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Mini-Workshop: Subvarieties in Projective Spaces and Their Projections

Organized by
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Abstract. The major goals of this workshop are to lay paths for a systematic study of geproci (and related, e.g., projecting to almost complete intersections or full intersections) sets of points in projective spaces, study algebraic properties of their ideals (e.g. in the spirit of the Cayley-Bacharach properties), and to identify the most promising new directions for study.

Mathematics Subject Classification (2020): 14C20, 13A15, 13C70, 14N20.

Introduction by the Organizers

The miniworkshop *Subvarieties in Projective Spaces and Their Projections*, organised by Thomas Bauer (Marburg), Giuseppe Favacchio (Palermo), Juan Migliore (Notre Dame) and Justyna Szpond (Warszawa) was attended by 14 participants in person and 2 online participants from Europe and North America. There was a diversity in experience level ranging from a PhD student to established, internationally recognized professors. Thanks to this diversity we were able to achieve considerable progress on topics highlighted at the workshop and to provide excellent training for early career participants. Workshop activities were divided between 11 talks and group research collaborations, the latter of which took place mostly in the afternoons. Activities commenced on the first day with an in-depth discussion of problems to be studied. There were also two short and one long formal progress report sessions, apart from informal discussions held throughout the workshop.

The focus of this workshop was around the so called geproci property. Despite the fact that the definition of geproci is very recent, it has some deep connections with several branches of research including unexpected hypersurfaces, and the Nagata and the SHGH Conjectures.

We say that a nondegenerate subscheme $V \subset \mathbb{P}^N$ has the *geproci property* if there exists M such that for the projection $\pi_\Lambda : \mathbb{P}^N \dashrightarrow \mathbb{P}^M$ from a general linear subspace Λ of $\mathbb{P}^N$ of dimension $N-M-1$, the image $\pi_\Lambda(V)$ is a complete intersection (and not a divisor).

A natural problem is to characterize subschemes of $\mathbb{P}^N$ with the geproci property. This seems very hard in the stated generality, even in the case of reduced zerodimensional subschemes of $\mathbb{P}^N$.

Since all the sets of points in $\mathbb{P}^1$ are a complete intersection, to avoid trivialities, we assume that $N \geq 3$ and $N-M-1 \geq 2$. Surprisingly, no examples of geproci sets of points in $\mathbb{P}^N$ are known for $N \geq 4$, over a field of characteristic zero. The situation is more multifaceted in $\mathbb{P}^3$. The first systematic study of geproci sets of points in $\mathbb{P}^3$ is due to Chiantini and Migliore [2] which is mostly focused on grids. Other examples of geproci sets of points in $\mathbb{P}^3$ are presented in the appendix of that work by Bernardi, Chiantini, Denham, Favacchio, Harbourne, Migliore, Szemberg and Szpond; by Pokora, Szemberg and Szpond in [5]; and by Wiśniewska and Zięba in [6].

A breakthrough has been achieved in the forthcoming monograph by Chiantini, Farnik, Favacchio, Harbourne, Migliore, Szemberg and Szpond where the authors provide a construction of non-grid (a,b)-geproci sets of points in $\mathbb{P}^3$ for all $4 \leq a \leq b$ and give a full classification of geproci sets with $a = 3$. In this book is also begun the exploration of numerous connections between geproci sets and various research areas, some of which are emphasized in some of the talks given and in the problems proposed during this workshop.

The research groups focused their efforts on three main problems, labeled A–C and described below.

A. Geproci sets and grids. Try to classify geproci sets. The study needs to begin with $(4,4)$ geproci.

Show that every geproci set of $n \geq 5$ points in $\mathbb{P}^3$ contains at least 3 collinear points. If this succeeds, the immediate consequence is that there are no nontrivial geproci sets of points in linear general position. As the first step one can try to prove that a set of ad points in linear general position on a smooth curve of degree d in $\mathbb{P}^3$ is never (a,d)-geproci.

The next problem is a considerably stronger version of the linear general position question for geproci sets. Every known (a,b)-geproci set in $\mathbb{P}^3$ contains an (a',b')-grid with $a' \geq \frac{a}{2}$ and $b' \geq \frac{b}{2}$. It would be extremely interesting to know if this is always the case.

B. Kochen-Specker sets and geproci property. Recall that a *Kochen-Specker set* (KS set) is a finite set of vectors V in $\mathbb{C}^n$ such that there does not exist a map $f : V \to \{0,1\}$ satisfying the following conditions.

(1) The product $f(\mathbf{u})f(\mathbf{v})$ is zero for every pair of orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$ in V.
(2) The sum $\sum_{i=1}^{n} f(\mathbf{u}_i)$ is one for every subset $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of V which gives an orthogonal basis for $\mathbb{C}^n$.

As discussed in [1, Chapter 8], KS sets have interesting connections to unexpected cones and to geproci sets.

The papers [4, 3] provide new and unexplored collections of KS sets. In particular, [4] provides a large but finite number of highly symmetric KS sets, while [3] provides an infinite family.

It would be interesting to explore these sets, looking in particular for examples which give unexpected cones and possibly even geproci sets. The first step would be to obtain the coordinates of the points in a form usable for Macaulay2 computations, either directly in characteristic 0 by working over some field extension $\mathbb{F}$ of $\mathbb{Q}$ containing all of the coordinates of all of the points of the given set, or indirectly by working over $\mathbb{Z}/p\mathbb{Z}$ for an appropriate prime p mimicking $\mathbb{F}$. The next step would be to test the KS sets using Macaulay2. The results of these tests would then guide the direction of the project thereafter.

C. Geproci sets and combinatorics. The collection of dual vectors to the reflecting hyperplanes of certain complex reflection groups (B_4, F_4, H_4 and G_{32}) form geproci sets. It would be interesting to know if the dual vectors to other complex reflection arrangements, particularly of higher rank, also provide examples of nontrivial geproci sets. Can the classification of irreducible complex reflection groups give some ideas towards a possible classification of nontrivial geproci sets?

It also seems natural to ask to what extent the combinatorics (specifically, the underlying matroid) of a collection of points determines the geproci property.

- Are there necessary combinatorial conditions, beyond trivial ones like a suitable factorization of the number of points?
- Are realizations of the uniform matroid (i.e., points in linear general position) ever geproci sets?
- If Z and Z' realize the same matroid, is it true that Z is geproci if and only if Z' is?

The question is analogous to Terao's freeness conjecture, and while a direct connection between the geproci and freeness property are not expected, it seems reasonable to hope that, via projective duality, Matlis duality or other algebraic considerations, some already-explored aspect of arrangement theory could shed some light on the geproci property.

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Mini-Workshop: Subvarieties in Projective Spaces and Their Projections

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Abstracts

Short history of geproci sets

Tomasz Szemberg

This presentation is based on a joint work with the POLITUS group, which was established shortly after the MFO workshop on Lefschetz Properties in Algebra, Geometry and Combinatorics held in the period September 27 – October 3, 2020 at the MFO. The members of the group are Luca Chiantini, Łucja Farnik, Giuseppe Favacchio, Brian Harbourne, Juan Migliore, Justyna Szpond and myself.

1. Origins

In a 2011 post on Math-Overflow [4] Francesco Polizzi asked when a general projection of d^2 points in $\mathbb{P}^3$ is a complete intersection (is a (d,d) geproci set in our terminology). This happens certainly when the points are already a complete intersection in a plane in $\mathbb{P}^3$. Dmitri Panov pointed out that there is another, non-degenerate class of sets of points enjoying this property formed by grids, i.e. all intersection points of two sets of lines with the property that all lines in each set are skew and all a lines in one set meet every line in the other set of b lines in a point. The union of these points is an (a,b) geproci.

2. First discoveries

In 2018 during the workshop Lefschetz Properties and Jordan Type in Algebra, Geometry and Combinatorics a working group discovered sets of 12, 16, 20 and 24 points in $\mathbb{P}^3$, which are geproci but are neither degenerate nor grids, see [2]. All these sets are subsets of the root system F_4 (or G_{28} in the Shephard-Todd classification, [5]). All of them are also half-grids, i.e. they can be covered by a disjoint union of lines whose projection to the plane is one of the curves defining the complete intersection.

In the Autumn of 2020 during one of few research in pairs projects held in the year of COVID-19 pandemic in person at MFO, Piotr Pokora, Justyna Szpond and myself [3] discovered new examples of geproci sets consisting of 30, 36, 42, 48, 54 and 60 elements. The largest of these sets is determined by the complex reflection group appearing as G_{31} in the Shephard-Todd classification. Also these sets of points are half-grids.

In Winter 2020/21 during a Lanckorona workshop in Poland the first non-half-grid example of a $(6,10)$ geproci set has been discovered by Paulina Wiśniewska and Maciej Zięba, see [6].

3. The Age of Exploration and Discovery

In the recent preprint [1] we summarize our almost 2 years long work devoted to geproci sets. Our main result is the following complete solution to the geography problem of non-grid geproci sets.

Theorem 1. There exists a non-grid (a, b) geproci (with $a \leq b$) if and only if either $a = 3$ and $b = 4$ or $a \geq 4$.

This theorem is established by means of the procedure baptized as the Standard Construction in [1].

Thus the smallest non-grid geproci set consists of 12 points and it is of type $(3, 4)$. In fact we show in [1] that the configuration of points corresponding to the D_4 root system is the unique set of such type.

All geproci sets obtained in the Standard Construction are half-grids. As of this writing there are only 3 known examples of non-half-grid geproci sets. The H_4 configuration of 60 points, a configuration of 40 points appearing in works of Penrose on Quantum Mechanics and an example of 120 points. It would be extremely interesting to know if only finitely many such examples exist and if so, to list them all.

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Comparing and contrasting the geproci property in characteristic 0 versus characteristic $p > 0$

Brian Harbourne

This talk is on recent work of Jake Kettinger [2] regarding geproci sets in characteristic $p > 0$.

Recall that a finite set $Z \subset \mathbb{P}^3$ is (a, b)*-geproci* if its image $\overline{Z} \subset H$ under projection to a plane $H \subset \mathbb{P}^3$ from a general point P is a complete intersection of type (a, b) (i.e., $\overline{Z}$ is the transverse intersection of two curves in H of degrees a and b, with $a \leq b$).

There are four types of (a, b)-geproci sets Z:

(1) Z is degenerate (i.e., Z is a complete intersection of type (a, b) contained in a plane);
(2) Z is an (a, b)-grid (i.e., there is a set A of a skew lines and a set of b skew lines B, each line of A meets each line B in exactly one point, and Z is the resulting set of ab points);

(3) Z is an (a,b)-geproci half-grid (i.e., $\overline{Z}$ is a complete intersection of type (a,b) where exactly one of the curves can always be taken to be a union of lines); or

(4) Z is a nondegenerate nongrid non-half-grid.

Cases (1) and (2) are well understood; we regard them as being trivial. Over the complex numbers, most examples in case (3) come from or are motivated by examples given by root systems, and only a few characteristic 0 examples are known for case (4). There is no currently known way to construct additional examples for case (4) in characteristic 0.

Before Kettinger, the geproci concept in positive characteristics has not been explored. Kettinger's work in characteristic $p > 0$ gives many new kinds of examples of half-grids and it gives an approach for constructing many new examples for case (4).

Here are some sample results Kettinger has obtained.

Theorem 1. Let $\mathbb{F}$ be a finite field of q elements and let k be its algebraic closure. Let Z consist of the q^3+q^2+q+1 points of $\mathbb{P}^3_k$ whose coordinates can all be written in $\mathbb{F}$. Then Z is a $(q+1, q^2+1)$-geproci half-grid.

We now recall the notion of *spreads*. Let Z be as in Theorem 1. Let $S = \{L_1,\dots,L_r\}$ be a set r disjoint sets of collinear points of Z such that $|L_i| = q+1$ for each i. Its deficiency is d_S where $r = q^2+1-d$. Then S is known as a *full spread* if $r = q^2+1$ (here $d_S = 0$); in this case $\cup L_i = Z'$. Otherwise S is a *partial spread*. If S is not full and not properly contained in a larger spread, it is called a *maximal partial spread* (here $d_S > 0$).

In the next result, we write $\{a,b\}$-geproci when we possibly may have $b < a$.

Theorem 2. Let $\mathbb{F}$ be a finite field of q elements and let k be its algebraic closure. Let Z' consist of the q^3+q^2+q+1 points of $\mathbb{P}^3_k$ whose coordinates can all be written in $\mathbb{F}$. Let S be a maximal partial spread of deficiency d. Let Z be the complement in Z' of the points occurring in S. Then Z is a nontrivial $\{q+1,d\}$-geproci set; if $d > q+1$, then Z is a non-half-grid.

It is known [1] for each prime $q \geq 7$ that there are maximal partial spreads S for each $d_S = d$ in the range

$$q-1 \leq d \leq \frac{q^2+1}{2} - 6. \tag{1}$$

Thus Theorem 2 gives a method for constructing many examples of non-half-grids. For example, from Theorem 2 and (1) we see there is a nontrivial non-half-grid $(q+1,d)$-geproci set for every print $q \geq 7$ and every d in the range $q+2 \leq d \leq \frac{q^2+1}{2} - 6$.

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Generalized Weddle loci

Luca Chiantini

(joint work with the POLITUS group)

In a footnote of a paper published in 1850 ([4]) Thomas Weddle corrected a previous claim by Chasles, and stated that the locus of vertexes of quadric cones passing through 6 general points is a quartic surface in the projective space $\mathbb{P}^3$. Quartic surfaces determined in this way are called *Weddle surfaces*. Weddle surfaces are linked with the study of projections of finite sets, since they determine the locus of points P which project a fixed finite set of cardinality 6 to a plane conic. The notion can be easily generalized by considering cones of any degree passing through a fixed finite set Z in a projective space $\mathbb{P}^n$. The resulting loci, called *Weddle loci*, can be formally defined in modern terms as follows.

Let Z be a set of distinct points in $\mathbb{P}^n$. Let P be a point not in Z and let d be a positive integer. The graded ideal $I = I(Z) \cap I(P)^d$ in the polynomial ring R contains forms vanishing at Z and vanishing at P with multiplicity at least d. Thus, the homogeneous compoonent $[I]_d$ of I is the vector space associated to the linear system of cones of degree d, vertex P, passing through Z. Let $\delta(Z, P, d, t) = \dim[I]_d$. For fixed Z and d, $\delta(Z, P, d, t)$ achieves its minimum as a function of P on an open set. Denote this minimum by $\delta(Z, d, t)$.

Definition. The *d-Weddle locus* $W_d(Z)$ of Z is the closure of the set of points $P \in \mathbb{P}^n \setminus Z$ (if any) for which $\delta(Z, P, d, d) > \delta(Z, d, d)$.

Weddle loci $W_d(Z)$ arise naturally when one considers polynomial interpolation problems connected with Z. $W_d(Z)$ can also be seen as the set of points P which project Z to a general hypeplane H so that the linear system of hypersurfaces in H containing the projection jumps of dimension with respect to a generic projection. Thus, Weddle loci play a fundamental role in comparing properties of sets of points and properties of their projections.

The easiest case of Weddle loci appear when Z is general of cardinality $\binom{d+n-1}{n-1}$. In this situation $W_d(Z)$ is a determinantal hypersurface of binomial degree $\binom{d+n-1}{n}$. The result was known classically to Emch ([1]) for $n = 3$, and the generalization to any $\mathbb{P}^n$ is contained in Section 2 of [3]. There are several ways to prove the result. One can determine the interpolation matrix associated to $[I]_d$ and study the locus where it drops rank. Equivalently, one can consider Macaulay duality and translate the problem to a question about the dimension of the cokernel of the multiplication by a general linear form in the quotient of the polynomial ring determined by the dual (apolar) ideal of $[I]_d$. In the latter view, the locus is connected to sets of linear forms for which the weak Lefschetz property fails in some quotients of polynomial rings.

A similar reduction can be performed when the cardinality of Z is different from the binomial $\binom{d+n-1}{n-1}$. In this generality the resulting interpolating matrix M has a rectangular shape, and it is not easy to find even the dimension of the Weddle locus, since the dimension of the vanishing set of the minors of M is not trivial. Known result concern the case of $\binom{d+2}{2}+1$ general points in $\mathbb{P}^3$, whose Weddle locus is an arithmetically Cohen-Macaulay curve of degree $\frac{1}{72}(2d^6+12d^5+17d^4-66d^3-271d^2+954d-648)$ (see Prop. 2.22 of [3], which corrects a claim by Emch). For other values of the cardinality of Z there are results only if the minors of the interpolating matrix drop rank in the expected dimension.
It follows that the question about the dimension of Weddle loci, even for general sets of points in projective spaces, is widely open.

Question 1. What is the dimension of the d-Weddle locus of a general set Z of r points in $\mathbb{P}^n$?

Even in $\mathbb{P}^3$ the question about $\binom{d+2}{2}-1$ general points is still open.
There is another way in which the problem can be interpreted. Going back to the original situation of 6 points in $\mathbb{P}^3$, the generators of the linear system of quadrics can be packed together to obtain a $4\times 4\times 4$ (partially symmetric) tensor. The interpolating matrix giving the Weddle locus comes from a generic contraction of the tensor given by a generic vector.
Using this point of view, the problem easily extends to linear systems $\mathcal{S}$ of quadrics of (projective) dimension 3, not necessarily obtained by 6 base points. The contraction of the tensor determines a 4×4 matrix of linear forms, whose determinant defines the Weddle locus $W_{\mathcal{S}}$ of the system, again a determinantal quartic surface.

While the Weddle surface determined by 6 points is quite special (e.g. it contains the 15 lines joining pair of the base points, see [2] for a detailed analysis of the geometry of Weddle surfaces), we do not know much about Weddle surfaces coming from general linear systems. So the following question arises.

Question 2. Which quartic surfaces is the Weddle surface of a linear system $\mathcal{S}$ of dimension 3 of quadrics in $\mathbb{P}^3$? More specifically, is a general determinantal quartic surface the Weddle surface of a suitable linear system of quadrics?

The interpolation matrix arising from general linear systems of quadrics is not necessarily symmetric, yet it is quite special since it comes out from a partially symmetric tensor. Thus the answers to Question 2 can be non-trivial.

One can easily generalize Question 2 to linear systems of higher degree in higher dimensional projective spaces.
Notice that when the degree grows, one translates the linear system $\mathcal{S}$ to a tensor T of high dimension, so that the interpolating matrix comes out by contracting T several times ($d-1$ times, exactly). From this point of view, the corresponding generalized Weddle loci can be seen as a first order generalization of the set of eigenvectors of the eigenvalue 0 of T.

Almost all questions on the geometry of generalized Weddle loci are wide open. Due to the applications of the theory of Weddle loci in the geometry of projections (but also in the analysis of tensors), we hope that the matter can stimulate further studies, able to clarify the structure of these interesting geometric objects.

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Generic projections of the H_4 configuration of points

Maciej Zięba

(joint work with Paulina Wiśniewska)

The purpose of this talk is to study properties of the set of 60 points in $\mathbb{P}^3$ determined by the H_4 root system. Our motivation comes from several directions. In [2], the authors observed that grids in $\mathbb{P}^3$ have the property that their general projection to a plane is a complete intersection. Such sets are said (after Definition 5.1 in [3]) to have the *geproci* property (from: **ge**neral **pro**jection **c**omplete **i**ntersection). We recall the fundamental Definition 3.2 from [2].

Definition (Grid)**.** Let a and b be positive integers. A set Z of $a \cdot b$ points in $\mathbb{P}^3$ is a $(a,b)-$*grid* if there exist two sets of lines $L_1, \dots, L_a$ and $M_1, \dots, M_b$ such that

- lines in each of the sets are pairwise skew;
- each pair of lines, one from one set and one from the other, intersects in a point of Z.

Thus,

$$Z = \{L_i \cap M_j,\ i = 1, \dots, a,\ j = 1, \dots, b\}.$$

A general projection π of an (a,b)–grid to a hyperplane $H \subset \mathbb{P}^3$ is a complete intersection. Indeed, the ideal of $\pi(Z)$ in H is generated by the equations of $C = \pi(L_1 \cup \ldots \cup L_a)$ and $D = \pi(M_1 \cup \ldots \cup M_b)$.

In Appendix to [2], it was observed that not only grids have the geproci property. The authors discovered that the set of 24 points in $\mathbb{P}^3$ determined by the F_4 root system does not form a grid, yet it has the geproci property. More precisely, its general projection is a complete intersection of a smooth curve of degree 4 and a curve of degree 6, which can be chosen to split totally in 6 lines.

The curve of degree 4 in contrary is uniquely determined by the projection of F_4 and does not split into lines. In this situation, we speak of a half-grid. This notion was introduced in [3] by Pokora, Szemberg and Szpond. They found a set of 60 points in $\mathbb{P}^3$, which, like F_4, is a half-grid rather than a grid.

Definition (Half-Grid)**.** Let a and b be positive integers. A set Z of $a \cdot b$ points in $\mathbb{P}^3$ is a (a,b)-*half-grid* if there exists a set of mutually skew lines $L_1, \ldots, L_a$ covering Z and a general projection of Z to a hyperplane is a complete intersection of images of the lines with a (possibly reducible) curve of degree b and it is not a grid.

The sets with the geproci property studied in [2] (F_4 and its subsets) and [3] (Klein configuration and its subsets) are half-grids. It is natural to wonder if all geproci sets of points in $\mathbb{P}^3$ are half-grids. The answer is no, although we only have a few examples of nontrivial geproci non-half grids (see [1]):

- A 60 point set coming from the H_4 root system.
- A 40 point set originally constructed by Penrose, who applied it to quantum mechanics.
- A 120 point set.

We demonstrate the properties of a set of 60 points derived from the H_4 root system, as well as proof that this is a (6,10)-geproci and not a half grid.

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Hyperplane arrangements from the commutative algebra viewpoint

Graham Denham

The objective of this talk is to give an overview of some classical algebraic constructions associated with a hyperplane arrangement, centered around the question of what role combinatorics plays. The goal is to bring to light ideas that could be useful in understanding the geproci property and its possible generalizations.

Notation and introduction. For $r \geq 1$, let $Z \subseteq \mathbb{P}^r$ be a (reduced) set of n points over a field $\Bbbk$, and let assume that the points Z are not contained in a proper projective linear subspace of $\mathbb{P}^r$.

Let $\mathcal{A}_Z$ denote the dual set of n hyperplanes in $(\mathbb{P}^r)^\vee$. This is a *hyperplane arrangement* of rank $r+1$. Hyperplane arrangements have an extensive literature, and the talk focusses on some aspects which particularly refer to their defining equations, with a view to establishing a connection with the ideal of points I_Z. Let us say an arrangement $\mathcal{A}_Z$ is *geproci* if the corresponding points Z have the property that their generic projection is a complete intersection.

Choosing coordinates, one may realize the points Z as (nonzero) columns of a $(r+1) \times n$ matrix A. Let W denote the span of the rows of A in the affine space $\Bbbk^n$, which we may regard as a point in the Grassmannian $\mathrm{Gr}(r+1, n)$. Let $\hat{H}_i$ denote the ith coordinate hyperplane in $\Bbbk^n$. Then $\mathcal{A}_Z$ consists of hyperplanes $\{H_i := \hat{H}_i \cap W\}$.

The torus $\mathbb{T} := (\Bbbk^*)^n$ acts on columns of A. In more invariant terms, then, specifying a hyperplane arrangement $\mathcal{A}_Z$ or point configuration Z is equivalent to choosing a torus orbit $W \cdot \mathbb{T} \subseteq \mathrm{Gr}(r+1, n)$, where W is a subspace not contained in any coordinate hyperplane $\hat{H}_i$. From this point of view, an arrangement or point configuration is the same notion as a (projective) linear realization of a matroid without loops.[3]

Remark 1. One can ask, then, what are the matroids that have a geproci realization. Are there necessary combinatorial conditions, beyond the obvious ones? The matroid of a generic projection of a set of points Z is a principal truncation of the matroid of Z. This leads to the more general question of which matroids admit a realization by points Z which (themselves) form a complete intersection. Another approach to the geproci question, then, would be to start with a complete intersection in $\mathbb{P}^{r-1}$ and consider the known problem in matroid theory of realizing that complete intersection as a principal truncation of a matroid of rank one higher.

Logarithmic derivations. Let $f\colon W \to \Bbbk^n$ denote the linear embedding given in coordinates by the matrix A. Then $H_i = \ker f_i$ for $1 \le i \le n$, regarding each f_i as a linear element of $R := \Bbbk[W] = \Bbbk[x_0, \ldots, x_r]$. The union of hyperplanes is defined by the product $Q := f_1 \cdots f_n$, and its Jacobian ideal $J_Z = (\partial Q/\partial x_i \colon 0 \le i \le r)$ can be presented as a cokernel,

$$R(-1)^{r+1} \longrightarrow R \longrightarrow J_Z \longrightarrow 0.$$

The kernel of the left map, denoted $\mathrm{Der}_0(\mathcal{A})$, is isomorphic to the module of derivations $\{\theta\colon R \to R \mid \theta(f_i) = 0 \text{ for all } i\}$. Since the Jacobian ideal has codimension 2, from the sequence

$$0 \longrightarrow \mathrm{Der}_0(\mathcal{A}) \longrightarrow R(-1)^{r+1} \longrightarrow R/J_Z \longrightarrow 0,$$

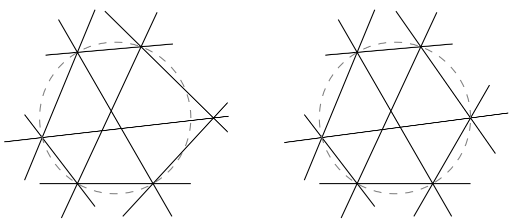
we see that R/J_Z is Cohen-Macaulay if and only if the R-module $\mathrm{Der}_0(\mathcal{A})$ is free. In this case, the arrangement $\mathcal{A}_Z$ is said to be a free arrangement. This property is the subject of an extensive literature: for references see, for example, the survey [8]. A motivating open question is whether or not the freeness of an arrangement is a combinatorial property: Terao famously conjectured an affirmative answer to this question. The analogous question for the geproci property is apparently still open.

Remark 2. Not all geproci arrangements are free: the $(3, 3)$ grid provides an easy counterexample. Not all free arrangements are geproci: the type A_4 reflection arrangement is free but not geproci. On the other hand, Terao [7] showed that the hyperplane arrangements associated with complex reflection groups are always free. Some interesting geproci arrangements (namely, B_4, F_4, H_4, G_{32}) are complex reflection arrangements.

While they do not coincide, the two properties seem to have a similar flavour. For example, realizations of a (nontrivial) uniform matroid are never free [5]. These correspond to point configurations in linear general position. Is it true that such point configurations are never geproci?

The reciprocal plane. Another construction from a hyperplane arrangement that gives rise to some interesting commutative algebra is the *reciprocal plane*: this is the Zariski closure of the image of W under the Cremona transformation $\mathbb{k}^n \dashrightarrow \mathbb{k}^n$ given (rationally) by $(p_1, \dots, p_n) \mapsto (1/p_1, \dots, 1/p_n)$. Its coordinate ring is known as the Orlik–Terao algebra [2]. Those authors give an explicit presentation by a homogeneous ideal of relations that depends on both matroid combinatorics and the equations $\{f_i\}$ of the hyperplanes themselves. For combinatorial considerations, we refer to [1] and the references therein; for some algebraic fundamentals, we mention [4].

Schenck and Tohăneanu [6] showed that the Orlik–Terao algebra's homological properties depend on the equations, and not just the combinatorics. The following pair of arrangements (originally due to Ziegler) realize the same matroid, but have non-isomorphic Orlik–Terao algebras:



The six triple points in the right-hand arrangement lie on a conic, while they do not in the left-hand arrangement. The same pair also exhibit different Betti numbers in their respective free resolutions of $\mathrm{Der}_0(\mathcal{A})$.

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Unexpected curves of type $(d + k, d)$

Grzegorz Malara, Halszka Tutaj-Gasińska

In 2016, Cook II, Harbourne, Migliore and Nagel in their groundbreaking paper [1] considered curves of degree $d + 1$ passing through a set $Z \subset \mathbb{P}^2$ of non-general – but imposing independent conditions on the curves of degree $d + 1$ – points, and having multiplicity d in a general point P. If the existence of such curves does not follow from the naïve dimension count, then the authors of [1] call them unexpected curves of type $(d + 1, d)$. Analogously one may define unexpected curves of type $(d + k, d)$.

In our talk we explain the existence of some unexpected curves of type $(d+k, d)$, along the lines of [1]. In [1] the existence of unexpected cures of type $(d + 1, d)$ is explained via degree d syzygies of the Jacobian ideal of an arrangement A_Z of lines dual to the points of the set Z. We show that this construction may be generalized, to produce curves (not necessarily unexpected) of degree $d + k$, passing through the points of Z and having a general point of multiplicity d.

In our work (but not in this talk) we prove also some conditions for the curve to be unexpected.

The idea of the construction is as follows. Let Z be a set of points in $\mathbb{P}^2$ and let L be a generic line on $\check{\mathbb{P}}^2$ with the equation $\alpha a + \beta b + \gamma c = 0$. Denote by $(g_{k,0,0}, \dots, g_{0,0,k}, g)$ a (reduced) syzygy of $J^k + (L)$ where g_{i_1,i_2,i_3} are all of degree d, i.e., for any $Q = (a : b : c) \in \check{\mathbb{P}}^2$ we have

$$g_{k,0,0}(Q)f_x(Q)^k+g_{k-1,1,0}(Q)f_x(Q)^{k-1}f_y(Q)+\cdots+g_{0,0,k}(Q)f_z(Q)^k+g(Q)L(Q) = 0.$$

Let S_Q be the curve of degree k in $\mathbb{P}^2$ given by the equation

$$S_Q(x, y, z) := g_{k,0,0}(Q)x^k + g_{k-1,1,0}(Q)x^{k-1}y + \cdots + g_{0,0,k}(Q)z^k = 0.$$

Let $Q = (a, b, c) \in L$. Consider the system of equations

$$(\star) \qquad \begin{cases} \alpha a + \beta b + \gamma c = 0 \\ ax + by + cz = 0 \\ g_{k,0,0}(a, b, c)x^k + g_{k-1,1,0}(a, b, c)x^{k-1}y + \cdots + g_{0,0,k}(a, b, c)z^k = 0. \end{cases}$$

We will say that this system is *not determined* in $Q = (a, b, c) \in L$ if for all (x, y, z) we have

$$g_{k,0,0}(a, b, c)x^k + g_{k-1,1,0}(a, b, c)x^{k-1}y + \cdots + g_{0,0,k}(a, b, c)z^k = (ax + by + cz)^k.$$

Let $P_L = \check{L} = (\alpha, \beta, \gamma)$.
Then:

(1) The system $(\star)$ may be not determined only for points Q on $A_Z \cap L$.
(2) The solutions (x, y, z) to the system $(\star)$ lie on a curve $C_L \subset \mathbb{P}^2$ of degree (at most) $d + k$.
(3) C_L passes through Z.
(4) C_L has a point of multiplicity at least d in P_L.
(5) The curve C_L may be treated as $C_L(x, y, z)$ with parameters (α, β, γ) and "dually" as $C_L(\alpha, \beta, \gamma)$ with parameters (x, y, z). The partial derivatives computed in point (α, β, γ) with respect to (x, y, z) and computed in point (x, y, z) with respect to (α, β, γ) are the same up to order d.

The main idea behind the proof is the elimination of the variables (a, b, c) from the system $(\star)$.

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Duality of asymptotic invariants

Alexandra Seceleanu
(joint work with Michael DiPasquale, Thái Nguyễn)

Let I denote a homogeneous ideal in a polynomial ring $R = k[x_0, \dots, x_N]$ with k a field of arbitrary characteristic (unless specified otherwise). The following families of ideals can be constructed from I:

- *the ordinary power* I^n is the ideal generated by n-fold products of elements of I;
- *the differential power* $I^{\langle n \rangle}$ is the ideal

$$I^{<n>} = \{f \in R \mid \frac{1}{a_0! \cdots a_N!} \frac{\partial}{\partial x_0^{a_0} \cdots \partial x_N^{a_N}}(f) \in I,\ \forall a_i \in \mathbb{N}, a_0 + \cdots a_N = n - 1\};$$

- *the symbolic powers* are described as $I^{(n)} = \bigcap_{P \in \mathrm{Ass}(I)} I^n R_P \cap R$ and they agree with the differential powers whenever I is radical and k is perfect, see [1, Proposition 2.14];
- if $\mathrm{char}(k) = p > 0$ and $I = (f_1, \dots, f_s)$, then one defines for each $q = p^e$ the *Frobenius power* $I^{[q]} = (f_1^q, \dots, f_s)^q$ and extends this definition to $n \in \mathbb{N}, n = n_0 + n_1 p + \cdots + n_t p^t$ by setting

$$I^{[n]} = I^{n_0}(I^{[p]})^{n_1} \cdots (I^{[p^t]})^{n_t}.$$

The latter *integral Frobenius powers* were defined in [3].

We write $I^{\boxed{n}}$ to refer to any of the above families in the sequel. The above are examples of graded families, meaning that $I^{\boxed{a}} I^{\boxed{b}} \subseteq I^{\boxed{a+b}}$ for all $a, b \in \mathbb{N}$.

Applying a function from ideals to natural numbers to a graded family we obtain a numerical sequence.

A sequence of real numbers $\{\alpha_n\}_{n\geq 0}$ is called

- **subadditive** if it satisfies $\alpha_{i+j} \leq \alpha_i + \alpha_j$ for all $i, j \geq n_0$;
- **superadditive** if it satisfies $\alpha_i + \alpha_j \leq \alpha_{i+j}$ for all $i, j \geq n_0$.

Fekete's lemma guarantees the existence of $\widehat{\alpha} := \lim_{n\to\infty} \frac{\alpha_n}{n}$ for any subadditive or superadditive sequence of real numbers $\{\alpha_n\}_{n\in\mathbb{N}}$, allowing for the value of the limit to be $-\infty$ in the subadditive case and ∞ in the superadditive case respectively.

Examples:

- Set $\alpha(I)$ to be the least degree of a nonzero element of I. Then set $\alpha_n = \alpha\left(I^{\boxed{n}}\right)$. This sequence is subadditive and the limit $\widehat{\alpha}(I)$ is called the *Waldschmidt constant* of I.
- Let I define points in $\mathbb{P}^{\mathbb{N}}$. Then the sequence $\{\operatorname{reg}(I^{(n)})\}_{n\geq 1}$ is subadditive and its limit $\widehat{\operatorname{reg}}(I)$ is the *asymptotic regularity* of I.

Let $\{\alpha_n\}_{n\geq n_0}$ be a sequence of natural numbers. Define dual sequences

$$\overleftarrow{\alpha}_n = \inf\{d \mid \alpha_d \geq n\},$$
$$\overrightarrow{\alpha}_n = \sup\{d \mid \alpha_d \leq n\}.$$

Under modest assumption on the sequence, these operations are mutual inverses.

Theorem 1. Let $\alpha = \{\alpha_n\}_{n\geq n_0}$ be a nondecreasing sequence of natural numbers.

(1) There are identities $\overleftarrow{\overrightarrow{\alpha}} = \alpha$ and $\overrightarrow{\overleftarrow{\alpha}} = \alpha$.
(2) If α is subadditive then $\{\overrightarrow{\alpha}_n\}_{n\geq\alpha_1}$ is superadditive with $\widehat{\overrightarrow{\alpha}} = \widehat{\alpha}^{-1}$.
(3) If α is superadditive, then $\{\overleftarrow{\alpha}_n\}_{n\geq 0}$ is subadditive with $\widehat{\overleftarrow{\alpha}} = \widehat{\alpha}^{-1}$.

We now identify dual sequences to those given in the example.

The sequence $s_n(I) := \overrightarrow{\operatorname{reg}}\left(I^{\boxed{n}}\right)$ is called the *jet separation sequence* of I.

Theorem 2. If $X = \{p_1, \ldots, p_r\}$ is a finite set of points in $\mathbb{P}^N$ with defining ideal I_X then the jet separation sequence $\{s_n(I_X)\}$ is eventually superadditive and satisfies

$$\lim_{n\to\infty} \frac{s_n}{n} = \varepsilon(X) = \inf_{C\text{curve}, C\cap X\neq\emptyset} \left\{ \frac{\deg C}{\operatorname{mult}_{p_i} C} \right\}.$$

The invariant $\varepsilon(X)$ defined by the rightmost expression of the above display is known as the *Seshadri constant* of X.

An interpretation of the dual sequence to $\left\{\alpha\left(I^{\boxed{n}}\right) - n\right\}_{n\in\mathbb{N}}$ in terms of Macaulay inverse systems can be found in [2].

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Homogeneous interpolation and moduli spaces of vector bundles

Jack Huizenga
(joint work with Izzet Coskun)

Moduli spaces of semistable vector bundles parameterize the isomorphism classes of vector bundles on a given projective variety X. In the case where X is a suitably nice surface, such as $\mathbb{P}^2$, del Pezzo surfaces, abelian surfaces, or K3 surfaces, a great deal is known about the geometry of the moduli space. For example, the space is often irreducible and generically smooth of the expected dimension. On the other hand, it is known that on surfaces of general type, e.g. on surfaces of high degree in $\mathbb{P}^3$, the moduli space can have arbitrarily many irreducible components and components can be everywhere nonreduced or have a higher dimension than the expected dimension [1]. However, general theorems of O'Grady [3] show that if the rank and first Chern class are fixed but the *discriminant* $\Delta = \frac{1}{2}\left(\frac{c_1}{r}\right)^2 - \frac{\mathrm{ch}_2}{r}$ is sufficiently large, then the moduli spaces again become nice.

In [2], we investigate certain moduli spaces of rank 2 vector bundles on blowups of the projective plane at 10 or more very general points. The spaces we consider have the smallest possible discriminant, so they should be expected to be some of the most pathological moduli spaces of vector bundles on these surfaces. Assuming the SHGH conjecture, we show that these spaces can have arbitrarily many components of arbitrarily high dimensions. In the case of 10 points, the components correspond to continued fractions of the square root of 10.

More precisely, let $X = \mathrm{Bl}_{p_1,\ldots,p_{10}}(\mathbb{P}^2)$ be the blowup of $\mathbb{P}^2$ at 10 very general points. For a rational number $m > \sqrt{10}$, we let $A_m = mH - E$, where H is the class of a line and E is the sum of the exceptional divisors in X. The SHGH conjecture implies the Nagata conjecture, which in turn implies that A_m is an ample divisor. We study the moduli space $M_{A_m}(2, K_X, 2)$ of A_m-semistable vector bundles of rank 2, first Chern class K_X, and Euler characteristic $\chi = 2$. For an effective divisor D, we say that a vector bundle V of these numerical invariants has *type* D if it fits into an exact sequence of the form

$$0 \to \mathcal{O}_X(D) \to V \to K_X(-D) \to 0.$$

We prove that every vector bundle V has a *unique* type. Furthermore, if V is A_m-semistable for some m then the type D must satisfy

$$2B \cdot D < B \cdot K_X,$$

where $B = \sqrt{10}H - E$ is the nef divisor in Nagata's conjecture. The effective divisors D satisfying this inequality can be classified by solving a Pell's equation, and the solutions correspond to the convergents in the continued fraction expansion of $\sqrt{10}$.

As m decreases from ∞ to $\sqrt{10}$, the moduli spaces $M_m := M_{A_m}(2, K_X, 2)$ become larger and more pathological, obtaining new disjoint components of rapidly increasing dimensions. Here we summarize the first few values of m where the moduli space changes.

(1) For $m > \frac{370}{117}$, the space M_m is empty.
(2) For $\frac{14050}{4443} < m < \frac{370}{117}$, the space M_m is isomorphic to $\mathbb{P}^8$ and parameterizes vector bundles of type $57H - 18E$.
(3) For $\frac{533530}{168717} < m < \frac{14050}{4443}$, the space M_m is isomorphic to a disjoint union of $\mathbb{P}^8$ (parameterizing bundles of type $57H - 18E$) and $\mathbb{P}^{359}$ (parameterizing bundles of type $2220H - 702E$).

As m continues to decrease, new disjoint projective spaces continue to be introduced to the moduli space. Similar results can be obtained for more than 10 blown up points, and the problem is studied systematically in [2].

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Hadamard products of symbolic powers and Hadamard fat grids

Elena Guardo

(joint work with Iman Bahamani Jafarloo, Cristiano Bocci, Grzegorz Malara)

During the CMO Workshop "Ordinary and Symbolic Powers of Ideals" (May 14-19, 2017, Oxaca, Mexico), Bocci proposed

Question 1. Is it true that for P, Q points in $\mathbb{P}^2$, $I(P)^m \star I(Q)^n = I(P \star Q)^{m+n-1}$?

We give a positive answers to Question 1 when the points P and Q have non-zero coordinates in Theorem 1.These results enlarge the known literature on the minimal graded resolution of sets of fat points in $\mathbb{P}^2$ with all the same multiplicities supported on a complete intersection ([2, 6]).

1. Hadamard Fat Grids

We work on the polynomial ring $S = \mathbb{K}[\mathbf{x}] = \mathbb{K}[x_0, x_1, x_2]$, over an algebraically closed field.

1.1. **Hadamard Product of two points.** Let $p, q \in \mathbb{P}^2$ be two points with coordinates $[p_0 : p_1 : p_2]$ and $[q_0 : q_1 : q_2]$ respectively. If $p_iq_i \neq 0$ for some i, the Hadamard product $p \star q$ of p and q, is defined as $p \star q = [p_0q_0 : p_1q_1 : p_2q_2]$. If $p_iq_i = 0$ for all $i = 0, 1, 2$ then we say $p \star q$ is not defined.

1.2. **Hadamard Product of two varieties.** Let X and Y be two varieties in $\mathbb{P}^N$. Then the Hadamard product $X \star Y$ is defined as

$$X \star Y = \overline{\{p \star q : p \in X, q \in Y, p \star q \text{ is defined}\}}.$$

From [1], the defining ideal is $I(X \star Y) = I(X) \star I(Y)$.

Let $P_M = \{P_1, \ldots, P_r\}$ and $Q_N = \{Q_1, \ldots, Q_s\}$ be two sets of collinear points in $\mathbb{P}^2 \setminus \Delta_1$ with assigned positive multiplicities, respectively, $M = \{m_1, \ldots, m_r\}$ with $m_1 \leq \cdots \leq m_r$ and $N = \{n_1, \ldots, n_s\}$ with $n_1 \leq \cdots \leq n_s$.

We set the ideals

$$I(P_M) = I(P_1)^{m_1} \cap \cdots \cap I(P_r)^{m_r} \text{ and } I(Q_N) = I(Q_1)^{n_1} \cap \cdots \cap I(Q_s)^{n_s}.$$

Theorem 1. Let P and Q be two points in $\mathbb{P}^2 \setminus \Delta_1$. Then for $m, n \geq 1$ one has $I(P)^m \star I(Q)^n = I(P \star Q)^{m+n-1}$.

1.3. **Hadamard fat grids.** Assume that $P_i \star Q_j \neq P_k \star Q_l$ for all $1 \leq i < k \leq r$ and $1 \leq j < l \leq s$. Then the set of fat points defined by $I(P_M) \star I(Q_N)$, is called *a Hadamard fat grid* and it is denoted by $HFG(P_M, Q_N)$. Its defining ideal is

$$\bigcap_{i \in [r]} \bigcap_{j \in [s]} I(P_i \star Q_j)^{m_i + n_j - 1}.$$

Using Lemma 3.1 in [1], we show that $HFG(P_M, Q_N)$ has the structure of a planar grid. Specifically, it is a set of fat points whose support is a complete intersection of type (r, s) in $\mathbb{P}^2$. And using some standard techniques and known results from [3, 4, 5], we prove

Theorem 2. Let X be a Hadamard fat grid $HFG(P_M, Q_N)$ in $\mathbb{P}^2$ and Z be an ACM set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ supported on an (r, s)-grid with the same multiplicities m_{ij} as the Hadamard fat grid X. Then X and Z share the same Betti numbers.

As a consequence of Theorem 2 we computed that:

- the Waldschmidt constant of the Hadamard fat grid $HFG(P_M, Q_N)$ is equal to the least degree of a minimal set of generators of its defining ideal, i.e., $\widehat{\alpha}(\mathcal{I}(P_M, Q_N)) = \alpha(\mathcal{I}(P_M, Q_N))$; and,
- the resurgence of $\mathcal{I}(P_M, Q_N)$ is $\rho(\mathcal{I}(P_M, Q_N)) = 1$.

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Schemes of eigenpoints in the plane

Francesco Galuppi

Whether we study mathematics or physics, engineering or computer science, data analysis or statistics, we often deal with eigenvectors of matrices. This presentation is about the much younger theory of eigenvectors of tensors. Just as we use matrices to encode linear maps, we can use higher-order tensors to encode polynomial maps and regard $T \in (\mathbb{C}^{n+1})^{\otimes d}$ as a tuple $(g_0, \ldots, g_n)$ of homogeneous polynomials of degree $d-1$. A non-zero vector v is an eigenvector if $(g_0(v), \ldots, g_n(v))$ is a scalar multiple of v. We can express it by a rank condition.

Definition 1. The *eigenscheme* of T is the scheme $E(T) \subseteq \mathbb{P}^n$ defined by

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ g_0 & g_1 & \cdots & g_n \end{pmatrix} \leq 1.$$

Listing all applications of tensor eigenvectors would be long and tedious, so we just mention the most relevant connection to the theme of this workshop: their link with Weddle loci. In addition, as we will soon see, eigenvectors define interesting zero-dimensional subschemes of $\mathbb{P}^n$. Our questions about eigenschemes are often the same we ask about geproci sets, and sometimes the techniques can be similar.

Now that we have a new scheme to play with, we want to know more about its geometry, starting from its dimension and degree.

Theorem 1. (Cartwright-Sturmfels, [5]) Let $d \geq 3$. If $T \in (\mathbb{C}^{n+1})^{\otimes d}$ is general, then $E(T)$ is a set of $D = \frac{(d-1)^{n+1}-1}{d-2}$ reduced points.

In [3] we determine the free resolution of their ideal and compute their Hilbert function. In this presentation I focus on a rather geometrical question: what configurations of points in $\mathbb{P}^n$ are the eigenscheme of a tensor? In [1], the authors prove that every set of d points in $\mathbb{P}^1$ is the eigenscheme of a tensor. Here is the complete characterization in the plane.

Theorem 2. (Beorchia-Galuppi-Venturello, [2]) A set Z of d^2-d+1 points in $\mathbb{P}^2$ is the eigenscheme of a tensor in $(\mathbb{C}^3)^{\otimes d}$ if and only if

(1) $\dim I_Z(d) = 3$,
(2) no $d+1$ points of Z are collinear, and
(3) no kd points of Z lie on a degree k curve.

This has been nicely generalized to $\mathbb{P}^3$ in [4], but the general problem is still open. We can ask a similar question in terms of moduli spaces: what is the variety parametrizing eigenschemes? We are especially interested in symmetric tensors.

Definition 2. Let $(\mathbb{P}^n)^{(D)}$ be the symmetric power of $\mathbb{P}^n$, parametrizing unordered sets of D points. Define

$$\begin{array}{rccc} \phi: & \mathbb{P}(\operatorname{Sym}^d \mathbb{C}^{n+1}) & \dashrightarrow & (\mathbb{P}^n)^{(D)} \\ & T & \mapsto & E(T). \end{array}$$

The first thing we would like to know is the dimension of the image or, equivalently, the dimension of the general fiber. Knowing that the map is generally injective would be great whenever we want to reconstruct a tensor from its eigenvectors. While this is not always the case, we now have an answer.

Theorem 3. If d is odd, then the general fiber of ϕ is a point. If d is even, then the general fiber of ϕ is a $\mathbb{P}^1$.

Once again, Theorem 3 is the result of the efforts of many people. In [1, Theorem 2.7] we find the correct answer for $n = 1$. The authors go on and present the solution for $n = 2$, but unfortunately their statement is correct only in the odd case. The even case has been fixed in [2, Theorem A] and widely generalized to any n in [6], thanks to a powerful vector bundle approach.

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An introduction to geometrically vertex decomposable ideals

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The goal of this talk is to introduce geometrically vertex decomposable ideals. Very roughly, these ideals can be viewed as generalizing the properties of those square-free monomial ideals whose associated simplicial complex via the Stanley-Reisner correspondence is vertex decomposable. Indeed, geometrically vertex decomposable ideals have many of the same properties as this family of square-free monomial ideals

Let k be a field of characteristic zero and fix a variable y of the polynomial ring $R = k[x_1, \ldots, x_n]$. A monomial ordering $<$ on R is said to be *y-compatible* if the initial term of f satisfies $\operatorname{in}_<(f) = \operatorname{in}_<(\operatorname{in}_y(f))$ for all $f \in R$. Here, $\operatorname{in}_y(f)$ is the *initial y-form* of f, that is, if $f = \sum_i \alpha_i y^i$ and $\alpha_d \neq 0$ but $\alpha_t = 0$ for all $t > d$, then $\mathrm{in}_y(f) = \alpha_d y^d$. The ideal generated by all the initial y-forms of an ideal I is denoted $\mathrm{in}_y(I) = \langle \mathrm{in}_y(f) \mid f \in I \rangle$.

Given an ideal $I \subseteq R$ and a y-compatible monomial ordering $<$, let $G(I) = \{g_1, \ldots, g_m\}$ be a Gröbner basis of I with respect to this ordering. For $i = 1, \ldots, m$, write g_i as $g_i = q_i y^{d_i} + r_i$, where y does not divide any term of q_i and $\mathrm{in}_y(g_i) = q_i y^{d_i}$. Given this setup, we define two ideals:

$$C_{y,I} = \langle q_1, \ldots, q_m \rangle \text{ and } N_{y,I} = \langle q_i \mid d_i = 0 \rangle.$$

With the above notation, Knutson, Miller, and Yong [3, Theorem 2.1] say an ideal I has a *geometric vertex decomposition with respect to* y if

$$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle).$$

The geometric vertex decomposition is *degenerate* if $C_{y,I} = \langle 1 \rangle$ or if $\sqrt{C_{y,I}} = \sqrt{N_{y,I}}$, and it is *nondegenerate* otherwise.

Using the notion of a vertex geometric vertex decomposition, Klein and Rajchgot [4] recursively defined geometrically vertex decomposable ideals. Recall that an ideal I is *unmixed* if all of its associated primes have the same height.

Definition 1. ([4, Definition 2.7]) An ideal I of $R = k[x_1, \ldots, x_n]$ is *geometrically vertex decomposable* if I is unmixed and

(1) $I = \langle 1 \rangle$, or I is generated by a (possibly empty) subset of variables of R, or

(2) there is a variable $y = x_i$ in R and a y-compatible monomial ordering $<$ such that I has a geometric vertex decomposition with respect to y, i.e.,

$$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

and the contractions of $C_{y,I}$ and $N_{y,I}$ to the ring $k[x_1, \ldots, \hat{y}, \ldots, x_n]$ are geometrically vertex decomposable.

The ideals $\langle 0 \rangle$ and $\langle 1 \rangle$ in the ring k are also considered geometrically vertex decomposable by convention.

In the case of square-free monomial ideals, we get following characterization:

Theorem 1. ([4, Proposition 2.9]) Let I be a square-free monomial ideal. Then I is geometrically vertex decomposable if and only if the simplicial complex $\Delta(I)$ associated to I via the Stanley-Reisner correspondence is vertex decomposable.

We direct the reader to [5] for the formal definition of a vertex decomposable simplicial complex.

Geometrically vertex decomposable ideals have some properties that may be of interest to those working in commuative algebra and algebraic geometry. Specifically, we have the following facts:

Theorem 2. ([4]) Let I be a geometrically vertex decomposable ideal in the polynomial ring $R = k[x_0, \ldots, x_n]$. Then

(a) I is radical and Cohen-Macaulay, and

(b) I is glicci, i.e., in the Gorenstein liaison class of a complete intersection.

Presently, there are very few known families of geometrically vertex decomposable ideals. Some families of ideals, including determinant ideals, can found in the original paper of Klein and Rajchgot [4]. More recently, M. Cummings, S. Da Silva, J. Rajchgot, and the author [1] proved that the toric ideal of a bipartite graph is always geometrically vertex decomposable.

As a final comment, to encourage future exploration, M. Cummings and the author [2] have created a *Macaulay2* package [2] to check if an ideal is geometrical vertex decomposable.

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