theorem B. *Let $\tau \in S_W$ be a simple reflection for a simply laced Hermitian symmetric pair (W, P). We explicitly construct the singular Hecke category $\mathcal{H}^{\tau}_{(W,P)}$ as a subcategory of $\mathcal{H}_{(W,P)}$. Furthermore, we prove that $\mathcal{H}^{\tau}_{(W,P)}$ is equivalent to the Hecke category of a simply laced Hermitian symmetric pair of strictly smaller rank.*

References

[1] C. Bowman, A. Cox and A. Hazi, *Path isomorphisms between quiver Hecke and diagrammatic Bott-Samelson endomorphism algebras*, arXiv:2005.02825

[2] C. Bowman, M. De Visscher, A. Hazi and E. Norton, *The anti-spherical Hecke category for Hermitian symmetric pairs*, arXiv:2208.02584

[3] V.V. Deodhar, *On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan–Lusztig polynomials*, J. Algebra, 111(2):483-506 (1987).

[4] F. Brenti, *Parabolic Kazhdan–Lusztig polynomials for Hermitian symmetric pairs*, Trans. Amer. Math. Soc. 361 no.4, 1703-1729 (2009).

[5] B. Elias and G. Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) 180(3): 1089-1136 (2014).

[6] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. 53(2):165-184 (1979).

[7] N. Libedinsky and G. Williamson, *The anti-spherical category*, Adv. in Math. 405 (2022).

[8] S. Riche and G. Williamson, *Tilting modules and the p-canonical basis*, Astérisque 397 (2018).

Generalised cores and atomic length

Thomas Gerber

(joint work with Nathan Chapelier and Emily Norton)

The Nakayama conjecture, proved in 1947 by Brauer and Robinson, states that the p-blocks of the symmetric group S_n are determined by partitions of n with the same p-core. In particular, the defect zero p-blocks of S_n, consisting of exactly one irreducible representation, are given by the p-core partitions. This has led to various generalisations of the notion of core, controlling block theory for other structures such as

- finite groups of classical Lie type [4],
- (cyclotomic) Hecke algebras [2], [3] [7].

The study of blocks for a given finite group has led to the following classification result: Every finite simple group has a defect zero p-block if $p \geq 5$. Moreover, for $p = 2, 3$, we know exactly which finite simple groups have a defect zero block. This task has been completed in 1996 by Granville and Ono [6] by studying the case of symmetric and alternating groups. More precisely, they proved the following result.

Theorem 1. Let $n \in \mathbb{Z}_{\geq 0}$. For all $e \geq 4$, there exists an e-core of size n.

In other terms, the size function defined from the set of e-cores to $\mathbb{Z}_{\geq 0}$ is surjective if $e \geq 4$. For $e \in \{2, 3\}$, on the contrary, it is easy to show that infinitely many values in $\mathbb{Z}_{\geq 0}$ are not attained by the size function.

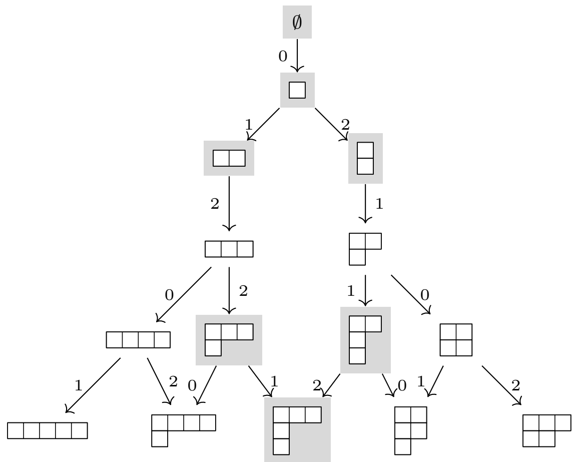


FIGURE 1. The beginning of the $\widehat{\mathfrak{sl}_3}$-crystal $B(\Lambda_0)$ and the 3-cores.

It is natural to ask a similar question for the different generalisations of cores. Here, we focus on the case of generalised cores arising from block theory for cyclotomic Hecke algebras (also known as Ariki-Koike algebras). These are constructed as follows. Let λ a dominant weight for the root system of type $A_{e-1}^{(1)}$, and let $B(\lambda)$ be the crystal graph of the highest weight $\widehat{\mathfrak{sl}_e}$-module with highest weight λ. There are several explicit realisations of this crystal via multipartitions/abaci combinatorics, see [5]. The generalised cores are then given by the orbit of the highest weight vertex of $B(\lambda)$ under the action of the Weyl group (given by "reversing the i-strings", see [1]). In the particular case $\lambda = \Lambda_0$, one takes the Weyl group orbit of the empty partition, which yields exactly the e-core partitions, as shown in Figure 1 (example for $e = 3$ where the e-cores are highlighted).

This construction of the e-cores suggests to introduce a statistic directly on (general) Weyl groups. Let W be the Weyl group of a symmetrisable Kac-Moody algebra and ρ (respectively $\rho^\vee$) denote the half-sum of the positive roots (respectively positive coroots).

Definition. Let $w \in W$ and λ be a dominant weight. The *atomic length* of w is the nonnegative integer $L_\lambda(w) = \langle \lambda - w(\lambda), \rho^\vee \rangle$.

In other words, if b_λ denotes the highest weight vertex in the crystal $B(\lambda)$, $L_\lambda(w)$ is the depth of $w(b_\lambda)$ in $B(\lambda)$. In particular, if W is of type $A_{e-1}^{(1)}$ and $\lambda = \Lambda_0$, $L_{\Lambda_0}(w)$ is the size of the partition $w(\emptyset)$. The terminology *atomic length* is justified by the following remark.

Remark. For $w \in W$, let $N(w)$ be the inversion set of w, that is, the set of all positive roots α such that $w^{-1}(\alpha)$ is a negative root. It is well-known that $|N(w)| = \ell(w)$, the length of w. Now, we have $\rho - w(\rho) = \sum_{\alpha \in N(w)} \alpha$. Thus, we have the alternative formula $L_\rho(w) = \sum_{\alpha \in N(w)} \mathrm{ht}(\alpha)$, where $\mathrm{ht}(\alpha)$ is the height of α, that is, the number of simple roots needed to decompose α.

In the spirit of Granville and Ono's result, we want to understand the image of L_λ. In joint work with N. Chapelier, we consider the case of finite Weyl groups. For $\lambda = \rho$, it is easy to see that the element with the largest atomic length is w_0, the longest element of W. Using a detailed study of the inversion sets and an inductive argument, we are able to prove the following finite analogue of Theorem 1.

Theorem 2. Let W be finite, and assume the rank of root system at least 3. Then $L_\rho : W \longrightarrow \{0, \ldots, L_\rho(w_0)\}$ is surjective.

In particular, in type A_{e-1}, we recover a result of [8]. Moreover, in general, L_λ is surjective whenever λ gives rise to a minuscule representation. The question of characterising the dominant weights λ for which L_λ is surjective (in rank not too small) remains open.

In another direction, we can consider affine type $A^{(1)}_{e-1}$. For $\lambda = \Lambda_0$ and $e \geq 4$, we know that $L_\lambda : W \longrightarrow \mathbb{Z}_{\geq 0}$ is surjective by Theorem 1. In joint work with E. Norton, we consider more general dominant weights. Write $\lambda = \sum_{i=0}^{e-1} m_i \Lambda_i$ with $m_i \in \mathbb{Z}_{\geq 0}$ and where Λ_i denote the fundamental weights. Instead of asking for complete surjectivity, we want to understand when

$$\mathbb{Z}_{\geq 0} \setminus L_\lambda(W) \text{ is finite.} \tag{1}$$

It is not hard to see that it is necessary that $\gcd(m_i) = 1$ for (1) to hold. Indeed, if $\gcd(m_i) = d > 1$, then $L_\lambda(W) \subseteq d\mathbb{Z}_{\geq 0}$. Finding a sufficient condition for (1) seems substantially more complicated. In the degenerate case $e = \infty$, however, we can show that the condition $\gcd(m_i) = 1$ is in fact often sufficient.

References

[1] D. Bump, A. Schilling, *Crystal Bases: Representations And Combinatorics*, World Scientific, 2017.

[2] M. Fayers, *Core blocks of Ariki-Koike algebras. II: The weight of a core block*, unpublished. Available at `http://www.maths.qmul.ac.uk/~mf/index1.html`.

[3] M. Fayers, *Simultaneous core multipartitions*, European J. Combin. **76** (2019), 138–158.

[4] P. Fong, B. Srinivasan, *The blocks of finite classical groups*, J. reine angew. Math. **396** (1989), 122–191.

[5] Meinolf Geck and Nicolas Jacon, *Representations of Hecke Algebras at Roots of Unity*, Springer, 2011.

[6] A. Granville, K. Ono, *Defect zero p-blocks for finite simple groups*, Trans. Amer. Math. Soc. **348** (1996), 331–347.

[7] N. Jacon, C. Lecouvey, *Cores of Ariki-Koike algebras*, Documenta Math. **103** (2021), 122–191.

[8] J. Sack, H. Úlfarsson, *Refined inversion statistics on permutations*, Electronic J. Comb. **19** (1) (2011).

Perverse sheaves on affine flag varieties and coherent sheaves on the dual Steinberg variety

Simon Riche
(joint work with R. Bezrukavnikov, L. Rider)

Geometric realizations of the extended affine Weyl group. Let G be a semisimple algebraic group of adjoint type over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let also $W = N_G(T)/T$ be the Weyl group of (G, T). The associated (extended) affine Weyl group is the semidirect product

$$W_{\mathrm{ext}} := W \ltimes X_*(T),$$

where $X_*(T)$ is the lattice of cocharacter of T.

This group arises in geometry in two very different contexts. First, consider the loop group LG associated with G, the arc group L^+G, and the Iwahori subgroup $I \subset L^+G$ associated with B. The affine flag variety Fl is the ind-scheme representing the (fppf) quotient LG/I. If $\ell \neq p$ is a prime number, and if $\Bbbk$ is an algebraic closure either of $\mathbb{Q}_\ell$ or of $\mathbb{F}_\ell$, then one can consider the I-equivariant derived category

$$D^{\mathrm{b}}_I(\mathsf{Fl}, \Bbbk)$$

of étale $\Bbbk$-sheaves on Fl. This category admits a monoidal product $\star$ (given by convolution), and it is a standard fact (dating back essentially to work of Iwahori–Matsumoto) that its Grothendieck ring $[D^{\mathrm{b}}_I(\mathsf{Fl}, \Bbbk)]$ identifies with the group algebra $\mathbb{Z}[W_{\mathrm{ext}}]$. This result is closely related with the decomposition

$$\mathsf{Fl} = \bigsqcup_{w \in W_{\mathrm{ext}}} \mathsf{Fl}_w \tag{1}$$

of Fl into I-orbits, which are naturally parametrized by W_{ext}.

On the other hand, let $G^\vee$ be the Langlands dual group over $\Bbbk$, let $B^\vee \subset G^\vee$ be the Borel subgroup corresponding to B and $T^\vee \subset B^\vee$ be the maximal torus corresponding to T. (In particular, $X_*(T)$ identifies with the lattice $X^*(T^\vee)$ of characters of $T^\vee$, and W with the Weyl group of $(G^\vee, T^\vee)$.) The associated Springer resolution is the cotangent bundle

$$\widetilde{\mathcal{N}} := T^*(G^\vee/B^\vee).$$

There exists a canonical morphism $\widetilde{\mathcal{N}} \to \mathrm{Lie}(G^\vee)^*$, and the Steinberg variety is the fiber product

$$\mathsf{St} := \widetilde{\mathcal{N}} \times_{\mathrm{Lie}(G^\vee)^*} \widetilde{\mathcal{N}}.$$

Kazhdan–Lusztig and Ginzburg have studied the equivariant K-theory of this variety; it results in particular from their work that the algebra $\mathsf{K}^{G^\vee}(\mathsf{St})$ (for the natural convolution product) also identifies with $\mathbb{Z}[W_{\mathrm{ext}}]$.

From this coincidence one can expend interesting relations between (appropriate variants of) the triangulated categories

$$D^{\mathrm{b}}_I(\mathsf{Fl}, \Bbbk) \quad \text{and} \quad D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\mathsf{St}).$$

Such relations have been established by Bezrukavnikov [1] in case $\Bbbk$ is an algebraic closure of $\mathbb{Q}_\ell$. It is now a project with him (and partly with Rider) to extend this construction to the case $\Bbbk$ is an algebraic closure of $\mathbb{F}_\ell$.

Some motivation for such a construction comes from expected applications in the Geometric Langlands Program and to representation theory of reductive algebraic groups.

The equivariant regular quotient. A first step in this direction has been established in [3]. Namely, consider the the category P_I of I-equivariant perverse sheaves on Fl. From (1) we obtain that the isomorphism classes of simple objects in P_I are in a canonical bijection with W_{ext}, and we define the "equivariant regular quotient" P_I^0 as the Serre quotient of the category P_I by the Serre subcategory generated by simple objects whose label w satisfies $\dim(\mathsf{Fl}_w) > 0$. The convolution product $\star$ on the triangulated category $D_I^{\mathrm{b}}(\mathsf{Fl}, \Bbbk)$ induces in a natural way an exact monoidal product $\star^0$ on P_I, and the main result of [3] is a description of the monoidal category $(\mathsf{P}_I^0, \star^0)$. More specifically, we construct an equivalence of monoidal categories

$$(\mathsf{P}_I^0, \star^0) \cong (\mathsf{Rep}(Z_{G^\vee}(u)), \otimes) \tag{2}$$

where $u \in G^\vee$ is a regular unipotent element. Here the appearance of the dual group in the geometry of Fl relies on the geometric Satake equivalence [5], via Gaitsgory's construction of the "central functor," see [4]. (This statement requires some technical assumptions on ℓ, which we do not explain here.)

The monodromic regular quotient. In [2] we "deform' this picture as follows. On the constructible side we consider the pro-unipotent radical I_u of I, and the ind-scheme $\widetilde{\mathsf{Fl}}$ representing the quotient LG/I_u. We can then consider the triangulated subcategory D_{I_u} of the category $D_{I_u}^{\mathrm{b}}(\widetilde{\mathsf{Fl}}, \Bbbk)$ generated by objects obtained by pullback from objects in $D_I^{\mathrm{b}}(\mathsf{Fl}, \Bbbk)$, and the heart P_{I_u} of the natural perverse t-structure on this category. As for P_I, the simple objects in P_{I_u} are naturally labeled by W_{ext}, and we can define the category $\mathsf{P}_{I_u}^0$ as the Serre quotient of P_{I_u} by the Serre subcategory generated by simple objects labeled by elements w such that $\dim(\mathsf{Fl}_w) > 0$. Once again, convolution produces a (right exact) monoidal product $\star^0$ on this category.

On the geometric side, consider a Steinberg section $\Sigma \subset G^\vee$ of the adjoint quotient, which identifies canonically with the adjoint quotient $T^\vee/W$. Let $\mathbf{J}_\Sigma$ be the restriction of the universal centralizer to Σ (a smooth affine group scheme), and set

$$\mathbf{I} := \mathbf{J}_\Sigma \times_{T^\vee/W} (T^\vee \times_{T^\vee/W} T^\vee).$$

The category $\mathsf{Rep}(\mathbf{I})$ of representations of $\mathbf{I}$ on coherent sheaves over $T^\vee \times_{T^\vee/W} T^\vee$ admits a natural convolution product $*$.

With this notation, the main result of [2] is an equivalence of monoidal categories

$$(\mathsf{P}_{I_u}^0, \star^0) \cong (\mathsf{Rep}(\mathbf{I}), *),$$

which is compatible with (2) in a natural way. (As above, this statement requires some technical assumptions on ℓ.)

The main interest of this version (compared to (2)) is that one can derive from it a "Soergel type" description of the category of tilting objects in (a pro-completion of) D_{I_u}. This implies in particular that stalks and costalks of tilting I_u-equivariant perverse sheaves on Fl are computed by ℓ-Kazhdan–Lusztig polynomials.

In future work we will show that the category of "Soergel bimodules" that we encounter here also describes an appropriate category of coherent sheaves on the Steinberg variety of $G^\vee$, which will complete our project.

References

[1] R. Bezrukavnikov, *On two geometric realizations of an affine Hecke algebra*, Publ. Math. Inst. Hautes Études Sci. **123** (2016), 1–67.

[2] R. Bezrukavnikov, S. Riche, *Modular affine Hecke category and regular centralizer*, preprint arXiv:2206.03738.

[3] R. Bezrukavnikov, S. Riche, and L. Rider, *Modular affine Hecke category and regular unipotent centralizer, I*, preprint arXiv:2005.05583.

[4] D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001), no. 2, 253–280.

[5] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.

Cyclotomic KLR algebras of affine types A and C

Andrew Mathas

(joint work with Anton Evseev)

In 1901 Young [9] gave an explicit construction of the ordinary irreducible representations of the symmetric groups. In doing this, he introduced content functions for partitions, which are now a key statistic in the semisimple representation theory of the symmetric groups.

My talk described joint work with Anton Evseev [2] that generalises of Young's ideas to the cyclotomic KLR algebras of affine types A and C. This is quite surprising because Young's seminormal forms are creatures from the semisimple world whereas the cyclotomic KLR algebras are rarely semisimple.

To construct seminormal forms for the cyclotomic KLR algebras it is necessary to first the algebras so that they become semisimple. The key point in constructing these deformations is that the "non-trivial" relations in the KLR algebras are given by Rouquier's Q-polynomials, which typically take the form:

$$Q_{i,j}(u,v) = \begin{cases} u-v & \text{if } i \longrightarrow j \\ (u-v)(v-u) & \text{if } i \leftrightarrows j \\ u-v^2 & \text{if } i \Longrightarrow j \end{cases}$$

where u and v are indeterminates, and i and j are vertices of the corresponding quiver. By deforming the Q-polynomials we deform the KLR algebras. Graded semisimple deformations of the KLR algebras can be defined using Q-polynomials of the following form:

$$Q^x_{i,j}(u,v) = \begin{cases} u+x-v & \text{if } i \longrightarrow j \\ (u+x-v)(v-u-x) & \text{if } i \leftrightarrows j \\ u-(v-x)^2 & \text{if } i \Longrightarrow j \end{cases}$$

where x is an indeterminate of degree 1. We actually allow much more general deformations, but this example gives the main idea. In the specialisation $x = 0$ we recover the "standard" Q–polynomials and, correspondingly, the deformed algebras specialise to the standard KLR algebras. The deformed cyclotomic KLR algebras are $\mathbb{Z}$-graded algebras and we think of them as being defined over graded rings, such as $\mathbb{Z}[x]$.

In affine types A and C the deformed cyclotomic KLR algebras are split semisimple algebras over *graded fields* like $\mathbb{Q}[x, x^{-1}]$, whereas these algebras are not semisimple over $\mathbb{Z}[x]$. (A graded field is a graded ring in which all nonzero homogeneous elements are invertible.) To construct the seminormal representations of the deformed algebras over a graded field we introduce (graded) *content systems* that consist of a pair of functions (c, r), which mirror the *content* and *residue* functions from the classical representation theory of the symmetric work — except that in our setting the content systems are recursively built from the (deformed) Q-polynomials. See [2, Definition 3A.1] for the precise definition.

With the content systems in hand, we obtain the following:

- we give an explicit description of the semisimple representations of the cyclotomic KLR algebras of affine types A and C. These representations take almost the same form as those originally constructed, except that are graded and give representations of the KLR algebras. In particular, it is every to do explicit calculations with these algebras and, in this way, we recover almost all of the classical semisimple representation theory of the symmetric groups for the KLR algebras of affine types A and C.
- The construction of two "dual" cellular bases for the cyclotomic KLR algebras of affine types A and C. As a corollary, this proves that these algebras are graded cellular algebras. In type A this recovers the main result of [3] and in type C this is new. The interplay between these two cellular bases, and the corresponding modules, which mirrors tensoring with the sign representation for the symmetric group, plays an role throughout the paper.
- We give explicit graded Specht filtrations for the modules obtained by inducing and restricting the graded Specht modules of these algebras. In turn, these results translate into giving explicit categorification results for the integrable highest weight modules of the corresponding Kac-Moody algebras
- Using the categorification theorem, and the representation theory of the KLR algebras, we can directly connect the representation theory of the cyclotomic KLR algebras with canonical bases. This shows that the simple

modules constructed from the cellular bases are non-trivial if and only if they are indexed by vertices in the corresponding crystal graphs. We also obtain a description of the MUllineux map in this setting.
- Finally, analogues of Kleshchev's modular branching rules [4] are obtained.

In type A all of these results were known previously, except that we prove stronger versions of these results for the deformed cyclotomic KLR algebras. In type C, these results are completely new. The arguments that we give rely our deformation. In particular, the proofs are all new and they are often more efficient than the existing arguments in the literature for type A.

Weighted KLRW algebras. In related work [5, 6], Daniel Tubbenhauer and I have used Webster's weighted KLRW algebras [7, 8] to construct cellular bases for cyclotomic KLR algebras of affine types A, B, C and D, generalising earlier work of Bowman's [1]. The weighted KLR algebras are "enhanced" versions of the KLR algebras that are described diagrammatically using braid-like diagrams, which have "red", "solid" and "ghost" strings. The key advantage of these algebras is that the analogue of the cyclotomic ideals are very easy to describe because they are spanned bu the "unsteady" diagrams. This makes it possible to construct cellular bases of these algebras by "pulling strings as far possible" to the right. This framework is proving to be extremely powerful and we expect that we soon generalise this approach to prove cellularity results for all (cyclotomic) KLR algebras of symmetrisable type.

References

[1] C. Bowman, *The many integral graded cellular bases of Hecke algebras of complex reflection groups*, Amer. J. Math., **144** (2022), 437–504. arXiv:1702.06579.
[2] A. Evseev and A. Mathas, *Content systems and deformations of cyclotomic KLR algebras of type A and C*, 2022. arXiv:2209.00134.
[3] J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov–Lauda–Rouquier algebras of type A*, Adv. Math., **225** (2010), 598–642. arXiv:0907.2985.
[4] A. Kleshchev, *Branching rules for modular representations of symmetric groups. III. Some corollaries and a problem of Mullineux*, J. London Math. Soc. (2), **54** (1996), 25–38.
[5] A. Mathas and D. Tubbenhauer, *Subdivision and cellularity for weighted KLRW algebras*, 2021. ArXiv preprint, arXiv:2111.12949.
[6] ———, *Cellularity for weighted KLRW algebras of types B, $A^{(2)}$, $D^{(2)}$*, 2022. ArXiv preprint, arXiv:2201.01998.
[7] B. Webster, *Rouquier's conjecture and diagrammatic algebra*, Forum of Mathematics, Sigma, **5** (2013). arXiv:1306.0074.
[8] B. Webster, *Weighted Khovanov-Lauda-Rouquier algebras*, Doc. Math., **24** (2019), 209–250. arXiv:1209.2463.
[9] A. Young, *On Quantitative Substitutional Analysis I*, Proc. London Math. Soc., **33** (1900), 97–145.

Broué's Abelian Defect Conjecture for double covers of symmetric groups

Jonathan Brundan
(joint work with Alexander Kleshchev)

Let K be an algebraically closed field of characteristic $p > 0$. Broué's Abelian Defect Conjecture formulated in 1990 asserts that certain blocks of the group algebras KG of finite groups G are derived equivalent to their Brauer correspondents. The conjecture was motivated in part by the much older MacKay conjecture—it gives a category-theoretic "explanation" for the expected equality between numbers of irreducible characters. However, unlike the McKay conjecture, Broué's conjecture requires a considerable restriction, namely, that the defect group of the block in question is Abelian. As we learnt in the talk of Gabriel Navarro, remarkably, the proof of the McKay conjecture is nearing completion. In this talk, I report on some other recent work which has completed the proof of Broué's conjecture for double covers of symmetric and alternating groups. We are still a long way from being able to approach Broué's conjecture for most other finite groups.

Broué's conjecture was proved for symmetric groups in a seminal paper by Chuang and Rouquier [5]. At that time, following an earlier conjecture of Rouquier, Chuang and Kessar [4] had already shown that certain generic blocks of symmetric groups now called *RoCK blocks* are derived equivalent to their Brauer correspondents assuming the underlying defect groups are Abelian. Thus, to complete the proof of Broué conjecture for symmetric groups, it remained to show that any block of a symmetric group (with Abelian defect) was derived equivalent to a RoCK block. In fact, Chuang and Rouquier proved this assertion for *all* blocks of symmetric groups—they did not require the assumption on Abelian defect. Their work initiated a new direction in representation theory known now as *categorification* of Kac-Moody algebras, with many rich connections to other areas of algebra and topology, which is now centered around the subsequent definition by Rouquier [12] and Khovanov and Lauda [9] of a remarkable family of 2-categories called *Kac-Moody 2-categories.*

The importance of categorification as pertains to symmetric groups arose a decade earlier, in another fundamental paper by Lascoux, Leclerc and Thibon [11]. Fitting several existing puzzle pieces together, they brought to prominence the idea that the Grothendieck group

$$\bigoplus_{n\geq 0} K_0(KS_n\text{-mod})$$

of the categories of finite-dimensional KS_n-modules for all n carries a natural structure as a representation of the Kac-Moody algebra $\mathfrak{g}$ of type $A^{(1)}_{p-1}$. The actions of the Chevalley generators e_i and f_i of $\mathfrak{g}$ are induced by certain biadjoint functors E_i and F_i known as *i-induction* and *i-restriction* functors. In this way, the complexified Grothendieck group become identified with the *basic representation* $V(-\Lambda_0)$ of $\mathfrak{g}$ of lowest weight $-\Lambda_0$. The blocks of the group algebras KS_n for all n correspond to the non-zero *weight spaces* of the $\mathfrak{g}$-module $V(-\Lambda_0)$.

As is well known, blocks of symmetric groups are parametrized naturally by their *p-core* and *p-weight*. The p-cores are in bijection with the extremal weights of $V(-\Lambda_0)$, that is, the weights in the same Weyl group orbit as the lowest weight $-\Lambda_0$. Then a block of p-weight d corresponds to a weight λ of $V(-\Lambda_0)$ of the form $\lambda = w(d\delta - \Lambda_0)$ for some $w \in W$, the Weyl group, where δ is the null root. The isomorphism type of defect group of the block is uniquely determined by this number d. Moreover, it is easy to see from the generic nature of the definition that there exist RoCK blocks of p-weight d for all $d \geq 0$. So now the problem becomes to show that two blocks corresponding to weights λ and μ of $V(-\Lambda_0)$ in the same W-orbit are derived equivalent. Chuang and Rouquier established this by constructing a derived equivalence in the case that $\mu = s_i\lambda$ for a simple Coxeter generator s_i of W. This equivalence is obtained from an explicit complex of functors written down earlier by Rickard, and known now as the *Rickard complex*. It is also related to what we call the *singular Rouquier complex* for Grassmannians, something which is of considerable interest also in the study of homological invariants of knots and links.

Now assume that p is odd and consider the double covers $\widehat{S}_n$ of the symmetric groups, denoting the canonical central involution by z. We have that

$$K\widehat{S}_n \cong KS_n \oplus S(n)$$

where $S(n) \cong KS_n/(z+1)$ is the *twisted group algebra*. In fact, it seems essential now to view $S(n)$ as a *superalgebra*—a $\mathbb{Z}/2$-graded algebra—with grading induced by the usual notion of parity of permutations. To prove Broué's conjecture for $\widehat{S}_n$, it suffices by the existing work of Chuang and Rouquier treating the ordinary group algebras KS_n to consider the (super)blocks of $S(n)$ for all n. The approriate categorification framework in this setting was worked out in [2]: the sum of the Grothendieck groups

$$\bigoplus_{n\geq 0} K_0(S(n)\text{-smod})$$

of the categories of finite-dimensional $S(n)$-*super*modules for all n carries the structure of a representation of the Kac-Moody algebra $\mathfrak{g}$ of the *twisted* type $A^{(2)}_{p-1}$. Moreover, the complexified Grothendieck group is naturally identified with the corresponding basic representation $V(-\Lambda_0)$, superblocks correspond to weight spaces, and so on.

We come finally to the new work. In [3], we established the analogous result to [5] for these twisted group algebras, showing that superblocks corresponding to weight spaces in the same W-orbit are derived superequivalent. Ebert, Lauda and Vera [7] independently gave a different approach—our papers were posted to the `arXiv` on the same day! This work involves an enormous amount of new mathematics, starting from a generalization of the Kac-Moody 2-categories mentioned earlier to the *super Kac-Moody 2-categories* as defined in [1]. The definition of these rests on earlier work of Ellis, Khovanov and Lauda [6] concerned with *odd symmetric functors* and the *odd nil-Hecke algebra*. Using this theory, we constructed an odd analog of the singular Rouquier complex built from *odd Grassmannian bimodules*, and extended many of Chuang and Rouquier's arguments to show that can be used to construct the missing derived equivalences in this setting. The double covers of symmetric groups actually only enter the story indirectly, since most of our work is carried out in the context of the super Kac-Moody 2-category associated to $\mathfrak{sl}_2$. The connection between this and $S(n)$ arises from a remarkable Morita equivalence between $S(n)$ and certain *quiver Hecke superalgebras* which was established by Kang, Kashiwara and Tsuchioka [8].

This is only part of the story, as one still needs to find appropriate analogs of RoCK blocks for the twisted group algebras and to prove an analog of the Chuang-Kessar theorem. This is also now in place due to another substantial recent piece work by Kleshchev and Livesey [10]—their manuscript is 200 pages long! Again, finite group theory is scarcely visible since this takes place largely in the context of quiver Hecke superalgebras, again using the Morita equivalences from [8]. The wide range of tools needed to approach Broué's conjecture even just for symmetric groups and their double covers is quite astonishing.

References

[1] J. Brundan and A. Ellis, *Super Kac-Moody 2-categories*, Proc. Lond. Math. Soc. **115** (2017), 925–973.

[2] J. Brundan and A. Kleshchev, *H*ecke-Clifford superalgebras, crystals of type A(2)2l and modular branching rules for $\widehat{S}_n$, Represent. Theory **5** (2001), 317–403.

[3] J. Brundan and A. Kleshchev *Odd Grassmannian bimodules and derived equivalences for spin symmetric groups*, `arXiv:2203.14149`.

[4] J. Chuang and R. Kessar, *Symmetric groups, wreath products, Morita equivalences, and Broué's abelian defect group conjecture*, Bull. London Math. Soc. **34** (2002), 174–184.

[5] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$ categorification*, Annals Math. **167** (2008), 245–298.

[6] A. Ellis, M. Khovanov and A. Lauda, *The odd nil-Hecke algebra and its diagrammatics*, IMRN, 2012.

[7] M. Ebert, A. Lauda and L. Vera, Derived superequivalence for spin symmetric groups and odd $\mathfrak{sl}_2$ categorification, `arXiv:2203:14153`.

[8] S.-J. Kang, M. Kashiwara and S. Tsuchioka, *Quiver Hecke superalgebras*, J. Reine Angew. Math. **711** (2016), 1–54.

[9] M. Khovanov and A. Lauda, *A categorification of quantum $\mathfrak{sl}(n)$*, Quantum Top. **1** (2010), 1–92.

[10] A. Kleshchev and M. Livesey, RoCK blocks for double covers of symmetric groups and quiver Hecke superalgebras, `arXiv:2201.06870`.

[11] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm. Math. Phys. **181** (1996), 205–263.

[12] R. Rouquier, *2-Kac-Moody algebras*, `arXiv:0812.5023`.

A recursive formula for plethysm coefficients and some applications

Stacey Law
(joint work with Y. Okitani)

Plethysm coefficients arise in the theory of symmetric functions, as the multiplicities $a^{\nu}_{\lambda,\mu}$ appearing in the decomposition of plethystic products of Schur functions $s_\lambda \circ s_\mu$ into non-negative integral linear combinations of Schur functions s_ν. Plethysm is one of the most important operations on symmetric functions that is still not fully understood combinatorially: indeed, the question of finding a combinatorial description of $a^{\nu}_{\lambda,\mu}$ in general is Problem 9 from Stanley's 1999 survey of positivity problems in algebraic combinatorics [13]. Although there has been much work done to understand plethysms in special cases, the full problem is still very much open, and the same can be said of another long-standing conjecture, due to Foulkes from 1950 [6]: that if $m \leq n$, then $a^{\nu}_{(m),(n)} \leq a^{\nu}_{(n),(m)}$ for all partitions ν.

Our main result is a recursive formula for plethysm coefficients of the form $a^{\nu}_{\lambda,(m)}$ for arbitrary partitions ν and λ [10, Theorem A]. We describe three collections of applications: to computing explicit plethysm coefficients, to stabilities of sequences of plethysm coefficients, and to Sylow branching coefficients of symmetric groups.

As a first consequence of our formula, we are able to determine explicitly the values of certain plethysm coefficients appearing in Foulkes' Conjecture. In particular, we prove two conjectures of de Boeck [4, Conjectures 6.5.1, 6.5.2].

Theorem (L.–Okitani [10]). *Let $m, n \in \mathbb{N}$ with m even.*

(a) Let $\lambda \vdash mn$ with $l(\lambda) \leq n$ and $\lambda_1 = m+2$. If λ has all parts even, then $a^{\lambda}_{(n),(m)} = 1$. Otherwise, $a^{\lambda}_{(n),(m)} = 0$.

(b) Suppose $m, n \geq 3$. Then the lexicographically smallest $\lambda \vdash mn$ such that $a^{\lambda}_{(n),(m)} > 0$ and λ has an odd part is $\lambda = (m+3, m^{n-2}, m-3)$.

Our second application centres on the behaviour of certain sequences of plethysm coefficients. While it is sometimes very difficult to calculate individual values explicitly, it can be more tractable to compare them with one another and to show that sequences exhibit properties such as stability or monotonicity, as results in [2, 3, 5] demonstrate, for example. We prove the stability of sequences in a new family of plethysm coefficients generalising those considered in [1]. To describe these, let $j, m \in \mathbb{N}$, $l \in \{0, 1, \ldots, m\}$ and α be a partition. We define partitions $\alpha^{l,m}\{j\}$ and $\alpha^{l,m}[j]$ by adding an 'arm' and/or 'legs' of certain lengths to the Young diagram of α: this is illustrated in Figure 1, and an explicit definition is given in [11].

Letting $\mathsf{A}_{l,m}(\nu,\lambda)$ denots the sequence of plethysm coefficients $\big(a^{\nu^{l,m}\{j\}}_{\lambda^{l,m}[j],(m)}\big)_{j\in\mathbb{N}}$, Bessenrodt, Bowman and Paget recently conjectured that for any partitions ν and λ, the sequence $\mathsf{A}_{1,2}(\nu,\lambda)$ is weakly increasing [1, Conjecture 1.2]. While we do not prove monotonicity, we were instead able to show that such sequences are always eventually constant [10, Proposition 5.3]. Wildon (in private communication) then conjectured the generalisation that $\mathsf{A}_{m-1,m}(\nu,\lambda)$ should stabilise, for all natural

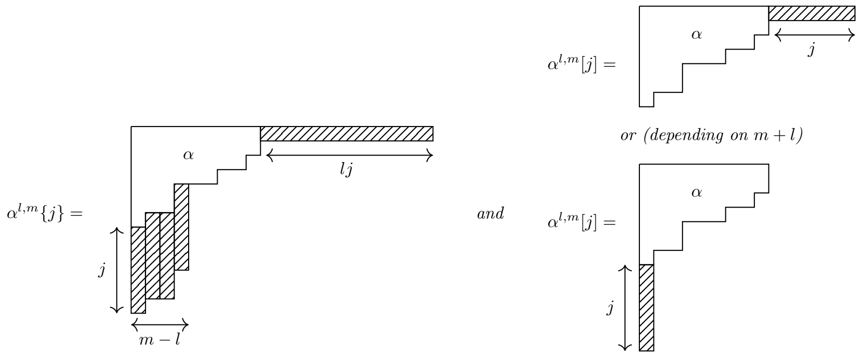


FIGURE 1. $\alpha^{l,m}\{j\}$ and $\alpha^{l,m}[j]$ in terms of Young diagrams.

m. Using our recursive formula, we were able to prove Wildon's conjecture, and in fact extend the parameters from $(m-1, m)$ to (l, m) for arbitrary $0 \leq l \leq m$.

Theorem (L.–Okitani [11])**.** *Let ν and λ be arbitrary partitions. Then the sequence of plethysm coefficients $\mathsf{A}_{l,m}(\nu, \lambda)$ is eventually constant, for all $m \in \mathbb{N}$ and $l \in \{0, 1, \ldots, m\}$.*

This leads to the following natural questions.

Question. *What is the limit of $\mathsf{A}_{l,m}(\nu, \lambda)$? When is it strictly positive?*

In fact, when $l = 0$ or $l = m$, we can deduce from [2, §2.6, Corollary 1] that $\mathsf{A}_{l,m}(\nu, \lambda)$ is weakly increasing, as well as stable. We therefore propose the following generalisation of [1, Conjecture 1.2].

Conjecture. *Let ν and λ be arbitrary partitions. Then the sequence of plethysm coefficients $\mathsf{A}_{l,m}(\nu, \lambda)$ is weakly increasing for all $m \in \mathbb{N}$ and $l \in \{0, 1, \ldots, m\}$.*

Indeed, Wildon's conjecture of the form of $\nu^{m-1,m}\{j\}$ appears related to certain plethystic semistandard tableaux, as defined in [5], of shape $(m-1, 1)^{(1^j)}$ of maximal weight. A question of Wildon from this workshop asks, could such stability results be generalised further to involve arbitrary maximal tableaux, where $(m-1, 1)$ is replaced by an arbitrary partition of m?

Finally, our third application concerns Sylow branching coefficients. For G a finite group, $\mathrm{Irr}(G)$ denotes the set of its complex irreducible characters. For p a prime and P a Sylow p-subgroup of G, the Sylow branching coefficient corresponding to $\chi \in \mathrm{Irr}(G)$ and $\phi \in \mathrm{Irr}(P)$ is simply the multiplicity $Z^\chi_\phi := [\chi_P, \phi]$. Sylow branching coefficients therefore describe the relationship between the character theory of G and its Sylow subgroups, a key theme in the representation theory of finite groups.

For instance, Malle and Navarro show in [12] that the normality of P can be characterised by the positivity of Sylow branching coefficients: that $P \trianglelefteq G$ if and only if $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ such that $Z^\chi_{\mathbb{1}} > 0$. In joint work with

E. Giannelli, J. Long and C. Vallejo [9], we prove another characterisation in terms of their divisibility properties: that $P \trianglelefteq G$ if and only if $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ such that $p \nmid Z_{\mathbb{1}}^{\chi}$. Turning specifically to symmetric groups $G = S_n$, we are able to describe, for instance, the positivity of Z_{ϕ}^{χ} for linear $\phi \in \mathrm{Irr}(P)$ in the case of odd primes in [7, 8]. However, relatively little is known about these coefficients when $p = 2$.

When translated into the context of characters of symmetric groups, plethysm governs induction and restriction between S_{mn} and the wreath product $S_m \wr S_n$. In particular, letting P_k denote a Sylow 2-subgroup of S_k, then $P_{2n} \cong P_2 \wr P_n \leq S_2 \wr S_n \leq S_{2n}$, so Sylow branching coefficients for S_{2n} may be computed using those for S_n along with knowledge of plethysm coefficients. This allows us to give a third class of applications of our recursive formula, namely new information on Sylow branching coefficients for symmetric groups when $p = 2$. In addition, we show that the proportion of $\chi \in \mathrm{Irr}(S_n)$ such that $Z_{\mathbb{1}}^{\chi} > 0$ tends to 1 as $n \to \infty$ when $p = 2$. While $\{\chi \in \mathrm{Irr}(S_n) \mid Z_{\mathbb{1}}^{\chi} > 0\}$ has been determined for all odd primes [7, Theorem A], an explicit description remains open in the case of $p = 2$.

References

[1] C. Bessenrodt, C. Bowman and R. Paget, The classification of multiplicity-free plethysms of Schur functions, *Trans. Amer. Math. Soc.*, **375** (2022), 5151–5194.

[2] M. Brion, Stable properties of plethysm: on two conjectures of Foulkes, *Manuscripta Math.* **80** (4) (1993), 347–371.

[3] C. Carré and J-Y. Thibon, Plethysm and vertex operators, *Adv. in Appl. Math.* **13** (4) (1992), 390–403.

[4] M. de Boeck, *On the structure of Foulkes modules for the symmetric group*, Ph.D. thesis, University of Kent, 2015.

[5] M. de Boeck, R. Paget and M. Wildon, Plethysms of symmetric functions and highest weight representations, *Trans. Amer. Math. Soc.* **374** (2021), 8013–8043.

[6] H. O. Foulkes, Plethysm of S-functions, *Philos. Trans. Royal Soc. A* **246** (1954), 555–591.

[7] E. Giannelli and S. Law, On permutation characters and Sylow p-subgroups of $\mathfrak{S}_n$, *J. Algebra* **506** (2018), 409–428.

[8] E. Giannelli and S. Law, Sylow branching coefficients for symmetric groups, *J. London Math. Soc.* (2) **103** (2021), 697–728.

[9] E. Giannelli, S. Law, J. Long and C. Vallejo, Sylow branching coefficients and a conjecture of Malle and Navarro, *Bull. London Math. Soc.* **54** (2022), 552–567.

[10] S. Law and Y. Okitani, On plethysms and Sylow branching coefficients, *Algebr. Comb.*, to appear.

[11] S. Law and Y. Okitani, Some stable plethysms, preprint.

[12] G. Malle and G. Navarro, Characterizing normal Sylow p-subgroups by character degrees, *J. Algebra* **370** (2012), 402–406.

[13] R. P. Stanley, Positivity problems and conjectures in algebraic combinatorics, in *Mathematics: Frontiers and Perspectives* (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.), American Mathematical Society, Providence, RI (2000), 295–319.

Weight conjectures for $\mathbb{Z}_\ell$-spetses

Gunter Malle

(joint work with Radha Kessar, Jason Semeraro)

Attached to a finite group G and a prime ℓ are various representation theoretic objects related to one another by deep and longstanding conjectures, like for example the Alperin weight conjecture. In our talk we described how to construct similarly looking objects starting out from a spetsial $\mathbb{Z}_\ell$-reflection group W by on the one hand side mimicking Lusztig's theory to obtain unipotent characters and the block theory of Cabanes and Enguehard to define a principal block, and on the other side using the fusion systems constructed by Broto–Møller from homotopy fixed points of ℓ-compact groups with Weyl group W.

It is then possible to formulate analogues of the Alperin weight conjecture as well as of Robinson's ordinary weight conjecture, relating invariants from these two seemingly unrelated sides of our construction, and to prove them in important special cases. Our proofs involve, among other ingredients, a generalisation of Yokonuma Hecke algebras to the setting of $\mathbb{Z}_\ell$-reflection groups for which we can show the analogue of a Morita equivalence due to Puig.

Characters of the Uniform Block Permutation Monoid

Rosa Orellana

(joint work with F. Saliola, A. Schilling, M. Zabrocki)

The polynomial representations of the general linear group, $GL_n(\mathbb{C}) = GL_n$, are indexed by partitions λ with at most n parts and the representations of the symmetric group S_n are indexed by partitions of n. S_n embeds in GL_n as the group of permutation matrices.

A longstanding open problem in representation theory is to give a combinatorial description of the coefficients when a polynomial representation of the general linear group is restricted to the symmetric group. If V^λ is a polynomial representation of GL_n indexed by λ, then

$$V^\lambda \downarrow_{S_n}^{GL_n} \cong \bigoplus_\mu (\mathbb{S}^\mu)^{\oplus r_{\lambda\mu}}$$

where $\mathbb{S}^\mu$ is the Specht module indexed by μ and $r_{\lambda\mu}$ is the *restriction coefficient.*

Let $V = \mathbb{C}^n$, then GL_n acts on $V^{\otimes k}$ diagonally and S_k acts by permuting the tensor factors; and these actions commute. This observation leads to the Schur-Weyl duality that says that $V^{\otimes k}$ is a $GL_n \otimes S_k$ module that decomposes multiplicity free as follows

$$V^{\otimes k} \cong \bigoplus_\mu V^\lambda \otimes \mathbb{S}^\lambda,$$

where the sum is over all partitions μ of k with at most n parts. When two groups (or algebras) satisfy the Schur-Weyl duality, we say that they form a *centralizer pair.*

Now, if we restrict the diagonal action of GL_n on $V^{\otimes k}$ to the permutation matrices, we obtain that the commutant of this action is the *partition algebra*, $P_k(n)$. This algebra was defined by Martin and Jones independently. Jones showed that S_n and $P_k(n)$ form a centralizer pair when acting on $V^{\otimes k}$. More precisely, as a $S_n \times P_k(n)$ module

$$V^{\otimes k} \cong \bigoplus_{\lambda} \mathbb{S}^{\lambda} \otimes V^{\lambda}_{P_k(n)},$$

where the sum is over all partition λ of n such that $\lambda_2 + \lambda_3 + \ldots \leq k$.

(GL_n, S_k) and $(S_n, P_k(n))$ are centralizer pairs both acting on $V^{\otimes k}$. Therefore, they form a *see-saw pair*, which can be viewed in a diagram as follows:

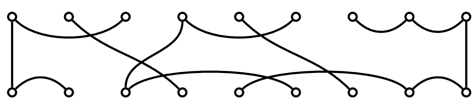
$$\begin{tikzcd}
GL_n \arrow[dr, dash] & & P_k(n) \arrow[dl, dash] \\
& V^{\otimes k} & \\
S_n \arrow[uu, hook] \arrow[ur, dash] & & S_k \arrow[uu, hook] \arrow[ul, dash]
\end{tikzcd}$$

The see-saw relations then say that

$$V^{\lambda}_{GL_n} \downarrow^{GL_n}_{S_n} = \bigoplus_{\mu} (\mathbb{S}^{\mu})^{\oplus r_{\lambda\mu}} \quad \text{and} \quad V^{\lambda}_{P_k(n)} \downarrow^{P_k(n)}_{S_k} = \bigoplus_{\lambda} (\mathbb{S}^{\lambda})^{\oplus r_{\lambda\mu}}.$$

This means that we can compute the restriction coefficients $r_{\lambda\mu}$ by restricting a $P_k(n)$-module to the symmetric group S_k contained in $P_k(n)$. In this talk we proposed to carry out this restriction in two steps, by first restricting form $P_k(n)$ to U_k, the monoid algebra of *uniform block permutations*, and then restricting from U_k to S_k. The restriction from $P_k(n)$ to U_k can be computing using the Littlewood–Richardson rule while the restriction from U_k to S_k can be computed from special cases of the plethysm.

For any nonnegative integer k, let $[k] = \{1, \ldots, k\}$. U_k is a subalgebra of $P_k(n)$ consisting of set partitions of $[k] \cup [\overline{k}]$ such that every block of the set partition such that the number of elements from $[k]$ is equal to the number of elements from $[\overline{k}]$. For example, if $k = 9$, then $\{\{1, 3, \overline{1}, \overline{2}\}, \{2, \overline{4}\}, \{4, 6, \overline{3}, \overline{6}\}, \{5, \overline{7}\}, \{7, 8, 9, \overline{5}, \overline{8}, \overline{9}\}\}$ is a uniform block permutation that can be visualized with the following diagram:



The product does not depend on n and U_k is always semisimple. The irreducible representations are indexed by sequences of partitions with at most k entries $(\lambda^{(1)}, \ldots, \lambda^{(k)})$ such that $k = \sum_i^k i|\lambda^{(i)}|$, where $|\lambda^{(i)}|$ denotes the size of the partition $\lambda^{(i)}$. We denote the set of all such sequences by $\mathcal{I}_k$. For a detailed description of the representation theory of U_k from a monoid algebra perspective see [1].

The main result presented in this talk is the following theorem which computes the characters of U_k in terms of the characters of the maximal subgroups contained in the underlying monoid.

Theorem 18 (OSSZ 2022). *$\vec{\lambda}, \vec{\mu} \in \mathcal{I}_k$, $a_i = |\lambda^{(i)}|$, $\lambda = (1^{a_1} 2^{a_2} \cdots k^{a_k})$*

$$\chi_{U_k}^{\vec{\lambda}}(d_{\vec{\mu}}) = \sum_{\substack{\vec{\nu} \in \mathcal{I}_k \\ |\nu^{(i)}| = a_i}} b_{\vec{\mu}}^{\vec{\nu}} \, \chi_{G_\lambda}^{\vec{\lambda}}(d_{\vec{\nu}})$$

where $d_{\vec{\nu}}$ is a generalized conjugacy class representative, G_λ is a maximal subgroup and $b_{\vec{\mu}}^{\vec{\nu}}$ is a nonnegative integer.

An explicit formula for the integer $b_{\vec{\mu}}^{\vec{\nu}}$ will be presented, for details see [1].

Let A_k be the block diagonal matrix containing the character tables of the maximal subgroups on the diagonal, and $B_k = (b_{\vec{\mu}}^{\vec{\nu}})$ for all $\vec{\mu}, \vec{\nu} \in \mathcal{I}_k$. A consequence of Theorem 1 is that the character table of U_k, X_k, has the following factorization $X_k = A_k B_k$, for details see [1].

References

[1] R. Orellana, F. Saliola, A. Schilling and M. Zabrocki, *Plethysm and the algebra of uniform block permutations*, to appear in Algebraic Combinatorics.

Heisenberg and Kac-Moody categorification

Ben Webster

(joint work with Jon Brundan, Alistair Savage)

One of the most powerful techniques in the study of symmetric groups and related algebras is that of categorification. The key notion underlying this is a *categorical action* of a semi-simple Lie algebra $\mathfrak{g}$. Such an action on a family of categories $(\mathcal{R}_\lambda)_{\lambda \in X}$ is the data of a strict 2-functor from $\mathcal{U}(\mathfrak{g})$ to the 2-category $\mathfrak{Cat}$ of categories sending λ to $\mathcal{R}_\lambda$ for each $\lambda \in X$. This means that there are functors $E_i : \mathcal{R}_\lambda \to \mathcal{R}_{\lambda+\alpha_i}$, $F_i : \mathcal{R}_{\lambda+\alpha_i} \to \mathcal{R}_\lambda$ corresponding to the Chevalley generators e_i, f_i ($i \in I$) of $\mathfrak{g}$ (where α_i is the ith simple root), and there are natural transformations which we can encapsulate in the conditions:

(KM1) there are prescribed adjunctions (E_i, F_i) for all $i \in I$;

(KM2) for $d \geq 0$ there is an action of the *quiver Hecke algebra* QH_d of the same Cartan type as $\mathfrak{g}$ on the dth power of the functor $E := \bigoplus_{i \in I} E_i$;

(KM3) there is an explicit isomorphism of functors lifting the familiar Chevalley relation $[e_i, f_j] = \delta_{i,j} h_i$ in the Lie algebra $\mathfrak{g}$.

In Cartan type A, Rouquier introduced a related notion of $\mathfrak{sl}'_I$*-categorification*, based on his previous work with Chuang [CR] treating the case of $\mathfrak{sl}_2$. Instead of the tower of quiver Hecke algebras, the definition of $\mathfrak{sl}'_I$-categorification involves a tower of *affine Hecke algebras* of type A (either quantum or degenerate). Let I be a subset of $\mathbb{K}$ closed under the automorphisms $i \mapsto i^\pm$ defined by $i^\pm := q^{\pm 2} i$ in the quantum case ($q \neq q^{-1}$), and $i^\pm := i \pm 1$ in the degenerate case ($q = q^{-1}$). The map

$i \mapsto i^+$ defines edges making the set I into a quiver with connected components of type A_∞ if $e = 0$ or $\mathrm{A}^{(1)}_{e-1}$ if $e \neq 0$, where e is the *quantum characteristic*, that is, the smallest positive integer such that $q^{e-1} + q^{e-3} + \cdots + q^{1-e} = 0$ or 0 if no such integer exists. Let $\mathfrak{g} = \mathfrak{sl}'_I$ be the corresponding (derived) Kac-Moody algebra. To have an $\mathfrak{sl}'_I$-categorification on an abelian category, we must have:

(SL1) an biadjoint pair (E, F) of endofunctors of $\mathcal{R}$;
(SL2) endomorphisms $x : E \Rightarrow E$ and $\tau : E^2 \Rightarrow E^2$ inducing an action of AH_d on the dth power E^d for all $d \geq 0$.

Assume moreover that all eigenvalues of $x : E \Rightarrow E$ belong to the given set I so that, by taking generalized eigenspaces, one obtains decompositions of E and its adjoint F into eigenfunctors: $E = \bigoplus_{i \in I} E_i$, $F = \bigoplus_{i \in I} F_i$. Then, we require that

(SL3) the induced maps $e_i := [E_i]$ and $f_i := [F_i]$ make the complexified Grothendieck group $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{R})$ into an integrable representation of the Lie algebra $\mathfrak{g}$, with the Grothendieck group of each block of $\mathcal{R}$ giving rise to an isotypic representation of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Under these hypotheses, there is an induced categorical action of $\mathfrak{g}$ on $(\mathcal{R}_\lambda)_{\lambda \in X}$ in the sense defined in the previous paragraph, for Serre subcategories $\mathcal{R}_\lambda$ of $\mathcal{R}$ defined so that $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{R}_\lambda)$ is the λ-weight space of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{R})$. This fundamental result is known as "control by K_0;" see [R1, Theorem 5.30] and [BD, Theorem 4.27]. In its proof, the property (SL1) obviously implies (KM1), and (SL2) implies (KM2) due to the isomorphism $\widehat{AH}_d \cong \widehat{QH}_d$ between completions of affine Hecke algebras and quiver Hecke algebras discovered in [R1, BK2]. Finally, and most interesting, to pass from (SL3) (which involves relations at the level of the Grothendieck group) to (KM3) (which involves "higher" relations), Rouquier applies the sophisticated structure theory developed in [CR], thereby reducing the proof to minimal $\mathfrak{sl}_2$-categorifications which are analyzed explicitly.

In the current literature, almost all examples of categorical actions of Kac-Moody algebras of Cartan type A on abelian categories are constructed via this theorem. In this talk, based on [BSW3], we discuss a new approach to constructing such Kac-Moody actions based instead on the *Heisenberg category* $\mathcal{H}eis_k$ of central charge $k \in \mathbb{Z}$. This is a monoidal category constructed from affine Hecke algebras in a way that is analogous to the construction of the Kac-Moody 2-category. It comes in two forms, degenerate or quantum, depending on the parameter $z = q - q^{-1}$ as fixed above. In the special case $k = -1$, the Heisenberg category was defined originally in the degenerate case by Khovanov [K] and in the quantum case by Licata and Savage [LS] and extended to arbitrary k much more recently in [MS, B2, BSW1]. A *categorical Heisenberg action* on a category $\mathcal{R}$ is the data of a strict monoidal functor $\mathcal{H}eis_k \to \mathcal{E}nd(\mathcal{R})$, where $\mathcal{E}nd(\mathcal{R})$ is the strict monoidal category consisting of endofunctors and natural transformations. Equivalently, this means that there are endofunctors $E, F : \mathcal{R} \to \mathcal{R}$ and natural transformations such that

(H1) there is a prescribed adjunction (E, F);
(H2) for $d \geq 0$ there is an action of AH_d on E^d;

(H3) there is an explicit isomorphism of functors lifting the relation $[e, f] = k$ in the Heisenberg algebra of central charge k.

The properties (H1)–(H3) exactly parallel (KM1)–(KM3), unlike (SL1)–(SL3).

Theorem ([BSW3, Th. A]). *Let $\mathcal{R}$ be either a suitably finite abelian $\mathbb{K}$-linear category equipped with a categorical Heisenberg action. Let I be the* spectrum *of $\mathcal{R}$, that is, the set of eigenvalues of the given endomorphism $x : E \Rightarrow E$. This set is closed under the maps $i \mapsto i^{\pm}$ defined above. Let $\mathfrak{g} = \mathfrak{sl}'_I$ be the corresponding Kac-Moody algebra with weight lattice X. For each $\lambda \in X$, there is a Serre subcategory $\mathcal{R}_\lambda$ of $\mathcal{R}$ defined in terms of the action of endomorphisms of the tensor unit ("bubbles"). Moreover, there is a canonically induced categorical action of $\mathfrak{g}$ on $(\mathcal{R}_\lambda)_{\lambda \in X}$ in the sense of (KM1)–(KM3).*

This theorem considerably simplifies the construction of the most important examples of categorical Kac-Moody actions. In these examples, the existence of a Heisenberg action is straightforward to demonstrate, so that the theorem above can be applied without any need to develop the theory to the point of being able to check relations on the Grothendieck group. For a more detailed discussion of examples, refer to the introduction of [BSW3]. This theorem applies to

(1) the representations of the symmetric groups: $\mathcal{R} := \bigoplus_{d \geq 0} \mathbb{K}\mathfrak{S}_d\text{-mod}_{\mathrm{fd}}$ with E given by induction and F by restriction. The set I (which is the spectrum in our language) is the image of $\mathbb{Z}$ in $\mathbb{K}$, so that $\mathfrak{sl}'_I$ is $\mathfrak{sl}_\infty(\mathbb{C})$ if $e = 0$ or $\widehat{\mathfrak{sl}}_e(\mathbb{C})'$ if $e > 0$. Khovanov [K] used this example to motivate his definition of the degenerate Heisenberg category $\mathcal{H}eis_{-1}$, making the existence of a categorical Heisenberg action on $\mathcal{R}$ almost tautological: the conditions (H1) and (H2) are immediate while (H3) follows from the Mackey isomorphism $F \circ E \cong E \circ F \oplus \mathrm{Id}$.

(2) There are many variations on this example, in which one replaces $\mathbb{K}\mathfrak{S}_d$ by higher level cyclotomic quotients of (degenerate or quantum) affine Hecke algebras or quiver Hecke algebras; see [A]. The Grothendieck groups in these cases give $\mathbb{Z}$-forms for the other integrable highest or lowest weight representations.

(3) The category $\mathcal{O}$ for rational Cherednik algebras of types $G(\ell, 1, d)$ for $d \geq 0$, which categorifies Fock space; see [G^{2}OR, S]. This also includes categories of modules over cyclotomic q-Schur algebras as a special case.

(4) Many variants of the representation theory of the general linear group, including rational representations of the algebraic group GL_n over $\mathbb{K}$, representations of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ in the BGG category $\mathcal{O}$, finite-dimensional representations of restricted enveloping algebras arising from the Lie algebra $\mathfrak{gl}_n(\mathbb{K})$ over a field of positive characteristic, and analogous categories for the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$, including situations in which q is a root of unity. The endofunctors E and F are defined by tensoring with the n-dimensional defining representation V and its dual V^*, respectively. The endomorphism $x : E \Rightarrow E$ arises from the action of the Casimir tensor, while $\tau : E^2 \Rightarrow E^2$ comes from the tensor flip

classically, or its braided analog defined by the R-matrix in the quantum case. The relations (H1)–(H3) are all easy to check, with (H3) amounting to the existence of a particular isomorphism $V \otimes V^* \cong V^* \otimes V$.

References

[A] S. Ariki, On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$, *J. Math. Kyoto Univ.* **36** (1996), 789–808.

[B1] J. Brundan, On the definition of Kac-Moody 2-category, *Math. Ann.* **364** (2016), 353–372.

[B2] J. Brundan, On the definition of Heisenberg category, *Alg. Comb.* **1** (2018), 523–544.

[BD] J. Brundan and N. Davidson, Categorical actions and crystals, *Contemp. Math.* **684** (2017), 116–159.

[BK2] J. Brundan and A. Kleshchev, Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras, *Invent. Math.* **178** (2009), 451–484.

[BSW1] J. Brundan, A. Savage and B. Webster, On the definition of quantum Heisenberg category, *Alg. Numb. Theory* **14** (2020) 275–321.

[BSW2] J. Brundan, A. Savage and B. Webster, The degenerate Heisenberg category and its Groth-endieck ring; arXiv:1812.03255v2.

[BSW3] J. Brundan, A. Savage and B. Webster,Heisenberg and Kac-Moody categorification, *Selecta Math. (N.S.)* **26** (2020), no. 5, Paper No. 74, 62 pp. arXiv:1812.03255v2.

[CR] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categ-orification, *Ann. Math.* **167** (2008), 245–298.

[G²OR] V. Ginzburg, N. Guay, E. Opdam and R. Rouquier, On the category $\mathcal{O}$ for rational Cherednik algebras, *Invent. Math.* **154** (2003), 617–651.

[K] M. Khovanov, Heisenberg algebra and a graphical calculus, *Fund. Math.* **225** (2014), 169–210.

[LS] A. Licata and A. Savage, Hecke algebras, finite general linear groups, and Heisenberg categorification, *Quantum Topology* **4** (2013), 125–185.

[MS] M. Mackaay and A. Savage, Degenerate cyclotomic Hecke algebras and higher level Heisenberg categorification, *J. Algebra* **505** (2018), 150–193.

[R1] R. Rouquier, 2-Kac-Moody algebras; arXiv:0812.5023v1.

[S] P. Shan, Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras, *Ann. Sci. Éc. Norm. Sup.* **44** (2010), 147–182.

[V] R. Virk, Derived equivalences and $\mathfrak{sl}_2$-categorifications for $U_q(\mathfrak{gl}_n)$, *J. Algebra* **346** (2011), 82–100.

Diagrammatic methods in p-Kazhdan–Lusztig theory

Amit Hazi

(joint work with Chris Bowman, Anton Cox, Maud De Visscher, Emily Norton)

A decade after the demise of the Lusztig and James conjectures, efforts to bridge the chasm between Kazhdan–Lusztig combinatorics and modular representation theory have come under the moniker "p-Kazhdan–Lusztig theory". Key to this approach is the diagrammatic Hecke category, which allows for effective diagrammatic computation of decomposition numbers in several Lie-theoretic settings. In this talk we discussed connections between the diagrammatic Hecke category and some of the oldest and most important ideas in representation theory.

Kazhdan–Lusztig theory. Let $\mathfrak{g}$ be semisimple Lie algebra over $\mathbb{C}$. The principal block $\mathcal{O}_0$ of category $\mathcal{O}$ gives the prototypical example of a non-semisimple representation-theoretic category in Lie theory. One of the most basic questions one can ask is to how calculate the decomposition numbers of this category. There are many ways to approach this problem, but perhaps the simplest is to use wall-crossing functors to construct (non-indecomposable) "Bott–Samelson" projectives, starting with the projective Verma module of dominant highest weight. We can write the characters of these Bott–Samelson projectives as elements in the Hecke algebra for the corresponding Weyl group, and we can decompose these elements in terms of a natural basis for the Hecke algebra called the Kazhdan–Lusztig basis.

The Kazhdan–Lusztig conjecture (proven independently by Beilinson–Bernstein and Brylinski–Kashiwara [1, 5]) then states that this decomposition lifts to category $\mathcal{O}$. More precisely, it gives decomposition numbers in $\mathcal{O}_0$ in terms of Kazhdan–Lusztig polynomials (the coefficients defining the Kazhdan–Lusztig basis) evaluated at 1. There are several generalizations and analogues of the Kazhdan–Lusztig conjecture in other settings, e.g. for Kac–Moody algebras, algebraic groups, diagram algebras, etc. The most important positive characteristic analogues (the Lusztig and James conjectures) are now known to be false, except when the characteristic p is extremely large [11].

The diagrammatic Hecke category. The success of the original Kazhdan–Lusztig conjecture and the failure of the Lusztig and James conjectures can ultimately be traced back to the their respective proofs, all of which use a categorification of the Hecke algebra. In characteristic 0, the geometric categorification of the Hecke algebra (arising from perverse sheaves on the flag variety) is particularly well behaved, but in positive characteristic many fundamental results including the decomposition theorem no longer hold.

To remedy this, Elias–Williamson (and later Libedinsky–Williamson) constructed a diagrammatic categorification of the Hecke category $\mathcal{H} := \mathcal{H}(W, P)$, which can be defined for all (parabolic) Coxeter types in any characteristic [7, 9]. We view $\mathcal{H}$ as a locally unital diagram algebra which is graded quasi-hereditary with a graded cellular basis given by (the diagrammatic analogue of) Libedinsky's light leaves construction. Elias–Williamson showed that the graded decomposition numbers of $\mathcal{H}$ are given by Kazhdan–Lusztig polynomials [6]. For $p > 0$ we subsequently take p-Kazhdan–Lusztig polynomials to be the graded decomposition numbers of $\mathcal{H}$ over a field of characteristic p.

Applications. There are relatively few examples where p-Kazhdan–Lusztig polynomials are known in complete generality. Since this problem is expected to be difficult in general, it is helpful to start with a combinatorial description of the ordinary Kazhdan–Lusztig polynomials. For example, Hermitian symmetric pairs are certain maximal parabolics inside finite Coxeter groups whose ordinary Kazhdan–Lusztig polynomials are all monic. In joint work with Chris Bowman, Maud De Visscher, and Emily Norton, we showed that in this case, p-Kazhdan–Lusztig polynomials and ordinary Kazhdan–Lusztig polynomials coincide.

Theorem 1 ([3])**.** *For (W, P) a Hermitian symmetric pair, the p-Kazhdan–Lusztig polynomials are independent of p.*

(Maud De Visscher spoke about this paper in much more detail in her talk.)

Another good example where everything is known combinatorially is the case of dominant or antidominant weights. This includes several versions of the Weyl character formula, which predates the Kazhdan–Lusztig conjecture by more than 50 years! In joint work with Chris Bowman and Emily Norton we generalized the BGG resolution from Lie theory (and the resulting character formula) to arbitrary characteristic and Coxeter type.

Theorem 2 ([4])**.** *The first column of the inverse matrix of p-Kazhdan–Lusztig polynomials is given by*

$$^{p}n_{\mathrm{id},w}^{-1} = (-v)^{\ell(w)}.$$

Even in cases where p-Kazhdan–Lusztig polynomials are difficult to compute, we can make direct connections to modular representation theory. In joint work with Chris Bowman and Anton Cox, we showed that the diagrammatic Hecke category in type $A_{h-1} \backslash \widehat{A}_{h-1}$ is Morita equivalent to a quotient of the symmetric group algebra.

Theorem 3 ([2])**.** *Suppose $p > h$. We have an explicit isomorphism*

$$F_n \mathcal{H}(A_{h-1} \backslash \widehat{A}_{h-1}) F_n \xrightarrow{\sim} f_n(\Bbbk\mathfrak{S}_n / \Bbbk\mathfrak{S}_n e_h \Bbbk\mathfrak{S}_n) f_n$$

of graded cellular $\Bbbk$-algebras, where

- *e_h is the idempotent corresponding to partitions with $> h$ rows*
- *f_n is a truncation of the principal block idempotent*
- *F_n is an idempotent corresponding to a subset of words of length $\leq n$*

The most important consequence of this result is that for partitions with at most h rows, the (graded) decomposition numbers of the symmetric group in characteristic p are given by antispherical p-Kazhdan–Lusztig polynomials of type $A_{h-1} \backslash \widehat{A}_{h-1}$. This was first shown by Riche–Williamson in their monograph on tilting characters for reductive algebraic groups [10]. Our proof is conceptually more elementary and explicit, in that we do not rely on categorification results. In our paper we more generally prove a level ℓ analogue of the isomorphism, thereby resolving the categorical blob conjecture of Libedinsky–Plaza [8].

References

[1] A. Beĭ linson and J. Bernstein. Localisation de g-modules. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):15–18, 1981.

[2] C. Bowman, A. Cox, and A. Hazi. Path isomorphisms between quiver Hecke and diagrammatic Bott-Samelson endomorphism algebras, May 2020. arXiv:2005.02825.

[3] C. Bowman, M. De Visscher, A. Hazi, and E. Norton. The anti-spherical Hecke categories for Hermitian symmetric pairs, Aug. 2022. arXiv:2208.02584.

[4] C. Bowman, A. Hazi, and E. Norton. The modular Weyl-Kac character formula. *Math. Z.*, 2022. To appear.

[5] J.-L. Brylinski and M. Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Invent. Math.*, 64(3):387–410, 1981.

[6] B. Elias and G. Williamson. The Hodge theory of Soergel bimodules. *Ann. of Math. (2)*, 180(3):1089–1136, 2014.
[7] B. Elias and G. Williamson. Soergel calculus. *Represent. Theory*, 20:295–374, 2016.
[8] N. Libedinsky and D. Plaza. Blob algebra approach to modular representation theory. *Proc. Lond. Math. Soc. (3)*, 121(3):656–701, 2020.
[9] N. Libedinsky and G. Williamson. The anti-spherical category. *Adv. Math.*, 405:Paper No. 108509, 34, 2022.
[10] S. Riche and G. Williamson. Tilting modules and the p-canonical basis. *Astérisque*, (397):ix+184, 2018.
[11] G. Williamson. Schubert calculus and torsion explosion. *J. Amer. Math. Soc.*, 30(4):1023–1046, 2017.

On the arithmetical structure of the irreducible character degrees of a finite group

Emanuele Pacifici
(joint work with S. Dolfi, L. Sanus, V. Sotomayor)

Character Theory is one of the fundamental tools in the theory of finite groups, and, given a finite group G, the study of the set $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ of all the degrees of the irreducible complex characters of G is a particularly intriguing aspect of this theory. This degree set has been extensively studied both on its own account as well as in relation to the structure of the group: in particular, starting with the work by Bertram Huppert and other researchers in the 80's (see [5]), many authors focused on the *arithmetical structure* of $\mathrm{cd}(G)$, i.e., on how the numbers in $\mathrm{cd}(G)$ decompose into prime factors.

One of the methods that have been devised to approach such degree set is to consider the prime graph attached to it. The *degree graph* $\Delta(G)$ is thus defined as the (simple, undirected) graph whose vertex set is the set of all the prime numbers that divide some $\chi(1) \in \mathrm{cd}(G)$, while a pair $\{p, q\}$ of distinct vertices belongs to the edge set if and only if pq divides an element in $\mathrm{cd}(G)$. Several results in the literature illustrate that graph-theoretical features of $\Delta(G)$ are significantly linked to the structure of G. For a detailed account on this subject, we refer to the expository paper [6].

As an important example, we mention that the number of connected components of a degree graph $\Delta(G)$ is in general at most three ([10]), and at most two if G is solvable. It is worth to highlight that non-adjacency between two vertices strongly restricts the structure of the group, and this property has been analyzed in several forms: for instance, the extreme situation when the degree graph is disconnected is studied in [7] for the solvable case, and in [9] for the non-solvable case.

As a further step in the study of the connectivity properties of the degree graph, it can be natural to consider the question about the existence of a *cut-vertex*, i.e., a vertex whose removal yields a resulting graph with more connected components than the original one. A graph that is connected and has a cut-vertex is said to have *connectivity degree* 1, which is the smallest degree of connectivity of a connected graph. The finite solvable groups G such that $\Delta(G)$ has connectivity degree 1 are investigated in [8], where, among other things, it is shown that $\Delta(G)$ has a unique cut-vertex in this case.

In the series of preprints on which we are reporting ([1], [2], [3]), we give a complete classification of all finite *non-solvable* groups whose degree graph has a cut-vertex (see Theorems A, B and C in Section 2 of [3]). It turns out that the structure of the relevant groups, as well as of the corresponding degree graphs, is significantly restricted and, quite surprisingly, does not fall too far from the structure of the finite non-solvable groups with a disconnected degree graph. As an interesting feature, we mention that in all cases (similarly to the situation for solvable groups) a degree graph with connectivity degree 1 has a unique cut-vertex ([3, Corollary D]).

The structure of the finite non-solvable groups whose character degree graph has connectivity degree 1 (with cut-vertex p, say) can be generally described as follows. Such a group G has a unique non-solvable composition factor S, belonging to the following short list of isomorphism types: $\mathrm{PSL}_2(t^a)$, $\mathrm{Sz}(2^a)$ where $2^a - 1 = p$, $\mathrm{PSL}_3(4)$, M_{11}, J_1. Note that, except for $\mathrm{PSL}_2(t^a)$ (whose degree graph is disconnected), the simple groups in this list are precisely the simple groups whose degree graph has a cut-vertex. Moreover, if $S \not\cong \mathrm{PSL}_2(t^a)$, then G has a (minimal) normal subgroup isomorphic to S. Finally, denoting by R the solvable radical of G (which turns out to have an abelian normal p-complement), the vertex set of $\Delta(G)$ consists exactly of the prime divisors of the order of the almost-simple group G/R and, if not already there, the cut-vertex of $\Delta(G)$. We also remark that in all cases, except possibly when the non-abelian simple section S is isomorphic to the Janko group J_1, the cut-vertex p of $\Delta(G)$ is a complete vertex (i.e., it is adjacent to all other vertices) of $\Delta(G)$; moreover, the graph obtained from $\Delta(G)$ by removing the vertex p (and all the edges incident to p) has exactly two connected components, which are complete graphs. Rather remarkably, one of them has a single vertex.

We refer the reader to Section 2 of [3] for the full statements of the results sketched above, and for a discussion concerning the degree graphs associated to the relevant groups. At any rate, to give an idea of what the graphs look like, some of them (namely, those related to the cases when the socle S of G/R is not isomorphic to a 2-dimensional projective special linear group) are shown in Table 1.

To close this report, we recall that results about the degree graph of finite groups often have a counterpart concerning the prime graph built on the set of *conjugacy class sizes* of groups. The analogue of the problem here described is studied in [4], where a classification of the finite groups whose prime graph on conjugacy class sizes has a cut-vertex is provided. However, the situation in that context turns out to be significantly different from the present one, since the relevant groups turn out to be solvable of Fitting height at most three and, moreover, there are cases in which the relevant graphs have two cut-vertices.

References

[1] S. Dolfi, E. Pacifici, L. Sanus, *Non-solvable groups whose character degree graph has a cut-vertex. II*, preprint 2022, submitted.

[2] S. Dolfi, E. Pacifici, L. Sanus, *Non-solvable groups whose character degree graph has a cut-vertex. III*, preprint 2022, submitted.

[3] S. Dolfi, E. Pacifici, L. Sanus, V. Sotomayor, *Non-solvable groups whose character degree graph has a cut-vertex. I*, preprint 2022, submitted.

[4] S. Dolfi, E. Pacifici, L. Sanus, V. Sotomayor, *Groups whose prime graph on class sizes has a cut vertex*, Israel J. Math. **244** (2021), 775–805.

[5] B. Huppert, *Research in representation theory at Mainz (1984–1990)*, Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), 17–36, Progr. Math. **95** (1991), Birkhäuser, Basel.

[6] M.L. Lewis, *An overview of graphs associated with character degrees and conjugacy classs sizes in finite groups*, Rocky Mountain J. Math. **38** (2008), 175–211.

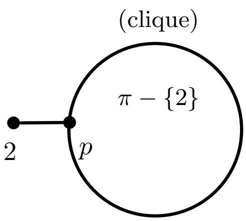
[7] M.L. Lewis, *Solvable groups whose degree graphs have two connected components*, J. Group Theory **4** (2001), 255–275.

[8] M.L. Lewis, Q. Meng, *Solvable groups whose prime divisor character degree graphs are 1-connected*, Monatsh. Math. **190** (2019), 541–548.

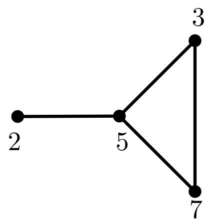
[9] M.L. Lewis, D.L. White, *Connectedness of degree graphs of nonsolvable groups*, J. Algebra **266** (2003), 51–76.

[10] O. Manz, R. Staszewski, W. Willems, *On the number of components of a graph related to character degrees*, Proc. Amer. Math. Soc. **103** (1988), 31–37.

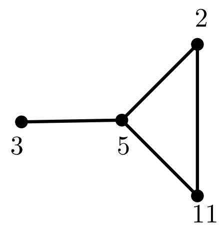
Table 1. $\Delta(G)$ for $S \not\cong \mathrm{PSL}_2(t^a)$



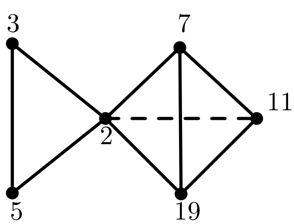
Case $S \cong \mathrm{Sz}(2^a)$
$(\pi = \pi(G/R) \cup \{p\})$



Case $S \cong \mathrm{PSL}_3(4)$



Case $S \cong \mathrm{M}_{11}$



Case $S \cong \mathrm{J}_1$ (the dashed edge is in the graph if and only if R is non-abelian)

Frobenius exact symmetric tensor categories

Victor Ostrik
(joint work with K. Coulembier, P. Etingof)

Let V be a finite dimensional representation of a finite group G over an algebraically closed field of characteristic $p > 0$. In this talk we discuss the asymptotical behavior of the number of non-negligible indecomposable summands in tensor power $V^{\otimes n}$ as n tends to infinity (recall that an indecomposable module is non-negligible if its dimension is not divisible by p). Namely, let d_n be the number of such non-negligible summands (counted with multiplicities) and let

$$\delta(V) = \lim_{n\to\infty} \sqrt[n]{d_n}.$$

Theorem 1.
(a) The limit in the definition of $\delta(V)$ exists;
(b) the invariant δ is additive and multiplicative:

$$\delta(V \oplus W) = \delta(V) + \delta(W),\ \delta(V \otimes W) = \delta(V)\delta(W);$$

(c) the invariant δ takes values in real cyclotomic integers, more precisely $\delta(V) \in \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ where ζ_p is a primitive complex $p-$th root of 1.

The proof of Theorem 1 is non-elementary and uses theory of symmetric tensor categories. Namely, for any Karoubian rigid symmetric monoidal category one defines its *semisimplification* as a quotient by the tensor ideal of negligible morphisms. When this construction is applied to representation category of a finite group, we get a semisimple tensor category; the simple objects of this category are naturally labeled by non-negligible indecomposable modules and tensor products control non-negligible summands in tensor products for the original group.

Of special importance is the special case when the finite group in question is cyclic of order p; the resulting semisimple category is called *Verlinde category* Ver_p (the name comes from similarity of tensor product rules for Ver_p with Verlinde algebra in physics). The main result is a characterization of symmetric tensor categories (assumed to be rigid and abelian) which admit a symmetric tensor functor to Ver_p similar to Deligne's characterization of Tannakian and super-Tannakian categories over fields of characteristic zero. This characterization requires the following ingredients:

(i) Let X be an object of symmetric tensor category. Then the symmetric group S_n acts naturally on $X^{\otimes n}$. Let us define *Frobenius twist* functor $Fr_+(X)$ as the image of the invariants of S_p acting on $X^{\otimes p}$ in the coinvariants of the same action. The functor Fr_+ is automatically additive. We say that the category in question is *Frobenius exact* if the functor Fr_+ is exact. Note that any semisimple category is automatically Frobenius exact.

(ii) We say that a symmetric tensor category is of *moderate growth* if for any object X there is a real number a_X such that the length of the object $X^{\otimes n}$ is bounded by a_X^n for any $n \in \mathbb{Z}_{\geq 0}$.

Theorem 2. For a symmetric tensor category $\mathcal{C}$ the following conditions are equivalent:

(a) $\mathcal{C}$ admits an exact symmetric tensor functor $\mathcal{C} \to \mathrm{Ver}_p$;

(b) $\mathcal{C}$ is Frobenius exact and of moderate growth.

In particular, any semisimple symmetric tensor category of moderate growth admits a symmetric tensor functor to Ver_p which can be used to prove Theorem 1.

References

[1] K. Coulembier, P. Etingof, V. Ostrik *On Frobenius exact symmetric tensor categories*, arxiv: 2107.02372.

Introduction to the τ-tilting theory for group theorists – the role of Schurian modules and a theorem on Schurian-finiteness

Susumu Ariki

Introduction. Examples in the classical tilting theory are mainly bound quiver algebras i.e. path algebras with relations. However, main objects of study for group theorists are blocks of group algebras or Hecke algebras, which are symmetric algebras, and a symmetric algebra has only one basic tilting module. In 2014, Adachi, Iyama and Reiten introduced the τ-tilting theory, which generalizes the classical tilting theory, and symmetric algebras can admit enough τ-tilting modules.

In this talk, we focus on the wide subcategories of the module category of a finite dimensional algebra. After explaining the relevant notions such as Schurian modules and support τ-tilting modules to describe the wide subcategories following [AIR] and [Asa], we turn to blocks of the Hecke algebras of symmetric groups, and we study the finiteness of the number of the wide subcategories. We present an answer in the final part.

The τ-tilting theory. In this talk, $\mathbb{F}$ is an algebraically closed field, A is a finite dimensional $\mathbb{F}$-algebra, and A-modules are finite dimensional left A-modules. We define the Nakayama functor $\nu(?) = D\operatorname{Hom}_A(?, A)$, where $D(?) = \operatorname{Hom}_{\mathbb{F}}(?, \mathbb{F})$ is the $\mathbb{F}$-dual. For an A-module M, we choose a minimal projective presentation

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and define the Auslander-Reiten translate $\tau(M)$ by $\tau(M) = \operatorname{Ker}(\nu(P_1) \to \nu(P_0))$. It is well-known that $\tau(M) = \Omega^2(M)$ if A is a symmetric algebra. That is,

$$0 \longrightarrow \tau(M) \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Definition 1. *An A-module M is called τ-rigid if $\operatorname{Hom}_A(M, \tau(M)) = 0$. A τ-rigid A-module M is called τ-tilting if the number of isoclasses of indecomposable summands of M is equal to the number of the isoclasses of simple A-modules. If there is an idempotent $e \in A$ such that M is a τ-tilting A/AeA-module, M is called support τ-tilting. We denote the set of isoclasses of basic support τ-tilting A-modules by $s\tau$-tilt(A).*

The next lemma [AIR] shows that τ-tilting modules generalize tilting modules.

Lemma 2. *If T is τ-tilting, T is a tilting $A/\operatorname{ann}_A(T)$-module. Moreover, T is a tilting A-module if and only if it is a faithful support τ-tilting A-module.*

We define wide subcategories of the category of A-modules as follows. Wide subcategories are abelian categories.

Definition 3. *A full subcategory $\mathcal{W}$ of the category of A-modules is called a wide subcategory if it is closed under isomorphisms, extensions, kernels and cokernels.*

Serre subcategories of the category of A-modules are in bijection with subsets of the set of isoclasses of simple A-modules, and they are wide subcategories. An old result by Ringel [Rin] generalizes this bijection.

Definition 4. *An A-module M is called Schurian or a brick if $\operatorname{End}_A(M) = \mathbb{F}$. A collection of Schurian modules is called a semibrick if $\operatorname{Hom}_A(M, N) = 0$ whenever $M \neq N$ are two elements of the semibrick. If the number of isoclasses of Schurian modules is finite, we say that the algebra A is Schurian-finite.*

Theorem 5. *Wide subcategories are in bijection with semibricks.*

- *For a wide subcategory, we assign the set of isoclasses of its simple objects.*
- *For a semibrick, we assign the full subcategory consisting of modules which admit filtration by Schurian modules in the semibrick.*

The following theorem by Asai [Asa] generalizes the correspondence between indecomposable τ-rigid modules and Schurian modules [DIJ]. The correspondence also implies that the algebra A is Schurian-finite if and only if it is τ-tilting finite, namely the number of isoclasses of τ-tilting A-modules is finite, or $s\tau$-tilt(A) is a finite set.

Theorem 6. *The map which sends $M \in s\tau$-tilt(A) to the semibrick which consists of indecomposable direct summands of $M/\operatorname{Rad}_{\operatorname{End}_A(M)}(M)$ is injective. Moreover, it is bijective if the algebra A is Schurian-finite.*

Suppose that a semibrick is in the image of the above map, M its preimage. Let $M = \oplus M_i$ be the indecomposable decomposition, and we denote by $e_i \in \operatorname{End}_A(M)$ the projector to M_i. We define J to be the two-sided ideal of $\operatorname{End}_A(M)$ generated by $\{e_i \mid M_i \in \operatorname{Fac}(M/M_i)\}$, where $\operatorname{Fac}(M/M_i)$ means the additive full subcategory of the category of A-modules whose objects are factor modules of finite direct sums of M/M_i. Then, the wide subcategory which corresponds to the semibrick is equivalent to the category of finite dimensional $\operatorname{End}_A(M)/J$-modules.

In general, a semibrick may consist of infinitely many Schurian modules. But, every semibrick consists of finitely many Schurian modules if the algebra is Schurian-finite. From a practical computational point of view, we may produce support τ-tilting modules by left mutations [AIR] starting from the basic projective τ-tilting module. Hence, we may also systematically produce semibricks by Asai's theorem.

Blocks of the Hecke algebra of the symmetric group. The Hecke algebra which is associated with the symmetric group of degree $n \in \mathbb{N}$ and $q \in \mathbb{F}^\times$ is the $\mathbb{F}$-algebra generated by $T_1, \dots, T_{n-1}$ subject to the quadratic relations $(T_i - q)(T_i + 1) = 0$ and the braid relations. The quantum characteristic is defined by $e = \min\{k \in \mathbb{N} \mid 1 + q + \cdots + q^{k-1} = 0\}$. The blocks of the Hecke algebra are labeled by pairs of an e-core partition κ and the weight $w = \frac{n-|\kappa|}{e} \in \mathbb{N}$.

We are interested in wide subcategories of the module category over a block of the Hecke algebra. The first question is whether there are finitely many wide subcategories or not. The following theorem gives a complete answer when $e \geq 3$. The theorem for principal blocks and blocks of weight less than or equal to three is proved in [AS], and the general case is proved in [LS].

Theorem 7. *Assume that the quantum characteristic satisfies* $e \geq 3$. *Then, blocks of Hecke algebras of symmetric groups have the property that it is representation-finite if and only if Schurian-finite if and only if the number of wide subcategories of the category of finite dimensional modules over the block is finite.*

We remark that there are representation-infinite algebras which are Schurian-finite. When $e \geq 3$, a block is representation-finite if and only if its weight is greater than or equal to two. Thus, we prove that blocks of weight greater than or equal to two are Schurian-infinite. The proof requires various results on graded and ungraded decomposition numbers.

References

[AIR] T. Adachi, O. Iyama and I. Reiten, *τ-tilting theory*, Compositio Math. **150** (2014), 415–452.

[AS] S. Ariki and L. Speyer, *Schurian-finiteness of blocks of type A Hecke algebras*, arXiv:2112.11148.

[Asa] S. Asai, *Semibricks*, Int. Math. Res. Not. IMRN (2020), 4993–5054.

[DIJ] L. Demonet, O. Iyama and G. Jasso, *τ-tilting finite algebras, bricks, and g-vectors*, Int. Math. Res. Not. IMRN (2019), 852–892.

[LS] S. Lyle and L. Speyer, *Schurian-finiteness of blocks of type A Hecke algebras II*, arXiv:2208.05711.

[Rin] C. M. Ringel, *Representations of K-species and bimodules*, J. Algebra **41** (1976), 269–302.

On the Endomorphism Algebra of Specht modules

Haralampos Geranios

(joint work with Adam Higgins)

We work in the context of the modular representation theory of the symmetric groups. One of the main objectives in this area is to shed light on the structure of the Specht modules. A standard result of Gordon James states that away from characteristic 2 the endomorphism algebra of the Specht modules is one dimensional and therefore these modules are indecomposable [J_1, Corollaries 13.17, 13.18]. In characteristic 2 the situation changes dramatically: here there exist decomposable Specht modules and this makes their structure much more complicated. The first example of such a module was found by James in the 70s [J_2, Example 1]. More recently, Donkin and I provided large families of decomposable Specht modules and described explicitly their decomposition into indecomposable summands [DG, Theorem 6.2]. An immediate corollary of our result is that there is no upper bound for the number of summands in these decompositions and so the dimension of the corresponding endomorphism algebra can be arbitrarily large, [DG, Example 6.3]. In fact, to date there is no closed formula available for the dimension of the endomorphism algebra of the Specht modules in characteristic 2. In this talk we will describe an effective way for working out the endomorphism algebra of the Specht modules in this characteristic and highlight several interesting applications of our result to the long-standing problem of the decomposability of the Specht modules.

References

[DG] S. Donkin, H. Geranios, *Decompositions of some Specht modules I*, J. Algebra **550**, (2020) 1–22.

[J_1] G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Mathematics, vol. **682**, Springer, NewYork/Heidelberg/Berlin, 1978.

[J_2] G. D. James, *Some Counterexamples in the Theory of Specht Modules*, J. Algebra **46** (1977), 457–461.

Cuspidal ribbon tableaux and skew cyclotomic Hecke algebras

Robert Muth

(joint work with Dina Abbasian, Lena DiFulvio, Thomas Nicewicz, Gabrielle Pasternak, Isabella Sholtes, Frances Sinclair, Liron Speyer, Louise Sutton)

Combinatorics. We fix some $e > 1$ and work with the affine positive root system Φ_+ of type $\mathtt{A}_{e-1}^{(1)}$. Motivated by the representation theory of Khovanov-Lauda-Rouquier (KLR) algebras studied in [5, 7, 9], we introduce the notion of *cuspidal* and *semicuspidal* skew shapes. Let τ be a skew shape such that $\mathrm{cont}(\tau) = m\beta$ for some $m \in \mathbb{N}$ and $\beta \in \Phi_+$. We say that τ is semicuspidal provided that whenever (λ_1, λ_2) is a tableau for τ, we may write $\mathrm{cont}(\lambda_1)$ as a sum of positive roots $\preceq \beta$, and $\mathrm{cont}(\lambda_2)$ as a sum of positive roots $\succeq \beta$. We say that τ is cuspidal provided that $m = 1$ and the comparisons above may be made strict. We describe a complete classification of cuspidal and semicuspidal skew shapes:

Theorem A. *Every cuspidal skew shape is a ribbon. There exists a unique cuspidal ribbon ζ^β of content β for all $\beta \in \Phi_+^{\mathrm{re}}$, and e distinct cuspidal ribbons $\zeta^{\overline{0}}, \dots, \zeta^{\overline{e-1}}$ of content δ.*

Theorem B. *Let $m \in \mathbb{N}$. There exists a unique semicuspidal skew shape $\zeta^{m\beta}$ of content $m\beta$, for all $\beta \in \Phi_+^{\mathrm{re}}$. The set of connected semicuspidal skew shapes of content $m\delta$ is in bijection with $\mathbb{Z}_e \times \mathcal{S}_{\mathsf{c}}(m)$, where $\mathcal{S}_{\mathsf{c}}(m)$ is the set of connected skew shapes of cardinality m.*

Imaginary semicuspidal skew shapes are constructed via an e-dilation process which is in some sense an inversion of the e-quotients defined in [8].

If Λ is a tiling for τ, we say a Λ-tableau $\mathtt{t} = (\lambda_1, \dots, \lambda_{|\Lambda|})$ is *Kostant* if there exist $m_1, \dots, m_k \in \mathbb{N}$, $\beta_1 \succeq \dots \succeq \beta_k \in \Phi_+$ such that $\mathrm{cont}(\lambda_i) = m_i\beta_i$ for all $1 \le i \le |\Lambda|$. We say it is *strict* Kostant if the comparisons above are strict. We say Λ is a *(strict) Kostant* tiling if a (strict) Kostant Λ-tableau exists.

If $\mathrm{cont}(\tau) = \theta$, then a Kostant tiling Λ of τ may naturally be associated with a Kostant partition $\boldsymbol{\kappa}^\Lambda$ of θ. The convex preorder $\succeq$ naturally induces a bilexicographic partial order $\trianglerighteq$ on the set $\Xi(\theta)$ of Kostant partitions of θ. Then we have the following theorem on Kostant tilings:

Theorem C. *Let τ be a nonempty skew shape.*

(1) There exists a unique cuspidal Kostant tiling Γ_τ for τ.
(2) There exists a unique semicuspidal strict Kostant tiling Γ_τ^{sc} for τ.
(3) If Λ is any Kostant tiling for τ, then we have $\boldsymbol{\kappa}^\Lambda \trianglelefteq \boldsymbol{\kappa}^{\Gamma_\tau} = \boldsymbol{\kappa}^{\Gamma_\tau^{sc}}$.

Application to representation theory. The above combinatorial study of cuspidality and Kostant tilings for skew shapes is motivated by a connection to the theory of cuspidal systems and Specht modules over Khovanov-Lauda-Rouquier (KLR) algebras, and has application in that setting.

For any field $\mathbb{F}$ and $\theta \in Q_+$, there is an associated KLR $\mathbb{F}$-algebra R_θ. This family of algebras categorifies the positive part of the quantum group $U_q(\widehat{\mathfrak{sl}}_e)$, see [3, 4, 11]. Associated to any skew shape τ of content θ is a *(skew) Specht R_θ-module* S^τ, as defined in [6, 10]. These Specht modules are key objects in the representation theory of cyclotomic KLR algebras, Hecke algebras and symmetric groups, via the connection between these objects proved in [1].

Under some restrictions on the ground field characteristic (see [7]), the category of finitely generated R_θ-modules is properly stratified, with strata labeled by $\Xi(\theta)$ and with the simple modules labeled $L(\boldsymbol{\kappa}, \boldsymbol{\lambda})$, where $\boldsymbol{\kappa} \in \Xi(\theta)$ and $\boldsymbol{\lambda}$ is an $(e-1)$-multipartition of the coefficient of δ in the Kostant partition $\boldsymbol{\kappa}$. *Cuspidal* and *semicuspidal* R_β-modules associated to every $\beta \in \Phi_+$ are the building blocks of this stratification theory, see [5, 9, 7].

An interesting objective is to find a combinatorial rule connecting the *cellular* structure in cyclotomic KLR algebra representation theory—built on Specht modules with simple modules labeled via multipartitions—to the *stratified* representation theory of the affine KLR algebra—built on cuspidal modules with simple modules labeled via Kostant partitions. Theorems A, B, C describe a rough step in this direction. In particular, Theorems A and B give a complete classification of all cuspidal and semicuspidal Specht modules over KLR algebras, and allows for a presentation of all simple cuspidal and semicuspidal modules associated to real positive roots.

The cuspidal Kostant tiling Γ_τ of Theorem C provides a tight upper bound, in the bilexicographic order on Kostant partitions, for the simple factors which occur in the skew Specht module S^τ:

Theorem D. *Let τ be a skew shape. Then the Specht module S^τ has a simple factor of the form $L(\boldsymbol{\kappa}^{\Gamma_\tau}, \boldsymbol{\lambda})$ for some $\boldsymbol{\lambda}$, and for every simple factor $L(\boldsymbol{\kappa}, \boldsymbol{\mu})$ of S^τ, we have that $\boldsymbol{\kappa}^{\Gamma_\tau} \trianglerighteq \boldsymbol{\kappa}$.*

This result is somewhat analogous to, but distinct from, James' regularization theorem [2]. In the talk I compare and contrast these theorems via some examples in the case $e = 2$.

Skew cyclotomic Hecke algebras. If ρ is the unique Young diagram of a given content and charge, then we say that ρ is a *core*. In the talk we define the quotient algebra $R_\theta^{\Lambda/\rho}$ of R_θ which is the homomorphic image under the composition of maps

$$R_\theta \hookrightarrow R_{\mathrm{cont}(\rho)} \otimes R_\theta \hookrightarrow R_{\mathrm{cont}(\rho)+\theta} \twoheadrightarrow R^\Lambda_{\mathrm{cont}(\rho)+\theta}.$$

We establish the following results:

Theorem E.

(1) *The algebra $R_\theta^{\Lambda/\rho}$ is graded cellular, with cells indexed by skew shapes λ/ρ with* $\mathrm{cont}(\lambda/\rho) = \theta$.

(2) *There is an exact functor*

$$\mathcal{T}_{\rho,\theta} : R^\Lambda_{\mathrm{cont}(\rho)+\theta}\text{-mod} \to R_\theta^{\Lambda/\rho}\text{-mod}$$

which sends S^λ to $S^{\lambda/\rho}$.

Of particular interest to this talk is the case $\theta = d\delta$. In this situation we have:

Theorem F. *Let $R^\Lambda_{\mathrm{cont}(\rho)+d\delta}$ be a RoCK block of weight d with core ρ.*

(1) *We have a Morita equivalence:*

$$\mathcal{T}_{\rho,d\delta} : R^\Lambda_{\mathrm{cont}(\rho)+d\delta}\text{-mod} \to R_{d\delta}^{\Lambda/\rho}\text{-mod}$$

which sends S^λ to $S^{\lambda/\rho}$.

(2) *If λ is e-restricted with has e-quotient $(\mu^{(1)}, \dots, \mu^{(e-1)}, \varnothing)$, then:*

$$\mathcal{T}_{\rho,d\delta} D^\lambda = \mathcal{T}_{\rho,d\delta}(\mathrm{head}(S^\lambda)) \cong \mathrm{head}(S^{\lambda/\rho}) \cong L(\boldsymbol{\mu}),$$

where $L(\boldsymbol{\mu})$ is the simple semicuspidal module corresponding to the $(e-1)$-*multipartition $\boldsymbol{\mu}$.*

The above theorem allows us to give a concrete presentation of all imaginary simple semicuspidal modules as heads of explicit skew Specht modules. Moreover, as decomposition formulas for RoCK blocks are well established, these may be used to explicitly compute characters of the imaginary simple semicuspidal modules in many cases.

Theorems A–D are joint work with Abbasian, DiFulvio, Pasternak, Sholtes, and Sinclair. Theorems E and F are joint work with Nicewicz, Speyer, and Sutton.

References

[1] J. Brundan and A. Kleshchev, Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras, *Invent. Math.* **178** (2009), 451–484.

[2] G. James, On the decomposition matrices of the symmetric groups II, *J. Algebra* **43** (1976), 45–54.

[3] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, *Represent. Theory* **13** (2009), 309–347.

[4] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups II, *Trans. Amer. Math. Soc.* **363** (2011), 2685–2700.

[5] A. Kleshchev, Cuspidal systems for affine Khovanov-Lauda-Rouquier algebras, *Math. Z.* **279**(3-4) (2014), 691–726.

[6] A. S. Kleshchev, A. Mathas and A. Ram, Universal graded Specht modules for cyclotomic Hecke algebras, *Proc. Lond. Math. Soc.* (3) **105** (2012), no. 6, 1245–1289.

[7] A. Kleshchev, R. Muth, Stratifying KLR algebras of affine ADE types, *J. Algebra* **475** (2017), 133–170.

[8] D. E. Littlewood, Modular Representations of Symmetric Groups. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **209**(1098) (1951), 333–353.

[9] P. McNamara, Representations of Khovanov-Lauda-Rouquier algebras III: symmetric affine type, `arXiv:1407.7304v2`.

[10] R. Muth, Graded Skew Specht Modules and Cuspidal Modules for Khovanov-Lauda-Rouquier Algebras of Affine Type A. *Algebr Represent Theor* **22** (2019), 977–1015.

[11] R. Rouquier, 2-Kac-Moody algebras; `arXiv:0812.5023`.

[12] S. Sheffield. Ribbon tilings and multidimensional height functions. *Trans. Amer. Math. Soc.*, **354**(12) (2002), 4789–4813.

[13] P. Tingley and B. Webster, Mirković-Vilonen polytopes and Khovanov-Lauda-Rouquier algebras, `arXiv:1210.6921`.

Quantum group perspective on the hypercube decomposition for Kazhdan–Lusztig polynomials

Max Gurevich

(joint work with Chuijia Wang)

A recent collaboration of Williamson with Google's DeepMind researchers [1] has produced a new inductive method for the construction of symmetric group Kazhdan–Lusztig polynomials. Their proof detects a combinatorial phenomenon, hypercube decomposition, which is then proved through analysis of perverse sheaves on Schubert varieties.

The resulting algorithm is conceptually novel in its reliance on the rank of the group as the recursion parameter rather than the length of a single permutation. Conjecturally, this algorithm is flexible enough to be independent of choices made based on input outside of the Bruhat graph of the group. This would imply the long-standing combinatorial invariance conjecture, in the case of S_n.

My talk presented a seemingly different algebraic approach for this decomposition. The geometric links between quiver representations and Schubert varieties in Lie type A allow us to place the Kazhdan–Lusztig theory in the context of canonical bases for quantum groups. The invariants of interest now appear as coefficients in transition matrices between much-studied bases of a non-commutative algebra. Since the rank of the corresponding permutation group designates the length of a product of algebra generators in the quantum group, its appearance as an induction parameter is most natural in this setting.

With Wang we were able to construct [2] an alternative proof for the hypercube decomposition. A motivating principle for it is the well known categorification of $U_q(\mathfrak{sl}_N)^+$ by quiver Hecke algebras. In this sense, the q-derived Kazhdan–Lusztig polynomials count the difference of graded multiplicities of simple constituents between proper standard and proper co-standard modules. The natural braid decompositions of the standard intertwiner operators between the standard and co-standard classes give rise to the algebraic interpretations of the hypercube decomposition. The Verdier duality inherent in the geometric arguments for the original proof, takes a new combinatorial form an involutive symmetry of the relevant categories of finite-dimensional graded modules.

References

[1] Charles Blundell, Lars Buesing, Alex Davies, Petar Veličković, and Geordie Williamson. Towards combinatorial invariance for Kazhdan–Lusztig polynomials. *arXiv preprint arXiv:2111.15161*, 2021.

[2] Maxim Gurevich, and Chuijia Wang. *in preparation*, 2022.

Characters of diagram algebras and symmetric functions

Mike Zabrocki

(joint work with Laura Colmenarejo, Rosa Orellana, Franco Saliola, Anne Schilling)

In his 1901 thesis, Schur identified a duality between the general linear group and symmetric group by considering the bi-action of $Gl_n(\mathbb{C}) \times S_k$ acting on the k-fold tensor of an n-dimensional vector space $V_n = span\{v_1, v_2, \ldots, v_n\}$. He showed that this implies

$$V_n^{\otimes k} \cong \bigoplus_{\lambda \vdash k} V_{Gl_n}^{\lambda} \otimes W_{S_k}^{\lambda} \tag{1}$$

where $V_{Gl_n}^{\lambda}$ is an irreducible polynomial Gl_n module and $W_{S_k}^{\lambda}$ is an irreducible S_k module.

Let A be an element of Gl_n with eigenvalues $x_1, x_2, \ldots, x_n$. The character of the action of $A \in Gl_n$ on a polynomial irreducible module $V_{Gl_n}^{\lambda}$ is equal to the Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_n)$ evaluated at these eigenvalues. For an element σ in S_k with cycle type μ, denote the character of of the action of σ on $W_{S_k}^{\lambda}$ by $\chi_{S_k}^{\lambda}(\mu)$.

Now let A be a diagonal matrix with entries $x_1, x_2, \ldots, x_n$ and take $(A, \sigma) \in Gl_n(\mathbb{C}) \times S_k$. The trace of the action of (A, σ) on a basis $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ can be computed in two different ways. On the one hand, the action on a single element is

$$(A, \sigma)(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = (x_{i_1} x_{i_2} \cdots x_{i_k}) v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(k)}}$$

and so if the cycle type of σ is again μ, then the trace of this action is (by combining it with Equation (1))

$$p_\mu(x_1, x_2, \ldots, x_n) = \sum_{\lambda \vdash k} \chi^\lambda_{S_k}(\mu) s_\lambda(x_1, x_2, \ldots, x_n)$$

where p_μ denotes the power sum element of the symmetric functions indexed by μ. With n sufficiently large we can ignore the variables and the values of the characters of the symmetric group $\chi^\lambda_{S_k}(\mu)$ are the coefficient of s_λ in the symmetric function p_μ.

In 1937 Brauer [2] introduced an algebra that extends Schur-Weyl duality to an action of the orthogonal group. If we denote Brauer's algebra $B_k(n)$ as the diagram algebra then the irreducible representations of $B_k(n)$ are indexed by partitions of size $k - 2r$ for $0 \leq r \leq \lfloor k/2 \rfloor$. We will again assume that n is 'sufficiently large' and denote the irreducible character indexed by a partition λ acting by an element of the Brauer algebra of type μ by $\chi^\lambda_{B_k(n)}(\mu)$.

In 1939, Weyl published [14] his treaty on the representation theory of the classical groups. The Weyl character formula from that reference gives a formula for the irreducible polynomial characters of the classical groups and Koike-Terada [7] formalized those elements as a basis of the symmetric functions, o_λ. The evaluation of these functions at the eigenvalues of an element of the orthogonal group is the value of the character of the irreducible module indexed by the partition λ. A similar calculation (see [11]) shows that

$$p_\mu = \sum_{r=0}^{\lfloor k/2 \rfloor} \sum_{\lambda \vdash k-2r} \chi^\lambda_{B_k(n)}(\mu) o_\lambda \ .$$

We thus far have two examples where the coefficients of one basis of symmetric functions in another are the values of the irreducible characters of a diagram algebra.

In this talk I will present several other such examples. In 2016, Rosa Orellana and I [12] introduced a basis of the symmetric functions that we called the irreducible symmetric group character basis, $\tilde{s}_\lambda$. Evaluating these symmetric functions at the eigenvalues of permutation matrices gives the irreducible characters of the symmetric groups similarly to the way that the Schur functions give the character values of the general linear groups. The algebra that is Schur-Weyl dual to the symmetric group as permutation matrices is known as the partition algebra and was introduced in the early 90s independently by Jones and Martin [6, 9].

In 2020, Assaf and Speyer [1] introduced an intermediate set of elements in the Grothendeick ring that Orellana and I [13] identified as another basis, $\tilde{x}_\lambda$, of the symmetric functions. We recently found that the evaluations of these symmetric functions are the characters of the rook monoid. The four bases s_λ, o_λ, $\tilde{s}_\lambda$, and $\tilde{x}_\lambda$ combined with various expressions of symmetric functions allow us to calculate the characters of various diagram algebras: the quasi-partition algebra of Daugherty and Orellana [3], the propagating partition algebra (also known as the dual symmetric inverse monoid algebra) [8, 10], partition algebras indexed by half integers [5] and the rook-Brauer algebra [4].

References

[1] S. Assaf, D. Speyer, *Specht modules decompose as alternating sums of restrictions of Schur modules*, Proc. Amer. Math. Soc., 148, (2020) pp. 1015–1029.

[2] R. Brauer, *On Algebras Which are Connected with the Semisimple Continuous Groups*, Annals of Mathematics, 38 (4) (1937) pp. 857–872.

[3] Z. Daugherty, R. Orellana, *The quasi-partition algebra*, Journal of Algebra. 404, (2014). p. 124–151.

[4] T. Halverson, E. delMas *Representations of the Rook-Brauer Algebra*, Communications in Algebra. 42 (1) (2014) pp. 423–443

[5] T. Halverson, A. Ram, *Partition algebras*, European Journal of Combinatorics, Volume 26, Issue 6, (2005) pp. 869–921.

[6] V. F. R. Jones, *The Potts model and the symmetric group*, in Subfactors (Kyuzeso, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 259–267.

[7] K. Koike, I. Terada, *Young-Diagrammatic Methods for the Representation Theory of the Classical Groups of Type B_n, C_n and D_n*, J. Algebra 107 (1987) pp. 466–511.

[8] V. Maltcev, *On a new approach to the dual symmetric inverse monoid I_X^**. arXiv:math/0703478v1.

[9] Paul Martin, *Potts models and related problems in statistical mechanics*, Series on Advances in Statistical Mechanics, vol. 5, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

[10] A. Mishra, S. Srivastava, *Jucys-Murphy elements of partition algebras for the rook monoid*, International Journal of Algebra and Computation. 31 (5) (2021) pp. 831–864.

[11] A. Ram, *Characters of Brauer's centralizer algebras*, Pac. J. of Math. Vol. 169, No. 1, (1995) pp. 173–200.

[12] R. Orellana, M. Zabrocki, *Products of symmetric group characters*, J. of Comb. Th., Series A, Volume 165 (2019) pp. 299–324.

[13] R. Orellana, M. Zabrocki, *The Hopf structure of symmetric group characters as symmetric functions*, Alg. Comb, Volume 4, no. 3 (2021) pp. 551–574.

[14] H. Weyl, *The Classical Groups. Their Invariants and Representations.* Princeton University Press, Princeton, N.J., 1939.

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