

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 27/2022

DOI: 10.4171/OWR/2022/27

## Universality: Random Matrices, Random Geometry and SPDEs

Organized by  
Martin Hairer, London  
Grégory Miermont, Lyon  
Horng-Tzer Yau, Cambridge MA

29 May – 4 June 2022

ABSTRACT. The postulate that large random systems can be described by limiting objects whose characteristic do not depend on the exact details of the models one started from is central in probability theory, under the name of universality. This workshop was aimed at uncovering the latest developments of this concept in the various topics where it is relevant, namely statistical physics, stochastic partial differential equations, random geometries and random matrices. It was in particular the occasion to feature some important recently introduced universal objects like the stochastic quantization of the Yang-Mills measure in dimensions 2 and 3, the KPZ fixed point, Liouville quantum gravity metrics and other objects connected to the Gaussian free field.

*Mathematics Subject Classification (2020)*: Primary: 60-XX; Secondary: 81-XX, 83-XX.

### Introduction by the Organizers

The workshop *Universality: Random Matrices, Random Geometry and SPDEs*, organised by Martin Hairer (London), Grégory Miermont (Lyon) and Horng-Tzer Yau (Cambridge, MA), was initially planned to happen in 2020 but had to be postponed due to the Covid pandemic. Faithful to a long tradition of workshops around the general theme of *stochastic analysis*, it aimed a presenting a panoramic view of modern probability theory and interacting topics. It focused on the important concept of *universality*, according to which a wide variety of large random systems are attracted to a small number of limiting random objects, which are often prone to explicit calculations and allow to shed light on the systems of interest.

The week started with a thematic morning on the Yang-Mills model. Ajay Chandra and Ilya Chevyrev reported their recent work with Hairer and Shen on the stochastic quantization of the Yang-Mills measure in dimensions 2 and 3. Among the highlights of this work are the fact that the renormalization of the stochastic Yang-Mills heat flow can be made compatible with the action of the gauge group, and the construction of a Markov process in the gauge orbit space onto which the stochastic Yang-Mills equation projects. Hao Shen closed this morning by discussing his work with Smith, Zhu and Zhu on the dynamical lattice Yang-Mills measure, allowing to obtain properties of the lattice Yang-Mills measure via spectral and geometric techniques from Markov process theory. The Monday afternoon talks focused on statistical physics. Christophe Garban presented his ongoing work with Aru and Sepulveda on vector-valued Gaussian free field and their connection with planar  $O(n)$  models, showing how this may help understand the absence of phase transition in  $O(3)$  spin models predicted by Polyakov via a connection with XY models. Alain Sznitman presented his work on the fine analysis of the spectral gap in the Kac-Luttinger model of a Bose gas interacting with a Poisson cloud of hard obstacles and its consequences for the density of states of the model, and on Bose-Einstein condensation above a critical density. Antti Knowles discussed his work with Fröhlich, Schlein and Sohinger on interacting Bose gas converging to a Euclidean  $\Phi_2^4$  theory, by making rigorous a number of formal field-theoretic integral representation in terms of models of Brownian bridges.

On Tuesday, we focused on the various guises of the KPZ universality class. Alessandra Occelli described the integrable aspects of new last-passage percolation growth models in  $\mathbb{Z}^2$ . Jeremy Quastel showed how Skorokhod renormalization allows to give Feynman-Kac representations of solutions of the stochastic heat equation, from which KPZ fluctuations can be easily deduced. Balint Virag gave an overview of the *directed landscape*, a universal object of crucial importance as it is one way to represent the universal *KPZ fixed point* as a random directed distance function on the plane. Amol Aggarwal presented his recent work with Gorin and Huang on the universality of lozenge tiling statistics, another model in the KPZ class, showing convergence to GUE corner process or Airy line ensembles for a wide class of tileable domains. In a related spirit, Benjamin Landon discussed his work with Noack and Sosoe on interacting diffusions via a class of potentials for which KPZ-type fluctuations can be observed, hinting again at the universality of these models. We closed this day with the different topic of random matrices. Tatyana Shcherbina, who gave the only remote talk of the conference, presented her work with Mariya Shcherbina on extreme singular values for deformed Ginibre ensembles via supersymmetry techniques, generalizing recent results of Cipolloni, Erdős and Schröder.

Wednesday started with a talk by Nicolas Curien on the parking model on the infinite binary tree, joint with Aldous, Contat and Hénard, showing a surprisingly general criterion for (sub)criticality of the model, and interesting connections with Tutte's quadratic equations typical of the theory of enumeration of maps. Ofer Zeitouni discussed directed polymers in weak disorder. In joint work with Cosco,

he shows high moment bounds for the partition function, as a first step towards the goal of showing convergence to Gaussian multiplicative chaos. Ewain Gwynne closed this morning session with his work with Ding on supercritical Liouville quantum gravity metrics, a family of random metrics on the complex plane that complete the subcritical LQG metrics. The afternoon was the opportunity to take a walk through the forest to Oberwolfach.

On Thursday, Scott Sheffield discussed an ongoing project with Chandgotia and Wolfram on the three-dimensional dimer model and the generalization of arctic curves phenomena supported by simulations. In a joint talk, Alice Guionnet and Raphaël Ducatez presented their recent results with Augeri and Cook dealing with large deviation principles for the edge of the spectrum of random Wigner matrices with sub-Gaussian entries, based on the analysis of spherical integrals and showing a delicate modification of the usual LDP functional in the form of a phase transition in the sub-Gaussianity constant. Giuseppe Cannizzaro then presented his work with Haunschmidt and Toninelli on the singular stochastic differential equation for a Brownian motion drifted by the curl of a GFF, showing the Tóth–Valkó conjecture that this process is superdiffusive with logarithmic divergence away from diffusivity. In the afternoon, Pierre-François Rodriguez presented exciting results with Drewitz and Prévost on percolation models with long-range dependence derived from GFF level lines on  $\mathbb{Z}^d$ . Jean-François Le Gall presented an intriguing property for the local times of Brownian motion indexed by the Brownian tree, showing that the joint process of local times and its derivative forms a Markov process with explicit transition probabilities. Finally, Jason Miller showed that chemical distances inherited from conformal loop ensembles rescale to a new family of random geodesic metrics in the plane, that are yet different from the LQG metrics he characterized with Gwynne.

Friday morning continued with random planar metrics with the talk by Ahmed Bou-Rabee, who showed in joint work with Gwynne how to construct a notion of harmonic balls for the LQG measures, i.e. sets  $E$  satisfying that averages of harmonic functions  $f$  on  $E$  have average  $f(0)$ , and motivated by the study of internal DLA in random maps. He showed how weak Harnack inequalities allow to uniquely construct a family of such balls, which in turn are the limits of DLA on Tutte-embedded mated-CRT maps. Dmitry Beliaev surveyed his recent joint work with McAuley and Muirhead on the number of level sets of smooth Gaussian field and their asymptotics. Florian Schweiger, in joint work with Zeitouni, extended the class of extreme theory for log-correlated fields by studying the case of log-correlated fields in random environment. To wrap up in a cyclic way, Massimiliano Gubinelli closed this conference by presenting his views on stochastic quantization in quantum field theory.

The workshop was well-attended, with about 35 participants on-site. About 20 participants registered online.

*Acknowledgement:* The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, “US Junior Oberwolfach Fellows”.



## Workshop: Universality: Random Matrices, Random Geometry and SPDEs

### Table of Contents

Dmitry Beliaev (joint with Michael McAuley, Stephen Muirhead)	
<i>On the number of level sets of smooth Gaussian fields</i> .....	1507
Ahmed Bou-Rabee (joint with Ewain Gwynne)	
<i>Harmonic balls in Liouville Quantum Gravity</i> .....	1509
Giuseppe Cannizzaro (joint with L. Haunschild, F. Toninelli)	
<i>Diffusion in the curl of the two-dimensional Gaussian Free Field</i> .....	1510
Ilya Chevyrev (joint with Ajay Chandra, Martin Hairer, Hao Shen)	
<i>Stochastic quantisation of Yang–Mills: state space and Markov process</i> ..	1513
Nicolas Curien (joint with David Aldous, Alice Contat and Olivier Hénard)	
<i>Parking on the infinite binary tree</i> .....	1515
Raphaël Ducatez, Alice Guionnet (joint with Nicholas Cook)	
<i>Large deviations for the spectrum of random matrices</i> .....	1516
Christophe Garban (joint with Juhan Aru, Avelio Sepúlveda)	
<i>Exit sets for vector valued GFF and interplay with planar <math>O(N)</math> models</i>	1518
Massimiliano Gubinelli	
<i>What is stochastic quantisation?</i> .....	1519
Ewain Gwynne (joint with Jian Ding)	
<i>The critical and supercritical Liouville quantum gravity metrics</i> .....	1522
Antti Knowles (joint with Jürg Fröhlich, Benjamin Schlein, Vedran Sohinger)	
<i>The Euclidean <math>\phi_2^4</math> theory as a limit of an interacting Bose gas</i> .....	1523
Benjamin Landon (joint with Christian Noack, Philippe Sosoe)	
<i>KPZ-type fluctuation exponents for interacting diffusions in equilibrium</i>	1526
Jean-François Le Gall	
<i>On the Markov property of local times of Brownian motion indexed by the Brownian tree</i> .....	1527
Alessandra Occelli	
<i>Stationarity and space-time covariance for KPZ growth models in half-space</i> .....	1528
Jeremy Quastel (joint with Alejandro Ramirez, Balint Virag)	
<i>KPZ fluctuations in the planar stochastic heat equation</i> .....	1530

---

Pierre-François Rodriguez (joint with Subhajit Goswami, Franco Severo, Alexander Drewitz, Alexis Prévost)	
<i>Critical exponents for three-dimensional percolation models with long-range dependence</i> .....	1533
Florian Schweiger (joint with Ofer Zeitouni)	
<i>The maximum of log-correlated Gaussian fields in random environments</i>	1535
Scott Sheffield (joint with Nishant Chandgotia, Catherine Wolfram)	
<i>Dimer model in 3D</i> .....	1537
Tatyana Shcherbina (joint with Mariya Shcherbina)	
<i>The least singular value of the deformed Ginibre ensemble</i> .....	1537
Hao Shen	
<i>A dynamical approach to lattice Yang-Mills</i> .....	1539
Alain-Sol Sznitman	
<i>On the spectral gap in the Kac-Luttinger model and Bose- Einstein condensation</i> .....	1542
Ofer Zeitouni (joint with Clement Cosco)	
<i>Directed polymers in the weak disorder regime, and exponential moments</i>	1543

## Abstracts

### On the number of level sets of smooth Gaussian fields

DMITRY BELIAEV

(joint work with Michael McAuley, Stephen Muirhead)

In a series of papers we have studied the number of level and excursion sets of smooth Gaussian fields. Throughout this text we assume that  $F(x)$  is a stationary Gaussian field defined in  $\mathbb{R}^n$ . We also assume that with probability one  $F$  is sufficiently smooth, we always require it to be at least  $C^2$ , but for some results we require more derivatives. In reality, all examples that we really care about are real analytic functions, so the smoothness assumption is not too restrictive.

The number of level sets could be thought of as a higher dimensional version of the number of level crossings of Gaussian processes. Indeed, in one-dimensional case the level set  $\{F(x) = \ell\}$  is just a collection of point and excursion sets  $\{F(x) \geq \ell\}$  are intervals between them. In higher dimensional case, level sets i.e. connected components of  $\{F(x) = \ell\}$  have non-trivial geometry and topology, so the question is more involved. In particular, the number of level sets inside a domain is not an additive function of a domain, which makes things even more complicated and makes many one-dimensional tools inapplicable.

The other reason for looking at the number of level sets is the connection to the percolation theory and works of physicists who have related the statistical behaviour of the number of level sets with quantum chaos questions.

The starting point in the investigation of the number of level/excursion sets is the breakthrough paper [1] where Nazarov and Sodin obtained the Law of Large Numbers for the number of nodal domains for spherical random harmonics. Later they extended their result to more general setting [2]. Although both papers are written in the case of nodal domains, the argument works verbatim for other levels. Roughly speaking, they have shown that under some mild non-degeneracy and regularity assumptions

$$\frac{N_{LS,ES}(R, \ell)}{R^n} \rightarrow c_{LS,ES}(\ell)$$

where  $N(R, \ell)$  is the number of either level sets ( $LS$ ) or excursion sets ( $ES$ ) of the field  $F$  inside the cube of side-length  $R$ .

In papers [4] and [5] we have studied how  $c(\ell)$  depends on the level  $\ell$  in the planar case. Usual Morse theory arguments give that the level set changes smoothly as long as  $\ell$  does not cross a critical level. There are also some boundary effects that we ignore for now. When we cross the level of a local maximum, then one component of the level set and one component of the excursion set disappear. When we cross the level of a local minimum, then one level set appears, but the number of excursion sets does not change. The behaviour at saddles is a bit more complicated, there are two types of saddles (we call them positively and negatively connected saddles). In the first case one excursion set is split into two and one level set is split into two (a ‘barbell’ domain is split into two ‘disc’ domains), in

the other case, the number of excursion sets does not change, and two level sets merge into one ( a ‘ring’ domain turns into a ‘horseshoe’ domain). We have shown that under mild regularity assumptions and assuming that the covariance kernel decays sufficiently fast all non-degenerate critical points can be split into these four categories and each type has a continuous density. We show that

$$\begin{aligned} c'_{ES}(\ell) &= p_{m^+}(\ell) - p_{s^-}(\ell) \\ c'_{LS}(\ell) &= p_{m^+}(\ell) - p_{s^-}(\ell) + p_{s^+}(\ell) - p_{m^-}(\ell), \end{aligned}$$

where  $p_{m^+}, p_{m^-}, p_{s^+}, p_{s^-}$  are densities of local maxima, minima, positive and negative connected saddles at level  $\ell$ .

Using similar ideas, we have obtained a lower bound on the variance of  $N(R, \ell)$ . We have shown in [6] that if  $F$  satisfies similar non-degeneracy and regularity assumptions, the Fourier transform of the covariance kernel (so called spectral measure) is bounded and bounded from zero in some neighbourhood of the origin, there is a certain estimate of ‘4-arm saddles’ and  $c'(\ell) \neq 0$ , then the variance of  $N(R, \ell)$  is of order at least  $R^2$ . It is not immediately clear how to verify all assumptions of the theorem, but it has been shown that the required ‘4-arm’ estimate hold for all fields with non-negative and sufficiently fast decaying correlation. It is believed to be true for a wide class of fields. Finally, it is believed that  $c'(\ell) \neq 0$  for all but a few exceptional levels. This is known for many fields and levels. Notably, by symmetry  $c'_{LS}(0) = 0$ , so our theorem is not applicable to nodal level sets. It is interesting to compare this result to a complementary result of Nazarov and Sodin. In [3] they have shown that under extremely mild conditions the variance of the number of nodal sets grows at least as  $R^\delta$  for some  $\delta > 0$ .

We also want to point out that our result is not applicable to a very important case of the random plane wave. Using slightly different methods we can show that in this case if  $\ell \neq 0$  and  $c'(\ell) \neq 0$  then the variance is of order at least  $R^3$ .

Finally, in [7] we have shown that for a field in  $\mathbb{R}^n$  which satisfies rather strong assumption on the decay of correlations and smoothness there is a Central Limit Theorem. Namely  $(N(R, \ell) - \mathbb{E}N(R, \ell))/R^{n/2}$  converges to the normal variable with variance  $\sigma(\ell)$ . This result holds both for level and excursion sets. Assuming additionally that the integral of the covariance kernel is strictly positive, we can show that  $\sigma > 0$  for all levels  $\ell$ .

## REFERENCES

- [1] F. Nazarov and M. Sodin. *On the number of nodal domains of random spherical harmonics*. American Journal of Mathematics 131.5 (2009): 1337–1357.
- [2] F. Nazarov and M. Sodin. *Asymptotic Laws for the Spatial Distribution and the Number of Connected Components of Zero Sets of Gaussian Random Functions*. Journal of Mathematical Physics, Analysis, Geometry 2016, **12**, No 3, pp. 205–278
- [3] F. Nazarov and M. Sodin. *Fluctuations in the number of nodal domains*. J. Math.Phys. **61** pp. 123302 (2020)
- [4] D. Beliaev M. McAuley S. Muirhead. *On the number of excursion sets of planar Gaussian fields*. Probab. Theory Relat. Fields **178**, 655–698 (2020).
- [5] D. Beliaev M. McAuley S. Muirhead. *Smoothness and monotonicity of the excursion set density of planar Gaussian fields*. Electron. J. Probab. **25**, 1–37, 2020.

- [6] D. Beliaev M. McAuley S. Muirhead. *Fluctuations of the number of excursion sets of planar Gaussian fields*. Probability and Mathematical Physics **3**, no. 1 (2022): 105–144.
- [7] D. Beliaev M. McAuley S. Muirhead. *A central limit theorem for the number of excursion set components of Gaussian fields*. arXiv preprint arXiv:2205.09085 (2022).

## Harmonic balls in Liouville Quantum Gravity

AHMED BOU-RABEE

(joint work with Ewain Gwynne)

Let  $\mu$  be a Radon measure on  $\mathbb{C}$ . A *harmonic ball* is an open set  $E \subset \mathbb{C}$  containing the origin which satisfies the mean-value property

$$(1) \quad f(0) = \frac{1}{\mu(E)} \int_E f(z)\mu(dz),$$

for all functions  $f : \mathbb{C} \rightarrow \mathbb{R}$  which are harmonic in  $E$ . We construct and analyze harmonic balls when  $\mu$  is the *Liouville Quantum Gravity measure* — a Radon measure informally given by

$$(2) \quad \mu_h(dz) = e^{\gamma h(z)} dz$$

where  $h$  is a Gaussian free field and  $\gamma \in (0, 2)$  is a constant.

Specifically, we construct a family of harmonic balls  $\{\Lambda_t\}_{t>0}$  satisfying the following properties:

- $\mu_h(\Lambda_t) = t$  and  $\mu_h(\partial\Lambda_t) = 0$  for each  $t$
- $\{\Lambda_t\}_{t>0}$  is continuous and monotone in  $t$ .

We show that this family is essentially unique; that is, any other harmonic balls  $\{\Lambda'_t\}_{t>0}$  satisfying the above properties coincide with  $\{\Lambda_t\}_{t>0}$  up to a set of  $\mu_h$ -measure zero. We also show that  $\Lambda_t$  is neither convex nor an LQG-metric ball for Lebesgue almost every  $t$ .

A key tool in proving these results is (a minor variation of) the following weak Harnack-type estimate:

$$(3) \quad \mu_h(\Lambda_t \cap B_r(x_0)) \leq \alpha \mu_h(B_r(x_0)) \implies \Lambda_t \cap B_{r/2}(x_0) = \emptyset,$$

where  $B_r(x_0)$  is the Euclidean ball of radius  $r$  centered at  $x_0$ ,  $t > 0$  is arbitrary, and  $\alpha \in (0, 1)$  is small. The estimate (3) may be thought of as a continuum analogue of the no-thin-tentacles lemma of Jerison-Levine-Sheffield (2012).

Our main motivation for studying harmonic balls is that they are the conjectured scaling limit of internal diffusion limited aggregation on random planar maps.

## Diffusion in the curl of the two-dimensional Gaussian Free Field

GIUSEPPE CANNIZZARO

(joint work with L. Haunschid, F. Toninelli)

In the present talk, we study the motion of a Brownian particle in  $\mathbb{R}^2$ , subject to a random, time-independent drift  $\omega$  given by the curl of the two-dimensional Gaussian Free Field (2d GFF). Namely, we look at the SDE which is (formally) given by

$$(1) \quad dX(t) = \omega(X(t))dt + dB(t),$$

where  $B(t)$  is a standard two-dimensional Brownian motion and the two-dimensional vector field  $\omega$  is defined as

$$(2) \quad x = (x_1, x_2) \mapsto \omega(x) = (\partial_{x_2}\xi(x), -\partial_{x_1}\xi(x)),$$

with  $\xi$  the 2d GFF. As written, (1) is ill-posed due to the singularity of the drift  $\omega$ . In fact, not only classical stochastic analytical tools would fail in characterising (even) its law but it would also be *critical* for the recent techniques established in [4] as its spatial regularity is way below the threshold identified therein. Nevertheless, we are interested in its large time behaviour and hence we regularise  $\xi$  by convolving it with a  $C^\infty$  bump function, so that  $\omega$  is well-defined pointwise and smooth. Note that the vector field  $\omega$  is everywhere orthogonal to the gradient of the field  $\xi$ , and therefore parallel to its level lines. As a consequence, the particle is subject to two very different mechanisms: the drift tends to push the motion *along the level lines of the GFF* (which are finite as we smoothed  $\xi$ , but have long tails), while the Brownian noise tends to make it diffuse *isotropically*. Our main theorem is a *sharp* superdiffusivity result: the mean square displacement  $\mathbf{E}[|X(t)|^2]$ , under the joint law of the Brownian noise and of the random drift, is of order  $t\sqrt{\log t}$  for  $t \rightarrow \infty$ , up to multiplicative loglog corrections. This proves a conjecture of B. Tóth and B. Valkó [13] and, in a broader perspective, it is the first proof of the  *$\sqrt{\log t}$ -superdiffusivity phenomenon* conjectured to occur in a large class of (self-)interacting diffusive systems in dimension  $d = 2$ , including self-repelling random walks and polymers [2, 13], lattice gas models [1, 10], and, more recently, the two-dimensional Anisotropic KPZ equation (2d AKPZ) [5].

To put the model and the result into context, let us observe first that the vector field  $\omega$  is divergence-free and that its law is translation-invariant and ergodic. Brownian diffusions in ergodic, divergence-free vector fields have been introduced in the Physics and Mathematics literature as a (toy) model for a tracer particle evolving in an incompressible turbulent flow. If the energy spectrum of the vector field (i.e. the Fourier transform  $e(p)$  of the trace of the covariance matrix  $R(x - y) = \{\mathbb{E}(\omega_a(x)\omega_b(y))\}_{a,b \leq d}$ , with  $d$  the space dimension) satisfies the integrability condition

$$(3) \quad \int_{\mathbb{R}^d} \frac{e(p)}{|p|^2} dp < \infty,$$

the behaviour of the particle is known to be diffusive on large scales [7, 12]. In the robustly superdiffusive case, where the integral in (3) has a power-law divergence

for small  $p$ , it turns out that  $\mathbf{E}[|X(t)|^2]$  grows like  $t^\nu$  for some  $\nu > 1$  [8]. The case under consideration in this work instead, where  $d = 2$  and  $\omega$  is the curl of the GFF, is precisely at the boundary between the two -  $e(p)$  is essentially constant for  $p$  small, the integral (3) diverges logarithmically at small momenta and logarithmic corrections to diffusivity are expected.

In order to state the main result, we need some notations. Let  $U: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a smooth function in  $C^\infty(\mathbb{R}^2)$ , radially symmetric, decaying sufficiently fast at infinity (say, exponentially) and such that  $\int_{\mathbb{R}^2} U(x)dx = 1$ , and set  $\tilde{\xi} := \xi * U$ , where  $\xi$  is a Gaussian free field on  $\mathbb{R}^2$ . Let  $X$  be the solution to (1) but with  $\omega$  defined according to (2) but with  $\tilde{\xi}$  replacing the  $\xi$  therein. For any  $\lambda > 0$ , we define the Laplace transform of the (bulk) diffusion coefficient of  $X$  according to

$$(4) \quad D(\lambda) := \int_0^\infty e^{-\lambda t} \mathbf{E}(|X(t)|^2) dt.$$

Then, we have the following theorem.

**Theorem 1.** [6, Theorem 2.2] *For every  $\varepsilon > 0$  there exists constants  $C_\pm(\varepsilon)$  such that, for every  $0 < \lambda < 1$ ,*

$$(5) \quad C_-(\varepsilon)(\log |\log \lambda|)^{-1-\varepsilon} \leq \lambda^2 \frac{D(\lambda)}{\sqrt{|\log \lambda|}} \leq C_+(\varepsilon)(\log |\log \lambda|)^{1+\varepsilon}.$$

where  $D$  is defined according to (4).

**Remark 1.** *By a well-established argument (see [11]) the upper bound in (5) implies an upper bound for the diffusivity in real time of the form*

$$\mathbf{E}(|X(t)|^2) \leq O\left(t\sqrt{\log t} (\log \log t)^{1+\varepsilon}\right).$$

*Deducing a pointwise (in time) lower bound on  $\mathbf{E}(|X(t)|^2)$  from the behaviour for  $\lambda \rightarrow 0$  of the Laplace transform is much more delicate. That said, one can easily get (applying for instance [3, Theorem 1.7.1]) the following*

$$\limsup_{t \rightarrow \infty} \frac{\mathbf{E}(|X(t)|^2)}{t\sqrt{\log t} (\log \log t)^{-1-\varepsilon}} > 0.$$

The argument we exploit in order to prove the theorem above is based on an iterative analysis of the resolvent of the generator of the Markov process given by the environment seen from the particle, i.e. the process  $t \mapsto \omega_t(\cdot) := \omega(X_t + \cdot)$ . The method is inspired by that introduced in [9] and later employed by H.-T. Yau [14] to prove  $(\log t)^{2/3}$  corrections to the diffusivity of the two-dimensional Asymmetric Simple Exclusion Process ( $2d$  ASEP) and, more closely, by the techniques developed in [5] to determine logarithmically superdiffusive behaviour for the  $2d$  AKPZ equation. Note that the exponent  $2/3$  of the logarithmic corrections of  $2d$  ASEP is different from the exponent  $1/2$  in the present setting, reflecting the fact that the two models belong to two different universality classes, as emphasized already in [10, 13]. From a technical point of view, a crucial difference between the two models is that for  $2d$  ASEP the iterative method in [14] provides, at each step  $k$  of the recursion, upper/lower bounds for  $D(t)$  of the form  $(\log t)^{\nu_k}$ , with

$\nu_k$  converging exponentially fast to  $2/3$  as  $k \rightarrow \infty$ . In our case, on the other hand, at step  $k$  the method naturally provides lower (resp. upper) bounds of order  $(\log \log t)^k/k!$  (resp.  $k! \log t/(\log \log t)^k$ ) and we have to run the iteration for a number of steps of order  $k = k(t) \approx \log \log t$  (instead of  $k(t) \approx \log \log \log t$  as in [14]) to reach the final result. As a consequence, in contrast with [14], we cannot afford to lose a multiplicative constant at each step of the iteration (such multiplicative constants are responsible for the sub-optimal result in the first version of [5]), and a much finer analysis of the resolvent is needed. Further, we get a significantly sharper control of sub-leading corrections to  $D(t)$  with respect to  $2d$  ASEP, namely, a multiplicative correction that is polynomial in  $\log \log t$ , to be compared with the corrections of order  $\exp((\log \log \log t)^2)$  for  $2d$  ASEP [14].

## REFERENCES

- [1] B. Alder, T. Wainwright, *Velocity autocorrelations for hard spheres*, Physical review letters **18** (1967), 988.
- [2] D.J. Amit, G. Parisi, L. Peliti, *Asymptotic behavior of the "true" self-avoiding walk.*, Physical Review B **27** (1983), 1635.
- [3] N.H. Bingham, C.M. Goldie, and J.L. Teugels, *Regular variation*. Encyclopedia of Mathematics and its Applications **27** (1989), Cambridge University Press, Cambridge.
- [4] G. Cannizzaro, K. Chouk, *Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential*, Ann. Probab. **46** (2018), 1710–1763.
- [5] G. Cannizzaro, D. Erhard, and F. Toninelli, *The stationary AKPZ equation: logarithmic superdiffusivity*, Comm. Pure Appl. Math. (2020) to appear. arXiv 2007.12203.
- [6] G. Cannizzaro, L. Haunschmid, and F. Toninelli,  $\sqrt{\log t}$  - superdiffusivity for a Brownian particle in the curl of the 2d GFF, Ann. Probab. (2021) to appear. arxiv.2106.06264
- [7] T. Komorowski, C. Landim, and S. Olla, *Fluctuations in Markov processes*, Grundlehrender mathematischen Wissenschaften **345** (2012), Springer, Heidelberg.
- [8] T. Komorowski, S. Olla, *On the superdiffusive behavior of passive tracer with a Gaussian drift*, J. Statist. Phys. **108** (2002), 647–668.
- [9] C. Landim, J. Quastel, M. Salmhofer, and H.-T. Yau, *Superdiffusivity of asymmetric exclusion process in dimensions one and two*, Comm. Math. Phys. **244** (2004), 455–481.
- [10] C. Landim, J.A. Ramirez, and H.-T. Yau, *Superdiffusivity of two dimensional lattice gas models.* J. Stat. Phys. **119** (2005), 963–995.
- [11] J. Quastel, B. Valkó, *A note on the diffusivity of finite-range asymmetric exclusion processes on  $\mathbb{Z}$ . In and out of equilibrium. 2*. Progr. Probab. **60** (2008), 543–549.
- [12] B. Tóth, *Quenched central limit theorem for random walks in doubly stochastic random environment*, Ann. Probab. **46** (2018), 3558–3577.
- [13] B. Tóth, B. Valkó, *Superdiffusive bounds on self-repellent Brownian polymers and diffusion in the curl of the Gaussian free field in  $d = 2$* , J.Stat.Phys. **147** (2012), 113-131.
- [14] H.-T. Yau,  *$(\log t)^{2/3}$  law of the two dimensional asymmetric simple exclusion process*, Ann. of Math. **159** (2004), 377–405.

**Stochastic quantisation of Yang–Mills: state space and Markov process**

ILYA CHEVYREV

(joint work with Ajay Chandra, Martin Hairer, Hao Shen)

We report on results from [3, 4] which analyse the stochastic quantisation equation of Yang–Mills (YM) on the 2- and 3-dimensional torus. Following ideas of Parisi–Wu, the mathematical goal is to make rigorous sense of the YM measure on  $\mathbb{T}^d$  for  $d = 2, 3$  by studying the associated Langevin dynamic

$$(SYM) \quad \partial_t A = -d_A F_A - d_A d^* A + \xi .$$

Here  $A = (A_1, \dots, A_d) : \mathbb{T}^d \rightarrow \mathfrak{g}^d$  is a  $\mathfrak{g}$ -valued 1-form,  $F_A$  is the curvature of  $A$ ,  $\mathfrak{g}$  is the Lie algebra of a compact Lie group  $G$ , and  $\xi$  is a canonical white noise.

One of the main results of [3, 4] is that there exists a natural solution to (SYM) obtained as the limit of renormalised smooth solutions when  $\xi$  is mollified; see the abstract of Ajay Chandra (see also [5] for a survey). This solution projects down to a Markov process on a suitable space of gauge orbits for which the YM measure is conjecturally invariant. We note that the construction of the YM measure on  $\mathbb{T}^2$  is well-known but that on  $\mathbb{T}^3$  is open.

In this abstract, we summarise the construction of the Markov process on gauge orbits along with a suitable state space for the solution to (SYM). In 2D, a natural state space is a Banach space  $(\Omega_\alpha^1, |\cdot|_\alpha)$  of distributional  $\mathfrak{g}$ -valued 1-forms that embeds into the Hölder–Besov space  $\mathcal{C}^{\alpha-1}$ , where  $\alpha \in (\frac{2}{3}, 1)$  is a regularity parameter. We define  $\Omega_\alpha^1$  as the closure of smooth 1-forms under the norm

$$(1) \quad |A|_\alpha = \sup_\ell |\ell|^{-\alpha} |A(\ell)| + \sup_P |P|^{-\alpha/2} |A(\partial P)| ,$$

where the first supremum is taken over line segments  $\ell$  in  $\mathbb{T}^2$  with length  $0 < |\ell| < 1$  and the second supremum is over oriented triangles  $P = (\ell_1, \ell_2, \ell_3)$  with area  $|P| > 0$ . Here  $A(\ell)$  is the line integral of  $A$  along  $\ell$ , and  $A(\partial P) = \sum_{i=1}^3 A(\ell_i)$  is the integral around the boundary of  $P$ .

The space  $\Omega_\alpha^1$  has a number of desirable properties, including that holonomies (and thus Wilson loops) are well-defined along  $\mathcal{C}^2$  curves for all  $A \in \Omega_\alpha^1$  and that the gauge group  $\mathcal{C}^\alpha(\mathbb{T}^2, G)$  acts continuously on  $\Omega_\alpha^1$  by  $(A, g) \mapsto A^g := gAg^{-1} - (dg)g^{-1}$ . Classical gauge equivalence  $\sim$  thus extends to  $\Omega_\alpha^1$  via  $A \sim B \Leftrightarrow \exists g \in \mathcal{C}^\alpha, A^g = B$ . The quotient space of gauge orbits  $\Omega_\alpha^1/\sim$  is furthermore Polish. Finally, one has a simple but useful estimate following from Young ODE theory

$$(2) \quad |g|_{\mathcal{C}^\alpha} \leq C_\alpha (|A|_\alpha + |A^g|_\alpha) .$$

The construction of the associated Markov process on  $\Omega_\alpha^1/\sim$  proceeds by considering two solutions  $A_t$  and  $B_t$  to (SYM) with gauge-equivalent initial conditions  $A_0 \sim B_0$ . It is possible to couple  $A_t$  and  $B_t$  so that  $B_t = A_t^{g_t}$ , where  $g$  solves

$$(3) \quad g^{-1} \partial_t g = \partial_j (g^{-1} \partial_j g) + [A_j, g^{-1} \partial_j g] .$$

One then defines a sequence of stopping times  $\tau_1, \tau_2, \dots$  which stop the processes  $A$  and  $B$  when one of them, say  $A$ , becomes significantly larger than  $\inf_{X \sim A} |X|_\alpha$ . At each stopping time, we find a representative of the equivalence class of  $A$  which

nearly attains this infimum and restart the process. By construction, it is clear that  $A_t^{g_t} = B_t$  for all  $t$  up until blow up of  $|A|_\alpha \wedge |B|_\alpha$ , where  $g_t$  solves (3) on each interval  $(\tau_i, \tau_{i+1})$ , which proves the Markov property for the projected process  $[A_t] \in \Omega_\alpha^1/\sim$ . Crucial in this construction is the estimate (2) which ensures that  $g$  does not blow up before  $A$  or  $B$  does.

We now turn to the 3D case. Unlike in 2D, even the Gaussian free field (GFF) in 3D cannot be integrated along lines, which means we need a different strategy. We instead first define a suitable space  $\mathcal{I}$  of initial conditions for  $\{\mathcal{F}_t\}_{t>0}$  which is the solution map of the DeTurck–YM heat flow – this flow is given by the PDE (SYM) but without  $\xi$ . We define  $\mathcal{I}$  as the closure of smooth 1-forms under the metric

$$\Theta(A, B) = |A - B|_{\mathcal{C}^{-\frac{1}{2}-\kappa}} + \sup_{t \in (0,1)} t^{1-\kappa} |e^{t\Delta} A \otimes \nabla e^{t\Delta} A - e^{t\Delta} B \otimes \nabla e^{t\Delta} B|_{\mathcal{C}^{-3\kappa}}$$

for  $\kappa > 0$  sufficiently small. One can show that  $\mathcal{F}$  is well-posed and locally Lipschitz on  $\mathcal{I}$ . By gauge covariance,  $\sim$  extends to  $\mathcal{I}$  via  $A \sim B \Leftrightarrow \exists t > 0, \mathcal{F}_t(A) \sim \mathcal{F}_t(B)$ .

The second term in the definition of  $\Theta$  controls the quadratic terms in (SYM). The GFF furthermore takes values in  $\mathcal{I}$ , which proves well-posedness of the YM heat flow for GFF. Similar well-posedness results and state space  $\mathcal{I}$  independently appeared in [1, 2]. Note that  $\mathcal{I}$  is non-linear, and this cannot be avoided: there is norm inflation, and thus ill-posedness, for  $\mathcal{F}$  in every Banach space that supports the GFF on  $\mathbb{T}^3$  [6] (and even the endpoint space  $\mathcal{C}^{-1/2}$ ).

Unfortunately, for the space  $\mathcal{I}$ , there appears to be no analogue of the estimate (2). We therefore define the final state space in 3D as the closure of smooth 1-forms under the strengthened metric

$$\Sigma(A, B) = \Theta(A, B) + \sup_{t \in (0,1)} \sup_{|\ell| < t^\kappa} |\ell|^{-\frac{1}{2}+\kappa} |e^{t\Delta} A(\ell) - e^{t\Delta} B(\ell)|.$$

The final term substitutes the first term in (1), with the natural weighting  $|\ell|^{-\frac{1}{2}+\kappa}$ . The catch is that we now need to restrict to short line segments  $|\ell| < t^\kappa$ . One should think of  $t^\kappa$  as a characteristic length scale on which  $e^{t\Delta} A$  integrates lines to give  $(\frac{1}{2} - \kappa)$ -Hölder continuous paths. One can then prove the crucial estimate  $|g|_{\mathcal{C}^\nu} \leq C(1 + \Sigma(A, 0) + \Sigma(A^g, 0))^q$ , for some  $\nu \in (0, \frac{1}{2})$  and  $C, q > 0$ , which substitutes (2). This allows one to use the same strategy as in 2D to build a Markov process associated to (SYM) on the space of gauge orbits  $\mathcal{S}/\sim$  in 3D.

REFERENCES

[1] Sky Cao, Sourav Chatterjee, *The Yang-Mills heat flow with random distributional initial data*, Preprint 2021, arXiv:2111.10652  
 [2] Sky Cao, Sourav Chatterjee, *A state space for 3D Euclidean Yang-Mills theories*, Preprint 2021, arXiv:2111.12813  
 [3] Ilya Chevyrev, Ajay Chandra, Martin Hairer, Hao Shen, *Langevin dynamic for the 2D Yang-Mills measure*, To appear in Publications mathématiques de l’IHÉS. arXiv:2006.04987  
 [4] Ilya Chevyrev, Ajay Chandra, Martin Hairer, Hao Shen, *Stochastic quantisation of Yang-Mills-Higgs in 3D*, Preprint 2022, arXiv:2201.03487  
 [5] Ilya Chevyrev, *Stochastic quantisation of Yang-Mills*, Preprint 2022, arXiv:2202.13359  
 [6] Ilya Chevyrev, *Norm inflation for a non-linear heat equation with Gaussian initial conditions*, Preprint 2022, arXiv:2205.14350

**Parking on the infinite binary tree**

NICOLAS CURIEN

(joint work with David Aldous, Alice Contat and Olivier Hénard)

Consider the full planar rooted binary tree  $\mathbb{B} = \cup_{n \geq 0} \{0, 1\}^n$ , with  $\{0, 1\}^0 = \emptyset$  being the root of the tree. Those vertices will be interpreted as free parking spots, each spot accommodating at most 1 car. On top of that tree, we consider a non-negative integer labeling  $(A_u : u \in \mathbb{B})$  representing the number of cars arriving on each vertex  $u \in \mathbb{B}$ . Each car tries to park on its arrival vertex, and if the spot is occupied, it travels downwards in direction of the root of the tree until it finds an empty vertex to park. If there is no such vertex on the path towards the root  $\emptyset$ , the car exits the tree, contributing to the *flux* of cars at the root that we denote by  $X$ . An easy Abelian property show that the value of  $X$  does not depend upon the order chosen to park the cars.

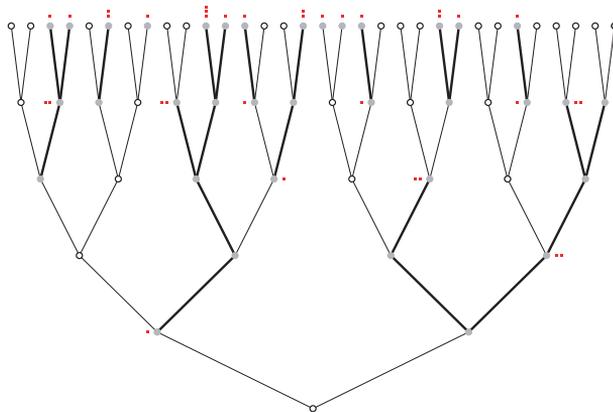


FIGURE 1. Illustration of the parking process in the first 5 levels of the full binary tree. The car arrivals are represented by red squares (including the cars that may come from higher levels on top of the tree). After the parking process, the vertices accommodating a car are displayed in gray, whereas the free spots are displayed in white. In this case the flux of outgoing cars at is null.

We suppose that the car arrivals  $(A_u : u \in \mathbb{B})$  are i.i.d. with a given distribution  $\mu = (\mu_k : k \geq 0)$  on  $\{0, 1, 2, 3, \dots\}$  with finite support (and suppose that  $\mu(\{0, 1\}) < 1$  to avoid trivialities). We let

$$G(x) = \sum_{k \geq 0} \mu_k x^k$$

be the generating function of the law  $\mu$ .

**Theorem 2.** *Then the flux of outgoing car  $X$  is almost surely finite if and only if at*

$$t_c = \min\{t \geq 0 : 2(G(t) - tG'(t))^2 = t^2G(t)G''(t)\} \in (0, \infty),$$

*we have*

$$(t_c - 2)G(t_c) \geq t_c(t_c - 1)G'(t_c).$$

*Otherwise  $X = \infty$  almost surely.*

REFERENCES

- [1] R. BAHL, P. BARNET, AND M. JUNGE, *Parking on supercritical Galton-Watson trees*, arXiv:1912.13062, (2019).
- [2] X. CHEN, V. DAGARD, B. DERRIDA, Y. HU, M. LIFSHTIS, AND Z. SHI, *The Derrida–Retaux conjecture on recursive models*, The Annals of Probability, 49 (2019), pp. 637–670.
- [3] C. GOLDSCHMIDT AND M. PRZYKUCKI, *Parking on a random tree*, Combinatorics, Probability and Computing, 28 (2019), pp. 23–45.

**Large deviations for the spectrum of random matrices**

RAPHAËL DUCATEZ, ALICE GUIONNET

(joint work with Nicholas Cook)

We consider  $H = \frac{1}{\sqrt{N}}X$  a Hermitian random matrix with entries  $(X_{ij})_{i < j}$  iid where  $\mathbb{E}X_{ij} = 0$ ,  $\mathbb{E}(|X_{ij}|^2) = 1$ . We denote the empirical spectral measure

$$\mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$$

where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$  are the eigenvalues of  $H$ . We have the well known Theorem of Wigner [10](Semi-circle law)

$$\lim_{N \rightarrow \infty} \mu_N([a, b]) = \sigma_{sc}([a, b]) = \frac{1}{2\pi} \int_a^b \sqrt{4 - y^2} \mathbb{1}_{|y| \leq 2} dy \quad a.s.$$

Moreover if  $\mathbb{E}X_{ij}^4 < \infty$  then we also have the convergence of the largest eigenvalue [7, 5]

$$\lim_{N \rightarrow \infty} \lambda_1 = 2 \quad a.s.$$

The goal of this presentation is to give large deviation estimates for these results.

**The Gaussian Case (Beta-Ensemble).** When the entries  $X_{ij}$  are gaussian, the system has much more symmetries and one can write the law of the eigenvalues explicitly

$$d\mathbb{P}_\beta^N = \frac{1}{Z_\beta^N} \exp \left( \beta \sum_{i < j} \log |\lambda_i - \lambda_j| - \frac{\beta}{4} \sum_{i=1}^N \lambda_i^2 \right) \prod_{1 \leq i \leq N} d\lambda_i.$$

We also define  $\xi(\mu) = \beta(J(\mu) - \inf J)$  where

$$J(\mu) := \frac{1}{8} \iint (x^2 + y^2 - 4 \log|x - y|) d\mu(x) d\mu(y)$$

Here it is possible to compute the large deviation functions and we have the following.

**Theorem 1.** [9, 2] (*Large deviation for the distribution of the eigenvalues*) For any “nice”  $A \subset \mathcal{P}(\mathbb{R})$

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbb{P}_\beta^N(\mu_N \in A) = - \inf_{\mu \in A} \xi(\mu).$$

[1] (*Large deviation for the largest eigenvalue*) For  $x \geq 2$ ,

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_\beta^N(|\lambda_1 - x| \leq \epsilon) = -I_{GOE}(x) := - \int_2^x \sqrt{y^2 - 4} dy$$

Remark that the rate of large deviation is  $N^2$  for spectral measure and  $N$  for the largest eigenvalue. This is not so surprising as one should “move”  $O(N)$  eigenvalues to have a  $O(1)$  perturbation on  $\mu_N$ .

**The general case.** The general non-gaussian case is more complicated. For the distribution of the eigenvalues the rate of the large deviation should still be  $N^2$  if  $(X_{ij})$  are bounded (or log-concave) and indeed we have

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbb{P}^N(d(\mu_N, \sigma_{sc}) > \delta) < 0.$$

However we do not have a large deviation function here. For the largest eigenvalue, the probability estimate depends on the tails of the law of the  $X_{ij}$ . Define  $A = 2 \sup \frac{\log \mathbb{E} e^{tX}}{t^2}$  and remark that  $A = 1$  in the Gaussian case. We say that the law of the entries is not sub-gaussian if  $A = \infty$ . In that case, the largest eigenvalue is created by a very large entry in the matrix and as a consequence the rate of the large correspond to the tail of the law [3, 6]. In the case  $A < \infty$ , one obtains that the rate is  $N$  as in the gaussian case and write an large deviation function.

**Theorem 2.** [8, 4][Cook, Ducatez, Guionnet] *There exists  $C \leq C' \in (2, \infty) \cup \{\infty\}$  such that for  $x \geq 2$ ,*

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}^N(|\lambda_1 - x| \leq \epsilon) = \begin{cases} -I_{GOE}(x) & \text{if } x < C \\ -I(x) > -I_{GOE}(x) & \text{if } x > C' \end{cases}$$

with  $C = \infty$  iff  $A = 1$ .

Here  $I(x)$  is not universal but depends on the law of  $X$ , In general it doesn't have a simple expression but can be formulated as the solution of a minimising problem. One has  $C = C'$  adding a few technical hypothesis.

## REFERENCES

- [1] G Ben Arous, Amir Dembo, and Alice Guionnet. Aging of spherical spin glasses. *Probability theory and related fields*, 120(1):1–67, 2001.
- [2] G Ben Arous and Alice Guionnet. Large deviations for wigner’s law and voiculescu’s non-commutative entropy. *Probability theory and related fields*, 108(4):517–542, 1997.
- [3] Fanny Augeri. Large deviations principle for the largest eigenvalue of Wigner matrices without Gaussian tails. *Electronic Journal of Probability*, 21:Paper No. 32, 49, 2016.
- [4] Fanny Augeri, Alice Guionnet, and Jonathan Husson. Large deviations for the largest eigenvalue of sub-gaussian matrices. *Communications in mathematical physics*, 383(2):997–1050, 2021.
- [5] Zhi-Dong Bai and Yong-Qua Yin. Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a wigner matrix. *The Annals of Probability*, pages 1729–1741, 1988.
- [6] Charles Bordenave and Pietro Caputo. A large deviation principle for Wigner matrices without Gaussian tails. *The Annals of Probability*., 42(6):2454–2496, 2014.
- [7] Zoltán Füredi and János Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.
- [8] Alice Guionnet and Jonathan Husson. Large deviations for the largest eigenvalue of rademacher matrices. *The Annals of Probability*, 48(3):1436–1465, 2020.
- [9] Dan Voiculescu. The analogues of entropy and of fisher’s information measure in free probability theory, i. *Communications in mathematical physics*, 155(1):71–92, 1993.
- [10] Eugene P Wigner. On the distribution of the roots of certain symmetric matrices. *Annals of Mathematics*, pages 325–327, 1958.

## Exit sets for vector valued GFF and interplay with planar $O(N)$ models

CHRISTOPHE GARBAN

(joint work with Juhan Aru, Avelio Sepúlveda)

The first motivation of this work is to understand the *exit sets* of (planar) vector-valued GFF. Given an instance of a vector valued GFF

$$\Phi = (\varphi_1, \dots, \varphi_N) : [-n, n]^2 \subset \mathbb{Z}^2 \rightarrow \mathbb{R}^N,$$

and a magnitude  $R > 0$ , the exit set  $A_R$  is obtained by exploring the field starting from the boundary and by stopping the exploration at each vertex  $x$  where  $\|\Phi_x\| > R$ . In the scalar case (i.e. ”one-component GFF”), such exit sets are now well understood (at least for the continuum GFF) thanks for example to the works [1, 2]. They happen to be intriguing fractal sets which can be described by versions of  $CLE_4$  loop ensembles.

The first main result of this work is that for the  $N$ -component GFF, as long as  $N \geq 2$ , such exit sets  $A_R$  are *degenerate* for all values of  $R$  in the sense that the exploration process starting from  $\partial[-n, n]^2$  does not enter the bulk of the domain further than  $o(n)$  away from the boundary.

When  $N = 2$ , the main idea behind the proof uses the absence of long-range order of a suitable  $XY$  model with random long-range conductances.

The second main motivation of this work is to revisit a series of works by Patrascioiu-Seiler ([3, 4]) which argued against Polyakov’s celebrated prediction in

1975 ([5]) that planar classical Heisenberg model should exhibit exponential decay at all positive temperatures.

We first prove that the (properly rescaled) transverse fluctuations of a classical Heisenberg model converge as the temperature goes to 0 to a two-component GFF  $\Phi = (\varphi_1, \varphi_2)$ . Using the first part of this work, it implies that if one projects the classical Heisenberg model down to a  $XY$  model in random environment (following [5]), then locally around pole-pointing spins, the “warm” regions of the induced random environment (where the  $XY$  model will decouple more rapidly) do not percolate. The percolation of such regions would have provided a convincing mechanism for exponential decay. In particular if Polyakov’s prediction were to hold, a mass needs to be generated by the  $XY$  model defined on a strongly percolating set of high conductances. This possible scenario is excluded in Patrascioiu-Seiler works and the last contribution of this present work is to build an explicit example of an  $XY$  model in random conductances which has a mass gap despite a percolating set of (arbitrary) high conductances.

#### REFERENCES

- [1] Aru, Sepúlveda, Werner (2019). On bounded-type thin local sets of the two-dimensional Gaussian free field. *Journal of the Institute of Mathematics of Jussieu*, 18(3), 591–618.
- [2] Aru, Sepúlveda (2018). Two-valued local sets of the 2D continuum Gaussian free field: connectivity, labels, and induced metrics. *Electronic Journal of Probability*, 23, 1–35.
- [3] Patrascioiu, Seiler (1992). Phase structure of two-dimensional spin models and percolation. *Journal of statistical physics*, 69(3), 573–595.
- [4] Patrascioiu, Seiler (1995). Superinstantons and the reliability of perturbation theory in non-abelian models. *Physical review letters*, 74(11), 1920.
- [5] Polyakov (1975). Interaction of goldstone particles in two dimensions. Applications to ferromagnets and massive Yang-Mills fields. *Physics Letters B*, 59(1), 79–81.

### What is stochastic quantisation?

MASSIMILIANO GUBINELLI

I regularly teach a course called “Stochastic Analysis”, in which the students are introduced to the use of stochastic integration and Itô’s formulas to study several problems in probability, including existence, conditioning, uniqueness, convergence of diffusion processes. However it is quite difficult for me to give a succinct definition of what stochastic analysis *is*. It is usually however quite easy to classify a result as a result in stochastic analysis or not. As D. Stroock puts it in the introduction to [16]:

*[...] there now exists a reasonably well-defined amalgam of probabilistic and analytic ideas and techniques that, at least among the cognoscenti, are easily recognized as stochastic analysis. Nonetheless, the term continues to defy a precise definition, and an understanding of it is best acquired by way of examples.*

According to Stroock [15], Itô arrived to his calculus while trying to understand Feller’s theory of diffusions an evolution in the space of probability measures and he

introduced stochastic differential equations as maps which send Wiener measure to the law of a diffusion. Malliavin's calculus, another example of stochastic analysis, is also based on reducing a probabilistic problem to the analysis of certain random maps and of their smoothness according to an infinite-dimensional calculus of variations. Lyons' rough path theory shift the focus on more robust properties of Itô's solution map and identify specific topologies in which it can be extended to a larger class of driving signals. Controlled rough paths are a variation on Itô's idea of using Brownian motion as "universal" building block for continuous diffusions, by replacing it with a general rough path in the sense of Lyons.

While technically all these approaches use an analysis based on one-dimensional random fields, the concerns of analysing the behaviour of certain maps which send a probability measure to another is not intrinsically limited to (possibly infinite-dimensional) systems evolving along a one-dimensional time parameter. Schramm's stochastic Loewner equation is a map which sends Wiener measure into a random growing domain in the complex plane whose law has natural probabilistic properties like diffeomorphism covariance and domain Markovianity. Hairer's regularity structures and paracontrolled calculus bring forward Itô's and Lyons' ideas to the realm of multidimensional random fields which are solutions to stochastic partial differential equations. In many cases these random fields are distribution-valued and the associated non-linear equation have to be understood to be posed in some kind of non-smooth jet space.

From this point of view, stochastic analysis is not really *just* a part of probability, in the same way as probability is not *just* a part of measure theory. While the technical background and the arena come from another discipline, both probability and stochastic analysis evolved quite characteristic features which sets them apart from their foundational predecessors.

This long introduction was meant to set the stage for an attempt to an answer to the question in the title. Stochastic quantisation has been introduced, from different perspectives, by Nelson and by Parisi–Wu to provide a construction of Euclidean quantum field theories (EQFTs) using diffusion processes. EQFTs are probability measures on the space of Schwarz distributions on the Euclidean space  $\mathbb{R}^d$  which enjoy particular properties (e.g. locality, reflection positivity, covariance under the Euclidean group, integrability). These properties allows to use them as building block to construct an Hilbert space and an algebra of local fields for a relativistic quantum theory in the sense of Wightman.

Parisi–Wu's construction uses the Itô map of a Langevin equation to push forward the law of a multidimensional white noise to a target EQFT, explicitly providing a stochastic construction of an (Euclidean) quantum theory. We can call it a *stochastic quantisation map*. The use of a Langevin equation framed, quite rapidly, the idea of stochastic quantisation in the area of the theory of Markov processes, to which stochastic analysis is naturally related. However I want to argue that stochastic quantisation is a stochastic analysis of Euclidean quantum fields in a more fundamental way, and that is in the very basic sense of an analysis

of maps which send some convenient *source* probability measures  $\mu$  to a target measure  $\nu$  which has the structure and properties of an EQFT.

In order to articulate this idea and “flesh it out”, together with various collaborators, we set out to explore this analysis along a range of directions. For example one can produce such a stochastic quantisation map in various ways, i.e. there are various kind of stochastic quantisation that we know of. One can use parabolic equations [10] or elliptic equations [2, 3, 11, 5], or wave equations [13, 12], or forward-backwards stochastic differential equations (FBSDEs) [7, 4] which in some cases are related to a stochastic control problem as in [6] and to renormalization group ideas, in the continuous formulation of Wilson and Polchinski.

Fermionic quantum field theories are associated to probability measures on Grassmann variables and there is an associated stochastic analysis [1] which give then rise to a stochastic quantisation, e.g. via FBSDEs coupled to an approximate RG flow as in [9]. In this case there is no measure theory or probability in conventional sense, but stochastic analysis retain is characteristic aspects and delivers useful results.

The stochastic quantisation map give rise to a natural coupling of the Gaussian free field and the “interacting” Euclidean quantum field which can be exploited to study pathwise properties of the EQFTs and related functional inequalities as in e.g. [8, 14].

These results are the beginning of a systematic exploration of these new examples of stochastic analysis which, I hope, could bring some new insights in constructive Euclidean quantum field theory at least in the superrenormalizable and asymptotically free cases where the Gaussian free field plays the role of a basic building block on small scales.

## REFERENCES

- [1] S. Albeverio, L. Borasi, F. C. De Vecchi and M. Gubinelli, *Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions*, Probab. Theory Relat. Fields **183** (2022), 909–995.
- [2] S. Albeverio, F. C. De Vecchi and M. Gubinelli, *Elliptic stochastic quantization*, Ann. Probab. **48** (2020), no. 4, 1693–1741.
- [3] S. Albeverio, F. C. De Vecchi and M. Gubinelli, *The elliptic stochastic quantization of some two dimensional Euclidean QFTs*, Ann. Inst. Henri Poincaré Probab. Stat. **57** (2021), no. 4, 2372–2414.
- [4] N. Barashkov, *A Stochastic Control Approach to Sine Gordon EQFT*, ArXiv:2203.06626 (2022).
- [5] N. Barashkov and F. C. De Vecchi, *Elliptic Stochastic Quantization of Sinh-Gordon QFT*. ArXiv:2108.12664 (2021).
- [6] N. Barashkov and M. Gubinelli, *A variational method for  $\Phi_3^4$* , Duke Math. J. **169** (2020), no. 17, 3339–3415.
- [7] N. Barashkov and M. Gubinelli, *On the Variational Method for Euclidean Quantum Fields in Infinite Volume*, ArXiv:2112.05562 (2021).
- [8] R. Bauerschmidt and M. Hofstetter, *Maximum and coupling of the sine-Gordon field*, Ann. Probab. **50** (2022), 455–508.
- [9] F. C. De Vecchi, L. Fresta and M. Gubinelli, *A stochastic analysis of subcritical Euclidean fermionic field theories*, ArXiv:2210.15047 (2022).

- [10] M. Gubinelli and M. Hofmanová, *A PDE construction of the Euclidean  $\phi_3^4$  quantum field theory*, *Comm. Math. Phys.* **384** (2021), no. 1, 1–75.
- [11] M. Gubinelli and M. Hofmanová, *Global solutions to elliptic and parabolic  $\Phi^4$  models in Euclidean space*, *Comm. Math. Phys.* **368** (2019), no. 3, 1201–1266.
- [12] M. Gubinelli, H. Koch and T. Oh, *Paracontrolled Approach to the Three-Dimensional Stochastic Nonlinear Wave Equation with Quadratic Nonlinearity*, *ArXiv:1811.07808* (2018).
- [13] M. Gubinelli, H. Koch and T. Oh, *Renormalization of the two-dimensional stochastic nonlinear wave equations*, *Trans. Amer. Math. Soc.* **370** (2018), no. 10, 7335–7359.
- [14] D. Mikulincer, Y. Shenfeld, *The Brownian Transport Map*, *ArXiv:2111.11521* (2021).
- [15] D. W. Stroock, *Markov processes from K. Itô's perspective*, *Annals of Mathematics Studies*, 155, Princeton University Press, Princeton, NJ, 2003.
- [16] D. W. Stroock, *Elements of stochastic calculus and analysis*, CRM Short Courses, Springer, Cham, 2018.

## The critical and supercritical Liouville quantum gravity metrics

EWAIN GWYNNE

(joint work with Jian Ding)

We discuss the construction of the **Liouville quantum gravity (LQG)** metric associated with the planar Gaussian free field in the subcritical ( $\gamma \in (0, 2)$ ), critical ( $\gamma = 2$ ) and supercritical ( $\gamma \in \mathbb{C}$ ,  $|\gamma| = 2$ ) regimes.

To define this metric, let  $h$  be the planar Gaussian free field and let  $\xi > 0$  be a parameter. For  $\varepsilon > 0$ , let  $h_\varepsilon$  be the convolution  $h * p_{\varepsilon^2/2}$ , where  $p_t(z) = \frac{1}{2\pi t} e^{-|z|^2/2t}$  is the planar heat kernel. We define a family of approximating metrics  $\{D_h^\varepsilon\}_{\varepsilon > 0}$  by

$$D_h^\varepsilon(z, w) := \inf_{P: z \rightarrow w} \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)| dt, \quad \forall z, w \in \mathbb{C}$$

where the infimum is over all piecewise continuously differentiable paths  $P : [0, 1] \rightarrow \mathbb{C}$  from  $z$  to  $w$ .

To extract a non-trivial limit of the metrics  $D_h^\varepsilon$ , we need to re-normalize. We (somewhat arbitrarily) define our normalizing factor by

$$\mathfrak{a}_\varepsilon := \text{median of } \inf \left\{ \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)| dt : P \text{ is a left-right crossing of } [0, 1]^2 \right\},$$

where a left-right crossing of  $[0, 1]^2$  is a piecewise continuously differentiable path  $P : [0, 1] \rightarrow [0, 1]^2$  joining the left and right boundaries of  $[0, 1]^2$ . We do not know the value of  $\mathfrak{a}_\varepsilon$  explicitly. Nevertheless, it was shown by Ding and Gwynne (2020) that for each  $\xi > 0$ , there exists  $Q = Q(\xi) > 0$  such that

$$\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q+o(1)} \quad \text{as } \varepsilon \rightarrow 0.$$

Furthermore,  $Q$  is a continuous, non-increasing function of  $\xi$ .

We define the **critical value**  $\xi_{\text{crit}}$  to be the unique  $\xi > 0$  for which  $Q(\xi) = 2$ . In the subcritical case  $\xi < \xi_{\text{crit}}$ , it was shown in works by Ding-Dunlap-Dubédat-Falconet (2019) and Gwynne-Miller (2019) that the re-scaled metrics  $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$  converge in probability to a limiting random metric  $D_h$  on  $\mathbb{C}$ . This limiting metric induces the same topology as the Euclidean metric, but has very different

geometric properties. For example, the Hausdorff dimension of  $(\mathbb{C}, D_h)$  is strictly greater than two. The metric  $D_h$  can be interpreted as the Riemannian distance function associated with  $\gamma$ -LQG, where  $\gamma \in (0, 2)$  satisfies  $Q(\xi) = 2/\gamma + \gamma/2 > 2$ .

In the supercritical and critical cases, i.e., when  $\xi \geq \xi_{\text{crit}}$ , the convergence of  $\alpha_\varepsilon^{-1} D_h^\varepsilon$  was established by Ding-Gwynne (2021). In the critical case  $\xi = \xi_{\text{crit}}$ , the limit is a random fractal metric on  $\mathbb{C}$  which induces the Euclidean topology (like in the subcritical case). It can be interpreted as the distance function associated with critical ( $\gamma = 2$ ) LQG.

In the supercritical case, however, the limiting metric  $D_h$  is allowed to take the value  $+\infty$ . More precisely, we say that  $z \in \mathbb{C}$  is a **singular point** if

$$D_h(z, w) = \infty, \quad \forall w \in \mathbb{C} \setminus \{z\}.$$

For  $\xi > \xi_{\text{crit}}$ , a.s. the set of singular points is uncountable and Euclidean-dense, but has zero Lebesgue measure. Furthermore, any two non-singular points lie at finite distance from each other. These properties imply that  $D_h$  does not induce the Euclidean topology and that  $D_h$ -metric balls have positive Lebesgue measure but empty Euclidean interior.

For  $\xi > \xi_{\text{crit}}$ , we have  $Q(\xi) \in (0, 2)$ . If we choose  $\gamma$  so that  $Q(\xi) = 2/\gamma + \gamma/2$  then  $\gamma \in \mathbb{C}$  with  $|\gamma| = 2$ . Hence, the limiting metric for  $\xi > \xi_{\text{crit}}$  can be interpreted as the distance function associated with  $\gamma$ -LQG for such a complex value of  $\gamma$ , or equivalently with **matter central charge**  $\mathbf{c}_M = 25 - 6(2/\gamma + \gamma/2)^2 \in (1, 25)$ .

The  $\mathbf{c}_M \in (1, 25)$  phase for LQG is rather mysterious, even at a physics level of rigor. Although we can now construct a metric in this phase, it remains an open problem to extend various other features of LQG from the case when  $\gamma \in (0, 2]$  ( $\mathbf{c}_M \leq 1$ ) to the case when  $\mathbf{c}_M \in (1, 25)$ . Such features include the construction of the volume measure, connections to SLE, and various formulas from conformal field theory.

### The Euclidean $\phi_2^4$ theory as a limit of an interacting Bose gas

ANTTI KNOWLES

(joint work with Jürg Fröhlich, Benjamin Schlein, Vedran Sohinger)

A Euclidean field theory of a scalar field on a the  $d$ -dimensional torus  $\Lambda := \mathbb{T}^d$  is specified, at least formally, by a probability measure on a space of fields  $\phi : \Lambda \rightarrow \mathbb{R}^N$  given by

$$(1) \quad \mu(d\phi) = \frac{1}{c} e^{-S(\phi)} \mathbf{D}\phi,$$

where  $\mathbf{D}\phi = \prod_{x \in \Lambda} d\phi(x)$  is the formal uniform measure on the space of fields, and  $S$  is the action. The latter is typically the integral over  $\Lambda$  of a local function of the field  $\phi$  and its gradient. One of the simplest field theories with nontrivial interaction is the  $N$ -component *Euclidean  $\phi_d^4$  theory*, whose action is given by

$$(2) \quad S(\phi) := - \int_{\Lambda} dx \phi(x) \cdot (\theta + \Delta/2)\phi(x) + \frac{\lambda}{2} \int_{\Lambda} dx |\phi(x)|^4,$$

where  $\theta$  is a constant,  $\lambda$  is a coupling constant,  $\Delta$  is the Laplacian on  $\Lambda$ , and  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{R}^N$ .

The rigorous construction of  $\mu$  is a classical topic in mathematical physics; we refer for instance to [4, 9] for seminal references. The starting point is to regard  $\mu$  as a perturbation of the Gaussian free field measure, obtained by setting  $\lambda = 0$  in the definition of  $\mu$ . For  $d = 1$ , under the Gaussian free field, the field  $\phi$  is almost surely a continuous function, so that the density  $e^{-S(\phi)}$  is well defined, and hence  $\mu$  can be rigorously constructed. For  $d > 1$ , under the Gaussian free field, the field  $\phi$  is almost surely a distribution of negative regularity, and hence the interaction term  $V(\phi) := \frac{\lambda}{2} \int_{\Lambda} dx |\phi(x)|^4$  in (2) is ill-defined. This is an *ultraviolet* problem: a divergence for large wave vectors (i.e. spatial frequencies) producing small-scale singularities in the field. As the dimension  $d = 1, 2, 3$  increases, the difficulty of making sense of the measure in (1) increases significantly.

To handle the ultraviolet problem, it is necessary to regularize the interaction  $V(\phi)$  by subtracting infinite mass and energy counterterms. Formally, the renormalized interaction is  $\frac{\lambda}{2} \int_{\Lambda} dx (|\phi(x)|^4 - \infty|\phi(x)|^2 + \infty)$ . More precisely, one replaces the field  $\phi$  with its mollified version  $\phi_{\eta}$  obtained by convolving it with an approximate delta function of range  $\eta$ . One then chooses  $M_{\eta}$  and  $E_{\eta}$  such that, as  $\eta \downarrow 0$ , the measure  $\mu_{\eta}$  with interaction  $V_{\eta}(\phi) := \frac{\lambda}{2} \int_{\Lambda} dx (|\phi(x)|^4 - M_{\eta}|\phi(x)|^2 + E_{\eta})$  converges to a limiting measure.

In [3], we prove that the complex (i.e.  $N = 2$ ) Euclidean  $\phi_2^4$  theory arises as a high density limit of an interacting Bose gas, in a regime where the range of the interactions is much smaller than the linear size of the system but much larger than the typical interparticle separation. This extends the recent results [1, 2, 5, 6, 7, 8] on the mean-field limit of the Bose gas at positive temperature to local interactions.

To describe our result more precisely, we recall that a quantum system of  $n$  spinless non-relativistic bosons of mass  $m$  in  $\Lambda$  is described by the Hamiltonian

$$\mathbb{H}_n := - \sum_{i=1}^n \frac{\Delta_i}{2m} + \frac{g}{2} \sum_{i,j=1}^n v(x_i - x_j)$$

acting on the space  $\mathcal{H}_n$  of square-integrable wave functions that are symmetric in their arguments  $x_1, \dots, x_n$  and supported in  $\Lambda^n$ . Here  $\Delta_i$  is the Laplacian in the variable  $x_i$ ,  $g$  is a coupling constant, and  $v$  is a repulsive (i.e. with nonnegative Fourier transform) two-body interaction potential. We consider a system in the grand canonical ensemble at positive temperature, characterized by the density matrix

$$(3) \quad \frac{1}{Z} \bigoplus_{n \in \mathbb{N}} e^{-\beta(\mathbb{H}_n - \theta n)}$$

acting on Fock space  $\mathcal{F} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$ , where  $\beta < \infty$  is the inverse temperature,  $\theta$  is the chemical potential, and  $Z$  is a normalization factor.

The limiting regime of this paper is obtained by introducing two parameters,  $\nu, \varepsilon > 0$ , where  $\nu = \frac{\beta}{m} = \sqrt{\beta g}$ , and the potential  $v$  is taken to be an approximate delta function of range  $\varepsilon$ . We suppose that  $\nu, \varepsilon \rightarrow 0$ . This can be interpreted as

a high-density limit, where the range of the interactions tends to zero. Our main result is the following.

**Theorem.** Let  $d = 2$ . Suppose that  $\nu, \varepsilon \rightarrow 0$  under the constraint

$$\varepsilon \geq \exp(-(\log \nu^{-1})^{1/2-c}),$$

for some constant  $c > 0$ . Then, there exists a suitable renormalization of the chemical potential  $\theta \equiv \theta_\nu^\varepsilon$  such that, for any  $p \in \mathbb{N}$ , the reduced  $p$ -particle density matrices of the quantum Bose gas in the grand canonical state (3) converge to the correlation functions of the complex  $\phi_d^4$  theory. The convergence holds in the  $L^p$ -norm for any  $p < \infty$ .

We also show convergence in the  $L^\infty$  norm of the renormalized density matrices to the corresponding renormalized correlation functions of  $\phi_2^4$ . This yields a link between the distribution of the local particle density of the Bose gas and the mass density of  $\phi_2^4$ .

The proof is based on three main ingredients: (a) a quantitative analysis of the infinite-dimensional saddle point argument for the functional integral introduced in [2] using continuity properties of Brownian paths, (b) a Nelson-type estimate for a general nonlocal field theory in two dimensions, and (c) repeated Gaussian integration by parts in field space to obtain uniform control on the renormalized correlation functions.

## REFERENCES

- [1] J. Fröhlich, A. Knowles, B. Schlein, and V. Sohinger, *Gibbs Measures of Nonlinear Schrödinger Equations as Limits of Many-Body Quantum States in Dimensions  $d \leq 3$* , Comm. Math. Phys. **356** (2017), no. 3, 883–980.
- [2] J. Fröhlich, A. Knowles, B. Schlein, and V. Sohinger, *The mean-field limit of quantum Bose gases at positive temperature*, J. Amer. Math. Soc. **35** (2022), no. 4, 955–1030.
- [3] J. Fröhlich, A. Knowles, B. Schlein, and V. Sohinger, *The Euclidean  $\phi_2^4$  theory as a limit of an interacting Bose gas*, Preprint arXiv:2201.07632.
- [4] J. Glimm and A. Jaffe, *Positivity of the  $\phi_3^4$  Hamiltonian*, Fortschritte der Physik **21** (1973), no. 7, 327–376.
- [5] M. Lewin, P.T. Nam, and N. Rougerie, *Derivation of nonlinear Gibbs measures from many-body quantum mechanics*, Journal de l'École polytechnique-Mathématiques **2** (2015), 65–115.
- [6] M. Lewin, P.T. Nam, and N. Rougerie, *Classical field theory limit of 2D many-body quantum Gibbs states*, Preprint arXiv:1810.08370v1 (2018).
- [7] M. Lewin, P.T. Nam, and N. Rougerie, *Gibbs measures based on 1D (an)harmonic oscillators as mean-field limits*, J. Math. Phys. **59** (2018), no. 4, 041901.
- [8] M. Lewin, P.T. Nam, and N. Rougerie, *Classical field theory limit of many-body quantum Gibbs states in 2D and 3D*, Invent. Math. **224** (2021), 315–444.
- [9] B. Simon, *The  $P(\Phi)_2$  Euclidean (Quantum) Field Theory*, Princeton University Press, 1974.

**KPZ-type fluctuation exponents for interacting diffusions  
in equilibrium**

BENJAMIN LANDON

(joint work with Christian Noack, Philippe Sosoe)

Consider the family of diffusions  $\{u_j(t)\}_{j=1}^N$  interacting via the following system of stochastic differential equations,

$$\begin{aligned} du_1 &= -V'(u_1)dt + dB_0 - \theta dt + dB_1 \\ du_j &= (V'(u_{j-1}) - V'(u_j))dt + dB_j - dB_{j-1}, \quad j \geq 2 \end{aligned}$$

where  $\{B_j\}_{j \geq 1}$  are a family of independent Brownian motions,  $V : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth, convex potential, and the initial data of the system  $\{u_j(0)\}_{j=1}^N$  are iid with density proportional to  $e^{-V(u)-\theta u}$  (which turns out to be the unique invariant measure for the above system).

We study the following *height function* of this class of interacting diffusions,

$$W_{N,t} := \sum_{j=1}^N u_j(t) - B_0(t) + \theta t.$$

For the special case  $V(u) = e^{-u}$ , this height function has the same distribution as the free energy of a stationary version of the O'Connell-Yor polymer [4], which is an important example of a random growth model lying in the KPZ universality class and one of the central models of integrable probability.

Due to its exactly solvable nature, much is known about the O'Connell-Yor polymer. For  $N$  and  $t$  tuned in certain *characteristic directions*, Seppäläinen-Valko [5] showed that the fluctuations of the free energy are of the order  $N^{1/3}$ . Borodin-Corwin-Ferrari [1] and Imamura-Sasamoto [3] found the limiting distribution of the O'Connell-Yor polymer free energy and its stationary analog (the Tracy-Widom and Baik-Rains distributions, respectively) confirming the conjecture that this polymer lies in the KPZ universality class.

While the case  $V(u) = e^{-u}$  admits exact formulas for its distribution, the KPZ universality of the above system is conjectured to hold for general classes of potentials going beyond the exactly solvable cases [2].

In our work we consider the above system for a general class of potentials  $V(u)$  which include, but are not limited to, the exactly solvable O'Connell-Yor case  $V(u) = e^{-u}$ . In the same characteristic directions as in [5], we show that the height function  $W_{N,t}$  obeys,

$$cN^{2/3} \leq \text{Var}(W_{N,t}) \leq CN^{2/3},$$

which are the expected exponents for models lying in the KPZ universality class. In particular, our methods are dynamical and analytic in nature, and make no use of exact formulas or the polymer representation associated to the O'Connell-Yor polymer.

REFERENCES

- [1] A. Borodin, I. Corwin, P. Ferrari, *Free energy fluctuations for directed polymers in random media in 1+ 1 dimension*. Communications on Pure and Applied Mathematics, **67**, no. 7, 2014.
- [2] P. Ferrari, H. Spohn, R. Weiss, *Reflected Brownian Motions in the KPZ Universality Class*, SpringerBriefs in Mathematical Physics, Volume 18, 2017.
- [3] T. Imamura, T. Sasamoto, *Free energy distribution of the stationary O’Connell-Yor directed random polymer*. Journal of Physics A, 50.28, 2017.
- [4] N. O’Connell, M. Yor, *Brownian Analogues of Burke’s Theorem*. Stochastic Processes and Applications, **96**, 2, 2001.
- [5] T. Seppäläinen, B. Valkó, *Bounds for scaling exponents for a 1+1 dimensional directed polymer in a Brownian environment*, ALEA **7**, 2010.

**On the Markov property of local times of Brownian motion indexed by the Brownian tree**

JEAN-FRANÇOIS LE GALL

The Brownian tree may be defined as the tree coded by a Brownian excursion. To make this precise, write  $\mathbf{n}(de)$  for the Itô measure of positive Brownian excursions, so that  $\mathbf{n}(de)$  is a  $\sigma$ -finite measure. Given an excursion  $(e(t), 0 \leq t \leq \sigma)$ , we introduce the equivalence relation  $\sim_e$  on  $[0, \sigma]$  defined by

$$s \sim_e t \text{ if and only if } e(s) = e(t) = \min\{e(r) : s \wedge t \leq r \leq s \vee t\}.$$

Then the tree coded by  $e$  is the quotient space  $\mathcal{T}_e := [0, \sigma] / \sim_e$ , which is equipped with the distance induced by

$$d_e(s, t) := e(s) + e(t) - 2 \min\{e(r) : s \wedge t \leq r \leq s \vee t\}.$$

The tree  $\mathcal{T}_e$  under the measure  $\mathbf{n}(de)$  is called the Brownian tree (under the conditional probability  $\mathbf{n}(de \mid \sigma = 1)$ , this is Aldous’ CRT, up to an unimportant scaling factor). The volume measure  $\text{Vol}$  on  $\mathcal{T}_e$  is the pushforward of Lebesgue measure on  $[0, \sigma]$  under the canonical projection.

In what follows, we assume that the excursion  $e$  is distributed according to  $\mathbf{n}(de)$ . To define Brownian motion indexed by the Brownian tree, we consider, conditionally on the excursion  $(e(t), 0 \leq t \leq \sigma)$ , the centered Gaussian process  $(Z_s, 0 \leq s \leq \sigma)$  whose distribution is characterized by the properties  $Z_0 = 0$  and

$$\mathbf{E}[(Z_s - Z_t)^2] = d_e(s, t), \quad \forall s, t \in [0, \sigma]$$

Then  $(Z_s, 0 \leq s \leq \sigma)$  has a modification with continuous sample paths. Moreover the property  $s \sim_e t$  implies that  $Z_s = Z_t$ . It follows that we may view  $Z$  as indexed by  $\mathcal{T}_e$ , just by setting  $Z_a = Z_t$  whenever  $a \in \mathcal{T}_e$  is the equivalence class of  $t \in [0, \sigma]$ .

The total occupation measure of the process  $(Z_a, a \in \mathcal{T}_e)$  is the measure  $\Theta$  on  $\mathbb{R}$  defined by

$$\langle \Theta, \varphi \rangle = \int_{\mathcal{T}_e} \varphi(Z_a) \text{Vol}(da),$$

for any continuous function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ .

**Proposition 1.** *Almost everywhere, the measure  $\Theta$  has a continuous density on  $\mathbb{R}$ , which is denoted by  $(\ell_x)_{x \in \mathbb{R}}$ , and the mapping  $x \mapsto \ell_x$  is continuously differentiable.*

This essentially follows from Sugitani [4] and the relations between super-Brownian motion and Brownian motion indexed by the Brownian tree (see also Bousquet-Mélou and Janson [2]). We write  $\dot{\ell}_x$  for the derivative of  $x \mapsto \ell_x$ .

In a way analogous to the classical Ray-Knight theorems for Brownian local times, one may expect the process  $(\ell_x)_{x>0}$  to be Markov. This is not true (informally, to predict the future of this process after “time”  $y$ , one needs the value of  $\ell_y$ , but clearly also the value of the derivative  $\dot{\ell}_y$ ). However, we have the following statement

**Theorem 2.** *The process  $(\ell_x, \dot{\ell}_x)_{x>0}$  is Markov.*

This makes sense because, although we are dealing with a  $\sigma$ -finite measure (the excursion  $e$  is distributed according to  $\mathbf{n}(de)$ ), the event where  $\sup\{\ell_x : x \geq \varepsilon\} > 0$  has finite measure, for every  $\varepsilon > 0$ .

The proof of the theorem heavily relies on the excursion theory developed in [1]. We also refer to [3] for closely related results in a discrete setting.

#### REFERENCES

- [1] C. Abraham, J.-F. Le Gall, *Excursion theory for Brownian motion indexed by the Brownian tree*, J. Eur. Math. Soc. (JEMS) **20** (2018), 2951–3016.
- [2] M. Bousquet-Mélou, S. Janson, *The density of the ISE and local limit laws for embedded trees*, Ann. Appl. Probab. **16** (2006), 1597–1632.
- [3] G. Chapuy, J.-F. Marckert, in preparation.
- [4] S. Sugitani, *Some properties for the measure-valued branching diffusion processes*. J. Math. Soc. Japan **41** (1989), 436–462.

## Stationarity and space-time covariance for KPZ growth models in half-space

ALESSANDRA OCCELLI

We consider two models of last passage percolation (LPP) with exponential weights in half-space geometry. A last passage percolation is a directed up-right path in a random environment that attempts to maximize its energy, which is given by the sum of the random variables visited by the path. We consider LPPs on the octant  $\{(i, j) \in \mathbb{Z}_{>0}^2, i \geq j\}$  and we place independent random variables  $\omega_{i,j}$  on each site of the lattice. Thus, the last passage time between  $(1, 1)$  and  $(n, m)$  is defined as

$$(1) \quad L_{(1,1) \rightarrow (n,m)} = \max_{\pi: (1,1) \rightarrow (n,m)} \sum_{(i,j) \in \pi} \omega_{i,j}.$$

Paths which maximize (1) are called geodesics.

As in full-space, also the half-space LPP has an interpretation in terms of an exclusion process, the totally asymmetric simple exclusion process (TASEP) on the half-line with a reservoir of particles in the origin. The connection between TASEP and LPP is as follows. If  $\omega_{i,j}$ ,  $i > j$  is an exponential random variable,

then it represents the waiting time of particle  $j$  to jump from site  $i - j - 1$  to site  $i - j$ , while the diagonal weights  $\omega_{i,i}$  regulate the rate of creation of particles in the origin. It follows that

$$\mathbb{P}(N_{m-n+1}(t) \leq n) = \mathbb{P}(L_{m,n} \geq t),$$

where  $N_x(t)$  is the current at site  $x$ .

The first model is a point-to-point last passage percolation in half-space, i.e. with weights being exponentially distributed of parameter 1 in the bulk and of parameter  $\rho$  on the diagonal. It was proved in [1],[4],[5] that the fluctuations of the last passage time are influenced by the diagonal weights and that the limit distribution interpolates between the GSE Tracy–Widom ( $\rho > 1/2$ ), the GOE Tracy–Widom ( $\rho = 1/2$ ), the Gaussian ( $\rho < 1/2$ ) and the GUE Tracy–Widom distributions (end point far away from the diagonal). The limit process for this model was defined in [1] as a marginal of Pfaffian Schur processes.

The second model is a stationary last passage percolation in half-space, which is defined starting from the point-to-point model by adding an extra boundary condition on the horizontal line, i.e. weights exponentially distributed of parameter  $1 - \rho$ , and by putting 0 in the origin. This model is stationary in the sense of [6]: the increments of the last passage time along the horizontal and vertical directions are given by sums of i.i.d. random variables. For this model, in [8] we determine the limiting distribution of the last passage time in a critical window close to the origin. The result is a new two-parameter family of distributions: one parameter for the strength of the diagonal bounding the half-space (strength of the source at the origin in the equivalent TASEP language) and the other for the distance of the point of observation from the origin. The distribution is expressed in terms of the product of a Fredholm pfaffian (limit distribution of the point-to-point model, in other words, the half-space equivalent of the GUE Tracy–Widom distribution) and certain explicit functions that can be written as contour integrals of Airy-like functions.

This result is a generalisation of what is observed in full space, i.e. the one-parameter family giving the Baik–Rains distributions [3], which depends only on the distance from the diagonal. We also show that far away from the characteristic line, the distributions indeed converge to the Baik–Rains family. The strategy to prove this results heavily relies on the integrable structure of the point-to-point model. In fact, we derive our results using a related inhomogeneous integrable model having Pfaffian correlations, together with careful analytic continuation, and steepest descent analysis.

In [9] we study the multi-point distribution of the same model. We derive both finite-size and asymptotic results for this distribution: we observe a new one-parameter process we call *half-space Airy<sub>stat</sub>*. It is a one-parameter generalization of the Airy<sub>stat</sub> process of [2], which is recovered far away from the diagonal.

We expect both the distribution and the process to be universal in the framework of stationary KPZ models in half-space geometry (e.g. compare with [7]).

In [11] we study several properties of the point-to-point half-space last passage percolation, in particular the two-time covariance. We extend the results obtained

for the full space model in [10] and we show that, when the two end points are at small macroscopic distance, then the first order correction to the covariance for the point-to-point model is the same as the one of the stationary model. In this case, our strategy is based on more probabilistic arguments, such as polymer localization. In order to obtain this result, we first derive comparison inequalities of the last passage increments with respect to the stationary increments. This is used to prove tightness of the point-to-point process as well as localization of the geodesics. Unlike for the full-space case, for half-space we have to overcome the difficulty that the point-to-point model in half-space with generic start and end points is not known.

#### REFERENCES

- [1] J. Baik, G. Barraquand, I. Corwin, and T. Suidan, *Pfaffian Schur processes and last passage percolation in a half-quadrant*, Ann. Probab. **46** (2018), 3015–3089.
- [2] J. Baik, P.L. Ferrari, and S. Péché, *Limit process of stationary TASEP near the characteristic line*, Comm. Pure Appl. Math. **63** (2010), 1017–1070.
- [3] J. Baik and E.M. Rains, *Limiting distributions for a polynuclear growth model with external sources*, J. Stat. Phys. **100** (2000), 523–542.
- [4] J. Baik and E.M. Rains, *Algebraic aspects of increasing subsequences*, Duke Math. J. **109** (2001), 1–65.
- [5] J. Baik and E.M. Rains, *The asymptotics of monotone subsequences of involutions*, Duke Math. J. **109** (2001), 205–281.
- [6] M. Balázs, E. Cator, and T. Seppäläinen, *Cube root fluctuations for the corner growth model associated to the exclusion process*, Electron. J. Probab. **11** (2006), 1094–1132.
- [7] G. Barraquand, A. Krajenbrink, and P. Le Doussal, *Half-space stationary Kardar–Parisi–Zhang equation*, J. Stat. Phys. **181** (2020), no. 4, 1149–1203.
- [8] D. Betea, P.L. Ferrari, and A. Occelli, *Stationary half-space last passage percolation*, Comm. Math. Phys. **377** (2020), 421–467.
- [9] D. Betea, P.L. Ferrari, and A. Occelli, *The half-space Airy stationary process*, Stoch. Process. Appl. **146** (2022), 207–263.
- [10] P.L. Ferrari and A. Occelli, *Time-time covariance for last passage percolation with generic initial profile*, Math. Phys. Anal. Geom. **22** (2019), 1.
- [11] P.L. Ferrari and A. Occelli, *Time-time covariance for last passage percolation in half-space*, arXiv:2204.06782 (2022).

### KPZ fluctuations in the planar stochastic heat equation

JEREMY QUASTEL

(joint work with Alejandro Ramirez, Balint Virag)

We study the stochastic heat equation

$$(4) \quad \partial_t u = \frac{1}{2} \Delta u + u \xi,$$

where  $\xi$  is a white noise on  $\mathbb{R}^2$  that does not depend on time. The equation is taken in the *Skorokhod* sense, motivated by the interpretation as a polymer free energy. By the Feynman-Kac formula one expects the solution of the stochastic

heat equation (4) to have a representation

$$(5) \quad u(t, x) = E_B \left[ : \exp : \left\{ \int_0^t \xi(B(s)) ds \right\} \right] p(t, x).$$

This is heuristic, and the colons around the exponential are simply to remind us that some sort of renormalization will be required. The  $x$  is hidden on the right side of equation in the fact that the expectation is over a 2d Brownian bridge  $B$  from 0 to  $x$ . If we tried to take the exponent literally, it would be  $\int_{\mathbb{R}^2} \xi m$  where  $m$  is the occupation measure of the Brownian bridge. But  $m \notin L^2(\mathbb{R}^2)$  so it doesn't make sense.

Let  $e_j$  be bounded functions forming an orthonormal basis of  $L^2(\mathbb{R}^2)$ ,  $\xi_j = \langle e_j, \xi \rangle$ ,  $m_j = \int_0^t e_j(B(s)) ds$  and

$$(6) \quad Z_n(\xi) = E_B e^{\sum_{j=1}^n m_j \xi_j - \frac{1}{2} \sum_{j=1}^n m_j^2}.$$

$Z_n$  is a positive martingale w.r.t.  $\sigma(\xi_1, \dots, \xi_n)$ . If it converges in  $L^1$ , i.e. is *uniformly integrable*, then

$$(7) \quad u(t, x) = p(t, x) \lim_{n \rightarrow \infty} Z_n$$

turns out to be the Skorokhod solution with Dirac initial data. Other initial data can be solved by convolving with it. The reader can consult [3] for a nice introduction to the Skorokhod integral.

Formally,  $\sum_{j=1}^\infty m_j^2 = \int_0^t \int_0^t \delta_0(B(s) - B(s')) ds ds'$ , the *self-intersection local time*. It does not exist without a renormalization. If  $\delta_0^\epsilon$  is a smoothed out Dirac, by convolving with a heat kernel at time  $\epsilon$ , then [2]

$$(8) \quad \int_0^t \int_0^t \delta_0^\epsilon(B(s) - B(s')) ds ds' - c |\log \epsilon| t \rightarrow \gamma_t(B)$$

is the *renormalized self-intersection local time*. A more popular notion of solution to (4) is called PAM (for parabolic Anderson model). If we try to make sense of (5) by smoothing out the white noise to get  $\int_{\mathbb{R}^2} \xi_\epsilon m$ , and then subtracting something in the exponent to compensate as we remove the smoothing, PAM means we subtract  $c |\log \epsilon| t$ , while Skorokhod means we subtract  $c |\log \epsilon| t + \gamma_t(B)$ . We will also construct polymer measures corresponding to the solution of the stochastic heat equation. An advantage of the Skorokhod interpretation is that the expectation of the resulting polymer measure is the original Brownian bridge. On the other hand, the PAM interpretation results in a solution operator with the semi-group property, while Skorokhod does not. The choice is reminiscent of Itô vs. Stratonovich for stochastic integrals and is more of a modelling issue.

As we saw above, to obtain a solution we need to show uniform integrability of the martingale  $Z_n$ . Typically, one might try to bound  $E[Z_n^2]$ . It is fairly easy to see that this is given by  $E_{B, B'} [e^{\sum_{j=1}^n m_j m'_j}]$ , where  $B'$  is an independent copy of  $B$ , and  $m'_j$  are the coefficients of its occupation measure. Now

$$(9) \quad \sum_{j=1}^\infty m_j m'_j = \int_0^t \int_0^t \delta_0(B(s) - B'(s')) ds ds',$$

the *mutual intersection local time* of the two Brownian bridges. Let's call it  $\alpha_t(B, B')$ . It is not hard to check that  $E_{B, B'}[e^{\alpha_t(B, B')}]$  is only finite for  $t \leq t_c$  (see [2]). The critical time  $t_c$  is related to the constant in the Gagliardo-Nirenberg-Sobolev inequality [1]. So we only have  $\sup_n E[Z_n^2(t, x)] < \infty$  for  $t \leq t_c$ . Because of this, a solution theory was previously only known up to this time  $t_c$ .

We show that for any  $c, \delta > 0$  there is a set of Brownian bridge paths  $A$  with  $P(A) > 1 - \delta$  such that

$$(10) \quad E_{B, B'}[\mathbf{1}_{B, B' \in A} e^{c\alpha_1}] < \infty.$$

The set  $A = \{\text{Holder norm}, M \leq L\}$  where

$$(11) \quad M = \frac{\epsilon^{-2}}{|\log \epsilon|} \max_{\mathbb{R}^2 = \cup \{\epsilon \times \epsilon \text{ boxes}\}} \delta \sum_{i=0}^{\lfloor 1/\delta \rfloor} \mathbf{1}(B(\delta i) \in \text{box}).$$

This means that for the expectation with respect to the white noise  $\xi$ ,

$$(12) \quad E_\xi[E_B[\mathbf{1}_{B \in A}(e^{\sum_{j=1}^n \xi_j m_j - \frac{1}{2} m_j^2})^2]] \leq E[\mathbf{1}_{B, B' \in A} e^{t\alpha_1}]$$

and

$$(13) \quad E_\xi[E_B[\mathbf{1}_{B \in A^c} |e^{\sum_{j=1}^n \xi_j m_j - \frac{1}{2} m_j^2}|]] = P(A^c)$$

which tells us that  $Z_n$  is uniformly integrable.

This produces a solution  $Z \in L^1(\xi)$  for all time. The construction also yields the polymer measure. The method is also very good for convergence. If  $m^n$  are occupation measures converging to a limiting occupation measure  $m$ , then under reasonable conditions the corresponding  $Z(m^n)$  converge in  $L^1$  to  $Z(m)$ . In fact, all one needs is that the corresponding mutual intersection local times converge in  $L^2$ .

One also sees that the law of  $\xi + m$ , averaged over  $m$ , is absolutely continuous with respect to the law of the white noise  $\xi$ , and  $Z$  is the Radon-Nikodym derivative.

The one dimensional stochastic heat equation with space-time white noise,

$$(14) \quad \partial_t z = \partial_x^2 z + \xi z$$

fits into the same picture. Recall that  $h = \log z$  is the Cole-Hopf solution of KPZ. Now the occupation measure is the occupation measure of the graph  $(s, B(s))$  of the one dimensional Brownian bridge.

In particular, we show that on appropriate scales, the solution  $u$  of the planar stochastic heat equation (4) has KPZ fluctuations: *For any  $t > 0$  and  $a \in \mathbb{R}$  as  $N \rightarrow \infty$ , we have*

$$(15) \quad P(u(Nt, 0, N^{3/2}t) \times N e^{N^{2t/2} \sqrt{2\pi t}} \leq a) \rightarrow F_{KPZ}(t, a),$$

where  $F_{KPZ}(t, \cdot)$  are the KPZ crossover distributions

$$F_{KPZ}(t, a) = P(z(t, 0) \leq a).$$

It is actually not hard to see why this is true. The above constructions are just to provide renormalizations which make the following heuristic scaling rigorous:

$$\begin{aligned}
 & E_{(0,0) \rightarrow (N^{3/2}t, N^{1/2}x)} \left[ e^{\int_0^{Nt} \xi(B_1(s), B_2(s)) ds} \right] \\
 &= E_{(0,0) \rightarrow (N^{3/2}t, N^{1/2}x)} \left[ e^{N \int_0^t \xi(B_1(Ns), B_2(Ns)) ds} \right] \\
 &= E_{(0,0) \rightarrow (t,x)}^{\text{cov}=\text{diag}(N^{-2}, 1)} \left[ e^{N \int_0^t \xi(N^{3/2}B_1(s), N^{1/2}B_2(s)) ds} \right] \\
 &= E_{(0,0) \rightarrow (t,x)}^{\text{cov}=\text{diag}(N^{-2}, 1)} \left[ e^{\int_0^t \tilde{\xi}(B_1(s), B_2(s)) ds} \right] \\
 &\sim E_{0 \rightarrow x} \left[ e^{\int_0^t \tilde{\xi}(s, B_2(s)) ds} \right].
 \end{aligned}$$

REFERENCES

[1] Chen, Xia, *Exponential asymptotics and law of the iterated logarithm for intersection local times of random walks*, Ann. Probab., 32, 3248–3300 2004.  
 [2] Le Gall, Jean-François, *Exponential moments for the renormalized self-intersection local time of planar Brownian motion*, Séminaire de Probabilités, XXVIII, Lecture Notes in Math., 1583, 172–180, Springer, Berlin, 1994.  
 [3] Nualart, David, *The Malliavin calculus and related topics*, Probability and its Applications (New York), Springer-Verlag, Berlin, 2006.

**Critical exponents for three-dimensional percolation models with long-range dependence**

PIERRE-FRANÇOIS RODRIGUEZ

(joint work with Subhajit Goswami, Franco Severo, Alexander Drewitz, Alexis Prévost)

Consider the Gaussian free field  $\varphi$  on  $\mathbb{Z}^3$ . Associated to  $\varphi$  are the following two percolation models:

- Model  $(M1)$ , introduced by Lebowitz and Saleur [4], consists of looking at the mountaintops in this landscape, i.e. study percolation of the excursion sets  $\{x \in \mathbb{Z}^3 : \varphi_x \geq h\}$  for a varying height parameter  $h \in \mathbb{R}$ .
- Model  $(\widetilde{M1})$  is the bond percolation model obtained by retaining each nearest-neighbor edge  $\{x, y\}$  in  $\mathbb{Z}^3$  conditionally on  $\varphi$  independently with probability  $1 - \exp\{- (\varphi_x - h)_+ (\varphi_y - h)_+\}$ , where  $a_+ = \max\{a, 0\}$  denotes the positive part of  $a$ .

Thus, it is harder to percolate in  $(\widetilde{M1})$ , for not only must the field value at the endpoints  $x$  and  $y$  exceed value  $h$ , but also the ensuing ‘disorder variable’ on the edge survive. In fact, one knows that the critical parameters  $h_*$  and  $\tilde{h}_*$  corresponding to percolation in  $(M1)$  and  $(\widetilde{M1})$ , respectively, satisfy

$$0 = \tilde{h}_* < h_* < \infty,$$

which reflects this intuitive picture, and also implies that the phase transition for both models is non-trivial. Note that the field  $\varphi$  induces long-range correlations,

whereby the covariance between occupation variables around 0 and  $x$  decays like  $|x|^{-a}$  with  $a = 1$  as  $|x| \rightarrow \infty$ . As it turns out, the exponent  $a$ , which comes as part of the data, plays a central role in determining the scaling behavior of the models at and near criticality.

The goal of this talk is to present two related results, one for each of the models  $(M1)$  and  $(\widetilde{M1})$ , aimed at exhibiting this scaling behavior. For simplicity, the focus will be on one specific quantity, the (truncated) one-arm observable  $\tau_h^{\text{tr}}(L)$ , which is the probability that the open cluster of the origin at level  $h$  is bounded and intersects the boundary of the ball of radius  $L \geq 1$  around 0. We write  $\tilde{\tau}_h^{\text{tr}}(L)$  for the corresponding quantity for  $(\widetilde{M1})$ . The first result is:

**Theorem 1** ([3]). *For  $(M1)$  and all  $h \in \mathbb{R}$ ,*

$$-\frac{\log L}{L} \log \tau_h^{\text{tr}}(L) \rightarrow \frac{\pi}{6} \xi(h)^{-1} \text{ as } L \rightarrow \infty,$$

where  $\xi(h) = |h - h_*|^{-\nu}$  with  $\nu = 2$ .

Among other things, Theorem 1 settles the leading-order decay of  $\tilde{\tau}_h^{\text{tr}}(\cdot)$  away from criticality, which is sub-exponential. The lower bounds are obtained by a change of measure argument, and the upper bounds require a delicate coarse-graining. The decay turns out to be entirely carried by deviations of harmonic averages on small macroscopic scales, as those derived in [6] in a different context, yielding a cost governed by the capacity induced by the Dirichlet form naturally associated to  $\varphi$ . Moreover, these bounds produce the sharp polynomial divergence of  $\xi(h)$  as  $h$  approaches  $h_*$ . We return to this below.

The results for  $(\widetilde{M1})$  refine this picture, by providing estimates which are quantitative in both  $h$  and  $L$ .

**Theorem 2** ([1]). *For  $(\widetilde{M1})$ , with  $\xi = \xi(h) = |h|^{-\nu} (= |h - \tilde{h}_*|^{-\nu})$  and  $\nu = 2$ ,*

$$\tilde{\tau}_h^{\text{tr}}(L) \asymp \tilde{\tau}_0^{\text{tr}}(L) \exp \left\{ -c \frac{L/\xi}{\log((L/\xi)^{\nu/2})} \right\}.$$

Here,  $\asymp$  means that the two sides are comparable up to numerical factors, with the upper bound “ $\leq$ ” holding for all  $|h| < 1$  and  $L \geq 1$  and the lower bound “ $\geq$ ” for all  $|h| < 1$  and  $L \leq \xi$  or  $L \geq \xi(\log \xi)^\delta$  where  $\xi = \xi(h)$ , for some  $\delta \in (0, 1)$ .

The results of Theorem 2 are nearly optimal. The bounds witness the transition from a near-critical picture  $L \lesssim \xi$  where  $\tilde{\tau}_h^{\text{tr}}(L) \approx \tilde{\tau}_0^{\text{tr}}(L)$ , to an off-critical one at scales  $L \gg \xi$  governed by the rapidly decaying factor. The system size is thus naturally measured in units of the (correlation) length scale  $\xi$ , which diverges polynomially at the critical point. The value of the associated exponent  $\nu$  fits with a prediction of Weinrib and Halperin [7] based on an extended Harris criterion, which suggests that  $\nu = \frac{2}{a}$  (recall that  $a = 1$  here).

The talk will highlight the mechanisms underlying the formation of long clusters in Theorems 1 and 2. In the latter case, the critical cost  $\tilde{\tau}_0^{\text{tr}}(L)$ ,  $L \lesssim \xi$ , emerges when producing a “blob,” i.e. forcing the cluster  $C^h$  of 0 to have large capacity. The latter is an integrable quantity for  $(\widetilde{M1})$ : as a result of [2], one knows that

$$\mathbb{P}[\text{cap}(C^0) \geq r] \sim r^{-\frac{1}{2}}, \text{ as } r \rightarrow \infty;$$

noteworthy, along with the fact that clusters in  $(\widetilde{M1})$  at levels  $h = -\varepsilon$  are carried by random interacements at intensity  $u = \varepsilon^2/2$ , which automatically percolate, the previous estimate implies continuity of the phase transition. In the cross-over regime  $L \approx \xi$ , the cluster in  $(\widetilde{M1})$  is carried by essentially one random walk trajectory, which emanates from the explored region and escapes to  $\infty$ , on account of a suitable signed isomorphism theorem [5]; forcing the capacity to be suitably large is exactly designed to give birth to this trajectory. Finally at scales  $L \gg \xi$ , one can prove a finite-size criterion to continue building the cluster by stacking good boxes. The criterion corresponds to a sharp quantitative version of some of the elements underlying the proof of Theorem 1.

Currently, all results generalize to any transient graph with polynomial volume growth of the form  $R^d$  for  $d \geq a+2$  and Green's function decay with exponent  $0 < a \leq 1$ , with  $\nu = 2/a$  and stretched exponential decay of the form  $\exp\{-c(L/\xi)^a\}$  when  $a < 1$ . The presence of  $\log((L/\xi) \vee 2)$  rather than the weaker  $\log(L \vee 2)$  for the upper bound in Theorem 2 is part of ongoing work to extend these results, notably to higher values of  $a$ .

#### REFERENCES

- [1] A. Drewitz, A. Prevoist and P.-F. Rodriguez, *Critical exponents for a percolation model on transient graphs*, Preprint, arXiv:2101.05801 (2021).
- [2] A. Drewitz, A. Prevoist and P.-F. Rodriguez, *Cluster capacity functionals and isomorphism theorems for Gaussian free fields*, Probability Theory and Related Fields **183** (2022), 255–313.
- [3] S. Goswami, P.-F. Rodriguez and F. Severo, *On the radius of Gaussian free field excursion clusters*, Annals of Probability, to appear (2022).
- [4] J. L. Lebowitz and H. Saleur, *Percolation in strongly correlated systems*, Physica A: Statistical Mechanics and its Applications **138** (1986), 194–205.
- [5] T. Lupu, *From loop clusters and random interacements to the free field*, Annals of Probability **44** (2016), 2117–2146.
- [6] A.-S. Sznitman, *Disconnection and level-set percolation for the Gaussian free field*, Journal of the Mathematical Society of Japan **67** (2015), 1801–1843.
- [7] A. Weinrib, *Long-range correlated percolation*, Physical Review B **29** (1984), 387–395.

### The maximum of log-correlated Gaussian fields in random environments

FLORIAN SCHWEIGER

(joint work with Ofer Zeitouni)

We study logarithmically correlated random fields on a  $d$ -dimensional lattice. It is expected that many features of such fields are universal in the sense that they do not depend on the precise law of the field. An example of such a feature is the maximum of the field, when restricted to a box of sidelength  $N$ , say. Here it is expected that the maximum is of order  $c_1 \log N - c_2 \log \log N$  for some explicit constants  $c_1, c_2$  and that the recentered maximum even converges in distribution to a non-degenerate random variable.

For general (non-Gaussian) log-correlated fields, such precise results are often out of reach. The situation is better for Gaussian log-correlated fields. In particular, it is known that the recentered maximum of the two-dimensional discrete Gaussian free field converges in distribution [1], and due to work by Biskup, Louidor and others we have even much finer results on the extremal process. Regarding universality, there are results by Madaule and by Ding, Roy and Zeitouni. In particular, in [3] it is proved that all  $d$ -dimensional log-correlated Gaussian fields satisfying certain assumptions on their covariances share the property that their recentered maximum converges in distribution. These assumptions are satisfied by the two-dimensional discrete Gaussian free field and also by the four-dimensional discrete membrane model [4].

The focus of the talk was on log-correlated Gaussian fields in (quenched) random environments, and based on [5]. Unfortunately, such fields typically do not satisfy the assumptions from [3], because uniform bounds on the covariances as required there cannot hold at points near which the environment is atypical. Motivated by this, we presented a generalization of the result in [3] where the uniform assumptions on the covariances are replaced by non-uniform assumptions in terms of some random scales, together with bounds on the number of points where those scales are large. The proof is an adaptation of the one in [3], with additional technicalities arising from the need to control the maximum of the field on the atypical points.

This general result should be applicable to various log-correlated Gaussian fields in random environments. One particularly interesting example is the discrete Gaussian free field on the infinite cluster of Bernoulli bond percolation with parameter  $p > \frac{1}{2}$  in two dimensions. Here we can prove that if  $p$  is sufficiently close to 1, then the aforementioned general result applies, and consequently the recentered maximum of the field on a box of sidelength  $N$  converges in distribution. Our proof uses recent results by Armstrong, Dario and Gu on quantitative homogenization on percolation clusters (in particular [2]), as well as various large-deviation results for the local behavior of the cluster. It is an interesting open question whether this result still holds true for any  $p > \frac{1}{2}$ .

#### REFERENCES

- [1] M. Bramson, J. Ding, O. Zeitouni, *Convergence in law of the maximum of the two-dimensional discrete Gaussian free field*, Comm. Pure Appl. Math. 69 (2016), no. 1, 62–123.
- [2] P. Dario, C. Gu, *Quantitative homogenization of the parabolic and elliptic Green's functions on percolation clusters*, Ann. Probab. 49 (2021), no. 2, 556–636.
- [3] J. Ding, R. Roy, O. Zeitouni, *Convergence of the centered maximum of log-correlated Gaussian fields*, Ann. Probab. 45 (2017), no. 6A, 3886–3928.
- [4] F. Schweiger, *The maximum of the four-dimensional membrane model*, Ann. Probab. 48 (2020), no. 2, 714–741.
- [5] F. Schweiger, O. Zeitouni, *The maximum of log-correlated Gaussian fields in random environment*, arXiv:2205.07210.

**Dimer model in 3D**

SCOTT SHEFFIELD

(joint work with Nishant Chandgotia, Catherine Wolfram)

The 3D dimer model is more complex than its 2D counterpart. Classical 2D tools that fail to apply in 3D include Kasteleyn matrices and determinants, spanning tree bijections, monotonicity, FKG inequalities, amoeba and Ronkin function constructions, and so on. Even the simplest “local move connectedness” results fall apart in dimensions higher than two, although several papers have been written with partial results.

Nonetheless, the 3D dimer model turns out to be surprisingly interesting as a model of a random divergence-free flow. There are few tools (such as Hall’s marriage theorem and the non-intersecting path interpretation) that continue to apply in three dimensions, and we build on these to establish a large deviation principle for the random flow which is analogous to the 2D results of Cohn, Kenyon and Propp, with a unique rate function minimizer. We also present several interesting simulations and animations, including for higher dimensional Aztec diamonds and variants. There are many open problems here on which we would welcome assistance. Example: “Is there a finite set of local moves that, for all  $n$ , connect the tilings of a  $2n$  by  $2n$  by  $2n$  box?”

The continuum analog of this model is also straightforward to describe. If one starts with a 3-vector-valued white noise, the orthogonal projection onto the space of “curl-free fields” gives the gradient of a Gaussian free field (GFF) while the orthogonal projection onto the space of “divergence-free fields” gives a very closely related object called the Gaussian divergence-free field (GDFF) which itself has many beautiful properties.

**The least singular value of the deformed Ginibre ensemble**

TATYANA SHCHERBINA

(joint work with Mariya Shcherbina)

We study the random  $n \times n$  matrices  $H = A + H_0$ , where  $A$  is some general  $n \times n$  matrix with complex entries, and  $H_0$  is drawn from the complex Ginibre ensemble, i.e.  $H_0$  has i.i.d. complex Gaussian entries  $\{h_{ij}^{(0)}\}_{i,j=1}^n$  such that

$$\mathbb{E}[h_{ij}^{(0)}] = 0, \quad \mathbb{E}[|h_{ij}^{(0)}|^2] = 1/n, \quad \mathbb{E}[(h_{ij}^{(0)})^2] = 0.$$

Deformation  $A$  can be deterministic or random (but in this case it is independent of  $H_0$ ).

Such matrices are important in communication theory, where  $A$  is considered as a *signal*, and  $H_0$  as a *noise* matrix. In particular, one is interested in effective numerical solvability of a large system of linear equations  $Hx = b$  which is determined by the behaviour of the smallest singular value  $\sigma_1(H)$  of  $H$ .

The classical bound of Sankar, Spielman and Teng [5] states that the smallest singular value  $\sigma_1(H)$  is of order not smaller than  $n^{-1}$  (equivalently, the smallest eigenvalue  $\lambda_1(HH^*)$  of  $HH^*$  is of order not smaller than  $n^{-2}$ ), i.e.

$$(16) \quad \mathbb{P}\left(\lambda_1(HH^*) = (\sigma_1(H))^2 \leq x/n^2\right) \lesssim x, \quad x > 0$$

up to logarithmic corrections, and uniformly in  $A$ .

The bound is proved to be optimal for the case of pure complex Ginibre ensemble (i.e.  $A = 0$ ), see [1].

The matrix  $HH^*$  can be considered as the so-called deformed Laguerre ensemble, and its limiting eigenvalue distribution is well-known, see [4]. Moreover, almost surely no eigenvalues lie outside of any finite support neighbourhood of the limiting measure. Therefore, there are three possible situations: 0 is away of the support of the limiting spectral measure, and then  $\lambda_1(HH^*)$  has a positive constant lower bound; 0 lies in the bulk of the spectrum, and then (16) is expected to be optimal; the intermediate regime when 0 is near the edge of the spectrum.

In the bulk regime the lower bounds on  $\lambda_1(HH^*)$  with quite general  $A$  and even with the non-Gaussian  $H_0$  (with iid elements) have been obtained in [7] (although not uniformly in  $A$ ).

The edge regime is much less studied. However, as it was shown in [2] for the case of the constant diagonal shift of the Ginibre ensemble, i.e.  $A = -zI$ , the bound (16) can be improved in the edge regime  $|z| \sim 1$ .

Another important source of motivation is that an effective lower tail bound on the least singular value of  $H - z$  is an essential ingredient for the study of eigenvalues distribution of large non-Hermitian matrices. In particular, the results of Cipolloni, Erdős, Schröder [2] was used in their subsequent work [3] to remove the four moment matching condition in the classical edge universality result for non-Hermitian random matrices with iid entries by Tao, Vu [8]. Better understanding of higher order correlation functions of the shifted Ginibre ensemble in the bulk is expected to help to do the same thing for the universality in the bulk.

The main result of the current presentation extends the result of [2] to the case  $A = A_0 - zI$  with a complex deformation  $A_0$  satisfying some rather general conditions ensuring that the support  $D$  of the limiting normalized counting measure of eigenvalues of  $H = H_0 + A_0$  takes the nice form

$$D = \{z : \limsup_{\varepsilon \rightarrow 0} n^{-1} \text{Tr} (Y_0(z) + \varepsilon^2)^{-1} \geq 1\}, \quad Y_0(z) = (A_0 - z)(A_0 - z)^*$$

and that  $\partial D$  is a set of piece-wise smooth closed curves enclosing  $\sigma_0 = \{z : 0 \in \text{supp } \nu_z\}$  (with a high probability, if  $A$  is random) where  $\nu_z$  is the limiting spectral distribution of  $Y_0(z)$  (see [6] for the technical details).

The main result is the following theorem

**Theorem 1.** *Under the conditions above, if  $z \in D$ ,  $\text{dist}\{z, \partial D\} = \tilde{\delta}^2 n^{-1/2}$  with  $\tilde{\delta} \leq C_0$ , then (16) can be improved to*

$$\mathbb{P}\left\{\lambda_1((H-z)(H-z)^*) \leq xn^{-3/2}\right\} \leq Cx(1 + \log x)$$

uniformly in  $\tilde{\delta} \leq C_0$ .

The proof is based on the asymptotic analysis of the supersymmetric integral representation of the average trace of the resolvent of  $(H-z)(H-z)^*$ .

#### REFERENCES

- [1] A. Edelman, *Eigenvalues and condition numbers of random matrices*, SIAM J. Matrix Anal. Appl. **9** (1988), 543–560 .
- [2] G. Cipolloni, L. Erdős, D. Schröder, *Optimal lower bound on the least singular value of the shifted Ginibre ensemble*, Probability and Mathematical Physics **1:1** (2020), 101–146.
- [3] G. Cipolloni, L. Erdős, D. Schröder, *Edge universality for non-Hermitian random matrices*, Prob.Theory and Related Fields **179** (2021), 1– 28.
- [4] R. B. Dozier, J. W. Silverstein, *On the empirical distribution of eigenvalues of large dimensional information-plus-noise-type matrices*, J. Multivariate Anal. **98** (2007), 678 – 694.
- [5] A. Sankar, D. A. Spielman, and S.-H. Teng, *Smoothed analysis of the condition numbers and growth factors of matrices*, SIAM J. Matrix Anal. Appl. **28** (2006), 446–476.
- [6] M. Shcherbina, T. Shcherbina. *The least singular value of the general deformed Ginibre ensemble*, arXiv:2204.06026
- [7] T. Tao, V. Vu, *Smooth analysis of the condition number and the least singular value*, Math. Comp. **79** (2010), 2333–2352 .
- [8] T. Tao, V. Vu, *Random matrices: universality of local spectral statistics of non- Hermitian matrices*, Ann. Probab. **43** (2015), 782–874 .

## A dynamical approach to lattice Yang-Mills

HAO SHEN

Lattice Yang–Mills is a model of a collection of matrices. We consider its Langevin dynamic (stochastic quantization) which is a system of SDEs. Using this SDE system, we give a simple derivation of the Makeenko–Migdal loop equations, prove at strong coupling the uniqueness of infinite volume limit and ergodicity, log-Sobolev and Poincare inequalities, and apply these inequalities to prove large N limits and exponential decay of correlations. These are based on joint works with Scott Smith, Rongchan Zhu and Xiangchan Zhu.

The dynamics in continuum in 2D and 3D have been recently constructed by Chandra, Chevyrev, Hairer and the speaker [1, 2] – see the summaries of Chandra and Chevyrev’s talk. Part of the motivation of the talk is to see if certain properties of the YM model can be extracted from the dynamics with a simplification of a lattice cutoff.

Let’s start by recalling the definition of the lattice Yang–Mills model. Consider a square lattice (finite or infinite) of dimension  $d$ . Each edge  $xy$ , where  $x$  and  $y$  are nearest neighbor vertices, is associated with a Lie group element  $Q_{xy}$ . For the talk we simply assume that the Lie group is  $SO(N)$ . Here edges are directed, and  $Q_{xy} = Q_{yx}^{-1}$ . The lattice Yang–Mills model is defined by the measure  $\exp(\mathcal{S}(Q))d\mu_{Haar}$

on  $\mathcal{Q} = \{Q_e : e \in E\}$ , where  $E$  is the set of all positively directed edges,  $\mu_{Haar}$  is the Haar measure, and

$$\mathcal{S}(Q) = \beta N \sum_p Tr(Q_{xy}Q_{yz}Q_{zw}Q_{wx}).$$

Here,  $p$  sums over all the plaquettes, and we wrote  $x, y, z, w$  for the four vertices of the plaquette  $p$ . On a finite lattice this is clearly a well-defined measure. The importance of the model is that it has gauge symmetry:  $\forall SO(N)$  valued function  $g$  on vertices,  $\mathcal{S}$  is invariant under

$$Q_{xy} \mapsto g_x Q_{xy} g_y^{-1}.$$

In our “dynamical approach” we study the dynamic which leaves the lattice YM measure invariant (so called “stochastic quantization”). This yields interesting stochastic processes. The study of these dynamics may also lead to proofs for properties of the measure. We consider the following SDE (Langevin dynamic)

$$dQ = \nabla \mathcal{S}(Q)dt + \sqrt{2}d\mathfrak{B}$$

where  $Q = (Q_e : e \in E)$  and  $\mathfrak{B} = (\mathfrak{B}_e : e \in E)$  where  $E$  is the set of all lattice edges. Each  $\mathfrak{B}_e$  is a Brownian motion on  $SO(N)$ , and the Brownian motions on different edges are independent. The measure  $\exp(\mathcal{S}(Q))d\mu_{Haar}$  is invariant.

We can give a more explicit form of this equation by some calculations, namely:

- (1) Recall  $\mathcal{S}(Q) = \beta N \sum_p Tr(Q_p)$  where we write  $Q_p \stackrel{def}{=} Q_{e_1} Q_{e_2} Q_{e_3} Q_{e_4}$  for the rest of this summary. So, we can calculate the gradient  $\nabla \mathcal{S}$  explicitly.
- (2) It is well-known that the Brownian motion  $\mathfrak{B}$  on  $SO(N)$  and the Brownian motion  $B$  on the Lie algebra  $so(N)$  are related by

$$d\mathfrak{B} = dB \circ \mathfrak{B} = dB \mathfrak{B} + \frac{c}{2} \mathfrak{B}dt,$$

where  $dB \circ \mathfrak{B}$  is the Stratonovich product and  $dB \mathfrak{B}$  is the Itô product, and  $c = -\frac{N-1}{2}$  (Itô–Stratonovich correction) for  $SO(N)$ .

With these we can obtain a more explicit SDE system parametrized by edges  $e$

$$dQ_e = -\frac{1}{2}N\beta \sum_{p \succ e} (Q_p - Q_p^*)Q_e dt - \frac{1}{2}(N-1)Q_e dt + \sqrt{2}dB_e Q_e$$

where  $p \succ e$  means plaquettes starting from the edge  $e$  (namely if  $e = xy$  then  $p$  must be of the form  $xyzw$  for some  $z, w$ ), and the last term is Itô.

The first application of this SDE is a simplified proof of Dyson–Schwinger equations. Fix a loop  $\gamma = e_1 e_2 \cdots e_n$  in the lattice. The observable  $W_\gamma = Tr(Q_{e_1} \cdots Q_{e_n})$  is called a Wilson loop. Expectations of  $W_\gamma$  satisfy recursive relations among different loops  $\gamma$ , which are called Dyson–Schwinger (a.k.a. Makeenko–Migdal equations or master loop equations). They state that  $\mathbf{E}[W_\gamma]$  is equal to a sum of terms of the form  $\sum_{\gamma'} \mathbf{E}[W_{\gamma'}]$  (with certain coefficients) where  $\gamma'$  are all possible ‘deformations’, ‘splittings’, ‘twists’, etc. of  $\gamma$ . This was first proved by Chatterjee in 2016 for  $SO(N)$  lattice YM using Stein’s method. Our new proof is extremely simple: since we have the SDE for  $Q_e$ , the proof just follows by applying Itô formula to  $\mathbf{E}W_\gamma$ , namely:

- The drift term  $\sum_{p \sim e} (Q_p - Q_p^*) Q_e dt$  yields deformations of  $\gamma$ , namely  $\gamma$  ‘deformed’ by one plaquette;
- The linear term  $Q_e dt$  yields the  $\mathbf{E}W_\gamma$  on left-hand side;
- The martingale term  $dB_e Q_e$  disappears upon taking  $\mathbf{E}$ ;
- Quadratic variation yields splittings and twists.

We refer to [3] for details.

For the following results, we write  $G = SO(N)$  and we assume  $|\beta| < \frac{N-2}{32(d-1)N}$ . We have

**Theorem.** ([4]) The SDE on entire  $\mathbf{Z}^d$  has a unique probabilistically strong solution in  $C([0, \infty); G^E)$ . Every tight limit of the finite volume YM measures is invariant under the above solution to the SDE (on  $\mathbf{Z}^d$ ). Under the above smallness assumption for  $\beta$  the invariant measure of the SDE (on  $\mathbf{Z}^d$ ) is unique.

The proof of the above results relies on a version on Kendall–Cranston coupling and applying Itô formula to a suitable distance between the SDE solutions with different initial conditions. See [4] for details.

Let’s discuss a few other results on lattice YM. Fix a finite volume. Under the above smallness assumption for  $\beta$ , the Bakry–Émery condition holds: for any tangent vector  $v$  (of the product Lie group)

$$\text{Ric}(v, v) - \text{Hess}_S(v, v) \geq K_S |v|^2, \quad K_S > 0.$$

It is a standard fact that  $\text{Ric}(v, v) = C_{\text{Ric}, N} |v|^2$  with  $C_{\text{Ric}, N} = \frac{N-2}{4}$  for  $SO(N)$ . By detailed calculations,  $|\text{Hess}_S(v, v)| \leq 8(d-1)N|\beta||v|^2$ . From these,  $K_S > 0$  leads to  $|\beta| < \frac{N-2}{32(d-1)N}$ . Note that this assumption is such that  $\beta$  still belongs to a non-empty open set when we take  $N \rightarrow \infty$  later.

Note that Bakry–Émery  $\Leftrightarrow$  Ergodicity  $\Leftrightarrow$  log-Sobolev  $\Rightarrow$  Poincaré inequality. Moreover, Ergodicity  $\Rightarrow$  uniqueness of invariant measure on finite lattice. The log-Sobolev and Poincaré inequalities pass to infinite volume, since they are dimension independent.

We show several applications of the Poincaré inequality: for  $F \in C^\infty(\mathcal{Q})$ ,

$$\text{Var}(F) \leq \frac{1}{K_S} \mathbf{E}(|\nabla F|^2).$$

Apply this inequality to  $F = \frac{1}{N} W_\gamma$ , and calculate the right-hand side. We get

$$\text{Var}\left(\frac{1}{N} W_\gamma\right) \leq \frac{1}{K_S} \frac{n(n-3)}{N} \rightarrow 0 \quad N \rightarrow \infty$$

which shows that the rescaled Wilson loop converges to a deterministic limit (so called master field). Studying fluctuations or higher order corrections would be interesting.

As another application of Poincaré inequality, one has

**Corollary (Mass gap):** Let  $f, g \in C_{\text{cyl}}^\infty(\mathcal{Q})$  with supports  $\Lambda_f \cap \Lambda_g = \emptyset$ . Then

$$\text{Cov}(f, g) \leq c_1 e^{-c_2 d(\Lambda_f, \Lambda_g)}$$

In particular  $f, g$  can be Wilson loops. The proof uses certain earlier ideas by Guionnet–Zegarlinski for other models.

#### REFERENCES

- [1] A. Chandra, I. Chevyrev, M. Hairer, H. Shen, *Langevin dynamic for the 2D Yang-Mills measure*, Publ. Math. IHÉS. To appear.
- [2] A. Chandra, I. Chevyrev, M. Hairer, H. Shen, *Stochastic quantisation of Yang-Mills-Higgs in 3D*, arXiv:2201.03487
- [3] H. Shen, S. Smith, R. Zhu, *A new derivation of the finite master loop equation for lattice Yang-Mills*, arXiv:2202.00880
- [4] H. Shen, R. Zhu, X. Zhu, *A stochastic analysis approach to lattice Yang-Mills at strong coupling*, arXiv:2204.12737

### On the spectral gap in the Kac-Luttinger model and Bose-Einstein condensation

ALAIN-SOL SZNITMAN

In this talk we report on the results of the recent preprint [1]. We consider the Dirichlet eigenvalues of the Laplacian among a Poissonian cloud of hard spherical obstacles of fixed radius in large boxes of  $R^d$ ,  $d \geq 2$ . In a large box of side-length  $2\ell$  centered at the origin, the lowest eigenvalue is known to be typically of order  $(\log \ell)^{-2/d}$ . We show in [1] that with probability arbitrarily close to 1 as  $\ell$  goes to infinity, the spectral gap stays bigger than  $\sigma(\log \ell)^{-(1+2/d)}$ , where the small positive number  $\sigma$  depends on how close to 1 one wishes the probability. Incidentally, the scale  $(\log \ell)^{-(1+2/d)}$  is expected to capture the correct size of the gap. Our result involves the proof of new deconcentration estimates. Combining this lower bound on the spectral gap with the results of Kerner-Pechmann-Spitzer [2], we infer a type-I generalized Bose-Einstein condensation in probability for a Kac-Luttinger system of non-interacting bosons among Poissonian spherical impurities, with the sole macroscopic occupation of the one-particle ground state when the density exceeds the critical value.

#### REFERENCES

- [1] A.S. Sznitman, *On the spectral gap in the Kac-Luttinger model and Bose-Einstein condensation*, available at arXiv:2203.08123 .
- [2] J. Kerner, M. Pechmann, and W. Spitzer, *On a condition for type-I Bose-Einstein condensation in random potentials in  $d$  dimensions.*, J. Math. Pures Appl. **143** (2020), 287–310.

**Directed polymers in the weak disorder regime, and exponential moments**

OFER ZEITOUNI

(joint work with Clement Cosco)

Given an (infinite) graph  $G = (V, E)$  and an i.i.d. collection of centered standard Gaussian random variables  $\{\omega(i, x)\}_{i \geq 0, x \in V}$ , the *directed polymer at inverse temperature  $\beta$*  is defined as

$$\frac{d\hat{P}^x}{dP^x} = \frac{\left[ e^{\sum_{n=1}^N \beta \omega(n, S_n) - N\beta^2/2} \right]}{W_N(x, \beta)},$$

where

$$W_N(x, \beta) = E^x \left[ e^{\sum_{n=1}^N \beta \omega(n, S_n) - N\beta^2/2} \right]$$

is the partition function, and  $P^x$  (with associated expectation  $E^x$ ) is the law of simple random walk  $(S_n)$  on the graph, started at  $x$ . For fixed  $\beta$ ,  $W_N := W_N(0, \beta)$  is a positive martingale and thus converges to a limit  $W_\infty(\beta)$ . The *weak disorder regime* is when  $W_\infty(\beta)$  is non-degenerate, i.e.  $W_\infty(\beta) > 0$ . The directed random polymer case has close relations with the stochastic heat equation (SHE) and the KPZ equation.

In the classical case,  $G = \mathbb{Z}^d$ , and then a weak disorder phase exists for  $d \geq 3$ . We refer to [2] for an extensive discussion and references, and to [4] for a discussion of the general case. For  $d = 2$ , such a weak disorder phase does not exist, but a meaningful rescaling was discovered in the context of the SHE by Bertini and Cancrini [1] and studied extensively in both the SHE and polymer setups by Caravenna, Sun and Zygouras [5, 6], see also [7], [8], [9] and [10].

To define the rescaling, introduce the mean intersection local time for two random walks  $(S_n^1, S_n^2)$  by

$$R_N = E^{0 \otimes 0} \left[ \sum_{n=1}^N \mathbf{1}_{S_n^1 = S_n^2} \right] \sim \frac{\log N}{\pi},$$

and set

$$(17) \quad \beta_N = \frac{\hat{\beta}}{\sqrt{R_N}}, \quad \hat{\beta} \geq 0.$$

Then, see [5], one has that

$$(18) \quad \forall \hat{\beta} < 1 : \quad \log W_N \xrightarrow{(d)} \mathcal{N} \left( -\frac{\lambda^2}{2}, \lambda^2 \right), \quad \text{with} \quad \lambda^2(\hat{\beta}) = \log \frac{1}{1 - \hat{\beta}^2}.$$

The convergence in (18) has recently been extended in [11] to the convergence of  $W_N$  to the exponential of a Gaussian, in all  $L^p$ .

The spatial behavior of  $W_N(\beta_N, x)$  is also of interest. Indeed, one has, see [6],

$$G_N(x) := \sqrt{R_N} \left( \log W_N(\beta_N, x\sqrt{N}) - \mathbb{E} \log W_N(\beta_N, x\sqrt{N}) \right) \xrightarrow{(d)} \sqrt{\frac{\hat{\beta}^2}{1 - \hat{\beta}^2}} G(x),$$

with  $G(x)$  a log-correlated Gaussian field on  $\mathbb{R}^2$ .

A natural question pertains to the study of extrema of  $G_N(x)$ . As a first step in this direction, we study the exponential moments of  $G_N = G_N(0)$ . Our main result reads as follows.

**Theorem [3]** *There exists  $\hat{\beta}_0 \leq 1$  so that if  $\hat{\beta} < \hat{\beta}_0$  and*

$$(19) \quad \limsup_{N \rightarrow \infty} \frac{3\hat{\beta}^2}{(1 - \hat{\beta}^2)} \frac{1}{\log N} \binom{q}{2} < 1,$$

then,

$$(20) \quad \mathbb{E}[W_N^q] \leq C e^{\binom{q}{2} \lambda^2 (1 + |\varepsilon_N|)},$$

where  $\varepsilon_N = \varepsilon(N, \hat{\beta}) \rightarrow 0$  as  $N \rightarrow \infty$ , and  $C = C(\hat{\beta}) > 0$ .

The proof shows that in the Theorem,  $\hat{\beta}_0$  can be taken as  $1/96$ , but we do not expect this to be optimal.

A complimentary lower bound, as well as multi-points exponential estimates, is part of work in progress.

## REFERENCES

- [1] Lorenzo Bertini and Nicoletta Cancrini. The two-dimensional stochastic heat equation: renormalizing a multiplicative noise. *Journal of Physics A: Mathematical and General*, 31(2):615–622, Jan 1998.
- [2] Francis Comets. *Directed polymers in random environments. École d'Été de Probabilités de Saint-Flour XLVI – 2016*. Cham: Springer, 2017.
- [3] Clement Cosco and Ofer Zeitouni. Moments of partition functions of 2D Gaussian polymers in the weak disorder regime - I *arXiv:2112.03767v3*, 2022.
- [4] Clement Cosco, I. Seroussi and O. Zeitouni. Directed polymers on infinite graphs. *Comm. Math. Phys.*, 386:395–432, 2021.
- [5] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. Universality in marginally relevant disordered systems. *Ann. Appl. Probab.*, 27(5):3050–3112, 10 2017.
- [6] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. The two-dimensional KPZ equation in the entire subcritical regime. *Annals Probab.*, 48:1086–1127, 2020.
- [7] Sourav Chatterjee and Alexander Dunlap. Constructing a solution of the (2+1)-dimensional KPZ equation. *Annals Probab.*, 48:1014–1055, 2020.
- [8] Yu Gu. Gaussian fluctuations of the 2d KPZ equation. *Stoch. Partial Differ. Equ. Anal. Comput.*, 8:150–185, 2020.
- [9] Jeremy Quastel, Yu Gu and Li-Cheng Tsai. Moments of the 2d SHE at criticality. *arXiv:1905.11310*, 2019.

- 
- [10] Shuta Nakajima and Makoto Nakashima. Fluctuation of two-dimensional stochastic heat equation and KPZ equation in subcritical regime for general initial conditions. *arXiv:2103.07243*, 2021.
- [11] Dimitris Lygkonis and Nikos Zygouras. Moments of the  $2d$  directed polymer in the subcritical regime and a generalization of the Erdős-Taylor theorem. *arXiv:2109.06115*, 2021.

## Participants

**Amol Aggarwal**

Department of Mathematics  
Harvard University  
Science Center 324 F  
1 Oxford Street  
Cambridge, MA 02138  
UNITED STATES

**Prof. Dr. Dmitry Belyaev**

Mathematical Institute, Oxford  
University  
Andrew Wiles Building, Woodstock  
Road  
OX2 6GG Oxford  
UNITED KINGDOM

**Federico Bertacco**

Imperial College London  
Department of Mathematics  
Huxley Building  
180 Queen's Gate  
London SW7 2AZ  
UNITED KINGDOM

**Ahmed Bou-Rabee**

Department of Mathematics  
The University of Chicago  
5734 South University Avenue  
Chicago, IL 60637-1514  
UNITED STATES

**Prof. Dr. Christian Brennecke**

Institute for Applied Mathematics  
University of Bonn  
Endenicher Allee 60  
Bonn 53115  
GERMANY

**Dr. Thomas Budzinski**

ENS Lyon  
UMPA  
46, Allee d'Italie  
69007 Lyon Cedex  
FRANCE

**Dr. Giuseppe Cannizzaro**

Department of Statistics  
University of Warwick  
Gibbet Hill Road  
Coventry CV4 7AL  
UNITED KINGDOM

**Baptiste Cerclé**

Bâtiment 307  
Laboratoire de Mathématiques d'Orsay,  
Université Paris-Saclay  
UMR 8628 CNRS  
Rue Michel Magat  
91405 Orsay Cedex  
FRANCE

**Dr. Ajay Chandra**

Imperial College London  
Department of Mathematics  
SW19 2LZ London  
UNITED KINGDOM

**Dr. Ilya Chevyrev**

School of Mathematics  
University of Edinburgh  
James Clerk Maxwell Bldg.  
Peter Guthrie Tait Road  
Edinburgh EH9 3FD  
UNITED KINGDOM

**Alice Contat**

Laboratoire de Mathématiques  
Université Paris Saclay  
Batiment 307  
91405 Orsay  
FRANCE

**Prof. Dr. Nicolas Curien**

Laboratoire de Mathématiques  
Université Paris Saclay  
Batiment 307  
91405 Orsay  
FRANCE

**Prof. Dr. Jian Ding**

Department of Statistics  
The Wharton School  
University of Pennsylvania  
3730 Walnut Street  
Philadelphia PA 19104-6340  
UNITED STATES

**Prof. Dr. Julien Dubedat**

Department of Mathematics  
Columbia University  
2990 Broadway  
NY 10027 New York  
UNITED STATES

**Sofia Dubova**

Department of Mathematics  
Harvard University  
Science Center  
One Oxford Street  
Cambridge MA 02138-2901  
UNITED STATES

**Dr. Raphaël Ducatez**

Unité de mathématiques pures et  
appliquées  
École normale supérieure de Lyon  
46, Allée d'Italie  
69364 Lyon Cedex 07  
FRANCE

**Prof. Dr. Christophe Garban**

Institut Camille Jordan  
Université de Lyon I  
43 Blvd. du 11 Novembre 1918  
69622 Villeurbanne Cedex  
FRANCE

**Prof. Dr. Massimiliano Gubinelli**

Institut für Angewandte Mathematik  
Universität Bonn  
Endenicher Allee 60  
53115 Bonn  
GERMANY

**Prof. Dr. Alice Guionnet**

Département de Mathématiques, UMPA  
École Normale Supérieure de Lyon  
46, allée d'Italie  
69364 Lyon Cedex 07  
FRANCE

**Dr. Ewain Gwynne**

University of Chicago  
University Avenue  
60637 Chicago  
UNITED STATES

**Prof. Dr. Martin Hairer**

Imperial College London  
Department of Mathematics  
SW72AZ London  
UNITED KINGDOM

**Dr. Nina Holden**

Institute for Theoretical Studies  
ETH Zürich, CLV Building  
Clausiusstrasse 47  
8092 Zürich  
SWITZERLAND

**Ruhong Jin**

Mathematical Institute  
Oxford University  
24-29 St. Giles  
Oxford OX1 3LB  
UNITED KINGDOM

**Prof. Dr. Antti Knowles**

Section de Mathématiques  
Université de Genève  
CP 64  
Rue du Conseil-Général 7-9  
1205 Genève 4  
SWITZERLAND

**Dr. Justin Ko**

ENS Lyon  
L'Unité de Mathématiques Pures et  
Appliquée  
46, Allée d'Italie  
69007 Lyon Cedex  
FRANCE

**Dr. Benjamin Landon**

Department of Mathematics  
University of Toronto  
40 St. George Street  
Toronto M5S2E4  
CANADA

**Prof. Dr. Jean-Francois Le Gall**

Département de Mathématiques  
Université de Paris-Sud  
Bât. 307  
91405 Orsay Cedex  
FRANCE

**Matthis Lehmkuhler**

Forschungsinstitut für Mathematik  
ETH Zürich  
Rämistrasse 101  
8092 Zürich  
SWITZERLAND

**Dr. Titus Lupu**

CNRS  
LPSM, Sorbonne Université  
75005 Paris  
FRANCE

**Prof. Dr. Grégory Miermont**

Unité de mathématiques pures et  
appliquées  
École normale supérieure de Lyon  
46, Allée d'Italie  
69364 Lyon Cedex 07  
FRANCE

**Prof. Dr. Jason P. Miller**

Department of Pure Mathematics  
and Mathematical Statistics  
University of Cambridge  
Wilberforce Road  
Cambridge CB3 0WB  
UNITED KINGDOM

**Dr. Alessandra Occeili**

Unité de mathématiques pures et  
appliquées  
École normale supérieure de Lyon  
46, Allée d'Italie  
69364 Lyon Cedex 07  
FRANCE

**Dr. Joonas Oikarinen**

Dept. of Mathematics & Statistics  
University of Helsinki  
P.O. Box 68  
Pietari Kalmin katu 5  
00014 Helsinki  
FINLAND

**Prof. Dr. Jeremy Quastel**

Department of Mathematics  
University of Toronto  
40 St George Street  
M5S1L2 Toronto  
CANADA

**Prof. Dr. Rémi Rhodes**

Centre de Mathématiques et  
d'Informatique  
Université de Provence  
39, Rue Joliot-Curie  
13453 Marseille Cedex 13  
FRANCE

**Dr. Armand Riera**

Institut für Mathematik  
Universität Zürich  
Winterthurerstr. 190  
8057 Zürich  
SWITZERLAND

**Prof. Dr. Hao Shen**

Department of Mathematics  
University of Wisconsin-Madison  
480 Lincoln Drive  
Madison, WI 53706-1388  
UNITED STATES

**Prof. Dr. Pierre-Francois  
Rodriguez**

Imperial College London  
Department of Mathematics  
Huxley Building  
180 Queen's Gate  
London SW7 2AZ  
UNITED KINGDOM

**Prof. em. Dr. Alain-Sol Sznitman**

Departement Mathematik  
ETH-Zentrum  
Rämistrasse 101  
8092 Zürich  
SWITZERLAND

**Dr. Dominik Schröder**

Institute for Theoretical Studies  
ETH Zürich  
Clausiusstrasse 47  
8096 Zürich  
SWITZERLAND

**Dr. Meltem Unel**

Department of Mathematical Sciences  
University of Copenhagen  
Universitetsparken 5  
2100 København  
DENMARK

**Dr. Florian Schweiger**

Department of Mathematics  
Weizmann Institute of Science  
P.O. Box 26  
Rehovot 7610001  
ISRAEL

**Prof. Dr. Bálint Virág**

Department of Mathematics  
University of Toronto  
Bahen Building  
40 St George Street  
Toronto ON M5S 2E4  
CANADA

**Dr. Tatyana Shcherbina**

Department of Mathematics  
University of Wisconsin - Madison  
480 Lincoln Drive  
Madison WI 53705  
UNITED STATES

**Prof. Dr. Wendelin Werner**

Departement Mathematik  
ETH-Zentrum  
Rämistrasse 101  
8092 Zürich  
SWITZERLAND

**Prof. Dr. Scott Sheffield**

Department of Mathematics  
Massachusetts Institute of Technology  
77 Massachusetts Avenue  
02139 Cambridge  
UNITED STATES

**Baojun Wu**

Mathématiques  
Faculté des Sciences  
Université d'Aix-Marseille  
23 Bis Boulevard du Louvain  
13008 Marseille Cedex 3  
FRANCE

**Prof. Dr. Horng-Tzer Yau**

Department of Mathematics  
Harvard University  
1 Oxford Street  
Cambridge MA 02138-2901  
UNITED STATES

**Prof. Dr. Rongchan Zhu**

School of Mathematics & Statistics  
Beijing Institute of Technology  
5 South Zhongguancun Street  
Beijing 100081  
CHINA

**Pu Yu**

Department of Mathematics  
MIT  
Cambridge, MA 02139  
UNITED STATES

**Prof. Dr. Xiangchan Zhu**

Chinese Academy of Science  
Beijing 100044  
CHINA

**Prof. Dr. Ofer Zeitouni**

Department of Mathematics  
Weizmann Institute of Science  
POB 26  
Rehovot 7610001  
ISRAEL